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A RIEMANN-SIEGEL FORMULA FOR THE HURWITZ ZETA FUNCTION

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ABSTRACT. Following H.M. Edwards [3], we compute a Riemann-Siegel formula for the Hurwitz zeta function and hence for Dirichlet series with periodic coefficients. By giving a complete estimate for the error term, we show that our Riemann-Siegel formula for the Hurwitz zeta function represents an asymptotic series.

1. Introduction

The numerical exploration of the Riemann Hypothesis for the Riemann zeta function, has heavily rested on the Riemann-Siegel formula for the Riemann zeta function, see [6]. A Riemann-Siegel formula for Dirichlet L-series has been obtained by C.L. Siegel [7] and M. Deuring [2]. It seems however, that these two authors do not render ready to use results from a computational point of view. Thus for example, in his numerical computations concerning the Extended Riemann Hypothesis, R. Rumely [5] has used an Euler-Maclaurin based computational procedure, instead of the more efficient Riemann-Siegel formula.

In this work, we follow closely the presentation of the Riemann-Siegel formula for the Riemann zeta function, as given in the known treatise [3] by H.M. Edwards, and extend these calculations to the Hurwitz zeta function, and hence to a general Dirichlet series with periodic coefficients. The Riemann-Siegel formula for the Hurwitz zeta function presented in this paper, allows to compute the numerical value of the $\varphi(q)$ distinct Dirichlet L-series modulo q with an order of $\varphi(q)\big(\sqrt{t/2\pi}+1\big)+\varphi^2(q)$ arithmetical operations.

A special case (corresponding to $J_0 = 0$ in Theorem (2.3) below) of the Riemann-Siegel formula proved in this paper, appears already in [4].

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2. The Riemann-Siegel formula

Let $0 < a \leq 1$ be a real number. The Hurwitz zeta function is defined by

$$\zeta(s,a) \;=\; \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad \text{for all} \qquad s=\sigma+it \qquad \text{with} \qquad \sigma>1.$$

The following Theorem (2.1) will give the first, main term of the Riemann-Siegel formula for the Hurwitz zeta function. Hence, it will set the stage for our

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main result of this paper, namely Theorem (2.3) below. Theorem (2.1) is easily derived (see chapter 12 in [1] and section 7.2 in [3]).

THEOREM (2.1). *Let N be a natural number. For all complex number s, let*

$$R_N(s,a) = \frac{\Gamma(1-s)}{2\pi i}\int_{C_N} \frac{e^{-(N+a)z}}{1-e^{-z}}\left(e^{-\pi i}z\right)^s \frac{dz}{z}$$

where C_N is the contour of integration starting at $+\infty$ and going down to $\pi(2N+1)$. Then C_N traverses (in the positive direction) a circular path with center at the origin and radius $\pi(2N+1)$. Finally C_N continues from $\pi(2N+1)$ to infinity. Then we have

$$\begin{aligned}\zeta(s,a) \;=\;& \sum_{k=0}^{N-1}\frac{1}{(k+a)^s}\\ &+\frac{\Gamma(1-s)}{(2\pi)^{1-s}}\Big\{e^{\frac{\pi i}{2}(1-s)}\sum_{k=1}^{N}\frac{e^{-2\pi ika}}{k^{1-s}}+e^{-\frac{\pi i}{2}(1-s)}\sum_{k=1}^{N}\frac{e^{2\pi ika}}{k^{1-s}}\Big\}\\ &+R_N(s,a)\,.\end{aligned}$$

We will apply this theorem with

$$N \;=\; \left[\sqrt{\frac{t}{2\pi}}\,\right] \qquad \text{and} \qquad \sqrt{\frac{t}{2\pi}} \notin \mathbb{Z}\,, \tag{2.2}$$

the second condition in (2.2) being necessary for the existence of the integral in $R_N(s,a)$.

Now we can state the main result of this paper.

THEOREM (2.3). *Let J_0 be a nonnegative integer. Let $s=\sigma+it$ be such that $0\le\sigma\le 1$. Let N be as in (2.2) above. Let $R_N(s,a)$ be as in Theorem (2.1). Then we have, as $t\to\infty$,*

$$R_N(s,a) \;=\; M_3\Big\{\sum_{j=0}^{J_0} w^j P_j(\varrho) + O\big(t^{-\frac{1}{2}(J_0+1)}\big)\Big\}$$

where

$$M_3 = M_0\left(\frac{2\pi}{t}\right)^{\frac{\sigma}{2}} e^{-\frac{3}{4}\pi i+\frac{it}{2}\left\{1+\log\left(\frac{2\pi}{t}\right)\right\}-i\pi B^2+2\pi ip(1-a)},$$

$$w \;=\; \sqrt{\frac{2\pi}{t}}, \qquad B \;=\; N+a-1 \qquad \textit{and} \qquad p \;=\; \sqrt{\frac{t}{2\pi}}-B.$$

The expression M_0 is equal to

$$e^{-\frac{i\sigma(\sigma-1)}{2t}-\frac{(\sigma-1)^2(2\sigma+1)}{12t^2}+\cdots}\left\{1+\frac{1}{12(1-\sigma-it)}+\frac{1}{288(1-\sigma-it)^2}+\cdots\right\}.$$

The quantities $P_j(\varrho)$ are to be computed according to the following rules. Let

$$P_0(z) \;=\; 1.$$

For $k\ge 1$, the polynomial $P_k(z)$ is given recursively by

$$P_k \;=\; \frac{1}{k}\sum_{j=1}^{k} jH_jP_{k-j} \qquad \textit{where} \qquad H_j \;=\; \left(\frac{1-\sigma}{j}+\frac{iz^2}{2\pi(j+2)}\right)\left(\frac{iz}{2\pi}\right)^j.$$

The formal power of the symbol ϱ in $P_j(\varrho)$ is understood to mean

$$\varrho^{\langle j\rangle} = \frac{j!}{2^j}\sum_{\ell=0}^{j}\frac{\Psi^{(\ell)}(p)}{\ell!}\sum_{h=0}^{\left[\frac{j-\ell}{2}\right]}\frac{(2\pi i)^{j-\ell-h}}{(j-\ell-2h)!\,h!}(1-a)^{j-\ell-2h}$$

where the function $\Psi(p)$ is given by

$$\Psi(p) = \frac{e^{\pi i(-\frac{a^2}{2}+a-\frac{5}{8})}}{\cos\pi(a+2p-1)}\cos\left\{\frac{\pi}{2}\Big(a+2p-2+\frac{\sqrt{5}}{2}\Big)\Big(a+2p-2-\frac{\sqrt{5}}{2}\Big)\right\}.$$

The first three terms of the sequence $\{P_k(z)\}$ are

$$\begin{aligned}
P_0(z) &= z^0\,,\\
P_1(z) &= \frac{iz^1}{2\pi}(1-\sigma)-\frac{z^3}{12\pi^2},\\
P_2(z) &= -\frac{z^2}{8\pi^2}(1-\sigma)(2-\sigma)-\frac{iz^4}{96\pi^3}(7-4\sigma)+\frac{z^6}{288\pi^4}\,.
\end{aligned}$$

For the sake of simplicity, now we list the first three terms of the sequence $\{P_k(\varrho)\}$ for the special case $\sigma = 1/2$,

$$\begin{aligned}
P_0(\varrho) &= \Psi(p)\\
P_1(\varrho) &= -\frac{1}{96\pi^2}\,\Psi^{(3)}(p)+\frac{i}{16\pi}(a-1)\,\Psi^{(2)}(p)\\
&\qquad +\frac{1}{8}(a-1)^2\,\Psi^{(1)}(p)-\frac{i\pi}{12}(a-1)^3\,\Psi(p)
\end{aligned}$$

while $P_2(\varrho)$ is equal to

$$\begin{aligned}
&\frac{1}{18432\pi^4}\,\Psi^{(6)}(p)-\frac{i(a-1)}{1536\pi^3}\,\Psi^{(5)}(p)-\frac{5(a-1)^2}{1536\pi^2}\,\Psi^{(4)}(p)+\frac{5i(a-1)^3}{576\pi}\,\Psi^{(3)}(p)\\
&\quad+\left\{\frac{1}{64\pi^2}+\frac{5}{384}(a-1)^4\right\}\Psi^{(2)}(p)-\left\{\frac{i(a-1)}{16\pi}+\frac{i\pi}{96}(a-1)^5\right\}\Psi^{(1)}(p)\\
&\qquad+\left\{\frac{i}{96\pi}-\frac{1}{16}(a-1)^2-\frac{\pi^2}{288}(a-1)^6\right\}\Psi(p)\,.
\end{aligned}$$

This procedure described in Theorem (2.3) can be continued in order to compute as many expressions $P_j(\varrho)$ as desired.

Remark. The quantity M_0 in Theorem (2.3), gives the first higher order terms in the Stirling formula for $\Gamma(1-s)$, see formula (3.4) below. These higher order terms are needed when using the Riemann-Siegel formula with the first three terms $P_0(\varrho)$, $P_1(\varrho)$ and $P_2(\varrho)$ in order to approximate $R_N(s,a)$. If the Riemann-Siegel formula is used with more than three terms for the approximation of $R_N(s,a)$, then M_0 has to include still higher order terms in the Stirling formula for $\Gamma(1-s)$.

3. Proof of Theorem (2.3)

Here we begin our study of the error term $R_N(s,a)$ in Theorem (2.1). For this purpose we have to consider the integral in

$$R_N(s,a) \;=\; \frac{\Gamma(1-s)}{2\pi i}\, e^{-\pi i s} \int_{C_N} \frac{e^{-(N+a-1)z}}{e^z-1}\, z^{s-1}\, dz \tag{3.1}$$

with $s=\sigma+it$. The integral in (3.1) is taken along the contour C_N of Theorem (2.1), where N is as in (2.2). Now, C_N crosses the imaginary axis at $\pi i(2N+1)$. We will consider the integral which results from taking (3.1) along a line segment L_1, making an angle of $\pi/4$ with respect to the real axis and crossing the imaginary axis at

$$\xi \;:=\; i\sqrt{2\pi t}\,. \tag{3.2}$$

Notice that $2\pi N < |\xi| < 2\pi(N+1)$, so that no crossing of poles has occurred.

The parameter $\gamma \in (0,1/6)$ in Lemma (3.3) below will remain free but constant. The choice of the value of γ will depend on the number of terms to be considered in the asymptotic formula for $R_N(s,a)$. For example, if three terms are required to approximate $R_N(s,a)$, then γ should be taken to be $1/24$.

Lemma (3.3). *Let ξ be as in (3.2). Let $s=\sigma+it$. Assume that $0\le\sigma\le 1$. Let $0<\gamma<1/6$. Let $\Upsilon = e^{\frac14 i\pi} t^{\gamma}$. Let L_1 be the line segment*

$$-L_1 \;:\; \begin{cases} z=\xi+y\Upsilon, \\ -1\le y\le 1\,. \end{cases}$$

Then, as $t\to\infty$, we have

$$R_N(s,a) \;=\; \frac{\Gamma(1-s)}{2\pi i}\, e^{-\pi i s} \int_{L_1} \frac{e^{-(N+a-1)z}}{e^z-1}\, z^{s-1}\, dz + O\Big(e^{-\frac{1}{5\pi}t^{2\gamma}}\Big).$$

The implied constant is independent of a.

Proof. From the known formula, with $s=\sigma+it$ and $t\to\infty$,

$$\Gamma(1-s) = t^{\frac12-\sigma-it}\, e^{-\frac{\pi t}{2}+it-\frac{i\pi}{2}(\frac12-\sigma)}\,\sqrt{2\pi}\,\Big\{1+O\big(\frac{1}{t}\big)\Big\} \tag{3.4}$$

(see section 4.12 of Titchmarsh [8]) we notice that

$$\Big|\frac{\Gamma(1-s)}{2\pi i}\, e^{-\pi i s}\Big| \;\ll\; t^{\frac12}\, e^{\frac12 \pi t}\,. \tag{3.5}$$

Let $\theta\in(0,\pi/2)$ be such that

$$\tan\theta \;=\; 2\sqrt{\pi}\, t^{\frac12-\gamma}+1\,.$$

We consider the three line segments

$$-L_0 \quad : \quad \begin{cases} z = \xi + \Upsilon + xe^{i\theta}, \\ 0 \le x < \infty. \end{cases}$$

$$L_2 \quad : \quad \begin{cases} z = \big(\xi - \Upsilon\big)(1-y) + y\big(-\Re\mathrm{e}\,\Upsilon - \pi i(2N+1)\big), \\ 0 \le y \le 1. \end{cases}$$

$$L_3 \quad : \quad \begin{cases} z = x - \Re\mathrm{e}\Upsilon - \pi i(2N+1), \\ 0 \le x < \infty\,. \end{cases}$$

Let $L^* = L_0 \cup L_1 \cup L_2 \cup L_3$. Then the integral of the lemma can be taken along L^* instead of C_N. It is known that $|1/(e^z-1)|$ is bounded in the region $S(1)$ which remains when we remove from the complex plane the interior of circles of radius 1 with centers at $s = 2k\pi i$, with $k \in \mathbb{Z}$. The three line segments L_0, L_2 and L_3 lie in this region $S(1)$. We will estimate the integral along L_0 first.

For $z \in L_0$ we note that $\arg z = \theta$. Hence we have

$$(3.6) \qquad \begin{aligned} |z^{s-1}| &= \Big|\exp\Big\{\big(\sigma - 1 + it\big)(\log|z| + i\arg z)\Big\}\Big| \\ &\ll \exp\big\{-t\arg z\big\} \;=\; e^{-t\theta}\,. \end{aligned}$$

On the other hand (for $z \in L_0$) we have

$$(3.7) \qquad \begin{aligned} \big|e^{-(N+a-1)z}\big| &\le \exp\Big\{-\Big(\sqrt{\frac{t}{2\pi}} - 2\Big)\Big(\frac{t^\gamma}{\sqrt{2}} + x\cos\theta\Big)\Big\} \\ &\ll \exp\Big\{-\frac{t^{\frac12+\gamma}}{2\sqrt{\pi}} + \sqrt{2}t^\gamma\Big\}\cdot e^{-x\cos\theta}\,. \end{aligned}$$

From (3.5), (3.6) and (3.7) we see that the contribution to $R_N(s,a)$ coming from L_0 is

$$\ll \frac{1}{\cos\theta}\,\exp\Big\{t\big(\frac{\pi}{2} - \theta\big) - \frac{t^{\frac12+\gamma}}{2\sqrt{\pi}} + \sqrt{2}t^\gamma\Big\} \;\ll\; e^{-\frac{1}{5\pi}t^{2\gamma}}, \qquad \text{say}\,,$$

because (with $x = 2\sqrt{\pi}\,t^{\frac12-\gamma} + 1$)

$$\begin{aligned} \frac{\pi}{2} - \theta &= \frac{1}{x} - \frac{1}{3}\frac{1}{x^3} + O\Big(\frac{1}{x^5}\Big) \\ &= \frac{t^{-\frac12+\gamma}}{2\sqrt{\pi}} - \frac{t^{-1+2\gamma}}{4\pi} + O\big(t^{-3(\frac12-\gamma)}\big). \end{aligned}$$

Now we consider the integral over L_2. For $z \in L_2$ we have

$$|e^{-(N+a-1)z}\,z^{s-1}| \;\ll\; \exp\Big\{\frac{t^{\frac12+\gamma}}{2\sqrt{\pi}} - t\arg z\Big\}.$$

On the other hand, for $z \in L_2$, we have

$$\begin{aligned} \arg z \quad &\ge \quad \frac{\pi}{2} + \arctan\frac{t^\gamma}{2\sqrt{\pi t} - t^\gamma} \\ &\ge \quad \frac{\pi}{2} + \arctan\Big\{\frac{t^{\gamma-\frac12}}{2\sqrt{\pi}} + \frac{t^{2\gamma-1}}{4\pi} + \cdots\Big\}\,. \end{aligned}$$

Therefore

$$-t\arg z \ll -\frac{\pi t}{2} - \frac{t^{\frac{1}{2}+\gamma}}{2\sqrt{\pi}} - \frac{t^{2\gamma}}{4\pi}.$$

Taking into account estimate (3.5), we see that the contribution to the quantity $R_N(s,a)$ coming from L_2 is

$$\ll t\exp\Big\{-\frac{t^{2\gamma}}{4\pi}\Big\}.$$

For $z \in L_3$, it is better to consider the contribution of the denominator $e^z - 1$. Assume first that $\Re\mathrm{e}\, z \geq \log 2$. Then we have

$$\Big|\frac{e^{(1-a)z}}{e^z-1}\Big| \leq \frac{e^{-a\,\Re\mathrm{e}\, z}}{1-e^{-\Re\mathrm{e}\, z}} \leq 2^{1-a}.$$

If $\Re\mathrm{e}\, z < \log 2$ then we have

$$|e^{(1-a)z}| = e^{(1-a)\Re\mathrm{e}\, z} \leq 2^{1-a}.$$

On the other hand, if $z \in L_3$, then we have $\arg z > \pi$, and therefore

$$\begin{aligned}|e^{-Nz}\, z^{s-1}| &\leq e^{-t\arg z}\exp\Big\{-\Big(\sqrt{\frac{t}{2\pi}}-1\Big)\big(x-\Re\mathrm{e}\,\Upsilon\big)\Big\}\\ &\ll e^{-\pi t}\exp\Big\{-x+\frac{t^{\frac{1}{2}+\gamma}}{2\sqrt{\pi}}\Big\}.\end{aligned}$$

Taking into account estimate (3.5), we see that the contribution to the quantity $R_N(s,a)$ coming from L_3 is

$$\ll \sqrt{t}\exp\Big\{-\frac{\pi}{2}t+\frac{t^{\frac{1}{2}+\gamma}}{2\sqrt{\pi}}\Big\}.$$

This finishes the proof of the lemma. □

Now we want to study the integral in Lemma (3.3). The numerator inside the integral is equal to

$$\exp\big\{-(N+a-1)z+(s-1)\log z\big\}.$$

With N as in (2.2) above, we now let (as in the statement of Theorem (2.3))

$$A = \sigma - 1 + it, \qquad B = N+a-1 \quad \text{and} \quad p = \sqrt{\frac{t}{2\pi}} - B. \tag{3.8}$$

Recall $\xi = i\sqrt{2\pi t}$. For $z \in L_1$ we have $|z-\xi| < |\xi|/2$, and hence

$$\begin{aligned}A\log(z) - Bz &= A\log\big((z-\xi)+\xi\big) - B(z-\xi) - B\xi\\ &= A\log\xi + A\log\big(1+\frac{z-\xi}{\xi}\big) - B(z-\xi) - B\xi\\ &= A\log\xi - B\xi + \big(\frac{A}{\xi}-B\big)(z-\xi) - \frac{A}{2\xi^2}(z-\xi)^2 + \Theta(\frac{z-\xi}{\xi})\end{aligned}$$

where

$$\Theta(z) = -A\sum_{j=3}^{\infty}\frac{(-1)^j}{j}z^j.$$

Thus, we are interested in the following expression

$$\frac{M_1}{2\pi i}\cdot\int_{L_1}\frac{1}{e^z-1}\,\exp\Big\{\big(\frac{A}{\xi}-B\big)(z-\xi)-\frac{A}{2\xi^2}(z-\xi)^2+\Theta\big(\frac{z-\xi}{\xi}\big)\Big\}\,dz$$

where

$$M_1 \;=\; \Gamma(1-s)\,e^{-\pi is}\,\xi^A\,e^{-B\xi}.$$

Notice from (3.8) that

$$\frac{A}{2\xi^2}\;=\;\frac{1-\sigma}{4\pi t}-\frac{i}{4\pi}\qquad\text{and}\qquad\frac{A}{\xi}-B\;=\;\frac{i(1-\sigma)}{\sqrt{2\pi t}}+p.$$

Let (as in the statement of Theorem (2.3))

$$w\;=\;\sqrt{\frac{2\pi}{t}}\qquad\text{so that}\qquad\xi\;=\;\frac{2\pi i}{w}\,. \tag{3.9}$$

An easy calculation shows that

$$\big(\frac{A}{\xi}-B\big)z-\frac{A}{2\xi^2}z^2+\Theta\big(\frac{z}{\xi}\big)\;=\;pz+\frac{iz^2}{4\pi}+\sum_{j=1}^{\infty}\Big(\frac{1-\sigma}{j}+\frac{iz^2}{2\pi(j+2)}\Big)\Big(\frac{izw}{2\pi}\Big)^j.$$

In the next lemma we consider the expression

$$\exp\Big\{\sum_{j=1}^{\infty}\Big(\frac{1-\sigma}{j}+\frac{iz^2}{2\pi(j+2)}\Big)\Big(\frac{izw}{2\pi}\Big)^j\Big\}.$$

Lemma (3.10). *Let z be such that $z+\xi\in L_1$ so that*

$$\max\big\{|z|:z+\xi\in L_1\big\}=\frac{t^\gamma}{\sqrt{2}}$$

and $0<\gamma<1/6$. Let w be as in (3.9). For $a,b\in\mathbb{N}\cup\{\infty\}$ we write

$$\mathrm{X}_a^b\;=\;\sum_{j=a}^{b}H_jw^j\qquad\textit{where}\qquad H_j\;=\;\Big(\frac{1-\sigma}{j}+\frac{iz^2}{2\pi(j+2)}\Big)\Big(\frac{iz}{2\pi}\Big)^j.$$

For $J\in\mathbb{N}\cup\{\infty\}$ let

$$\mathrm{E}_J(x)\;=\;\sum_{r=0}^{J}\frac{x^r}{r!}.$$

Then we have

$$\mathrm{E}_\infty\big(\mathrm{X}_1^\infty\big)\;=\;\mathrm{E}_J\big(\mathrm{X}_1^J\big)+O\big(t^{(3\gamma-\frac12)(J+1)}\big).$$

Moreover, there exist polynomials $Q_j(z)$ of the form

$$Q_j(z)\;=\;z^jq_j(z)\qquad\textit{with}\qquad\deg\big(q_j\big)\;\le\;2j \tag{3.11}$$

such that

$$\mathrm{E}_J\big(\mathrm{X}_1^J\big)\;=\;\sum_{j=0}^{J^2}Q_j(z)w^j\,. \tag{3.12}$$

Finally, for $0 \le j \le J$ *we have* $Q_j = P_j$ *where the polynomials* $P_j(z)$ *are given recursively by*

$$P_j \;=\; \frac{1}{j}\sum_{\ell=1}^{j} \ell\, H_\ell P_{j-\ell} \qquad \textit{for} \qquad j \ge 1 \qquad \textit{and} \qquad P_0 = 1.$$

Proof. Let $L \in \mathbb{N}$. On the one hand, we have

$$\mathrm{E}_\infty\big(\overset{\infty}{\underset{1}{\mathrm{X}}}\big) - \mathrm{E}_\infty\big(\overset{L}{\underset{1}{\mathrm{X}}}\big) \;=\; \mathrm{E}_\infty\big(\overset{L}{\underset{1}{\mathrm{X}}}\big)\Big\{\mathrm{E}_\infty\big(\overset{\infty}{\underset{L+1}{\mathrm{X}}}\big) - 1\Big\}$$

$$\ll\; |z|^2\Big(|zw|\Big)^{L+1} \;\ll\; t^{(3\gamma-\frac{1}{2})(L+1)}$$

because $|z| \ll t^\gamma$ and $\gamma < 1/6$. On the other hand,

$$\mathrm{E}_\infty\big(\overset{L}{\underset{1}{\mathrm{X}}}\big) - \mathrm{E}_J\big(\overset{L}{\underset{1}{\mathrm{X}}}\big) \;\ll\; \Big(|z|^3\,|w|\Big)^{J+1} \;\ll\; t^{(3\gamma-\frac{1}{2})(J+1)}.$$

Putting $L = J$ we obtain the first claim of the lemma. It is clear that there exist polynomials $Q_j(z)$ such that equation (3.12) is true. Assertion (3.11) follows by considering expressions of the form $w^a z^b$ in

$$\Big\{\sum_{j=1}^{J}\Big(\frac{1-\sigma}{j} + \frac{iz^2}{2\pi(j+2)}\Big)\Big(\frac{izw}{2\pi}\Big)^j\Big\}^r$$

and noticing that $b = a + 2\kappa$ for some $\kappa \in \mathbb{N} \cup \big\{0\big\}$. That for $0 \le j \le J$ we have $Q_j = P_j$ follows from fact that

$$\mathrm{E}_\infty\big(\overset{\infty}{\underset{1}{\mathrm{X}}}\big) \;=\; \sum_{j=0}^{\infty} P_j(z)\,w^j.$$

$\square$

With $Q_j(z)$ as in Lemma (3.10), now we consider the expression

$$\frac{M_1}{2\pi i}\cdot\int_{L_1}\frac{1}{e^z-1}\,\exp\Big\{\frac{i}{4\pi}\,(z-\xi)^2 + p(z-\xi)\Big\}\,Q_j(z-\xi)\,dz\,. \tag{3.13}$$

Let $z = u + 2\pi i B$. Then we have

$$z - \xi \;=\; u - 2\pi i\Big\{\sqrt{\frac{t}{2\pi}} - B\Big\} \;=\; u - 2\pi i p\,.$$

Therefore

$$\frac{i}{4\pi}(z-\xi)^2 \;=\; \frac{iu^2}{4\pi} + pu \quad \overbrace{-\,\pi i p^2}^{\text{out of integral}}$$

$$p(z-\xi) \;=\; pu \quad \underbrace{-\,2\pi i p^2}_{\text{out of integral}}.$$

Let $\widetilde{L}_1$ be the path given by $\widetilde{L}_1 = \big\{z - 2\pi i B : z \in L_1\big\}$. Then expression (3.13) can be written as

$$\frac{M_2}{2\pi i}\cdot\int_{\widetilde{L}_1}\frac{1}{e^{2\pi i a}\,e^u - 1}\,\exp\Big\{\frac{iu^2}{4\pi} + 2pu\Big\}\,Q_j(u - 2\pi i p)\,du \tag{3.14}$$

where now the multiplier M_2 is given by

$$M_2 \;=\; \Gamma(1-s)\,\xi^A\,\exp\big\{-B\xi - \pi i s - 3\pi i p^2\big\}\,.$$

Now we want to extend the line segment $\widetilde{L}_1$ to a line going to infinity in both directions. Let $\Upsilon = e^{\frac{1}{4}i\pi}\,t^{\gamma}$ be as in Lemma (3.3). We notice that

$$\int_1^{\infty} \exp\Big\{\frac{i}{4\pi}\big(\Upsilon y\big)^2\Big\}\,y^{3J^2}\,dy \;\ll\; \exp\Big\{-\frac{1}{5\pi}\,t^{2\gamma}\Big\}\,. \tag{3.15}$$

Hence, expression (3.14) is

$$\frac{M_2}{2\pi i}\Big\{\int_L \frac{e^{\frac{iu^2}{4\pi}+2pu}}{e^{2\pi i a}\,e^u - 1}\,Q_j(u-2\pi i p)\,du + O\big(e^{-\frac{1}{5\pi}t^{2\gamma}}\big)\Big\}\,. \tag{3.16}$$

The path of integration in (3.14) has been changed from $\widetilde{L}_1$ to an infinite straight line L, making an angle of $\pi/4$ with the real axis and crossing the imaginary axis from right to left at $2\pi i p$.

LEMMA (3.17). *Let $0 < a \le 1$. Let β be a real number such that*

$$\frac{\beta}{2} + a - 1 \;<\; 1 \;<\; \frac{\beta}{2} + a.$$

Let

$$\Phi_a(\beta) \;=\; \frac{1}{2\pi i}\int_L \frac{1}{e^{2\pi i a}\,e^u - 1}\,\exp\Big\{\frac{iu^2}{4\pi} + \beta u\Big\}\,du$$

where L is the straight line making an angle of $\pi/4$ with the real axis and crossing the imaginary axis at $\pi i\beta$. Then we have

$$\Phi_a(\beta) \;=\; \frac{e^{\pi i(\frac{\beta^2}{2}+\beta(1-a)-\frac{a^2}{2}+a-\frac{5}{8})}}{\cos\pi(a+\beta-1)}\,\cos\frac{\pi}{2}\Big(a+\beta-2+\frac{\sqrt5}{2}\Big)\Big(a+\beta-2-\frac{\sqrt5}{2}\Big).$$

Proof. Let $\lambda = e^{2\pi i a}$. Let us consider the difference

$$\begin{aligned}
\lambda\Phi_a(\beta+1) - \Phi_a(\beta) \;&=\; \frac{1}{2\pi i}\int_L \frac{e^{\frac{iu^2}{4\pi}}}{\lambda e^u - 1}\,\big(\lambda e^{(\beta+1)u} - e^{\beta u}\big)\,du \\
&=\; \frac{1}{2\pi i}\int_L \exp\Big\{\frac{iu^2}{4\pi} + \beta u\Big\}\,du \\
&=\; e^{i\pi\beta^2}\,\frac{1}{2\pi i}\int_L \exp\Big\{\frac{i(u-2\pi i\beta)^2}{4\pi}\Big\}\,du\,.
\end{aligned}$$

Now we change the line L to a parallel line such that $u - 2\pi i\beta$ passes through the origin. Thus, we obtain

$$\begin{aligned}
\lambda\Phi_a(\beta+1) - \Phi_a(\beta) \;&=\; -e^{i\pi\beta^2}\,\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \exp\Big\{\frac{i}{4\pi}\big(te^{\pi i/4}\big)^2\Big\}\,e^{\pi i/4}\,dt \\
&=\; \exp\Big\{\pi i\big(\beta^2 + \frac{3}{4}\big)\Big\}\,.
\end{aligned}$$

Let L^* the line parallel to L and crossing the imaginary axis 2π below L. Then

$$\frac{1}{2\pi i}\Big\{\int_L - \int_{L^*}\Big\} \frac{e^{\frac{iu^2}{4\pi}+\beta u}}{\lambda e^u - 1}\, du \;=\; -\exp\Big\{2\pi i\big(a - \frac{a^2}{2} + \beta(1-a)\big)\Big\}.$$

On the other hand

$$\begin{aligned}\frac{1}{2\pi i}\int_{L^*} \frac{e^{\frac{iu^2}{4\pi}+\beta u}}{\lambda e^u - 1}\, du &= \frac{1}{2\pi i}\int_L \frac{1}{\lambda e^u - 1}\, \exp\Big\{\frac{i}{4\pi}\big(u - 2\pi i\big)^2 + \beta(u - 2\pi i)\Big\}\, du \\ &= -e^{-2\pi i\beta}\frac{1}{2\pi i}\int_L \frac{e^{\frac{iu^2}{4\pi}+u(\beta+1)}}{\lambda e^u - 1}\, du \\ &= -e^{-2\pi i\beta}\, \Phi_a(\beta+1)\,.\end{aligned}$$

Hence we have

$$\begin{aligned} e^{2\pi i a}\, \Phi_a(\beta+1) - \Phi_a(\beta) &= e^{\pi i(\beta^2 + \frac{3}{4})} \\ e^{-2\pi i\beta}\, \Phi_a(\beta+1) + \Phi_a(\beta) &= -e^{2\pi i(a - \frac{a^2}{2} + \beta(1-a))}\,. \end{aligned}$$

Adding both equations we obtain

$$\begin{aligned}\Phi_a(\beta+1) \;&=\; \frac{e^{2\pi i(\frac{\beta^2}{2}+\frac{3}{8})} + e^{2\pi i(a - \frac{a^2}{2} + \beta(1-a) + \frac{1}{2})}}{e^{2\pi i a} + e^{-2\pi i\beta}} \\ &= \frac{e^{\pi i(\frac{\beta^2}{2} + a - \frac{a^2}{2} + \beta(1-a) + \frac{7}{8})}\, \cos\pi\big(\frac{\beta^2}{2} - a + \frac{a^2}{2} - \beta(1-a) - \frac{1}{8}\big)}{e^{\pi i(a-\beta)}\, \cos\pi(a+\beta)} \\ &= e^{\pi i(\frac{\beta^2}{2} - \frac{a^2}{2} + \beta(2-a) + \frac{7}{8})}\, \frac{\cos\frac{\pi}{2}\big(a+\beta-1+\frac{\sqrt{5}}{2}\big)\big(a+\beta-1-\frac{\sqrt{5}}{2}\big)}{\cos\pi(a+\beta)}\,.\end{aligned}$$

The lemma follows from making the substitution $\beta \mapsto \beta - 1$. $\square$

Since

$$\sqrt{2\pi t}\, B \;=\; 2\pi(p+B)B \;=\; \pi(B^2 - p^2) + \frac{t}{2}$$

then we see that the second of the following two relations holds,

$$\begin{aligned}\xi^A \;&=\; \frac{e^{\frac{\pi i}{2}(\sigma-1)}}{\big(2\pi t\big)^{\frac{1}{2}(1-\sigma)}}\; e^{-\frac{\pi t}{2} + \frac{it}{2}\log(2\pi t)} \\ e^{-B\xi - \pi i s} \;&=\; e^{-\pi i\sigma + \pi t - \pi i(B^2-p^2) - \frac{it}{2}}\,.\end{aligned}$$

These, together with (3.4), imply

$$M_2 \;=\; \Big(\frac{2\pi}{t}\Big)^{\frac{\sigma}{2}}\, e^{-\frac{3}{4}\pi i + \frac{it}{2}\left\{1+\log(\frac{2\pi}{t})\right\} - i\pi B^2 - 2\pi i p^2}\Big\{1 + O\big(\frac{1}{t}\big)\Big\}\,.$$

Let us write

$$\varrho^{\langle j\rangle} \;=\; e^{-2\pi i(p^2+p(1-a))}\,\frac{1}{2\pi i}\int_L \frac{e^{\frac{iu^2}{4\pi}+2pu}}{e^{2\pi i a}e^u-1}\,\bigl(u-2\pi i p\bigr)^j\,du\,.$$

If we agree to understand by $\varrho^{\langle j\rangle}$ the formal power of the symbol ϱ in the polynomial $Q_j(\varrho)$ then expression (3.16) can be written as

$$M_3\Bigl\{Q_j(\varrho)+O\bigl(e^{-\frac{1}{5\pi}t^{2\gamma}}\bigr)\Bigr\} \tag{3.18}$$

where $M_3 = M_2\exp\bigl\{2\pi i\bigl(p^2+p(1-a)\bigr)\bigr\}$, so that

$$M_3 \;=\; \Bigl(\frac{2\pi}{t}\Bigr)^{\frac{\sigma}{2}}\,e^{-\frac{3}{4}\pi i+\frac{it}{2}\left\{1+\log(\frac{2\pi}{t})\right\}-i\pi B^2+2\pi i p(1-a)}\Bigl\{1+O\bigl(\frac{1}{t}\bigr)\Bigr\}\,.$$

THEOREM (3.19). *Let $\Phi_a(\beta)$ be as in Lemma (3.17). Let*

$$\Psi(p) \;=\; e^{-2\pi i(p^2+p(1-a))}\;\Phi_a(2p)\,.$$

Then the quantities $\varrho^{\langle j\rangle}$ can be expressed in terms of $\Psi(p)$ and its derivatives $\Psi^{(j)}(p)$. In fact, we have

$$\varrho^{\langle j\rangle} \;=\; \frac{j!}{2^j}\sum_{\ell=0}^{j}\frac{\Psi^{(\ell)}(p)}{\ell!}\sum_{h=0}^{\left[\frac{j-\ell}{2}\right]}\frac{(2\pi i)^{j-\ell-h}}{(j-\ell-2h)!\,h!}(1-a)^{j-\ell-2h}\,.$$

Proof. Let $\tau = 1-a$. From Lemma (3.17), we see that

$$\Psi(p) \;=\; e^{-2\pi i(p^2+\tau p)}\,\frac{1}{2\pi i}\int_L\frac{e^{\frac{iu^2}{4\pi}+2pu}}{e^{2\pi i a}\,e^u-1}\,du\,.$$

Notice that $\varrho^{\langle 0\rangle} = \Psi(p)$. Then we have

$$\Psi(p+y) \;=\; e^{-2\pi i\left(p^2+y^2+(p+y)\tau\right)}\,\frac{1}{2\pi i}\int_L\frac{e^{\frac{iu^2}{4\pi}+2pu}}{e^{2\pi i a}\,e^u-1}\,e^{2y(u-2\pi i p)}\,du\,.$$

Therefore,

$$e^{2\pi i(y^2+\tau y)}\,\Psi(p+y) \;=\; \sum_{j=0}^{\infty}\frac{(2y)^j}{j!}\,\varrho^{\langle j\rangle}\,. \tag{3.20}$$

Now we write the left hand side of this equation as power series in y,

$$\begin{aligned}
e^{2\pi i(y^2+\tau y)} \;&=\; \sum_{\ell=0}^{\infty}\frac{(2\pi i y)^\ell}{\ell!}\sum_{h=0}^{\ell}\binom{\ell}{h}\,y^h\tau^{\ell-h}\\
&=\; \sum_{h=0}^{\infty}\sum_{k=2h}^{\infty}\frac{(2\pi i)^{k-h}}{(k-h)!}\binom{k-h}{h}\,y^k\tau^{k-2h}\\
&=\; \sum_{k=0}^{\infty}y^k\,Q_k
\end{aligned}$$

where we have set

$$Q_k \;=\; \sum_{h=0}^{[k/2]}q(k,h) \qquad\text{and}\qquad q(k,h) \;=\; \frac{(2\pi i)^{k-h}}{(k-2h)!\,h!}\,\tau^{k-2h}\,.$$

From the Taylor series for $\Psi(p+y)$ we see that

$$e^{2\pi i(y^2+\tau y)}\,\Psi(p+y) \;=\; \sum_{k=0}^{\infty} y^k \sum_{j=0}^{k} \frac{\Psi^{(j)}}{j!}\,Q_{k-j}\,. \tag{3.21}$$

The lemma follows from (3.20) and (3.21). □

It is easy to verify that, for all $0 \le j \le J^2$,

$$\int_L \frac{e^{\frac{iu^2}{4\pi}+2pu}}{e^{2\pi i a+u}-1}\,Q_j(u-2\pi i p)\,du \;\ll\; 1\,.$$

This estimate is uniform in p and a because the contour L can always be deformed in order to avoid poles of the integrand.

Since, for $0 \le j \le J$ we have $Q_j = P_j$, then we can sum expression (3.18) and write $R_N(s,a)$ as

$$M_3\Big\{\sum_{j=1}^{J} P_j(\varrho)\,w^j + \underbrace{\sum_{j=J+1}^{J^2} Q_j(\varrho)\,w^j}_{\ll\; t^{-(J+1)/2}} + O\big(t^{(3\gamma-\frac{1}{2})(J+1)}\big) + O\big(e^{-\frac{1}{5\pi}t^{2\gamma}}\big)\Big\}\,.$$

Theorem (2.3) follows from this estimate by choosing $J = J_0+1$ and γ such that

$$\frac{1}{2}(J_0+1) \;\le\; \big(\frac{1}{2}-3\gamma\big)(J+1) \qquad \text{i.e.,} \qquad \gamma \;\le\; \frac{1}{6}\big(J_0+2\big)^{-1}\,.$$

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LIFTING POLYNOMIALS OVER A LOCAL FIELD

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ABSTRACT. Lifting polynomials over a local field K have been introduced in [10] in order to study the structure of irreducible polynomials in one variable over K. We investigate the distances between the roots of a lifting polynomial, and provide sufficient conditions under which two lifting polynomials define the same field extension.

1. Introduction

The problem of describing the structure of irreducible polynomials in one variable over a local field K has been studied in [10]. In the process, the notion of a saturated distinguished chain of polynomials over K was defined in [10], and later studied also in [3], [8] and [9]. Knowing a saturated distinguished chain for a given element $\alpha \in \bar{K}$, where $\bar{K}$ denotes a fixed algebraic closure of K, can be helpful in various problems. One reason is that we can use such a chain to construct an integral basis of $K(\alpha)$ over K, following the procedure explained in [10], Remark 4.7. The shape of such a basis may be useful in practice, for instance it has been used in [7] in order to show that the Ax-Sen constant vanishes for deeply ramified extensions (in the sense of Coates-Greenberg [5]). A constructive way to produce all the irreducible polynomials in one variable over K is described in [10], via a repeated operation of "lifting". As we shall see below, from the same given data, the actual lifting process produces infinitely many lifting polynomials. A natural problem that arises is to try to classify these polynomials, and in particular to investigate which lifting polynomials produce the same field extension of K. This problem has been investigated in [11], in the particular case when the lifting procedure is applied with respect to an unramified irreducible polynomial f over K. In the present paper we consider the case when the lifting is done with respect to a general irreducible polynomial f over K. Section 2 below contains some general notions, definitions and results. In Section 3 we estimate the distances between the roots of a given lifting polynomial. In the last section we consider two lifting polynomials obtained from the same initial data, and we provide sufficient conditions under which these polynomials define the same field extension.

2. Notations, definitions and general results

In what follows K will be a field of characteristic zero, complete with respect to a rank one and discrete valuation v (see [4], [6], [12]). Let $\bar{K}$ be a fixed algebraic closure of K and denote also by v the unique extension of v to $\bar{K}$. If $K \subseteq L \subseteq \bar{K}$ is an intermediate field, denote $G(L) = \{v(x) \colon x \in L\}$. As usual,

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$G(K)$ will be identified with the ordered group $\mathbf{Z}$ of rational integers, and for every intermediate field L, $G(L)$ will be viewed as a subgroup of the additive group $\mathbf{Q}$. Denote $A(L) = \{x \in L\colon v(x) \geq 0\}$, the ring of integers of L. Let $M(L) = \{x \in L\colon v(x) > 0\}$, and denote by π_L a uniformizing element of L. Let $R(L) = A(L)/M(L)$, the residue field of L. If $x \in A(L)$, denote by x^* the canonical image of x in $R(L)$. As usual, $R(L)$ will be viewed canonically as a subfield of $R(\bar{K})$. Moreover, $R(\bar{K})$ is an algebraic closure of $R(K)$. In the following we assume that $R(K)$ is a perfect field.

Let $K \subseteq L_1 \subseteq L_2 \subseteq \bar{K}$ be intermediate fields such that L_2 is a finite extension of K. Then $R(L_2)/R(L_1)$ is a finite extension, and the number $f(L_2/L_1) = [R(L_2)\colon R(L_1)]$ is called the inertial degree of L_2 relative to L_1. The quotient group $G(L_2)/G(L_1)$ is finite. Its index, denoted by $e(L_2/L_1)$, is called the ramification index of L_2 relative to L_1. It is well known (see [4], Ch. IV) that $f(L_2/L_1)e(L_2/L_1) = [L_2\colon L_1]$.

If $K \subseteq L \subseteq \bar{K}$ and $a \in \bar{K}$, then the degree $[L(a)\colon L]$ of a relative to L will be denoted by $\deg_L a$, or simply by $\deg a$ when $L = K$.

If $f \in A(K)[X]$, $f = a_0X^n + a_1X^{n-1} + \cdots + a_n$, we denote

$$\bar{f} = a_0^*X^n + a_1^*X^{n-1} + \cdots + a_n^* \in R(K)[X],$$

the canonical image of f in $R(K)[X]$.

If $a \in \bar{K}$ and $\delta \in \mathbf{Q}$, we define, for any $F(X) = c_0+c_1(X-a)+\cdots+c_n(X-a)^n \in \bar{K}[X]$,

$$w(F) := \inf_{0\leq i\leq n}\{v(c_i) + i\delta\}.$$

In this way one obtains a valuation w on $\bar{K}[X]$, which extends canonically to a valuation on $\bar{K}(X)$, and which is a residual transcendental (r.t. for short) extension of $(\bar{K}, v)$, in the sense that the residual field of w is transcendental over $R(\bar{K})$. This valuation w is called the r.t. extension of $(\bar{K}, v)$ defined by inf, a and δ.

An element $(a, \delta) \in \bar{K} \times \mathbf{Q}$ is said to be minimal with respect to K if for every $b \in \bar{K}$ the condition $v(a - b) \geq \delta$ implies $[K(a)\colon K] \leq [K(b)\colon K]$.

If $a \in \bar{K} \setminus K$, we denote $\omega(a) = \sup\{v(a - a')\}$ where a' runs over the set of conjugates of a over K, $a' \neq a$. Krasner's Lemma (see [4], p. 66) shows that for any $a \in \bar{K} \setminus K$ and any $\delta > \omega(a)$, the pair (a, δ) is a minimal pair.

Let (a, δ) be a minimal pair, and let f be the monic polynomial of a over K. Let $a_1 = a, a_2, \ldots, a_n$ be all the roots of f, and let us put

$$\gamma = \sum_{i=1}^{n} \min\{v(a - a_i), \delta\}.$$

If $F \in K[X]$, we write F in the form

$$F = F_0 + F_1 f + \cdots + F_t f^t, \quad \deg F_i < \deg f,\ \ i = 0, 1, \ldots, t.$$

Then we define

$$w(F) = \inf_{0\leq i\leq t}(v(F_i(a)) + i\gamma).$$

In [2] it is proved the following result.

THEOREM (2.1). *Let (a, δ) be a minimal pair with respect to K. Then w is a valuation on $K[X]$, and canonically on $K(X)$, which coincides with the restriction of the valuation on $\bar{K}(X)$ defined by inf, a and δ. Moreover one has:*

(i) *The value group of w is canonically isomorphic to $G(K(a)) + \mathbf{Z}\gamma$.*

(ii) *Let e be the smallest non-zero positive integer such that $e\gamma \in G(K(a))$. Let $h \in K[X]$, $\deg h < \deg f$ such that $w(h(X)) = v(h(a)) = e\gamma$. Then $r = f^e/h$ is an element of $K(X)$ for which $w(r) = 0$, the image r^* of r in the residue field k_w of w is transcendental over $R(K)$ and k_w is isomorphic to $R(K(a))(r^*)$. This isomorphism is canonical: for any $F \in K[X]$ with $\deg F < \deg a$ we have $w(F(X)) = w(F(a))$, $(F(X)/F(a))^* = 1$ and the above isomorphism becomes an equality in the residue field of w.*

Moreover, if w' is an r. t. extension of v to $K[X]$, then there exists a pair (a, δ) which is minimal with respect to K and such that w' coincides with the r. t. extension defined by the minimal pair (a, δ).

Let now $(a, \delta) \in \bar{K} \times \mathbf{Q}$ be a minimal pair, and denote by w the corresponding r. t. extension of v to $K(X)$. We identify the residue field $k_w = R(K(a))(r^*)$ of w with the field of rational functions $R(K(a))(Y)$ in one variable Y over the field $R(K(a))$, i.e. we shall write $r^* = Y$.

Let $G \in R(K(a))[Y]$ be monic and let $m = \deg G$. A monic polynomial $g \in K[X]$ is said to be a lifting of G with respect to w (or with respect to a, δ and h) provided one has

$$\deg g = em \deg f,$$

$$w(g(X)) = mw(h(X)) = me\gamma,$$

and

$$\left(\frac{g}{h^m}\right)^* = G.$$

One says that the lifting g of G is trivial if $\deg g = \deg f$. This situation appears exactly when $\deg G = 1$ and $\gamma = w(f) \in G(K(a))$.

In [10], Theorem 2.1 it is shown that if $G \in R(K(a))[Y]$, $G \neq Y$, G monic and irreducible, then any lifting g of G in $K[X]$ is irreducible over K.

A pair (a, b) of elements from $\bar{K}$ is said to be a distinguished pair, provided one has

$$\deg a > \deg b,$$

$$v(a - c) \leq v(a - b)$$

for any $c \in \bar{K}$ with $\deg c < \deg a$, and

$$v(a - c) < v(a - b)$$

for any $c \in \bar{K}$ with $\deg c < \deg b$.

Given two irreducible polynomials $f, g \in K[X]$, one says that (g, f) is a distinguished pair if there exist a root a of g and a root b of f such that (a, b) is a distinguished pair. It is easy to see that if (g, f) is a distinguished pair of polynomials, then for any root a of g there exists a root b of f such that (a, b) is a distinguished pair, and for any root b of f there exists a root a of g such that (a, b) is a distinguished pair.

The connection between lifting polynomials and distinguished pairs of polynomials has been investigated in [10]. In Theorem 3.1 of [10] it is shown that if $G \in R(K(a))[Y]$, $G \neq Y$, G monic and irreducible, and if g is a nontrivial lifting of G in $K[X]$, then (g, f) is a distinguished pair. A converse of this result was obtained in Theorem 3.2 of [10].

3. Distances between the roots of a lifting polynomial

Let K be a field of characteristic zero, complete with respect to a rank one, discrete valuation v, and having a perfect residue field $R(K)$. Fix an element $a \in \bar{K}$ and denote by f its minimal polynomial over K. Let $a_1 = a, a_2, \dots, a_n$ be the roots of f in $\bar{K}$. Choose any rational number δ such that $\omega(a) < \delta$. Then (a, δ) is a minimal pair with respect to K. For any $i \in \{1, \dots, n\}$, let us denote

$$M(a_i) = \{x \in \bar{K} : v(x - a_i) \geq \delta\}.$$

The sets $M(a_i)$ and $M(a_j)$ are clearly disjoint for $i \neq j$. Let w be the r. t. extension of v to $K(X)$ defined by the minimal pair (a, δ). We use the same notations as in Theorem (2.1). Thus we let

$$\gamma = \sum_{i=1}^{n} \min\{v(a - a_i), \delta\},$$

and denote by e the smallest non-zero positive integer such that $e\gamma \in G(K(a))$. Then we fix a polynomial $h \in K[X]$ for which $\deg h < \deg f = n$ and $w(h(X)) = v(h(a)) = e\gamma$, and we let $r = f^e/h$. Next we select an irreducible polynomial

$$G = Y^m + c_1 Y^{m-1} + \cdots + c_m \in R(K(a))[Y],$$

$G \neq Y$. Therefore $c_m \neq 0$. Let now $g \in K[X]$ be any nontrivial lifting of G with respect to a, δ and h. Say

$$g = f^{em} + B_1 f^{em-1} + \cdots + B_{me},$$

with $B_1, \dots, B_{me} \in K[X]$, $\deg B_j < \deg f$ for any $j \in \{1, \dots, me\}$. We know from Theorem 2.1 of [10] that g is irreducible over K. We proceed to investigate the distances between the roots of g. Denote by M the set of roots of g in $\bar{K}$. We need the following result of Aghigh and Khanduja (see [1], Lemma 2.1).

LEMMA (A). *Let (a, δ) be a minimal pair (with respect to K and $\bar{v}$) and θ be an element of $\bar{K}$ with $\bar{v}(\theta - a) \geq \delta$. Let $h(x) \in K[x]$ be a polynomial of degree strictly less than $[K(a) : K]$. Then $\bar{v}(h(\theta) - h(a)) > \bar{v}(h(a))$.*

We now prove the following result.

PROPOSITION (3.1). (a) *For any $i \in \{1, \dots, n\}$,*

$$\#\{M(a_i) \cap M\} = em.$$

Also,

$$M = \cup_{i=1}^{n} (M \cap M(a_i)).$$

(b) *If $\theta \in M(a_i) \cap M$, then $K(a_i) \subseteq K(\theta)$, and the conjugates of θ over $K(a_i)$ are exactly the elements of $M(a_i) \cap M$.*

(c) *If $\theta \in M(a_i) \cap M$, then $v(a_i - \theta) = \delta$.*

(d) *For any $\theta \in M$ one has $v(f(\theta)) = \gamma$.*

Proof. (a) First, note that for any $i \in \{1, \ldots, n\}$,

$$\prod_{\theta\in M}(a_i-\theta)=g(a_i)=B_{me}(a_i),$$

and so keeping in view that v has a unique prolongation to $\bar{K}$, we have

$$v(g(a_i))=\sum_{\theta\in M}v(a_i-\theta)=v(B_{me}(a_i))=v(B_{me}(a)).$$

Since

$$\begin{aligned} em\gamma &= w(g)=w(f^{me}+B_1f^{me-1}+\cdots+B_{me})\\ &=\min_{0\le j\le me}\{v(B_j(a))+(me-j)\gamma\}\le v(B_{me}(a)), \end{aligned}$$

where $B_0(X):=1$, we deduce that

$$v(g(a_i))=\sum_{\theta\in M}v(a_i-\theta)\ge em\gamma. \tag{3.2}$$

Using the fact that both polynomials f and g are irreducible, we see that for any $i \in \{1, \ldots, n\}$ and any $\eta \in M$

$$\frac{1}{\deg g}\sum_{\theta\in M}v(a_i-\theta)=\frac{1}{n\deg g}\sum_{\substack{1\le j\le n\\ \theta\in M}}v(a_j-\theta)=\frac{1}{n}\sum_{1\le j\le n}v(a_j-\eta).$$

It now follows from (3.2) that

$$\sum_{1\le j\le n}v(a_j-\eta)=\frac{n}{\deg g}\sum_{\theta\in M}v(a_i-\theta)=\frac{1}{em}\sum_{\theta\in M}v(a_i-\theta)\ge\gamma. \tag{3.3}$$

We apply this inequality to a root η of g which is chosen such that

$$v(a-\eta)=\max_{1\le j\le n}v(a_j-\eta).$$

Then for any $1 \le j \le n$ one has

$$v(a_j-\eta)=\min\{v(a-a_j),v(a-\eta)\}.$$

As a consequence using (3.3), we have

$$\sum_{1\le j\le n}\min\{v(a-a_j),v(a-\eta)\}=\sum_{1\le j\le n}v(a_j-\eta)\ge\gamma=\sum_{1\le j\le n}\min\{v(a-a_j),\delta\},$$

which further gives $v(a-\eta) \ge \delta$, i.e. $\eta \in M(a)$. For any automorphism $\sigma \in \mathrm{Gal}(\bar{K}/K)$, one has $v(\sigma(a)-\sigma(\eta)) = v(a-\eta) \ge \delta$. Therefore any element of M belongs to one of the sets $M(a_1), \ldots, M(a_n)$. Also, for any $1 \le i, j \le n$ and any $\sigma \in \mathrm{Gal}(\bar{K}/K)$ for which $\sigma(a_i) = a_j$, we have $\sigma(M \cap M(a_i)) = M \cap M(a_j)$. Hence the sets $M \cap M(a_1), \ldots, M \cap M(a_n)$ have the same number of elements, and this completes the proof of (a).

(b) If $\theta \in M \cap M(a_i)$, then $v(\theta - a_i) \ge \delta > \omega(a)$, and by Krasner's Lemma it follows that $K(a_i) \subseteq K(\theta)$. Here

$$[K(\theta)\colon K(a_i)]=\frac{\deg g}{\deg f}=em.$$

On the other hand, each of the em conjugates of θ over $K(a_i)$ lie in $M(a_i) \cap M$, because they are all at the same distance from a_i. Since $M(a_i) \cap M$ has exactly

em elements, it follows that these elements are all the conjugates of θ over $K(a_i)$, which proves (b).

(c) It is enough to consider the case $i = 1$, that is, the case $a_i = a$. Then the general case will follow by applying an appropriate automorphism $\sigma \in \mathrm{Gal}(\bar{K}/K)$, which sends a to a_i. Let now $\theta \in M(a) \cap M$. We know that $v(a-\theta) \geq \delta$. Suppose to the contrary that $v(a-\theta) > \delta$. Then one would have

$$\begin{aligned}(3.4)\qquad v(f(\theta)) &= \sum_{1\leq j\leq n} v(a_j - \theta) = v(a-\theta) + \sum_{2\leq j\leq n} \min\{v(a-a_j), v(a-\theta)\}\\ &> \delta + \sum_{2\leq j\leq n} \min\{v(a-a_j), \delta\} = \gamma.\end{aligned}$$

Let us recall that

$$em\gamma = w(g) = w(f^{me} + B_1 f^{me-1} + \cdots + B_{me}) = \min_{0\leq s\leq me} \left(v(B_{me-s}(a)) + s\gamma\right).$$

Then, since $\deg B_s < \deg f$ for any $0 \leq s \leq me$, it follows from Lemma (A) that $v(B_s(a)) = v(B_s(\theta))$. Combining these results, we see that

$$em\gamma = \min_{0\leq s\leq me} \left(v(B_{me-s}(\theta)) + s\gamma\right).$$

Let $0 \leq s_0 \leq me$, s_0 as small as possible, for which the above minimum is attained. Since $em\gamma \in G(K(a))$, and for $0 \leq s \leq me$ one has $v(B_s(\theta)) = v(B_s(a)) \in G(K(a))$, while $s\gamma \in G(K(a))$ only if s is a multiple of e, it is clear that s_0 must be a multiple of e. If $s_0 = 0$, then we obtain by using (3.4) and Lemma (A) for any $0 \leq j < me$,

$$\begin{aligned}v(B_{me}(\theta)) &\leq v(B_j(\theta)) + (me-j)\gamma\\ &< v(B_j(\theta)) + v(f^{me-j}(\theta)) = v(B_j(\theta) f^{me-j}(\theta)).\end{aligned}$$

This is false, because from $g(\theta) = 0$ it follows that

$$B_{me}(\theta) = -\sum_{0\leq j\leq me-1} B_j(\theta) f^{me-j}(\theta),$$

which further gives

$$v(B_{me}(\theta)) \geq \min_{0\leq j\leq me-1} v(B_j(\theta) f^{me-j}(\theta)),$$

and we obtain a contradiction. Therefore $s_0 = ei_0$, with $i_0 \geq 1$. Then

$$w(B_{me}(X)) = v(B_{me}(a)) > em\gamma,$$

and so the image of $\left(B_{me}(X)/h^m(X)\right)$ in the residue field k_w of w vanishes. As a consequence, the image of $\left(g/h^m\right)$ in k_w will not have a constant term, and in fact this image will be a polynomial in r^*, divisible by $(r^*)^{i_0}$. Since g is a lifting of $G(Y)$, this would imply that Y^{i_0} divides $G(Y)$, which is not the case, because $i_0 \geq 1$, $G(Y)$ is irreducible and $G(Y) \neq Y$. In conclusion one can not have $v(a-\theta) > \delta$, and this proves (c).

(d) We know that $f(\theta)$ has the same valuation for all the elements $\theta \in M$. Choosing θ in $M \cap M(a)$, and using part (c) above, we see that

$$v(f(\theta)) = \sum_{1\leq j\leq n} v(a_j - \theta) = \sum_{1\leq j\leq n} \min\{v(a-a_j), v(a-\theta)\}$$

$$= \sum_{1\le j\le n} \min\{v(a-a_j), \delta\} = \gamma.$$

This completes the proof of the proposition. □

The following corollary is an immediate consequence of Lemma (A) and assertion (d) of the above proposition.

COROLLARY. *With the notations as in the above proposition, for any polynomial $H(x)$ belonging to $K[x]$ and any $\rho \in M$, the inequality $v(H(\rho)) \ge w(H(x))$ holds.*

We now derive some consequences of the above result. We are interested in the distances between any two roots of g. First, for any $1 \le i \ne j \le n$, and any elements $\theta \in M \cap M(a_i)$, $\eta \in M \cap M(a_j)$, we have $v(\theta - a_i) = v(\eta - a_j) = \delta$, while $v(a_i - a_j) < \delta$. This implies that

$$v(\theta - \eta) = v(a_i - a_j),$$

for any $\theta \in M \cap M(a_i)$ and $\eta \in M \cap M(a_j)$, with $1 \le i \ne j \le n$. It remains to estimate the distances between the elements of a set of the form $M \cap M(a_i)$, for a fixed i. Note that since each automorphism $\sigma \in \mathrm{Gal}(\bar{K}/K)$ is an isometry, and since for any i, j, the sets $M \cap M(a_i)$ and $M \cap M(a_j)$ are sent to one another via a suitable automorphism σ, the mutual distances between the elements of the set $M \cap M(a_i)$ are the same as those between the elements of $M \cap M(a_j)$. Thus it is enough to estimate the distances between the elements of one of these sets. To make a choice, we consider the set $M \cap M(a)$. For any $\eta_1, \eta_2 \in M \cap M(a)$ one has $v(\eta_1 - a) = v(\eta_2 - a) = \delta$, therefore $v(\eta_1 - \eta_2) \ge \delta$.

Let now $\theta \in M \cap M(a)$. We already know that $K(a) \subseteq K(\theta)$ and that (θ, a) is a distinguished pair. As a consequence, for any non-zero polynomial $F(X) \in K[X]$ with $\deg F < \deg a$, we have $v(F(\theta)) = v(F(a)) = w(F(X))$, and moreover, in the residue field k_w of the r. t. extension of v to $\bar{K}(X)$ defined by inf, a and δ, which we continue to denote by w, one has by virtue of Theorem (2.1) and Lemma (A)

$$\left(\frac{F(\theta)}{F(a)}\right)^* = \left(\frac{F(\theta)}{F(X)}\right)^* = 1.$$

Recall that

$$g(X) = f^{em}(X) + B_1(X)f^{em-1}(X) + \cdots + B_{me}(X),$$

with $B_1(X), \ldots, B_{me}(X) \in K[X]$ and $\deg B_i(X) < \deg f$ for $1 \le i \le me$. Hence

$$\left(\frac{B_i(\theta)}{B_i(X)}\right)^* = 1, \quad 1 \le i \le me.$$

Also, $v(f(\theta)) = \gamma = w(f(X))$, and $\big(h(\theta)/h(X)\big)^* = 1$. On combining the above relations with the fact that $g(X)$ is a lifting of G, we deduce that

$$\left(\frac{g(\theta)}{h^m(\theta)}\right)^* = G(r^*(\theta)).$$

But $g(\theta) = 0$, so $\big(g(\theta)/h^m(\theta)\big)^* = 0$, and hence $r^*(\theta)$ is a root of G. Here $r^*(\theta) \in R(K(\theta))$, and since G is irreducible and of degree m over $R(K(a))$, it follows that the inertial degree $[R(K(\theta))\colon R(K(a))]$ is $\ge m$. On the other hand

$v(f(\theta)) = \gamma$, thus the ramification index $e(K(\theta)/K(a))$ is $\geq e$. But $[K(\theta)\colon K(a)] = em$. We deduce that $e(K(\theta)/K(a)) = e$, $[R(K(\theta))\colon R(K(a))] = m$, and moreover $R(K(\theta)) = R(K(a))(r^*(\theta))$, and there exists a uniformizing element π_θ of $K(\theta)$ of the form $\pi_\theta = f(\theta)^s z$, with $z \in K(a)$ and s a positive integer. Denote by U the maximal unramified extension of $K(a)$ which is contained in $K(\theta)$. Therefore $[U\colon K(a)] = m$ and $[K(\theta)\colon U] = e$. Denote also by $\beta_1, \dots, \beta_m \in R(\bar{K})$ the roots of G. We know that for any $\eta \in M \cap M(a)$, $r^*(\eta)$ is a root of G. So we have a map from $M \cap M(a)$ to the set $\{\beta_1, \dots, \beta_m\}$, given by $\eta \mapsto r^*(\eta)$. For any root β of $G(Y)$ set

$$M(a, \beta) := \{\eta \in M | v(\eta - a) \geq \delta, r^*(\eta) = \beta\}. \tag{3.5}$$

For any $1 \leq i \neq j \leq m$, the sets $M(a, \beta_i)$ and $M(a, \beta_j)$ are sent to one another by an appropriate automorphism $\sigma \in \mathrm{Gal}(\bar{K}/K(a))$. As a consequence, all these sets have the same number of elements. The set $M \cap M(a)$ has exactly em elements, so we obtain

$$\# M(a, \beta_i) = e, \quad 1 \leq i \leq m.$$

For simplicity, denote by β the element of the set $\{\beta_1, \dots, \beta_m\}$ for which we have $r^*(\theta) = \beta$. We claim that for any $\beta_j \neq \beta$, and any $\eta \in M(a, \beta_j)$, one has

$$v(\theta - \eta) = \delta.$$

For let us assume that there exist $\beta_j \neq \beta$ and an element $\eta \in M(a, \beta_j)$, for which $v(\theta - \eta) > \delta$. One has

$$\frac{f(\eta)}{f(\theta)} = \prod_{1 \leq i \leq n} \frac{\eta - a_i}{\theta - a_i} = \prod_{1 \leq i \leq n} \left(1 + \frac{\eta - \theta}{\theta - a_i}\right).$$

Since $v(\theta - a_i) \leq \delta < v(\eta - \theta)$, we find that $\left(\frac{\eta - \theta}{\theta - a_i}\right)^* = 0$, and so

$$\left(\frac{f(\eta)}{f(\theta)}\right)^* = \prod_{1 \leq i \leq n} \left(1 + \frac{\eta - \theta}{\theta - a_i}\right)^* = 1.$$

We also know that

$$\left(\frac{h(\eta)}{h(\theta)}\right)^* = \left(\frac{h(\eta)}{h(a)}\right)^* \left(\frac{h(a)}{h(\theta)}\right)^* = 1.$$

On combining the above equalities we derive

$$\frac{\beta_j}{\beta} = \frac{r^*(\eta)}{r^*(\theta)} = \left(\frac{f^e(\eta)}{h(\eta)}\right)^* \left(\frac{h(\theta)}{f^e(\theta)}\right)^* = \left(\frac{f^e(\eta)}{f^e(\theta)}\right)^* \left(\frac{h(\theta)}{h(\eta)}\right)^* = 1,$$

which is not the case. We obtained a contradiction, which proves the claim.

Finally we estimate the distance between two roots of $g(X)$ which belong to the same set $M(a, \beta_j)$. As before, any two such sets are sent to one another via an automorphism σ, which is an isometry. Therefore the mutual distances between the elements of $M(a, \beta_j)$ are the same for any such set. We may then restrict to only consider one of these sets, which we choose to be $M(a, \beta)$. We know that $v(\theta - \eta) \geq \delta$ for any $\eta \in M(a, \beta)$. We now proceed to derive an upper bound for $v(\theta - \eta)$, uniformly for $\theta \neq \eta \in M(a, \beta)$. Note first that $M(a, \beta)$ coincides with the set of conjugates of θ over U. Indeed, on the one hand, $\# M(a, \beta) = e$ and θ has exactly e conjugates over U (recall that $[K(\theta)\colon U] = e$).

On the other hand, any automorphism $\sigma \in \mathrm{Gal}(\bar{K}/U)$ sends $r(\theta)$ to $r(\sigma(\theta))$, and if we choose $u \in U$ with $v(u) = 0$ and $u^* = \beta$, then

$$v(u - r(\sigma(\theta))) = v(\sigma(u) - \sigma(r(\theta))) = v(u - r(\theta)) > 0.$$

So $(r(\sigma(\theta)))^* = u^* = \beta$, and hence $\sigma(\theta) \in M(a, \beta)$ for any $\sigma \in \mathrm{Gal}(\bar{K}/U)$. Therefore $M(a, \beta)$ coincides with the set of conjugates of θ over U.

Let now $\eta \in M(a, \beta)$ and choose an automorphism $\sigma \in \mathrm{Gal}(\bar{K}/U)$ for which $\sigma(\theta) = \eta$. Then $\sigma(f(\theta)) = f(\eta)$, and

$$\sigma(\pi_\theta) = \sigma\big(f^s(\theta)z\big) = f^s(\eta)z.$$

Thus the elements $\pi_\eta := f^s(\eta)z$, with $\eta \in M(a, \beta)$, are the conjugates of π_θ over U. Consider the minimal polynomial $P(X)$ of π_θ over U, which is an Eisenstein polynomial (see [6, Ch. 14, Section 2]). Let

$$P(X) = X^e + u_1 X^{e-1} + \cdots + u_{e-1}X + u_e = \prod_{\eta \in M(a,\beta)} (X - \pi_\eta).$$

We compute the valuation of $P'(\pi_\theta)$ in two ways. First, one has

$$P'(\pi_\theta) = \prod_{\eta \neq \theta} (\pi_\theta - \pi_\eta),$$

hence

$$v(P'(\pi_\theta)) = \sum_{\eta \neq \theta} v(\pi_\theta - \pi_\eta). \tag{3.6}$$

Second, keeping in mind that e is the smallest positive integer such that $ev(\pi_\theta) \in G(K(U))$, we conclude that

$$v(P'(\pi_\theta)) = v(e\pi_\theta^{e-1} + (e-1)u_1\pi_\theta^{e-2} + \cdots + u_{e-1}) = \min\{v(ju_{e-j}\pi_\theta^{j-1})\}.$$

In particular

$$v(P'(\pi_\theta)) \leq v(e\pi_\theta^{e-1}) = v(e) + (e-1)v(\pi_\theta). \tag{3.7}$$

By combining (3.6) and (3.7) we find that

$$\sum_{\eta \neq \theta} v(\pi_\theta - \pi_\eta) \leq v(e) + (e-1)v(\pi_\theta),$$

which can also be written in the form

$$\sum_{\eta \neq \theta} v\left(1 - \frac{\pi_\eta}{\pi_\theta}\right) \leq v(e).$$

Here the left hand side is a sum of non-negative terms. It follows that

$$v\left(1 - \frac{\pi_\eta}{\pi_\theta}\right) \leq v(e)$$

for any $\eta \in M(a, \beta)$, $\eta \neq \theta$. Note that

$$v\left(1 - \frac{\pi_\eta}{\pi_\theta}\right) = v\left(1 - \frac{f^s(\eta)}{f^s(\theta)}\right)$$
$$= v\left(1 - \frac{f(\eta)}{f(\theta)}\right) + v\left(1 + \frac{f(\eta)}{f(\theta)} + \cdots + \frac{f^{s-1}(\eta)}{f^{s-1}(\theta)}\right) \geq v\left(1 - \frac{f(\eta)}{f(\theta)}\right).$$

Hence
$$v\left(1-\frac{f(\eta)}{f(\theta)}\right) \leq v(e).$$
Next we have
$$1-\frac{f(\eta)}{f(\theta)} = 1-\prod_{1\leq i\leq n}\frac{\eta-a_i}{\theta-a_i} = 1-\prod_{1\leq i\leq n}\left(1+\frac{\eta-\theta}{\theta-a_i}\right).$$
If $v(\eta-\theta) > \delta + v(e)$, then since $v(\theta-a_i) \leq \delta$ for any i, one would have $v\left(\frac{\eta-\theta}{\theta-a_i}\right) > v(e)$, and then
$$v\left(1-\prod_{1\leq i\leq n}\left(1+\frac{\eta-\theta}{\theta-a_i}\right)\right) > v(e),$$
which is false. In conclusion, one has
$$v(\eta-\theta) \leq \delta + v(e),$$
for any $\eta \in M(a,\beta)$, $\eta \neq \theta$.

We gather the above results in the following theorem.

THEOREM (3.8). *Let K be a field of characteristic zero, complete with respect to a rank one, discrete valuation v, and having a perfect residue field. Let $a \in \bar{K}$, $\delta \in \mathbf{Q}$, with δ strictly larger than the Krasner constant $\omega(a)$, and denote by w the r. t. extension of v to $K(X)$ defined by inf, a and δ. Let $G(Y) \neq Y$ be a monic, irreducible polynomial over the residue field of $K(a)$, and let g be a nontrivial lifting of G with respect to w. Let e, $f(X)$ and $r(X)$ be as in Theorem (2.1). Choose a root β of $G(Y)$ and a root θ of $g(X)$, with $\theta \in M(a,\beta)$ defined by* (3.5). *Then $K(a) \subseteq K(\theta)$, and the ramification index and inertial degree of $K(\theta)$ over $K(a)$ are equal to e and m respectively. We also have*

(i) *For any conjugate a_i of a over K, $a_i \neq a$, and any root η of $g(X)$ which belongs to $M(a_i)$,*
$$v(\theta-\eta) = v(a-a_i).$$
(ii) *For any conjugate β_j of β over $K(a)$, $\beta_j \neq \beta$, and any $\eta \in M(a,\beta_j)$,*
$$v(\theta-\eta) = \delta.$$
(iii) *For any $\eta \in M(a,\beta)$, $\eta \neq \theta$,*
$$\delta \leq v(\theta-\eta) \leq \delta + v(e).$$

4. Lifting polynomials defining the same field extension

As a consequence of the above results, we obtain the following theorem, which provides sufficient conditions under which two different liftings obtained from the same initial data, define the same field extension.

THEOREM (4.1). *Let $K, v, a, \delta, w, e, G, g$ be as in the statement of Theorem (3.8), and let g_1 be another nontrivial lifting of G with respect to w, such that $w(g_1(X)-g(X)) > ev(e)+w(g(X))$. Then, for any root ρ of g_1 there exists a root θ of g such that $K(\rho) = K(\theta)$, and conversely, for any root θ of g there exists a root ρ of g_1 such that $K(\rho) = K(\theta)$.*

Note that if $v(e) = 0$, then any nontrivial lifting g_1 of G with respect to w satisfies the conditions from the statement of the theorem. We always have $v(e) = 0$ in case $R(K)$ has characteristic zero. If $R(K)$ has characteristic $p \neq 0$, then $v(e) = 0$ if e is not a multiple of p. So we have the following corollary.

COROLLARY (4.2). *Let K, v, a, δ, w, G be as in the statement of Theorem (3.8), and assume that char $R(K) = 0$, or char $R(K) = p \neq 0$ and e is not a multiple of p. Let g, g_1 be any nontrivial liftings of G with respect to w. Then, for any root ρ of g_1 there exists a root θ of g such that $K(\rho) = K(\theta)$, and conversely, for any root θ of g there exists a root ρ of g_1 such that $K(\rho) = K(\theta)$.*

Proof of Theorem (4.1). We use the same notations as in Theorem (3.8). Let ρ be a root of g_1, and let θ be a root of g such that $v(\rho - \theta)$ is as large as possible. Then for any other root η of g, we have

$$v(\rho - \eta) = \min\{v(\rho - \theta), v(\theta - \eta)\}.$$

To simplify the notation, we assume in what follows that $\theta \in M(a, \beta)$. Using corollary to Proposition (3.1) and the hypothesis, we have

$$\begin{aligned} v(g(\rho)) &= v(g_1(\rho) - g(\rho)) \geq w(g_1(X) - g(X)) \\ &> ev(e) + w(g(X)) = ev(e) + em\gamma. \end{aligned} \tag{4.3}$$

We claim that from this relation it follows that

$$v(\rho - \theta) > \delta + v(e).$$

For let us assume that $v(\rho - \theta) \leq \delta + v(e)$. Then for any root $\eta \neq \theta$ of g,

$$v(\rho - \eta) \leq \min\{\delta + v(e), v(\theta - \eta)\} = v(\theta - \eta),$$

by Theorem (3.8). Then the above inequality gives

$$v(g(\rho)) = v(\rho - \theta) + \sum_{\eta \neq \theta} v(\rho - \eta) \leq \delta + v(e) + \sum_{\eta \neq \theta} v(\theta - \eta).$$

It now follows from (4.3) that

$$\sum_{\eta \neq \theta} v(\theta - \eta) \geq v(g(\rho)) - \delta - v(e) > (e-1)v(e) + em\gamma - \delta. \tag{4.4}$$

We now evaluate the contribution of each term on the left side of the above inequality. For any $\eta \in M \cap M(a_i)$ with $a_i \neq a$, we know from Theorem (3.8) (i) that $v(\theta - \eta) = v(a - a_i)$. For each fixed $a_i \neq a$, there are exactly em elements η in $M \cap M(a_i)$. Therefore

$$\sum_{\eta \in M \cap M(a_i)} v(\theta - \eta) = emv(a - a_i).$$

The total contribution of these terms is

$$\sum_{a_i \neq a} \sum_{\eta \in M \cap M(a_i)} v(\theta - \eta) = em \sum_{a_i \neq a} v(a - a_i) = em(\gamma - \delta), \tag{4.5}$$

since

$$\gamma = \sum_{i=1}^{n} \min\{v(a - a_i), \delta\} = \delta + \sum_{a_i \neq a} v(a - a_i).$$

Next for any $\eta \in M(a, \beta_j)$ with $\beta_j \neq \beta$ we have $v(\theta - \eta) = \delta$ by Theorem (3.8) (ii), hence

$$\sum_{\eta \in M(a,\beta_j)} v(\theta - \eta) = \delta \# M(a, \beta_j) = e\delta,$$

which gives in turn

$$\sum_{\beta_j \neq \beta} \sum_{\eta \in M(a,\beta_j)} v(\theta - \eta) = (m-1)e\delta. \tag{4.6}$$

Finally, from Theorem (3.8) (iii) we know that for any $\eta \in M(a, \beta)$, $\eta \neq \theta$, one has $v(\theta - \eta) \leq \delta + v(e)$, therefore

$$\sum_{\substack{\eta \in M(a,\beta) \\ \eta \neq \theta}} v(\theta - \eta) \leq (\delta + v(e))\big(\# M(a, \beta) - 1\big) = (e-1)(\delta + v(e)). \tag{4.7}$$

Combining (4.5), (4.6) and (4.7) we deduce that

$$\begin{aligned} \sum_{\eta \neq \theta} v(\theta - \eta) &\leq em(\gamma - \delta) + (m-1)e\delta + (e-1)(\delta + v(e)) \\ &= (e-1)v(e) + em\gamma - \delta. \end{aligned} \tag{4.8}$$

Relations (4.4) and (4.8) contradict each other, and this proves our claim that $v(\rho - \theta) > \delta + v(e)$. This further implies that

$$\omega(\theta) = \max_{\substack{\eta \in M \\ \eta \neq \theta}} v(\theta - \eta) \leq \delta + v(e) < v(\rho - \theta).$$

It follows now from Krasner's Lemma that $K(\theta) \subseteq K(\rho)$, and since both θ and ρ have the same degree $em \deg f$ over K, we conclude that $K(\theta) = K(\rho)$. Similarly, if we start with a root θ of g then we can find a root ρ of g_1 such that $K(\theta) = K(\rho)$, and this completes the proof of the theorem. □

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NOETHER-LEFSCHETZ FOR K_1 OF A CERTAIN CLASS OF SURFACES

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Abstract. We first give an elementary new proof of the vanishing of the regulator on $K_1(Z)$ where $Z \subset \mathbb{P}^3$ be a general surface of degree $d \geq 5$, using a Lefschetz pencil argument. By a similar argument we then show the triviality of the regulator for K_1 of a general product of two curves.

1. Statement of results.

Let Z be a smooth quasiprojective variety over $\mathbb{C}$, and for given nonnegative integers k, m, let $\mathrm{CH}^k(Z,m)$ be the higher Chow group as introduced in [Blo1]. In [Blo2], Bloch constructs a cycle class map into any suitable cohomology theory. In our setting, the corresponding map is:

$$\mathrm{cl}_{k,m}\colon \mathrm{CH}^k(Z,m) \to H^{2k-m}_{\mathcal{D}}(Z,\mathbb{Q}(k)),$$

where $H^{2k-m}_{\mathcal{D}}(Z,\mathbb{Q}(k))$ is Deligne-Beilinson cohomology, which fits in a short exact sequence

$$0 \to \frac{H^{2k-m-1}(Z,\mathbb{C})}{F^kH^{2k-m-1}(Z,\mathbb{C}) + H^{2k-m-1}(Z,\mathbb{Q}(k))} \to H^{2k-m}_{\mathcal{D}}(Z,\mathbb{Q}(k))$$

$$\to F^kH^{2k-m}(Z,\mathbb{C}) \bigcap H^{2k-m}(Z,\mathbb{Q}(k)) \to 0.$$

Our primary interest is when Z is also complete, and $m = 1$. Thus one has the corresponding map:

$$\mathrm{cl}_{k,1}\colon \mathrm{CH}^k(Z,1) \to \frac{H^{2k-2}(Z,\mathbb{C})}{F^kH^{2k-2}(Z,\mathbb{C}) + H^{2k-2}(Z,\mathbb{Q}(k))}.$$

Let $\mathrm{Hg}^{k-1}(Z) := H^{2k-2}(Z,\mathbb{Q}(k-1)) \cap F^{k-1}H^{2k-2}(Z,\mathbb{C})$ be the Hodge group. Then one has an induced map

$$\underline{\mathrm{cl}}_{k,1}\colon \mathrm{CH}^k(Z,1) \to \frac{H^{2k-2}(Z,\mathbb{C})}{F^kH^{2k-2}(Z,\mathbb{C}) + \mathrm{Hg}^{k-1}(Z)\otimes\mathbb{C} + H^{2k-2}(Z,\mathbb{Q}(k))}.$$

It is known that $\underline{\mathrm{cl}}_{k,1}$ is trivial for Z a sufficiently general complete intersection and of sufficiently high multidegree. This is a consequence of the work of Nori [No], together with a technique similar to that given in [G-S]. The argument is

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presented in [MS]. Further, it is noted in [MS], based on an effective bound in [Pa], that

$$\underline{\mathrm{cl}}_{2,1}\colon \ \mathrm{CH}^2(Z,1) \to \frac{H^2(Z,\mathbb{C})}{F^2H^2(Z,\mathbb{C}) + \mathrm{Hg}^1(Z)\otimes\mathbb{C} + H^2(Z,\mathbb{Q}(2))},$$

is trivial for sufficiently general surfaces $Z \subset \mathbb{P}^3$ of degree $d \geq 5$. The method of Nori involves passing to the universal family of complete intersections of a given multidegree, in a given projective space. A similar point of view appears in [Na]. In this paper, we give an elementary and direct proof of the triviality of $\underline{\mathrm{cl}}_{2,1}$ for a general surface $Z \subset \mathbb{P}^3$ of degree ≥ 5, by working with a Lefschetz pencil of degree $d \geq 5$ surfaces in $\mathbb{P}^3$. Thus our first main result is an elementary new proof of the following:

Theorem (1.1). *For a sufficiently general surface $Z \subset \mathbb{P}^3$ of degree $d \geq 5$, the map $\underline{\mathrm{cl}}_{2,1}$ is trivial.*

We remark that the theorem is trivially true, without the generic hypothesis, if $\deg Z \leq 3$, as $H^2(Z)$ is algebraic. From the works of Collino, Voisin, S. Müller-Stach, *et al,* and more recently the authors [C-L], it is false if $\deg Z = 4$. Since our method requires only a Lefschetz pencil as opposed to the universal family of surfaces of degree d in $\mathbb{P}^3$, and that it provides a rather simple proof of a counterexample of the Hodge-$\mathcal{D}$-conjecture of Beilinson [Bei1], we believe that this approach has some merit. In particular, a variant of this argument leads to our next result:

Theorem (1.2). *Let $X = C_1 \times C_2$ be a product of two general curves (resp. general hyperelliptic curves), where the genus $g(C_1) \geq 1$ and $g(C_2) \geq 2$. Then $\underline{\mathrm{cl}}_{2,1}$ is trivial.*

We remark that story is *false* if C_1 and C_2 are general elliptic curves ([C-L]).

Question (1.3). Consider a smooth projective surface X and the corresponding regulators

$$r_{2,1}\colon \ \mathrm{CH}^2(X,1;\mathbb{R}) \to H^{1,1}(X,\mathbb{R}(1)),$$

$$\underline{\mathrm{cl}}_{2,1}\colon \ \mathrm{CH}^2_{\mathrm{ind}}(X,1;\mathbb{R}) \to \frac{H^3_{\mathcal{D}}(X,\mathbb{Q}(1))}{\mathrm{Hg}^1(X)\otimes\mathbb{C}/\mathbb{Q}(1)}.$$

Let $\kappa(X)$ be the Kodaira dimension. If $\kappa(X) = -1$, then $r_{2,1}$ is surjective. If $\kappa(X) = 0$, then $r_{2,1}$ is surjective for "general" X (see [C-L]). If X is "general" and if $\kappa(X) \geq 1$, is it the case that $\underline{\mathrm{cl}}_{2,1}$ is trivial?

2. Some definitions

(1) *Deligne cohomology.* A good source for the definition of Deligne cohomology can be found in [EV]. For our narrow purposes, the following will suffice. Let X be a projective algebraic manifold, and for a subring $\mathbb{A} \subset \mathbb{R}$, put $\mathbb{A}(j) = \mathbb{A}(2\pi\sqrt{-1})^j$. Consider the Deligne complex

$$\mathbb{A}(j)_{\mathcal{D}}\colon \quad \mathbb{A}(j) \to \mathcal{O}_X \to \Omega^1_X \to \cdots \to \Omega^{j-1}_X.$$

Definition (2.1). Deligne cohomology is given by $H^i_{\mathcal{D}}(X,\mathbb{A}(j)) := \mathbb{H}^i(\mathbb{A}(j)_{\mathcal{D}})$ (hypercohomology).

One has a short exact sequence

$$0 \to \frac{H^{i-1}(X,\mathbb{C})}{F^jH^{i-1}(X,\mathbb{C}) + H^{i-1}(X,\mathbb{A}(j))} \to H^i_{\mathcal{D}}(X,\mathbb{A}(j))$$
$$\to F^jH^i(X,\mathbb{C}) \bigcap H^i(X,\mathbb{A}(j)) \to 0.$$

We are mainly interested in the cases where $i = 2j-1$ and where $\mathbb{A} = \mathbb{Q}$ and $\mathbb{A} = \mathbb{R}$. In these cases we have

$$H^{2j-1}_{\mathcal{D}}(X,\mathbb{Q}(j)) \simeq \frac{H^{2j-2}(X,\mathbb{C})}{F^jH^{2j-2}(X,\mathbb{C}) + H^{2j-2}(X,\mathbb{Q}(j))},$$
$$H^{2j-1}_{\mathcal{D}}(X,\mathbb{R}(j)) \simeq H^{j-1,j-1}(X,\mathbb{R}(j-1)).$$

(2) *Higher Chow groups.* For X given in (1), the following abridged definition of $\mathrm{CH}^k(Z,1)$ will suffice (see [La] or [MS]).

Definition (2.2). $\mathrm{CH}^k(X,1)$ is the homology of the middle term in the complex

$$\coprod_{\mathrm{cd}_X Y = k-2} K_2(\mathbb{C}(Y)) \xrightarrow{\text{Tame}} \coprod_{\mathrm{cd}_X Y = k-1} K_1(\mathbb{C}(Y)) \xrightarrow{\text{Div}} \coprod_{\mathrm{cd}_X Y = k} K_0(\mathbb{C}(Y)),$$

where we recall that $K_1(\mathbb{F}) = \mathbb{F}^\times$ and $K_0(\mathbb{F}) = \mathbb{Z}$, for a field $\mathbb{F}$, and Tame, Div are respectively the Tame symbol and divisor maps.

Thus classes in $\mathrm{CH}^k(X,1)$ can be represented by cycles of the form

$$\left\{\xi = \sum_{j=1}^N (f_j, Z_j) \;\middle|\; f_j \in \mathbf{C}(Z_j)^\times,\ \mathrm{cd}_{X Z_j} = k-1,\ \sum_{j=1}^N \mathrm{div}_{Z_j}(f_j) = 0\right\}.$$

Note: For the most part, we will identify $\mathrm{CH}^k(-,m)$ with $\mathrm{CH}^k(-,m)\otimes\mathbb{Q}$, unless there is a specific reason to work with $\mathrm{CH}^k(-,m)$ (and in which case the interpretation will be clear).

(3) *Regulators.* There are cycle class maps

$$\mathrm{cl}^{\mathbb{A}}_{k,1} \colon \mathrm{CH}^k(X,1) \to H^{2k-1}_{\mathcal{D}}(X,\mathbb{A}(k)).$$

In the case where $\mathbb{A} = \mathbb{R}$, we put $r_{k,1} = \mathrm{cl}^{\mathbb{R}}_{k,1}$. Let $n = \dim X$. The map

$$r_{k,1} \colon \mathrm{CH}^k(X,1) \to H^{2k-2}_{\mathcal{D}}(X,\mathbb{R}(k)) \simeq H^{k-1,k-1}(X,\mathbb{R}(k-1))$$
$$\simeq H^{n-k+1,n-k+1}(X,\mathbb{R}(n-k+1))^\vee,$$

is given explicitly as follows (see [Bei1] or [Ja]):

$$r_{k,1}(\xi)(\omega) = \frac{1}{(2\pi\sqrt{-1})^{n-k+1}} \sum_{j=1}^N \int_{Z_j} \omega \log|f_j|.$$

(4) *Horizontal displacement.* Let $h\colon W \to S$ be a proper smooth morphism of quasiprojective varieties over $\mathbb{C}$, where say for simplicity $\dim S = 1$, with smooth projective fiber $W_t := h^{-1}(t)$. Fix a reference point $t_0 \in S$ and consider a disk Δ centered at t_0. It is well known that there is a diffeomorphism $h^{-1}(\Delta) \approx \Delta \times W_{t_0}$. Thus for a cohomology class $\gamma := \gamma_{t_0} \in H^\bullet(W_{t_0})$, one can talk about its horizontal displacement $\gamma_t \in H^\bullet(W_t)$, for $t \in \Delta$ and more generally for $t \in S$. Consider the Hodge decomposition $H^\bullet(W_t,\mathbb{C}) = \bigoplus_{p+q=\bullet} H^{p,q}(W_t)$, $\gamma_t = \oplus_{p+q=\bullet}\gamma^{p,q}$. We say that the Hodge (p,q) components deform horizontally

if $\gamma_t^{p,q} = (\gamma^{p,q})_t$ for all $t \in \Delta$. By analytic considerations of Hodge subbundles, this is equivalent to saying that $\gamma_t^{p,q} = (\gamma^{p,q})_t$ for all $t \in S$.

(5) The word "general" in this paper will have the following meaning. In the notation of (1.1), a point $t \in \mathcal{T}$ is general if t belongs to the complement of a *countable* union of proper subvarieties of $\mathcal{T}$ governed by a certain property.

3. Proof of Theorem (1.1)

Let $\{X_t\}_{t\in\mathbb{P}^1}$ be a Lefschetz pencil of surfaces of degree $d \geq 5$ in $\mathbb{P}^3$, i.e. the general fiber X_t is smooth, and each singular fiber has an ordinary double point singularity. We will think of this pencil in the form $X \subset \mathbb{P}^3 \times \mathbb{P}^1$, i.e. where X is the blowup of $\mathbb{P}^3$ along the base locus $\cap_{t\in\mathbb{P}^1} X_t$. Suppose that for a general $t \in \mathbb{P}^1$, the cycle class map $\mathrm{cl}_{2,1}\colon \mathrm{CH}^2(X_t, 1) \to H^3_{\mathcal{D}}(X_t, \mathbb{Q}(2))$ is nontrivial. We can assume that X is defined over an algebraically closed field L of finite transcendence degree over $\mathbb{Q}$, i.e. $X/_{\mathbb{C}} = X_L \times \mathbb{C}$. Let η be the generic point of $\mathbb{P}^1_L$. For some finite algebraic extension $K \supset L(\eta)$, and via a suitable embedding $K \hookrightarrow \mathbb{C}$, there is a class $\xi_K \in \mathrm{CH}^2(X_K := X_\eta \times K, 1)$ such that $\mathrm{cl}_{2,1}(\xi_K) \neq 0$ in $H^3_{\mathcal{D}}(X_K(\mathbb{C}), \mathbb{Q}(2))$. The situation here is not unlike than that found in ([Lew] p. 191). There is a smooth projective curve Γ_L with function field $L(\Gamma) = K$. Then after a base change $Y = X \times_{\mathbb{P}^1} \Gamma$, ξ_K defines a cycle in $\xi \in \mathrm{CH}^2(Y_U, 1)$, where $U \subset \Gamma$ is a Zariski open subset of Γ and $Y_U = \cup_{t\in U} Y_t$. This uses the fact that

$$\mathrm{CH}^2(X_K, 1) = \mathrm{CH}^2(Y_{\tilde{\eta}}, 1) = \varinjlim_{U} \mathrm{CH}^2(Y_U, 1),$$

where $Y_{\tilde{\eta}}$ is the generic fiber of Y over Γ_L. We want to spread ξ to *all* of Γ. However, there is obstruction preventing us to do it; rather we can extend it after a suitable modification of ξ. That is, we will show that there exists $\xi' \in \mathrm{CH}^2(Y, 1)$ such that $\mathrm{cl}_{2,1}(\xi_t) = \mathrm{cl}_{2,1}(\xi'_t)$ for every $t \in U$. Our main tool is the localization sequence. (Strictly speaking, we don't really need the localization sequence in this paper. Rather, it is used out of convenience).

$$\mathrm{CH}^2(Y, 1) \to \mathrm{CH}^2(Y_U, 1) \to \mathrm{CH}^1(Y_B) \to \mathrm{CH}^2(Y) \to \mathrm{CH}^2(Y_U) \to 0 \tag{3.1}$$

over $\mathbb{Q}$, where $B = \Gamma\backslash U$ and $Y_B = \cup_{t\in B} Y_t$.

Note that the map $\mathrm{CH}^1(Y_B) \to \mathrm{CH}^2(Y)$ might not be injective if $|B| > 1$, so there is obstruction to extend ξ directly.

Let H be a plane in $\mathbb{P}^3$ and $\pi^*H \subset Y$ be the pullback of H under the projection $\pi\colon Y \to \mathbb{P}^3$. Let $C_b = \pi^*H \cap Y_b$ for $b \in B$ and $C_B = \cup_{b\in B} C_b$. Let us first extend ξ to $Y\backslash C_B$. We look at the localization sequence

$$\mathrm{CH}^2(Y\backslash C_B, 1) \to \mathrm{CH}^2(Y_U, 1) \to \mathrm{CH}^1(Y_B\backslash C_B) \to \mathrm{CH}^2(Y\backslash C_B) \tag{3.2}$$

Note that

$$\mathrm{CH}^1(Y_B\backslash C_B) = \bigoplus_{b\in B} \mathrm{CH}^1(Y_b\backslash C_b) \tag{3.3}$$

We claim that $\mathrm{CH}^1(Y_t\backslash C_t) \otimes \mathbb{Q} = 0$ for every $t \in \Gamma$.

The classical Noether-Lefschetz theorem tells us that a general surface of degree $d \geq 4$ in $\mathbb{P}^3$ has Picard rank 1. This statement was refined by Mark Green [G] to the following. Let $M = \mathbb{P}^N$ be the space parameterizing surfaces of degree d in $\mathbb{P}^3$ and $M_2 \subset M$ be the subset parameterizing surfaces with Picard rank ≥ 2. Then $\operatorname{codim}_M M_2 = d-3$. So when $d \geq 5$, M_2 has codimension at least 2 in M and a general pencil will avoid this locus. Thus $\operatorname{Pic}(Y_t)\otimes\mathbb{Q} = \mathbb{Q}$ for every $t \in \Gamma$. Note that Y_t might be singular, i.e., Y_t has an ordinary double point. Since an ordinary double point is a quotient singularity, every Weil divisor of Y_t is $\mathbb{Q}$-Cartier. Therefore, $\mathrm{CH}^1(Y_t)\otimes\mathbb{Q} = \operatorname{Pic}(Y_t)\otimes\mathbb{Q}$. In any case, we have

$$\mathrm{CH}^1(Y_t)\otimes\mathbb{Q} = \operatorname{Pic}(Y_t)\otimes\mathbb{Q} = \operatorname{Pic}(\mathbb{P}^3)\otimes\mathbb{Q} = \mathbb{Q}. \tag{3.4}$$

Obviously, $\mathrm{CH}^1(Y_t)$ is generated by $C_t = \pi^*H \cap Y_t$ over $\mathbb{Q}$. Consequently,

$$\mathrm{CH}^1(Y_t\backslash C_t)\otimes\mathbb{Q} = 0 \tag{3.5}$$

and there is no obstruction to extend ξ to $Y\backslash C_B$. So we may regard ξ as a class in $\mathrm{CH}^2(Y\backslash C_B, 1)$ from now on.

There might be obstruction to further extend ξ to all of Y by the localization sequence

$$\mathrm{CH}^2(Y,1) \to \mathrm{CH}^2(Y\backslash C_B, 1) \xrightarrow{\phi} \mathrm{CH}^0(C_B) \xrightarrow{\gamma} \mathrm{CH}^2(Y) \tag{3.6}$$

where

$$\mathrm{CH}^0(C_B) = \bigoplus_{b\in B} \mathrm{CH}^0(C_b) = \mathbb{Q}^{\oplus\beta} \tag{3.7}$$

with $\beta = |B|$.

Let $\xi = \sum_\alpha (f_\alpha, D_\alpha)$ where D_α is a divisor on $Y\backslash C_B$ and f_α is a rational function on D_α. We have

$$\sum_\alpha \operatorname{div}(f_\alpha) = 0. \tag{3.8}$$

Let $\overline{D}_\alpha$ be the closure of D_α in Y and f_α naturally extends to a rational function $\overline{f}_\alpha$ on $\overline{D}_\alpha$. Let $\overline{\xi} = \sum_\alpha(\overline{f}_\alpha, \overline{D}_\alpha)$. We no longer have (3.8). Instead,

$$\sum_\alpha \operatorname{div}(\overline{f}_\alpha) = \sum_{b\in B} m_b C_b \tag{3.9}$$

for some $m_b \in \mathbb{Z}$. Actually, the RHS of (3.9) is exactly the image of ξ under the map ϕ: $\mathrm{CH}^2(Y\backslash C_B, 1) \to \mathrm{CH}^0(C_B)$ in (3.6), i.e.,

$$\phi(\xi) = \sum_{b\in B} m_b C_b. \tag{3.10}$$

Note that $\phi(\xi)$ lies in the kernel of γ: $\mathrm{CH}^0(C_B) \to \mathrm{CH}^2(Y)$ and there is a natural map $\mathrm{CH}^0(C_B) \to \mathrm{CH}^1(\Gamma)$ via

$$\mathrm{CH}^0(C_B) \xrightarrow{\gamma} \mathrm{CH}^2(Y) \to \mathrm{CH}^3(\mathbb{P}^3\times\Gamma) \to \mathrm{CH}^1(\Gamma). \tag{3.11}$$

Note that the map $\mathrm{CH}^3(\mathbf{P}^3\times\Gamma) \to [\mathbb{P}^1]\otimes\mathrm{CH}^1(\Gamma) = \mathrm{CH}^1(\Gamma)$, comes from the projective bundle formula. Of course, the map $\mathrm{CH}^0(C_B) \to \mathrm{CH}^1(\Gamma)$ simply sends C_b to Nb, where $N = d$. Thus $\phi(\xi)$ maps to zero under this map, i.e. the divisor $\sum m_b b$ is N-torsion in $\mathrm{CH}^1(\Gamma) = \operatorname{Pic}(\Gamma)$.

Note that π^*H is a fibration of curves over Γ. So the fact $\sum m_b b$ is torsion in $\mathrm{CH}^1(\Gamma)$ implies that $\sum m_b C_b$ is N-torsion in $\mathrm{CH}^1(\pi^*H)$. Consequently, there exists a rational function f_H on π^*H such that

$$\operatorname{div}(f_H) = N\sum_{b\in B} m_b C_b. \tag{3.12}$$

So we may simply modify $\overline{\xi}$ as follows

$$\xi' = \overline{\xi} - \frac{1}{N}(f_H, \pi^* H). \tag{3.13}$$

Now $\xi' \in \mathrm{CH}^2(Y,1)$ and $\underline{\mathrm{cl}}_{2,1}(\xi'_t) = \underline{\mathrm{cl}}_{2,1}(\xi_t)$ for all $t \in U$, where we recall that

$$\underline{\mathrm{cl}}_{2,1}\colon\ \mathrm{CH}^2(Y_t,1) \to \frac{H^3_{\mathcal{D}}(Y_t,\mathbb{Q}(2))}{\mathrm{Hg}^1(Y_t)\otimes\big(\mathbb{C}/\mathbb{Q}(1)\big)}$$

is the induced map. This is due to the fact that the restrictions f_H to Y_t are obviously constants. Thus we can now replace ξ by ξ'. Next observe that even though Y is complete, it may be singular. It is worthwhile pointing out that we can further pull back ξ to a desingularization $\widetilde{Y}$ of Y. More precisely,

Claim (3.14). There exists $\widetilde{\xi} \in \mathrm{CH}^2(\widetilde{Y},1)$ such that $\widetilde{\xi}$ and ξ agree on the open set where $\widetilde{Y}$ and Y are isomorphic.

The usefulness of this claim is as follows. The (cohomological) cycle class map $\mathrm{cl}_{2,1}\colon\ \mathrm{CH}^2(Y,1) \to H^3_{\mathcal{D}}(Y,\mathbb{Q}(2))$ is only defined if Y is smooth. Granting the existence of this cycle class map, the remaining argument only requires the completeness of Y. There is a short exact sequence:

$$0 \to \frac{H^2(Y,\mathbb{C})}{F^2H^2(Y,\mathbb{C}) + H^2(Y,\mathbb{Q}(2))} \to H^3_{\mathcal{D}}(Y,\mathbb{Q}(2)) \to F^2 \cap H^3(Y,\mathbb{Q}(2)) \to 0.$$

But since Y is complete, a weight argument gives $F^2 \cap H^3(Y,\mathbb{Q}(2)) = 0$. Thus for $t \in U$, $\underline{\mathrm{cl}}_{2,1}(\xi_t)$ is given by the restriction $\underline{\mathrm{cl}}_{2,1}(\xi)\big|_{Y_t}$, i.e. induced by the restriction

$$\frac{H^2(Y,\mathbb{C})}{F^2H^2(Y,\mathbb{C}) + H^2(Y,\mathbb{Q}(2))} \to \frac{H^2(Y_t,\mathbb{C})}{F^2H^2(Y_t,\mathbb{C}) + H^2(Y_t,\mathbb{Q}(2))}.$$

Thus as $t \in U$ varies, the class $\underline{\mathrm{cl}}_{2,1}(\xi_t)$ varies by *horizontal* displacement; further, the restriction $H^2(Y) \to H^2(Y_t)$ is a morphism of mixed Hodge structures. Thus $\underline{\mathrm{cl}}_{2,1}(\xi_t)$ is induced by a class in $H^2(Y_t)$, whose Hodge (p,q) components displace *horizontally*, i.e. preserving the given Hodge type. But over the set where $\Gamma \to \mathbb{P}^1$ ramifies, one can find open sets $\Delta_\Gamma \subset U \subset \Gamma$, $\Delta \subset \mathbb{P}^1$, in the strong topology, such that $\Delta_\Gamma \simeq \Delta$. Thus $\underline{\mathrm{cl}}_{2,1}(\xi_t) = 0$, by virtue of:

LEMMA (3.15). *Consider a Lefschetz pencil $\{Z_t\}_{t\in\mathbb{P}^1}$ of surfaces in $\mathbb{P}^3$ of degree $d \geq 1$, and let $U_0 \subset \mathbb{P}^1$ be the smooth set. Further, let $\Delta \subset U_0$ be a disk, and assume given $\gamma_t \in H^2(Z_t,\mathbb{C})$, a horizontal displacement of a class γ for $t \in \Delta$. If the (p,q) components of γ_t also horizontally displace, then $\gamma_t \in \mathrm{Hg}^1(Z_t)$.*

Proof. This follows from a standard monodromy argument, together with the analyticity of Hodge subbundles. □

Finally, we attend to:

Proof of claim. It turns out that the singularities of Y are quite mild. Note that the singularities of Y are introduced during the base change $\Gamma \to \mathbb{P}^1$; Y becomes singular when the map $\Gamma \to \mathbb{P}^1$ ramifies over a point $t \in \mathbb{P}^1$ where X_t is singular, i.e., it has an ordinary double point. Therefore, the singularities of Y have the type of $x^2 + y^2 + z^2 + t^m = 0$. Let $p \in Y$ be such a singularity. We may resolve p by a sequence of blowups:

$$\widetilde{Y} = Y_\mu \xrightarrow{\varphi_\mu} Y_{\mu-1} \xrightarrow{\varphi_{\mu-1}} \dots \xrightarrow{\varphi_1} Y_0 = Y \tag{3.16}$$

where $\mu = \lfloor m/2 \rfloor$. The exceptional divisor $E_k \subset Y_k$ of φ_k is a quadric in $\mathbb{P}^3$; it is a cone over a conic curve if $2k < m$ and it is a smooth quadric if $m = 2k$. Let $p_0 = p$ and $p_k \in E_k$ be the vertex of the cone E_k for $2k < m$. It is obvious that Y_k is locally given by $x^2 + y^2 + z^2 + t^{m-2k} = 0$ at p_k and $\varphi_{k+1} \colon Y_{k+1} \to Y_k$ is the blowup of Y_k at p_k.

In order to pull back ξ to $\widetilde{Y}$, we do it step by step, i.e., we first pull it back to Y_1, then Y_2 and so on. We will show that there exists a sequence of cycles $\{\xi_k \in \mathrm{CH}^2(Y_k, 1)\}$ with all of them agreeing on the open set $Y \backslash \{p\}$.

By induction, it suffices to pull back the cycle $\xi_{k-1} \in \mathrm{CH}^2(Y_{k-1}, 1)$ to $\xi_k \in \mathrm{CH}^2(Y_k, 1)$.

Since $\varphi_k \colon Y_k \to Y_{k-1}$ is the blowup of Y_{k-1} at p_{k-1},

$$Y_k \backslash E_k \cong Y_{k-1} \backslash \{p_{k-1}\}. \tag{3.17}$$

So the question is again to extend a class in $\mathrm{CH}^2(Y_k \backslash E_k, 1)$ to $\mathrm{CH}^2(Y_k, 1)$. We look at the localization sequence

$$\mathrm{CH}^2(Y_k, 1) \to \mathrm{CH}^2(Y_k \backslash E_k, 1) \to \mathrm{CH}^1(E_k) \xrightarrow{\gamma} \mathrm{CH}^2(Y_k)\,. \tag{3.18}$$

If E_k is a cone over a conic curve, then $\mathrm{CH}^1(E_k) = \mathbb{Q}$ (see [Ha, Appendix A, Example 1.1.2, p. 428]) and $\gamma\colon\ \mathrm{CH}^1(E_k) \to \mathrm{CH}^2(Y_k)$ is obviously injective.

Suppose that E_k is a smooth quadric. This happens in the last step of blowups, i.e., when $k = \mu$ and $m = 2\mu$ is even. Now

$$\mathrm{CH}^1(E_k) = \mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Q} \oplus \mathbb{Q}. \tag{3.19}$$

Let $L_1, L_2 \subset E_k$ be the two rulings of E_k which generate $\mathrm{CH}^1(E_k)$. We claim that L_1 and L_2 are numerically independent on Y_k, i.e., there exist divisors $D_1, D_2 \subset Y_k$ such that $D_i \cdot L_j = 0$ if $i = j$ and $D_i \cdot L_j \neq 0$ if $i \neq j$. This certainly implies that γ is injective.

Note that Y_{k-1} has an ordinary double point $x^2 + y^2 + z^2 + t^2 = 0$ at p_{k-1}. It is well known that there exist two small resolutions of Y_{k-1}. That is, we may blow down Y_k along either of the two rulings L_1 and L_2. Let $g\colon Y_k \to Y_k'$ be the blowdown of Y_k along L_1. Let D be an ample divisor on Y_k'. Then $g^*D \cdot L_2 \neq 0$ since D is ample on Y_k' and $g^*D \cdot L_1 = 0$ since $g_* L_1 = 0$. We are done. □

4. Proof of Theorem (1.2)

We see that the very essence of the proof for Theorem (1.1), i.e., the fact that the argument works for quintic surfaces but not quartic surfaces, lies in the result of M. Green that those quintic surfaces with Picard rank two lie in a subset of codimension two in the moduli space of quintic surfaces, while the same is not true for quartic surfaces. For a product of curves $C_1 \times C_2$ with $g(C_1)g(C_2) > 1$, we have a similar situation. It is well known that $\mathrm{Pic}(C_1 \times C_2) = \mathrm{Pic}(C_1) \oplus \mathrm{Pic}(C_2)$ for a general pair (C_1, C_2). Moreover, we have the following

PROPOSITION (4.1). *Let $g_1 = g(C_1)$ and $g_2 = g(C_2)$ with $g_1 g_2 > 1$ and $\mathcal{M}_{g_i}$ be the moduli space of curves of genus g_i. Let $W \subset \mathcal{M}_{g_1} \times \mathcal{M}_{g_2}$ be the locus of the products $C_1 \times C_2$ with $\rho(C_1 \times C_2) > 2$, where $\rho(S)$ is the rank of the Neron-Severi group of S. Then* $\operatorname{codim} W \geq 2$.

We believe that the above proposition is well known. But since we cannot locate a reference to it, we will give a proof at the end of this section.

Fix $E = C_2$ and let $Y \to \Gamma$ be a one parameter family of curves of genus $g_1 = g(C_1)$. For a general choice of Y, we assume that

$$\mathrm{CH}^1(Y_t \times E) = \mathrm{CH}^1(Y_t) \oplus \mathrm{CH}^1(E) \tag{4.2}$$

for every $t \in \Gamma$ by Proposition (4.1), where Y_t is the fiber over a point $t \in \Gamma$. Here we have to be a little careful about the singular fibers of Y as Proposition (4.1) does not say anything about one of C_i being singular. Let $C = Y_s$ be a singular fiber of Y. For a general choice of Y, C has one node and its normalization $\widetilde{C}$ is a general curve of genus $g_1 - 1$. Therefore, $\mathrm{CH}^1(\widetilde{C} \times E) = \mathrm{CH}^1(\widetilde{C}) \oplus \mathrm{CH}^1(E)$ and $\mathrm{CH}^1(C \times E) \cong \mathrm{CH}^1(\widetilde{C} \times E)/\{F_p \sim F_q\}$. where F_p and F_q are the fibers of $\widetilde{C} \times E$ over $p, q \in \widetilde{C}$ which are the two points over the node of C. Hence (4.2) follows for singular fibers Y_s. This is still true after a base change of Y followed by a semi-stable reduction, in which case Y_s is a union of curves $R_0 \cup R_1 \cup ... \cup R_n$ with $g(R_0) = g_1 - 1$, $g(R_1) = ... = g(R_n) = 0$ and $R_i R_{i+1} = R_0 R_n = 1$.

For a subset $U \subset \Gamma$, we use the notation Y_U for $Y_U = \cup_{t \in U} Y_t$.

Let $\xi \in \mathrm{CH}^2(Y_U \times E, 1)$ for some open set $U \subset \Gamma$. We claim that ξ can be extended to everywhere on Y, i.e., there exists $\overline{\xi} \in \mathrm{CH}^2(Y \times E, 1)$ such that $\underline{\mathrm{cl}}_{2,1}(\xi_t) = \underline{\mathrm{cl}}_{2,1}(\overline{\xi}_t)$ for a general point $t \in \Gamma$, where ξ_t and $\overline{\xi}_t$ are the restrictions ξ and $\widetilde{\xi}$ to the fiber $Y_t \times E$, respectively.

Let $B = \Gamma \backslash U$. We have the localization sequence

$$\mathrm{CH}^2(Y \times E, 1) \to \mathrm{CH}^2(Y_U \times E, 1) \xrightarrow{\phi} \mathrm{CH}^1(Y_B \times E) \xrightarrow{\gamma} \mathrm{CH}^2(Y \times E)\,. \tag{4.3}$$

Let $\xi = \sum_\alpha (D_\alpha, f_\alpha)$, where D_α is a divisor on $Y_U \times E$ and f_α is a rational function on D_α. We have

$$\mathrm{div}(\xi) = \sum_\alpha \mathrm{div}(f_\alpha) = 0. \tag{4.4}$$

Let $\overline{D}_\alpha$ be the closure of D_α in $Y \times E$ and f_α naturally extends to a rational function $\overline{f}_\alpha$ on $\overline{D}_\alpha$. Let $\overline{\xi} = \sum_\alpha (\overline{D}_\alpha, \overline{f}_\alpha)$. We no longer have $\mathrm{div}(\overline{\xi}) = 0$. Instead,

$$\mathrm{div}(\overline{\xi}) = \sum_\alpha \mathrm{div}(\overline{f}_\alpha) \in Z^1(Y_B \times E) \tag{4.5}$$

where $Z^k(X)$ is the free abelian group generated by the codimension-k algebraic cycles of X. Obviously, $\mathrm{div}(\overline{\xi})$ is exactly the image of ϕ in the localization sequence (4.3).

By our assumption,

$$\mathrm{CH}^1(Y_B \times E) = \left(\mathrm{CH}^1(Y_B) \otimes \mathrm{CH}^0(E)\right) \oplus \left(\mathrm{CH}^0(Y_B) \otimes \mathrm{CH}^1(E)\right). \tag{4.6}$$

Therefore,

$$\sum_{\alpha} \mathrm{div}(\overline{f}_{\alpha}) \sim_{rat} D_Y \otimes E + \sum_{b \in B} Y_b \otimes D_b \tag{4.7}$$

where $\sim_{rat}$ is the rational equivalence relation, $D_Y \in \mathrm{CH}^1(Y_B)$ and $D_b \in \mathrm{CH}^1(E)$. Therefore, there exist rational functions g_b on $Y_b \times E$ such that

$$\sum_{\alpha} \mathrm{div}(\overline{f}_{\alpha}) + \sum_{b \in B} \mathrm{div}(g_b) = D_Y \otimes E + \sum_{b \in B} Y_b \otimes D_b \tag{4.8}$$

Hence we may replace $\overline{\xi}$ by $\overline{\xi} + \sum_{b \in B}(Y_b \times E, g_b)$ and assume that

$$\mathrm{div}(\overline{\xi}) = D_Y \otimes E + \sum_{b \in B} Y_b \otimes D_b\,. \tag{4.9}$$

Since $\gamma(\mathrm{div}(\overline{\xi})) = 0$,

$$D_Y \otimes E + \sum_{b \in B} Y_b \otimes D_b \sim_{rat} 0 \tag{4.10}$$

in $\mathrm{CH}^2(Y \times E)$. Now choose any point $p \in (E \setminus \cup_{b \in B} D_b)$. Then $D_Y \times p$ is the expression in (4.9) intersected with $Y \times p$. But the expression in (4.9) being $\sim_{rat}$ 0 implies that the intersection cycle is $\sim_{rat}$ 0, i.e. $\{D_Y \times p\} \sim_{rat} 0$. Thus $D_Y = Pr_{Y,*}(D_Y \times p) \sim_{rat} 0$ in $\mathrm{CH}^2(Y)$. Next, by definition of rational equivalence, if $D \in Z^2(Y)$ and $D \sim_{rat} 0$, then there exists a pre-higher Chow cycle $\varepsilon = \sum_{\beta}(D_{\beta}, f_{\beta})$ on Y with $\mathrm{div}(\varepsilon) = D$. Therefore, there exists $\varepsilon = \sum_{\beta}(D_{\beta}, f_{\beta})$ on Y with $\mathrm{div}(\varepsilon) = D_Y$. So we may replace $\overline{\xi}$ by

$$\overline{\xi} - \varepsilon \otimes E = \overline{\xi} - \sum_{\beta}(D_{\beta} \times E, f_{\beta} \times E) \tag{4.11}$$

with resulting $\overline{\xi}$ satisfying

$$\mathrm{div}(\overline{\xi}) = \sum_{b \in B} Y_b \otimes D_b\,. \tag{4.12}$$

Obviously,

$$\sum_{b \in B} Y_b \otimes D_b = \pi^*(\delta) \tag{4.13}$$

for some $\delta \in Z^2(\Gamma \times E)$, where $\pi \colon Y \times E \to \Gamma \times E$ is the projection induced by $Y \to \Gamma$.

Note that if $\pi \colon X \to Y$ is a surjective morphism between two smooth projective varieties. Then

$$\mathrm{CH}^k(Y) \otimes \mathbb{Q} \xrightarrow{\pi^*} \mathrm{CH}^k(X) \otimes \mathbb{Q} \tag{4.14}$$

is injective. This follows by reducing to the case where $\dim Y = \dim X$, and using the fact that $\pi_* \circ \pi^* = \deg \pi$.

Thus there exists a pre-higher Chow cycle ε on $\Gamma \times E$ with $\mathrm{div}(\varepsilon) = N\delta$. Finally, we replace $\overline{\xi}$ by

$$\overline{\xi} - \frac{1}{N}\pi^* \varepsilon \tag{4.15}$$

and obtain a higher Chow cycle $\overline{\xi} \in \mathrm{CH}^2(Y \times E, 1)$. It is easy to check that $\underline{\mathrm{cl}}_{2,1}(\xi_t) = \underline{\mathrm{cl}}_{2,1}(\overline{\xi}_t)$ for a general $t \in \Gamma$.

Next we use a monodromy argument just like in the proof of Theorem (1.1). Applying the same monodromy action considerations to $\pi(\{s \in \Gamma \colon Y_s \text{ is smooth}\}) \to \mathrm{Aut}\big(H^1(Y_t)\big)$, any class in $H^2(Y_t \times E)$ whose Hodge (p,q) components deform horizontally, must be algebraic. But since $\underline{\mathrm{cl}}_{2,1}(\overline{\xi}_t)$ is induced by restriction from a class $\underline{\mathrm{cl}}_{2,1}(\overline{\xi})$, and hence from a cohomology class in $H^2(Y \times E)$, it is clear that the Hodge (p,q) components of $\underline{\mathrm{cl}}_{2,1}(\overline{\xi}_t)$ deform horizontally as the restriction $H^2(Y \times E) \to H^2(Y_t \times E)$ is a morphism of Hodge structures, *a fortiori* $\underline{\mathrm{cl}}_{2,1}(\overline{\xi}_t) = 0$ for general t. We are done.

It remains to give a proof for Proposition (4.1). We will use a deformation-theoretic argument.

LEMMA (4.16). *Let X/Δ be a family of smooth projective surfaces over disk Δ with central fiber $S = X_0$ and let $D \subset S$ be an effective divisor on S. Suppose that D can be extended to X, i.e., there exists a flat family Y/Δ with the commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow \\ \Delta & \longrightarrow & \Delta \end{array} \tag{4.17}$$

such that Y_0 embeds into X_0 with image D. For each $w \in H^0(K_S)$, let μ_w be the map

$$\mu_w \colon H^1(\Omega_S) \xrightarrow{\otimes w} H^1(\Omega_S(K_S)) \tag{4.18}$$

where K_S is the canonical class of S. Then the Kodaira-Spencer class $\mathrm{ks}(\partial/\partial t) \in H^1(T_S)$ *of X lies in the subspace*

$$\{v \in H^1(T_S) \colon \langle v, \mu_w(c_1(D))\rangle = 0 \textit{ for all } w \in H^0(K_S)\} \tag{4.19}$$

where $\langle\cdot,\cdot\rangle$ is the pairing $H^1(T_S) \times H^1(\Omega_S(K_S)) \to \mathbb{C}$ given by Serre duality.

Proof. The pushforward $\pi_* Y$ is a divisor on X whose restriction to X_0 is a multiple of D, say nD. Fix a sufficiently ample divisor A of X and we embed X to $\mathbb{P}^g \times \Delta$ with the linear series $|A + \pi_* Y|$. Let N_S be the normal bundle of S in $\mathbb{P}^g$. Then the Kodaira-Spencer map $\mathrm{ks}\colon T_{\Delta,0} \to H^1(T_S)$ factors through $H^0(N_S)$. Note the exact sequence

$$H^0(N_S) \to H^1(T_S) \xrightarrow{f} H^1(T_{\mathbb{P}^g}|_S). \tag{4.20}$$

We claim that the kernel of f is contained in the space

$$\ker(f) \subset \{v \in H^1(T_S) \colon \langle v, \mu_w(c_1(A_0 + nD))\rangle = 0\} \tag{4.21}$$

where $A_0 + nD$ is the restriction of $A + \pi_* Y$ to S. Then $\mathrm{ks}(\partial/\partial t)$ lies in the space (4.21). The same argument with $A + \pi_* Y$ replaced by A will produce that $\mathrm{ks}(\partial/\partial t)$ lies in

$$\{v \in H^1(T_S) \colon \langle v, \mu_w(c_1(A_0))\rangle = 0\}\,. \tag{4.22}$$

Then it follows that $\mathrm{ks}(\partial/\partial t)$ lies in the space (4.19). So it suffices to justify (4.21).

Consider the dual map

$$f^\vee\colon H^1(\Omega_{\mathbb{P}^g}(K_S)|_S) \to H^1(\Omega_S(K_S))\,. \tag{4.23}$$

Obviously, (4.21) is equivalent to the statement that the image of $f^\vee$ contains $\mu_w(c_1(A_0+nD))$.

From the Euler sequence, we see that $H^1(\Omega_{\mathbb{P}^g}|_S) = H^1(\Omega_{\mathbb{P}^g})$ is generated by $c_1(H)$, where H is the hyperplane divisor of $\mathbb{P}^g$. Of course, $c_1(H)|_S = c_1(A_0 + nD)$. Hence the image of $H^1(\Omega_{\mathbb{P}^g}|_S) \to H^1(\Omega_S)$ is generated by $c_1(A_0+nD)$. From the commutative diagram

$$\begin{array}{ccc} H^1(\Omega_{\mathbb{P}^g}|_S) & \xrightarrow{\otimes w} & H^1(\Omega_{\mathbb{P}^g}(K_S)|_S) \\ \downarrow & & \downarrow f^\vee \\ H^1(\Omega_S) & \xrightarrow{\otimes w} & H^1(\Omega_S(K_S)) \end{array} \tag{4.24}$$

we see that the image of $f^\vee$ contains $\mu_w(c_1(A_0+nD))$ for all $w \in H^0(K_S)$. $\square$

Now let us finish the proof of Proposition (4.1). First, let us deal with the case that $g_1, g_2 > 1$.

Let $D \subset S = C_1 \times C_2$ be an effective divisor with $D \notin \mathrm{CH}^1(C_1) \oplus \mathrm{CH}^1(C_2)$. Then under the decomposition

$$\begin{aligned} H^1(\Omega_S) = H^1(K_{C_1}) \otimes H^0(\mathcal{O}_{C_2}) \oplus H^0(K_{C_1}) \otimes H^1(\mathcal{O}_{C_2}) \\ \oplus H^0(\mathcal{O}_{C_1}) \otimes H^1(K_{C_2}) \oplus H^1(\mathcal{O}_{C_1}) \otimes H^0(K_{C_2}) \end{aligned} \tag{4.25}$$

the projection $\omega_1 + \omega_2$ of $c_1(D)$ to

$$H^0(K_{C_1}) \otimes H^1(\mathcal{O}_{C_2}) \oplus H^1(\mathcal{O}_{C_1}) \otimes H^0(K_{C_2}) \tag{4.26}$$

does not vanish, where

$$\omega_1 \in H^0(K_{C_1}) \otimes H^1(\mathcal{O}_{C_2}) \text{ and } \omega_2 \in H^1(\mathcal{O}_{C_1}) \otimes H^0(K_{C_2}). \tag{4.27}$$

The restriction of μ_w to the subspace (4.26) is

$$\mu_w\colon H^0(K_{C_1}) \otimes H^1(\mathcal{O}_{C_2}) \to H^0(2K_{C_1}) \otimes H^1(K_{C_2}) \tag{4.28}$$

and

$$\mu_w\colon H^1(\mathcal{O}_{C_1}) \otimes H^0(K_{C_2}) \to H^1(K_{C_1}) \otimes H^0(2K_{C_2}). \tag{4.29}$$

It is easy to see that the space spanned by $\mu_w(\omega_i)$ for $w \in H^0(K_S)$ has dimension g_i if $\omega_i \neq 0$. Therefore, the space spanned by $\mu_w(c_1(D))$ has dimension at least $\min(g_1, g_2)$. By Lemma (4.16), the deformations of S preserving D have codimension at least $\min(g_1, g_2)$ in the versal deformation space of S (the deformations of S are unobstructed when $g_1, g_2 > 1$).

Suppose that one of C_1 and C_2 is elliptic. Let $E = C_2$ be elliptic and let $B = C_1$. Let L be a line bundle on $S = B \times E$. For each $b \in B$, let S_b be the fiber of S over $b \in B$ and L_b be the restriction L to $S_b \cong E$. This gives a map $B \to \mathrm{Pic}(E)$ by sending $b \to L_b$. By fixing a base point on E, we obtain a map $\phi\colon B \to J(E) = E$. If ϕ is constant, then it is easy to see that $L \in \mathrm{Pic}(B) \oplus \mathrm{Pic}(E)$ and we are done. If not, we have a nontrivial map from B to E. Fix E and we see that the locus of the curves of genus g_1 that dominates E has dimension $2g_1 - 3$ (ϕ has $2g_1 - 2$ ramification points; -1 for the automorphism of E). Proposition (4.1) now follows.

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POINCARÉ'S REDUCIBILITY THEOREM WITH G-ACTION

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ABSTRACT. A finite group G acting on an abelian variety A induces a decomposition of A up to isogeny. In this paper we prove an equivariant version of Poincaré's reducibility theorem saying that up to isogeny A decomposes into a product of G-simple abelian subvarieties. This decomposition is unique up to isogeny.

Let G be a finite group acting on an abelian variety A defined over an algebraically closed field k of arbitrary characteristic. The abelian variety A is called *G-simple*, if there is no G-equivariant isogeny $A_1 \times A_2 \to A$ with nontrivial abelian varieties A_1 and A_2 with G-action. It is the aim of this note to show that any abelian variety A with G-action admits a G-equivariant isogeny $\varphi : A_1 \times \ldots \times A_r \to A$ with G-simple abelian varieties A_i. Since the image of A_i under such an isogeny φ is a G-simple abelian subvariety of A, it suffices to decompose A into a sum of G-simple abelian subvarieties. Our main result is the following theorem which might be called Poincaré's reducibility theorem with G-action.

THEOREM (1). *Let A be an abelian variety with G-action.*

(a) *There are G-simple abelian subvarieties $B_1, \ldots, B_r$ of A such that the addition map gives a G-equivariant isogeny*

$$\mu\colon B_1 \times \ldots \times B_r \to A\,.$$

(b) *This decomposition is unique in the following sense: Let $B_1', \ldots, B_s'$ be another set of G-simple abelian subvarieties such that the addition map induces a G-equivariant isogeny $B_1' \times \cdots \times B_s' \to A$. Then $r = s$ and there is a permutation σ of degree r such that for $i = 1, \ldots, r$ there is a G-isogeny $B_i \to B'_{\sigma(i)}$.*

In the special case where G is the trivial group this is just Poincaré's original reducibility theorem (see [3]). For the proof we need some preliminaries.

Remark (2). If L denotes a polarization on A, that is the first Chern class of an ample line bundle L on A, we say that *G acts on the polarized abelian variety* (A, L), and L is called a *G-equivariant polarization*, if $g^*L \approx L$ (algebraically equivalent) for any $g \in G$. Note that for any abelian variety A with G-action there exists a polarization L on A, such that G acts on (A, L).

To see this let $G = \{g_1, \ldots, g_n\}$ and let M denote an arbitrary ample line bundle on A. Then $L = g_1^*M \otimes \cdots \otimes g_n^*M$ defines a polarization on A with $g_\nu^*L \approx L$ for all $\nu = 1, \ldots, n$.

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Remark (3). If $\varphi\colon A \to B$ is a G-isogeny of abelian varieties with G-action, it is easy to see that there is a G-isogeny $\psi\colon B \to A$ such that $\psi\varphi = n_A$ and $\varphi\psi = n_B$ for some positive integer n. Notice however, that if $\varphi\colon (A, L) \to (B, M)$ is a G-isogeny of polarized abelian varieties, there is in general no G-isogeny $\psi\colon (B, M) \to (A, L)$.

Now fix a G-equivariant polarization L on A and let $\phi_L\colon A \to \hat{A}$ denote the associated isogeny and "$\,'\,$" the corresponding Rosati-involution of $\mathrm{End}_{\mathbb{Q}}(A)$, i.e. $\varphi' = \phi_L^{-1}\hat{\varphi}\phi_L$ for any endomorphism φ of A.

For an abelian subvariety B of A let $\iota_B\colon B \to A$ denote the canonical embedding. Let $e(B)$ denote the exponent of the induced polarization ι^*L. By definition $e(B)$ is the smallest positive integer such that $\psi_{\iota^*L} := e(B)\phi_{\iota^*L}^{-1}\colon \hat{B} \to B$ is an isogeny. Define the norm endomorphism $N_B \in \mathrm{End}(A)$ of B to be the composition

$$N_B := \iota_B\psi_{\iota^*L}\hat{\iota}_B\phi_L.$$

Then

$$\varepsilon_B = \frac{1}{e(B)}N_B$$

is a symmetric idempotent of $\mathrm{End}_{\mathbb{Q}}(A)$, i.e. $\varepsilon_B^2 = \varepsilon_B = \varepsilon'_B$. So to every abelian subvariety B of A we associated a symmetric idempotent ε_B of $\mathrm{End}_{\mathbb{Q}}(A)$. Conversely if $\varepsilon \in \mathrm{End}_{\mathbb{Q}}(A)$ is a symmetric idempotent, then $n\varepsilon \in \mathrm{End}(A)$ for some $n > 0$ and $\mathrm{Im}\,(n\varepsilon)$ is an abelian subvariety. Thus the polarization L induces a bijection between the sets of abelian subvarieties of A and symmetric idempotents of $\mathrm{End}_{\mathbb{Q}}(A)$ (see [1], Theorem 5.3.2, where it is assumed that A is a complex abelian variety. The proof however works for arbitrary algebraically closed fields). For the proof of Theorem 1 we need 2 observations.

(i) For an automorphism g of A we have $g^*L \approx L$ if and only if $\phi_{g^*L} = \phi_L$. Since $\phi_{g^*L} = \hat{g}\phi_L g$ this is equivalent to

$$\hat{g}\phi_L = \phi_L g^{-1} \tag{4}$$

(ii) Let $\iota_B\colon B \to A$ be the canonical embedding of the abelian subvariety B. Then gB is an abelian subvariety with canonical embedding

$$\iota_{gB} = g\iota_B g^{-1} \tag{5}$$

Lemma (6). $\varepsilon_{gB} = g\varepsilon_B g^{-1}$.

Proof. Since $g^*L \approx L$ we have $e(gB) = e(B)$. Hence the assertion follows from

$$\begin{aligned}
N_{gB} &= e(B)g\iota_B g^{-1}\phi^{-1}_{(g\iota_B g^{-1})^*L}\hat{g}^{-1}\hat{\iota}_B\hat{g}\phi_L \qquad \text{(using (5))}\\
&= e(B)g\iota_B g^{-1}g\phi^{-1}_{\iota_B^*L}\hat{g}\hat{g}^{-1}\hat{\iota}_B\hat{g}\phi_L \quad (\text{since } \phi_{(g\iota_B g^{-1})^*L} = \hat{g}^{-1}\phi_{\iota_B^*L}g^{-1})\\
&= e(B)g\iota_B\phi^{-1}_{\iota_B^*L}\hat{\iota}_B\phi_L g^{-1} \quad \text{(by (4))}\\
&= gN_B g^{-1}
\end{aligned}$$

□

Lemma (7). *If $\varepsilon \in End_{\mathbb{Q}}(A)$ is a symmetric idempotent, so is $g\varepsilon g^{-1}$.*

Proof.

$$\begin{aligned}(g\varepsilon g^{-1})' &= \phi_L^{-1}(g\varepsilon g^{-1})^\wedge\phi_L \\ &= (\phi_L g^{-1})^{-1}\hat{\varepsilon}(\phi_L g^{-1}) \qquad \text{(by (4))} \\ &= g\phi_L^{-1}\hat{\varepsilon}\phi_L g^{-1} \\ &= g\varepsilon g^{-1}\end{aligned}$$

□

Hence the action of G on the set of abelian subvarieties corresponds to conjugation on the set of symmetric idempotents of $\mathrm{End}_{\mathbb{Q}}(A)$. In particular we have with the notation as above

Corollary (8). *$gB = B$ if and only if $g\varepsilon_B g^{-1} = \varepsilon_B$.*

Theorem 1 (a) follows by induction from the subsequent lemma.

Lemma (9). *Let B be a G-stable abelian subvariety of A. There exists a G-stable abelian subvariety C of A, such that the addition map induces an isogeny $\mu : B \times C \to A$.*

Proof. Since B is G-stable, ε_B is G-invariant by Corollary (8). Hence $1-\varepsilon_B$ is a G-invariant symmetric idempotent. Hence again by Corollary (8) the abelian subvariety C corresponding to $1-\varepsilon_B$ is G-stable. Moreover $\varepsilon_B + \varepsilon_C = 1$ implies that $\mu : B \times C \to A$ is an isogeny. □

Proof of Theorem (1) (b). Suppose $B_1 \times \ldots \times B_r \to A$ and $B_1' \times \cdots \times B_s' \to A$ are 2 isogenies with G-simple abelian subvarieties B_i, B_i' of A. By Remark 3 there is a surjective G-homomorphism $B_1 \times \ldots \times B_r \to B_1'$. Now B_j being G-simple for all j there is an B_i, which without loss of generality we may assume to be B_1, such that the restriction $B_1 \to B_1'$ is a G-isogeny.

Renumerate the B_i and B_j' such that $B_1, \ldots, B_{r_1}$ are and $B_{r_1+1}, \ldots, B_r$ are not G-isogenous to B_1 and similarly $B_1', \ldots B_{s_1}'$ are and $B_{s_1+1}', \ldots, B_s'$ are not G-isogenous to B_1. Then the G-isogeny $B_1 \times \ldots \times B_r \to B_1' \times \ldots \times B_s'$ splits into the product of 2 isogenies $f_1 : B_1 \times \ldots \times B_{r_1} \to B_1' \times \ldots \times B_{s_1}'$ and $f_2 : B_{r_1+1} \times \ldots \times B_r \to B_{s_1+1}' \times \ldots \times B_s'$. Since all of $B_1, \ldots, B_{r_1}, B_1', \ldots B_{s_1}'$ are G-isogenous to each other we get $r_1 = s_1$ and the assertion follows by induction. □

The following example shows that the G-simple abelian subvarieties B_i in Theorem 1 are not uniquely determined in general.

Example (10). Let $E_i = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ be the elliptic curve with automorphism group $\mathrm{Aut}E_i =< i >\simeq \mathbb{Z}/4\mathbb{Z}$ and $A = E_i \times E_i$ with the principal product polarization L. Then $< i >$ acts on (A, L) and

$$\mu\colon (E_i \times \{0\}) \times (\{0\} \times E_i) \to A$$

is a decomposition into G-simple abelian subvarieties. On the other hand the diagonal Δ and the antidiagonal $\Xi = \{(x, -x) \in A \,|\, x \in E_i\}$ are G-simple abelian subvarieties giving a G-decomposition

$$\tilde{\mu}\colon \Delta \times \Xi \to A.$$

Note that the induced polarizations $\iota_\Delta^* L$ and $\iota_\Xi^* L$ are of degree 2.

Next we will show that the abelian subvariety of A generated by all abelian subvarieties, which are isogenous to a given one, is uniquely determined.

Let $\mu\colon B_1 \times \cdots \times B_r \to A$ be as in Theorem (1). Renumbering we may assume that there is a sequence of integers $0 = r_0 < r_1 < r_2 < \ldots < r_{s-1} < r_s = r$ such that $B_{r_i+1}, \ldots, B_{r_{i+1}}$ are G-isogenous to each other but not to the other B_j for $i = 0, \ldots, s-1$. Defining A_i to be the abelian subvariety of A generated by $B_{r_{i-1}+1}, \ldots, B_{r_i}$ for $i = 1, \ldots, s$ we obtain a G-decomposition

$$\nu : A_1 \times \ldots \times A_s \to A.$$

with $\mathrm{Hom}_G(A_i, A_j) = 0$ for $i \neq j$. Then we have

Proposition (11). *The decomposition $\nu : A_1 \times \ldots \times A_s \to A$ is uniquely determined.*

Proof. Suppose $\nu' : A_1' \times \ldots \times A_t' \to A$ is a second such decomposition. According to Remark (3) there is a G-isogeny $f\colon A_1' \times \ldots \times A_t' \to A_1 \times \ldots \times A_s$. Now $\mathrm{Hom}_G(A_i, A_j) = 0$ for $i \neq j$ implies that $f(A_1')$ is exactly one of the components, say A_1 of $A_1 \times \ldots \times A_s$. Similarly one obtains $f(A_1') \cap f(A_i') = 0$ for $i \geq 2$. Hence f splits into a product

$$f = f_1 \times f_2$$

with G-isogenies $f_1\colon A_1' \to A_1$ and $f_2\colon A_2' \times \cdots \times A_t' \to A_2 \times \cdots \times A_s$. Hence $A_1' \sim A_1 \sim B_1^{r_1}$ with a G-simple abelian subvariety B_1. If $A_1 \neq A_1'$ as abelian subvarieties of A, we would have

$$A_1' + A_1 \sim B_1^{s_1}$$

with $s_1 > r_1$, contradicting the uniqueness statement of Theorem (1). Now the assertion follows by induction. □

The next proposition says that the abelian subvarieties A_i above are of a special isogeny type.

Proposition (12). *For any A_i in Proposition* (11) *there is a simple abelian variety C_i uniquely determined up to isogeny and a positive integer n_i such that*

$$A_i \sim C_i^{n_i}.$$

Proof. It suffices to prove the analogous statement for any G-simple abelian subvariety B_i of A_i. Let C_i denote any simple abelian subvariety of B_i. The sum $\sum_{g \in G} gC_i$ is a G-simple abelian subvariety of C_i and hence coincides with B_i. This implies the assertion. □

Note that C_i and C_j in Proposition (12) might be isogenous for $i \neq j$.

Let us finally observe that some of the symmetric idempotents can be described in purely algebraic terms. The action of the group G on the abelian variety A induces an algebra homomorphism

$$\rho : \mathbb{Q}[G] \to \mathrm{End}_{\mathbb{Q}}(A)$$

where $\mathbb{Q}[G]$ denotes the rational group algebra of G. If α is any element in $\mathbb{Q}[G]$, we define

$$\mathrm{Im}\ (\alpha) := \mathrm{Im}\ (\rho(m\alpha)) \subset A$$

where m is some positive integer such that $m\alpha \in \mathbb{Z}[G]$. Im (α) is an abelian subvariety of A, which certainly does not depend on the chosen integer m. In order to obtain proper abelian subvarieties we have to choose suitable elements α of $\mathbb{Q}[G]$.

For this recall that $\mathbb{Q}[G]$ is a semisimple $\mathbb{Q}$-algebra of finite dimension. Let

$$1 = e_1 + \cdots + e_r$$

denote the decomposition of the unit 1 into the sum of the central primitive idempotents of $\mathbb{Q}[G]$. Now consider the abelian subvariety $A_i = \mathrm{Im}\ (e_i)$ for $i = 1, \ldots, r$. The symmetric idempotent ε_{A_i} does not make sense in $\mathbb{Q}[G]$. However considering e_i as an element of $\mathrm{End}_{\mathbb{Q}}(A)$ (instead of writing $\rho(e_i)$), we have

PROPOSITION (13). $\varepsilon_{A_i} = e_i$.

Proof. This is a consequence of the uniqueness of the central primitive idempotents in a semisimple algebra. It can also been seen directly using the explicit description of the idempotents e_i as given in [2]. □

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A NOTE ON THE ISOMORPHISM OF MODULAR GROUP ALGEBRAS OF GLOBAL WARFIELD ABELIAN GROUPS

PETER DANCHEV

Abstract. Let G be a global Warfield Abelian group and F a field of char $F = p > 0$. It is shown that if for some group H the group algebras FH and FG are F-isomorphic, then H is isomorphic to G modulo their special torsion subgroups that are subgroups of elements of finite orders which are invertible in F, namely $H/\coprod_{q\neq p} H_q \cong G/\coprod_{q\neq p} G_q$. This extends a result due to W. L. May obtained when G is p-local Warfield Abelian (Proc. Amer. Math. Soc., 1988).

Throughout the text, RG denotes the group algebra of a multiplicative Abelian group G over a commutative ring R with identity and prime characteristic p. All notions and notations which we will further use are standard. For instance, G_p shall designate the p-component of G, $V(RG)$ the group of normed units in RG, and for every prime number p the ranks (=dimensions) of the vector spaces $G^{p^\alpha}[p]/G^{p^{\alpha+1}}[p]$ and $G^{p^\alpha}/(G^{p^{\alpha+1}}G_p^{p^\alpha})$ are the Ulm-Kaplansky and Warfield p-invariants respectively. As usual, the letter F is reserved for a field of characteristic $p \neq 0$.

Following [U], $\mathrm{inv}(R)$ is the set of all rational primes which invert in R, *i.e.* primes q such that $q \cdot 1_R$ is a unit of R. Put $G_R = \{\coprod G_q : q \in \mathrm{inv}(R) \subseteq G_t\}$, the maximal torsion subgroup of G. Evidently $G_R = \coprod_{q\neq p} G_q$.

In 1988, May proved in [May] that when G is a p-local Warfield group the R-isomorphism $RG \cong RH$ for any group H gives $G \cong H$, thus improving a result appeared in [BRW] in the case when G and H are both p-local Warfield groups.

Our aim in this short note is to explore the isomorphism problem of commutative modular group algebras of global Warfield groups, arguing in this way that the above cited May's theorem may be generalized to such a more large class of mixed abelian groups.

Before formulating the major statement, we need a few technical conventions beginning with

Proposition (1). (a) *The Ulm-Kaplansky p-invariants of G may be recaptured from FG* [M],

(b) *The Warfield p-invariants of G can be retrieved by FG.*

Proof. (a) By definition, we elementarily see that the Ulm-Kaplansky p-functions of G are precisely those of G_p. Consequently, exploiting [M], we are done.

(b) Follows directly owing to [K] and to the method described by us in [D]. ☐

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Next, we emphasize some facts necessary for proving the main affirmation.

Proposition (2). *The isomorphism $FG \cong FH$ as F-algebras implies:*
(a) *H is torsion provided G is torsion* [M],
(b) $F(G/G_F) \cong F(H/H_F)$ [U]. *As a consequence, when G is torsion, $FG_p \cong FH_p$.*

So, we concentrate now on the verification of the central attainment (see [Dan] too).

Isomorphism Theorem. *Supppose G is a Warfield Abelian group. Then $RH \cong RG$ as R-algebras for any group H yields $H/H_R \cong G/G_R$. In particular when G is p-mixed Warfield Abelian, $H \cong G$.*

Proof. If we set $F = R/M$ for some maximal ideal M containing p, then F is a field of characteristic p and $RG \cong RH$ implies $FG \cong FH$ as F-algebras. It is well known that the latter isomorphism may be assumed to preserve augmentation. We therefore write $FG = FH$ and regard H as a subgroup of $V(FG) = V(FH)$ that is a linear basis for FG.

Foremost, presume G is p-mixed. It now easily follows that H is also p-mixed. Let X' be a nice decomposition basis for G. Because $V(FH)/H$ is a p-group, there is a subordinate X of X' that is contained in H, and it follows that X is a nice decomposition basis for both G and H with $G/\langle X\rangle$ simply presented torsion. Moreover, $F(G/\langle X\rangle) \cong F(H/\langle X\rangle)$ so that $F((G/\langle X\rangle)_p) \cong F((H/\langle X\rangle)_p)$ and $H/\langle X\rangle$ is torsion by Proposition (2). Whence, by the isomorphism theorem in [May], $(G/\langle X\rangle)_p \cong (H/\langle X\rangle)_p$.

We claim that $(H/\langle X\rangle)_q$ is simply presented for all primes $q \neq p$, and thereby establish that H is a global Warfield group. Since H has trivial q-torsion, the q-height of an element of H must be either a finite ordinal or ∞, and the q-height sequence contains no gaps. Therefore, the q-height sequence of an element of H is of the form $(n, n+1, n+2, \ldots)$, where n is finite, or else $(\infty, \infty, \infty, \ldots)$. Write $X = \{x_i : i < \alpha\}$ for some ordinal α and, for each $i < \alpha$, let n_i denote the q-height of x_i. Now take

$P = \langle y \in H : y^{q^r} = x_i$ for some $i < \alpha$ and nonnegative integer $r\ \rangle$.

Since H has no q-torsion, it is clear that $P/\langle X\rangle \cong \oplus_{i<\alpha}\mathbf{Z}(q^{n_i})$, a direct sum of cocyclic (cyclic and quasi-cyclic) groups and hence simply presented. Furthermore, to establish the claim, we need only show that $(H/\langle X\rangle)_q = P/\langle X\rangle$. Because the reverse inclusion is plain, it is enough to show that $(H/\langle X\rangle)_q \subseteq P/\langle X\rangle$. To this end, suppose that $h\langle X\rangle \in (H/\langle X\rangle)_q$ for some $h \in H\setminus\langle X\rangle$. Then, $h^{q^s} \in \langle X\rangle$ for some positive integer s, and we can write $h^{q^s} = x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}\ldots x_{i_t}^{\varepsilon_t}$, where $i_1, i_2, \ldots, i_t$ are distinct indices $< \alpha$, and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$ are positive integers. Since $\langle X\rangle = \oplus_{i<\alpha}\langle x_i\rangle$ is a valuated coproduct and h^{q^s} has q-height at least s, $x_{i_j}^{\varepsilon_j}$ has q-height at least s in H for $j = 1, 2, \ldots, t$. Consequently, $h_j^{q^s} = x_{i_j}^{\varepsilon_j}$ for some $h_j \in H$. Note however that we may select the h_j so that $h_j \in P$ for all j. Indeed, if $n_{i_j} = \infty$, we get $y_j^{q^s} = x_{i_j}$ for some $y_j \in P$ and thus $h_j = y_j^{\varepsilon_j}$ has the desired property. On the other hand, if n_{i_j} is finite, select $y_j \in P$ with $y_j^{q^{n_{i_j}}} = x_{i_j}$. Since the q-height sequence of x_{i_j} has no gaps, note that $u_j = q^{s-n_{i_j}}$ must divide ε_j, in which case $h_j = y_j^{\varepsilon_j/u_j}$ possesses the wanted property. Recalling once again that H has trivial q-torsion,

$$h^{q^s} = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \dots x_{i_t}^{\varepsilon_t} = h_1^{q^s} h_2^{q^s} \dots h_t^{q^s}$$

implies that $h = h_1 h_2 \dots h_t \in P$. Therefore, $h\langle X\rangle \in P/\langle X\rangle$ and this establishes the claim that $(H/\langle X\rangle)_q = P/\langle X\rangle$ is simply presented for all primes $q \neq p$. So, H is a p-mixed Warfield group as is G and, by Proposition (1), these groups have the same Ulm-Kaplansky and Warfield invariants. That $G \cong H$ now follows from the uniqueness theorem of [HR].

After this, when G is arbitrary Warfield, we observe that $G/\coprod_{q\neq p} G_q$ is p-mixed Warfield. In order to infer this, we know by definition from ([HR]) that G is a direct factor of some simply presented group A, hence $G/\coprod_{q\neq p} G_q$ is an outer direct factor for $A/\coprod_{q\neq p} A_q$. Since $\coprod_{q\neq p} A_q$ is torsion p-divisible, it is routine checked that $A/\coprod_{q\neq p} A_q$ is simply presented, that substantiates our claim. But Proposition (2) ensures $F(G/\coprod_{q\neq p} G_q) \cong F(H/\coprod_{q\neq p} H_q)$ and so the first step guarantees our general statement, as expected.

The proof is finished after all. □

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STABLE PARTITIONS AND ALPERIN'S WEIGHT CONJECTURE FOR THE SYMMETRIC GROUPS IN CHARACTERISTIC TWO

RADHA KESSAR AND LUIS VALERO-ELIZONDO

Abstract. Alperin's weight conjecture for the symmetric groups has been proved using an enumeration of the weights and the simple modules (see [2]), but so far there is no explicit way to associate weights with simple modules. Based on data obtained using an algorithm for finding weights for small symmetric groups in characteristic two (see [16]), we put forward a combinatorial conjecture which, if true, would provide explicit bijections between weights and irreducible modules for the symmetric groups in characteristic two. We prove some results towards the proof of this combinatorial conjecture.

1. Introduction

Alperin's weight conjecture is one of the most important and difficult open problems in the representation theory of finite groups. This conjecture has already been established for several families of groups, but some of these proofs are just enumerations of the sets of weights and irreducibles, with no explicit correspondence between them. Such is the case of the symmetric groups. Alperin and Fong proved in [2] that Alperin's conjecture holds for the symmetric groups, so we know that the number of weights for kS_n equals the number of simple kS_n-modules, where k is a field of characteristic $p > 0$. In [16] the second author used Brauer quotients to assign weights to the simple modules parameterized by three infinite families of 2-regular partitions. Using computer software written in GAP (see [9]), he used Brauer quotients to give an explicit bijection bewteen weights and irreducible modules for kS_n for $n \leq 9$. This information was gathered in one Table of Partitions, whose rows are indexed by the weight subgroups of all the symmetric groups, and whose columns are indexed by the triangular partitions. This table is in Section 4. In this section we also observe some remarkable properties of this Table of Partitions, and put forward a conjecture. This conjecture is a stronger version of a reformulation of Alperin's conjecture for the symmetric groups in characteristic two.

In Section 2 we define weights and state Alperin's conjecture. In Section 3 we give James' construction of the irreducible kS_n-modules in characteristic p. At the end of this section we also define skew-hooks and cores of partitions, which are needed to determine the blocks of the irreducibles. Section 4 has the Table

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of Partitions and our proposed conjecture. Section 5 has the results we have towards a proof of this conjecture.

2. Alperin's Conjecture

We give the definition of weight and formulate Alperin's Conjecture in its most general form. We mention some classes of groups for which it is known to be valid (including the symmetric groups) and we note the possible advantages of a combinatorial proof, that is, an explicit bijection between weights and irreducible modules.

Throughout this section, G will be a finite group, p a prime number, and k a splitting field for G in characteristic p. All our modules will be finite dimensional over k.

Definition (2.1). A *weight* for G is a pair (Q, S) where Q is a p-subgroup and S is a simple module for $k[N_G(Q)]$ which is projective when regarded as a module for $k[N_G(Q)/Q]$.

Remark (2.2). Since S is $k[N(Q)]$-simple and Q is a p-subgroup of $N_G(Q)$, it follows that Q acts trivially on S, so S is also a $k[N_G(Q)/Q]$-module and the definition makes sense. Moreover, S is $k[N_G(Q)/Q]$-simple as well.

Remark (2.3). If we replace S by an isomorphic $k[N_G(Q)]$-module we consider this the same weight, and we make the same identification when we replace Q by a conjugate subgroup (so that the normalizers will be conjugate, too).

Now we can formulate the main problem that we shall discuss in this section.

Conjecture (2.4) (Alperin's conjecture). The number of weights for G equals the number of simple kG-modules.

A stronger version of the preceding statement is that there is a bijection within each block of the group algebra.

Definition (2.5). If (Q, S) is a weight for G, then S belongs to a block b of $N_G(Q)$ and this block corresponds with a block B of G via the Brauer correspondence; hence we can say that the weight (Q, S) *belongs* to the block B of G so the weights are partitioned into blocks.

Conjecture (2.6) (Alperin's Conjecture, Block Form). The number of weights in a block of G equals the number of simple modules in the block.

This version of the conjecture implies the original one, as it can be obtained by summing the equalities from the stronger conjecture over the blocks. This stronger conjecture has been proved when G is a:

Finite group of Lie type and characteristic p (Cabanes, [8]).
Soluble group (Okuyama, [14]).
Symmetric group (Alperin and Fong, [2]).
$GL(n, q)$, p odd and p does not divide q (Alperin and Fong, [2]).
$GL(n, q)$, $p = 2$ and q odd (An, [3]).

The conjecture has also been checked in a variety of other cases (see [4], [5], [6], [7], etc.).

Alperin and Fong's proof in the case of symmetric groups was just an observation of a numerical equality which did not suggest a deeper reason for the relationship. For finite groups in general one does not expect to have any canonical bijection between weights and simple modules; as a matter of fact, Alperin himself says this is unlikely (see [1], p 369). For groups of Lie type in their defining characteristic there is a canonical bijection (described in [1]). Since symmetric groups and groups of Lie type have such strong connections in their representation theory, it is reasonable to ask whether there is some canonical bijection in the case of symmetric groups.

If true, Alperin's conjecture would imply a number of known results, until now unrelated (see [1]). It is also reasonable to expect that if an explicit bijection can be given to prove it, this may reveal new connections between simple kG-modules and weights; there are many results known about the former, and the latter are related to the blocks of defect zero, which are not as easy to deal with as the simple modules. In fact, this is really the true importance of Alperin's conjecture in that it provides a connection between the blocks of defect zero and the set of all simple modules. More specifically, Alperin's conjecture has been shown by Knörr and Robinson [13] to be equivalent to a statement which expresses the number of blocks of defect zero of a group in terms of the number of p-modular irreducibles of sections of the group of the form $N_G(P)/P$, $P \leq G$ a p-subgroup. These latter numbers are easy to compute, since by a theorem of Brauer the number of p-modular irreducibles of a group equals the number of p-regular conjugacy classes.

3. Some important kS_n-modules

We define the modules M^λ, S^λ and D^λ following James [11]. The simple kS_n-modules, as is well known, can be parameterized by certain partitions of n called p-regular, where p is the characteristic of the field k. Moreover, it is possible to construct each simple module from its associated partition. We end this section with the definition of p-core. In this section n is a natural number, k is a field of characteristic $p > 0$ and λ is a partition of n.

Definition (3.1). A λ-*tableau* is one of the $n!$ arrays of integers obtained by replacing each node in the partition λ by one of the integers $1, 2, \ldots, n$, allowing no repeats. If t is a tableau, its *row stabilizer*, R_t, is the subgroup of S_n consisting of the elements which fix all rows of t setwise. The *column stabilizer* of t, denoted C_t, is the subgroup of S_n consisting of the elements which fix all columns of t setwise. The *signed column sum* of t, denoted κ_t, is the element of kS_n given by

$$\kappa_t := \sum_{\pi \in C_t} (-1)^{\mathrm{sign}(\pi)} \pi.$$

We define an equivalence relation on the set of λ-tableaux by $t_1 \sim t_2$ if and only if $\pi t_1 = t_2$ for some $\pi \in R_{t_1}$. The *tabloid*, $\{t\}$ containing t is the equivalence class of t under this relation. The kS_n-module $M^\lambda = M_k^\lambda$ is the vector space over k whose basis elements are the various λ-tabloids. The *polytabloid*, e_t, associated with the tableau t is given by

$$e_t := \kappa_t \{t\}.$$

The *Specht module*, $S^\lambda = S^\lambda_k$ for the partition λ is the submodule of M^λ spanned by polytabloids (this is indeed a kS_n-module).

We also define an S_n-invariant, symmetric, non-singular *bilinear form* $\langle\ ,\ \rangle$ on M^λ, whose values on pairs of tabloids is given by

$$\langle t_1, t_2\rangle := \begin{cases} 1 & \text{if } t_1 = t_2, \\ 0 & \text{if } t_1 \neq t_2. \end{cases}$$

The partition λ is *p-singular* if it has at least p rows of the same size; otherwise, λ is *p-regular*. The module $D^\lambda = D^\lambda_k$ is defined as

$$D^\lambda := S^\lambda / (S^\lambda \cap S^{\lambda\perp})$$

where λ is a p-regular partition.

THEOREM (3.2) (James). *As λ varies over p-regular partitions of n, D^λ varies over a complete set of inequivalent irreducible kS_n-modules. Each D^λ is self-dual and absolutely irreducible. Every field is a splitting field for S_n.*

For a proof of this result, see [11].

Definition (3.3). Let λ be a partition. A *skew-hook* is a connected part of the rim of λ which can be removed to leave a proper diagram. The *r-core* of λ is the partition obtained by removing all possible skew-hooks of size r from λ (this is a well-defined partition, that is, the order in which we remove the skew-hooks does not matter). A *brick* is a skew-hook of size 2 (compare to the definition of *domino* in [10]). Recall that if $\lambda = (\lambda_1, \ldots, \lambda_t)$ is a 2-regular partition, then its rows must be of different sizes, i.e. $\lambda_1 > \lambda_2 > \cdots > \lambda_t$.

THEOREM (3.4) (Nakayama's Conjecture). *Let α and β be p-regular partitions of n, and let k be a field of characteristic $p > 0$. Then D^α and D^β lie in the same block of kS_n if and only if α and β have the same p-cores.*

For a proof of this result see [12], Thm 6.1.21.

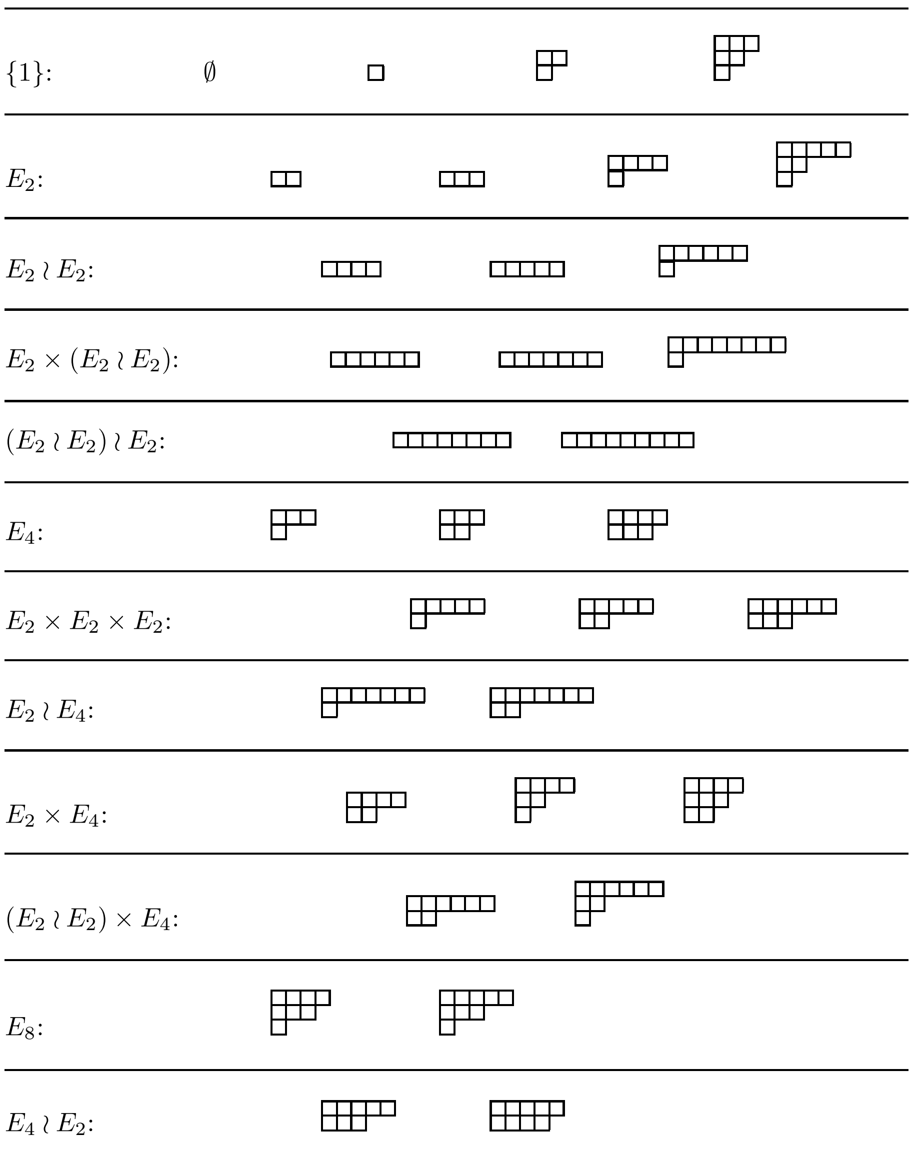
4. Table of partitions

Since the number of weights for the symmetric group S_n equals the number of simple kS_n-modules, one can define explicit bijections between weights and irreducibles. Given any such possible bijection, it is natural to ask whether there is a pattern hidden in its construction. The following table of partitions shows the pattern in the case of the partial correspondence described in [16].

Notice that the fact that the characteristic is 2 implies that each weight (Q, S) for S_n is uniquely determined by its weight subgroup Q. Indeed, the quotient $N_{S_n}(Q)/Q$ is in general the semidirect product of some copies of $GL(m_i, 2)$ and S_m (see [2]). Since any simple projective module for the direct product of $GL(m_i, 2)$ is S_m-invariant and since S_m has at most one simple projective module, Clifford theory tells that (Q, S) is determined by Q.

Each weight subgroup of S_n is used to index one row of the table. In order to determine the weight associated to a simple S_n-module V, one should first look up in the table the partition that parameterizes V, and locate the subgroup Q of S_n that indexes its row. By the previous remark, this subgroup can be completed to a unique weight (Q, S) for S_n. This is the weight that corresponds to the irreducible module V.

The correspondence for these small values of n was determined using Brauer quotients. For these specific irreducible modules and weight subgroups, we constructed their Brauer quotients (see [16] for their definition) and determined which ones were simple and projective. This algorithm worked for all but one of the simple modules of kS_n for $n \leq 9$ and k a field of characteristic two. We used software written in GAP (see [9]) to make these computations. The routines we used were written by Peter Webb and Luis Valero-Elizondo.



Two-regular partitions

Notice the following facts about this table of partitions:

1. The trivial subgroup indexes a row that consists of all triangular partitions (we included the triangular partitions of size 0 and 1 for completeness).
2. Each weight subgroup Q appears for the first time inside a symmetric group S_n where n is such that Q has no fixed points on the set $\{1, \ldots, n\}$. Moreover, if a 2-regular partition of size m appears in a row indexed by the group Q, then Q is a weight subgroup of S_m.
3. The first partition of every row has empty 2-core. The second partition has 2-core of size 1, the third has 2-core of size 3 and the fourth has 2-core of size 6. In other words, the 2-core of every partition along the i-th column is the i-th triangular partition (where $\emptyset$ is the first triangular partition).
4. For every row, all partitions λ in that row are such that the difference of the size of λ minus the size of its 2-core is constant.
5. Along every row, each partition is contained in the one to its right.

Item 1 is just stating the well-known fact that in characteristic 2, the only symmetric groups with simple projective modules are the S_t with t a triangular number, and that such modules are parameterized by the corresponding triangular partitions. Item 2 is proved implicitly in [2].

It is rather straightforward to come up with the following conjecture:

Conjecture (4.1). It is possible to arrange all 2-regular partitions in an infinite table satisfying the five conditions mentioned above.

Note that the existence of an infinite table of partitions satisfying conditions 1, 2, 3 and 4 is equivalent to the block version of Alperin's conjecture for the symmetric groups in characteristic two. Indeed, all we have to do is choose arbitrary bijections between weights (or rather, weight subgroups) and irreducibles in their blocks, which are parameterized by partitions with appropriate 2-cores.

In this paper we prove that if a table of matrices satisfying all five conditions exists, then most of its data is completely determined by a few entries.

5. 2-stability

We define the main concept of this paper.

Definition (5.1). Let λ be a partition. We call λ *2-stable* if it has the same number of rows as its 2-core.

Proposition (5.2). *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ be a partition. The following are equivalent:*

(i) *λ is 2-stable.*

(ii) *$\lambda_t \equiv 1 (\operatorname{mod} 2)$ and $\lambda_i \not\equiv \lambda_{i+1} (\operatorname{mod} 2)$ for all $i = 1, \ldots, t-1$.*

(iii) *$\lambda_i \equiv t - i + 1 (\operatorname{mod} 2)$ for all $i = 1, \ldots, t$.*

(iv) *λ is obtained from its 2-core by adjoining horizontal bricks to the non-empty rows of the core.*

Proof. (i) implies (ii): If λ_t were even, then the t-th row would be a string of horizontal bricks, so we could remove it and the 2-core would have at most $t-1$ rows. Thus λ_t must be odd. If λ_{t-1} were also odd, then we would be able to remove all nodes but one from λ_t, all nodes but one from its neighbour and then remove a vertical brick, which means the 2-core would have at most $t-2$

rows. Now assume that the parities of $\lambda_t, \lambda_{t-1}, \ldots, \lambda_{i+1}$ alternate. We must show that $\lambda_i \not\equiv \lambda_{i+1} (\bmod 2)$. Without loss of generality, we may assume that $\lambda_t = 1, \lambda_{t-1} = 2, \ldots, \lambda_{i+1} = t - i$ (by removing all possible horizontal bricks from the bottom row λ_t and working our way up). Note that the 2-core of λ must have t rows, so it must be the triangular partition $(t, t-1, \ldots, 1)$, and in particular, its $i+1$ row has size $t-i$. If λ_i had the same parity as λ_{i+1}, then we would be able to remove horizontal bricks from λ_i until we have $t-i$ nodes left, and then we would be able to remove a vertical brick, so that the row $i+1$ of the 2-core of λ would have at most $t-i-1$ nodes, contradicting the fact that it had exactly $t-i$ nodes.

(ii) implies (iii): We have $\lambda_t \equiv 1 (\bmod 2)$, so (iii) holds when $i = t$. Now use induction going down from $i = t$ to $i = 1$.

(iii) implies (iv): Since $\lambda_t \equiv 1 (\bmod 2)$, we can remove horizontal bricks from the last row to leave one node, then proceed to remove horizontal bricks from the previous row to leave two nodes, and work our way up until we get the triangular partition $(t, t-1, \ldots, 1)$, which is the 2-core of λ.

(iv) implies (i): If we remove the horizontal bricks that were adjoined we shall obtain the 2-core of λ, so both partitions must have the same number of rows (no bricks were added to form new rows). □

Corollary (5.3). *If λ is 2-stable, then it is also 2-regular.*

Proof. Since $\lambda_i \not\equiv \lambda_{i+1} (\bmod 2)$, no two consecutive rows can have the same size. □

Corollary (5.4). *If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ is 2-stable, then the partition given by $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_t + 1, 1)$ is also 2-stable.*

Proof. This follows from Proposition (5.2), part (ii). □

Remark (5.5). Note that the rows of a 2-stable partition have the same parity as the rows of its 2-core. A possible way to measure how far a partition is from being 2-stable is to count the number of its "mismatched rows", that is, the rows that have a different parity from the corresponding rows of the 2-core. The following lemma gives an estimate of how many of these rows an arbitrary partition can have.

Lemma (5.6). *Let $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ be a partition (not necessarily 2-regular) with 2-core $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$. Let $\Lambda = \{i \mid \gamma_i \not\equiv \mu_i (\bmod 2),\ 1 \leq i \leq k\}$ be the set of "mismatched" rows of μ. Then $s \geq k + 2|\Lambda|$.*

Proof. We use induction on $|\mu|$. If $|\mu| = 0$ or 1 then μ is its own 2-core and $\Lambda = \emptyset$. Now assume the result holds for all partitions of size smaller than $|\mu|$. If μ is a 2-core, then once again $\Lambda = \emptyset$ and the result holds. If μ is not a 2-core, let ν be any partition obtained from μ by removing a brick, so $|\nu| < |\mu|$ and the 2-core of ν is also γ. Let s_1 be the number of rows of ν, and $\Lambda_1 = \{i \mid \gamma_i \not\equiv \nu_i (\bmod 2),\ 1 \leq i \leq k\}$ the set of mismatched rows of ν. By the induction hypothesis, $s_1 \geq k + 2|\Lambda_1|$. There are two cases:

Case 1: We removed a horizontal brick to obtain ν from μ. Then $\Lambda_1 = \Lambda$, so $s \geq s_1 \geq k + 2|\Lambda_1| = k + 2|\Lambda|$.

Case 2: We removed a vertical brick to obtain ν from μ. Let $i, i+1$ be the rows where the vertical brick was removed. Then $\nu_i = \nu_{i+1}$. If $\nu_{i+1} > \nu_{i+2}$, then continue to remove all possible vertical bricks from rows $i, i+1$ until rows $i+1$ and $i+2$ have the same size. If $\nu_{i+2} > \nu_{i+3}$ then remove all possible vertical bricks from rows $i+1, i+2$, and continue in this manner until you reach the last two rows, $s-1, s$ (which could have been the original $i, i+1$), and simply remove them both (using vertical bricks). Let $\alpha = (\alpha_1, \ldots, \alpha_{s-2})$ be the resulting partition. Notice that α and μ have the same 2-core, $|\alpha| < |\mu|$ and α has exactly two fewer rows than μ. Let $\Lambda_2 = \{i \mid \gamma_i \neq \alpha_i (\mathrm{mod}\, 2),\ 1 \leq i \leq k\}$. All vertical bricks removed from any of the first k rows kept the size of two consecutive rows equal, so exactly one out of each such pair contributed to the set of mismatched rows, and the number of mismatched rows remained the same. Similarly, no vertical bricks removed from any of the rows $k+1$ through s changed the size of the set of mismatched rows (because these rows do not appear in the 2-core). The only time when the number of mismatched rows could have changed was while removing vertical bricks from the rows k and $k+1$, and this number cannot have been changed by more than one unit (depending on whether the k-th row kept its mismatched status or not). This means that $||\Lambda| - |\Lambda_2|| \leq 1$, and since α satisfies the induction hypothesis, we have

$$s - 2 \geq k + 2|\Lambda_2| \geq k + 2(|\Lambda| - 1),$$

and the result is valid for μ. □

Now we can prove our main result.

THEOREM (5.7). *Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a 2-stable partition, and let $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ be a 2-regular partition containing λ. If $|\mu| = |\lambda| + t + 1$ and the 2-core of μ is $(t+1, t, \ldots, 1)$, then $\mu = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_t + 1, 1)$, and μ is 2-stable.*

Proof. Since λ is a subpartition of μ, it is possible to write $\mu = (\lambda_1 + \alpha_1, \ldots, \lambda_t + \alpha_t, \alpha_{t+1}, \ldots, \alpha_s)$. It suffices to show that $\alpha_i \geq 1$ for all $1 \leq i \leq t+1$ since then the equality $|\mu| - |\lambda| = t+1$ forces $\alpha_i = 1$ where $1 \leq i \leq t+1 = s$. Note that $\alpha_{t+1} \geq 1$ since the 2-core of μ has $t+1$ rows. Let $\Gamma = \{i \mid \alpha_i = 0,\ 1 \leq i \leq t\}$. We must show that $\Gamma = \emptyset$. Suppose $|\Gamma| \geq 1$. Note that

$$\sum_{i=1}^{t} \alpha_i = \sum_{i \in \{1, \ldots, t\} - \Gamma} \alpha_i \geq t - |\Gamma|.$$

Since λ is 2-stable, by Proposition (5.2) (iii) it has no mismatched rows. However, the 2-core of μ is the next triangle, and all the rows of the smaller triangle must change parity, so all the rows of λ that kept their parity will be mismatched rows of μ, so Γ is a subset of the set Λ of mismatched rows of μ. By Lemma (5.6) we have $s \geq (t+1) + 2|\Lambda| \geq t + 1 + |\Gamma|$, so

$$s - t \geq |\Gamma| + 1\,.$$

Since μ is 2-regular, $1 \leq \alpha_s < \alpha_{s-1} < \cdots < \alpha_{t+1}$, so $\sum_{i=t+1}^{s} \alpha_i \geq \sum_{i=1}^{s-t} i = \frac{(s-t)(s+1-t)}{2}$, and

$$\begin{aligned} t+1 = \sum_{i=1}^{s} \alpha_i = \sum_{i=t+1}^{s} \alpha_i + \sum_{i \in \{1,\ldots,t\}-\Gamma} \alpha_i \geq \frac{(s-t)(s+1-t)}{2} + t - |\Gamma| \\ \geq t + \frac{(|\Gamma|+1)(|\Gamma|+2)}{2} - |\Gamma| = t + \frac{|\Gamma|^2 + 3|\Gamma| + 2}{2} - |\Gamma| \\ = t + 1 + \frac{|\Gamma|^2 + |\Gamma|}{2} > t+1 \end{aligned}$$

which is a contradiction. It is now immediate that μ is 2-stable. □

If an infinite table of partitions satisfying the five conditions from Section 4 existed, then by Theorem (5.7) we see that for any 2-stable partition λ in this table, the partitions on the same row and to the right of λ are completely determined. Now we shall prove that in any row of such a table of partitions there are only finitely many partitions which are not 2-stable.

Lemma (5.8). *Let λ be a 2-regular partition with 2-core γ, and let t be the number of rows of γ. If λ is not 2-stable, then $|\lambda| - |\gamma| \geq t + 1$.*

Proof. Let $\gamma = (\gamma_1 > \gamma_2 > \cdots > \gamma_t > 0 = \gamma_{t+1})$, $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_s)$. Since λ is not 2-stable, then $\lambda_{t+1} \geq 1 = 1 + \gamma_{t+1}$. The partition λ is 2-regular, so $\lambda_t \geq \lambda_{t+1} + 1 \geq 2 = 1 + \gamma_t$. Inductively we have that $\lambda_i \geq 1 + \lambda_{i+1} \geq 1 + (1 + \gamma_{i+1}) = 1 + \gamma_i$ for all $i = 1, \ldots, t+1$. Therefore, $|\lambda| - |\gamma| \geq \sum_{i=1}^{t+1} (\lambda_i - \gamma_i) \geq t + 1$. □

Corollary (5.9). *Let n be a positive integer. Then there exist only finitely many non 2-stable partitions λ such that $|\lambda| - |\gamma| \leq n$, where γ is the 2-core of λ. In particular, in any table of partitions satisfying condition 4 from Section 4, in any row of this table there are only finitely many non 2-stable partitions.*

Proof. Assume λ and γ are as above. By Lemma (5.8), the number of rows of γ is less than or equal to $n-1$, and since γ is a triangular partition, then $|\gamma| \leq (n-1)n/2$, so $|\lambda| \leq |\gamma| + n \leq \frac{(n-1)n}{2} + n = \frac{n^2+n}{2}$, and there are only finitely many partitions whose size is bounded above. The last part follows because for any row of the table, the difference between the size of any partition in that row and its 2-core is constant. □

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ON A CAUCHY-TYPE INTEGRAL RELATED TO THE HELMHOLTZ OPERATOR IN THE PLANE

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ABSTRACT. Vector fields and quaternionic α-hyperholomorphic functions in a domain of $\mathbb{R}^2$ generalizing the notion of solenoidal and irrotational vector fields are considered. Sufficient conditions are established for the corresponding Cauchy-type integral along a closed Jordan rectifiable curve to be continuously extendable to the closure of a domain. Sokhotski-Plemelj-type formulas are proved as well.

1. Introduction

The role of the Cauchy-type integral in holomorphic function theory of one complex variable is widely known, and it would be trivial to explain its importance. Its study has led to numerous important results not only in holomorphic function theory itself but in many other areas such as potential theory, singular integrals and the corresponding operators, function spaces such as the Hardy spaces, boundary value problems for some elliptic operators, etc. Thus one can expect that any reasonably close generalization of the theory should possess an adequate analogue of the Cauchy-type integral, and the richness of the properties of the analogue can serve as a "measure of proximity" between the generalized theory and the original one. In particular, classical holomorphic function theory of several complex variables has either reproducing holomorphic kernels which depend strongly on a domain or a single reproducing (the Bochner-Martinelli) kernel which serves for domains of any shape but which lacks the property of being holomorphic.

There exists a generalization of holomorphic function theory suggested for both mathematical and physical reasons, namely, considering vector-fields satisfying the system

$$\begin{cases} \operatorname{div} \boldsymbol{f} = 0, \\ \mathbf{rot}\ \boldsymbol{f} = 0, \end{cases} \tag{1.1}$$

see exact notations and definitions in the next section. Solutions to the system (1.1) are called solenoidal and irrotational vector-fields, and the interested reader can find a very good introduction to their theory in the first chapter of the book [Zh], where some applications to geophysics are given. It is known that solutions of (1.1) satisfy the Laplace equation and are sometimes called Laplacian or harmonic vector fields.

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In [Zh] there is introduced the notion of Cauchy-type integral for the system (1.1), and some of its properties are established for integration along Liapunov surfaces with a finite number of conical points. It appears that in order to preserve a deep analogy with the complex case, it is necessary to consider not arbitrary Hölder or L_p densities but special subspaces of those space; the mathematical reasons for these restrictions are given in the Appendix of [Zh] written by the author together with M. Shapiro and N. Vasilevski; see also [VZS], where it is explained why and how the Cauchy-type integral for (1.1) can be studied via its counter-part for the version of quaternionic analysis defined by the Moisil-Theodoresco operator. This approach is presented in much more detail in [Sh] and in the introduction to [KS] and allows one to make use of the deep structural similarity between quaternionic analysis and holomorphic function theory.

Early results on solenoidal and irrotational vector-fields can be found, for instance, in [M] and [B].

Solutions of (1.1) generalize holomorphic functions in such a way that their components continue to be solutions of the Laplace equation. There is another way of generalizing in which the components become solutions of the Helmholtz rather than the Laplace equations; this way interests us here. The most direct generalization, the system (2.1) below is obtained formally from (1.1) when the right-hand side term in the second equation, becomes a (complex) vector collinear to $\boldsymbol{f}$. It is a direct verification to establish that a solution of (2.1) satisfies the Helmholtz equation with wave number defined by the coefficient of collinearity. It is interesting to note that such a mathematically formal generalization has an immediate physical interpretation; see, for instance, [LS] and the book [Zh]; such vector fields are sometimes called force-free fields.

The main aim of this paper consists of introducing the Cauchy-type integral for solutions of the system (2.1), in case where the integration curve is only rectifiable and the density is assumed to be a continuous function. For that integral we establish its asymptotic behaviour near the boundary, thus generalizing N. Davydov's theorem for the complex case, in particular, the analogues of the Sokhotski-Plemelj formulas are established; see Theorems (2.9) and (4.3) and Corollary (2.15). This means of course that we have constructed the starting point of many developments which can be obtained along usual lines, and using our and other techniques.

The proofs are based on the idea of imbedding the theory of vector fields into the corresponding version of quaternionic analysis for which the concept of the Cauchy-type integral arises naturally because of a deep similarity with holomorphic function theory. See the book [KS], the list of references there, and further explanation in Section 1 below. Observe that in vectorial terms the above version of quaternionic analysis is described by the system (2.2) whose solutions are referred to as $\alpha-$hyperholomorphic functions.

The class of rectifiable curves is quite wide and includes as proper sub-classes many other important classes of curves. First of all, we would like to emphasize that the study of the vectorial Cauchy-type integral, i.e., that of the system (2.1), has practically no antecedents, at least as far as we are aware. At the same time, the study of the Cauchy-type integral for α-hyperholomorphic functions has been the subject of research in a number of papers. The case of smooth (Liapunov) curves has been treated in [ST1], [ST2], see Appendix 1 in [KS] as well, and the treatment has been extended to piece-wise Liapunov curves in [GSS]. Recently much work has been done in the setting of Lipschitz curves (and surfaces) which are also rectifiable. A good reference for the part of that work closely related to the present paper, is [McM], in which are considered the Cauchy-type integrals for the Clifford analysis version of α-hyperholomorphic function theory in Lipschitz domains but in the L_p setting; thus the authors are interested in the global behaviour of their Cauchy-type integrals (i.e., almost everywhere) ignoring their local behaviour, i.e., near any specific point of the surface or curve. The latter is exactly what distinguishes what we are interested in from the Lipschitz domains case. In [McM] the Cauchy-type integrals serve as an efficient tool in the study of the Hardy-type spaces of α-hyperholomorphic functions of Clifford analysis; our results could be adapted as well to study the analogous Hardy-type spaces in domains with rectifiable boundary. One may find it interesting to read about both the rectifiable and non-rectifiable cases in complex and hypercomplex analysis in [BA1], [BA2], [BA3].

It is instructive to make comments in terms of the theory of partial differential equations. Indeed, the system (2.1) is overdetermined, and the quaternionic theory suggests how to complement it to the system (2.2) which is already determined. The latter possesses a fundamental matrix generating the corresponding Cauchy formula and thus, the Cauchy-type integral with "more predictable" behaviour. It is essential to note that such an explanation reveals only a part of the truth, namely, it omits an important and not so easy question: how to find such a complement of an overdetermined system? It is the rich multiplicative structure of the quaternions, both real and complex, which allows to construct the quaternionic theory which then shows explicitly how to obtain the complement.

2. A generalization of holomorphy in $\mathbb{R}^2$

Given a domain $\Omega \in \mathbb{R}^2$, consider a $\mathbb{C}^3$-valued function $\boldsymbol{f} : \Omega \mapsto \mathbb{C}^3$. Let $\boldsymbol{i_1}, \boldsymbol{i_2}, \boldsymbol{i_3}$ be a canonical basis in $\mathbb{C}^3$, so $\boldsymbol{f}$ is of the form $\boldsymbol{f} = f_1\boldsymbol{i_1} + f_2\boldsymbol{i_2} + f_3\boldsymbol{i_3}$; moreover, we let $\mathbb{R}^2$ be a real linear space with the basis $\boldsymbol{i_1}, \boldsymbol{i_2}$. Here f_1, f_2, f_3 are complex-valued functions in Ω and $\|\boldsymbol{f}\|^2 := |f_1|^2 + |f_2|^2 + |f_3|^2$.

In this paper we are interested in those vector-functions $\boldsymbol{f}$ which are solutions of the following system:

$$\begin{cases} \operatorname{div} \boldsymbol{f} = 0, \\ \mathbf{rot}\ \boldsymbol{f} = -\alpha \boldsymbol{f}, \end{cases} \tag{2.1}$$

where α is a given complex number and for $\boldsymbol{z} := x\boldsymbol{i_1} + y\boldsymbol{i_2} \in \Omega$,

$$\begin{aligned} \operatorname{div} \boldsymbol{f} &:= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}, \\ \mathbf{rot}\ \boldsymbol{f} &:= \frac{\partial f_3}{\partial y}\boldsymbol{i_1} - \frac{\partial f_3}{\partial x}\boldsymbol{i_2} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\boldsymbol{i_3}. \end{aligned}$$

Solutions of the system (2.1) form a natural generalization of the notion of a solenoidal and irrotational vector field, which corresponds to $\alpha = 0$. There are many papers about the cases $\alpha = 0$ and $\alpha \in \mathbb{C} \setminus \{0\}$, see for instance [M], [B], [KS], [RST]. The system (2.1) can be naturally considered in a domain of $\mathbb{R}^3$, but since the two–dimensional case has its essential peculiarities, we present it here; the former will be considered elsewhere.

There are deep mathematical reasons for considering a more general system, namely, the following one:

$$\begin{cases} \operatorname{div} \boldsymbol{f} = \alpha f_0, \\ \mathbf{rot}\ \boldsymbol{f} + \alpha \boldsymbol{f} = -\,\mathbf{grad}\ f_0, \end{cases} \tag{2.2}$$

where f_0 is a $\mathbb{C}$-valued function and $\mathbf{grad}\ f_0 := \dfrac{\partial f_0}{\partial x}\boldsymbol{i_1} + \dfrac{\partial f_0}{\partial y}\boldsymbol{i_2}$. Thus we shall be working with pairs $\mathcal{F} := (f_0, \boldsymbol{f})$ with norm $\|\mathcal{F}\|^2 := |f_0|^2 + \|\boldsymbol{f}\|^2$ and satisfying (2.2) from where certain conclusions will be made for vector–functions satisfying (2.1).

The particular case $\alpha = 0$ of the system (2.2) has been considered in [VZS]. Another generalization of the system (2.1) (different from (2.2)) is found in [O].

Let $H_n^{(p)}$ be the Hankel function of the kind $p \in \{1, 2\}$ and order $n \in \{0, 1, 2\}$ (see [GR]), and introduce the following notation:

$$K_{\alpha,0}(\boldsymbol{z}) := \begin{cases} (-1)^p \dfrac{i\alpha}{4} H_0^{(p)}(\alpha\|\boldsymbol{z}\|), & \text{if } \alpha \neq 0, \\ 0, & \text{if } \alpha = 0, \end{cases}$$

$$\boldsymbol{K}_{\alpha}(\boldsymbol{z}) := \begin{cases} (-1)^p \dfrac{i\alpha}{4} H_1^{(p)}(\alpha\|\boldsymbol{z}\|) \dfrac{\boldsymbol{z}}{\|\boldsymbol{z}\|}, & \text{if } \alpha \neq 0, \\ -\dfrac{\boldsymbol{z}}{2\pi\|\boldsymbol{z}\|^2}, & \text{if } \alpha = 0, \end{cases}$$

where $\boldsymbol{z} \in \mathbb{R}^2 \setminus \{(0; 0)\}$, i is the complex imaginary unit in $\mathbb{C}$, and

$$p = \begin{cases} 1, & \text{if } Im(\alpha) > 0 \text{ or } \alpha > 0, \\ 2, & \text{if } Im(\alpha) < 0 \text{ or } \alpha < 0. \end{cases} \tag{2.3}$$

Now the following pair

$$K_{\alpha}(\boldsymbol{z}) := \Big(K_{\alpha,0}(\boldsymbol{z}),\ \boldsymbol{K}_{\alpha}(\boldsymbol{z}) \Big) \tag{2.4}$$

will play the role, in a sense, of the Cauchy kernel for the system (2.2). It generates an analog of the Cauchy-type integral for a continuous pair $\mathcal{F} = (f_0, \boldsymbol{f})$, $f_0 : \Gamma \mapsto \mathbb{C}$, $\boldsymbol{f} : \Gamma \mapsto \mathbb{C}^3$; by the formulas: if $\boldsymbol{\zeta} := \xi\boldsymbol{i_1} + \eta\boldsymbol{i_2}$, $\boldsymbol{\sigma} := d\eta\boldsymbol{i_1} - d\xi\boldsymbol{i_2}$, and if Γ is a closed rectifiable Jordan curve which is the boundary of a bounded domain $\Omega^+ := \Omega$, and $\Omega^- := \mathbb{R}^2 \setminus (\Omega^+ \cup \Gamma)$ is its complement, then we define the pair

$$\Phi_{\alpha}[\mathcal{F}](\boldsymbol{z}) := \Big(\Phi_{\alpha,0}[\mathcal{F}](\boldsymbol{z}), \boldsymbol{\Phi}_{\alpha}[\mathcal{F}](\boldsymbol{z}) \Big), \qquad \boldsymbol{z} \in \mathbb{R}^2 \setminus \Gamma, \tag{2.5}$$

with

$$\begin{gathered} \Phi_{\alpha,0}[\mathcal{F}](\boldsymbol{z}) := \\ -\int_{\Gamma} \bigg(\big\langle \boldsymbol{K}_{\alpha}(\boldsymbol{\zeta} - \boldsymbol{z}), \boldsymbol{\sigma} \big\rangle f_0(\boldsymbol{\zeta}) + \Big\langle \big[\boldsymbol{K}_{\alpha}(\boldsymbol{\zeta} - \boldsymbol{z}), \boldsymbol{\sigma}\big] + K_{\alpha,0}(\boldsymbol{\zeta} - \boldsymbol{z})\,\boldsymbol{\sigma}, \boldsymbol{f}(\boldsymbol{\zeta}) \Big\rangle \bigg), \end{gathered} \tag{2.6}$$

$$\boldsymbol{\Phi}_{\alpha}[\mathcal{F}](\boldsymbol{z}) := \int_{\Gamma} \bigg(\Big[[\boldsymbol{K}_{\alpha}(\boldsymbol{\zeta} - \boldsymbol{z}), \boldsymbol{\sigma}] + K_{\alpha,0}(\boldsymbol{\zeta} - \boldsymbol{z})\,\boldsymbol{\sigma}, \boldsymbol{f}(\boldsymbol{\zeta}) \Big] -$$

$$\Big\langle \boldsymbol{K}_\alpha(\boldsymbol{\zeta}-\boldsymbol{z}), \boldsymbol{\sigma}\Big\rangle \boldsymbol{f}(\boldsymbol{\zeta}) + f_0(\boldsymbol{\zeta})\Big(\big[\boldsymbol{K}_\alpha(\boldsymbol{\zeta}-\boldsymbol{z}), \boldsymbol{\sigma}\big] + K_{\alpha,0}(\boldsymbol{\zeta}-\boldsymbol{z})\,\boldsymbol{\sigma}\Big)\Big), \tag{2.7}$$

where $<\cdot,\cdot>$ and $[\cdot,\cdot]$ denote, respectively, the scalar and the vector products in $\mathbb{C}^3$; i.e., for $\{\boldsymbol{a}, \boldsymbol{b}\}$, $\boldsymbol{a} = \sum_{k=1}^3 a_k \boldsymbol{i_k}$, $\boldsymbol{b} = \sum_{k=1}^3 b_k \boldsymbol{i_k}$, $<\boldsymbol{a}, \boldsymbol{b}>:= \sum_{k=1}^3 a_k b_k$;

$$[\boldsymbol{a}, \boldsymbol{b}] := \begin{vmatrix} \boldsymbol{i_1} & \boldsymbol{i_2} & \boldsymbol{i_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

We shall write $\Phi_\alpha^+[f]$ and $\Phi_\alpha^-[f]$ for the respective restrictions of $\Phi_\alpha[f]$ onto Ω^+ and Ω^-.

If, in particular, the scalar component of the pair $\mathcal{F}$ is identically zero, $\mathcal{F} = \{0; \boldsymbol{f}\}$, this does not mean, in general, that $\Phi_\alpha[\mathcal{F}]$ is vector-valued, which follows directly from (2.6). Thus, thinking of an analog of the Cauchy-type integral for the system (2.1) as a purely vectorial object, we must exclude the scalar component of $\Phi_\alpha[\mathcal{F}]$ in the following sense. Let $C(\Gamma; \mathbb{C}^3)$ be the complex linear space of all $\mathbb{C}^3$-valued continuous vector–functions $\boldsymbol{f}$ on Γ, and introduce

$$\mathcal{M}(\Gamma; \mathbb{C}^3) := C(\Gamma; \mathbb{C}^3) \cap \Big\{ \boldsymbol{f} : \int_\Gamma \big\langle [\boldsymbol{K}_\alpha(\boldsymbol{\zeta}-\boldsymbol{z}), \boldsymbol{\sigma}] + K_{\alpha,0}(\boldsymbol{\zeta}-\boldsymbol{z})\,\boldsymbol{\sigma}, \boldsymbol{f}(\boldsymbol{\zeta})\big\rangle = 0,\ \boldsymbol{z} \notin \Gamma \Big\}. \tag{2.8}$$

Now for $\boldsymbol{f} \in \mathcal{M}(\Gamma; \mathbb{C}^3)$, the analog of the Cauchy-type integral for the system (2.1) is given by the formula

$$\boldsymbol{\Phi}_\alpha[\boldsymbol{f}](\boldsymbol{z}) := \int_\Gamma \Big(\Big[[\boldsymbol{K}_\alpha(\boldsymbol{\zeta}-\boldsymbol{z}), \boldsymbol{\sigma}] + K_{\alpha,0}(\boldsymbol{\zeta}-\boldsymbol{z})\,\boldsymbol{\sigma}, \boldsymbol{f}(\boldsymbol{\zeta}) \Big] - \Big\langle \boldsymbol{K}_\alpha(\boldsymbol{\zeta}-\boldsymbol{z}), \boldsymbol{\sigma}\Big\rangle \boldsymbol{f}(\boldsymbol{\zeta}) \Big).$$

One may verify that $\boldsymbol{\Phi}_\alpha[\boldsymbol{f}]$ is a solution to (2.1).

THEOREM (2.9) (Analogue of N. A. Davydov's theorem (see [D]) for the system (2.2)). *Let Γ be a closed rectifiable Jordan curve, $f_0 : \Gamma \mapsto \mathbb{C}$ and $\boldsymbol{f} : \Gamma \mapsto \mathbb{C}^3$ be continuous functions, $\mathcal{F} := (f_0, \boldsymbol{f})$, and let the integral*

$$\Psi_\alpha[\mathcal{F}](\boldsymbol{t}) := \lim_{\delta \to 0} \int_{\Gamma \setminus \Gamma_{\boldsymbol{t},\delta}} \|K_\alpha(\boldsymbol{\zeta}-\boldsymbol{t})\| \cdot \|\boldsymbol{\sigma}\| \cdot \|\mathcal{F}(\boldsymbol{\zeta}) - \mathcal{F}(\boldsymbol{t})\|, \qquad \boldsymbol{t} \in \Gamma, \tag{2.10}$$

exist uniformly with respect to $\boldsymbol{t} \in \Gamma$ where $\Gamma_{\boldsymbol{t},\delta} := \{\boldsymbol{\zeta} \in \Gamma : \|\boldsymbol{\zeta} - \boldsymbol{t}\| \leqslant \delta\}$. Then there exists the pair of integrals

$$F_\alpha[\mathcal{F}](\boldsymbol{t}) := \Big(F_{\alpha,0}[\mathcal{F}](\boldsymbol{t}), \boldsymbol{F}_\alpha[\mathcal{F}](\boldsymbol{t}) \Big), \qquad \boldsymbol{t} \in \Gamma,$$

where

$$F_{\alpha,0}[\mathcal{F}](\boldsymbol{t}) := -\lim_{\delta \to 0} \int_{\Gamma \setminus \Gamma_{\boldsymbol{t},\delta}} \big(\langle \boldsymbol{K}_\alpha(\boldsymbol{\zeta}-\boldsymbol{t}), \boldsymbol{\sigma}\rangle\,(f_0(\boldsymbol{\zeta}) - f_0(\boldsymbol{t})) + \big\langle [\boldsymbol{K}_\alpha(\boldsymbol{\zeta}-\boldsymbol{t}), \boldsymbol{\sigma}] + K_{\alpha,0}(\boldsymbol{\zeta}-\boldsymbol{t})\,\boldsymbol{\sigma}, (\boldsymbol{f}(\boldsymbol{\zeta}) - \boldsymbol{f}(\boldsymbol{t}))\big\rangle \big), \tag{2.11}$$

(2.12)

$$\boldsymbol{F}_\alpha[\mathcal{F}](\boldsymbol{t}) := \lim_{\delta\to 0} \int\limits_{\Gamma\setminus\Gamma_{\boldsymbol{t},\delta}} \Big(\big[[\boldsymbol{K}_\alpha(\boldsymbol{\zeta}-\boldsymbol{t}),\boldsymbol{\sigma}] + K_{\alpha,0}(\boldsymbol{\zeta}-\boldsymbol{t})\,\boldsymbol{\sigma}, (\boldsymbol{f}(\boldsymbol{\zeta})-\boldsymbol{f}(\boldsymbol{t}))\big] - \langle \boldsymbol{K}_\alpha(\boldsymbol{\zeta}-\boldsymbol{t}),\boldsymbol{\sigma}\rangle\,(\boldsymbol{f}(\boldsymbol{\zeta})-\boldsymbol{f}(\boldsymbol{t})) + (f_0(\boldsymbol{\zeta})-f_0(\boldsymbol{t}))\,([\boldsymbol{K}_\alpha(\boldsymbol{\zeta}-\boldsymbol{t}),\boldsymbol{\sigma}] + K_{\alpha,0}(\boldsymbol{\zeta}-\boldsymbol{t})\,\boldsymbol{\sigma})\Big);$$

moreover, the functions $\Phi^{\pm}_{\alpha}[\mathcal{F}]$ *extend continuously onto* Γ *and the following analogues of the Sokhotski-Plemelj formulas hold:*

(2.13)

$$\Phi^+_{\alpha,0}[\mathcal{F}](\boldsymbol{t}) = (I_{\alpha,\Gamma,0}(\boldsymbol{t})+1)f_0(\boldsymbol{t}) - \langle \boldsymbol{I}_{\alpha,\Gamma}(\boldsymbol{t}),\boldsymbol{f}(\boldsymbol{t})\rangle + F_{\alpha,0}[\mathcal{F}](\boldsymbol{t}), \quad \boldsymbol{t}\in\Gamma,$$

$$\boldsymbol{\Phi}^+_{\alpha}[\mathcal{F}](\boldsymbol{t}) = [\boldsymbol{I}_{\alpha,\Gamma}(\boldsymbol{t}),\boldsymbol{f}(\boldsymbol{t})] + (I_{\alpha,\Gamma,0}(\boldsymbol{t})+1)\boldsymbol{f}(\boldsymbol{t}) + f_0(\boldsymbol{t})\boldsymbol{I}_{\alpha,\Gamma}(\boldsymbol{t}) + \boldsymbol{F}_\alpha[\mathcal{F}](\boldsymbol{t}),\ \boldsymbol{t}\in\Gamma,$$

(2.14)

$$\Phi^-_{\alpha,0}[\mathcal{F}](\boldsymbol{t}) = I_{\alpha,\Gamma,0}(\boldsymbol{t})f_0(\boldsymbol{t}) - \langle \boldsymbol{I}_{\alpha,\Gamma}(\boldsymbol{t}),\boldsymbol{f}(\boldsymbol{t})\rangle + F_{\alpha,0}[\mathcal{F}](\boldsymbol{t}), \quad \boldsymbol{t}\in\Gamma,$$

$$\boldsymbol{\Phi}^-_{\alpha}[\mathcal{F}](\boldsymbol{t}) = [\boldsymbol{I}_{\alpha,\Gamma}(\boldsymbol{t}),\boldsymbol{f}(\boldsymbol{t})] + I_{\alpha,\Gamma,0}(\boldsymbol{t})\boldsymbol{f}(\boldsymbol{t}) + f_0(\boldsymbol{t})\boldsymbol{I}_{\alpha,\Gamma}(\boldsymbol{t}) + \boldsymbol{F}_\alpha[\mathcal{F}](\boldsymbol{t}), \quad \boldsymbol{t}\in\Gamma,$$

where the following notation is used:

$$I_{\alpha,\Gamma,0}(\boldsymbol{t}) := -\alpha \iint\limits_{\Omega^+} K_{\alpha,0}(\boldsymbol{\zeta}-\boldsymbol{t})d\xi d\eta,$$

$$\boldsymbol{I}_{\alpha,\Gamma}(\boldsymbol{t}) := -\alpha \iint\limits_{\Omega^+} \boldsymbol{K}_{\alpha}(\boldsymbol{\zeta}-\boldsymbol{t})d\xi d\eta,$$

and $\Phi^{\pm}_{\alpha}[\mathcal{F}](\boldsymbol{t}) := \lim\limits_{\Omega^{\pm}\ni\boldsymbol{z}\to\boldsymbol{t}} \Phi_\alpha[\mathcal{F}](\boldsymbol{z})$.

The proof will be given after some preparatory work which is of independent interest.

Corollary (2.15) (Analogue of N. A. Davydov's theorem for the system (2.1)). *Let* Γ *be a closed rectifiable Jordan curve,* $\boldsymbol{f}\in\mathcal{M}(\Gamma;\mathbb{C}^3)$*, and let the integral*

(2.16)
$$\Psi_\alpha[\boldsymbol{f}](\boldsymbol{t}) := \lim_{\delta\to 0}\int\limits_{\Gamma\setminus\Gamma_{\boldsymbol{t},\delta}} \|K_\alpha(\boldsymbol{\zeta}-\boldsymbol{t})\|\cdot\|\boldsymbol{\sigma}\|\cdot\|\boldsymbol{f}(\boldsymbol{\zeta})-\boldsymbol{f}(\boldsymbol{t})\|$$

exist uniformly with respect to $\boldsymbol{t}\in\Gamma$*. Then there exists the integral*

(2.17)

$$\boldsymbol{F}_\alpha[\boldsymbol{f}](\boldsymbol{t}) := \lim_{\delta\to 0} \int\limits_{\Gamma\setminus\Gamma_{\boldsymbol{t},\delta}} \Big(\big[[\boldsymbol{K}_\alpha(\boldsymbol{\zeta}-\boldsymbol{t}),\boldsymbol{\sigma}] + K_{\alpha,0}(\boldsymbol{\zeta}-\boldsymbol{t})\,\boldsymbol{\sigma}, (\boldsymbol{f}(\boldsymbol{\zeta})-\boldsymbol{f}(\boldsymbol{t}))\big] - \langle \boldsymbol{K}_\alpha(\boldsymbol{\zeta}-\boldsymbol{t}),\boldsymbol{\sigma}\rangle\,\big(\boldsymbol{f}(\boldsymbol{\zeta})-\boldsymbol{f}(\boldsymbol{t})\big)\Big);$$

moreover, the functions $\boldsymbol{\Phi}^{\pm}_{\alpha}[\boldsymbol{f}](\boldsymbol{z})$ *extend continuously onto* Γ *and the following analogues of the Sokhotski-Plemelj formulas hold:*

(2.18)
$$\boldsymbol{\Phi}^+_{\alpha}[\boldsymbol{f}](\boldsymbol{t}) = [\boldsymbol{I}_{\alpha,\Gamma}(\boldsymbol{t}),\boldsymbol{f}(\boldsymbol{t})] + (I_{\alpha,\Gamma,0}(\boldsymbol{t})+1)\boldsymbol{f}(\boldsymbol{t}) + \boldsymbol{F}_\alpha[\boldsymbol{f}](\boldsymbol{t}), \quad \boldsymbol{t}\in\Gamma,$$

(2.19)
$$\boldsymbol{\Phi}^-_{\alpha}[\boldsymbol{f}](\boldsymbol{t}) = [\boldsymbol{I}_{\alpha,\Gamma}(\boldsymbol{t}),\boldsymbol{f}(\boldsymbol{t})] + I_{\alpha,\Gamma,0}(\boldsymbol{t})\boldsymbol{f}(\boldsymbol{t}) + \boldsymbol{F}_\alpha[\boldsymbol{f}](\boldsymbol{t}), \quad \boldsymbol{t}\in\Gamma,$$

where $\boldsymbol{\Phi}^{\pm}_{\alpha}[\boldsymbol{f}](\boldsymbol{t}) := \lim\limits_{\Omega^{\pm}\ni\boldsymbol{z}\to\boldsymbol{t}} \boldsymbol{\Phi}_\alpha[\boldsymbol{f}](\boldsymbol{z})$.

3. Quaternions and quaternion-valued α-hyperholomorphic functions in $\mathbb{R}^2$

We shall denote as usual by $\mathbb{H} = \mathbb{H}(\mathbb{R})$ and $\mathbb{H}(\mathbb{C})$ the sets of real and complex quaternions, i.e., each quaternion is of the form

$$a = \sum_{k=0}^{3} a_k i_k$$

with $\{a_k\} \subset \mathbb{R}$ for real quaternions and $\{a_k\} \subset \mathbb{C}$ for complex quaternions; $i_0 = 1$ stands for the unit and i_1, i_2, i_3 stand for imaginary units; the complex imaginary units in $\mathbb{C}$ will be denoted by i. $\mathbb{H}$ has the structure of a real non-commutative, associative algebra without zero divisors. $\mathbb{H}(\mathbb{C})$ is a complex non-commutative, associative algebra with zero divisors.

For a complex quaternion $a = \sum\limits_{k=0}^{3} a_k i_k$ the quaternionic conjugate is defined by $\overline{a} := a_0 - \sum\limits_{k=1}^{3} a_k i_k$. The modulus of a quaternion a coincides with its Euclidean norm: $|a| = \|a\|_{\mathbb{R}^8}$. In particular, for $a \in \mathbb{H}$ we have $|a| = \|a\|_{\mathbb{R}^4}$ and further $|a|^2 = a\,\overline{a} = \overline{a}a$ while for a complex quaternion $|a|^2 \neq a\overline{a}$. What is more, for a, b from $\mathbb{H}$ there holds $|ab| = |a|\,|b|$, which is extremely important in working with real quaternions. For complex quaternions the situation is different.

Lemma (3.1) (see also [HL]). *$|ab| \leqslant \sqrt{2}\,|a|\,|b|$ for all $a, b \in \mathbb{H}(\mathbb{C})$.*

Proof. Let

$$a := \sum_{k=0}^{3} a_k i_k,\ a_k := \alpha_k + i\lambda_k, \qquad \alpha_k, \lambda_k \in \mathbb{R},$$

$$b := \sum_{k=0}^{3} b_k i_k,\ b_k := \beta_k + i\gamma_k, \qquad \beta_k, \gamma_k \in \mathbb{R}\,.$$

We have $a = a' + ia''$ and $b = b' + ib''$, where a', a'', b', b'' are real quaternions. Since

$$|a|^2 = \sum_{k=0}^{3}(\alpha_k^2 + \lambda_k^2) = |a'|^2 + |a''|^2, \quad |b|^2 = \sum_{k=0}^{3}(\beta_k^2 + \gamma_k^2) = |b'|^2 + |b''|^2,$$

then

$$(|a|\,|b|)^2 = |a'|^2|b'|^2 + |a'|^2|b''|^2 + |a''|^2|b'|^2 + |a''|^2|b''|^2.$$

Therefore

(3.2)
$$\begin{aligned}|ab|^2 &= |a'b' - a''b'' + i(a'b'' + a''b')|^2 = |a'b' - a''b''|^2 + |a'b'' + a''b'|^2 = \\ &= \overline{(a'b' - a''b'')}\,(a'b' - a''b'') + \overline{(a'b'' + a''b')}\,(a'b'' + a''b') = \\ &= (\overline{a'b'} - \overline{a''b''})\,(a'b' - a''b'') + (\overline{a'b''} + \overline{a''b'})\,(a'b'' + a''b') = (|a|\,|b|)^2 + d,\end{aligned}$$

where $\mathbb{H} \ni d = \overline{a'b''}a''b' + \overline{a''b'}a'b'' - \overline{a'b'}a''b'' - \overline{a''b''}a'b'$, and

$$|d| \leqslant 2|a'b'|\,|a''b''| + 2|a'b''|\,|a''b'| \leqslant (|a|\,|b|)^2. \tag{3.3}$$

Combining (3.2) and (3.3), we obtain the assertion of the Lemma. $\square$

Let Ω be a domain in the plane $\mathbb{R}^2$. We consider $\mathbb{H}(\mathbb{C})$-valued functions defined in Ω. On the bi-$\mathbb{H}(\mathbb{C})$-module $C^2(\Omega;\mathbb{H}(\mathbb{C}))$ we introduce the two-dimensional Helmholtz operator with wave number $\lambda \in \mathbb{C}$:

$$\Delta_\lambda := \Delta_{\mathbb{R}^2} + M^\lambda,$$

where $\Delta_{\mathbb{R}^2} = \partial_1^2 + \partial_2^2$, $\partial_k = \frac{\partial}{\partial x_k}$. For $a \in \mathbb{H}(\mathbb{C})$ we denote by M^a the operator of multiplication by a on the right-hand side; of course, for $a \in \mathbb{C}$ this coincides with the operator of the left-hand side multiplication ${}^aM = a$. The operators $\overline{\partial} := \partial_1 + i\partial_2$ and $\partial := \partial_1 - i\partial_2$ determine, respectively, classes of holomorphic and anti-holomorphic functions of a complex variable, and the following factorization holds:

$$\partial \circ \overline{\partial} = \overline{\partial} \circ \partial = \Delta_{\mathbb{R}^2}.$$

Consider the following partial differential operators with quaternionic coefficients:

$$_{st}\partial := i_1\,\partial_1 + i_2\,\partial_2;\ \ _{st}\overline{\partial} := \overline{i_1}\,\partial_1 + \overline{i_2}\,\partial_2;$$

$$\partial_{st} := \partial_1 \circ M^{i_1} + \partial_2 \circ M^{i_2};\ \ \overline{\partial}_{st} := \partial_1 \circ M^{\overline{i_1}} + \partial_2 \circ M^{\overline{i_2}}.$$

The following equalities can be easily verified:

$$\partial_{st} \circ \overline{\partial}_{st} = \overline{\partial}_{st} \circ \partial_{st} = \Delta_{\mathbb{R}^2} = {}_{st}\partial \circ {}_{st}\overline{\partial} = {}_{st}\overline{\partial} \circ {}_{st}\partial,$$

which mean that

$$_{st}\partial^2 = \partial_{st}^2 = -\Delta_{\mathbb{R}^2}.$$

For $\alpha \in \mathbb{C}$ a complex square root of $\lambda \in \mathbb{C}$, i.e. $\alpha^2 = \lambda$, set

$$_\alpha\partial := \partial_{st} + \alpha;\ \ \partial_\alpha := {}_{st}\partial + \alpha.$$

Then we have the following factorizations of the Helmholtz operator:

$$\Delta_\lambda = -\partial_\alpha \circ \partial_{-\alpha} = -\partial_{-\alpha} \circ \partial_\alpha = -{}_\alpha\partial \circ {}_{-\alpha}\,\partial = -{}_{-\alpha}\partial \circ {}_\alpha\partial.$$

In analogy with the usual notion of a holomorphic function, consider the following definition of α-hyperholomorphic functions in $\mathbb{R}^2$.

Definition (3.4) ([ST1]). A function $f \in C^1(\Omega, \mathbb{H}(\mathbb{C}))$ is called α-hyperholomorphic if $_\alpha\partial f \equiv 0$ in Ω.

More precisely, such functions could be called, for instance, left-α-hyperholomorphic because there is a "symmetric" definition for ∂_α, as well as for $_\alpha\overline{\partial}$ and $\overline{\partial}_\alpha$. We shall deal with the above case only.

This definition for α-hyperholomorphic functions was introduced in [ST1] both for complex and quaternionic values of α, and some essential properties were established there. The main integral formulas for α-hyperholomorphic functions were constructed in [ST2]. All proofs and details can be found in these papers, see also [KS, Appendix 4]. Some developments of the topic are presented in [RST] and [GSS]. One can find many other relevant bibliographical references in all these papers.

In what follows we shall need some properties of the Hankel functions $H_n^{(p)}(t)$, $t \in \mathbb{C}$, (see [GR]) which we concentrate in this section for the reader's convenience. The following equalities are valid:

$$\frac{d}{dt}H_1^{(p)}(t) = \frac{1}{2}\left(H_0^{(p)}(t) - H_2^{(p)}(t)\right), \tag{3.5}$$

$$\frac{d}{dt}H_0^{(p)}(t) = -H_1^{(p)}(t), \tag{3.6}$$

$$tH_2^{(p)}(t) = 2H_1^{(p)}(t) - tH_0^{(p)}(t), \tag{3.7}$$

and the following series expansions of the Hankel functions $H_0^{(p)}(t)$ and $H_1^{(p)}(t)$ hold:

(3.8)

$$H_0^{(p)}(t) = \left(1 - (-1)^p \frac{2i}{\pi}(\log\frac{t}{2} + \boldsymbol{C})\right) \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{2^{2k}(k!)^2} + \frac{2i}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+p} t^{2k}}{2^{2k}(k!)^2} \sum_{m=1}^{k} \frac{1}{m},$$

$$H_1^{(p)}(t) = \left(1 - (-1)^p \frac{2i}{\pi}(\log\frac{t}{2} + \boldsymbol{C})\right) \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{2^{2k+1}k!(k+1)!} + (-1)^p \left(\frac{2i}{\pi t} + \frac{it}{2\pi}\right) +$$

$$\frac{i}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+p} t^{2k+1}}{2^{2k+1}k!(k+1)!} \left(\sum_{m=1}^{k+1} \frac{1}{m} + \sum_{m=1}^{k} \frac{1}{m}\right), \tag{3.9}$$

where $\boldsymbol{C}$ is the Euler constant.

4. Quaternionic generalization of the Cauchy-type integral

Given $\alpha \in \mathbb{C}$ and real quaternions $z := xi_1 + yi_2$, $\zeta := \xi i_1 + \eta i_2$, considered as points of Euclidean space $\mathbb{R}^2$ equipped with the additional structure of quaternionic multiplication, introduce the notation

$$\theta_\alpha(z) := \begin{cases} (-1)^p \frac{i}{4} H_0^{(p)}(\alpha|z|), & \text{if } \alpha \neq 0, \\ \frac{1}{2\pi} \log|z|, & \text{if } \alpha = 0, \end{cases}$$

where p depends on α via formula (2.3). It is a well known fact (see e.g. [V]) that the function θ_α is the fundamental solution of the Helmholtz operator $\Delta_{\alpha^2} := \Delta_{\mathbb{R}^2} + M^{\alpha^2}$ for all values of α.

The α-hyperholomorphic Cauchy kernel, i.e., the fundamental solution to the operators ${}_\alpha\partial$ and ∂_α, is defined as

$$K_\alpha(z) := -{}_{-\alpha}\partial[\theta_\alpha](z) = -\partial_{-\alpha}[\theta_\alpha](z).$$

Hence one has explicitly

$$K_\alpha(z) = \begin{cases} (-1)^p \frac{i\alpha}{4}\left(H_1^{(p)}(\alpha|z|)\frac{z}{|z|} + H_0^{(p)}(\alpha|z|)\right), & \text{if } \alpha \neq 0, \\ -\frac{z}{2\pi|z|^2}, & \text{if } \alpha = 0. \end{cases} \tag{4.1}$$

Now, for a continuous function $f : \Gamma \to \mathbb{H}(\mathbb{C})$ and $\sigma := d\eta i_1 - d\xi i_2$, the Cauchy-type integral of f is given by the formula

$$\Phi_\alpha[f](z) := \int_\Gamma K_\alpha(\zeta - z)\, \sigma\, f(\zeta), \qquad z \in \mathbb{R}^2 \setminus \Gamma; \tag{4.2}$$

as in the previous section, Γ is a closed rectifiable Jordan curve.

THEOREM (4.3). *Let* Γ *be a closed rectifiable Jordan curve,* $f : \Gamma \to \mathbb{H}(\mathbb{C})$ *be a continuous function, and let the integral*

$$\Psi_\alpha[f](t) := \lim_{\delta\to 0} \int\limits_{\Gamma\setminus\Gamma_{t,\delta}} |K_\alpha(\zeta - t)|\,|\sigma|\,|f(\zeta) - f(t)|, \qquad t \in \Gamma, \tag{4.4}$$

exist uniformly with respect to $t \in \Gamma$, *where* $\Gamma_{t,\delta} := \{\zeta \in \Gamma : |\zeta - t| \leqslant \delta\}$. *Then there exists the integral*

$$F_\alpha[f](t) := \lim_{\delta\to 0} \int\limits_{\Gamma\setminus\Gamma_{t,\delta}} K_\alpha(\zeta - t)\,\sigma\,(f(\zeta) - f(t)), \qquad t \in \Gamma; \tag{4.5}$$

moreover, the functions $\Phi^\pm_\alpha[f]$ *extend continuously onto* Γ, *and the following analogues of the Sokhotski-Plemelj formulas hold:*

$$\Phi^+_\alpha[f](t) = (I_{\alpha,\Gamma}(t) + 1)f(t) + F_\alpha[f](t), \quad t \in \Gamma, \tag{4.6}$$

$$\Phi^-_\alpha[f](t) = I_{\alpha,\Gamma}(t)f(t) + F_\alpha[f](t), \quad t \in \Gamma, \tag{4.7}$$

where $\Phi^\pm_\alpha[f](t) := \lim\limits_{\Omega^\pm \ni z \to t} \Phi_\alpha[f](z)$, *and*

$$I_{\alpha,\Gamma}(t) := -\alpha \iint\limits_{\Omega^+} K_\alpha(\zeta - t) d\xi d\eta.$$

The proof is based on several lemmas which are of interest independently. Observe that the proof involves laborious computations and estimations similar to those which can be found, for instance, in [MP] and [P-R]; we have included many details for the reader's convenience and for completeness.

LEMMA (4.8). *The limit (4.5) exists uniformly with respect to* $t \in \Gamma$, *and* $F_\alpha[f]$ *is a continuous function on* Γ.

LEMMA (4.9).

$$\Phi_\alpha[1](z) = \begin{cases} I_{\alpha,\Gamma}(z) + 1, & z \in \Omega^+, \\ I_{\alpha,\Gamma}(z), & z \in \Omega^-. \end{cases} \tag{4.10}$$

LEMMA (4.11). $I_{\alpha,\Gamma}$ *is a continuous function in* $\mathbb{R}^2$.

5. Proof of the results of Section 4

Proof of Lemma (4.8). Denote

$$\Psi_\alpha(\delta, t) := \int\limits_{\Gamma\setminus\Gamma_{t,\delta}} |K_\alpha(\zeta - t)|\,|\sigma|\,|f(\zeta) - f(t)|\,.$$

$$F_\alpha(\delta, t) := \int\limits_{\Gamma\setminus\Gamma_{t,\delta}} K_\alpha(\zeta - t)\,\sigma\,(f(\zeta) - f(t))\,.$$

We have $F_\alpha(\delta, t) = \sum\limits_{k=0}^{3} (F^{(1)}_{\alpha,k}(\delta, t) + iF^{(2)}_{\alpha,k}(\delta, t))i_k$, where $F^{(1)}_{\alpha,k}$ and $F^{(2)}_{\alpha,k}$ are real-valued functions.

Under the conditions of Theorem (4.3) the function $\Psi_\alpha(\delta, t)$ tends to the finite limit $\Psi_\alpha(t)$, when $\delta \to 0$, uniformly with respect to $t \in \Gamma$. Using the criterion of uniform convergence for the integral and Lemma (3.1), we get that for $\forall\varepsilon > 0\ \exists\,\delta(\varepsilon) > 0\ \forall t \in \Gamma$:

$$0 < \delta_1 < \delta_2 < \delta(\varepsilon) \Rightarrow$$

$$\Psi_\alpha(\delta_1, t) - \Psi_\alpha(\delta_2, t) = \int\limits_{\Gamma_{t,\delta_2}\setminus\Gamma_{t,\delta_1}} |K_\alpha(\zeta - t)|\,|\sigma|\,|f(\zeta) - f(t)| < \varepsilon \Rightarrow$$

$$|F_\alpha(\delta_1, t) - F_\alpha(\delta_2, t)| = \left| \int\limits_{\Gamma_{t,\delta_2}\setminus\Gamma_{t,\delta_1}} K_\alpha(\zeta - t)\,\sigma\,(f(\zeta) - f(t)) \right| \tag{5.1}$$

$$\leqslant 2 \int\limits_{\Gamma_{t,\delta_2}\setminus\Gamma_{t,\delta_1}} |K_\alpha(\zeta - t)|\,|\sigma|\,|f(\zeta) - f(t)| < 2\varepsilon \Rightarrow$$

$$\left|F^{(j)}_{\alpha,k}(\delta_1, t) - F^{(j)}_{\alpha,k}(\delta_2, t)\right| < 2\varepsilon \quad (j = 1, 2;\ k = 0, \dots, 3).$$

Therefore for each fixed $t \in \Gamma$ there exist limits

$$F^{(j)}_{\alpha,k}(t) := \lim_{\delta\to 0} F^{(j)}_{\alpha,k}(\delta, t) \quad (j = 1, 2;\ k = 0, \dots, 3)$$

and consequently there exists

$$F_\alpha[f](t) = \lim_{\delta\to 0} F_\alpha(\delta, t). \tag{5.2}$$

Proceeding to the limit as $\delta_1 \to 0$ in inequality (5.1) we obtain that for $\forall\varepsilon > 0\ \exists\delta(\varepsilon) > 0\ \forall t \in \Gamma$:

$$0 < \delta < \delta(\varepsilon) \Rightarrow \left|F^{(j)}_{\alpha,k}(\delta, t) - F^{(j)}_{\alpha,k}(t)\right| \leqslant 2\varepsilon \quad (j = 1, 2;\ k = 0, \dots, 3) \Rightarrow$$
$$|F_\alpha(\delta, t) - F_\alpha[f](t)| \leqslant 4\sqrt{2}\varepsilon,$$

which is all that is required. □

Proof of Lemma (4.9). Let $\alpha \neq 0$. We have

$$\Phi_\alpha[1](z) = \int\limits_\Gamma K_\alpha(\zeta - z)\,\sigma = (-1)^{p-1}\frac{i}{4}\alpha(i_1 + I_2 i_3 - I_3), \tag{5.3}$$

where

$$I_1 = \int\limits_\Gamma \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|}\left((\xi - x)d\eta - (\eta - y)d\xi\right),$$

$$I_2 = \int\limits_\Gamma \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|}\left((\xi - x)d\xi + (\eta - y)d\eta\right),$$

$$I_3 = \int\limits_\Gamma H_0^{(p)}(\alpha|\zeta - z|)(d\eta\boldsymbol{i}_1 - d\xi i_2).$$

Let $z \in \Omega^+$, let $\rho > 0$ be such that $B(z, \rho) := \{\zeta \in \mathbb{C} : |\zeta - z| \leqslant \rho\}$ is contained in Ω^+, and let γ_ρ be the boundary of $B(z, \rho)$.

Using the Green formula and the equalities (3.5), (3.7) we get:

$$\int\limits_{\Gamma\setminus\gamma_\rho} \frac{H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|}\,((\xi-x)d\eta-(\eta-y)d\xi)=$$

$$\iint\limits_{\Omega^+\setminus B(z,\rho)} \left(\frac{\partial}{\partial\xi}\left(\frac{H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|}(\xi-x)\right)+\frac{\partial}{\partial\eta}\left(\frac{H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|}(\eta-y)\right)\right)d\xi\,d\eta=$$

$$\frac{1}{2}\iint\limits_{\Omega^+\setminus B(z,\rho)} \left(\alpha H_0^{(p)}(\alpha|\zeta-z|)-\alpha H_2^{(p)}(\alpha|\zeta-z|)+2\frac{H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|}\right)d\xi\,d\eta=$$

$$\iint\limits_{\Omega^+\setminus B(z,\rho)} \alpha H_0^{(p)}(\alpha|\zeta-z|)d\xi\,d\eta,$$

$$\int\limits_{\gamma_\rho} \frac{H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|}\,((\xi-x)d\eta-(\eta-y)d\xi)=2\pi\rho H_1^{(p)}(\alpha\rho)=$$

$$(-1)^p\frac{4i}{\alpha}+o(1)\text{ as }\rho\to 0.$$

Hence

$$I_1=\lim_{\rho\to 0}\left(\int\limits_{\Gamma-\gamma_\rho}+\int\limits_{\gamma_\rho}\right)\frac{H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|}((\xi-x)d\eta-(\eta-y)d\xi)= \tag{5.4}$$
$$\iint\limits_{\Omega^+}\alpha H_0^{(p)}(\alpha|\zeta-z|)d\xi\,d\eta+(-1)^p\frac{4i}{\alpha}.$$

Furthermore,

$$\int\limits_{\Gamma\setminus\gamma_\rho} \frac{H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|}\,((\xi-x)d\xi+(\eta-y)d\eta)$$

$$=\iint\limits_{\Omega^+\setminus B(z,\rho)} \left(\frac{\partial}{\partial\xi}\left(\frac{H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|}(\eta-y)\right)-\frac{\partial}{\partial\eta}\left(\frac{H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|}(\xi-x)\right)\right)d\xi\,d\eta$$

$$=\iint\limits_{\Omega^+\setminus B(z,\rho)} \frac{\partial}{\partial|\zeta-z|}\left(\frac{H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|}\right)\left(\frac{\partial|\zeta-z|}{\partial\xi}(\eta-y)-\frac{\partial|\zeta-z|}{\partial\eta}(\xi-x)\right)d\xi\,d\eta$$
$$=0,$$

$$\int\limits_{\gamma_\rho} \frac{H_1^{(p)}(\alpha|\zeta-z|)}{|\zeta-z|}\,((\xi-x)d\xi+(\eta-y)d\eta)=0$$

and, consequently,

$$I_2=0. \tag{5.5}$$

Analogously, using the equality (3.6), we have

$$\int\limits_{\Gamma\setminus\gamma_\rho} H_0^{(p)}(\alpha|\zeta - z|)(d\eta i_1 - d\xi i_2)$$

$$= \iint\limits_{\Omega^+\setminus B(z,\rho)} \left(\frac{\partial H_0^{(p)}(\alpha|\zeta - z|)}{\partial\xi} i_1 + \frac{\partial H_0^{(p)}(\alpha|\zeta - z|)}{\partial\eta} \boldsymbol{i}_2 \right) d\xi\, d\eta$$

$$= - \iint\limits_{\Omega^+\setminus B(z,\rho)} \frac{\alpha H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|}(\zeta - z) d\xi\, d\eta,$$

$$\int\limits_{\gamma_\rho} H_0^{(p)}(\alpha|\zeta - z|)(d\eta \boldsymbol{i}_1 - d\xi i_2) = 0$$

and

$$I_3 = - \iint\limits_{\Omega^+} \frac{\alpha H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|}(\zeta - z) d\xi\, d\eta. \tag{5.6}$$

Thus, from (5.3) – (5.6) we have

$$\Phi_\alpha[1](z) = 1 + (-1)^{p-1}\frac{i\alpha^2}{4} \iint\limits_{\Omega^+} \left(H_0^{(p)}(\alpha|\zeta - z|) + \frac{H_1^{(p)}(\alpha|\zeta - z|)}{|\zeta - z|}(\zeta - z) \right) d\xi\, d\eta = I_{\alpha,\Gamma}(z) + 1.$$

Now let $\alpha = 0$. We have

$$\Phi_0[1](z) = \int\limits_\Gamma K_0(\zeta - z)\,\sigma = -\frac{1}{2\pi}\int\limits_\Gamma \frac{\zeta - z}{|\zeta - z|^2}\,\sigma =$$
$$\frac{1}{2\pi}\int\limits_\Gamma \frac{(\xi - x)d\eta - (\eta - y)d\xi}{|\zeta - z|^2} + \frac{1}{2\pi}\int\limits_\Gamma \frac{(\xi - x)d\xi + (\eta - y)d\eta}{|\zeta - z|^2} i_3.$$

Continuing as in the computation of the integrals I_1 and I_2, and using the Green formula, we obtain that $\Phi_0[1](z) = 1$.

In the case of $z \in \Omega^-$ the proof of (4.10) is simplified because of the continuity of the kernel K_α on Ω^+. □

Proof of Lemma (4.11). Making use of the series expansions of the Hankel functions (3.8), (3.9) we obtain

$$I_{\alpha,\Gamma}(z) = \frac{i\alpha}{8}\left((-1)^{p-1} I_{\alpha,\Gamma}^{(1)}(z) + \frac{2i}{\pi} I_{\alpha,\Gamma}^{(2)}(z) + (-1)^p \frac{4i}{\pi} I_\Gamma^{(3)}(z) \right), \tag{5.7}$$

where

$$I^{(1)}_{\alpha,\Gamma}(z) := \iint\limits_{\Omega^+} \sum_{k=0}^{\infty} \alpha^{2k+1}|\zeta - z|^{2k}(a_{k,p} + b_{k,p}\alpha(\zeta - z))d\xi\, d\eta,$$

$$I^{(2)}_{\alpha,\Gamma}(z) := \iint\limits_{\Omega^+} \log|\zeta - z| \sum_{k=0}^{\infty}(-1)^k \frac{\alpha^{2k+1}|\zeta - z|^{2k}}{2^{2k}k!(k+1)!}(2(k+1) + \alpha(\zeta - z))d\xi\, d\eta,$$

$$I^{(3)}_{\Gamma}(z) := \iint\limits_{\Omega^+} \frac{\zeta - z}{|\zeta - z|^2} d\xi\, d\eta,$$

and $a_{k,p}$, $b_{k,p}$ are complex coefficients.

The continuity of $I^{(1)}_{\alpha,\Gamma}$ follows from the continuity of the integrand. Let us prove the continuity of $I^{(3)}_{\Gamma}$. For an arbitrary $z \in \mathbb{C}$ and a measurable $E \subset \mathbb{C}$ set

$$I_E(z) := \iint\limits_{E} \frac{\zeta - z}{|\zeta - z|^2} d\xi\, d\eta.$$

Let us fix any point $z_0 \in \mathbb{C}$. For an arbitrary $z \in \mathbb{C}$ we have

$$\begin{aligned} I^{(3)}_{\Gamma}(z_0) - I^{(3)}_{\Gamma}(z) = {} & I_{\Omega^+ \cap B(z_0,\rho)}(z_0) + I_{(\Omega^+ \setminus B(z_0,\rho)) \cap B(z,\rho)}(z_0) + \\ & I_{\Omega^+ \setminus (B(z,\rho) \cup B(z_0,\rho))}(z_0) - I_{\Omega^+ \cap B(z_0,\rho)}(z) - \\ & I_{(\Omega^+ \setminus B(z_0,\rho)) \cap B(z,\rho)}(z) - I_{\Omega^+ \setminus (B(z,\rho) \cup B(z_0,\rho))}(z). \end{aligned}$$

Fix an arbitrary $\varepsilon > 0$. Since $|I_{E \cap B(z_1,\rho)}(z_2)| \leqslant 16\rho$ for arbitrary $\rho > 0$, $z_1 \in \mathbb{C}$, $z_2 \in \mathbb{C}$, $E \subset \mathbb{C}$, there exists $\rho(\varepsilon) > 0$ such that $|I_{E \cap B(z_1,\rho)}(z_2)| \leqslant \frac{\varepsilon}{6}$. Therefore

$$\begin{aligned} \left| I^{(3)}_{\Gamma}(z_0) - I^{(3)}_{\Gamma}(z) \right| \leqslant {} & \frac{2\varepsilon}{3} + |I_{\Omega^+ \setminus (B(z,\rho) \cup B(z_0,\rho))}(z_0) - I_{\Omega^+ \setminus (B(z,\rho) \cup B(z_0,\rho))}(z)| \leqslant \\ & \frac{2\varepsilon}{3} + \frac{4}{\pi}|z_0 - z| \iint\limits_{\Omega^+ \setminus (B(z,\rho) \cup B(z_0,\rho))} \frac{d\xi\, d\eta}{|\zeta - z_0||\zeta - z|}. \end{aligned}$$

Under the condition $|z_0 - z| < \frac{\rho(\varepsilon)}{2}$ we get:

$$\iint\limits_{\Omega^+ \setminus (B(z,\rho) \cup B(z_0,\rho))} \frac{d\xi\, d\eta}{|\zeta - z_0||\zeta - z|} \leqslant 4\pi \log \frac{d(z_0, \Gamma)}{\rho(\varepsilon)},$$

where $d(z_0, \Gamma) = \max\limits_{t \in \Gamma} |z_0 - t|$. By choosing $|z_0 - z| < \min\left\{\frac{\rho(\varepsilon)}{2}; \varepsilon\left(48 \log \frac{d(z_0,\Gamma)}{\rho(\varepsilon)}\right)^{-1}\right\}$ we obtain

$$\left| I^{(3)}_{\Gamma}(z_0) - I^{(3)}_{\Gamma}(z) \right| < \varepsilon.$$

The integrand in $I^{(2)}_{\alpha,\Gamma}(z)$ is the sum of the function $2\alpha \log|\zeta - z|$ and a function which is continuous at $\zeta = z$ and which has a removable discontinuity for $\zeta \neq z$. Therefore it suffices to prove the continuity of the integral

$$I^{(4)}_{\Gamma}(z) := \iint\limits_{\Omega^+} \log|\zeta - z| d\xi\, d\eta.$$

Fix any point $z_0 \in \mathbb{C}$. For any $z \in \mathbb{C}$ denote $\delta := |z - z_0|$, $\Omega_1^+ := B(z_0, 3\delta) \cap \Omega^+$, $\Omega_2^+ := \Omega^+ \setminus \Omega_1^+$.

Then

$$|I_\Gamma^{(4)}(z) - I_\Gamma^{(4)}(z_0)| \leqslant \left| \iint\limits_{\Omega_1^+} \log|\zeta - z_0| d\xi\, d\eta \right| + \left| \iint\limits_{\Omega_1^+} \log|\zeta - z| d\xi\, d\eta \right| +$$
$$\left| \iint\limits_{\Omega_2^+} \log \frac{|\zeta - z|}{|\zeta - z_0|} d\xi\, d\eta \right| =: I_4 + I_5 + I_6,$$

and setting $4\delta < 1$ we get

$$I_4 \leqslant 2\pi \int\limits_0^{3\delta} \rho \log \frac{1}{\rho}\, d\rho = o(1) \ as \ \delta \to 0,$$

$$I_5 \leqslant 2\pi \int\limits_0^{4\delta} \rho \log \frac{1}{\rho} d\rho = o(1) \ as \ \delta \to 0.$$

Using the inequality

$$\left| \log \frac{|\zeta - z|}{|\zeta - z_0|} \right| \leqslant \frac{2\delta}{|\zeta - z_0|}, \qquad \zeta \in \Omega_2^+,$$

we have

$$I_6 \leqslant 2\delta \iint\limits_{\Omega_2^+} \frac{d\xi\, d\eta}{|\zeta - z_0|} \leqslant 4\pi d(z_0, \Gamma)\, \delta.$$

□

Proof of Theorem (4.3). Let us prove (4.6) (the relation (4.7) is proved similarly). We consider a sequence $z_n \in \Omega^+$, $z_n \to t \in \Gamma$, and denote by ζ_n the point of the curve Γ nearest to z_n.

Applying formula (4.10) we have that

$$\begin{aligned} &|\Phi_\alpha[f](z_n) - (I_{\alpha,\Gamma}(t) + 1) f(t) - F_\alpha[f](t)| = \\ &|\Phi_\alpha[f](z_n) - \Phi_\alpha[f(\zeta_n)](z_n) + \Phi_\alpha[f(\zeta_n)](z_n) - F_\alpha[f](\zeta_n) + F_\alpha[f](\zeta_n) - \\ &\qquad (I_{\alpha,\Gamma}(t) + 1) f(t) - F_\alpha[f](t)| \leqslant M_1 + M_2, \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} M_1 &= |\Phi_\alpha[f - f(\zeta_n)](z_n) - F_\alpha[f](\zeta_n)|, \\ M_2 &= |(I_{\alpha,\Gamma}(z_n) + 1) f(\zeta_n) + F_\alpha[f](\zeta_n) - (I_{\alpha,\Gamma}(t) + 1) f(t) - F_\alpha[f](t)|. \end{aligned}$$

Let $\alpha \neq 0$. On the basis of the relations (3.8) – (4.1) we have the representation

$$K_\alpha(z) = S_\alpha(z) + \varphi_\alpha(z), \tag{5.9}$$

where $\varphi_\alpha(z)$ is a continuous function in $\mathbb{C}$ and

$$S_\alpha(z) := -\frac{1}{2\pi} \left(\frac{z}{|z|^2} + \alpha \log|z| \right).$$

Then

$$
\begin{aligned}
M_1 \leqslant & \left| \int_\Gamma S_\alpha(\zeta - z_n)\, \sigma\, (f(\zeta) - f(\zeta_n)) - \int_\Gamma S_\alpha(\zeta - \zeta_n)\, \sigma\, (f(\zeta) - f(\zeta_n)) \right| + \\
& \left| \int_\Gamma \varphi_\alpha(\zeta - z_n)\, \sigma\, (f(\zeta) - f(\zeta_n)) - \int_\Gamma \varphi_\alpha(\zeta - \zeta_n)\, \sigma\, (f(\zeta) - f(\zeta_n)) \right| =: \\
& M_3 + M_4.
\end{aligned}
$$

By virtue of continuity of the functions $F_\alpha[f]$ (Lemma (4.8)), $I_{\alpha,\Gamma}$ (Lemma (4.11)), φ_α and f we get that $M_2 \to 0$ and $M_4 \to 0$, when $z_n \to t$.

Let us fix an arbitrary $\varepsilon > 0$. For any given $\delta > 0$ we have

$$
M_3 \leqslant M_5 + M_6 + M_7, \tag{5.10}
$$

where

$$
\begin{aligned}
M_5 &= \left| \int_{\Gamma_{\zeta_n,\delta}} S_\alpha(\zeta - \zeta_n)\, \sigma\, (f(\zeta) - f(\zeta_n)) \right|, \\
M_6 &= \left| \int_{\Gamma_{\zeta_n,\delta}} S_\alpha(\zeta - z_n)\, \sigma\, (f(\zeta) - f(\zeta_n)) \right|, \\
M_7 &= \left| \int_{\Gamma \setminus \Gamma_{\zeta_n,\delta}} \left(S_\alpha(\zeta - z_n) - S_\alpha(\zeta - \zeta_n) \right)\, \sigma\, (f(\zeta) - f(\zeta_n)) \right|.
\end{aligned}
$$

By virtue of the equality (5.9) it follows from the uniform existence of $F_\alpha[f]$ (Lemma (4.8)) that for all sufficiently small δ and for all $\zeta_n \in \Gamma$ the inequality $M_5 < \frac{\varepsilon}{3}$ is valid.

Let us estimate M_6. For any $\delta > 0$ let us take z_n near to t and so that $|\zeta_n - z_n| < \frac{\delta}{3}$. We have

$$
\begin{aligned}
M_6 \leqslant & \left| \int_{\Gamma_{\zeta_n, 3|\zeta_n - z_n|}} S_\alpha(\zeta - z_n)\, \sigma\, (f(\zeta) - f(\zeta_n)) \right| + \\
& \left| \int_{\Gamma_{\zeta_n,\delta} \setminus \Gamma_{\zeta_n, 3|\zeta_n - z_n|}} S_\alpha(\zeta - z_n)\, \sigma\, (f(\zeta) - f(\zeta_n)) \right| =: M_8 + M_9.
\end{aligned} \tag{5.11}
$$

Let us estimate M_8. Using the inequalities $|\zeta-\zeta_n| \leqslant 3|\zeta-z_n| < 4\delta$, we obtain for sufficiently small $\delta < \frac{3}{4}$,

$$\frac{|S_\alpha(\zeta-z_n)|}{|S_\alpha(\zeta-\zeta_n)|} = \frac{\left|\frac{\zeta-z_n}{|\zeta-z_n|^2} + \alpha\log|\zeta-z_n|\right|}{\left|\frac{\zeta-\zeta_n}{|\zeta-\zeta_n|^2} + \alpha\log|\zeta-\zeta_n|\right|} \leqslant$$

$$\frac{\frac{3}{|\zeta-\zeta_n|} + |\alpha|\,|\log|\zeta-\zeta_n|| + |\alpha|\log 3}{\frac{1}{|\zeta-\zeta_n|} - |\alpha|\,|\log|\zeta-\zeta_n||} \leqslant 4,$$

$$|S_\alpha(\zeta-z_n)| = |S_\alpha(\zeta-\zeta_n)|\,\frac{|S_\alpha(\zeta-z_n)|}{|S_\alpha(\zeta-\zeta_n)|} \leqslant 4\,|S_\alpha(\zeta-\zeta_n)|. \tag{5.12}$$

Due to the uniform existence of the integral (4.4) and the equality (5.9) it follows from (5.12) that for all sufficiently small δ and for all $|\zeta_n - z_n| < \frac{\delta}{3}$ there holds

$$M_8 \leqslant 4 \int\limits_{\Gamma_{\zeta_n,3|\zeta_n-z_n|}} |S_\alpha\,(\zeta-\zeta_n)\,|\,|\sigma|\,|f(\zeta)-f(\zeta_n)| < \frac{\varepsilon}{6}. \tag{5.13}$$

Let us estimate M_9. We get

$$|S_\alpha(\zeta-z_n)| \leqslant |S_\alpha(\zeta-\zeta_n)| + |S_\alpha(\zeta-z_n) - S_\alpha(\zeta-\zeta_n)|. \tag{5.14}$$

As long as $|z_n-\zeta_n| \leqslant \frac{1}{2}|\zeta-z_n|$, $3|\zeta_n-z_n| < |\zeta-\zeta_n| \leqslant \delta < \frac{3}{4}$ and $\frac{1}{|\zeta-\zeta_n|} \leqslant 2\pi|S_\alpha(\zeta-\zeta_n)|$ we have

$$\begin{aligned}
&|S_\alpha(\zeta-z_n) - S_\alpha(\zeta-\zeta_n)| \leqslant \\
&\frac{1}{2\pi}\left|\frac{\zeta-z_n}{|\zeta-z_n|^2} - \frac{\zeta-\zeta_n}{|\zeta-\zeta_n|^2}\right| + \frac{|\alpha|}{2\pi}\left|\log\frac{|\zeta-z_n|}{|\zeta-\zeta_n|}\right| = \\
&\frac{|z_n-\zeta_n|}{2\pi\,|\zeta-z_n|\,|\zeta-\zeta_n|} + \frac{|\alpha|}{2\pi}\left|\log\frac{|\zeta-z_n|}{|\zeta-\zeta_n|}\right| \leqslant \\
&\frac{1}{4\pi\,|\zeta-\zeta_n|} + \frac{|\alpha|}{2\pi}\log\frac{3}{2} \leqslant \frac{1+|\alpha|}{4\pi\,|\zeta-\zeta_n|} \leqslant \frac{1+|\alpha|}{2}|S_\alpha(\zeta-\zeta_n)|.
\end{aligned} \tag{5.15}$$

From (5.14), (5.15) we get

$$M_9 \leqslant \frac{3+|\alpha|}{2} \int\limits_{\Gamma_{\zeta_n,\delta}\setminus\Gamma_{\zeta_n,3|\zeta_n-z_n|}} |S_\alpha(\zeta-z_n)|\,|\sigma|\,|f(\zeta)-f(\zeta_n)| < \frac{\varepsilon}{6} \tag{5.16}$$

for sufficiently small δ and for $|\zeta_n - z_n| < \frac{\delta}{3}$.

We have from (5.11), (5.13, (5.16) that $M_6 < \frac{\varepsilon}{3}$.

In order to estimate M_7 fix any δ satisfying all the conditions stated above and take z_n near t such that $|\zeta_n - z_n| \leqslant \frac{\delta}{3}$.

We have $\delta < |\zeta-\zeta_n|$, $\frac{2}{3}\delta < |\zeta-z_n|$. Therefore

$$\frac{|z_n-\zeta_n|}{|\zeta-z_n|\,|\zeta-\zeta_n|} \leqslant \frac{3}{2\delta^2}|z_n-\zeta_n|, \tag{5.17}$$

and by Lagrange's theorem

$$\left|\log\frac{|\zeta-z_n|}{|\zeta-\zeta_n|}\right| = \frac{1}{\mu}\,||\zeta-z_n| - |\zeta-\zeta_n|| \leqslant \frac{3}{2\delta}|z_n-\zeta_n|,$$

where μ lies between $|\zeta - z_n|$ and $|\zeta - \zeta_n|$. Then using the relation (5.15) we get

$$|S_\alpha(\zeta - z_n) - S_\alpha(\zeta - \zeta_n)| \leqslant 3\frac{1+|\alpha|\delta}{4\pi\delta^2}|z_n - \zeta_n|,$$

and taking into account the boundedness of f, we obtain for the above fixed δ and for z_n sufficiently near to t

$$M_7 \leqslant \int\limits_{\Gamma\setminus\Gamma_{\zeta_n,\delta}} |S_\alpha(\zeta - z_n) - S_\alpha(\zeta - \zeta_n)|\,|\sigma|\,|f(\zeta) - f(\zeta_n)| \leqslant$$
$$\frac{1+|\alpha|\delta}{2\delta^2}\, l(\Gamma) \max_{t\in\Gamma}|f(t)|\,|\zeta_n - z_n| < \frac{\varepsilon}{3},$$

where $l(\Gamma)$ denotes the length of Γ. Thus we have $M_3 < \varepsilon$ and, consequently, the relation (4.6) is proved.

The continuity of $\Phi^{\pm}_{\alpha}[f]$ on Γ now follows from Lemmas (4.8) and (4.11). This completes the proof of Theorem (4.3). □

6. Proof of main results

We identify a complex quaternion $a = \sum_{k=0}^{3} a_k \boldsymbol{i}_k$ with the scalar-vector pair $(a_0, \boldsymbol{a})$, where $\boldsymbol{a} = \sum_{k=1}^{3} a_k \boldsymbol{i}_k$ is a vector of the complex linear space $\mathbb{C}^3$ with the canonical basis $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, $\boldsymbol{i}_3$. Then a quaternionic function $f = \sum_{k=0}^{3} f_k i_k$ is interpretable as a pair $\mathcal{F} = (f_0, \boldsymbol{f})$, and the operator ∂_α as a pair $(\alpha, {}_{st}\boldsymbol{\partial})$ where ${}_{st}\boldsymbol{\partial} := \partial_1 \boldsymbol{i}_1 + \partial_2 \boldsymbol{i}_2$. Using the vectorial representation of the multiplication of any complex quaternions $a = (a_0, \boldsymbol{a})$ and $b = (b_0, \boldsymbol{b})$ (see [KS], p. 24):

$$ab = (a_0 b_0 - <\boldsymbol{a}, \boldsymbol{b}>, [\boldsymbol{a}, \boldsymbol{b}] + a_0 \boldsymbol{b} + b_0 \boldsymbol{a}), \tag{6.1}$$

we obtain

$$\partial_\alpha f = (\alpha f_0 - \operatorname{div} \boldsymbol{f}, \mathbf{rot}\, \boldsymbol{f} + \alpha \boldsymbol{f} + \mathbf{grad}\ f_0),$$

arriving at the system (2.2) as the vector form of Definition (3.4) of an α-hyperholomorphic function.

Proof of Theorem (2.9). The representation (2.4) follows from the formula (4.1) and we obtain (2.5) from (4.2) by using the equality (6.1). Combining the vector form of the functions F_α, $I_{\alpha,\Gamma}$, $\Phi^{\pm}_{\alpha}$ in Theorem (4.3) with the equality (6.1), we arrive at Theorem (2.9) as a vector reformulation of Theorem (4.3). □

Proof of Corollary (2.15). Applying Theorem (2.9) to the pair $\mathcal{F} = (0, \boldsymbol{f})$, we obtain the desired conclusion. Because of the condition $\boldsymbol{f} \in \mathcal{M}(\Gamma; \mathbb{C}^3)$ the Cauchy-type integral $\Phi_\alpha[\mathcal{F}]$ is purely vectorial and therefore its boundary values $\Phi^{\pm}_{\alpha}[\mathcal{F}]$ are also purely vectorial. □

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ON THE EXISTENCE OF POSITIVE SOLUTIONS OF A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. We present some results on the existence of bounded positive solutions to a class of nonlinear second order ordinary diferential equations by using the Schauder-Tikhonov fixed point theorem. An application to the existence of bounded positive solutions to certain quasilinear elliptic equations in two-dimensional exterior domains is also given.

1. Introduction

We consider the second order nonlinear ordinary differential equation

$$u'' + \frac{u'}{t} + f(t, u, u') = 0, \quad t \geq 1, \tag{1.1}$$

where the nonlinear function $f : [1, +\infty) \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

The goal of the paper is to prove the existence of positive bounded solutions to (1.1) under general conditions on the nonlinear function f. Problems of this type were already considered several decades ago (see e.g. [1], [10] and the citations therein) but this is still a very active area of research, cf. the discussions in [4], [9], [12]. Our results are obtained by applying the Schauder-Tikhonov theorem to an integral form of (1.1). While the fixed point approach to this type of problems is extensively used in the mathematical literature, by working in function spaces where the derivative is also involved, we are able to obtain a considerable improvement with respect to previous works. In particular, our results cover more general situations than the recent studies [4], [8], [9], [12], [17]. Moreover, our main result can be applied to show the existence of bounded positive solutions to certain quasilinear elliptic equations in two-dimensional exterior domains, improving and enhancing some earlier investigations (see [3], [11], [13]).

2. Main results

In this section, we shall prove the existence of bounded positive global solutions to (1.1) under certain conditions on the nonlinearity f.

We first introduce a function space and present two propositions which are used in the proof of Theorem (2.3) in the sequel. Define the set

$$X = \{u \in C^1([1, \infty), \mathbb{R}) : \lim_{t\to\infty} u(t) \text{ exists} \quad \text{and} \quad \lim_{t\to\infty} u'(t) \text{ exists}\},$$

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endowed with the usual linear operation for C^1 function. Then, we have the following two propositions:

PROPOSITION (2.1). *The space X is Banach space under the norm*

$$\| u \| = \max\{\sup_{t\geq 1} | u(t) |, \quad \sup_{t\geq 1} | u'(t) |\}.$$

Proof. Obviously, X is a linear space, so we only need to prove its completeness. By means of the completeness of $C([1,\infty),\mathbb{R})$ endowed with the supremum norm and the integral representation of $u(t)$, the statement follows at once (see also Proposition 10 in [9]). □

PROPOSITION (2.2). *Let K be subset of X. Then K is relatively compact in X if and only if the following three properties hold:*

(i) *K is bounded, that is, there exists a number $M > 0$ such that for all $t \geq 1$ and all $u \in K$,*

$$| u(t) | \leq M \quad \textit{and} \quad | u'(t) | \leq M;$$

(ii) *K is equicontinuous, that is, for every $\varepsilon > 0$, there exists a $\sigma > 0$ such that for all $| t_1 - t_2 | < \sigma$ and all $u \in K$,*

$$| u(t_1) - u(t_2) | \leq \varepsilon \quad \textit{and} \quad | u'(t_1) - u'(t_2) | \leq \varepsilon;$$

(iii) *K is equiconvergent, that is, for every $\varepsilon > 0$, there exists a $t_\varepsilon > 1$ such that for all $t, s \geq t_\varepsilon$ and all $u \in K$,*

$$| u(t) - u(s) | \leq \varepsilon \quad \textit{and} \quad | u'(t) - u'(s) | \leq \varepsilon\,.$$

Proof. The statement is a consequence of the Arzela-Ascoli theorem (see also Proposition 12 in [9]). □

We present now a result on the existence of bounded positive solutions to the equation (1.1).

THEOREM (2.3). *Assume that f satisfies the inequality*

$$|f(t,u,u')| \leq F(t, | u |, | u' |), \tag{2.4}$$

where $F \in C([1,\infty) \times [0,\infty) \times [0,\infty), [0,\infty))$ and $F(t,r,s)$ is nondecreasing in both r and s for each fixed $t \in [1,\infty)$.

(i) *If F satisfies for some $c > 0$,*

$$\int_1^\infty t \, \max\{1, \ln t\} \, F(t, 2c, c) dt < c, \tag{2.5}$$

then there exists a $\delta \in (0,c)$ such that (1.1) has at least a bounded positive solution u_c which satisfies $\delta \leq u_c(t) \leq 2c - \delta$ for $t \geq 1$, and such that $\lim_{t\to\infty} u_c(t)$ exists.

(ii) *If F satisfies for some $c > 0$,*

$$\int_1^\infty t \, \ln t \, F(t, 2c, c) dt < \infty, \tag{2.6}$$

then there exist $t_0 \geq 1$ and $\delta \in (0,c)$ such that (1.1) has at least a bounded positive solution u_c which satisfies $\delta \leq u_c(t) \leq 2c - \delta$ for $t \geq t_0$, and such that $\lim_{t\to\infty} u_c(t)$ exists.

Proof. We first prove part (i). Using the hypothesis (2.5) and the monotonicity property of the function F, by applying the Lebesgue dominated convergence theorem we obtain that there exists an $\delta = \delta(c) \in (0, c)$ such that

$$\int_1^\infty t \max\{1, \ln t\} F(t, 2c - \delta, c - \delta) dt \leq c - \delta. \tag{2.7}$$

Set $K = \{v \in X : \delta \leq v(t) \leq 2c - \delta, \mid v'(t) \mid\leq c - \delta\}$. Define the operator $T : K \to X$ by

$$(Tv)(t) = c + \ln t \int_t^\infty sf(s, v, v')ds + \int_1^t s \ln s\, f(s, v, v')ds, \tag{2.8}$$

for $t > 1$, with $(Tv)(1) = c$. We shall apply the Schauder-Tikhonov theorem to prove that there exists a fixed point for the operator T in the nonempty closed bounded convex set K.

(1) We check that $T : K \to K$. Note that from the inequalities (2.4) and (2.7) and the monotonicity property of F, we have that for every $v \in K$,

$$\begin{aligned} \mid (Tv)(t) - c \mid &=\mid \ln t \int_t^\infty sf(s, v, v')ds + \int_1^t s \ln s\, f(s, v, v')ds \mid \\ &\leq \int_t^\infty (\frac{\ln t}{\ln s}) s \ln s \mid f(s, v, v') \mid ds + \int_1^t s \ln s \mid f(s, v, v') \mid ds \\ &\leq \int_1^\infty s \ln s\, F(s, 2c - \delta, c - \delta)ds \leq c - \delta, \quad t \geq 1. \end{aligned} \tag{2.9}$$

It follows that $\delta \leq (Tv)(t) \leq 2c - \delta$ for $t \geq 0$. On the other hand, differentiating both sides of (2.8) with respect to t, we get

$$(Tv)'(t) = \frac{1}{t} \int_t^\infty s\, f(s, v, v')ds, \quad t \geq 1. \tag{2.10}$$

Thus, by (2.7) and (2.10),

$$\begin{aligned} \mid (Tv)'(t) \mid &=\mid \frac{1}{t} \int_t^\infty s\, f(s, v, v')ds \mid\leq \int_t^\infty s\, F(s, v, v')ds \\ &\leq \int_1^\infty s\, F(s, 2c - \delta, c - \delta)ds \leq c - \delta, \quad t \geq 1. \end{aligned} \tag{2.11}$$

It follows that $\mid (Tv)'(t) \mid\leq c - \delta$. Thus, $T : K \to K$ is well-defined.

(2) We check that $T(K)$ is relatively compact in X. If $\{v_n\}_{n\geq 1}$ is an arbitrary sequence in K, set $M = c - \delta$. We have by (2.11) that

$$\mid (Tv_n)'(t) \mid\leq c - \delta = M, \qquad t \geq 1, n \geq 1. \tag{2.12}$$

An application of the mean value theorem yields

$$\mid (Tv_n)(t_1) - (Tv_n)(t_2) \mid\leq M \mid t_1 - t_2 \mid, \quad t_1, t_2 \geq 1,\ n \geq 1. \tag{2.13}$$

On the other hand, from (2.10) we infer that for $t_2 > t_1 \geq 1$,

$$\begin{aligned}
&| (Tv_n)'(t_1) - (Tv_n)'(t_2)) | \\
&= \Big| \frac{1}{t_1} \int_{t_1}^{\infty} s\, f(s, v_n, (v_n)')ds - \frac{1}{t_2} \int_{t_2}^{\infty} s\, f(s, v_n, (v_n)')ds \Big| \\
&\leq \Big| \frac{1}{t_1} - \frac{1}{t_2} \Big| \int_{t_1}^{\infty} sF(s, v_n, (v_n)')ds + \frac{1}{t_2} \int_{t_1}^{t_2} s\, F(s, v_n, (v_n)')ds \\
&\leq | t_2 - t_1 | \int_{t_1}^{\infty} s\, F(s, 2c, c)ds + \int_{t_1}^{t_2} s\, F(s, 2c, c)ds.
\end{aligned}$$

Taking into account (2.13), by (2.5) and the above inequality, we infer that $\{Tv_n\}_{n\geq 1}$ is equicontinuous in X.

Note that

$$\int_1^t | s \ln s\, f(s, v, v') | \, ds \leq \int_1^{\infty} s \ln s\, F(s, 2c - \delta, c - \delta)ds \leq c - \delta,$$

so that $\lim_{t\to\infty} \int_1^t s \ln s\, f(s, v, v')ds$ exists. Set

$$\alpha = \lim_{t\to\infty} \int_1^t s \ln s\, f(s, v, v')ds.$$

By (2.8), we have

$$| (Tv)(t) - c - \alpha | = \Big| \ln t \int_t^{\infty} sf(s, v, v')ds + \int_1^t s \ln s\, f(s, v, v')ds - \alpha \Big| . \tag{2.14}$$

This shows that for every $\varepsilon > 0$ there exists $t_0(\varepsilon) > 1$ such that

$$| (Tv_n)(t) - c - \alpha | \leq \varepsilon, \quad t \geq t_0(\varepsilon),\ n \geq 1. \tag{2.15}$$

Since by (2.10) we have

$$| (Tv_n)'(t) | \leq \int_t^{\infty} s\, F(s, 2c - \delta, c - \delta)ds, \quad t \geq 1,$$

we deduce that for every $\varepsilon > 0$ there exists $t_1(\varepsilon) > 1$ such that

$$| (Tv_n)'(t) | \leq \varepsilon, \quad t \geq t_1(\varepsilon), n \geq 1. \tag{2.16}$$

The relations (2.15) and (2.16) show that $\{Tv_n\}_{n\geq 1}$ is equiconvergent in X. Since $Tv_n \in K$, we also know that $\{Tv_n\}_{n\geq 1}$ is bounded in X. Thus, applying Proposition (2.2), we obtain that $\{Tv_n\}_{n\geq 1}$ is relatively compact in X.

(3) We check that $T : K \to K$ is continuous. Fix an $\varepsilon > 0$. In view of (2.5), there exists some $t_* > 1$ such that

$$\int_{t_*}^{\infty} s \max\{1, \ln s\} F(s, 2c, c)ds < \frac{\varepsilon}{3}. \tag{2.17}$$

Since $f : [1, t_*] \times [\delta, 2c - \delta] \times [-(c - \delta), (c - \delta)] \longrightarrow \mathbb{R}$ is uniformly continuous, there exists an $\sigma > 0$ such that

$$| f(\tau, r_1, s_1) - f(\tau, r_2, s_2) | < \frac{\varepsilon}{3(\frac{1}{2}t_*^2 \ln t_* + \frac{1}{4}t_*^2 + \frac{1}{4})} \tag{2.18}$$

for all $\tau \in [1, t_*]$, all $r_1, r_2 \in [\delta, 2c - \delta]$ with $| r_1 - r_2 |< \sigma$, and all $s_1, s_2 \in [-(c-\delta), (c-\delta)]$ with $| s_1 - s_2 |< \sigma$. A straightforward computation using (2.8) shows that, for all $v_1, v_2 \in K$ with $\| v_1 - v_2 \|< \sigma$, we have

$$\begin{aligned}
&| (Tv_1)(t) - (Tv_2)(t) |\leq \int_1^\infty s \max\{1, \ln s\} \mid f(s, v_1, v_1') - f(s, v_2, v_2') \mid ds \\
&\leq \int_1^{t_*} s \max\{1, \ln s\} \mid f(s, v_1, v_1') - f(s, v_2, v_2') \mid ds \\
&\quad + \int_{t_*}^\infty s \max\{1, \ln s\} \big[\mid f(s, v_1, v_1') \mid + \mid f(s, v_2, v_2') \mid\big] ds \\
&\leq \frac{\varepsilon}{3(\frac{1}{2}t_*^2 \ln t_* + \frac{1}{4}t_*^2 + \frac{1}{4})} \int_1^{t_*} s(\ln s + 1) ds + 2 \int_{t_*}^\infty s \max\{1, \ln s\} F(s, 2c, c) ds \\
&\leq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon,
\end{aligned}$$

in view of (2.17)-(2.18). Similarly, from (2.10) we infer that

$$\begin{aligned}
&| (Tv_1)'(t) - (Tv_2)'(t) | \\
&\leq \frac{1}{t} \int_t^\infty s \mid f(s, v_1, v_1') - f(s, v_2, v_2') \mid ds \\
&\leq \int_1^\infty s \max\{1, \ln s\} \mid f(s, v_1, v_1') - f(s, v_2, v_2') \mid ds \leq \varepsilon.
\end{aligned}$$

It follows that

$$\begin{aligned}
\| Tv_1 - Tv_2 \| \leq \max\{&\sup_{t\geq 1} | Tv_1(t) - Tv_2(t) |, \\
&\sup_{t\geq 1} | (Tv_1)'(t) - (Tv_2)'(t) |\} \leq \varepsilon.
\end{aligned}$$

Hence, $T : K \to K$ is a continuous operator.

We have verified that $T : K \to K$ satisfies all assumptions of the Schauder-Tikhonov theorem [6]. Hence there exists $u_c \in K$ such that $Tu_c = u_c$. Therefore $\delta \leq u_c(t) \leq 2c - \delta$ for $t \geq 1$, and

$$u_c(t) = c + \ln t \int_t^\infty s f(s, u_c(s), u_c'(s)) ds + \int_1^t s \ln s f(s, u_c(s), u_c'(s)) ds, \ t \geq 1.$$

Differentiating both sides of the above equation with respect to t, we get that $u_c(t)$ is a solution to (1.1) which satisfies

$$\lim_{t\to\infty} (u_c(t) - c - \alpha) = 0,$$

in view of (2.14). This proves part (i) of Theorem (2.3).

Let us now prove part (ii) of Theorem (2.3). In view of (2.6), if we take $t_0 \geq 1$ sufficiently large, then we have

$$\int_{t_0}^\infty t \max\{1, \ln t\} F(t, 2c, c) dt < c.$$

For the equation (1.1) on $[t_0, \infty)$, set

$$X_1 = \{v \in C^1([t_0, \infty), \mathbb{R}) : \lim_{t\to\infty} v(t) \text{ exists} \quad \text{and} \quad \lim_{t\to\infty} \{v'(t)\} \text{ exists}\},$$

and let $K_1 = \{v \in X_1 : \delta \leq v(t) \leq 2c - \delta, \mid v'(t) \mid \leq c - \delta, \text{ for } t \geq t_0\}$. Define the operator $T_1 : K_1 \to X_1$ by

$$(T_1 v)(t) = c + \frac{\ln t}{t_0} \int_t^\infty s\, f(s, v, v')ds + \int_{t_0}^t s \ln s\, f(s, v, v')ds, \ t > t_0,$$

with $(T_1 v)(t_0) = c$.

Following the proof for the part (i) of Theorem (2.3), we verify that the operator $T_1 : K_1 \to K_1$ satisfies all assumptions of the Schauder-Tikhonov theorem. Thus there exists a fixed point for the operator T_1 in the nonempty closed bounded convex set K_1 and the statement (ii) of Theorem (2.3) is true. This completes the proof. $\square$

Example (2.19). Consider (1.1) with $f(t, u, u') = q(t)\, g(u)$, where $g \in C(\mathbb{R}, \mathbb{R})$ and $q \in C([1, \infty), [0, \infty))$. Theorem (2.3) guarantees the existence of a solution to (1.1) that is bounded and positive in a neighborhood of infinity if

$$\int_1^\infty t\, q(t)\, \ln(t)\, dt < \infty. \tag{2.20}$$

For $q(t) = t^\beta, \ t \geq 1$, note that (2.20) holds if and only if $\beta < -2$. To see that this result is sharp, it suffices to restrict our attention to the particular case $g(u) \equiv u$. Then the substitution $v(t) = \sqrt{t}\, u(t)$ transforms (1.1) into

$$v'' + \left(t^\beta + \frac{1}{4t^2}\right) v = 0, \quad t \geq 1. \tag{2.21}$$

It is known (see [7], page 461) that the necessary and sufficient condition for the existence of a solution to (2.21) that is positive in a neighborhood of infinity is precisely $\beta < -2$. Let us also note that the condition (2.20), the sharpness of which we just established, can not be obtained in the same general setting as above by using the results in [4], [5], [8], [9], [17].

Remark. The existence of non-oscillatory solutions to (1.1) has also been recently investigated in [9]. However, the approach devised in [9] relies on the use of nonlinear integral inequalities. For this reason, the global existence of all solutions to (1.1) has to be ensured (we refer to [2] for a general discussion of the global existence issue) and this leads to conditions that are more restrictive than ours. Within our setting we allow certain solutions to blow-up in finite time, as one can see from the particular case

$$f(t, u, u') = -\frac{(n+1)(n+t)}{t^{n+2}}\, u^2 - \frac{2}{t^{2n}}\, u^3$$

with $n \geq 2$. In this case Theorem (2.3) applies despite the fact that the solution $u(t) = \frac{t^n}{2-t}$, $t \in [1, 2)$, blows-up in finite time. $\square$

3. Application to quasilinear elliptic equations

In this section, we shall apply the comparison method and Theorem (2.3) to prove that there exists a bounded positive solution to the quasilinear elliptic equation in two-dimensional exterior domains,

$$\Delta u + f_1(x, u) + f_2(x, u) x \cdot \nabla u + f_3(x, u)(x \cdot \nabla u)^2 = 0, \quad \mid x \mid \geq 1. \tag{3.1}$$

We first recall the comparison method. Consider the elliptic equation

$$\Delta u + \phi(x, u, \nabla u) = 0, \quad x \in G_A, \tag{3.2}$$

where $G_A = \{x \in \mathbb{R}^2 :| x |> A\}$ for some $A > 0$. Fix some $\alpha \in (0, 1)$. Let $\phi \in C^\alpha(\overline{M} \times \overline{J} \times \overline{N}, \mathbb{R})$ for every bounded domain $M \subset G_A$, every bounded interval $J \subset \mathbb{R}$, and every bounded domain $N \subset \mathbb{R}^2$. Assume that for every bounded domain $M \subset G_A$ there exists a nonnegative continuous function θ_M such that

$$| \phi(x, t, p) |\leq \theta_M(| t |)(1+ | p |^2), \quad x \in M, \ t \in \mathbb{R}, \ p \in \mathbb{R}^2.$$

A solution $u(x)$ of (3.2) in G_B for some $B \geq A$ is defined to be a function $u \in C^{2+\alpha}(\overline{M})$ for every bounded domain $M \subset G_B$, such that $u(x)$ satisfies (3.2) at every point $x \in G_B$. A subsolution of (3.2) is defined to be a function u of the same regularity that satisfies $\Delta u + \phi(x, u, \nabla u) \geq 0$. Similarly, a supersolution of (3.2) satisfies $\Delta u + \phi(x, u, \nabla u) \leq 0$. Set $S_B = \{x \in \mathbb{R}^2 :| x |= B\}$ for $B \geq A$.

The following lemma encompasses the version of the comparison method that will be used in the sequel.

LEMMA (3.3). [11] *Assume that ϕ satisfies the above assumptions. If for some $B \geq A \geq 0$ there exists a positive subsolution w and a positive supersolution v to (3.2) in G_B such that $w(x) \leq v(x)$ for all $x \in G_B \cup S_B$, then (3.2) has a solution u in G_B such that $w(x) \leq u(x) \leq v(x)$ throughout $G_B \cup S_B$ and $u(x) = v(x)$ for $x \in S_B$.*

We present now the main theorem of this section

THEOREM (3.4). *Assume that there exists a number $\alpha \in (0, 1)$ such that f_1, f_2, and $f_3 \in C^\alpha(\overline{M} \times \overline{J}, \mathbb{R})$ for every bounded domain $M \subset \mathbb{R}^2$, every bounded interval $J \in \mathbb{R}$, and these functions satisfy the following conditions*

$$\begin{aligned} &0 \leq f_1(x, t), && x \in G_1 \subset \mathbb{R}^2, \ t \in [0, \infty); \\ &| f_i(x, u) | \leq F_i(| x |, | u |), \quad i = 1, 2, 3, && x \in G_1 \subset \mathbb{R}^2, \ u \in \mathbb{R}, \end{aligned} \tag{3.5}$$

where for every $i = 1, 2, 3$, $F_i : [1, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ is Hölder continuous and $F_i(r, s)$ is non-decreasing in s for every fixed $r \in [1, \infty)$.

(i) *If for some $c > 0$ we have*

$$\int_1^\infty s \max\{1, \ln s\} \left(F_1(s, 2c) + cs\, F_2(s, 2c) + (cs)^2 F_3(s, 2c)\right) ds < c \tag{3.6}$$

then (3.1) has a bounded solution u in G_B, with $u(x) > 0$ for $| x |\geq 1$.

(ii) *If for some $c > 0$ we have*

$$\int_1^\infty s \ln s \left(F_1(s, 2c) + cs\, F_2(s, 2c) + (cs)^2 F_3(s, 2c)\right) ds < \infty$$

then there is some $B \geq 1$ such that (3.1) has a bounded solution u in G_B, with $u(x) > 0$ in G_B.

Proof. We first prove part (i) of Theorem (3.4). Let us consider the following differential equation

$$\begin{aligned} \Delta u + F_1(| x |, u) + F_2(| x |, u) \,| (x \cdot \nabla u) | \\ + F_3(| x |, u)(x \cdot \nabla u)^2 = 0, \ | x |\geq 1. \end{aligned} \tag{3.7}$$

The change of variables $t = \mid x \mid,\ u(x) = y(\mid x \mid)$, transforms the above equation into

$$(3.8)\quad y''(t) + \frac{y'}{t} + F_1(t, y(t)) + tF_2(t, y(t)) \mid y' \mid + t^2 F_3(t, y(t))(y')^2 = 0,\ t \geq 1.$$

Applying Theorem (2.3), in view of hypothesis (3.6), we obtain that there exists a $\delta \in (0, c)$ such that (3.8) has at least a bounded positive solution $y(t)$ which satisfies $\delta \leq y(t) \leq 2c - \delta$ for $t \geq 1$, and $\lim_{t\to\infty} y(t)$ exists. If we set $v(x) = y(t)$, then we have that $\delta \leq v(x) \leq 2c - \delta$ for $\mid x \mid \geq 1$, $\lim_{|x|\to\infty} v(x)$ exists, and

$$\begin{aligned}
&\Delta v + f_1(x, v) + f_2(x, v)x \cdot \nabla v + f_3(x, v)(x \cdot \nabla v)^2 \\
&\leq \Delta v + F_1(\mid x \mid, v) + F_2(\mid x \mid, v) \mid x \cdot \nabla v \mid + F_3(\mid x \mid, v)(x \cdot \nabla v)^2 \\
&= y''(t) + \frac{y'}{t} + F_1(t, y(t)) + tF_2(t, y(t)) \mid y' \mid + t^2 F_3(t, y(t))(y')^2 = 0
\end{aligned}$$

Hence, $v(x)$ is a supersolution to (3.1) on $\mid x \mid \geq 1$. In addition, $w(x) \equiv \delta$ satisfies obviously

$$\Delta w(x) + f_1(x, w(x)) + f_2(x, w(x))x \cdot \nabla w(x) + f_3(x, w(x))(x \cdot \nabla w(x))^2 \geq 0$$

for $\mid x \mid \geq 1$. The above inequality shows that $w(x) \equiv \delta$ is a subsolution to (3.1) on $\mid x \mid \geq 1$. Applying Lemma (3.3) with $B = A = 1$, we deduce that there exists a solution u to (3.1) such that $0 < \delta \equiv w(x) \leq u(x) \leq v(x)$ for all $\mid x \mid > 1$, and $u(x) = v(x) > 0$ on S_1. This proves (i) of the theorem.

In addition, by part (ii) of Theorem (2.3) and the previous considerations, we deduce that the statement for (ii) of Theorem (3.4) is true. This completes the proof. □

Example (3.9). For the equation in $G_1 \cup S_1 \subset \mathbb{R}^2$

$$\Delta u + \frac{u^2}{12(1 + \mid x \mid^2)^2} - \frac{x \cdot \nabla u}{3(1 + \mid x \mid^2)^3} - \frac{(x \cdot \nabla u)^2}{3(1 + \mid x \mid^2)^3} = 0,$$

with $F_1(\mid x \mid, \mid u \mid) = \frac{u^2}{12(1+|x|^2)^2}$, $F_2(\mid x \mid, \mid u \mid) = \frac{1}{3(1+|x|^2)^3}$, $F_3(\mid x \mid, \mid u \mid) = \frac{1}{3(1+|x|^2)^3}$. A straightforward computation yields

$$\int_1^\infty s \max\{1, \ln s\} \Big(F_1(s, 2) + s\, F_2(s, 2) + s^2 F_3(s, 2)\Big)\, ds < 1.$$

Therefore, Theorem (3.4) ensures that the above equation has a bounded positive solution $u(x)$ with $u(x) > 0$ for $\mid x \mid \geq 1$. Observe that the results from [3], [5], [11], [13], [16] are powerless.

Let us consider the particular case of (3.1),

$$(3.10)\qquad \Delta u + f(x, u) + g(x)x \cdot \nabla u = 0, \quad G_1 \cup S_1 \subset \mathbb{R}^2.$$

As a consequence of Theorem (3.4), we have

COROLLARY (3.11). *Assume that g is class of C^1 and there exists a number $\alpha \in (0,1)$ such that $f \in C^\alpha(\overline{M} \times \overline{J}, \mathbb{R})$ for every bounded domain $M \subset \mathbb{R}^2$, every bounded interval $J \in \mathbb{R}$ and satisfies the following conditions*

$$0 \leq f(x,k), \qquad\qquad | x | \geq 1, \; k \in [0,\infty);$$
$$| f(x,u) | \leq F(| x |, | u |), \; | x | \geq 1, \; u \in \mathbb{R},$$

where $F : [1,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ is Hölder continuous and $F(r,s)$ is non-decreasing in s for every fixed $r \in [1,\infty)$.

(i) *If for some $c > 0$ we have*

$$\int_1^\infty s \max\{1, \ln s\} \Big(F(s,2c) + cs \mid g(s) \mid \Big) \, ds < c$$

then (3.10) has a bounded positive solution $u(x)$ with $u(x) > 0$ for $| x | \geq 1$.

(ii) *If for some $c > 0$ we have*

$$\int_1^\infty s \ln s \Big(F(s,2c) + cs \mid g(s) \mid \Big) \, ds < \infty$$

then there is some $B \geq 1$ such that (3.10) has a bounded positive solution $u(x)$ with $u(x) > 0$ in $G_B \cup S_B$.

Example (3.12). For the equation in $G_1 \cup S_1 \subset \mathbb{R}^2$

$$\Delta u + \frac{\sqrt{1+u^2}}{6(1+ \mid x \mid^2)^2} - \frac{x \cdot \nabla u}{2(1+ \mid x \mid^2)^3} = 0,$$

with $F(| x |, | u |) = \frac{\sqrt{1+u^2}}{6(1+|x|^2)^2}$ and $g(x) = -\frac{1}{2(1+|x|^2)^3}$, a straightforward computation yields

$$\int_1^\infty s \max\{1, \ln s\} \, (F(s,2) + s \mid g(s) \mid) \, ds < 1.$$

Therefore, Corollary (3.11) ensures that there exists a bounded positive solution $u(x)$ with $u(x) > 0$ for $| x | \geq 1$. Observe that the results from [3], [11], [13] are not conclusive.

Consider the particular case of (3.1),

$$\Delta u + f(x,u) = 0, \quad | x | \geq 1. \tag{3.13}$$

As a consequence of Theorem (3.4), we have

COROLLARY (3.14). *Assume that f is locally Hölder continuous in $(G_1 \cup S_1) \times \mathbb{R}$ and satisfies the following conditions*

$$0 \leq f(x,k), \qquad\qquad | x | \geq 1, \; k \in [0,\infty);$$
$$| f(x,u) | \leq a(| x |) w(| u |), \quad | x | \geq 1, \; u \in \mathbb{R},$$

where $w(r)$ is non-decreasing for all $r \geq 0$, $a \in C([1,\infty),[0,\infty))$, and $w \in C([0,\infty),[0,\infty))$.

(i) *If for some $c > 0$ we have*

$$\int_1^\infty s \max\{1, \ln s\} \, a(s) w(2c) \, ds < c$$

then (3.13) has a bounded positive solution $u(x)$ with $u(x) > 0$ for $| x | \geq 1$.

(ii) *If for some* $c > 0$ *we have*

$$\int_1^\infty s \ln s\, a(s)\, ds < \infty$$

then there is some $B \geq 1$ *such that (3.13) has a bounded positive solution* $u(x)$ *with* $u(x) > 0$ *in* $G_B \cup S_B$.

Example (3.15). Among the equations of the form (3.13), we have the Emden-Fowler equation

$$\Delta u + p(x) \mid u \mid^r sign(u) = 0, \quad r > 0,\ G_1 \cup S_1 \subset \mathbb{R}^2,$$

where $p(x)$ is nonnegative and Hölder continuous in $\mathbb{R}^n$.

(i) For the sublinear ($0 < r < 1$) or superlinear ($r > 1$) Emden-Fowler equations, if

$$\int_1^\infty s\ \max\{1, \ln s\} \max_{|x|=s}\{p(x)\} ds < \infty, \tag{3.16}$$

then there exists $c > 0$ large enough or $c > 0$ small enough, such that

$$\int_1^\infty s\ \max\{1, \ln s\} \max_{|x|=s}\{p(x)\}\ \mid 2c \mid^r ds < c,$$

Consequently, applying Corollary (3.14), we deduce that if (3.16) holds, then the sublinear and the superlinear Emden-Fowler equation has a bounded solution u in G_1, with $u(x) > 0$ in $G_1 \cup S_1 \subset \mathbb{R}^2$.

(ii) For the linear Emden-Fowler equation ($r = 1$), if

$$\int_1^\infty s\ \max\{1, \ln s\} \max_{|x|=s}\{p(x)\}\, ds < \frac{1}{2}, \tag{3.17}$$

then, applying Corollary (3.14), we obtain the existence of a bounded solution u in G_1, with $u(x) > 0$ in $G_1 \cup S_1 \subset \mathbb{R}^2$.

Note that under the same conditions (3.16) or (3.17), the investigations in [3], [11], [13] show only that there is some $B > 1$ such that the Emden-Fowler equation has a solution u in G_B, with $u(x) > 0$ for $\mid x \mid \geq B$. □

Remark: Our approach is typically 2-dimensional ($n = 2$). For the n-dimensional case with $n \geq 3$, we refer to [14], [15]. □

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NEW CHARACTERIZATION OF $BMO(\mathbb{R}^n)$ SPACE

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ABSTRACT. The authors generalize the result of David, Journé and Semmes about the $L^p(\mathbb{R}^n)$, $1 < p < \infty$, boundedness of the Littlewood-Paley g function associated to a para-accretive function to the case where $p = \infty$ and, therefore, give a new characterization of the classical $BMO(\mathbb{R}^n)$ space by using an approximation to the identity which satisfies more general cancellation condition adapted to para-accretive functions.

Introduction

It is well-known that the remarkable $T1$ theorem given by David and Journé provides a general criterion for the $L^2(\mathbb{R}^n)$-boundedness of generalized Calderón-Zygmund singular integral operators; (see [1], [8]). The $T1$ theorem, however, cannot be directly applied to the Cauchy integral on Lipschitz curves. Meyer in [6] (see also [7]) observed that if the function 1 in the $T1$ theorem is allowed to be replaced by a bounded complex-valued function b satisfying $0 < \delta \le \operatorname{Re} b(x)$ almost everywhere, then this result would imply the $L^2(\mathbb{R}^n)$ boundedness of the Cauchy integral on all Lipschitz curves. Replacing the function 1 by an accretive function, McIntosh and Meyer in [6] proved the Tb theorem, where b is an accretive function. David, Journé, and Semmes in [2] introduced a more general class of $L^\infty(\mathbb{R}^n)$ functions b, namely, the so-called para-accretive functions. They proved that the function 1 in the $T1$ theorem can be replaced by para-accretive functions, which is by now called the Tb theorem. Moreover, they showed that the para-accretivity is also necessary in the sense that the Tb theorem holds for a bounded function b, then b is para-accretive.

The $L^p(\mathbb{R}^n)$, $1 < p < \infty$, boundedness of operators which satisfy the Tb theorem follows from the Calderón-Zygmund operator theory. In general, however, such operators are not bounded from the classical Hardy spaces $H^p(\mathbb{R}^n)$ to itself even if T satisfies $T(b) = T^*(b) = 0$. Meyer in [7] observed that if $b(x)$ is a bounded function and $1 \le \operatorname{Re} b(x)$, one can then define the modified Hardy space $H^1_b(\mathbb{R}^n)$ simply via the classical Hardy space $H^1(\mathbb{R}^n)$. More precisely, the space $H^1_b(\mathbb{R}^n)$ is defined by the collection of all functions f such that bf is in the classical Hardy space $H^1(\mathbb{R}^n)$. This space has the advantage of the cancellation adapted to the complex measure $b(x)\,dx$ and is closely related to the Tb theorem, where b is an accretive function. More recently, Lee, Lin and the first author of this paper in [5] proved that if $T^*(b) = 0$, where b is a para-accretive function, then the Calderón-Zygmund operator T is bounded from classical $H^p(\mathbb{R}^n)$ to a new Hardy space $H^p_b(\mathbb{R}^n)$ for $n/(n+\epsilon) < p \le 1$, where $\epsilon \in (0, 1]$ is some positive

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constant which depends on para-accretive function b. In fact, they generalize the result of David, Journé and Semmes in [2] on the $L^p(\mathbb{R}^n)$, $1 < p < \infty$, boundedness of the Littlewood-Paley g function associated to a para-accretive function to the case where $n/(n+\epsilon) < p \le 1$.

1. Proof of the main result

The main purpose of this paper is to generalize the result of David, Journé and Semmes in [2] to the classical space $BMO(\mathbb{R}^n)$. More precisely, we will establish a new characterization of the space $BMO(\mathbb{R}^n)$ via the so-called discrete Carleson maximal function defined by an approximation to the identity having the cancellation condition adapted to para-accretive functions. See also [3].

We first recall some definitions and notation. A function $f \in BMO(\mathbb{R}^n)$ if f is a locally integrable function on $\mathbb{R}^n$ and satisfies

$$\|f\|_{BMO(\mathbb{R}^n)} = \sup_Q \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx < \infty,$$

where the supremum is taken over all cubes Q whose sides are parallel to the axes and

$$f_Q = \frac{1}{|Q|}\int_Q f(x)dx.$$

The following definition of the para-accretive function was given in [2].

Definition (1.1). A bounded complex-valued function b defined on $\mathbb{R}^n$ is said to be para-accretive if there exist constants $C,\ \gamma > 0$ such that, for all cubes $Q \subseteq \mathbb{R}^n$, there is a cube $Q' \subseteq Q$ with $\gamma|Q| \le |Q'|$ satisfying

$$\frac{1}{|Q|}\left|\int_{Q'} b(x)\,dx\right| \ge C > 0.$$

Remark (1.2). It is easy to deduce, by the Lebesgue differential theorem, that for any given para-accretive function b as in definition (1.1), $|b(x)| \ge C > 0$ *a.e.* in $\mathbb{R}^n$.

We now recall the space of "test functions" in [4].

Definition (1.3). Let b be a para-accretive function. Fix two exponents $0 < \beta \le 1$ and $\gamma > 0$. A function f defined on $\mathbb{R}^n$ is said to be a test function of type (β, γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$ if f satisfies

(i) $|f(x)| \le C\dfrac{d^\gamma}{(d+|x-x_0|)^{n+\gamma}}$,

(ii) $|f(x) - f(x')| \le$

$$C\left(\frac{|x-x'|}{d+|x-x_0|}\right)^\beta \frac{d^\gamma}{(d+|x-x_0|)^{n+\gamma}} \text{ for } \ |x-x'| \le \frac{d+|x-x_0|}{2}, \text{ and}$$

(iii) $\int_{\mathbb{R}^n} f(x)b(x)\,dx = 0$.

We denote by $\mathcal{M}^{(\beta,\gamma)}(x_0, d)$ the collection of all test functions of type (β, γ) centered at $x_0 \in \mathbb{R}^n$ with width $d > 0$. If $f \in \mathcal{M}^{(\beta,\gamma)}(x_0, d)$, then the norm of f in $\mathcal{M}^{(\beta,\gamma)}(x_0, d)$ is defined by

$$\|f\|_{\mathcal{M}^{(\beta,\gamma)}(x_0,d)} = \inf\{C: \text{ (i) and (ii) hold}\}.$$

We denote $\mathcal{M}^{(\beta,\gamma)}(0,1)$ simply by $\mathcal{M}^{(\beta,\gamma)}$. It is easy to see that $\mathcal{M}^{(\beta,\gamma)}$ is a Banach space under the norm $\|f\|_{\mathcal{M}^{(\beta,\gamma)}}$. The dual space $\left(\mathcal{M}^{(\beta,\gamma)}\right)'$ consists of all linear functionals $\mathcal{L}$ from $\mathcal{M}^{(\beta,\gamma)}$ to $\mathbb{C}$ satisfying

$$|\mathcal{L}(f)| \leq C\|f\|_{\mathcal{M}^{(\beta,\gamma)}}$$

for all $f \in \mathcal{M}^{(\beta,\gamma)}$. We denote by $\langle h, f\rangle$ the natural pairing of elements $h \in \left(\mathcal{M}^{(\beta,\gamma)}\right)'$ and $f \in \mathcal{M}^{(\beta,\gamma)}$. It is easy to check that for any $x_0 \in \mathbb{R}^n$ and $d > 0$, $\mathcal{M}^{(\beta,\gamma)}(x_0, d) = \mathcal{M}^{(\beta,\gamma)}$ with the equivalent norms. Thus, for all $h \in \left(\mathcal{M}^{(\beta,\gamma)}\right)'$, $\langle h, f\rangle$ is well defined for all $f \in \mathcal{M}^{(\beta,\gamma)}(x_0, d)$ with any $x_0 \in \mathbb{R}^n$ and $d > 0$.

As usual, we write

$$b\mathcal{M}^{(\beta,\gamma)} = \{f : \ f = bg \text{ for some } g \in \mathcal{M}^{(\beta,\gamma)}\}.$$

If $f \in b\mathcal{M}^{(\beta,\gamma)}$ and $f = bg$ for $g \in \mathcal{M}^{(\beta,\gamma)}$, then the norm of f is defined by

$$\|f\|_{b\mathcal{M}^{(\beta,\gamma)}} = \|g\|_{\mathcal{M}^{(\beta,\gamma)}}.$$

We need the definition of an approximation to the identity in [4], see also [2].

Definition (1.4). Let b be a para-accretive function. A sequence of operators $\{S_k\}_{k\in\mathbb{Z}}$ is called to be an *approximation to the identity associated to* b if the kernels $S_k(x, y)$ of S_k are functions from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{C}$ such that there exist constant C and some $0 < \varepsilon \leq 1$ satisfying that for all $k \in \mathbb{Z}$ and all $x,\ x',\ y$, and $y' \in \mathbb{R}^n$,

(i) $|S_k(x, y)| \leq C\dfrac{2^{-k\varepsilon}}{(2^{-k} + |x - y|)^{n+\varepsilon}}$,

(ii) $|S_k(x, y) - S_k(x', y)| \leq C\left(\dfrac{|x - x'|}{2^{-k} + |x - y|}\right)^{\varepsilon}\dfrac{2^{-k\varepsilon}}{(2^{-k} + |x - y|)^{n+\varepsilon}}$

$$\text{for } |x - x'| \leq \frac{1}{2}\left(2^{-k} + |x - y|\right),$$

(iii) $|S_k(x, y) - S_k(x, y')| \leq C\left(\dfrac{|y - y'|}{2^{-k} + |x - y|}\right)^{\varepsilon}\dfrac{2^{-k\varepsilon}}{(2^{-k} + |x - y|)^{n+\varepsilon}}$,

$$\text{for } |y - y'| \leq \frac{1}{2}\left(2^{-k} + |x - y|\right),$$

(iv) $\left|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]\right|$

$$\leq C\left(\frac{|x - x'|}{2^{-k} + |x - y|}\right)^{\varepsilon}\left(\frac{|y - y'|}{2^{-k} + |x - y|}\right)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k} + |x - y|)^{n+\varepsilon}}$$

$$\text{for } |x - x'| \leq \frac{1}{2}\left(2^{-k} + |x - y|\right) \text{ and } |y - y'| \leq \frac{1}{2}\left(2^{-k} + |x - y|\right),$$

(v) $\int_{\mathbb{R}^n} S_k(x, y)b(y)dy = 1$ for all $k \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

(vi) $\int_{\mathbb{R}^n} S_k(x, y)b(x)dx = 1$ for all $k \in \mathbb{Z}$ and $y \in \mathbb{R}^n$.

The main result of this paper is the following theorem.

THEOREM (1.5). *Let b be a para-accretive function. Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity as in Definition (1.4) with the regularity exponent ε and set $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$. Let $l_0 \in \mathbb{Z}$. Then $f \in BMO(\mathbb{R}^n)$ if and only if f is in $\left(b\mathcal{M}^{(\beta,\gamma)}\right)'$ for some $0<\beta,\ \gamma<\varepsilon$ and*

$$\|f\|_{\widetilde{BMO}(\mathbb{R}^n)} = \sup_{P\ dyadic}\left[\frac{1}{|P|}\int_P \sum_{k=-\log_2 l(P)+l_0}^{\infty} |D_k(bf)(x)|^2\,dx\right]^{1/2} < \infty,$$

where the supremum is taken over all dyadic cubes P and $l(P)$ is the side length of the dyadic cube P.

Moreover, there is a constant $C>0$ only depending on l_0, n and b such that for all $f \in BMO(\mathbb{R}^n)$,

$$C^{-1}\|f\|_{BMO(\mathbb{R}^n)} \le \|f\|_{\widetilde{BMO}(\mathbb{R}^n)} \le C\|f\|_{BMO(\mathbb{R}^n)}.$$

To prove this theorem, we need the following continuous version of Calderón reproducing formula provided in [4].

THEOREM (A). *Let b be a para-accretive function. Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity defined as in Definition (1.4) with the regularity exponent ε and set $D_k = S_k - S_{k-1}$ for $k\in\mathbb{Z}$. Then there exists a family $\{\bar{D}_k\}$ of operators such that for all $f \in b\mathcal{M}^{(\beta',\gamma')}$,*

$$f(x) = \sum_{k=-\infty}^{\infty} bD_k b\bar{D}_k(f)(x)\,, \tag{1.6}$$

where the series converge in the L^p-norm, $1<p<\infty$, in the $b\mathcal{M}^{(\beta',\gamma')}$-norm for $0<\beta'<\beta<\varepsilon$ and $0<\gamma'<\gamma<\varepsilon$; and for all $f\in\left(\mathcal{M}^{(\beta',\gamma')}\right)'$, (1.6) also holds in $\left(\mathcal{M}^{(\beta',\gamma')}\right)'$ for $0<\beta<\beta'<\varepsilon$ and $0<\gamma<\gamma'<\varepsilon$. Moreover, $\bar{D}_k(x,y)$, the kernel of $\bar{D}_k$, satisfies the following estimates: for $0<\varepsilon'<\varepsilon$, where ε is the regularity exponent of S_k, there exists a constant $C>0$ such that

(i) $|\bar{D}_k(x,y)| \le C\dfrac{2^{-k\varepsilon'}}{(2^{-k}+|x-y|)^{n+\varepsilon'}}$,

(ii) $|\bar{D}_k(x,y)-\bar{D}_k(x,y')| \le C\left(\dfrac{|y-y'|}{(2^{-k}+|x-y|)}\right)^{\varepsilon'}\dfrac{2^{-k\varepsilon'}}{(2^{-k}+|x-y|)^{n+\varepsilon'}}$

$$for\ |x-x'| \le \left(2^{-k}+|x-y|\right)/2,$$

(iii) $\int_{\mathbb{R}^n}\bar{D}_k(x,y)b(y)\,dy = 0$ *for all $k\in\mathbb{Z}$ and $x\in\mathbb{R}^n$,*
(iv) $\int_{\mathbb{R}^n}\bar{D}_k(x,y)b(x)\,dx = 0$ *for all $k\in\mathbb{Z}$ and $y\in\mathbb{R}^n$.*

Proof of Theorem (1.5). Let $f\in BMO(\mathbb{R}^n)$. By using Theorem 4.3 in [5] and some trivial computation, it is easy to verify that $b\mathcal{M}^{(\beta,\gamma)}\subset H^1(\mathbb{R}^n)$. From this fact and the fact that the space $BMO(\mathbb{R}^n)$ is the dual space of $H^1(\mathbb{R}^n)$ (see [8]), we easily see that $f\in\left(b\mathcal{M}^{(\beta,\gamma)}\right)'$.

Let P be any dyadic cube. We denote by P^* the cube with the same center as P and $4\sqrt{n}$ times the side length of P. We then decompose f into

$$f = f_1 + f_2 + f_3,$$

where $f_1 = (f - f_{P^*})\chi_{P^*}$, $f_2 = (f - f_{P^*})\chi_{\mathbb{R}^n \setminus P^*}$ and $f_3 = f_{P^*}$. Let $\{D_k\}_{k\in\mathbb{Z}}$ be as in the theorem. Then, for all $k \in \mathbb{Z}$,

$$D_k(bf)(x) = D_k(bf_1)(x) + D_k(bf_2)(x) + D_k(bf_3)(x) = D_k(bf_1)(x) + D_k(bf_2)(x),$$

since

$$\int_{\mathbb{R}^n} D_k(x,y)b(y)\,dy = 0.$$

The result of David, Journé and Semmes in [2] states that for $p \in (1,\infty)$,

$$\left\| \left\{ \sum_{k=-\infty}^{\infty} |D_k(f)|^2 \right\}^{1/2} \right\|_{L^p(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)},$$

which yields

$$\begin{aligned}
&\left[\frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)+l_0}^{\infty} |D_k(bf_1)(x)|^2 \, dx \right]^{1/2} \\
&\qquad \le \frac{1}{|P|^{1/2}} \left\{ \int_{\mathbb{R}^n} \sum_{k=-\infty}^{\infty} |D_k(bf_1)(x)|^2 \, dx \right\}^{1/2} \\
&\qquad \le C \frac{1}{|P|^{1/2}} \left\{ \int_{\mathbb{R}^n} |b(x) f_1(x)|^2 \, dx \right\}^{1/2} \\
&\qquad \le C \frac{1}{|P^*|^{1/2}} \left\{ \int_{P^*} |f(x) - f_{P^*}|^2 \, dx \right\}^{1/2} \\
&\qquad \le C \|f\|_{BMO(\mathbb{R}^n)} .
\end{aligned} \tag{1.7}$$

Let x_0 be the center of P. To obtain a desired estimate for f_2, we first note that if $y \in \mathbb{R}^n \setminus P^*$ and $x \in P$, then

$$|y - x_0| \ge Cl(P) \quad \text{and} \quad |x - y| \ge C(l(P) + |y - x_0|).$$

From this, Definition (1.4) (i) and some basic properties of $BMO(\mathbb{R}^n)$ functions (see [8]), it follows that for $x \in P$,

$$\begin{aligned}
|D_k(bf_2)(x)| &= \left| \int_{\mathbb{R}^n} D_k(x,y) b(y) f_2(y)\,dy \right| \\
&\le C \int_{\mathbb{R}^n \setminus P^*} \frac{2^{-k\varepsilon}}{(2^{-k} + |x-y|)^{n+\varepsilon}} |f(y) - f_{P^*}| \, dy \\
&\le C 2^{-k\varepsilon} \int_{|y-x_0| \ge Cl(P)} \frac{1}{(l(P) + |y - x_0|)^{n+\varepsilon}} |f(y) - f_{P^*}| \, dy \\
&\le C \frac{2^{-k\varepsilon}}{l(P)^{\varepsilon}} \|f\|_{BMO(\mathbb{R}^n)} .
\end{aligned} \tag{1.8}$$

By (1.8), we obtain

$$\left[\frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)+l_0}^{\infty} |D_k(bf_2)(x)|^2 \, dx \right]^{1/2}$$

$$\begin{aligned}(1.9)\qquad &\leq C\|f\|_{BMO(\mathbb{R}^n)}\left\{\frac{1}{l(P)^{2\varepsilon}}\sum_{k=-\log_2 l(P)+l_0}^{\infty}2^{-2k\varepsilon}\right\}^{1/2}\\ &\leq C\|f\|_{BMO(\mathbb{R}^n)}.\end{aligned}$$

The estimates (1.7) and (1.9) tell us that

$$\|f\|_{\widetilde{BMO}(\mathbb{R}^n)}\leq C\|f\|_{BMO(\mathbb{R}^n)}.$$

We now establish the converse. Without loss of generality, we may assume that $l_0=3$. Let $f\in\left(b\mathcal{M}^{(\beta,\gamma)}\right)'$ and $g\in\left(\mathcal{M}^{(\beta,\gamma)}\right)'$ and $\tau>0$. We define

$$\mathcal{J}_b(f)(x)=\sup_{\substack{P\ni x\\ P\ dyadic}}\left[\frac{1}{|P|}\int_P\sum_{k=-\log_2 l(P)+3}^{\infty}|D_k(bf)(y)|^2\,dy\right]^{1/2},$$

which is called the discrete Carleson maximal function, where the supremum is taken over all dyadic cubes containing x;

$$\bar{S}(g)(x)=\left\{\sum_{k=-\infty}^{\infty}\int_{|x-y|\leq 2^{-k}}2^{kn}\left|\bar{D}_k(g)(y)\right|^2\,dy\right\}^{1/2};$$

$$S_b(f)(x)=\left\{\sum_{k=-\infty}^{\infty}\int_{|x-y|\leq 2^{-k}}2^{kn}\left|D_k(bf)(y)\right|^2\,dy\right\}^{1/2};$$

and

$$S_b^{\tau}(f)(x)=\left\{\sum_{k=-[\log_2\tau]}^{\infty}\int_{|x-y|\leq 2^{-k}}2^{kn}\left|D_k(bf)(y)\right|^2\,dy\right\}^{1/2},$$

where $[x]$ for $x\in\mathbb{R}$ is the maximal integer no more than x. It is easy to verify that $S_b^{\tau}(f)(x)$ is increasing with τ and $S_b^{\infty}(f)(x)=S_b(f)(x)$.

For every fixed f and $x\in\mathbb{R}^n$, we define the "stopping-time" $\tau(x)$ by

$$\tau(x)=\inf\left\{\nu\in\mathbb{Z}:\ S_b^{2^{-\nu}}(f)(x)\leq A\mathcal{J}_b(f)(x)\right\},$$

where $A>0$ is a large constant to be determined later, and we will fix A which depends only on n.

We first prove that for all $y\in\mathbb{R}^n$ and any $j\in\mathbb{Z}$, if we suitably choose A, then there is a constant $C_1>0$ such that

$$\left|\left\{x\in\mathbb{R}^n:\ |x-y|\leq 2^{-j},\ j\geq\tau(x)\right\}\right|\geq C_1 2^{-jn}\,. \tag{1.10}$$

Let $B_0=B(y,2^{-j})$. It is easy to see that we can find a dyadic cube P such that P contains all balls $B(x,2^{-k})$ with $x\in B_0$ and $k\geq j$, moreover, $l(P)=2^{-j+3}$. Thus, for this dyadic cube P, we have

$$\frac{1}{|B_0|}\int_{B_0}\left[S_b^{2^{-j}}(f)(x)\right]^2\,dx$$

$$
\begin{aligned}
&\le \sum_{k=j}^{\infty} \frac{1}{|B_0|} \int_{B_0} \int_{|x-z|\le 2^{-k}} 2^{kn} \left|D_k(bf)(z)\right|^2 \, dz\, dx \\
&\le \sum_{k=j}^{\infty} \frac{C}{|P|} \int_{P} \int_{B_0} 2^{kn} \left|D_k(bf)(z)\right|^2 \chi_{B(z,2^{-k})}(x)\, dx\, dz \\
&= C_2 \frac{1}{|P|} \int_{P} \sum_{k=j}^{\infty} \left|D_k(bf)(z)\right|^2 \, dz \\
&\le C_2 \inf_{B_0 \ni x} \mathcal{J}_b(f)(x)^2,
\end{aligned}
$$

where $C_2 > 0$ depends only on n. From this, it follows that if $A^2 > C_2$, then

$$
\left| \left\{ x \in B_0 : \ S_b^{2^{-j}}(f)(x) > A\mathcal{J}_b(f)(x) \right\} \right| \le \frac{C_2}{A^2} |B_0|,
$$

which, in turn, tells us (1.10) if we choose $A > 0$ large enough such that $C_1 = 1 - C_2/A^2 > 0$.

Let $g \in b\mathcal{M}^{(\beta,\gamma)} \subset H^1(\mathbb{R}^n)$ and $\|g\|_{H^1(\mathbb{R}^n)} \le 1$. By Theorem (A), (1.10), the Fubini theorem and the Hölder inequality, we obtain

(1.11)

$$
\begin{aligned}
|\langle f, g\rangle| &= \left| \left\langle f, \sum_{k=-\infty}^{\infty} bD_k b\bar{D}_k g \right\rangle \right| \\
&= \left| \sum_{k=-\infty}^{\infty} \left\langle D_k bf, b\bar{D}_k g \right\rangle \right| \\
&\le \sum_{k=-\infty}^{\infty} \int_{\mathbb{R}^n} \left|D_k(bf)(y)\right| \left| \left(b\bar{D}_k g\right)(y) \right| \, dy \\
&\le C_1^{-1} \int_{\mathbb{R}^n} \left[\sum_{k=\tau(x)}^{\infty} \int_{|x-y|\le 2^{-k}} 2^{kn} \left|D_k(bf)(y)\right| \left| \left(b\bar{D}_k g\right)(y) \right| \, dy \right] dx \\
&\le C_1^{-1} \int_{\mathbb{R}^n} \left[\sum_{k=\tau(x)}^{\infty} \int_{|x-y|\le 2^{-k}} 2^{kn} \left|D_k(bf)(y)\right|^2 \, dy \right]^{1/2} \\
&\quad \times \left[\sum_{k=\tau(x)}^{\infty} \int_{|x-y|\le 2^{-k}} 2^{kn} \left| \left(b\bar{D}_k g\right)(y) \right|^2 \, dy \right]^{1/2} dx \\
&= C_1^{-1} \int_{\mathbb{R}^n} S_b^{2^{-\tau(x)}}(f)(x) \bar{S}^{2^{-\tau(x)}}(g)(x)\, dx \\
&\le C_1^{-1} A \int_{\mathbb{R}^n} \mathcal{J}_b(f)(x) \bar{S}(g)(x)\, dx \\
&\le C_1^{-1} A \left\|\mathcal{J}_b(f)\right\|_{L^\infty(\mathbb{R}^n)} \left\|\bar{S}(g)\right\|_{L^1(\mathbb{R}^n)} \\
&\le C \|f\|_{\widetilde{BMO}(\mathbb{R}^n)} \|g\|_{H^1(\mathbb{R}^n)} \\
&\le C \|f\|_{\widetilde{BMO}(\mathbb{R}^n)},
\end{aligned}
$$

where we used the fact that

$$
\left\|\bar{S}(g)\right\|_{L^1(\mathbb{R}^n)} \le C \|g\|_{H^1(\mathbb{R}^n)},
$$

which is a simple corollary of Theorem 3.3 and Theorem 4.3 in [5]. By Theorem 5.4 in [5], we know that $b\mathcal{M}^{(\beta,\gamma)}$ is dense in $H^1(\mathbb{R}^n)$. From this, (1.11) and the fact that the space $BMO(\mathbb{R}^n)$ is the dual space of the space $H^1(\mathbb{R}^n)$, it follows that $f \in BMO(\mathbb{R}^n)$ and

$$\|f\|_{BMO(\mathbb{R}^n)} \leq C \, \|f\|_{\widetilde{BMO}(\mathbb{R}^n)} .$$

This finishes the proof of Theorem (1.5). □

Similar to the definition of $H^1_b(\mathbb{R}^n)$, if b is a para-accretive function, we define

$$BMO_b(\mathbb{R}^n) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \ f = bg \text{ for some } g \in BMO(\mathbb{R}^n) \right\}$$

and

$$\|f\|_{BMO_b(\mathbb{R}^n)} = \|g\|_{BMO(\mathbb{R}^n)} .$$

Then it is easy to verify that $BMO_b(\mathbb{R}^n)$ is the dual space of $H^1_b(\mathbb{R}^n)$. Moreover, from Theorem (1.5), we can easy to deduce the following consequence.

COROLLARY (1.12). *Let b be a para-accretive function. Suppose that $\{S_k\}_{k\in\mathbb{Z}}$ is an approximation to the identity as in Definition (1.4) with the regularity exponent ε and $\{D_k\}_{k\in\mathbb{Z}}$ as in Theorem (A). Let $l_0 \in \mathbb{Z}$. Then $f \in BMO_b(\mathbb{R}^n)$ if and only if f is in $\left(\mathcal{M}^{(\beta,\gamma)}\right)'$ for some $0 < \beta, \ \gamma < \varepsilon$ and*

$$\|f\|_{\widetilde{BMO}_b(\mathbb{R}^n)} = \sup_{P \ dyadic} \left[\frac{1}{|P|} \int_P \sum_{k=-\log_2 l(P)+l_0}^{\infty} |D_k(f)(x)|^2 \, dx \right]^{1/2} < \infty,$$

where the supremum is taken over all dyadic cubes P and $l(P)$ is the side length of the dyadic cube P.

Moreover, there is a constant $C > 0$ depending on l_0 such that for all $f \in BMO_b(\mathbb{R}^n)$,

$$C^{-1}\|f\|_{BMO_b(\mathbb{R}^n)} \leq \|f\|_{\widetilde{BMO}_b(\mathbb{R}^n)} \leq C\|f\|_{BMO_b(\mathbb{R}^n)}.$$

Finally, we remark that if T is a Calderón-Zygmund operator and $T(b) = 0$, where b is a para-accretive function, then T is bounded from $BMO_b(\mathbb{R}^n)$ into $BMO(\mathbb{R}^n)$, which can be proved by the result in [5] via a dual argument. We leave the details to the reader. This result together with the results in [5] completes the theory of the Calderón-Zygmund operators.

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ON THE ALGEBRA GENERATED BY THE BERGMAN PROJECTION AND A SHIFT OPERATOR II

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ABSTRACT. Let $G \subset \mathbb{C}$ be a domain with smooth boundary and let α be a C^2-diffeomorphism on $\overline{G}$ satisfying the Carleman condition $\alpha \circ \alpha = \mathrm{id}_{\overline{G}}$. We denote by $\mathcal{R}$ the C^*-algebra generated by the Bergman projection of G, all multiplication operators aI ($a \in C(\overline{G})$) and the operator $W\varphi = \sqrt{|\det J_\alpha|}\,\varphi\circ\alpha$, where $\det J_\alpha$ is the Jacobian of α. Index formulae for Fredholm operators in $\mathcal{R}$ are given.

1. Introduction

Let $G \subset \mathbb{C}$ be a bounded domain whose boundary is a finite union of non-intersecting simple closed curves of class C^1. Let α be a C^2-diffeomorphism of $\overline{G}$ satisfying the Carleman condition $\alpha \circ \alpha = \mathrm{id}_{\overline{G}}$, and let $J_\alpha(z)$ stand for the Jacobian matrix of α with respect to the real variables x and y, where $z = x+iy$. The mapping α induces the unitary operator W on $L_2(G)$ defined by

$$(W\varphi)(z) = \sqrt{|\det J_\alpha(z)|}\varphi(\alpha(z)).$$

Let K stand for the orthogonal projection of $L_2(G)$ onto the Bergman space $\mathcal{A}^2(G)$, which consists of all analytic functions of $L_2(G)$ [1].

We denote by $\mathcal{R} = \mathcal{R}(C(\overline{G})I; K, W)$ the C^*-algebra generated by K, W and $C(\overline{G})I$. Analogously we define $\mathcal{R}_0 = \mathcal{R}(C(\overline{G})I; K, WKW)$. Let $\mathcal{C}$ stand for the ideal of all compact operators in $L_2(G)$ and let π be the natural mapping from $\mathcal{R}$ onto the Calkin algebra $\widehat{\mathcal{R}} := \mathcal{R}/\mathcal{C}$. This work is an application of the paper [13] which describes, in an abstract setting, the C^*-algebra generated by an orthogonal projection and a shift operator. Using local techniques [10], it is proved in [9] that $\widehat{\mathcal{R}}$ and also $\widehat{\mathcal{R}}_0$ are isomorphic to the C^*-algebra of all continuous sections of a C^*-bundle with $\overline{G}$ as a base space. For every $z_0 \in \overline{G}$ the local algebra (fiber) of $\mathcal{R}$ at z_0 is isomorphic to a subalgebra of $M_2 \otimes \mathcal{R}_0(z_0)$, where $\mathcal{R}_0(z_0)$ denotes the local algebra of $\mathcal{R}_0$ at z_0. When z_0 belongs to the boundary of G, $\mathcal{R}_0(z_0)$ is isomorphic to the C^*-algebra generated by the non-zero local images of the orthogonal projections K and WKW; but $\mathcal{R}_0(z_0)$ is isomorphic to $\mathbb{C}$ if $z_0 \in G$. It is well known (see, for example, [6, 11]) that the C^*-algebra generated by two orthogonal projections P_1 and P_2 is isomorphic to a subalgebra of $M_2(C(\Delta))$, where $\Delta = \mathrm{sp}(P_1 - P_2)^2$. Thus, most of the work developed in [9] was devoted to the computation of $\Delta_{z_0} = \mathrm{sp}_{\mathrm{loc}-z_0}(K - WKW)^2$ (the local spectrum of $(K-WKW)^2$ at z_0). It is interesting to note that the local spectrum of $K \pm WKW$ is independent of α when $\det J_\alpha$ is negative; whereas $\Delta z_0 = \{0, \beta(z_0)\}$ if $\det J_\alpha$

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is positive, where β is the function defined in (1.4). We point out that the local principle theory for singular integral operators played a very important role in the computation of Δ_{z_0}. See for instance [8].

This paper is devoted to the index calculation of Fredholm operators in $\mathcal{R}$ and is organized as follows. Theorems (1.8) and (1.9) below give index formulae for Fredholm operators in $\mathcal{R}_0$. The proofs of these theorems are relegated to Section 2. Index formulae for Fredholm operators in $\mathcal{R}$ are given in Subsection 3.9. Finally, some simple examples are considered in Section 4.

The following two theorems describe the (Fredholm) symbol algebra of $\mathcal{R}_0$.

THEOREM (1.1) ([9]). *If* $\det J_\alpha < 0$*, then every operator in* $\mathcal{R}_0$ *has the form*

$$A = a_1(I - K - WKW) + a_2K + a_3WKW + T, \tag{1.2}$$

where T *is compact and* $a_j \in C(\overline{G})$*. The Calkin algebra of* $\mathcal{R}_0$ *is isomorphic to* $C(\overline{G}) \times (C(\partial G))^2$*. Under the identification* $\widehat{\mathcal{R}}_0 = C(\overline{G}) \times (C(\partial G))^2$ *we have*

$$\pi(A) = (a_1, a_2|_{\partial G}, a_3|_{\partial G}).$$

Define $A_0 = I - K,\ \ A_1 = K,\ \ A_2 = (I-K)WKW(I-K)$ and $A_4 = A_3^*$, where

$$A_3 = \frac{1}{\sqrt{1-\beta}} K(WKW)(I-K), \tag{1.3}$$

$$\beta(z) = \frac{(\|J_\alpha(z)\|_2)^2 - 2\det J_\alpha(z)}{(\|J_\alpha(z)\|_2)^2 + 2\det J_\alpha(z)} \tag{1.4}$$

and $\|J_\alpha(z)\|_2$ is the Euclidean norm of $J_\alpha(z)$; that is, $\|J_\alpha(z)\|_2^2$ is the sum of the squares of all its entries. The simplest representation of the function β is given by

$$\beta(z) = \left| \frac{\partial\alpha/\partial\overline{z}}{\partial\alpha/\partial z} \right|^2.$$

Let Γ be the C^*-algebra consisting of all pairs $(a, \sigma) \in C(\overline{G}) \times M_2(C(\partial G))$ with the following property: if $\beta(z) = 0$, then $\sigma(z)$ is diagonal and $a(z) = \sigma_{22}(z)$.

THEOREM (1.5) ([9]). *If* $\det J_\alpha > 0$*, then the algebra (not necessarily closed) of all operators of the form*

$$A = a_0A_0 + a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 + T, \tag{1.6}$$

with $T \in \mathcal{C}$ *and* $a_j \in C(\overline{G})$*, is dense in* $\mathcal{R}_0$*. The Calkin algebra of* $\mathcal{R}_0$ *is isomorphic to* Γ*. Under the identification* $\widehat{\mathcal{R}}_0 = \Gamma$ *we have*

$$\pi(A) = \left(a_0, \begin{pmatrix} a_1|_{\partial G} & (\sqrt{\beta}a_3)|_{\partial G} \\ (\sqrt{\beta}a_4)|_{\partial G} & (\beta a_2 + a_0)|_{\partial G} \end{pmatrix} \right).$$

Remark (1.7). If $\beta = 0$, then $\widehat{\mathcal{R}}_0 = C(\overline{G}) \times C(\partial G)$ and $\pi(A) = (a_0, a_1|_{\partial G})$. If β does not vanish on ∂G, then $\widehat{\mathcal{R}}_0 = C(\overline{G}) \times M_2(C(\partial G))$.

The group of all invertible elements of a Banach algebra $\mathcal{A}$ will be denoted by $\mathcal{A}^{-1}$.

Let $\widehat{\mathcal{R}}_{00}^{-1}$ be the connected component of $\widehat{\mathcal{R}}_0^{-1}$ which contains the identity. By definition the *abstract index group* of $\widehat{\mathcal{R}}_0$ is $\Lambda_{\widehat{\mathcal{R}}_0} = \widehat{\mathcal{R}}_0^{-1}/\widehat{\mathcal{R}}_{00}^{-1}$, and the *abstract index* of $\widehat{\mathcal{R}}_0$ is the natural mapping ind : $\widehat{\mathcal{R}}_0^{-1} \to \Lambda_{\widehat{\mathcal{R}}_0}$.

To obtain an index formula for all Fredholm operators in $\mathcal{R}_0$ first of all we have to determine the group $\Lambda_{\widehat{\mathcal{R}}_0}$. In Section 2 we will see that $\Lambda_{\widehat{\mathcal{R}}_0}$ is isomorphic to a direct sum of $\mathbb{Z}$. The desired index formula will be obtained by considering the index mapping as a composition of the natural homomorphism from Fred $\mathcal{R}_0$ to $\Lambda_{\widehat{\mathcal{R}}_0}$ with a homomorphism from $\Lambda_{\widehat{\mathcal{R}}_0}$ to $\mathbb{Z}$, where Fred $\mathcal{R}_0$ denotes the semigroup of all Fredholm operators in $\mathcal{R}_0$.

Let $\bigcup_{k=0}^{n} \gamma_k$ be the (positively oriented) boundary of G, where $\gamma_0, ..., \gamma_n$ are non-intersecting simple closed curves. We can assume that γ_k is inside γ_0 for each $k \geq 1$. We define the k-th winding number of $a \in C(\partial G)^{-1}$ as follows:

$$\kappa_k = \kappa_k(a) = \frac{1}{2\pi}[\arg a]_{\gamma_k}.$$

As usual, the *index* of a Fredholm operator A is defined by Ind A = dim ker A − dim ker A^*.

THEOREM (1.8). *If the Jacobian of α is negative, then $\Lambda_{\widehat{\mathcal{R}}_0} \cong \mathbb{Z}^{3n+2}$. If operator (1.2) of Theorem (1.1) is Fredholm, then*

1) $\operatorname{ind}\pi(A) = (\kappa_1(a_1), ..., \kappa_n(a_1), \kappa_0(a_2), ..., \kappa_n(a_2), \kappa_0(a_3), ...\kappa_n(a_3))$ *and*
2) $\operatorname{Ind} A = \dfrac{1}{2\pi}\displaystyle\sum_{k=0}^{n}[\arg a_3]_{\gamma_k} - \dfrac{1}{2\pi}\sum_{k=0}^{n}[\arg a_2]_{\gamma_k}.$

THEOREM (1.9). *If the Jacobian of α is positive, then $\Lambda_{\widehat{\mathcal{R}}_0} \cong \mathbb{Z}^{2n+1}$. If the operator (1.6) is Fredholm, then*

$$\operatorname{Ind} A = \frac{1}{\pi}\sum_{k=0}^{n}[\arg a_0]_{\gamma_k} - \frac{1}{2\pi}\sum_{k=0}^{n}[\arg\{a_1(a_0+\beta a_2) - \beta a_3 a_4\}]_{\gamma_k}.$$

COROLLARY (1.10) ([12]). *If $A = a(I-K) + bK$ is Fredholm, then*

$$\operatorname{Ind} A = \frac{1}{2\pi}\sum_{k=0}^{n}[\arg a]_{\gamma_k} - \frac{1}{2\pi}\sum_{k=0}^{n}[\arg b]_{\gamma_k}.$$

Note that Corollary (1.10) does not depend on the sign of det J_α.

2. Proofs of theorems (1.8) and (1.9)

We denote by $[\widehat{\mathcal{R}}_0^{-1}]$ the group of all homotopy classes of elements in $\widehat{\mathcal{R}}_0^{-1}$. Actually $\Lambda_{\widehat{\mathcal{R}}_0} = [\widehat{\mathcal{R}}_0^{-1}]$, see for instance [2].

Let $\pi_1(X)$ denote the fundamental group of $X \subset \mathbb{C}$ [5]. Assume that det $J_\alpha < 0$. By Theorem (1.1), the group $\widehat{\mathcal{R}}_0^{-1}$ is isomorphic to $C(\overline{G})^{-1} \times [C(\partial G)^{-1}]^2$. Since the fundamental group of a simple closed curve equals $\mathbb{Z}$, we have

$$(2.1)\qquad [C(\partial G)^{-1}] \cong \pi_1(\bigcup_{k=0}^{n}\gamma_k) \cong \pi_1(\gamma_0)\times\cdots\times\pi_1(\gamma_n) \cong \mathbb{Z}^{n+1}.$$

On the other hand, $\overline{G}$ is homotopically equivalent to the suspension of $n+1$ points [5], thus

$$[C(\overline{G})^{-1}] \cong \pi_1(\bigcup_{k=1}^{n} \gamma_k) \cong \pi_1(\gamma_1) \times \cdots \times \pi_1(\gamma_n) \cong \mathbb{Z}^n. \tag{2.2}$$

By (2.1) and (2.2) we have that $\Lambda_{\widehat{\mathcal{R}}_0} = \mathbb{Z}^{3n+2}$, consequently the abstract index is given by

$$\operatorname{ind} : \widehat{\mathcal{R}}_0^{-1} \ni (a_1, a_2, a_3) \mapsto (\kappa_1(a_1), ..., \kappa_n(a_1), \kappa_0(a_2), ..., \kappa_n(a_2), \kappa_0(a_3), ...\kappa_n(a_3)).$$

Proof of Theorem (1.8). Let $A_0' = I - K + (z - z_0)K$, where $z_0 \in G$. By Theorem (1.1), the operator A_0' is Fredholm. The equality $\operatorname{Ind} A_0' = -1$ was established in [12], although it is easy to see that A_0' is injective and that $ker A_0'^* = V^\perp = \mathbb{C}$, where $V = \{\psi \in L_2(G) : (K\psi)(z_0) = 0\}$.

It is well known that an operator $A \in \mathcal{R}_0$ is Fredholm if and only if $\pi(A) \in \widehat{\mathcal{R}}_0^{-1}$. If $\mathcal{D}$ is a connected component of $\widehat{\mathcal{R}}_0^{-1}$, then all operators in $\pi^{-1}(D)$ have a common index. This fact implies that there exists a linear mapping $\Xi : \Lambda_{\widehat{\mathcal{R}}_0} \to \mathbb{Z}$ such that the following diagram is commutative:

$$\begin{tikzcd} \operatorname{Fred}\mathcal{R}_0 \arrow[r, "\pi"] \arrow[drr, "\operatorname{Ind}"'] & \widehat{\mathcal{R}}_0^{-1} \arrow[r, "\operatorname{ind}"] & \Lambda_{\widehat{\mathcal{R}}_0} = \mathbb{Z}^{3n+2} \arrow[d, "\Xi"] \\ & & \mathbb{Z} \end{tikzcd}$$

Since Ξ is an homomorphism, there exist integers $l_{k,j}$ such that the index of every Fredholm operator of the form (1.2) is given by

$$\operatorname{Ind} A = \sum_{k=1}^{n} l_{k,1}\kappa_k(a_1) + \sum_{j=2}^{3}\sum_{k=0}^{n} l_{k,j}\kappa_k(a_j). \tag{2.3}$$

The equality $l_{0,2} = -1$ can be obtained by applying formula (2.3) to the operator A_0'. For each $k \in \{1, ..., n\}$, let z_k be a complex number inside γ_k. It is easy to see that $A_k' = I - K + (z - z_k)K$ is invertible, consequently $\operatorname{Ind} A_k' = 0$. Applying formula (2.3) to A_k' we obtain $l_{0,2} - l_{k,2} = 0$. Therefore $l_{k,2} = -1$ for $k = 0, ..., n$. To simplify our arguments we assume that $\alpha(\gamma_0) = \gamma_0$. The assumption $\det J_\alpha < 0$ ensures that α reverses the orientation of ∂G. If $\alpha(\gamma_k) = \gamma_k$, then one more application of formula (2.3) to the invertible operator

$$WA_k'W = I - WKW + (\alpha(z) - z_k)WKW$$

results in $-l_{0,3} + l_{k,3} = 0$. However, if $\alpha(\gamma_k) = \gamma_l$ with $k \neq l$, then we get $-l_{0,3} + l_{l,3} = 0$. Since $\bigcup_{k=1}^{n} \alpha(\gamma_k) = \bigcup_{k=1}^{n} \gamma_k$ we have $-l_{0,3} + l_{k,3} = 0$ for every $k \in \{1, ..., n\}$.

From $-l_{0,3} = \operatorname{Ind} WA_0'W = -1$ it follows that $l_{k,3} = 1$. This equality is valid even if $\alpha(\gamma_0) = \gamma_{k_o}$ with $k_o \neq 0$. Now if operator (1.2) is Fredholm, then a_1 does not vanish on $\overline{G}$ (Theorem 1.1). This means that $a_1 I$ is invertible, thus

$$\operatorname{Ind} A = \operatorname{Ind}(I - K - WKW + \frac{a_2}{a_1}K + \frac{a_3}{a_1}WKW)$$

$$= \quad \sum_{k=0}^{n}(-\kappa_k(\frac{a_2}{a_1}) + \kappa_k(\frac{a_3}{a_1})).$$

The equality $\kappa_k(a_j/a_1) = \kappa_k(a_j) - \kappa_k(a_1)$ completes the proof. □

Proof of Theorem (1.9). Suppose that the operator (1.6) is Fredholm. By Theorem (1.5) the coefficient a_0 does not vanish on $\overline{G}$. Without lost of generality we can assume that a_1 does not vanish on ∂G. Let $\tilde{a}_1 \in C(\overline{G})$ be an extension of $(1/a_1)|_{\partial G}$. Setting $\pi(A) = (a_0, M)$, the mapping

$$H_1(t) = (a_0, \begin{pmatrix} a_1 & \sqrt{\beta}(1-t)a_3 \\ \sqrt{\beta}(1-t)a_4 & a_0 + \beta a_2 - t(2-t)\beta a_3 a_4/a_1 \end{pmatrix}\Big|_{\partial G})$$

is a homotopy between $\pi(A)$ and

$$\pi(D_1)\pi(D_2) = (a_0, \begin{pmatrix} a_1 & 0 \\ 0 & (\det M)/a_1 \end{pmatrix}),$$

where $D_1 = a_0(I-K) + a_1 K$ and $D_2 = I + (a_1 a_2 - a_3 a_4)(\tilde{a}_1/a_0)A_2$.

Let

$$U(t) = (1, \begin{pmatrix} \sqrt{1-t} & \sqrt{t}i \\ \sqrt{t}i & \sqrt{1-t} \end{pmatrix}).$$

The mapping H_2 defined by $H_2(t) = U(t)\pi(D_2)U(t)^* \in \widehat{\mathcal{R}}_0^{-1}$ is a homotopy between $\pi(D_2)$ and $\pi(D_3) = (1, \operatorname{diag}\{(\det M)/(a_0 a_1), 1\})$, where $D_3 = I - K + ((\det M)\tilde{a}_1/a_0)K$. Therefore, $\pi(A)$ is homotopic to $\pi(D_1)\pi(D_3) = \pi(a_0(I-K) + (\det M/a_0)K)$. Corollary (1.10) completes the proof. □

3. Index formulae for Fredholm operators in $\mathcal{R}$

The symbol algebra of $\mathcal{R}$ given in this section differs a little from that given in [9]. This new approach is convenient because it allows us to understand the algebraic structure of the group $[\mathcal{R}^{-1}]$.

Let $C_\alpha(\overline{G})$ be the space of all functions $a \in C(\overline{G})$ such that $a \circ \alpha = a$. Let M_α denote the quotient space induced by the following equivalence relation on $\overline{G}$: $z' \sim z$ if and only if either $z' = z$ or $z' = \alpha(z)$. Then $C_\alpha(\overline{G})$ is isomorphic to $C(M_\alpha)$.

Let J_{z_0} be the maximal ideal of $Z_0 = \pi(C(\overline{G})I)$ corresponding to $z_0 \in \overline{G}$, and let $J(z_0) = \widehat{\mathcal{R}}_0 \cdot J_{z_0}$ be the ideal of $\widehat{\mathcal{R}}_0$ generated by J_{z_0}. We refer to $\mathcal{R}_0(z_0) = \widehat{\mathcal{R}}_0/J(z_0)$ as the local algebra of $\mathcal{R}_0$ at z_0. Using $Z = \pi(C_\alpha(\overline{G})I)$, the construction of $\mathcal{R}[z_0]$ is similar to that of $\mathcal{R}_0(z_0)$. We denote by ν_{z_0} the natural mapping from $\mathcal{R}$ into $\mathcal{R}[z_0]$.

In [9] a symbol algebra of $\mathcal{R}_0$ ($\mathcal{R}$) was obtained by means of local techniques and using Z_0 (Z) as central subalgebra of $\widehat{\mathcal{R}}_0$ ($\widehat{\mathcal{R}}$), see also [13] for details. The local algebra $\mathcal{R}[z_0]$ is a subalgebra of $M_2(\mathcal{R}_0(z_0))$, and the natural images of the generators of $\mathcal{R}$ into $\mathcal{R}[z_0]$ are given by

$$\nu_{z_0}(aI + bK + dW) = \begin{pmatrix} a(z_0)e + b(z_0)p_1 & d(z_0)e \\ d(\alpha(z_0))e & a(\alpha(z_0))e + b(\alpha(z_0))p_2 \end{pmatrix}, \tag{3.1}$$

where e, p_1 and p_2 are the images of I, K and WKW into $\mathcal{R}_0(z_0)$, respectively.

Let π_{z_0} be the natural mapping from $\mathcal{R}_0$ into $\mathcal{R}_0(z_0)$. For $C = A + BW \in \mathcal{R}$, with $A, B \in \mathcal{R}_0$, the equality

$$\nu_{z_0}(C) = \begin{pmatrix} \pi_{z_0}(A) & \pi_{z_0}(B) \\ \pi_{z_0}(WBW) & \pi_{z_0}(WAW) \end{pmatrix} \tag{3.2}$$

holds.

Theorems (1.1), (1.5), and relation (3.2) establish a symbol algebra for $\mathcal{R}$ in a natural way as we will see below.

(3.3) Case $\det J_\alpha < 0$**.** Let Ω_1 be the C^*-subalgebra of $M_2(C(\overline{G})) \times M_2(C(\partial G))$ consisting of all pairs of the form

$$\left(\begin{pmatrix} a_1 & b_1 \\ b_1 \circ \alpha & a_1 \circ \alpha \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ b_3 \circ \alpha & a_3 \circ \alpha \end{pmatrix} \right), \tag{3.4}$$

where $a_1, b_1 \in C(\overline{G})$ and $a_2, b_2, a_3, b_3 \in C(\partial G)$.

Remark (3.5). Since $\alpha|_{\partial G}$ is an automorphism of ∂G, the second matrix in (3.4) has no restriction on its entries. We have used the above notation for convenience.

THEOREM (3.6). *If* $\det J_\alpha < 0$*, then the Calkin algebra of* $\mathcal{R}$ *is isomorphic to* Ω_1*. Under the identification* $\widehat{\mathcal{R}} = \Omega_1$*, the element (3.4) is the natural image in* Ω_1 *of the operator*

$$C = A + BW + T, \tag{3.7}$$

where

$$\begin{aligned} A &= a_1(I - K - WKW) + a_2K + a_3WKW, \\ B &= b_1(I - K - WKW) + b_2K + b_3WKW, \end{aligned}$$

$a_k, b_k \in C(\overline{G})$ *and* T *is compact. If* C *is Fredholm, then* $D = a_1I + b_1W$ *is invertible and*

$$D^{-1} = \frac{1}{a_1(a_1 \circ \alpha) - b_1(b_1 \circ \alpha)}(a_1 \circ \alpha I - b_1W).$$

THEOREM (3.8). *If the operator (3.7) is Fredholm, then*

$$\operatorname{Ind} C = \frac{1}{2\pi}\sum_{k=0}^{n}[\arg\{a_1(a_1\circ\alpha) - b_1(b_1\circ\alpha)\}]_{\gamma_k} - \frac{1}{2\pi}\sum_{k=0}^{n}[\arg\{a_2(a_3\circ\alpha) - b_2(b_3\circ\alpha)\}]_{\gamma_k}.$$

Proof of Theorem (3.6). From (3.2) and Theorem (1.1) we obtain the following symbol for C:

$$\widetilde{C} = \left(\begin{pmatrix} a_1 & b_1 \\ b_1 \circ \alpha & a_1 \circ \alpha \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ b_3 \circ \alpha & a_3 \circ \alpha \end{pmatrix}\Bigg|_{\partial G}, \begin{pmatrix} a_3 & b_3 \\ b_2 \circ \alpha & a_2 \circ \alpha \end{pmatrix}\Bigg|_{\partial G} \right).$$

Let μ be the isomorphism on $M_2(C(\partial G))$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d \circ \alpha & c \circ \alpha \\ b \circ \alpha & a \circ \alpha \end{pmatrix}.$$

The application of $I \oplus I \oplus \mu$ to $\widetilde{C}$ shows that $\widehat{\mathcal{R}} \cong \Omega_1$.

Suppose that C is a Fredholm operator. Then its symbol is invertible in Ω_1. Thus the matrix

$$M = \begin{pmatrix} a_1 & b_1 \\ b_1 \circ \alpha & a_1 \circ \alpha \end{pmatrix}$$

is invertible in $M_2(C(\overline{G}))$, which means that $a_1(a_1 \circ \alpha) - b_1(b_1 \circ \alpha)$ does not vanish on $\overline{G}$. A simple computation proves the last part of the theorem. $\square$

Proof of Theorem (3.8). Suppose that the operator (3.7) is Fredholm. By Theorem (3.6), the operator $D = a_1 I + b_1 W$ is invertible. It is easy to see that the symbol of D equals

$$(M, M|_{\partial G}),$$

where M is the matrix given in the proof of Theorem (3.6). Actually M is the first matrix appearing in the symbol of C. Therefore the symbol of $D^{-1}C$ is as follows:

$$\left(I_2, M^{-1} \begin{pmatrix} a_2 & b_2 \\ b_3 \circ \alpha & a_3 \circ \alpha \end{pmatrix}\right),$$

where I_2 denotes the 2×2 identity matrix.

Taking into account Remark (3.5), it is easy to see that the symbol of $D^{-1}C$ is homotopic to

$$M_0 = \left(I_2, \begin{pmatrix} 1/\det M & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_2(a_3 \circ \alpha) - b_2(b_3 \circ \alpha) & 0 \\ 0 & 1 \end{pmatrix}\right).$$

Note that M_0 is the symbol of the operator

$$C_0 = I - K + \frac{1}{\det M}(a_2(a_3 \circ \alpha) - b_2(b_3 \circ \alpha))K.$$

Since the symbol of $D^{-1}C$ is homotopic to the symbol of C_0, we have $\operatorname{Ind} C = \operatorname{Ind} D^{-1}C = \operatorname{Ind} C_0$. Now Corollary (1.10) completes the proof. $\square$

(3.9) Case $\det J_\alpha > 0$**.** We will obtain an index formula for Fredholm operators in $\mathcal{R}$ via an index formula for Fredholm operators in $\mathcal{R}_0 \otimes M_2(\mathbb{C})$. Let $\widehat{\Pi}$ be the natural mapping from $\mathcal{R}_0 \otimes M_2(\mathbb{C})$ into $\widehat{\mathcal{R}}_0 \otimes M_2(\mathbb{C}) = \Gamma \otimes M_2(\mathbb{C})$ (see Theorem (1.5)).

Consider $A = (A_{jk})_{j,k=1,2} \in \mathcal{R}_0 \otimes M_2(\mathbb{C})$.

Setting $\pi(A_{jk}) = (a^0_{jk}, \begin{pmatrix} a^1_{jk} & a^3_{jk} \\ a^4_{jk} & a^2_{jk} \end{pmatrix})$, we obtain

$$\widehat{\Pi}(A) = (\begin{pmatrix} a^0_{11} & a^0_{12} \\ a^0_{21} & a^0_{22} \end{pmatrix}, \begin{pmatrix} a^1_{11} & a^3_{11} & a^1_{12} & a^3_{12} \\ a^4_{11} & a^2_{11} & a^4_{12} & a^2_{12} \\ a^1_{21} & a^3_{21} & a^1_{22} & a^3_{22} \\ a^4_{21} & a^2_{21} & a^4_{22} & a^2_{22} \end{pmatrix}).$$

Introduce $\widetilde{V} = (I_{2\times 2}, \widetilde{U})$, where $I_{2\times 2}$ is the 2×2 identity matrix and

$$\widetilde{U} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\Pi(A) := \widetilde{V}(\widehat{\Pi}(A))\widetilde{V} = (\begin{pmatrix} a_{11}^0 & a_{12}^0 \\ a_{21}^0 & a_{22}^0 \end{pmatrix}, \begin{pmatrix} a_{11}^1 & a_{12}^1 & a_{11}^3 & a_{12}^3 \\ a_{21}^1 & a_{22}^1 & a_{21}^3 & a_{22}^3 \\ a_{11}^4 & a_{12}^4 & a_{11}^2 & a_{12}^2 \\ a_{21}^4 & a_{22}^4 & a_{21}^2 & a_{22}^2 \end{pmatrix}). \tag{3.10}$$

By Theorem (1.5) $(a_{jk}^0) = (a_{jk}^2)$, and the second matrix in (3.10) is 2×2 block diagonal at each point $z \in \partial G$ for which $\beta(z) = 0$.

Let Υ be the C^*-algebra consisting of all pairs

$$(N, M) \in M_2(C(\overline{G})) \times M_4(C(\partial G))$$

such that both $M(\zeta)$ is 2×2 block diagonal and $N(\zeta) = M_{22}(\zeta)$ whenever $\beta(\zeta) = 0$, where M_{22} is the 2×2 block $(m_{jk})_{j,k=3,4}$ and m_{jk} is the (j, k) entry of M.

Note that $\Upsilon = M_2(C(\overline{G})) \times M_4(C(\partial G))$ when β does not vanish on ∂G.

THEOREM (3.11). *If* $\det J_\alpha > 0$, *then the Calkin algebra of* $\mathcal{R}_0 \times M_2(\mathbb{C})$ *is isomorphic to* Υ. *Under the identification* $\widehat{\mathcal{R}}_0 \times M_2(\mathbb{C}) = \Upsilon$, *the element (3.10) is the image of* A *into* Υ. *If* A *is Fredholm and* β *does not vanish on* ∂G, *then*

$$\operatorname{Ind} A = \frac{1}{\pi}\sum_{k=0}^{n}[\arg\det N_A]_{\gamma_k} - \frac{1}{2\pi}\sum_{k=0}^{n}[\arg\det M_A]_{\gamma_k},$$

where $(N_A, M_A) = \Pi(A)$.

Introduce the unitary self-adjoint matrix

$$V = \begin{pmatrix} \sqrt{1-\beta} & \sqrt{\beta} \\ \sqrt{\beta} & -\sqrt{1-\beta} \end{pmatrix},$$

where $\beta(= \beta \circ \alpha)$ is defined in (1.4). Since $\pi(WKW) = (1, V)\pi(K)(1, V)$, the isomorphism $\Psi : \pi(A) \mapsto \pi(WAW)$ acts on the symbol algebra $\widehat{\mathcal{R}}_0$ as follows:

$$\Psi(\pi(A)) = (a_0 \circ \alpha,\ V\begin{pmatrix} a_1 \circ \alpha & \sqrt{\beta}(a_3 \circ \alpha) \\ \sqrt{\beta}(a_4 \circ \alpha) & \beta(a_2 \circ \alpha) + a_0 \circ \alpha \end{pmatrix}V),$$

where A is the operator (1.6).

Let Ω_2 be the C^*-subalgebra of $M_2(C(\overline{G})) \times M_4(C(\partial G))$ generated by all pairs of the form

$$(\begin{pmatrix} a_0 & b_0 \\ b_0 \circ \alpha & a_0 \circ \alpha \end{pmatrix}, M), \tag{3.12}$$

where

$$M = \tag{3.13}$$

$$\begin{pmatrix} \begin{pmatrix} a_1 & \sqrt{\beta}a_3 \\ \sqrt{\beta}a_4 & \beta a_2 + a_0 \end{pmatrix} & \begin{pmatrix} b_1 & \sqrt{\beta}b_3 \\ \sqrt{\beta}b_4 & \beta b_2 + b_0 \end{pmatrix} \\ V\begin{pmatrix} b_1 \circ \alpha & \sqrt{\beta}(b_3 \circ \alpha) \\ \sqrt{\beta}(b_4 \circ \alpha) & \beta(b_2 \circ \alpha) + b_0 \circ \alpha \end{pmatrix}V & V\begin{pmatrix} a_1 \circ \alpha & \sqrt{\beta}(a_3 \circ \alpha) \\ \sqrt{\beta}(a_4 \circ \alpha) & \beta(a_2 \circ \alpha) + a_0 \circ \alpha \end{pmatrix}V \end{pmatrix}.$$

Since $\alpha \circ \alpha = \mathrm{id}_{\overline{G}}$ and α preserves the orientation of ∂G, we have that either α is the identity function on γ_k or it does not have fixed points on γ_k [7]. We will assume the latter case for all k.

THEOREM (3.14). *If* $\det J_\alpha > 0$, *then the algebra (not necessarily closed) of all operators of the form*

$$C = \sum_{j=0}^{4}(a_j A_j + b_j A_j W) + T, \tag{3.15}$$

with $T \in \mathcal{C}$, *is dense in* $\mathcal{R}$. *The Calkin algebra* $\widehat{\mathcal{R}}$ *is isomorphic to* Ω_2. *Under the identification* $\widehat{\mathcal{R}} = \Omega_2$, *the element (3.12) is the natural projection of* C *into* Ω_2. *If* β *does not vanish on* ∂G *and* C *is Fredholm, then*

$$\operatorname{Ind} C = \frac{1}{2\pi}\sum_{k=0}^{n}[\arg\{a_0(a_0\circ\alpha) - b_0(b_0\circ\alpha)\}]_{\gamma_k} - \frac{1}{4\pi}\sum_{k=0}^{n}[\arg\det M]_{\gamma_k}. \tag{3.16}$$

Proof. Suppose that C is Fredholm. Without loss of generality we can assume that $a_0 = 1$ and $b_0 = 0$ (see the proof of Theorem (3.8)). Let 0_2 denote the 2×2 zero matrix, and let $\tilde{d} \in C(\overline{G})^{-1}$ be an extension of $d|_{\partial G}$, where $d(z) = \alpha(z) - z$. We have

$$\pi(\tilde{d}I) = (\begin{pmatrix} \tilde{d} & 0 \\ 0 & \tilde{d}\circ\alpha \end{pmatrix}, d\begin{pmatrix} I_2 & 0_2 \\ 0_2 & -I_2 \end{pmatrix}).$$

Introduce the operator

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ W & -W \end{pmatrix}.$$

Setting $A = \sum a_k A_k$ and $B = \sum b_k A_k$, we get $C = A + BW + T$. It is easy to see that $\pi(\tilde{d}IC\tilde{d}^{-1}I) = \pi(A - BW)$. This implies that C and $A - BW$ have the same index. Let

$$X = U\begin{pmatrix} A + BW & 0 \\ 0 & A - BW \end{pmatrix}U = \begin{pmatrix} A & B \\ WBW & WAW \end{pmatrix}.$$

Since $X \in \mathcal{R}_0 \otimes M_2(\mathbb{C})$, formula (3.16) follows from the equality $2\operatorname{Ind} C = \operatorname{Ind} X$ and Theorem (3.11). The proof is complete. □

4. Examples

Example (4.1). Let $G = \mathbb{D}$ be the unit disk, and $\alpha(z) = \bar{z}$. Then $(W\varphi)(z) = \varphi(\bar{z})$. In this particular case the operator $\overline{K} = WKW$ is the orthogonal projection from $L_2(\mathbb{D})$ onto the subspace of all anti-analytic functions. The following integral representations are well known [3]:

$$(K\varphi)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)}{(1 - z\bar{\zeta})^2}\, d\mu(\zeta), \qquad (\overline{K}\varphi)(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{\varphi(\zeta)}{(1 - \bar{z}\zeta)^2}\, d\mu(\zeta).$$

By Theorem (1.1) the Calkin algebra of $\mathcal{R}_0 = \mathcal{R}(C(\overline{\mathbb{D}})I; K, \overline{K})$ is isomorphic to $C(\overline{\mathbb{D}}) \times C(\partial\mathbb{D})^2$, and every operator in $\mathcal{R} = \mathcal{R}(C(\overline{\mathbb{D}})I; K, W)$ has the form

$$C = a_1 I + a_2 K + a_3\overline{K} + [b_1 I + b_2 K + b_3\overline{K}]W + T.$$

The operator C is Fredholm if and only if both $a_1(a_1 \circ \alpha) - b_1(b_1 \circ \alpha)$ does not vanish on $\mathbb{D}$ and $f(g \circ \alpha) - p(q \circ \alpha)$ does not vanish on $\mathbb{T} = \partial\mathbb{D}$, where $f = a_1 + a_2$, $g = a_1 + a_3$, $p = b_1 + b_2$ and $q = b_1 + b_3$. If C is Fredholm, then

$$\operatorname{Ind} C = -\frac{1}{2\pi}[\arg\{f(g \circ \alpha) - p(q \circ \alpha)\}]_{\mathbb{T}}.$$

Example (4.2). Now consider the annulus $G = \{z : 1 < |z| < 2\}$. Let $f : [1, 2] \to [1, 2]$ be a smooth function with negative derivative. If $f \circ f$ is the identity function on $[1, 2]$, then

$$\alpha(z) = f(|z|)\frac{\overline{z}}{|z|}$$

is a C^2-diffeomorphism on G satisfying the inequality $\det J_\alpha > 0$ and the Carleman condition. Let W be the shift operator induced by α. A straightforward computation shows that

$$\beta(z) = \left(\frac{f(|z|) + |z|f'(|z|)}{f(|z|) - |z|f'(|z|)}\right)^2.$$

In particular, consider the function $f = g \circ h \circ g^{-1}$, where $h(t) = 3 - t$ and

$$g(t) = g_s(t) = \frac{1}{2^s - 1}(t^s - 1) + 1, \quad s \neq 0. \tag{4.3}$$

In this example the function defined in (1.4) is constant on the boundary of G. For $|z| = 1$ we have

$$\beta(z) = \beta(2z) = \left(\frac{g'(2) - 2g'(1)}{g'(2) + 2g'(1)}\right)^2 = \left(\frac{1 - 2^{s-2}}{1 + 2^{s-2}}\right)^2.$$

For a fixed $r \in (-1, 1)$, we have $\beta(1) = \beta(2) = r^2$ whenever $2^{s-2} = (1 - r)/(1 + r)$.

In case $r \neq 0$ we have

$$\widehat{\mathcal{R}}_0 = C(\overline{G}) \times M_2(C(S^1)) \times M_2(C(S^2)),$$
$$\widehat{\mathcal{R}} = M_2(C(\overline{G})) \times M_4(C(S^1)) \times M_4(C(S^2)),$$

where $S^a = \{z : |z| = a\}$.

In case $r = 0$ ($s = 2$), the Calkin algebra $\widehat{\mathcal{R}}_0$ is isomorphic to $C(\overline{G}) \times C(S^1) \times C(S^2)$. Furthermore, every operator in $\mathcal{R}$ has the form

$$C = a_0(I - K) + a_1K + [b_0(I - K) + b_1K]W + T, \quad T \in \mathcal{C}.$$

By Remark (1.7) and the equality $S^2 = \alpha(S^1)$, the symbol of C (see (3.12)) can be simplified to

$$\pi(C) = (\begin{pmatrix} a_0 & b_0 \\ b_0 \circ \alpha & a_0 \circ \alpha \end{pmatrix}, \begin{pmatrix} a_1 & b_1 \\ b_1 \circ \alpha & a_1 \circ \alpha \end{pmatrix}\Bigg|_{S^1}). \tag{4.4}$$

Thus $\widehat{\mathcal{R}}$ contains $M_2(S^1)$ as a subalgebra. Note that this example is not contemplated in Theorem (3.14) because $\beta = 0$. However the index formula for Fredholm operators in $\mathcal{R}$ can be computed as follows. Assume that C is Fredholm and that $a_0 = 1$, $b_0 = 0$. Then $\pi(C)$ is homotopic to $\pi(I - K + aK)$, where a is any continuous function such that $a = a_1(a_1 \circ \alpha) - b_1(b_1 \circ \alpha)$ on S^1 and $a = 1$ on S^2. Corollary (1.10) gives the index of C. If $a_0 \neq 1$ and $b_0 \neq 0$, then we consider the operator $(a_0 I + b_0 W)^{-1} C$ to complete the proof of the following formula:

$$\operatorname{Ind} C = \frac{1}{2\pi}[\arg\{a_0(a_0 \circ \alpha) - b_0(b_0 \circ \alpha)\}]_{S^1} - \frac{1}{2\pi}[\arg\{a_1(a_1 \circ \alpha) - b_1(b_1 \circ \alpha)\}]_{S^1},$$

where S^1 is negatively oriented.

Example (4.5). Let $s = s(\theta) \neq 0$ be an even smooth function in the variable $\theta = \arg z$. Once again consider $G = \{z : 1 < |z| < 2\}$ but define

$$\alpha(z) = f(|z|, s(\theta))\frac{\overline{z}}{|z|},$$

where

$$f(t, s) = g_s(h(g_s^{-1}(t))),$$

$h(t) = 3 - t$ and g_s is the function on $[1, 2]$ defined in (4.3). The mapping α, as defined above, has all required properties. The function β can be determined by using the equality $\beta = |\alpha_{\overline{z}}/\alpha_z|^2$, where $\alpha_z = \partial\alpha/\partial z$ and $\alpha_{\overline{z}} = \partial\alpha/\partial\overline{z}$. Setting

$$r(\theta) = \frac{1 - 2^{s(\theta)-2}}{1 + 2^{s(\theta)-2}},$$

a straightforward computation leads to

$$\beta(z) = \beta(2z) = (r(\theta))^2, \quad |z| = 1\,.$$

In particular, consider

$$s(\theta) = \delta\cos(m\theta) + \tau + 2,$$

with $0 < \delta \leq \tau$ and $m \in \mathbb{Z}$. As in case $s \neq 2$ in Example 2, for $\delta < \tau$ the algebra $\widehat{\mathcal{R}}_0$ is isomorphic to $C(\overline{G}) \times M_2(C(S^1)) \times M_2(C(S^2))$ but the symbols of two operators that have the same form are not equal to each other, for example, the symbol of WKW changes from one example to another.

Define $S(\theta) = 2^{s(\theta)-2}$. Now suppose that $\delta = \tau$. In such a case $s(\theta)$ takes the value 2 at $\theta = (2k + 1)\pi/m$ for $k = 0, ..., m - 1$. This means that $r(\theta)$ vanishes at $\theta = (2k + 1)\pi/m$. Since $\beta(z) = \beta(2z) = (r(\arg z))^2$, the symbol of operator (1.6) in Theorem (1.5) becomes diagonal at each point z satisfying $\arg z = (2k + 1)\pi/m$, i.e.; each of these points induces only one-dimensional irreducible representations of $\mathcal{R}_0$. The rest of the points in ∂G generate two-dimensional irreducible representations of $\mathcal{R}_0$. Thus the set $M_\alpha = \overline{G} \cup M_\beta \cup M_{1\alpha}$ can be identified with the space of all irreducible representations of $\widehat{\mathcal{R}}_0$, where

$$M_\beta = \{(z, \beta(z)) : z \in \partial G\} \quad \text{and} \quad M_{1\alpha} = \{(z, 1) : \beta(z) = 0, z \in \partial G\}.$$

The figure shows M_α for $m = 3$ and $\delta = \tau = 1$. The subspace $\overline{G} \cup M_{1\alpha}$ is isomorphic to the space of all one-dimensional irreducible representations of $\widehat{\mathcal{R}}_0$; whereas M_β is isomorphic to the space of all two-dimensional irreducible representations.

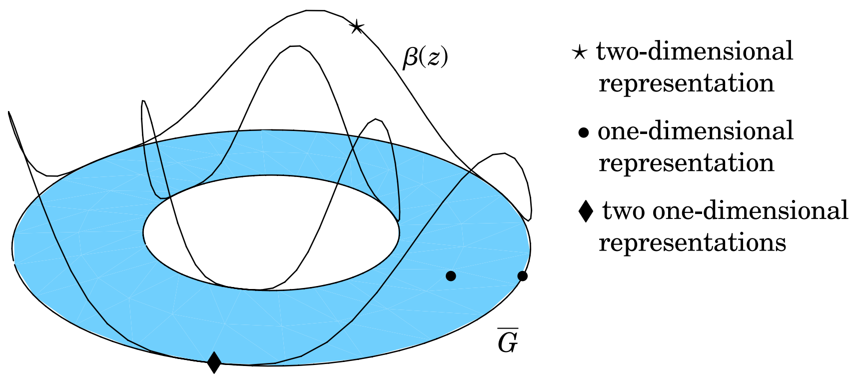

Figure 1. Space of irreducible representations

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DYNAMICS OF PROPERTIES OF TOEPLITZ OPERATORS ON THE UPPER HALF-PLANE: HYPERBOLIC CASE

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Dedicated to the fond memory of Olga Grudskaia, who generously assisted in the preparation of the figures in this paper.

Abstract. We consider Toeplitz operators $T_a^{(\lambda)}$ acting on the weighted Bergman spaces $\mathcal{A}^2_\lambda(\Pi)$, $\lambda \in [0,\infty)$, over the upper half-plane Π, whose symbols depend on $\theta = \arg z$. Motivated by the Berezin quantization procedure we study the dependence of the properties of such operators on the parameter of the weight λ and, in particular, under the limit $\lambda \to \infty$.

1. Introduction

This is a part of the two-paper set devoted to the study of Toeplitz operators acting on weighted Bergman spaces on the upper half-plane. Both are motivated by the same ideas and are a continuation of our research started in [6]. We have mentioned in [6] the papers [1, 2, 3, 9, 10], where Toeplitz operators with smooth (or continuous) symbols acting on the weighted Bergman spaces, as well as C^*-algebras generated by such operators, appear naturally in the context of problems in mathematical physics. In particular, recall that given a smooth symbol $a = a(z)$, the family of Toeplitz operators $T_a = \{T_a^{(h)}\}$, with $h \in (0,1)$, is considered under the Berezin quantization procedure [1, 2]. For a fixed h the Toeplitz operator $T_a^{(h)}$ acts on the weighted Bergman space $\mathcal{A}^2_h$. In the special quantization procedure each Toeplitz operator $T_a^{(h)}$ is represented by its Wick symbol $\widetilde{a}_h$, and the correspondence principle says that for smooth symbols one has

$$\lim_{h\to 0} \widetilde{a}_h = a.$$

Moreover by [8] the above limit remains valid in the L_1-sense for a wider class of symbols.

The same, as in a quantization procedure, weighted Bergman spaces appear naturally in many questions in complex analysis and operator theory. In the last cases a weight parameter is normally denoted by λ and runs through $(-1,+\infty)$. In the sequel we will consider weighted Bergman spaces $\mathcal{A}^2_\lambda$ parameterized by

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$\lambda \in (-1, +\infty)$ which is connected with $h \in (0,1)$, used as the parameter in the quantization procedure, by the rule $\lambda + 2 = \frac{1}{h}$.

At this stage an important problem emerges: study of the behavior of different properties (boundedness, compactness, spectral properties, etc.) of $T_a^{(\lambda)}$ in dependence on λ, and comparison of their limit behavior under $\lambda \to \infty$ with corresponding properties of the initial symbol a.

It seems to be quite impossible to get a reasonably complete answer to the above problem for general (smooth) symbols, even for the simplest case of the weighted Bergman spaces on the unit disk (hyperbolic plane). At the same time the recently discovered classes of commutative *-algebras of Toeplitz operators on the unit disk suggest classes of symbols for which a satisfactory complete answer can be given. Recall in this connection (for details see [11, 12]) that all known cases of commutative *-algebras of Toeplitz operators on the unit disk are classified by pencils of (hyperbolic) geodesics of the following three possible types: geodesics intersecting in a single point (elliptic pencil), parallel geodesics (parabolic pencil), and disjoint geodesics, i.e., all geodesics orthogonal to a given one (hyperbolic pencil). Symbols which are constant on the cycles, the orthogonal trajectories to the geodesics forming a pencil, generate in each case a commutative *-algebra of Toeplitz operators. Moreover these commutative properties of Toeplitz operators do not depend at all on smoothness properties of symbols, the symbols can be merely measurable.

The model case for elliptic pencils, Toeplitz operators on the unit disk with radial symbols, was considered in [6]. In the present paper we consider the model case for hyperbolic pencils, while another paper [5] of this two-paper set is devoted to the study of the model case for parabolic pencils. Both papers together cover the part remaining after [6]. The results for other (non model) cases can be easily obtained by means of Möbius transformations.

We study Toeplitz operators on the upper half-plane equipped with the hyperbolic metric, where the model case for hyperbolic pencils is realized as Toeplitz operators with symbols depending only on $\theta = \arg z$.

The key feature of symbols constant on cycles, which permits us to obtain much more complete information than when studying general symbols, is as follows. In each case of a commutative *-algebra generated by Toeplitz operators the Toeplitz operators admit a spectral type representation, i.e., they are unitary equivalent to multiplication operators, by a certain sequence in the elliptic case and by certain functions on $\mathbb{R}_+$ and $\mathbb{R}$ in the parabolic and hyperbolic cases, respectively.

We mention a difference between the previously studied elliptic case [6] and the remaining cases. In particular, in the elliptic case the Toeplitz operators have a discrete spectrum and can be compact even having symbols unbounded near the boundary, while in both the parabolic and hyperbolic cases the Toeplitz operators always have only a continuous spectrum and, being nonzero, can not be compact.

As in the preceding paper [6], the word "dynamics" in the title stands for the emphasis on our main theme: what happens to properties of Toeplitz operators acting on weighted Bergman spaces when the weight parameter varies.

In the paper, as is a custom in operator theory, we consider weighted Bergman spaces depending on a real parameter $\lambda \in (-1, \infty)$.

Denote by Π the upper half-plane in $\mathbb{C}$, and introduce the weighted Hilbert space $L_2(\Pi, d\mu_\lambda)$ which consists of measurable functions f on Π for which the norm

$$\|f\|_{L_2(\Pi,d\mu_\lambda)} = \left(\int_\Pi |f(z)|^2 d\mu_\lambda(z) \right)^{1/2}$$

is finite. Here $d\mu_\lambda(z) = \mu_\lambda(z) dv(z)$ with

$$\mu_\lambda(z) = (\lambda+1)(2\mathrm{Im}\, z)^\lambda, \quad dv(z) = \frac{1}{\pi}\, dxdy, \; z = x+iy.$$

Let further $\mathcal{A}^2_\lambda(\Pi)$ denote the weighted Bergman space defined to consist of functions belonging to $L_2(\Pi, d\mu_\lambda)$ and analytic in the upper half-plane Π.

It is well known (see, for example, [10]) that the orthogonal Bergman projection $B_{\Pi,\lambda}$ of $L_2(\Pi, d\mu_\lambda)$ onto the weighted Bergman space $\mathcal{A}^2_\lambda(\Pi)$ has the form

$$\begin{aligned}(B_{\Pi,\lambda} f)(z) &= (\lambda+1) \int_\Pi f(\zeta) \left(\frac{\zeta - \overline{\zeta}}{z - \overline{\zeta}} \right)^{\lambda+2} \frac{dv(\zeta)}{(2\,\mathrm{Im}\,\zeta)^2} \\ &= i^{\lambda+2} \int_\Pi \frac{f(\zeta)}{(z-\overline{\zeta})^{\lambda+2}}\, d\mu_\lambda(\zeta).\end{aligned}$$

Given a function (symbol) $a = a(z)$, $z \in \Pi$, the Toeplitz operators $T_a^{(\lambda)}$ acting on $\mathcal{A}^2_\lambda(\Pi)$ is defined as follows

$$T_a^{(\lambda)} f = B_{\Pi,\lambda} a f, \qquad f \in \mathcal{A}^2_\lambda(\Pi).$$

The key result, which gives an easy access to the properties of Toeplitz operators studied in the paper, is established in Section 2. Namely, we prove that the Toeplitz operator $T_a^{(\lambda)}$ with symbol $a(\theta)$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I$ acting on $L_2(\mathbb{R})$, where

$$\gamma_{a,\lambda}(\xi) = 2^\lambda \; (\lambda+1) \vartheta^2_\lambda(\xi) \int_0^\pi a(\theta)\, e^{-2\xi\theta} \, \sin^\lambda \theta \; d\theta \qquad \xi \in \mathbb{R},$$

where the function $\vartheta_\lambda(\xi)$ is given by (2.2).

We mention in this context (see, for example, [1, 3]) the Wick (or covariant, or Berezin) symbol $\widetilde{a}_\lambda(z, \overline{z})$, $z \in \Pi$, of the Toeplitz operator $T_a^{(\lambda)}$, which together with the so-called star product carries many essential properties of the corresponding Toeplitz operator. Recall that given a bounded operator A acting on a Hilbert space H which has a system of coherent states $\{k_g\}_{g\in G}$, its Wick symbol is defined as

$$\widetilde{a}_A(g, g) = \frac{\langle A k_g, k_g \rangle}{\langle k_g, k_g \rangle}, \qquad g \in G.$$

In our particular case we have $A = T_a^{(\lambda)}$, $H = \mathcal{A}^2_\lambda(\Pi)$, and $k_g = k_z(\zeta) = i^{\lambda+2}(\zeta - \overline{z})^{-(\lambda+2)}$, where $z, \zeta \in \Pi$. The star product defines the composition of two Wick symbols $\widetilde{a}_A$ and $\widetilde{a}_B$ of the operators A and B, respectively, as the Wick symbol of the composition AB, i.e., $\widetilde{a}_A \star \widetilde{a}_B = \widetilde{a}_{AB}$.

In Section 3 we give the formulas for the Wick symbols of Toeplitz operators $T_a^{(\lambda)}$, whose symbols depend only on θ, and the formulas for the star product in terms of our function $\gamma_{a,\lambda}$.

An interesting and important feature of Toeplitz operators on the (weighted) Bergman spaces is that such operators can be bounded even when they have symbols unbounded near the boundary. In Section 4 we study in details boundedness properties of Toeplitz operators with such unbounded symbols. We give several separate sufficient and necessary boundedness conditions, as well as a number of illustrating examples. It turns out that for unbounded symbols, the behaviour of certain means of a symbol, rather than the behaviour of the symbol itself, plays a crucial role in the boundedness properties. Given a symbol a, it is natural to introduce the set $B(a)$ of values $\lambda \in [0, \infty)$ for which the corresponding Toeplitz operator $T_a^{(\lambda)}$ is bounded on $\mathcal{A}^2_\lambda(\Pi)$. We show that being nonempty the set $B(a)$ may have only one of the following three forms: $[0, \infty)$, $[0, \nu)$, or $[0, \nu]$.

Section 5 is devoted to the spectral properties. The (continuous) spectrum of each $T_a^{(\lambda)}$ coincides with the closure of the image of the corresponding continuous function $\gamma_{a,\lambda}$. For each fixed λ the spectrum seems to be quite unrestricted, as the definite tendency starts appearing only as λ tends to infinity. The correspondence principle suggests that the limit set of the spectra has to be somehow connected with the range of the initial symbol a. This is definitely true for continuous symbols. Given a continuous symbol a, the limit set of the spectra, which we will denote by $M_\infty(a)$, does coincide with the range of a. As in [6], the new effects appear when we consider more complicated symbols. To understand the impact of each type of a discontinuity of a symbol we consider two model cases, piecewise continuous and oscillating symbols. In particular, in the case of piecewise continuous symbols the limit set $M_\infty(a)$ coincides with the range of a together with the line segments connecting the one-sided limit points of our piecewise continuous symbol.

Proofs of various theorems and construction of examples in the section are analogous to those of [5] and we omit them. On the other hand side to diminish somehow an imbalance with [5] we give a few illustrating graphical examples.

2. Representations of the weighted Bergman space

We start with the description of the weighted Bergman space $\mathcal{A}^2_\lambda(\Pi))$, where $\lambda \in (-1, +\infty)$, which is compatible with the polar coordinates in Π. Passing to polar coordinates we have

$$L_2(\Pi, d\mu_\lambda) = L_2(\mathbb{R}_+, r^{\lambda+1}dr) \otimes L_2([0,\pi], 1/\pi 2^\lambda(\lambda+1)\sin^\lambda \theta d\theta),$$

Rewriting the equation $\frac{\partial}{\partial \overline{z}}\varphi = 0$ in polar coordinates, we have that the Bergman space $\mathcal{A}^2_\lambda(\Pi)$ as the set of all functions satisfying the equation

$$\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta}\right)\varphi(r,\theta) = 0.$$

Introduce the unitary operator

$$\begin{aligned} U_1 &= 1/\sqrt{\pi}(M \otimes I) : L_2(\Pi, d\mu_\lambda) \\ &= L_2(\mathbb{R}_+, r^{\lambda+1}dr) \otimes L_2([0,\pi], 1/\pi 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta) \\ &\qquad \longrightarrow L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta), \end{aligned}$$

where the Mellin transform $M : L_2(\mathbb{R}_+, r^{\lambda+1}dr) \longrightarrow L_2(\mathbb{R})$ is given by

$$(M\psi)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}_+} r^{-i\xi+\lambda/2}\ \psi(r)\, dr.$$

The inverse Mellin transform $M^{-1} : L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}_+, r^{\lambda+1}dr)$ has the form

$$(M^{-1}\psi)(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} r^{i\xi-\lambda/2-1}\, \psi(\xi)\ d\xi.$$

It is easy to see that

$$U_1\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial\theta}\right)U_1^{-1} = i(\xi + (\lambda/2+1)i)I + i\frac{\partial}{\partial\theta}.$$

Thus, the image of the Bergman space $\mathcal{A}^2_{1,\lambda} = U_1(\mathcal{A}^2_\lambda(\Pi))$ can be described as the (closed) subspace of $L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta)$ which consists of all functions $\varphi(\xi,\theta)$ satisfying the equation

$$\left((\xi + (\lambda/2+1)i)I + \frac{\partial}{\partial\theta}\right)\varphi(\xi,\theta) = 0.$$

The general $L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta)$ solution of this equation has the form

$$\varphi(\xi,\theta) = f(\xi)\,\vartheta_\lambda(\xi)\, e^{-(\xi+(1+\lambda/2)i)\theta}, \qquad f(\xi) \in L_2(\mathbb{R}), \tag{2.1}$$

where (see, for example, [4] formula 3.892)

$$\begin{aligned} \vartheta_\lambda(\xi) &= \left(2^\lambda(\lambda+1)\int_0^\pi e^{-2\xi\theta}\sin^\lambda\theta d\theta\right)^{-1/2} \\ &= \frac{\mathrm{B}(\frac{\lambda+2}{2}+i\xi, \frac{\lambda+2}{2}-i\xi)^{1/2}}{\sqrt{\pi}}\, e^{\pi\xi/2} = \frac{|\Gamma(\frac{\lambda+2}{2}+i\xi)|}{\sqrt{\pi}\,\Gamma(\lambda+2)^{1/2}}\, e^{\pi\xi/2}, \end{aligned} \tag{2.2}$$

and

$$\|\varphi(\xi,\theta)\|_{L_2(\mathbb{R})\otimes L_2([0,\pi],2^\lambda(\lambda+1)\sin^\lambda\theta d\theta)} = \|f(\xi)\|_{L_2(\mathbb{R})}.$$

Lemma (2.3). *The unitary operator $U_1 = 1/\sqrt{\pi}(M \otimes I)$ is an isometric isomorphism of the space $L_2(\Pi, d\mu_\lambda)$, where $\lambda \in (-1, +\infty)$, onto $L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta)$ under which the Bergman space $\mathcal{A}^2_\lambda(\Pi)$ is mapped onto*

$$\mathcal{A}^2_{1,\lambda} = \left\{\varphi(\xi,\theta) = f(\xi)\,\vartheta_\lambda(\xi)\,e^{-(\xi+(1+\lambda/2)i)\theta} : \quad f(\xi) \in L_2(\mathbb{R})\right\}.$$

As above, let $R_0 : L_2(\mathbb{R}) \longrightarrow \mathcal{A}^2_{1,\lambda}(\Pi) \subset L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta)$ be the isometric imbedding given by

$$(R_0 f)(\xi,\theta) = f(\xi)\,\vartheta_\lambda(\xi)\,e^{-(\xi+(1+\lambda/2)i)\theta}.$$

The adjoint operator $R_0^* : L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta) \longrightarrow L_2(\mathbb{R})$ has the form

$$(R_0^*\psi)(\xi) = 2^\lambda(\lambda+1)\vartheta_\lambda(\xi)\int_0^\pi \psi(\xi,\theta)\, e^{-(\xi-(1+\lambda/2)i)\theta}\, \sin^\lambda\theta\, d\theta,$$

and

$$\begin{aligned} R_0^*R_0 = I &\quad : \quad L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}), \\ R_0R_0^* = B_1 &\quad : \quad L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta) \longrightarrow \mathcal{A}^2_{1,\lambda}, \end{aligned}$$

where $B_1 = U_1 B_\Pi^\lambda U_1^{-1}$ is the orthogonal projection of $L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta)$ onto $\mathcal{A}^2_{1,\lambda}$.

Now the operator $R_\lambda = R_0^* U_1$ maps the space $L_2(\Pi, d\mu_\lambda)$ onto $L_2(\mathbb{R})$, and its restriction

$$R_\lambda|_{\mathcal{A}^2_\lambda(\Pi)} : \mathcal{A}^2_\lambda(\Pi) \longrightarrow L_2(\mathbb{R})$$

is an isometric isomorphism. The adjoint operator

$$R_\lambda^* = U_1^* R_0 : L_2(\mathbb{R}) \longrightarrow \mathcal{A}^2_\lambda(\Pi) \subset L_2(\Pi, d\mu_\lambda)$$

is an isometric isomorphism of $L_2(\mathbb{R})$ onto $\mathcal{A}^2_\lambda(\Pi)$.

Remark (2.4). We have

$$\begin{aligned} R_\lambda R_\lambda^* = I &\quad : \quad L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}), \\ R_\lambda^* R_\lambda = B_\Pi^\lambda &\quad : \quad L_{2,\lambda}(\Pi) \longrightarrow \mathcal{A}^2_\lambda(\Pi). \end{aligned}$$

THEOREM (2.5). *The isometric isomorphism*

$$R_\lambda^* = U_1^* R_0 : L_2(\mathbb{R}) \longrightarrow \mathcal{A}^2_\lambda(\Pi)$$

is given by

$$(R_\lambda^* f)(z) = \frac{1}{\sqrt{2}}\int_{\mathbb{R}} z^{i\xi-(1+\lambda/2)}\, \vartheta_\lambda(\xi)\, f(\xi)\, d\xi. \tag{2.6}$$

Proof. Calculate

$$\begin{aligned} (R_\lambda^* f)(z) &= (U_1^* R_0 f)(z) \\ &= \sqrt{\pi}(M^{-1}\otimes I) f(\xi)\, \vartheta_\lambda(\xi)\, e^{-(\xi+(1+\lambda/2)i)\theta} \\ &= \frac{1}{\sqrt{2}}\int_{\mathbb{R}} r^{i\xi-(1+\lambda/2)} f(\xi)\, \vartheta_\lambda(\xi)\, e^{-(\xi+(1+\lambda/2)i)\theta}\, d\xi \\ &= \frac{1}{\sqrt{2}}\int_{\mathbb{R}} z^{i\xi-(1+\lambda/2)}\, \vartheta_\lambda(\xi)\, f(\xi)\, d\xi. \end{aligned}$$

□

COROLLARY (2.7). *The inverse isomorphism*

$$R_\lambda : \mathcal{A}^2_\lambda(\Pi) \longrightarrow L_2(\mathbb{R})$$

is given by

$$(R_\lambda\varphi)(\xi) = \frac{\vartheta_\lambda(\xi)}{\sqrt{2}}\int_\Pi (\overline{z})^{-i\xi-(1+\lambda/2)}\, \varphi(z)\, \mu_\lambda(z)\, dv(z). \tag{2.8}$$

The above representation of the Bergman space $\mathcal{A}^2_\lambda(\Pi)$ is especially important in the study of the Toeplitz operators with symbols depending only on $\theta = \arg z$.

THEOREM (2.9). *Given $a = a(\theta) \in L_1(0,\pi)$, the Toeplitz operator $T_a^{(\lambda)}$ acting on $\mathcal{A}_\lambda^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I = R_\lambda T_a^{(\lambda)} R_\lambda^*$, acting on $L_2(\mathbb{R})$. The function $\gamma_{a,\lambda}(\xi)$ is given by*

$$\begin{aligned}\gamma_{a,\lambda}(\xi) &= 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi)\int_0^\pi a(\theta)\, e^{-2\xi\theta}\,\sin^\lambda\theta\,d\theta \\ &= \left(\int_0^\pi e^{-2\xi\theta}\sin^\lambda\theta\,d\theta\right)^{-1}\int_0^\pi a(\theta)\,e^{-2\xi\theta}\sin^\lambda\theta\,d\theta,\ \ \xi\in\mathbb{R}.\end{aligned} \tag{2.10}$$

Proof. Calculate

$$\begin{aligned} R_\lambda T_a^{(\lambda)} R_\lambda^* &= R_\lambda B_{\Pi,\lambda} a B_{\Pi,\lambda} R_\lambda^* = R_\lambda (R_\lambda^* R_\lambda) a (R_\lambda^* R_\lambda) R_\lambda^* \\ &= (R_\lambda R_\lambda^*) R_\lambda a R_\lambda^* (R_\lambda R_\lambda^*) = R_\lambda a R_\lambda^* \\ &= R_0^* U_1 a(\theta) U_1^{-1} R_0 \\ &= R_0^* a(\theta) R_0. \end{aligned}$$

Thus

$$\begin{aligned}(R_0^* a(\theta) R_0 f)(\xi) &= 2^\lambda(\lambda+1)\vartheta_\lambda(\xi)\int_0^\pi a(\theta)\,e^{-(\xi-(1+\lambda/2)i)\theta}\,f(\xi) \\ &\times\ \vartheta_\lambda(\xi)\ e^{-(\xi+(1+\lambda/2)i)\theta}\,\sin^\lambda\theta\,d\theta \\ &= \gamma_{a,\lambda}(\xi) f(\xi),\end{aligned}$$

where

$$\gamma_{a,\lambda}(\xi) = 2^\lambda\ (\lambda+1)\vartheta_\lambda^2(\xi)\int_0^\pi a(\theta)\,e^{-2\xi\theta}\,\sin^\lambda\theta\ d\theta \qquad \xi\in\mathbb{R}.$$

Here the function $\vartheta_\lambda(\xi)$ is given by (2.2). □

The above theorem suggests considering not only L_∞-symbols, but unbounded ones as well. Note that given a bounded symbol $a(z)$, the Toeplitz operator $T_a^{(\lambda)}$ is bounded on all spaces $\mathcal{A}_\lambda^2(\Pi)$, for $\lambda \in (-1,\infty)$, and the corresponding norms are uniformly bounded by $\sup_z |a(z)|$. That is, all spaces $\mathcal{A}_\lambda^2(\Pi)$, where $\lambda \in (-1,\infty)$, are natural and appropriate for Toeplitz operators with bounded symbols. As one of our aims is a systematic study of unbounded symbols, we wish to have a sufficiently large class of them common to all admissible λ; moreover, we are especially interested in properties of Toeplitz operators for large values of λ. Thus it is convenient for us to consider λ belonging only to $[0,\infty)$, which we will always assume in what follows.

We have obviously:

COROLLARY (2.11). *The Toeplitz operator $T_a^{(\lambda)}$ with symbol $a(\theta)$ is bounded on $\mathcal{A}_\lambda^2(\Pi)$ if and only if the corresponding function $\gamma_{a,\lambda}(\xi)$ is bounded.*

3. Toeplitz operators with symbols depending on $\theta = \arg z$

Reverting the statement of Theorem (2.9) we come to the following spectral-type representation of a Toeplitz operator.

THEOREM (3.1). *Let $a = a(\theta) \in L_1(0,\pi)$. Then the Toeplitz operator $T_a^{(\lambda)}$ acting on $\mathcal{A}^2_\lambda(\Pi)$ admits the representation*

$$(T_a^{(\lambda)}\varphi)(z) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} z^{i\xi-(1+\lambda/2)}\, \vartheta_\lambda(\xi)\, \gamma_{a,\lambda}(\xi)\, f(\xi)\, d\xi, \tag{3.2}$$

where $f(\xi) = (R_\lambda \varphi)(\xi) \in L_2(\mathbb{R})$.

Proof. Follows directly from Theorems (2.9), and (2.5), and Corollary (2.7). □

THEOREM (3.3). *Given $a = a(\theta) \in L_1(0,\pi)$, the Wick symbol $\widetilde{a}_\lambda(z,\overline{z})$ of the Toeplitz operator $T_a^{(\lambda)}$ depends only on $\theta\,(=\arg z\,)$ and has the form*

$$\widetilde{a}_\lambda(\theta) = \widetilde{a}_\lambda(z,\overline{z}) = 2^{\lambda+1} \sin^{\lambda+2}\theta \int_{\mathbb{R}} e^{-2\xi\theta}\, \vartheta^2_\lambda(\xi)\, \gamma_{a,\lambda}(\xi)\, d\xi \tag{3.4}$$

and the corresponding Wick function is given by formula

$$\begin{aligned} \widetilde{a}_\lambda(z,\overline{w}) &= \frac{\langle T_a^{(\lambda)} k_w, k_z\rangle}{\langle k_w, k_z\rangle} \\ &= (z-\overline{w})^{\lambda+2}\, (z\overline{w})^{-(\lambda+2)/2}\, \frac{i^{-(\lambda+2)}}{2} \int_{\mathbb{R}} (\frac{z}{\overline{w}})^{i\xi}\, \vartheta^2_\lambda(\xi)\, \gamma_{a,\lambda}(\xi)\, d\xi. \end{aligned} \tag{3.5}$$

Proof. Consider $k_z(w) = i^{2+\lambda}(w-\overline{z})^{-(\lambda+2)} = i^{2+\lambda}(\rho e^{i\alpha} - re^{-i\theta})^{-(\lambda+2)}$ and calculate

$$(U_1 k_z)(\xi,\alpha) = \frac{i^{2+\lambda}}{\pi\sqrt{2}} \int_{\mathbb{R}_+} \rho^{-i\xi+\lambda/2} (\rho e^{i\alpha} - \overline{z})^{-(\lambda+2)} d\rho.$$

Using formula 3.194.3 from [4] and (2.2), we have

$$\begin{aligned} (U_1 k_z)(\xi,\alpha) &= \frac{\mathrm{B}(\frac{\lambda+2}{2} - i\xi, \frac{\lambda+2}{2} + i\xi)}{\sqrt{2}\pi}\, e^{\pi\xi}\, e^{-\xi\alpha - i\frac{\lambda+2}{2}\alpha}\, (\overline{z})^{-i\xi - \frac{\lambda+2}{2}} \\ &= \frac{\vartheta^2_\lambda(\xi)}{\sqrt{2}}\, e^{-\xi\alpha - i\frac{\lambda+2}{2}\alpha}\, (\overline{z})^{-i\xi - \frac{\lambda+2}{2}}. \end{aligned}$$

Thus

$$\begin{aligned} \langle T_a^{(\lambda)} k_z, k_z\rangle &= \langle a k_z, k_z\rangle = \langle U_1 a k_z, U_1 k_z\rangle = \langle a U_1 k_z, U_1 k_z\rangle \\ &= \frac{1}{2} \int_{\mathbb{R}} \int_0^\pi a(\alpha)\, \vartheta^4_\lambda(\xi) e^{-2\xi\alpha} (\overline{z})^{-i\xi - \frac{\lambda+2}{2}}\, z^{i\xi - \frac{\lambda+2}{2}}\, 2^\lambda(\lambda+1) \sin^\lambda \alpha\, d\xi d\alpha \\ &= \frac{r^{-(\lambda+2)}}{2} \int_{\mathbb{R}} \vartheta^2_\lambda(\xi)\, e^{-2\xi\theta} d\xi\ 2^\lambda(\lambda+1) \vartheta^2_\lambda(\xi) \int_0^\pi a(\alpha)\, e^{-2\xi\alpha}\, \sin^\lambda \alpha\, d\alpha \\ &= \frac{r^{-(\lambda+2)}}{2} \int_{\mathbb{R}} \vartheta^2_\lambda(\xi)\, e^{-2\xi\theta}\, \gamma_{a,\lambda}(\xi) d\xi. \end{aligned}$$

Similarly

$$\langle T_a^{(\lambda)} k_w, k_z\rangle = \frac{(z\overline{w})^{-(\lambda+2)/2}}{2} \int_{\mathbb{R}} (\frac{z}{\overline{w}})^{i\xi}\, \vartheta^2_\lambda(\xi)\, \gamma_{a,\lambda}(\xi)\, d\xi.$$

Furthermore $\langle k_w, k_z\rangle = k_w(z) = i^{\lambda+2}(z-\overline{w})^{-(\lambda+2)}$, and $\langle k_z, k_z\rangle = k_z(z) = (2\mathrm{Im}\, z)^{-(\lambda+2)}$. Thus we have both (3.4) and (3.5). ☐

Remark (3.6). Given a symbol $a = a(\theta) \in L_1(0,\pi)$, writing the Toeplitz operator $T_a^{(\lambda)}$ in terms of its Wick symbol we obtain formula (3.2). Indeed

$$\begin{aligned}
(T_a^{(\lambda)}\varphi)(z) &= \int_\Pi \widetilde{a}(z,\overline{w})\, \frac{\varphi(w)\, i^{\lambda+2}}{(z-\overline{w})^{\lambda+2}}\, \mu_\lambda(w)\, dv(w) \\
&= \frac{1}{2}\int_\Pi (z\overline{w})^{-(\lambda+2)/2}\varphi(w)\, \mu_\lambda(w)\, dv(w) \int_{\mathbb{R}} (\frac{z}{\overline{w}})^{i\xi}\, \vartheta_\lambda^2(\xi)\, \gamma_{a,\lambda}(\xi)\, d\xi \\
&= \frac{1}{\sqrt{2}}\int_{\mathbb{R}} z^{i\xi-\frac{\lambda+2}{2}}\, \vartheta_\lambda(\xi)\, \gamma_{a,\lambda}(\xi)\, d\xi \\
&\times \frac{\vartheta_\lambda(\xi)}{\sqrt{2}} \int_\Pi (\overline{w})^{-i\xi-\frac{\lambda+2}{2}}\, \varphi(w)\, \mu_\lambda(w)\, dv(w) \\
&= \frac{1}{\sqrt{2}}\int_{\mathbb{R}} z^{i\xi-\frac{\lambda+2}{2}}\, \vartheta_\lambda(\xi)\, \gamma_{a,\lambda}(\xi)\, (R_\lambda\varphi)(\xi)\, d\xi.
\end{aligned}$$

Corollary (3.7). *Let $T_a^{(\lambda)}$ and $T_b^{(\lambda)}$ be two Toeplitz operators with symbols $a(\theta), b(\theta) \in L_1(0,\pi)$ respectively, and let $\widetilde{a}_\lambda(\theta)$ and $\widetilde{b}_\lambda(\theta)$ be their Wick symbols. Then the Wick symbol $\widetilde{c}(\theta)$ of the composition $T_a^{(\lambda)}T_b^{(\lambda)}$ is given by*

$$\widetilde{c}_\lambda(\theta) = (\widetilde{a}_\lambda \star \widetilde{b}_\lambda)(\theta) = 2^{\lambda+1}\sin^{\lambda+2}\theta \int_{\mathbb{R}} e^{-2\xi\theta}\, \vartheta_\lambda^2(\xi)\, \gamma_{a,\lambda}(\xi)\, \gamma_{b,\lambda}(\xi)\, d\xi.$$

Proof. This can be verified directly from the formula for the star product, and also follows immediately from Theorems (2.9) and (3.3). ☐

4. Boundedness of Toeplitz operators with symbols depending on $\theta = \arg z$.

Recall (Corollary (2.11)) that the function

$$\gamma_{a,\lambda}(\xi) = \left(\int_0^\pi e^{-2\xi\theta}\ \sin^\lambda\theta\, d\theta\right)^{-1} \int_0^\pi a(\theta)\ e^{-2\xi\theta}\ \sin^\lambda\theta\, d\theta, \qquad \xi\in\mathbb{R} \tag{4.1}$$

is responsible for the boundedness of a Toeplitz operator with symbol $a(\theta)$ ($\in L_1(0,\pi)$). If the symbol $a(\theta) \in L_\infty(0,\pi)$, then the operator $T_a^{(\lambda)}$ is obviously bounded on $\mathcal{A}_\lambda^2(\Pi)$ for each λ, and $\|T_a^{(\lambda)}\| \le \text{ess-sup}|a(\theta)|$.

For $a(\theta) \in L_1(0,\pi)$ the function $\gamma_{a,\lambda}(\xi)$ is continuous at all finite points $\xi\in\mathbb{R}$. For a "very large ξ" ($\xi\to+\infty$) the exponent $e^{-2\xi\theta}$ has a very sharp maximum at the point $\theta = 0$, and thus the major contribution to the integral containing $a(\theta)$ in (4.1) for these "very large ξ" is determined by the values of $a(\theta)$ in a neighborhood of the point 0. The major contribution for a "very large negative ξ" ($\xi\to-\infty$) is determined by the values of $a(\theta)$ at a neighborhood of π, due to a very sharp maximum of $e^{-2\xi\theta}$ at $\theta=\pi$ for these values of ξ. In particular, if $a(\theta)$ has limits at the points 0 and π, then

$$\begin{aligned}
\lim_{\xi\to+\infty}\gamma_{a,\lambda}(\xi) &= \lim_{\theta\to 0} a(\theta), \\
\lim_{\xi\to-\infty}\gamma_{a,\lambda}(\xi) &= \lim_{\theta\to \pi} a(\theta).
\end{aligned}$$

As a matter of fact, 0 and π are the only worrying points for unbounded symbols $a(\theta) \in L_1(0,\pi)$. Moreover, the behaviour of certain means of a symbol, rather than the behaviour of the symbol itself, plays a crucial role under the study of boundedness properties.

Given $\lambda \in [0,\infty)$ and a function $a(\theta) \in L_1(0,\pi)$ introduce the following means:

$$\begin{aligned}
C^{(1)}_{a,\lambda}(\sigma) &= \int_0^\sigma a(\theta)\sin^\lambda\theta d\theta,\\
D^{(1)}_{a,\lambda}(\sigma) &= \int_\sigma^\pi a(\theta)\sin^\lambda\theta d\theta,\\
C^{(j)}_{a,\lambda}(\sigma) &= \int_0^\sigma C^{(j-1)}_{a,\lambda}(\theta)d\theta, \quad j=2,3,\dots,\\
D^{(j)}_{a,\lambda}(\sigma) &= \int_\sigma^\pi D^{(j-1)}_{a,\lambda}(\theta)d\theta, \quad j=2,3,\dots.
\end{aligned}$$

THEOREM (4.2). *Let $a(\theta) \in L_1(0,\pi)$. If for certain $\lambda_0 \in [0,\infty)$ and $j_0, j_1 \in \mathbb{N}$ the following conditions hold*

$$C^{(j_0)}_{a,\lambda_0}(\sigma) = O(\sigma^{j_0+\lambda_0}), \qquad \sigma \to 0, \tag{4.3}$$

$$D^{(j_1)}_{a,\lambda_0}(\sigma) = O((\pi-\sigma)^{j_1+\lambda_0}), \qquad \sigma \to \pi, \tag{4.4}$$

then the corresponding Toeplitz operator $T^{(\lambda)}_a$ is bounded on $\mathcal{A}^2_\lambda(\Pi)$ for each $\lambda \in [\lambda_0,\infty)$.

Proof. Note that the function $\gamma_{a,\lambda}(\xi)$ is continuous at finite points. Let $\xi \to +\infty$ and the condition (4.3) holds with $j_0 = 1$. Then

$$\begin{aligned}
\gamma_{a,\lambda}(\xi) &= 2^\lambda(\lambda+1)\vartheta^2_\lambda(\xi)\int_0^\pi \sin^{\lambda-\lambda_0}(\theta)e^{-2\xi\theta}dC^{(1)}_{a,\lambda_0}(\theta)\\
&= 2^\lambda(\lambda+1)\vartheta^2_\lambda(\xi)\left|\int_0^\pi C^{(1)}_{a,\lambda_0}(\theta)[(\lambda-\lambda_0)\sin^{\lambda-\lambda_0-1}\theta\cos\theta\right.\\
&\quad \left. -\, 2\xi\sin^{\lambda-\lambda_0}\theta]e^{-2\xi\theta}d\theta\right|\\
&\le \mathrm{const}\, 2^\lambda(\lambda+1)\vartheta^2_\lambda(\xi)\left[(\lambda-\lambda_0)\int_0^\infty \theta^\lambda e^{-2\xi\theta}d\theta + 2\xi\int_0^\infty \theta^{\lambda+1}e^{-2\xi\theta}d\theta\right]\\
&\le \mathrm{const}\,\vartheta^2_\lambda(\xi)\left[(\lambda-\lambda_0)(2\xi)^{-(\lambda+1)}\Gamma(\lambda+1) + (2\xi)^{-(\lambda+1)}\Gamma(\lambda+2)\right]\\
&\le \mathrm{const}\,(2\lambda-\lambda_0+1)2^\lambda(\lambda+1)\vartheta^2_\lambda(\xi)(2\xi)^{-(\lambda+1)}\Gamma(\lambda+1).
\end{aligned}$$

It is easy to get the asymptotic representation of the function $\vartheta^2_\lambda(\xi)$. According to (2.2) we have

$$\begin{aligned}
2^{-\lambda}(\lambda+1)^{-1}\vartheta^{-2}_\lambda(\xi) &= \int_0^\pi e^{-2\xi\theta}\sin^\lambda\theta d\theta\\
&= \int_0^\pi \theta^\lambda e^{-2\xi\theta}d\theta[1+\theta(\xi^{-1})]\\
&= (2\xi)^{-(\lambda+1)}\Gamma(\lambda+1)[1+O(\xi^{-1})]\,.
\end{aligned} \tag{4.5}$$

Thus we finally have

$$|\gamma_{a,\lambda}(\xi)| \le \mathrm{const}\,(2\lambda-\lambda_0+1).$$

The case $\xi \to -\infty$ (and $j_1 = 1$) is reduced to the previous one using the change of variable $\theta = \pi - \theta'$ in the integral for $\gamma_{a,\lambda}(\xi)$.

The cases $j_{0,1} > 1$ are considered analogously using integration by parts. $\square$

The proof of the following statement is analogous to that of Theorem 4.3 in [5].

THEOREM (4.6). 1. *Let conditions (4.3), (4.4) hold for $j_0 = j_0'$, $j_1 = j_1'$, and some λ_0. Then these conditions hold for $j_0 = j_0' + 1$, $j_1 = j_1' + 1$, and the same λ_0.*

2. *Let conditions (4.3), (4.4) hold for $j_0 = j_0'$, $j_1 = j_1'$, and some λ_0. Then these conditions hold for $j_0 = j_0'$, $j_1 = j_1'$, and λ_0 replaced by any $\lambda_1 \geq \lambda_0$.*

Example (4.7). Consider the following family of unbounded symbols

$$a(\theta) = (\sin\theta)^{-\beta} \sin[(\sin\theta)^{-\alpha}].$$

As in Example 4.4 in [5] it can be proved that for all $\lambda \geq 0$ the operator $T_a^{(\lambda)}$ is bounded for each $\beta \in (0,1)$ and $\alpha > 0$.

THEOREM (4.8). *Let the Toeplitz operator $T_a^{(\lambda)}$, with $a(\theta) \in L_1(0,\pi)$, be bounded on some $\mathcal{A}^2_{\lambda_0}(\Pi)$. Then it is bounded on each $\mathcal{A}^2_\lambda(\Pi)$, with $\lambda \in [0,\lambda_0]$.*

Proof. Let $\sup_{\xi\in\mathbb{R}} |\gamma_{a,\lambda_0}(\xi)| < \infty$. We split $a(\theta)$ in two functions which vanish on neighborhoods of 0 and π, respectively. The study of these two cases is quite similar, thus we suppose that $a(\theta)$ vanishes in a neighborhood of π, for example. Suppose also that $\xi \to \infty$. A similar argument is applicable for the study of the behavior of $\gamma_{a,\lambda}(\xi)$ when $\xi \to -\infty$. For $\lambda \in [0,\lambda_0)$, write

$$\gamma_{a,\lambda}(\xi) = \frac{2^{2\lambda-\lambda_0}\ (\lambda+1)\vartheta_\lambda^2(\xi)}{\Gamma(\lambda_0-\lambda)} \int_0^\infty y^{\lambda_0-\lambda-1} dy \int_0^\pi a(\theta) e^{-2\theta(\xi+\frac{\sin\theta}{\theta}y)}\ \sin^{\lambda_0}\theta d\theta.$$

Using $\frac{\sin\theta}{\theta} = 1 + O(\theta^2)$, as $\theta \to 0$, for some $c_\lambda \neq 0$, we have

$$\begin{aligned} \gamma_{a,\lambda}(\xi) &= (c_\lambda + o(1))\vartheta_\lambda^2(\xi) \int_0^\infty y^{\lambda_0-\lambda-1} dy \int_0^\pi a(\theta) e^{-2\theta(\xi+y)}\ \sin^{\lambda_0}\theta d\theta \\ &= \frac{(c_\lambda + o(1))\vartheta_\lambda^2(\xi)}{2^{\lambda_0}(\lambda_0+1)} \int_0^\infty y^{\lambda_0-\lambda-1} \frac{\gamma_{a,\lambda_0}(\xi+y)}{\vartheta_{\lambda_0}^2(\xi+y)} dy. \end{aligned}$$

Using (4.5) and $\sup_{\xi\in\mathbb{R}} |\gamma_{a,\lambda_0}(\xi)| < \infty$ we have

$$\begin{aligned} |\gamma_{a,\lambda}(\xi)| &\leq \text{const } \xi^{\lambda+1} \int_0^\infty y^{\lambda_0-\lambda-1} (\xi+y)^{-(\lambda_0+1)} dy \\ &= \text{const } \int_0^\infty u^{\lambda_0-\lambda-1}(1+u)^{-(\lambda_0+1)} du < \infty, \end{aligned}$$

since $\lambda < \lambda_0$ and $\lambda_0 + 1 > 1$. $\square$

As an immediate corollary of Theorems (4.2) and (4.8) we have now.

THEOREM (4.9). *Under the hypothesis of Theorem (4.2) the Toeplitz operator $T_a^{(\lambda)}$ is bounded on $\mathcal{A}^2_\lambda(\Pi)$ for each $\lambda \in [0,\infty)$.*

The proof of the next theorem is analogous to one of Theorem 4.8 in [5].

Theorem (4.10). 1. *Assume that* $a(\theta) \in L_1(0,\pi)$ *and* $a(\theta) \geq 0$ *almost everywhere. Let the operator* $T_a^{(\lambda')}$ *be bounded on* $\mathcal{A}^2_{\lambda'}(\Pi)$ *for some* $\lambda' > 0$*. Then the conditions (4.3) and (4.4) hold for* $j_0 = j_1 = 1$*,* $\lambda_0 = 0$ *and consequently the operator* $T_a^{(\lambda)}$ *is bounded on* $\mathcal{A}^2_\lambda(\Pi)$ *for arbitrary* $\lambda \in [0,\infty)$*.*

2. *Assume that the means satisfy* $C^{(j_0)}_{a,\mu_0}(\sigma) \geq 0$ *and* $D^{(j_1)}_{a,\mu_1}(\sigma) \geq 0$ *almost everywhere for some* $j_0 \geq 1$*,* $j_1 \geq 1$ *and* $\mu_0 \geq 0$*,* $\mu_1 \geq 0$*, and that the operator* $T_a^{(\lambda')}$ *is bounded on* $\mathcal{A}^2_{\lambda'}(\Pi)$ *for some* $\lambda' \geq 0$*. Then the operator* $T_a^{(\lambda)}$ *is bounded on* $\mathcal{A}^2_\lambda(\Pi)$ *for arbitrary* $\lambda \in [0,\infty)$*.*

For a nonnegative $a(\theta)$ we set

$$\begin{aligned} m_{a,0}(\sigma) &= \text{ess-inf}_{\theta\in(0,\sigma)}a(\theta), \\ m_{a,\pi}(\sigma) &= \text{ess-inf}_{\theta\in(\sigma,\pi)}a(\theta). \end{aligned}$$

Corollary (4.11). *Given a nonnegative symbol, if either* $\lim_{\sigma\to 0} m_{a,0}(\sigma) = \infty$ *or* $\lim_{\sigma\to\pi} m_{a,\pi}(\sigma) = \infty$*, then the Toeplitz operator* $T_a^{(\lambda)}$ *is unbounded on each* $\mathcal{A}^2_\lambda(\Pi)$*, with* $\lambda \in [0,\infty)$*.*

For a symbol $a(\theta) \in L_1(0,\pi)$ we denote by $\widetilde{B}(a)$ the set of points $\lambda \in [0,\infty)$ for which the corresponding Toeplitz operator $T_a^{(\lambda)}$ is bounded on $\mathcal{A}^2_\lambda(\Pi)$. Like in the parabolic case we have the following result, the proof of which is analogous to one result in [5].

Theorem (4.12). *There exists a family of symbols* $a_{\nu,\beta}(\theta)$*, where* $\nu \in (0,1)$*,* $\beta \in \mathbb{R}$*, such that*

a) $\widetilde{B}(a_{\nu,0}) = [0,\nu]$, $\quad \beta = 0$*;*
b) $\widetilde{B}(a_{\nu,\beta}) = [0,\nu)$, $\quad \beta > 0$.

5. Spectra of Toeplitz operators with symbols depending on $\theta = \arg z$

Continuous symbols. Let E be a subset of $\mathbb{R}$ having $+\infty$ as a limit point (typically $E = (0,+\infty)$), and suppose that, for each $\lambda \in E$, we are given a set $M_\lambda \subset \mathbb{C}$. Define the set M_∞ as the set of all $z \in \mathbb{C}$ for which there exists a sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ such that

(i) for each $n \in \mathbb{N}$ there exists $\lambda_n \in E$ such that $z_n \in M_{\lambda_n}$,
(ii) $\lim_{n\to\infty} \lambda_n = +\infty$,
(iii) $z = \lim_{n\to\infty} z_n$.

We will write

$$M_\infty = \lim_{\lambda\to+\infty} M_\lambda,$$

and call M_∞ the (partial) limit set of the family $\{M_\lambda\}_{\lambda\in E}$ when $\lambda \to +\infty$.

For the case when E is a discrete set with a unique limit point at infinity, the above notion coincides with the partial limit set introduced in [7], Section 3.1.1. Following the arguments of Proposition 3.5 in [7] one can show that

$$M_\infty = \bigcap_\lambda \text{clos}\left(\bigcup_{\mu\geq\lambda} M_\mu\right).$$

Note that obviously

$$\lim_{\lambda\to+\infty} M_\lambda = \lim_{\lambda\to+\infty} \overline{M}_\lambda = M_\infty .$$

The *a priori* spectral information for L_∞-symbols (see, for example, [1], [2]) says that for each $a \in L_\infty(\Pi)$ and each $\lambda \geq 0$

$$\operatorname{sp} T_a^{(\lambda)} \subset \operatorname{conv}(\text{ess-Range}\, a). \tag{5.1}$$

Given a symbol $a = a(\theta)$, the Toeplitz operator $T_a^{(\lambda)}$ acting on the space $\mathcal{A}^2_\lambda(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I$, where the function $\gamma_{a,\lambda}(\xi)$, $\xi \in \mathbb{R}$, is given by (2.10). Thus we have obviously

$$\operatorname{sp} T_a^{(\lambda)} = \overline{M_\lambda(a)},$$

where $M_\lambda(a) = \operatorname{Range} \gamma_{a,\lambda}$.

THEOREM (5.2). *Let $a = a(\theta) \in C[0,\pi]$. Then*

$$\lim_{\lambda\to\infty} \operatorname{sp} T_a^{(\lambda)} = \operatorname{Range} a.$$

Proof. We find the asymptotic of the function $\gamma_{a,\lambda}(\xi)$ when $\lambda \to \pm\infty$ using the Laplace method. Introduce the large parameter $L = \sqrt{\lambda^2 + (2\xi)^2}$ and represent $\gamma_{a,\lambda}(\xi)$ in the form

$$\gamma_{a,\lambda}(\xi) = 2^\lambda (\lambda+1) \vartheta_\lambda^2(\xi) \int_0^\pi a(\theta) e^{-LS(\theta,\varphi)} d\theta\,, \tag{5.3}$$

where

$$S(\theta,\varphi) = \sin\varphi \ln(\sin\theta)^{-1} + (\cos\varphi)\theta,$$
$$\sin\varphi = \lambda/L, \quad \cos\varphi = 2\xi/L \quad \text{with } \varphi \in [0,\pi).$$

To find the point of minimum of $S(\theta,\varphi)$ calculate

$$S'_\theta(\theta,\varphi) = -(\sin\varphi)\cot\theta + \cos\varphi.$$

It is obvious that $S'_\theta(\theta_\varphi,\varphi) = 0$, for $\theta_\varphi \in (0,\pi)$, if and only if $\theta_\varphi = \varphi$.

Rewrite (5.3) in the form

$$\begin{aligned} \gamma_{a,\lambda}(\xi) - a(\varphi) \;&=\; 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi) \Bigg[\int_{U(\varphi)\cap[0,\pi]} (a(\theta) - a(\varphi)) e^{-LS(\theta,\varphi)} d\theta \\ &+ \int_{[0,\pi]\setminus U(\varphi)} (a(\theta) - a(\varphi)) e^{-LS(\theta,\varphi)} d\theta \Bigg] \equiv I_1(L) + I_2(L) \end{aligned}$$

where $U(\varphi)$ is a neighborhood of φ such that $\sup_{\theta\in U(\varphi)} |a(\theta) - a(\varphi)| < \varepsilon$ for sufficiently small ε. We have used

$$2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi) \int_0^\pi a(\varphi) e^{-LS(\theta,\varphi)} d\theta = a(\varphi).$$

Further,

$$\left| \int_{U(\varphi)} (a(\theta) - a(\varphi)) e^{-LS(\theta,\varphi)} d\theta \right| \leq \varepsilon \int_{U(\varphi)} e^{-LS(\theta,\varphi)} d\theta \leq \varepsilon \left(2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi) \right)^{-1}$$

and finally,

$$\left| \int_{[0,\pi]\setminus U(\varphi)} (a(\theta) - a(\varphi)) e^{-LS(\theta,\varphi)} d\theta \right| \leq 2 \sup_{\theta\in[0,\pi]} |a(\theta)| \int_{[0,\pi]\setminus U(\varphi)} e^{-LS(\theta,\varphi)} d\theta$$

$$\leq \left(2M \sup_{\theta\in[0,\pi]} |a(\theta)| e^{-L\sigma(\varepsilon)}\right) \left(2^{\lambda}(\lambda+1)\vartheta_{\lambda}^{2}(\xi)\right)^{-1},$$

where $\sigma(\varepsilon) = \min_{\theta\in[0,\pi]\setminus U(\varphi)}(S(\theta,\varphi) - S(\varphi,\varphi))$. We note that $\sigma(\varepsilon)$ and M can be taken independent on $\varphi \in (0,\pi)$.

Since ε can be arbitrary small uniformly for $\varphi \in (0,\pi)$, we have

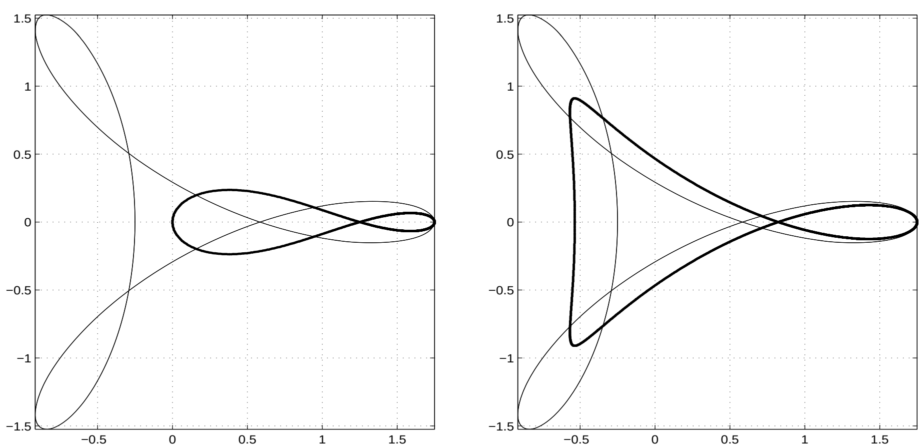
$$\gamma_{a,\lambda}(u) = a(\varphi)(1+\alpha(L)) \tag{5.4}$$

where $\lim\limits_{L\to\infty} \alpha(L) = 0$ uniformly for $\varphi \in (0,\pi)$, which proves the theorem. □

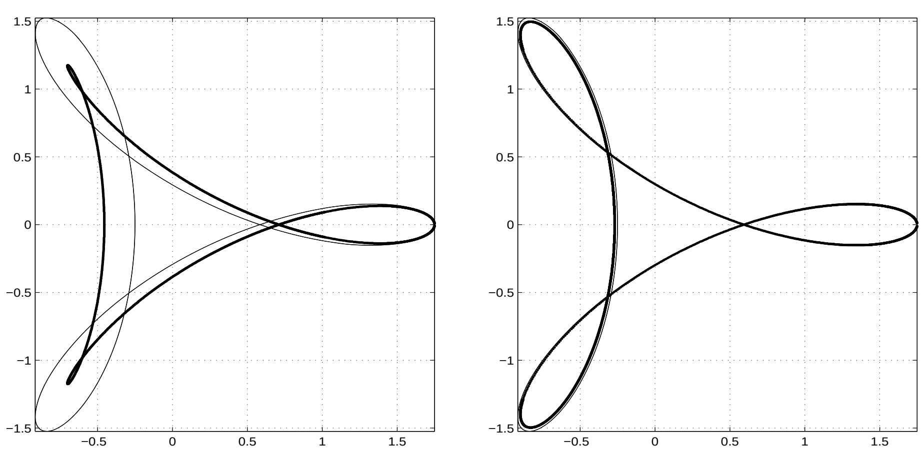
We illustrate the theorem on the continuous symbol (hypocycloid)

$$a(\theta) = \frac{3}{4}e^{4i\theta} + e^{-2i\theta},$$

and show the image of $\gamma_{a,\lambda}$ for the following values of λ: 0, 5, 12, and 200.



The function $\gamma_{a,\lambda}$ for $\lambda = 0$ and $\lambda = 5$



The function $\gamma_{a,\lambda}$ for $\lambda = 12$ and $\lambda = 200$

Piecewise continuous symbols. Let $a(\theta)$ be a piecewise continuous function having jumps on a finite set of points $\{\theta_j\}_{j=1}^m$ where

$$\theta_0 = 0 < \theta_1 < \theta_2 < \ldots < \theta_m < \pi = \theta_{m+1},$$

and $a(\theta_j \pm 0), j = 1, \ldots, m,$ exist. Introduce the sets

$$J_j(a) := \{z \in \mathbb{C} : z = a(\theta),\ \theta \in (\theta_j, \theta_{j+1})\}$$

where $j = 0, \ldots, m$, and let $I_j(a)$ be the segment with the endpoints $a(\theta_j - 0)$ and $a(\theta_j + 0)$, $j = 1, 2, \ldots m$. We set

$$\widetilde{R}(a) = \left(\bigcup_{j=0}^{m} J_j(a)\right) \cup \left(\bigcup_{j=1}^{m} I_j(a)\right).$$

THEOREM (5.5). *Let $a(\theta)$ be a piecewise continuous function. Then*

$$\lim_{\lambda\to\infty} \operatorname{sp} T_a^{(\lambda)} = M_\infty(a) = \widetilde{R}(a).$$

Proof. We use the Laplace method as in Theorem (5.2). For any $\varepsilon > 0$ we take $\delta > 0$ such that for each interval $I \subset (\theta_j, \theta_{j+1})$ with length less then δ, $j = 1, 2, \ldots, m$, the following inequality holds

$$\sup_{s_1, s_2 \in I} |a(s_1) - a(s_2)| < \varepsilon.$$

Suppose first that the minimum point $s_\varphi = \varphi$ satisfies the condition

$$\inf_{j=1,2,\ldots,m} |\varphi - \theta_j| > \delta.$$

We have

$$\begin{aligned} \gamma_{a,\lambda}(\xi) &= a(\varphi) + 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi) \int_{\varphi-\delta}^{\varphi+\delta} (a(\theta) - a(\varphi)) e^{-LS(\theta,\varphi)} d\theta \\ &\quad + 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi) \int_{[0,\pi]\setminus(\varphi-\delta,\varphi+\delta)} (a(\theta) - a(\varphi)) e^{-LS(\theta,\varphi)} d\theta \\ &\quad = a(\varphi) + O(\varepsilon) + O(e^{-\sigma L}) \end{aligned} \tag{5.6}$$

where

$$\sigma = \min_{[0,\pi]\setminus(\varphi-\delta,\varphi+\delta)} (S(\theta,\varphi) - S(\varphi,\varphi)).$$

Thus varying $\varphi \in \cup_{j=0}^m (\theta_j, \theta_{j+1})$ we have that

$$J_j(a) \subset M_\infty(a), \quad j = 0, 1, \ldots, m.$$

Now suppose that there exist j such that $|\varphi - \theta_j| < \delta$. Then we have

$$\begin{aligned}
\gamma_{a,\lambda}(\xi) &= 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi)\left(a(\theta_j - 0)\int_{\varphi-\delta}^{\theta_j} e^{-LS(\theta,\varphi)}d\theta\right. \\
&+ \left. a(\theta_j+0)\int_{\theta_j}^{\varphi+\delta} e^{-LS(\theta,\varphi)}d\theta\right) \\
&+ 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi)\left(\int_{\varphi-\delta}^{\theta_j}(a(\theta)-a(\theta_j-0))e^{-LS(\theta,\varphi)}d\theta\right. \\
&+ \int_{\theta_j}^{\varphi+\delta}(a(\theta)-a(\theta_j+0))e^{-LS(\theta,\varphi)}d\theta \\
&+ \left.\int_{(0,\pi)\setminus(\varphi-\delta,\varphi+\delta)} a(\theta)e^{-LS(\theta,\varphi)}d\theta\right).
\end{aligned}$$

Taking δ small enough we suppose that

$$\frac{\theta_1}{2} < s_\varphi(=\varphi) < \frac{\pi+\theta_m}{2}.$$

Thus the function

$$(S''_{\theta,\theta}(\varphi,\varphi))^{-1} = -\sin\varphi$$

is uniformly bounded on φ and the following asymptotic calculations are uniform on φ:

$$\begin{aligned}
[2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi)]^{-1} &= \int_0^\pi e^{-LS(\theta,\varphi)}d\theta \\
&= e^{-LS(\varphi,\varphi)}\int_0^\pi e^{-\frac{L}{2}(\sin^{-1}\varphi)(\theta-\varphi)^2}d\theta(1+O(1)) \\
&= e^{-LS(\varphi,\varphi)}\int_{-\varphi}^{\pi-\varphi} e^{-\frac{L}{2}(\sin^{-1}\varphi)u^2}du(1+O(1)) \\
&= \sqrt{2\sin\varphi}\,\frac{e^{-LS(\varphi,\varphi)}}{L^{1/2}}\int_{-\infty}^{\infty} e^{-v^2}dv(1+O(1)).
\end{aligned} \tag{5.7}$$

Analogously

$$\int_{\theta_j}^{\varphi+\delta} e^{-LS(\theta,\varphi)}d\theta = \sqrt{2\sin\varphi}\,\frac{e^{-LS(\varphi,\varphi)}}{L^{1/2}}\int_{x_j}^{\infty} e^{-v^2}dv(1+O(1)) \tag{5.8}$$

and

$$\int_{\varphi-\delta}^{\theta_j} e^{-LS(\theta,\varphi)}d\theta = \sqrt{2\sin\theta}\,\frac{e^{-LS(\varphi,\varphi)}}{L^{1/2}}\int_{-\infty}^{x_j} e^{-v^2}dv(1+O(1)), \tag{5.9}$$

where

$$x_j = \left(\frac{L}{2\sin\varphi}\right)^{1/2}(\theta_j - \varphi).$$

Thus from (5.7)–(5.9) we have

$$\gamma_{a,\lambda}(\xi) = (a(\theta_j-0)t + a(\theta_j+0)\tau)(1+O(1)+O(\varepsilon)+O(e^{-i\sigma})), \tag{5.10}$$

where

$$t = (\int_{-\infty}^{x_j} e^{-v^2} dv)/(\int_{-\infty}^{\infty} e^{-v^2} dv) \quad \text{and} \quad \tau = (\int_{x_j}^{\infty} e^{-v^2} dv)/(\int_{-\infty}^{\infty} e^{-v^2} dv).$$

Now it is evident that $t, \tau \in [0,1]$ and $\tau + t = 1$, which implies $I_j(a) \subset M_\infty(a)$. Thus

$$\widetilde{R}(a) \subset M_\infty(a).$$

Representations (5.6) and (5.10) imply the inverse inclusion

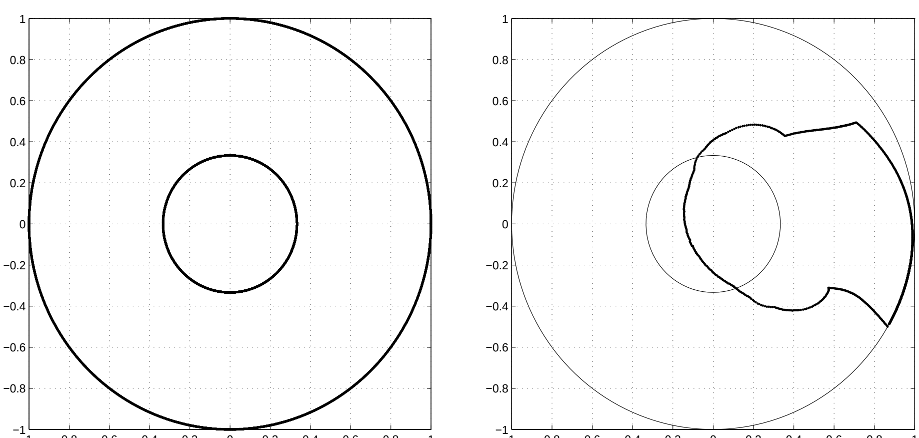
$$\widetilde{R}(a) \supset M_\infty(a).$$

□

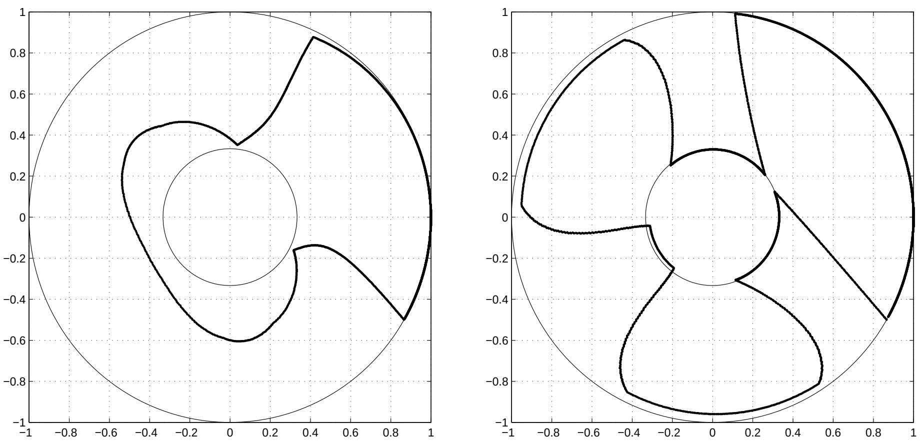
We illustrate the theorem on the following piece-wise continuous symbol which has six jump points,

$$a(\theta) = \begin{cases} \exp i\left[-\frac{\pi}{6} + \frac{2\pi}{3} \cdot \frac{7\theta}{\pi}\right], & \theta \in \left[0, \frac{\pi}{7}\right) \\ \frac{1}{3}\exp i\left[\frac{\pi}{6} + \frac{2\pi}{3} \cdot \left(\frac{7\theta}{\pi} - 1\right)\right], & \theta \in \left[\frac{\pi}{7}, \frac{2\pi}{7}\right) \\ \exp i\left[-\frac{\pi}{6} + \frac{2\pi}{3} \cdot \left(\frac{7\theta}{\pi} - 2\right)\right], & \theta \in \left[\frac{2\pi}{7}, \frac{3\pi}{7}\right) \\ \frac{1}{3}\exp i\left[-\frac{\pi}{6} + \frac{2\pi}{3} \cdot \left(\frac{7\theta}{\pi} - 3\right)\right], & \theta \in \left[\frac{3\pi}{7}, \frac{4\pi}{7}\right) \\ \exp i\left[-\frac{\pi}{6} + \frac{2\pi}{3} \cdot \left(\frac{7\theta}{\pi} - 4\right)\right], & \theta \in \left[\frac{4\pi}{7}, \frac{5\pi}{7}\right) \\ \frac{1}{3}\exp i\left[-\frac{\pi}{6} + \frac{2\pi}{3} \cdot \left(\frac{7\theta}{\pi} - 5\right)\right], & \theta \in \left[\frac{5\pi}{7}, \frac{6\pi}{7}\right) \\ \exp\left(-i\frac{\pi}{6}\right), & \theta \in \left[\frac{6\pi}{7}, \pi\right] \end{cases}.$$

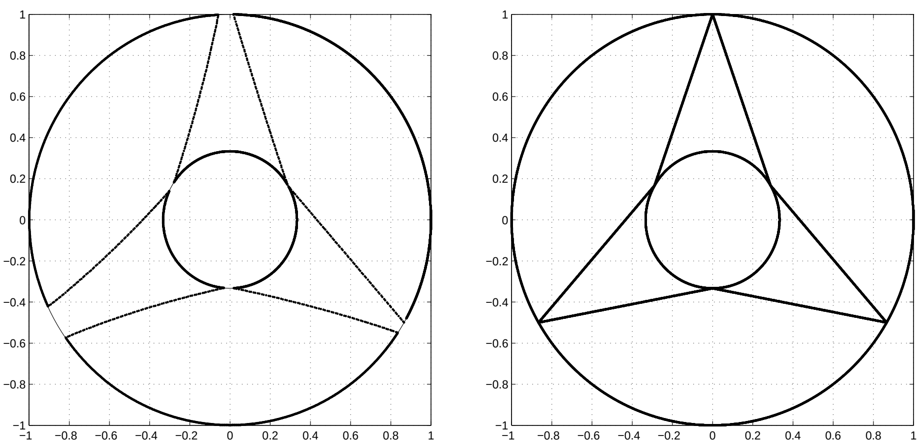
We show the image of the symbol $a = a(\theta)$, the image of $\gamma_{a,\lambda}$ for the following values of λ: 1, 10, 70, and 500, as well as the limit set $M_\infty(a)$.



The symbol $a(\theta)$ and the function $\gamma_{a,\lambda}$ for $\lambda = 1$



The function $\gamma_{a,\lambda}$ for $\lambda = 10$ and $\lambda = 70$



The function $\gamma_{a,\lambda}$ for $\lambda = 500$ and the limit set $M_\infty(a)$

We have obviously

$$\lim_{\lambda\to\infty} \operatorname{sp} T_a^{(\lambda)} = M_\infty(a) \subset \operatorname{conv}(\operatorname{ess\,Range} a). \tag{5.11}$$

To illustrate the possible interrelations among these sets we can repeat the arguments of Examples 5.3 - 5.6 in [5] and construct the (piecewise continuous) symbols $a = a(\theta)$ to realize the following possibilities:

$$\begin{aligned} M_\infty(a) &= \operatorname{Range} a \quad (= \operatorname{ess\,Range} a), \\ M_\infty(a) &= \operatorname{conv}(\operatorname{ess\,Range} a) \quad (= \operatorname{conv}(\operatorname{Range} a)), \\ M_\infty(a) &\subset \partial \operatorname{conv}(\operatorname{Range} a), \\ M_\infty(a) &= \partial \operatorname{conv}(\operatorname{Range} a). \end{aligned}$$

Unbounded symbols.

THEOREM (5.12). *Let $a(\theta) \in L_1(0,\pi) \cap C(0,1)$. Then*

$$\operatorname{Range} a \subset M_\infty(a).$$

Proof. We apply the Laplace method as in Theorem (5.2). Fix any point $\varphi \in (0,\pi)$ and consider for each ξ large enough the value $\lambda = 2\xi \arctan \varphi$. Then by (5.4) we have

$$\gamma_{a,\lambda}(\xi) = a(\varphi)(1+\alpha(\lambda\sqrt{1+(2\arctan\varphi)^{-2}})$$

where $\lim\limits_{L\to\infty} \alpha(L) = 0$. Thus if $\xi \to \infty$ then $\lambda \to \infty$ as well and we have

$$a(\varphi) \in M_\infty(a).$$

□

The next theorem, whose proof is analogous to that of Theorem 5.11 in [5], shows that the property (5.11), previously established for bounded symbols, remains valid for summable symbols.

THEOREM (5.13). *Let $a(\theta) \subset L_1(0,\pi)$. Then*

$$M_\infty(a) \subset \operatorname{conv}(\operatorname{ess\,Range} a).$$

Note that for functions $a(\theta) \in L_1(0,\pi) \cap C(0,\pi)$ Theorems (5.12) and (5.13) imply that

$$\operatorname{Range} a \subset M_\infty(a) \subset \operatorname{conv}(\operatorname{Range} a),$$

and we show that $\operatorname{Range} a$ can coincide with each of these extreme sets.

Example (5.14). For each $j \in \mathbb{N}$ define $I_j = [j^{-1} - j^{-3}, j^{-1}]$ and let $\overline{\{\xi_j\}_{j\in\mathbb{N}}} = [0, 2\pi]$. Define the symbol as follows

$$a(\theta) = \begin{cases} je^{i\xi_j}, & \theta \in I_j, \quad j \in \mathbf{N}; \\ 0, & \theta \in (0,\pi) \setminus \bigcup_{j=1}^\infty I_j. \end{cases}$$

It can be easily shown that

$$M_\infty(a) = \mathbb{C} = \operatorname{conv}(\operatorname{Range} a).$$

Example (5.15). Given $\alpha \in [0,1)$, introduce $a(\theta) = (\sin\theta)^{i-\alpha}$, which is unbounded for $\alpha \in (0,1)$, but bounded and oscillating for $\alpha = 0$. Calculate using [4], formula 3.892.1,

$$\begin{aligned}
\gamma_{a,\lambda}(\xi) &= \frac{\int_0^\pi (\sin\theta)^{\lambda+i-\alpha} e^{-2\xi\theta} d\theta}{\int_0^\pi (\sin\theta)^{\lambda} e^{-2\xi\theta} d\theta} \\
&= \frac{2^{\alpha-i}(\lambda+1)}{\lambda+i-\alpha+1} \frac{B\left(\frac{\lambda}{2}+1+i\xi, \frac{\lambda}{2}+1-i\xi\right)}{B\left(\frac{\lambda+i-\alpha}{2}+1+i\xi, \frac{\lambda+i-\alpha}{2}+1-i\xi\right)} \\
&= \frac{2^{\alpha-i}(\lambda+1)}{\lambda+i-\alpha+1} \frac{\Gamma(\lambda+2+i-\alpha)}{\Gamma(\lambda+2)} \frac{\Gamma\left(\frac{\lambda}{2}+1+i\xi\right)}{\Gamma\left(\frac{\lambda+i-\alpha}{2}+1+i\xi\right)} \\
&\times \frac{\Gamma\left(\frac{\lambda}{2}+1-i\xi\right)}{\Gamma\left(\frac{\lambda+i-\alpha}{2}+1-i\xi\right)}.
\end{aligned}$$

Applying the asymptotic formulas for the Γ-function (see [4], formulas 8.327 and 8.328.2) we have

$$\gamma_{a,\lambda}(\xi) = \left[\left(\frac{(\lambda+2)^2}{(\lambda+2)^2+4\xi^2}\right)^{\frac{1}{2}}\right]^{i-\alpha} \left(1+O\left(\frac{1}{\lambda+1}\right)\right).$$

Given any $v \in (0, \pi)$, we can take ξ and λ such that

$$\left(\frac{(\lambda+2)^2}{(\lambda+2)^2 + 4\xi^2} \right)^{\frac{1}{2}} = \sin v.$$

Thus

$$\gamma_{a,\lambda}(\xi) = (\sin v)^{i-\alpha} \left(1 + O\left(\frac{1}{\lambda+1} \right) \right),$$

and in this case $M_\infty(a) = \operatorname{Range} a$.

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WEYL'S THEOREM THROUGH FINITE ASCENT PROPERTY

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ABSTRACT. If T is a Banach space operator, with spectrum $\sigma(T)$ and Weyl spectrum $\sigma_w(T)$, then a necessary condition for T to satisfy Weyl's theorem is that $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T) = \pi^{\sharp}_{00}(T)$, where $\pi_{00}(T)$ denotes the set of eigenvalues of T of finite geometric multiplicity and $\pi^{\sharp}_{00}(T) = \{\lambda \in \pi_{00}(T) : (T-\lambda)$ has both finite ascent and descent$\}$. The condition $\pi_{00}(T) = \pi^{\sharp}_{00}(T)$ by itself is not sufficient for T to satisfy Weyl's theorem: however, if T has SVEP at points $\lambda \in \sigma(T) \setminus \sigma_w(T)$, then this condition is sufficient too. This generalizes the result (proved by a number of authors in the recent past) that a sufficient condition for T to satisfy Weyl's theorem is that the quasinilpotent part $H_0(T-\lambda)$ of $(T-\lambda)$ equal $(T-\lambda)^{-p}(0)$ for some integer $p \geq 1$.

1. Introduction

Given a Banach space X, let $B(X) = B(X, X)$ denote the algebra of bounded linear transformations (equivalently, operators) on X into itself. $T \in B(X)$ is said to be Fredholm if $T(X)$ is closed and both the *deficiency indices* $\alpha(T) = \dim T^{-1}(0)$ and $\beta(T) = \dim(X/T(X))$ are finite, and then the *index* of T, $\mathrm{ind}(T)$, is defined to be $\mathrm{ind}(T) = \alpha(T) - \beta(T)$. The ascent of T, $asc(T)$, is the least non-negative integer n such that $T^{-n}(0) = T^{-(n+1)}(0)$ and the descent of T, $dsc(T)$, is the least non-negative integer n such that $T^n(X) = T^{n+1}(X)$. (We shall, henceforth, shorten $T - \lambda I$ to $T - \lambda$.) The operator T is *Weyl* if it is Fredholm of index zero, and T is said to be *Browder* if it is Fredholm "of finite ascent and descent". Let $\mathbb{C}$ denote the set of complex numbers. The (Fredholm) essential spectrum $\sigma_e(T)$, the Browder spectrum $\sigma_b(T)$ and the Weyl spectrum $\sigma_w(T)$ of T of are the sets

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \quad \text{is not Fredholm}\};$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \quad \text{is not Browder}\}$$

and

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \quad \text{is not Weyl}\}.$$

If we let $\sigma(T)$ denote the usual spectrum of T and $\mathrm{acc}\ \sigma(T)$ denote the set of accumulation points of $\sigma(T)$, then:

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T) \subseteq \sigma_e(T) \cup \mathrm{acc}\ \sigma(T).$$

Let $\pi_0(T)$ denote the set of Riesz points of T (i.e., the set of isolated eigenvalues of T of finite algebraic multiplicity) and let $\pi_{00}(T)$ denote the set of eigenvalues of T of finite geometric multiplicity. Also, let $\pi_{a0}(T)$ be the set of $\lambda \in \mathbb{C}$ such that λ is an isolated point of $\sigma_a(T)$ and $0 < \dim\ker(T-\lambda) < \infty$, where $\sigma_a(T)$

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denotes the approximate point spectrum of the operator $T \in B(X)$. Clearly, $\pi_{00}(T) \subseteq \pi_{a0}(T)$. We say that *Browder's theorem holds for* $T \in B(X)$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_0(T),$$

Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

and *a-Weyl's theorem holds for* T

$$\sigma_{wa}(T) = \sigma_a(T) \setminus \pi_{a0}(T),$$

where $\sigma_{wa}(T)$ denote *the essential approximate point spectrum* (i.e., $\sigma_{wa}(T) = \cap\{\sigma_a(T+K) : K \in K(X)\}$ with $K(X)$ denoting the ideal of compact operators in $B(X)$). Recall that a-Weyl's theorem for $T \Longrightarrow$ Weyl's theorem for T [16].

An operator $T \in B(X)$ has *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$, SVEP at $\lambda_0 \in \mathbb{C}$ for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \to X$ which satisfies

$$(T-\lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. Trivially, every operator T has SVEP at points of the resolvent $\mathbb{C} \setminus \sigma(T)$; also T has SVEP at $\lambda \in iso\sigma(T)$. *We say that T has SVEP if it has SVEP at every $\lambda \in \mathbb{C}$.*

The study of operators satisfying Weyl's theorem was initiated by Hermann Weyl [17], who showed that self-adjoint Hilbert space operators T satisfy $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$. This study has since been carried out by a large number of authors (see [5], [8] and [9] for further references). A particularly useful, though fairly demanding, criterion for a Banach space operator T to satisfy Weyl's theorem is that the *quasi-nilpotent part* $H_0(T-\lambda)$ of $T-\lambda$ equal $(T-\lambda)^{-p}(0)$ for all complex λ and some integer $p \geq 1$ (see [3], [5] and [14]). Although the condition $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ is satisfied by a number of the commonly considered classes of Banach space operators (see [14], [3]), it is certainly not satisfied by a number of equally as commonly considered classes (such as *k-quasihyponormal operators* on a Hilbert space and *paranormal operators* on a Banach space). The property $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ implies finite ascent (and hence, SVEP) at all λ, and finite descent at all λ such that $\lambda \notin \sigma_w(T)$. We prove that T satisfies Weyl's theorem if and only if T has SVEP at all $\lambda \in \sigma(T) \setminus \sigma_w(T)$ and $\pi_{00}(T) = \pi_{00}^{\sharp}(T) = \{\lambda \in \pi_{00}(T) : asc(T-\lambda) = dsc(T-\lambda) < \infty\}$. Furthermore, if T^* has SVEP and $\pi_{00}(T) = \pi_{00}^{\sharp}(T)$ (or, T has SVEP and $\pi_{00}(T^*) \subseteq \pi_{00}(T) = \pi_{00}^{\sharp}(T)$), then T satisfies a-Weyl's theorem (resp., T^* satisfies a-Weyl's theorem).

2. Main results.

SVEP is enough to guarantee Browder's theorem, but SVEP (even the finite ascent property) is not enough to guarantee Weyl's theorem. Let

$$\lambda \in \pi_{00}^{\sharp}(T) = \{\lambda \in \pi_{00}(T) : asc(T-\lambda) = dsc(T-\lambda) < \infty\}.$$

Then $\lambda \in \pi_{00}^{\sharp}(T) \Longrightarrow \lambda$ is a pole of T, of some finite order $p \geq 1$, of finite rank $\Longrightarrow \lambda \in \pi_0(T) \subseteq \pi_{00}(T)$. The set $\pi_{00}^{\sharp}(T)$ coincides with the set $\pi_{00}(T)$ in the case in which $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$ for all $\lambda \in \mathbb{C}$, where

$$H_0(T-\lambda) = \{x \in X : \lim_{n\to\infty} \|(T-\lambda)^n\|^{1/n} = 0\}$$

is the *quasi-nilpotent part* of $(T-\lambda)$. (Recall that $(T-\lambda)^{-n}(0) \subseteq H_0(T-\lambda)$ for all $n = 0, 1, 2, \ldots$ [13].) To see this, let

$$\begin{aligned} K(T-\lambda) = \{x \in X &: \text{there exists a sequence } \{x_n\} \subset X \text{ and } \delta > 0 \\ &\text{for which } x = x_0, (T-\lambda)x_{n+1} = x_n \text{ and } \|x_n\| \le \delta^n \|x\| \text{ for all } n = 1, 2, \ldots\} \end{aligned}$$

denote the *analytical core* of $(T-\lambda)$ (see [13]). Observe that $(T-\lambda)K(T-\lambda) = K(T-\lambda)$. Let $\lambda \in \pi_{00}(T)$. Then $\lambda \in iso\sigma(T)$, and it follows that

$$\begin{aligned} X &= H_0(T-\lambda) \oplus K(T-\lambda) = (T-\lambda)^{-p}(X) \oplus K(T-\lambda) \\ &\Longrightarrow (T-\lambda)^p(X) = (T-\lambda)^p K(T-\lambda) = K(T-\lambda). \end{aligned}$$

Hence $X = (T-\lambda)^{-p}(0) \oplus (T-\lambda)^p(X)$, i.e. λ is a finite rank pole of order $p \Longrightarrow \lambda \in \pi_{00}^{\sharp}(T)$.

In the sequel we shall denote the set of Riesz points of T by $\pi_{00}^{\sharp}(T)$ (rather than by $\pi_0(T)$): our reason for this preference is partly explained by our Remark 2.7 *infra*. The following lemma says that the conditions T has SVEP at points $\lambda \in \sigma(T) \setminus \sigma_w(T)$ and $\sigma(T) \setminus \sigma_w(T) = \pi_{00}^{\sharp}(T)$ are necessary for T to satisfy Weyl's theorem. (Note that the condition $\sigma(T) \setminus \sigma_w(T) = \pi_{00}^{\sharp}(T)$ implies T satisfies Browder's theorem.)

LEMMA (2.1). *If $T \in B(X)$ satisfies Weyl's theorem, then $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T) = \pi_{00}^{\sharp}(T)$.*

Proof. If T satisfies Weyl's theorem, then $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T) \supseteq \pi_{00}^{\sharp}(T)$. Let $\lambda \in \pi_{00}(T)$. Then λ is an isolated eigenvalue of T such that $T-\lambda$ is a Fredholm operator of index 0. The point λ being isolated in $\sigma(T)$, T has SVEP at λ. This, taken together with the fact that $T-\lambda$ is a Fredholm operator of index 0, implies that both $asc(T-\lambda)$ and $dsc(T-\lambda)$ are finite [1, Corollary 2.10], and hence equal (see [10, Proposition 38.6 (i)]). $\square$

The condition $\sigma(T) \setminus \sigma_w(T) = \pi_{00}^{\sharp}(T)$ is not sufficient for T to satisfy Weyl's theorem. Consider for example the operator $T = T_1 \oplus T_2$, where T_1 is the operator $T_1(x_1, x_2, \ldots) = (\frac{x_1}{2}, \frac{x_2}{3}, \ldots)$ on a Hilbert space and T_2 is a p-nilpotent on a finite dimensional space. Then $\pi_{00}(T) = \{0\}$, $\sigma(T) = \sigma_w(T) = \{0\}$, $\pi_{00}^{\sharp}(T) = \emptyset$ and T does not satisfy Weyl's theorem. Again, the condition $\pi_{00}(T) = \pi_{00}^{\sharp}(T)$ is not sufficient for T to satisfy Weyl's theorem. Thus, let $T = U \oplus U^*$, where U is the simple unilateral shift on a Hilbert space. Then, $\sigma(T)$ is the closed unit disc $\overline{\mathbb{D}}$, $\sigma_w(T)$ is the boundary ∂D of $\mathbb{D}$, $\pi_{00}(T) = \pi_{00}^{\sharp}(T) = \emptyset$ and T fails to satisfy Weyl's theorem. However, the condition $\pi_{00}(T) = \pi_{00}^{\sharp}(T)$ does become sufficient in the case in which T has SVEP, as the following lemma shows.

LEMMA (2.2). *If $T \in B(X)$ has SVEP at all points $\lambda \in \sigma(T) \setminus \sigma_w(T)$ and $\pi_{00}(T) = \pi_{00}^{\sharp}(T)$, then T satisfies Weyl's theorem.*

Proof. Let $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then T has SVEP at λ and $T-\lambda$ is a Fredholm operator with index zero. Arguing as in the proof of Lemma (2.1) it follows that $asc(T-\lambda) = dsc(T-\lambda) = p < \infty$, for some $p > 0$ which taken together with fact that $T-\lambda$ is a Fredholm operator implies that $T-\lambda$ is a Browder operator, i.e. $\lambda \in \pi_0(T) \subseteq \pi_{00}(T)$. Hence, $\sigma(T) \setminus \sigma_w(T) \subseteq \pi_{00}(T)$.

For the reverse inequality, let $\lambda \in \pi_{00}(T) = \pi^{\sharp}_{00}(T)$. Then λ is an isolated point in $\sigma(T)$ such that $0 < \alpha(T-\lambda) < \infty$ and $asc(T-\lambda) = dsc(T-\lambda) < \infty$, which implies (by [10, Proposition 38.6]) that $0 < \alpha(T-\lambda) = \beta(T-\lambda) < \infty$, i.e. $T-\lambda$ is a Fredholm operator of index 0. Hence, $\lambda \in \sigma(T) \setminus \sigma_w(T)$. □

Putting Lemmas (2.1) and (2.2) together, we have the following necessary and sufficient conditions for $T \in B(X)$ to satisfy Weyl's theorem.

THEOREM (2.3). *$T \in B(X)$ satisfies Weyl's theorem if and only if it has SVEP at all points $\lambda \in \mathbb{C} \setminus \sigma_w(T)$ and $\pi_{00}(T) = \pi^{\sharp}_{00}(T)$.*

Proof. The points $\lambda \in \pi_{00}(T)$ being isolated in $\sigma(T)$, if T satisfies Weyl's theorem then T has SVEP at all points $\lambda \in \mathbb{C} \setminus \sigma_w(T)$ and $\pi_{00}(T) = \pi^{\sharp}_{00}(T)$ (by Lemma (2.1)). The proof being clear from Lemma (2.2), the theorem is proved. □

The following corollary shows that the condition $\pi_{00}(T) = \pi^{\sharp}_{00}(T)$ of Theorem (2.3) is sufficient for T to satisfy Weyl's theorem in the case in which T^* has SVEP.

COROLLARY (2.4). *Let $T \in B(X)$. If T or T^* has SVEP at all points $\lambda \in \sigma(T) \setminus \sigma_w(T)$and $\pi_{00}(T) = \pi^{\sharp}_{00}(T)$, then T satisfies Weyl's theorem.*

Proof. The proof being clear from Lemma (2.2) in the case in which T has SVEP at all $\lambda \in \sigma(T) \setminus \sigma_w(T)$, we consider the case in which T^* has SVEP. The argument of the proof of Lemma (2.2) show that $\pi_{00}(T) = \pi^{\sharp}_{00}(T) \subseteq \sigma(T) \setminus \sigma_w(T)$. Since $\lambda \in \sigma(T) \setminus \sigma_w(T)$ implies $T-\lambda$ is Fredholm with $\mathrm{ind}(T-\lambda) = 0$, T^* has SVEP at λ implies that $dsc(T-\lambda) < \infty$ [1, Theorem 2.9]. Taken together with $\alpha(T-\lambda) = \beta(T-\lambda) < \infty$ this implies that $asc(T-\lambda) < \infty$ [10, Proposition 38.6]. Consequently, $\sigma(T) \setminus \sigma_w(T) \subseteq \pi_{00}(T) = \pi^{\sharp}_{00}(T)$, which implies that $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$, and T satisfies Weyl's theorem. □

Theorem (2.3) contains information about T^* satisfying Weyl's theorem. One has:

COROLLARY (2.5). *If T or T^* has SVEP and $\pi_{00}(T) = \pi^{\sharp}_{00}(T)$, then the following conditions are equivalent:*

(i) *T^* satisfies Weyl's theorem;*

(ii) $\pi_{00}(T^*) \subseteq \pi_{00}(T)$*;*

(iii) $\pi_{00}(T^*) = \pi_{00}(T)$.

Proof. Suppose, to start with, that *T has SVEP*. Then T satisfies Weyl's theorem (by Corollary (2.4)). Since $\sigma(T^*) = \sigma(T)$ and $\sigma_w(T^*) = \sigma_w(T)$, $\sigma(T^*) \setminus \sigma_w(T^*) = \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T) = \pi^{\sharp}_{00}(T)$. This proves the equivalence $(i) \Longleftrightarrow$

(ii). The implication $(iii) \Longrightarrow (ii)$ being obvious, it remains to prove the implication $(ii) \Longrightarrow (iii)$. Suppose $\pi_{00}(T^*) \subseteq \pi_{00}(T)$. Then, since $\pi_{00}(T) = \pi^{\sharp}_{00}(T)$,

$$\begin{aligned} & \lambda \in \pi_{00}(T) \Longrightarrow \lambda \in \pi^{\sharp}_{00}(T) \\ \Longrightarrow \quad & asc(T-\lambda) = dsc(T-\lambda) < \infty \text{ and } X = (T-\lambda)^{-p}(0) \oplus (T-\lambda)^p(X) \end{aligned}$$

for some integer $p \geq 1$. By duality,

$$\begin{aligned} & X^* = (T^*-\lambda)^{-p}(0) \oplus (T^*-\lambda)(X^*) \\ \Longrightarrow \quad & asc(T^*-\lambda) = dsc(T^*-\lambda) < \infty \Longrightarrow \lambda \in \pi^{\sharp}_{00}(T) \subseteq \pi_{00}(T^*). \end{aligned}$$

Hence $\pi_{00}(T^*) = \pi_{00}(T)$.

Suppose now that T^* *has SVEP*. If T^* satisfies Weyl's theorem, then $\sigma(T^*) \setminus \sigma_w(T^*) = \pi_{00}(T^*)$. Since $\lambda \in \pi_{00}(T^*)$ implies $T^* - \lambda I^*$ is Fredholm and $asc(T^* - \lambda I^*) = dsc(T^* - \lambda I^*) < \infty$, $T - \lambda$ is Fredholm with $asc(T-\lambda) = dsc(T-\lambda) < \infty$. Thus $\lambda \in \pi_{00}(T)$, and $(i) \Longrightarrow (ii)$. For the reverse implication $(ii) \Longrightarrow (i)$, observe that the hypotheses T^* has SVEP and $\pi^{\sharp}_{00}(T) = \pi_{00}(T)$ imply that T satisfies Weyl's theorem (see Corollary (2.4)). Since $\pi_{00}(T^*) \subseteq \pi_{00}(T)$,

$$\sigma(T^*) \setminus \sigma_w(T^*) \subseteq \pi_{00}(T^*) \subseteq \pi_{00}(T) = \sigma(T) \setminus \sigma_w(T),$$

which implies that T^* satisfies Weyl's theorem. Hence $(i) \Longleftrightarrow (ii)$. Obviously $(iii) \Longrightarrow (ii)$. To complete the proof we have to show that $(ii) \Longrightarrow (iii)$. But this is immediate since (by duality) $\sigma(T) \setminus \sigma_w(T) = \sigma(T^*) \setminus \sigma_w(T^*)$, and $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$ and $\sigma(T^*) \setminus \sigma_w(T^*) = \pi_{00}(T^*)$ (since both T and T^* satisfy Weyl's theorem). $\square$

Remark (2.6). T^* may fail to satisfy Weyl's theorem in the absence of the hypothesis $\pi_{00}(T^*) \subseteq \pi_{00}(T)$ in Corollary (2.5). Thus let $T \in B(\ell^2(\mathbb{N}))$ be the operator $T(x_1, x_2, x_3, ...) = (0, \frac{x_1}{2}, \frac{x_2}{3}, ...)$. Then $T^*(x_1, x_2, x_3, ...) = (\frac{x_2}{2}, \frac{x_3}{3}, ...)$. Both T and T^* have SVEP (since they are quasi-nilpotent), $\pi_{00}(T) = \pi^{\sharp}_{00}(T) = \emptyset$ and T satisfies Weyl's theorem, but $\pi_{00}(T^*) = \{0\}$ and T^* does not satisfy Weyl's theorem.

Remark (2.7). The credit for Theorem (2.3) goes to Karl Gustafson [8, Theorem 1]: what we have done in Lemmas (2.1) and (2.2) is to use techniques from local spectral theory to single out the role played by SVEP to identity points of $\pi_{00}(T)$ as being points of $\pi^{\sharp}_{00}(T)$ and to show that T^* also satisfies Weyl's theorem under the additional hypothesis that $\pi_{00}(T^*) \subseteq \pi_{00}(T)$.

In the presence of SVEP, the hypotheses of Theorem (2.3) ensure a-Weyl's theorem for T^*; an additional hypothesis is required for an a-Weyl's theorem for T. Let

$$\Phi_+(X) = \{T \in B(X) : T(X) \text{ is closed and } \alpha(T) < \infty \}$$

denote the class of *upper semi-Fredholm* operators,

$$\Phi_-(X) = \{T \in B(X) : \operatorname{codim} T(X) < \infty\}$$

denote the class of *lower semi-Fredholm* operators, and let

$$\mathcal{B}_+(X) = \{T \in \Phi_+(X) : asc(T) < \infty\}$$

denote the class of *upper semi-Browder* operators. Let $\sigma_{ub}(T)$ denote the *upper Browder spectrum* of T. It is then clear that $\sigma_{wa}(T)$ is the complement of all those $\lambda \in \mathbb{C}$ for which $(T-\lambda) \in \Phi_+(X)$ and $\operatorname{ind}(T-\lambda) \leq 0$ (see [15]; see also [2] for further information about SVEP and semi-Browder spectrum). Clearly, $\sigma_{wa}(T) \subseteq \sigma_{ub}(T)$.

THEOREM (2.8). *If $T^* \in B(X)$ has SVEP and $\pi_{00}(T) = \pi_{00}^{\sharp}(T)$(resp., T has SVEP and $\pi_{00}(T^*) \subseteq \pi_{00}(T) = \pi_{00}^{\sharp}(T)$), then T satisfies a-Weyl's theorem (resp., T^* satisfies a-Weyl's theorem).*

Proof. Assume to start with that T^* has SVEP. Then $\sigma(T) = \sigma_a(T)$ [12, p. 35] and $\pi_{00}^a(T) = \pi_{00}(T)$. Since T satisfies Weyl's theorem (by Corollary (2.4)), $\lambda \in \pi_{00}^a(T) = \pi_{00}(T) \Longrightarrow \lambda \in \sigma(T) \setminus \sigma_w(T) \subseteq \sigma_a(T) \setminus \sigma_{wa}(T)$. Hence to prove $\sigma_a(T) \setminus \sigma_{wa}(T) = \pi_{00}^a(T)$ we have to prove that $\sigma_a(T) \setminus \sigma_{wa}(T) \subseteq \pi_{00}^a(T)$. If $\lambda \in \sigma_a(T) \setminus \sigma_{wa}(T)$, then $(T-\lambda) \in \Phi_+(X)$ and $\operatorname{ind}(T-\lambda) \leq 0$. Since T^* has SVEP, $(T-\lambda) \in \Phi_+(X) \Longrightarrow dsc(T-\lambda) < \infty$ [1, Theorem 2.9], which in turn implies that $\operatorname{ind}(T-\lambda) \geq 0$ [10, Proposition 38.5]. Hence $\operatorname{ind}(T-\lambda) = 0$. Since $\alpha(T-\lambda) < \infty$ and $\beta(T-\lambda) < \infty$, it follows that $asc(T-\lambda) = dsc(T-\lambda) < \infty$ [10, Proposition 38.6], which implies that $\lambda \in \pi_{00}^{\sharp}(T) = \pi_{00}(T) = \pi_{00}^a(T)$. This completes the proof for the case in which T^* has SVEP.

If T has SVEP, then $\sigma(T^*) = \sigma_a(T^*)$ [12, p. 35]. If also $\pi_{00}(T^*) \subseteq \pi_{00}(T) = \pi_{00}^{\sharp}(T)$), then T^* satisfies Weyl's theorem (by Corollary (2.5)), which implies that $\sigma(T^*) \setminus \sigma_w(T^*) = \pi_{00}(T^*) = \pi_{00}^{\sharp}(T^*) = \pi_{00}^a(T^*)$. Let $\lambda \in \sigma_a(T^*) \setminus \sigma_{wa}(T^*)$. Then $(T-\lambda) \in \Phi_-(X)$ and $\operatorname{ind}(T-\lambda) \geq 0$. Because T has SVEP, it follows that $asc(T-\lambda) < \infty$ and $\operatorname{ind}(T-\lambda) \geq 0$ [1, Theorem 2.6] $\Longrightarrow asc(T-\lambda) < \infty$ and $\operatorname{ind}(T-\lambda) = 0$ [10, Proposition 38.5], which in turn implies that $asc(T-\lambda) = dsc(T-\lambda) < \infty$ and $\operatorname{ind}(T-\lambda) = 0$ [10, Proposition 38.6], and hence that $\lambda \in \pi_{00}^{\sharp}(T^*) = \pi_{00}^a(T^*)$. Since $\lambda \in \pi_{00}^a(T^*) = \pi_{00}(T^*) = \pi_{00}^{\sharp}(T^*)$ trivially implies $\lambda \in \sigma_a(T^*) \setminus \sigma_{wa}(T^*)$, the proof is complete. □

Remark (2.9). The example of Remark (2.6) shows that the hypotheses T, T^* have SVEP and $\pi_{00}(T) = \pi_{00}^{\sharp}(T)$ are not sufficient for T and T^* to satisfy an a-Weyl theorem. It is however clear from Theorem (2.8) that if both T and T^* have SVEP and $\pi_{00}(T^*) \subseteq \pi_{00}(T) = \pi_{00}^{\sharp}(T)$, then both T and T^* satisfy a-Weyl's theorem.

3. Applications.

The hypotheses of Theorem (2.3) are satisfied by a large number of classes of operators, among them generalized scalar and totally paranormal operators on a Banach space, multipliers of semi-simple Banach algebras, and hyponormal, p-hyponormal $(0 < p < 1)$, log-hyponormal and M-hyponormal operators on a Hilbert space. All these classes of operators satisfy the property that

$$(H) \qquad H_0(T-\lambda) = (T-\lambda)^{-p}(0)$$

for all $\lambda \in \mathbb{C}$ and some integer $p \geq 1$. (See [3], [5] and [14] for the definitions and other properties of these classes of operators.) We prove in the following proposition that if hypothesis (H) is satisfied, then the hypotheses of Corollary (2.5) are satisfied.

PROPOSITION (3.1). *Let $T \in B(X)$. If T has property (H) for all $\lambda \in \mathbb{C}$ and some integer $p \geq 1$, then $asc(T-\lambda) \leq p$ for all $\lambda \in \mathbb{C}$, $dsc(T-\lambda) \leq p$ for all*

$\lambda \in \sigma(T) \setminus \sigma_w(T)$ and $\pi_{00}(T) = \pi^{\sharp}_{00}(T)$. Moreover, both T and T^* satisfy Weyl's theorem, and T^* satisfies a-Weyl's theorem.

Proof. That $asc(T-\lambda) \le p$ for all $\lambda \in \mathbb{C}$ and hence T has SVEP is obvious. Let $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then $T-\lambda$ is a Fredholm operator and $\mathrm{ind}(T-\lambda) = 0$. Since T has SVEP, it follows from [1, Corollary 2.10] that $dsc(T-\lambda) < \infty$, and this since $asc(T-\lambda) \le p$ implies that $asc(T-\lambda) = dsc(T-\lambda) \le p$ for all $\lambda \in \sigma(T) \setminus \sigma_w(T)$. As already seen (see the beginning of Section 2), if T satisfies property (H) then $\pi_{00}(T) = \pi^{\sharp}_{00}(T)$. Again, if $\lambda \in \pi_{00}(T^*)$, then $\lambda \in iso\sigma(T) \Longrightarrow X = (T-\lambda)^{-p}(0) \oplus K(T-\lambda) = (T-\lambda)^{-p}(0) \oplus (T-\lambda)^p(X)$, which implies that $\lambda \in \pi_{00}(T)$. The proof is completed by appealing to Corollaries (2.4), (2.5) and Theorem (2.8). □

Property (H) implies much more. Recall that $T \in B(X)$ is said to be *reguloid* if for every $\lambda \in iso\ \sigma(T)$ the operator $T-\lambda$ is *relatively regular*, i.e. there exists an operator $S_\lambda \in B(X)$ such that

$$(T-\lambda)S_\lambda(T-\lambda) = (T-\lambda).$$

Reguloid operators are *isoloid* (i.e., isolated points of the spectrum are eigenvalues of the operator). Property (H) (with $p = 1$) implies that T and T^* are reguloid (see [3, Theorem 2.5]). The reguloid property of T implies that $X = T^{-1}(0) \oplus T(X)$. Thus, if $H_0(T-\lambda) = (T-\lambda)^{-1}(0)$ for all $\lambda \in \mathbb{C}$, then

$$X = T^{-1}(0) \oplus T(X) \Longrightarrow T^{-1}(0) \perp T(X),$$

where by $T^{-1}(0) \perp T(X)$ we mean "kernel T is orthogonal to range T"and "orthogonality" is defined in the sense of Garret Birkhoff (see [7, pg. 93]) by

$$\|x\| \le \|x+y\|, \qquad \text{for all } x \in T^{-1}(0) \text{ and } y \in T(X).$$

This property fails for certain operators which satisfy Weyl's theorem. An example of such operators is provided by the class of k-quasihyponormal operators on a Hilbert space $\mathcal{H}$ (i.e., operators $T \in B(\mathcal{H})$ such that $T^{*k}(T^*T - TT^*)T^k \ge 0$ for some integer $k \ge 1$): if $T \in B(\mathcal{H})$ is k-quasihyponormal then T satisfies Weyl's theorem, $asc(T-\lambda) = k$, $iso\sigma(T) = \sigma_p(T)$, eigenspaces corresponding to non-zero eigenvalues of T are reducing, but the eigenspace corresponding to the eigenvalue 0 of T is not reducing (see [4] for these results). Another example is provided by paranormal operators (i.e., operators that satisfy $\|Tx\|^2 \le \|T^2x\|$ for all unit vectors $x \in X$, see for example [10, p. 229]). Paranormal operators do not satisfy property (H) (see [6, Remark following Lemma 3]: we prove that paranormal operators satisfy the conditions of Theorem (2.3).

PROPOSITION (3.2). *If $T \in B(X)$ is paranormal, then T has SVEP at all points $\lambda \in \mathbb{C} \setminus \sigma_w(T)$ and $\pi_{00}(T) = \pi^{\sharp}_{00}(T)$.*

Proof. Clearly T has SVEP at all points of the resolvent set of T. If $\lambda \in \sigma(T) \setminus \sigma_w(T)$, then $T-\lambda$ is a Fredholm operator of index 0 (which implies that $T-\lambda$ is Kato type [1, Remark 2.2(iv)]). We prove that points $\lambda \in \sigma(T) \setminus \sigma_w(T)$ are isolated in $\sigma(T)$: this would then imply that λ is simple pole of the resolvent of T (see [6, Lemma 2.1]; paranormal operators from a sub-class of the "heredetarily normaloid" operators of [6]), and hence that T has SVEP at all $\lambda \in \mathbb{C} \setminus \sigma_w(T)$

and that $\pi_{00}(T) = \pi^\sharp_{00}(T)$. Since a paranormal operator T has finite ascent, it follows from [1, Corollary 2.10] that if $0 \in \sigma(T) \setminus \sigma_w(T)$ then $\sigma(T)$ does not cluster at 0. Assume now that $(0 \neq)\lambda \in \sigma(T) \setminus \sigma_w(T)$. Let $\mathcal{N}(T)$ and $\gamma(T)$ denote the null space and the minimal modulus function of T, and let $d(x, \mathcal{N}(T)) = \inf_{y\in\mathcal{N}(T)} ||x-y||$ denote the distance of $x \in X$ from $\mathcal{N}(T)$. If $\lambda \notin iso\sigma(T)$, then, λ being a non-isolated eigenvalue of T, there exists a sequence of non-zero eigenvalues of T converging to λ. Recall from [6, Lemma 2.2] that eigenspaces corresponding to distinct non-zero eigenvalues of a paranormal operator are *orthogonal* (in the sense of G. Birkhoff [7, pp. 93]). Hence $d(x_n, \mathcal{N}(T-\lambda)) \geq 1$ for all $x_n \in \mathcal{N}(T-\lambda_n)$ such that $||x_n|| = 1$. We have:

$$\delta(\lambda_n, \lambda) = sup\{d(x_n, \mathcal{N}(T-\lambda)) : x_n \in \mathcal{N}(T-\lambda_n), ||x_n|| = 1\} \geq 1$$

for all n, which implies that

$$|\lambda_n - \lambda|/\delta(\lambda_n, \lambda) \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty.$$

But then

$$\gamma(T-\lambda) = |\lambda_n - \lambda|/\delta(\lambda_n, \lambda) \longrightarrow 0 \ \text{ as } \ n \longrightarrow \infty.$$

Since $(T-\lambda)(X)$ is closed, this is a contradiction [10, Proposition 36.1]. Consequently, points $(0 \neq)\lambda \in \sigma(T) \setminus \sigma_w(T)$ are isolated in $\sigma(T)$. □

As yet another example of an operator T which satisfies the hypotheses of Theorem (2.3) but fails to satisfy $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$, consider the operator $T \in B(H)$ (H is a Hilbert space) defined by

$$T = DU^* \oplus U,$$

where U is the unilateral shift and D is the diagonal operator with diagonal $(1, 1/2, 1/3, \ldots)$. Then T does not satisfy $H_0(T-\lambda) = (T-\lambda)^{-p}(0)$, since DU^* is a quasi-nilpotent operator. However, $\sigma(T) = \sigma_w(T)$ and hence $asc(T-\lambda) = dsc(T-\lambda) = 0$ for all $\lambda \in \mathbb{C} \setminus \sigma_w(T)$. (We are grateful to Professor Woo Young Lee for supplying us with this example, *albeit* in another context.)

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THE ORDER OF REAL LINE BUNDLES

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Abstract. It is shown that for any real line bundle ξ over a space X, such that $n\xi$ admits $r \geq 1$ independent sections, there is a natural upper bound on the order of $[\xi]$ as an element of the real K-theory of X, for an arbitrary space X. Previous work of Antoniano et al. [2], Barufatti and Hacon [3], [4] furnishes examples where this upper bound is best possible, and this theorem then enables a sharpening of some of the results in these papers. Applications are made to classifying spaces, the Alexandrov line, Stiefel manifolds, and projective Stiefel manifolds.

1. Introduction

For $\mathbb{F}$ equal to either $\mathbb{R}$ or $\mathbb{C}$, and for α an $\mathbb{F}$-vector bundle of rank r over a space X, the order of α is the least positive integer m (if such exists) such that $m\alpha \sim 0$, i.e. $m\alpha$ is stably trivial. Equivalently, $m([\alpha] - r) = 0 \in \widetilde{K_{\mathbb{F}}}(X)$, where $K_{\mathbb{R}}(X) = KO(X)$, $K_{\mathbb{C}}(X) = K(X)$ (and similarly for the reduced K-theory). We remark that $m\alpha$ need not be trivial, however it is usually the case, for X a finite CW-complex, that $m \gg \dim(X)$, and one can then use standard stability properties of vector bundles (cf. [8], Ch.9,§1) to conclude that $m\alpha$ is in fact trivial (in the real case $mr \geq \dim(X) + 1$ suffices, and in the complex case $mr \geq [(\dim(X) + 1)/2]$).

Suppose now that X is a space of the homotopy type of a finite CW-complex. If ξ is any $\mathbb{R}$-line bundle over X, then it is classified by a map $f : X \to BO(1) = \mathbb{R}P^\infty$. By cellular approximation f factors through a finite skeleton as $g : X \to \mathbb{R}P^n$ for some n, with $g^*(\xi_n) \approx \xi$, where ξ_n is the Hopf line bundle over $\mathbb{R}P^n$. Hence the order of ξ (respectively $c\xi$) divides the order of ξ_n (respectively $c\xi_n$), which from [1] is respectively $2^{\phi(n)}, 2^{[n/2]}$ (where as usual $\phi(n)$ is the number of integers j such that $1 \leq j \leq n, j \equiv 0, 1, 2, 4 (\bmod 8)$). In particular ξ has order a power of 2.

The bounds on the order of ξ obtained by the above method are generally not very sharp. Our main result (Theorem (1.2) below) gives much sharper bounds once information about sections of some multiple $n\xi$ of ξ is known, in the real case $\mathbb{F} = \mathbb{R}$, and under no additional hypothesis about X. To state this theorem first recall that for any positive integer t, $\nu_2(t)$ is the largest non-negative integer such that $2^{\nu_2(t)}$ divides t. Next, for any $1 \leq r \leq n$, set $n-r = s, c = [s/2], n = 2m$ or $n = 2m + 1$.

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Definition (1.1). With n, r, m, s, and c as above, set

$$a_0(n,r) = \begin{cases} \min\{2j-1+\nu_2\binom{m}{j} : c+1 \le j\}, nr \text{ even}, \\ \min\{2c+\nu_2\binom{m}{c}, 2j-1+\nu_2\binom{m}{j} : c+1 \le j\},\ nr \ \text{ odd}. \end{cases}$$

Note that in the above definition it is assumed, for $j > m$, that $\binom{m}{j} = 0$ and also that $\nu_2(0) = \infty$, so in fact these terms have no effect on $a_0(n,r)$.

THEOREM (1.2). *Let ξ be a real line bundle over a space X such that $n\xi$ admits $r \ge 1$ independent sections. Then $2^{a_0(n,r)}\xi \sim 0$.*

The proof of this theorem will be given in §2, using only elementary techniques from multilinear algebra involving exterior powers and combinatorial identities with binomial coefficients. For further results on sections of multiples of vector bundles, in a somewhat different direction, cf. [10].

A somewhat less general version of this theorem appears in [21]. There, under the extra restriction that X be a finite CW-complex, a proof is given that uses representation theory as well as previous calculations of the (complex) K-theory of the projective Stiefel manifolds [2], [4]. The present proof is shorter and more elementary, does not involve the projective Stiefel manifolds, and imposes no requirements on the base space X (not even paracompactness). As a consequence it serves as an independent verification for some of the difficult calculations in these papers. Also, in this paper, we explore a few applications of Theorem (1.2).

For any real vector bundle α having (finite) real order k, the fact that the composition $\widetilde{KO}(X) \xrightarrow{c} \widetilde{K}(X) \xrightarrow{r} \widetilde{KO}(X)$ of complexification with realification is multiplication by 2 implies that $k = 2^\epsilon \ell$, where ℓ is the order of $c\alpha$ in $\widetilde{K}(X)$ and $\epsilon \in \{0,1\}$. The exact value of ϵ can be difficult to determine, as for example in (part of) the Adams conjecture ([8] , p.240, 14.2(3)). A second example of this type occurs in the K-theory of the projective Stiefel manifolds $X_{n,r}$ (cf. [2], [3], [4]), where $X_{n,r}$ is defined to be the $\mathbb{Z}/2$ quotient of the Stiefel manifold $V_{n,r}$ obtained by identifying any r-frame with its antipodal r-frame. There is an evident Hopf (or canonical) line bundle $\xi_{n,r}$ associated to this double covering, and these authors show that the order of its complexification $c\xi_{n,r}$ equals $2^{a(n,r)}$, where $a(n,r) = \min\{[(n-1)/2], a_0(n,r)\}$, and thus the real order of $\xi_{n,r}$ equals $2^{a(n,r)+\epsilon(n,r)}$, with $\epsilon(n,r) \in \{0,1\}$.

Definition (1.3). When $a(n,r) = a_0(n,r)$,(i.e. $a_0(n,r) \le [(n-1)/2]$), we say that r is in the upper range of n.

Section 3 of this paper explores a few technicalities related to this definition, which should help to elucidate the somewhat complicated definition of $a_0(n,r)$ as well as the meaning of the "upper range". As in [21], we now state an easy consequence of Theorem (1.2) and the above definition, which settles the value of $\epsilon(n,r)$ as equal to 0 approximately 70% of the time. More precisely, we have the following.

THEOREM (1.4). *For $n \equiv 0, \pm 1 (\operatorname{mod} 8)$, or for r in the upper range of n, $\epsilon(n,r) = 0$.*

Finally, as remarked above, §4 explores for the first time some of the applications of these theorems. These are to line bundles over the classifying spaces

$BO(k)$, to the tangent bundle of the Alexandrov line, to equivariant maps of Stiefel manifolds, and to span and immersions of projective Stiefel manifolds.

We remark that for any real line bundle ξ with order 2^a, the multiplicative height of $y = [\xi] - 1 \in \widetilde{KO}(X)$ is then determined as equal to $a + 1$. This follows because $\xi \otimes \xi \approx \varepsilon$, the trivial line bundle, implies $y^2 = -2y$ and thus $y^i = (-2)^{i-1}y, 1 \leq i$. We also remark that for the cases not covered by Theorem (1.4) it seems likely that $\epsilon(n,r) = 1$, cf. the Lower Range Conjecture (3.8).

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2. Proof of Theorem (1.2)

To begin the proof of Theorem (1.2), suppose that ξ is a real (resp. complex or quaternionic) line bundle over an arbitrary space X such that $n\xi$ admits $r \geq 1$ independent sections.

LEMMA (2.1). *With the above hypotheses on ξ, it admits a euclidean (resp. hermitian) metric and*

$$\beta \oplus r\varepsilon \approx n\xi \tag{2.2}$$

for some vector bundle β of rank $n - r$.

Proof. We consider the case $\mathbb{F} = \mathbb{R}$, the other cases have identical proofs with "euclidean" changed to "hermitian". It will be useful to recall (cf. [8], Ch. 3, Theorem 9.6, or [18], Theorem 3.3) that any short exact sequence $\alpha \rightarrowtail \gamma \twoheadrightarrow \beta$ of vector bundles (over an arbitrary base space X), where γ has a euclidean metric, splits (i.e. $\gamma \approx \alpha \oplus \beta$). In particular it follows that any subbundle and any quotient bundle of a euclidean vector bundle will also be euclidean. For the case at hand, denote the line bundle ξ by $p : E \to X$ and the trivial bundle $n\varepsilon$ by $p_1 \colon X \times \mathbb{R}^n \to X$. Since $r \geq 1$, $n\xi$ admits a nowhere zero section $s \colon X \to E(n\xi)$. Writing $E(n\xi) = \{(x, v_1, \dots, v_n) \colon x \in X, p(v_i) = x\}$, we have $s(x) = (x, s_1(x), \dots, s_n(x))$ where each s_i is a section of ξ. Although $s_i(x)$ may well vanish for various x, the fact that s is nowhere zero implies that $\{s_1(x), \dots, s_n(x)\}$ span the (1-dimensional) fibre $p^{-1}(x)$, for all $x \in X$. Thus the vector bundle morphism $n\varepsilon \to \xi$ defined by

$$f \colon X \times \mathbb{R}^n \longrightarrow E, \quad f(x, t_1, \dots, t_n) = \sum_{i=1}^{n} t_i s_i(x)\,,$$

has constant rank 1 and is therefore surjective. This implies that ξ is a quotient bundle of the trivial bundle $n\varepsilon$, and since the latter admits a euclidean metric so does ξ (using the remark at the beginning of this proof).

Finally, this also implies that $n\xi$ is euclidean, and therefore (2.2) follows by another application of the splitting principle. $\square$

Furthermore, using Exercise 3E in [18], we have the following.

COROLLARY (2.3). *With ξ as above and $\mathbb{F} = \mathbb{R}$, $\xi^2 = \xi \otimes \xi \approx \varepsilon$.*

Remarks. Lemma (2.1) above is peculiar to line bundles. Taking $\operatorname{rank}(\xi) = 2$ and $r \geq 2$, it is not hard to produce a counterexample to the corresponding statement of this lemma. The use of the term "hermitian metric" for the quaternionic case does not seem to be in the standard texts such as [8], [18], but can be found e.g. in [15]. Finally, as in Corollary (2.3), we take $\mathbb{F} = \mathbb{R}$ for the remainder of the paper.

Now recall the "total exterior power" $\lambda_t\colon R \to 1 + R[[t]]^+$, given by $\lambda_t(x) = \sum_{i\geq 0} \lambda^i(x)t^i$, for a λ-semiring (R, λ^i) and its extension to the ring completion $\lambda_t\colon \widetilde{R} \to 1 + \widetilde{R}[[t]]^+$, as defined in [8], Ch. 13, §1. We shall frequently apply the basic properties of λ_t as given in this reference, without further mention, and use the λ-ring $\widetilde{R} = KO(X)$. Also, we shall henceforth abuse notation slightly and write ξ for $[\xi] \in KO(X)$ (e.g. we write $\xi^2 = 1$ in $KO(X)$). So, from (2.2) we now obtain

$$(1 + \xi t)^n = \lambda_t(\beta) \cdot (1 + t)^r \,. \tag{2.4}$$

Also recall that in §1 we defined $y = \xi - 1$, and that $y^i = \pm 2^{i-1}y$.

Lemma (2.5). *In $KO(X)$ one has $0 = \binom{n}{i}2^{i-1}y$, $n - r < i$.*

Proof. Starting from (2.4), we use a trick due to Atiyah (cf. [8], p.175) and replace t by $t/(1 - t)$. With some minor simplifications we get

$$(1 + ty)^n/(1 - t)^n = \lambda_{t/(1-t)}(\beta) \cdot (1 - t)^{-r} \,.$$

Clearing the denominators by multiplying through by $(1 - t)^n$, this gives

$$(1 + ty)^n = \lambda_{t/(1-t)}(\beta) \cdot (1 - t)^{n-r} \,. \tag{2.6}$$

Now observe that $\lambda_{t/(1-t)}(\beta)$ is a polynomial of degree $n - r$ in $t/(1 - t)$, so after it is multiplied by $(1 - t)^{n-r}$ the right hand side of (2.6) becomes a polynomial in t of degree at most $n - r$. Hence all coefficients of the left hand side in powers $t^i, i > n - r$, must equal zero. This gives

$$0 = \binom{n}{i} y^i = \pm\binom{n}{i} 2^{i-1}y, \quad i > n - r \,.$$

□

The proof of Theorem (1.2) is now very easy. First note that it is equivalent to showing $2^{a_0(n,r)}y = 0 \in KO(X)$, as remarked in the first paragraph of the Introduction. Taking $i = n$ in Lemma (2.5) gives $2^{n-1}y = 0$, hence the order of y is 2^b for some $b \leq n - 1$. Indeed, applying the full strength of Lemma (2.5), we have

$$b \leq \min\{\nu_2\binom{n}{i} + i - 1 : n - r + 1 \leq i\} \,. \tag{2.7}$$

So the proof will be completed by simply showing that the right hand side of (2.7) equals $a_0(n, r)$.

Recall, from §1, the notational conventions (depending on parities)

$$n = \begin{cases} 2m \\ 2m + 1 \end{cases}, \quad n - r = s = \begin{cases} 2c \\ 2c + 1. \end{cases}$$

Equivalently, $m = [n/2], c = [s/2]$.

It will now be useful to recall Kummer's rule [14], that $\nu_2\binom{p}{q} = \alpha(q) + \alpha(p-q) - \alpha(p)$, where $\alpha(t)$ equals the number of 1's in the binary expansion of a positive integer t. Using this it is readily seen that $\nu_2\binom{m}{j} = \nu_2\binom{2m}{2j} = \nu_2\binom{2m+1}{2j}$. We shall always set $i = 2j$ in the following four cases.

For $n = 2m$ even and r odd (so $s = 2c+1$), $\nu_2\binom{m}{j} = \nu_2\binom{n}{i}$ and the range $c+1 \le j$ is equivalent to $2c+2 = s+1 = n-r+1 \le 2j = i$. Thus the right hand side of (2.7) is identical to $a_0(n,r)$ in this case.

The proof for $n = 2m$ even and r even (so now $s = 2c$) is identical except that one now finds $n-r+2 \le i$, i.e. the term for $i = n-r+1$ is missing. However, this is immaterial since Kummer's rule shows that

$$\nu_2\binom{2m}{2c+2} + 2c + 1 \le \nu_2\binom{2m}{2c+1} + 2c\,,$$

so the missing term does not affect the minimum.

For $n = 2m+1$ odd and r even (so $s = 2c+1$), using $\nu_2\binom{m}{j} = \nu_2\binom{n}{i}$ shows that the two minimums are identical, as in the n even, r odd case.

Finally, for $n = 2m+1$ odd and r odd (so $s = 2c$), we have $2j - 1 + \nu_2\binom{m}{j} = i - 1 + \nu_2\binom{n}{i}$, and $c+1 \le j$ is equivalent to $n-r+2 \le i$. However, in the right hand side of (2.7) the term for $i = n-r+1$ must also be taken into account, namely

$$\nu_2\binom{n}{n-r+1} + n - r = \nu_2\binom{2m+1}{2c+1} + 2c = \nu_2\binom{m}{c} + 2c\,.$$

So again this equals $a_0(n,r)$ in this (final) case. □

3. Lower and upper range

Throughout this section $1 \le r \le n$ and $m, s, c, a_0(n,r)$ are defined as in (1.1). The definition of $a_0(n,r)$ given in §1 is not quite identical to the corresponding numerical functions defined in [2] and [4], but the following proposition implies that the two definitions are in fact the same. The proof involves an elementary application of Kummer's rule, and is omitted.

Proposition (3.1). *One has*

$$a_0(n,r) = \begin{cases} \min\{2j-1+\nu_2\binom{m}{j} : c+1 \le j \le s\}, & nr \ \text{ even},\\ \min\{2c+\nu_2\binom{m}{c}, 2j-1+\nu_2\binom{m}{j} : c+1 \le j \le s\}, & nr \ \text{ odd}.\end{cases}$$

Remark. Proposition (3.1) is vacuously true in case $m \le s$.

Proposition (3.2). *For $1 \le r \le n-2$, one has $a_0(n,r+1) \le a_0(n,r)$.*

Proof. From Definition (1.1) this is trivial for n (and hence nr) even, since increasing r by 1 either fixes c or decreases c by 1. The proof is completed by observing that for n, r both odd we have

$$\begin{aligned} a_0(n,r+1) &= \min\{2j-1+\nu_2\binom{m}{j} : c \le j\}\\ &\le \min\{2c+\nu_2\binom{m}{c}, 2j-1+\nu_2\binom{m}{j}\} = a_0(n,r)\,,\end{aligned}$$

while for n odd and r even we have

$$\begin{aligned} a_0(n,r+1) &= \min\{2c+\nu_2\binom{m}{c}, 2j-1+\nu_2\binom{m}{j} : c+1 \le j\} \\ &\le \min\{2j-1+\nu_2\binom{m}{j} : c+1 \le j\} = a_0(n,r)\,. \end{aligned}$$

□

Recalling now that $a(n,r) = \min\{[(n-1)/2], a_0(n,r)\}$ and Definition (1.3) of the upper range, we have the following three corollaries, the latter two justifying the nomenclature "upper range" and "lower range".

Corollary (3.3). $a(n,r+1) \le a(n,r)$.

Corollary (3.4). *If r is in the upper range of n, then so are $r+1, r+2, \ldots, n$.*

Similarly, defining the lower range of n as the complement of the upper range, we have the following.

Corollary (3.5). *If r is in the lower range of n, then so are $r-1, r-2, \ldots, 1$.*

The next lemma gives an obvious lower bound for $a_0(n,r)$. The proof is clear and is omitted.

Lemma (3.6). *One has*

$$a_0(n,r) \ge \begin{cases} 2c+1, & nr \text{ even}, \\ 2c, & nr \text{ odd}. \end{cases}$$

A few observations follow that should give a better intuitive grasp of the upper range, i.e. when $a_0(n,r) \le [(n-1)/2]$. Notice that $a_0(n,r) \le 2c+1+\nu_2\binom{m}{c+1}$ always holds. Neglecting the $\nu_2\binom{m}{c+1}$ term, which is generally relatively small, we see that $2c+1 \le [(n-1)/2]$ approximately suffices, which is in turn readily seen to be implied by $r \ge n/2+2$. Thus the upper range, in general, starts at approximately $r = [n/2]+3$, and Lemma (3.7) below shows that it cannot start as low as $r = [(n+1)/2]$. As an example, for $n = 199, 200, 201, 202, 203, 204$, the upper range is respectively $r \ge 102, 105, 105, 105, 105, 105$.

Our final lemma gives a few further properties of the upper and lower ranges. Since it is not needed in the sequel the proofs (which are simple except for (d), which involves some number theory) are not given.

Lemma (3.7). (a) *The lower range always contains $r = 1$,*

(b) *The upper range, apart from $n = 4$, always contains $r = n-1$,*

(c) *for any n, $r = m$ is in the lower range, while for $n = 2m+1 \ne 2^t - 1$, $r = m+1$ is also in the lower range,*

(d) *if $n \ge 2^{c+2}+1$, then r is in the upper range.*

The proof of Theorem (1.4) is based on Theorem (1.2), the definitions of the upper and lower ranges, and the fact that the projective Stiefel manifold $X_{n,r}$ together with its associated Hopf line bundle $\xi_{n,r}$ are universal for spaces having a line bundle such that n times the line bundle admits r independent sections. We omit this proof since it can be found in [21], and close this section with a conjecture on the lower range.

Lower Range Conjecture (3.8). For $n \not\equiv 0, \pm 1 (\bmod 8)$ and for r in the lower range of n, $\epsilon(n, r) = 1$.

This conjecture has been verified for $n \leq 8$, is true for $r = 1$ (cf. [1]), and is also obviously true in the Hurwitz-Radon range $r \leq \rho(n)$ since the fibration $X_{n,r} \to \mathbb{R}P^{n-1}$ has a cross-section in this range. Other cases have been verified using the Atiyah-Hirzebruch spectral sequence, but the information at present is far from complete.

4. Applications

In this section we explore various applications of Theorems (1.2) and (1.4), as well as the idea of the lower and upper range. The first application is really just a corollary of Theorem (1.2) and is stated without proof.

PROPOSITION (4.1). *Let ξ be a real line bundle over a space X, and suppose $[\xi] - 1$ has infinite order in $KO(X)$ (equivalently $[c\xi] - 1$ has infinite order in $K(X)$). Then, for any $n \geq 1$, $n\xi$ admits no nowhere-zero section.*

Remark. Examples of such bundles are the Hopf line bundles over the infinite Grassmann manifolds $BO(k) = G_k(\mathbb{R}^\infty)$ (cf. [8], p.95). In this case, Proposition (4.1) could also be proved using Stiefel-Whitney classes.

The second application has the same conclusion as Proposition (4.1), but here the line bundle is the tangent bundle of the Alexandrov ("long") line. The definition of this space is given in most topology books, for our purposes it suffices to recall that it is a smooth connected 1-dimensional manifold that is not paracompact.

PROPOSITION (4.2). *Let ξ be the tangent bundle of the Alexandrov line $\mathcal{L}$. Then for any $n \geq 1$, $n\xi$ admits no nowhere-zero section.*

Proof. If, on the contrary, $n\xi$ admitted a nowhere-zero section, then by Lemma (2.1) ξ admits a euclidean metric, which in this case is the same as a riemannian metric since we are dealing with the tangent bundle of $\mathcal{L}$. But this implies that $\mathcal{L}$ is metrizable (cf. [16], pp.45-46, or [17]), which is impossible since $\mathcal{L}$ is not paracompact (also see [18], Problem 2-D). □

We also remark here that since ξ admits no euclidean metric, ξ^2 must be non-trivial (Problem 3E in [18]). Furthermore, both this observation and Proposition (4.2) remain true for any differentiable structure on $\mathcal{L}$ (there are uncountably many inequivalent ones, cf. [9], [19]).

The third application is related to Randall's work [20] on equivariant maps of Stiefel manifolds. Recall that a map $V_{m,r} \xrightarrow{f} V_{n,s}$ is $(\mathbb{Z}/2)$-equivariant if $f(-a_1, \ldots, -a_r) = -f(a_1, \ldots, a_r)$ for any r-frame $(a_1, \ldots, a_r) \in V_{m,r}$. The main result of [20] asserts that there is no $(\mathbb{Z}/2)$-equivariant map of this type if $m > n$ and $r \leq s$. The following theorem is very easy to prove and strengthens Randall's result when r is in the lower range for m (recall from Lemma (3.7) that this includes all $r \leq m/2$, i.e. at least half the r-values for any m).

Theorem (4.3). *If r is in the lower range of m then there exists no $(\mathbb{Z}/2)$-equivariant map $V_{m,r} \xrightarrow{f} V_{n,s}$ when*
(i) *m odd and $m > n$,*
(ii) *m even and $m > n+1$.*

Proof. Any f as above induces a map $X_{m,r} \xrightarrow{g} X_{n,s}$ with $g^*(\xi_{n,s}) \approx \xi_{m,r}$. To get a contradiction we simply compare the (complex) order of these two line bundles, in fact $a(m,r) = [(m-1)/2]$ by hypothesis and $a(n,s) \leq [(n-1)/2]$ always holds. Either assumption (i) or (ii) then implies $a(n,s) \leq [(n-1)/2] < [(m-1)/2] = a(m,r)$, which is impossible if such a g exists. □

Remark. Assuming the Lower Range Conjecture one could also add the case (iii): $m \equiv 2 \pmod 8$ and $m > n$. To see this one simply notes that $\phi(m-1) > \phi(m-2)$ here, and $2^{\phi(m-1)}$ is the real order of $\xi_{m,r}$ (by the Lower Range Conjecture) while the real order of $\xi_{n,s}$ divides $2^{\phi(m-2)}$, giving a contradiction.

Our final two applications are to the projective Stiefel manifolds $X_{n,r}$, in particular to the span and immersions of these manifolds (introductory accounts of the basic properties of these manifolds can be found in [11], [12], and [23]). First let us consider $\mathrm{span}(X_{n,r})$, where by the span of any smooth manifold M is meant the maximal number of pointwise linearly independent tangent vector fields on M, i.e. the maximal number of independent sections of its tangent bundle τ_M. By the stable span of M, denoted $\mathrm{span}^0(M)$, we mean one less than the maximal number of independent sections of $\tau_M \oplus \varepsilon$. Clearly $\mathrm{span}(M) \leq \mathrm{span}^0(M)$. It will also be convenient to use the geometric dimension and stable geometric dimension of a vector bundle here, in particular, letting M have dimension d, we set $\mathrm{gd}(M) = d - \mathrm{span}(M) \geq \mathrm{gd}^0(M) = d - \mathrm{span}^0(M)$.

It would be too lengthy to explore all the implications of the theorems in §1 to $\mathrm{span}(X_{n,r})$ in any amount of detail here, but we shall illustrate their power with one easy lemma. Further applications of these theorems, in which the exact span of $X_{n,r}$ is calculated for $n - r \leq 3$, may be found in [24].

Lemma (4.4). *Let $nr \equiv q \ (\mathrm{mod} 2^{a_0(n,r)})$, then $\mathrm{gd}^0(X_{n,r}) \leq q$.*

Proof. It is well known that there is a stable equivalence $\tau_{n,r} \sim nr\xi_{n,r}$ (cf. [15], [22]). By Theorem (1.2), $2^{a_0(n,r)}\xi_{n,r} \sim 0$, and it follows that $\tau_{n,r} \sim q \cdot \xi_{n,r}$, which proves the lemma. □

Remark. This lemma can be sharpened slightly by using $a(n,r) + \epsilon(n,r)$ (cf. the remark before Definition (1.3)) instead of $a_0(n,r)$. Often, it can also be sharpened by using the "orthogonal complement bundle" $\beta_{n,r}$ (of rank $n-r$), cf. [21], [24].

We now turn to our final application, immersions of $X_{n,r}$. For this application, according to the Hirsch theorem (cf. [7] [18]), one needs lower (respectively upper) bounds on $\mathrm{gd}^0(\nu)$, where $\nu = \nu_{n,r}$ is the (stable) normal bundle. Using the isomorphism $\tau \oplus \lambda^2\beta \approx \binom{n}{2}\varepsilon$ (cf. [15]), or equivalently a theorem of Hiller [6], we see $\mathrm{gd}^0\nu \leq \mathrm{rank}(\lambda^2\beta) = \binom{n-r}{2}$, and this furnishes strong upper bounds for the immersion codimension, especially for r close to n. We shall call this the

Hiller bound. For example, for $r = n - 2$, it gives an immersion of $X_{n,n-2}$ in codimension 1, which is clearly best possible (the same is true for $X_{n,n-1}$, which is parallelizable for all n). Extensive calculations by the first named author in [3] and elsewhere, of upper bounds for $\mathrm{gd}^0\nu$, using both Stiefel-Whitney classes and the γ-operations in K-theory, suggest that the Hiller bound is not best possible even for small $n - r$ (as $n - r$ becomes larger in comparision to n it is clearly an inefficient bound). We now state a lemma quite analogous to Lemma (4.4) and will give two examples that show that the Hiller bound is indeed not best possible, even for $n - r \leq 5$. In fact, in both cases the Stiefel-Whitney classes will suffice to show that the best possible immersion dimension has been found.

Lemma (4.5). *Let* $-nr \equiv q(\mathrm{mod} 2^{a_0(n,r)}), 0 \leq q < 2^{a_0(n,r)}$. *Then* $X_{n,r}$ *immerses in codimension* $\max\{1, q\}$.

Proof. The proof is similar to that of Lemma (4.4), this time using the Hirsch theorem and the stable equivalences $\nu \sim -\tau \sim -nr\xi$. □

Example (4.6). (a) $X_{n,n-4}$, $n = 2^t(2p+1) + 2$, $p \geq 0$, $t \geq 3$, immerses in codimension 4.

(b) $X_{n,n-5}$, $n = 2^t(4p+1) + 6$, $p \geq 0$, $t \geq 3$, immerses in codimension 2.

Note that the Hiller bounds in these examples are $6, 10$ respectively. The proof in either case is an elementary application of Lemma (4.5) to get an upper bound for $\mathrm{gd}^0\nu$ (i.e. the immersion codimension), and use of the Stiefel-Whitney classes for the lower bound. We omit the details, and simply remark that in (a) one finds $a_0 = t+3, N = 2^t, w = 1 + x^4$,while in (b) $a_0 = 5, N = 6$, and $w = 1 + x^2$, where x generates $H^1(X_{n,r}; \mathbb{Z}/2)$ and N is its height ($N = \min\{j : j \geq n - r + 1, \binom{n}{j}$ is odd$\}$, cf. [5]).

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LARGE DEVIATIONS FOR MARKOV CHAINS IN A RANDOM SCENERY WITH COMPACT SUPPORT

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ABSTRACT. Let (Z_n) be a Markov chain in a random scenery, namely $Z_n \equiv \sum_{k=1}^n \Theta_{J_k}$ where $J = (J_n)$ is an irreducible Markov chain taking values in a finite set E and $\Theta = (\Theta_x)_{x \in E}$ is a family of $\mathbb{R}^p$-valued random variables (for some $p \geq 1$) assumed to be independent of J. Then, assuming that Θ takes values on a compact set, we prove the large deviations principle of $(\frac{Z_n}{n})$ and, when $p = 1$, a large deviations estimate for the level crossing probabilities of (Z_n).

1. Introduction

This paper concerns large deviations so that let us start by recalling the basic definition of large deviations principle (LDP for short). More precisely we adapt the definition in the literature (see [2], page 4-5) to a sequence of T-valued random variables (Y_n), where T is a topological space: given a lower semicontinuous function $I : T \to [0, \infty]$ (called rate function), (Y_n) satisfies the LDP if the two following inequalities hold:

$$\liminf_{n\to\infty} \frac{1}{n} \log P(Y_n \in O) \geq - \inf_{x\in O} I(x) \quad (\forall O \text{ open});$$

$$\limsup_{n\to\infty} \frac{1}{n} \log P(Y_n \in C) \leq - \inf_{x\in C} I(x) \quad (\forall C \text{ closed}).$$

Let $J = (J_n)$ be an irreducible Markov chain with state space $E = \{1, \ldots, s\}$ and let (L_n) be the empiric laws defined by

$$L_n \equiv (L_n(x))_{x\in E} \quad \text{with} \quad L_n(x) \equiv \frac{1}{n} \sum_{k=1}^n 1_{J_n = x}.$$

Furthermore let $\Theta = (\Theta_x)_{x\in E}$ be a family of $\mathbb{R}^p$-valued random variables (for some $p \geq 1$) assumed to be independent of J; these random variables play the role of the random scenery (also called random landscape) and let us denote the law of Θ by μ. Finally let (Z_n) be defined by

$$Z_n \equiv \sum_{k=1}^n \Theta_{J_k} \equiv n \sum_{x\in E} \Theta_x L_n(x).$$

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In [6] there is a wide source of references concerning random sceneries and their asymptotic behaviour. In particular Theorem 1 in [6] provides the LDP of $(\frac{Z_n}{n\sqrt{n}})$ when Θ is a suitable centered Gaussian random field.

In this paper we recall some preliminaries in section 2 and we present the results in section 3. The first result (Proposition (3.1)) is the LDP of $(\frac{Z_n}{n})$ and it is proved as a consequence of Theorem 2.3 in [3]; in particular we refer to the concept of exponential equivalence and we employ a related result (see [2], Theorem 4.2.13). The second result (Proposition (3.5)) provides a large deviations estimate for the level crossing probabilities of (Z_n) when $p = 1$; such level crossing probabilities are

$$(P(T_b < \infty))_{b>0} \ \text{ defined by } \ T_b \equiv \inf\{n \geq 1 : Z_n > b\}.$$

In some sense the infimum of the family of Lundberg parameters which appears in Proposition (3.5) in this paper is the analogous of the infimum of the family of large deviations rate functions which appears in Theorem 2.3 in [3]. This analogy leads the author to think that Proposition (3.5) could be extended to a wide class of mixtures satisfying the hypotheses of Theorem 2.3 in [3].

We point out that all the results presented in this paper agree with the well known large deviations results for Markov additive processes when Θ is a family of constant random variables. Indeed in such a case the support of μ is reduced to a single point $\overline{\theta}$ where

$$\overline{\theta} = (\overline{\theta}_x)_{x\in E} \in (\mathbb{R}^p)^E$$

and we have a particular discrete time Markov additive process with finite environment's state space according to the description in [1] (page 40, discrete time case). More precisely, as far as the conditional distribution of Z_1 given (J_0, J_1) is concerned, we have

$$P(Z_1 \in dz | J_0 = x, J_1 = y) \equiv \delta_{\overline{\theta}_y}(dz)$$

where $\delta_{\overline{\theta}_y}$ is the Dirac delta concentrated on $\overline{\theta}_y$. We also recall that Corollary 1 in [5] provides a generalization of the description in [1] to the case in which E is not necessarily finite and the additive component takes values in a Hilbert space.

2. Preliminaries

Throughout this paper we denote the scalar product in $\mathbb{R}^p$ by $\langle \cdot, \cdot \rangle$ and the infinity norm in $\mathbb{R}^p$ by $\| \cdot \|$; namely, for $\theta, \eta \in \mathbb{R}^p$ with $\theta = (\theta(i))_{i\in\{1,\ldots,p\}}$ and $\eta = (\eta(i))_{i\in\{1,\ldots,p\}}$, let us set

$$\langle \theta, \eta \rangle = \sum_{i=1}^{p} \theta(i)\eta(i) \ \text{ and } \ \|\theta\| = \max\{|\theta(i)| : i \in \{1, \ldots, p\}\}.$$

(2.1) On large deviations. Let $\Pi = (\pi(x, y))_{x,y\in E}$ be the transition matrix of J which is an irreducible matrix since J is an irreducible Markov chain. Then, for each $\alpha \in \mathbb{R}^p$ and for each $\theta = (\theta_x)_{x\in E} \in (\mathbb{R}^p)^E$, let us consider the matrix $\Pi^\theta_\alpha = (\pi^\theta_\alpha(x, y))_{x,y\in E}$ defined by

$$\pi^\theta_\alpha(x, y) \equiv \pi(x, y)e^{\langle \alpha, \theta_y \rangle}.$$

In general these matrices are irreducible since Π is irreducible. Thus let $\rho(\Pi_\alpha^\theta)$ be the Perron Frobenius eigenvalue of the matrix Π_α^θ and let Λ_θ be the function defined as follows:

$$\Lambda_\theta(\alpha) \equiv \log \rho(\Pi_\alpha^\theta).$$

Then it is known (see [2], Theorem 3.1.2) that $(\frac{1}{n}\sum_{k=1}^n \theta_{J_k})$ satisfies the LDP with rate function λ_θ defined by ($z \in \mathbb{R}^p$)

$$\lambda_\theta(z) \equiv \sup_{\alpha \in \mathbb{R}^p} [\langle \alpha, z\rangle - \Lambda_\theta(\alpha)]. \tag{2.1.1}$$

(2.2) On level crossing probabilities. In this subsection we recall some preliminaries in view of the next Proposition (3.5). Then we assume $p = 1$ and let $\theta = (\theta_x)_{x\in E} \in \mathbb{R}^E$ be arbitrarily fixed.

For each fixed $\alpha \in \mathbb{R}$ we can still apply Perron Frobenius theorem for the matrix Π_α^θ as before. Then there exists a (unique up to a constant multiple) positive eigenvector $(h_x^{(\theta)}(\alpha))_{x\in E}$ of Π_α^θ corresponding to the eigenvalue $\rho(\Pi_\alpha^\theta)$ and, for each $n \geq 1$, we have

$$\sum_{y\in E} \mathbb{E}[e^{\alpha \sum_{k=1}^n \theta_{J_k}} 1_{J_n=y} | J_0 = x] h_y^{(\theta)}(\alpha) = (\rho(\Pi_\alpha^\theta))^n h_x^{(\theta)}(\alpha) = e^{n\Lambda_\theta(\alpha)} h_x^{(\theta)}(\alpha)$$
$$(\forall x \in E)\,.$$

Then we can introduce a family of probability measures $(P_\alpha^\theta)_{\alpha\in\mathbb{R}}$ defined as follows. In general, for each $n \geq 1$, P is absolutely continuous with respect to P_α^θ on the σ-field generated by $\{J_0, J_1, \ldots, J_n\}$ with density $\ell_{\alpha,\theta}^{(n)}$ defined by

$$\ell_{\alpha,\theta}^{(n)} \equiv \exp(-\alpha \sum_{k=1}^n \theta_{J_k} + n\Lambda_\theta(\alpha)) \frac{h_{J_0}^{(\theta)}(\alpha)}{h_{J_n}^{(\theta)}(\alpha)}. \tag{2.2.1}$$

Now let us concentrate our attention on the level crossing probabilities of $(\sum_{k=1}^n \theta_{J_k})$, namely the probabilities $(P(T_b^{(\theta)} < \infty))_{b>0}$ defined by

$$T_b^{(\theta)} \equiv \inf\{n \geq 1 : \sum_{k=1}^n \theta_{J_k} > b\}.$$

Furthermore let us consider the condition

$$(\mathbf{H})_\theta: \text{ there exists } w_\theta > 0 \text{ such that } \Lambda_\theta(w_\theta) = 0 \text{ and } \Lambda_\theta'(w_\theta) > 0.$$

Then Theorem 3.1 in [8] shows that, if $(\mathbf{H})_\theta$ holds, we have

$$\lim_{b\to\infty} \frac{1}{b} \log P(T_b^{(\theta)} < \infty) = -w_\theta = -\inf_{z>0} \frac{\lambda_\theta(z)}{z}. \tag{2.2.2}$$

We remark that (2.2.2) agrees with the more general results in [4] and, in general, the value w_θ is called Lundberg parameter of $(\sum_{k=1}^n \theta_{J_k})$. Furthermore, if we denote the unique stationary distribution of J by $(\mathcal{L}(x))_{x\in E}$, we can say that $(\mathbf{H})_\theta$ holds if and only if $\sum_{x\in E} \theta_x \mathcal{L}(x) < 0$ and $\theta_{x_0} > 0$ for some $x_0 \in E$.

3. The results

Let $S(\mu)$ be the support of μ. The idea is to employ Theorem 2.3 of [3] so that in the next propositions we always assume that $S(\mu)$ is a compact set.

Proposition (3.1). Assume $S(\mu)$ is compact. Then $(\frac{Z_n}{n})$ satisfies the LDP with rate function λ defined by

$$\lambda(z) \equiv \inf\{\lambda_\theta(z) : \theta \in S(\mu)\}.$$

Proof. We prove the proposition employing Theorem 2.3 of [3], so that we have to check the two following conditions:

(EC) let $(\theta^{(n)})$ be a sequence in $(\mathbb{R}^p)^E$ (where in general $\theta^{(n)} = (\theta_x^{(n)})_{x\in E}$) such that

$$\lim_{n\to\infty} \max_{x\in E} \|\theta_x^{(n)} - \theta_x\| = 0 \tag{3.2}$$

for some $\theta = (\theta_x)_{x\in E} \in (\mathbb{R}^p)^E$; then $(\frac{1}{n}\sum_{k=1}^n \theta_{J_k}^{(n)})$ satisfies the LDP with rate function λ_θ.

(JLSC) the function $(\theta, z) \mapsto \lambda_\theta(z)$ is lower semicontinuous.

Proof of **(EC)**. By Theorem 4.2.13 in [2], we only need to check that $(\frac{1}{n}\sum\limits_{k=1}^n \theta_{J_k}^{(n)})$ and $(\frac{1}{n}\sum\limits_{k=1}^n \theta_{J_k})$ are exponentially equivalent in the sense of Definition 4.2.10 in [2], namely

$$\limsup_{n\to\infty} \frac{1}{n} \log P(\|\frac{1}{n}\sum_{k=1}^n \theta_{J_k}^{(n)} - \frac{1}{n}\sum_{k=1}^n \theta_{J_k}\| > \delta) = -\infty \quad (\forall \delta > 0). \tag{3.3}$$

Thus let $\delta > 0$ be arbitrarily fixed. Then we have

$$\|\frac{1}{n}\sum_{k=1}^n \theta_{J_k}^{(n)} - \frac{1}{n}\sum_{k=1}^n \theta_{J_k}\| = \|\sum_{x\in E}(\theta_x^{(n)} - \theta_x)L_n(x)\| \le$$

$$\sum_{x\in E}\|\theta_x^{(n)} - \theta_x\|L_n(x) \le \max_{x\in E}\|\theta_x^{(n)} - \theta_x\|$$

whence we obtain the following inclusion between events:

$$\{\|\frac{1}{n}\sum_{k=1}^n \theta_{J_k}^{(n)} - \frac{1}{n}\sum_{k=1}^n \theta_{J_k}\| > \delta\} \subset \{\max_{x\in E}\|\theta_x^{(n)} - \theta_x\| > \delta\}.$$

Moreover for n large enough the deterministic event

$$\{\max_{x\in E}\|\theta_x^{(n)} - \theta_x\| > \delta\}$$

does not occur by (3.2). In conclusion for n large enough we have

$$0 \le P(\|\frac{1}{n}\sum_{k=1}^n \theta_{J_k}^{(n)} - \frac{1}{n}\sum_{k=1}^n \theta_{J_k}\| > \delta) \le P(\max_{x\in E}\|\theta_x^{(n)} - \theta_x\| > \delta) = 0$$

and (3.3) holds since $\delta > 0$ is arbitrarily fixed.

Proof of **(JLSC)**. Let $(\theta_0, z_0) \in (\mathbb{R}^p)^E \times \mathbb{R}^p$ be arbitrarily fixed. Then we have to check

$$\liminf_{(\theta,z)\to(\theta_0,z_0)} \lambda_\theta(z) \geq \lambda_{\theta_0}(z_0). \tag{3.4}$$

We point out that, in general, we have the following inequality

$$\lambda_\theta(z) \geq \langle \alpha, z\rangle - \Lambda_\theta(\alpha)$$

by (2.1.1). We remark that the latter right hand side is a regular function of (θ, z); indeed, reasoning as in the final part of proof of Theorem 3.1.2 in [2] which explains the regularity of Λ_θ for each fixed θ (see Theorem 7.7.1 in [7] for some detail), $\Lambda_\theta(\alpha)$ is a regular function of θ for each fixed α. Then we have

$$\liminf_{(\theta,z)\to(\theta_0,z_0)} \lambda_\theta(z) \geq \langle \alpha, z_0\rangle - \Lambda_{\theta_0}(\alpha) \quad (\forall \alpha \in \mathbb{R}^p);$$

thus (3.4) holds by taking the supremum with respect to α in the right hand side. □

Proposition (3.5). Assume the following conditions hold: $p = 1$; $S(\mu)$ is compact; $(\mathbf{H})_\theta$ holds for each $\theta \in S(\mu)$;

$$\sup_{\theta\in S(\mu)} \max_{x,y\in E} \frac{h_x^{(\theta)}(w_\theta)}{h_y^{(\theta)}(w_\theta)} < \infty. \tag{3.6}$$

Then

$$\lim_{b\to\infty} \frac{1}{b} \log P(T_b < \infty) = -\inf_{\theta\in S(\mu)} w_\theta.$$

Proof. We have to check

$$\liminf_{b\to\infty} \frac{1}{b} \log P(T_b < \infty) \geq -\inf_{\theta\in S(\mu)} w_\theta$$

and

$$\limsup_{b\to\infty} \frac{1}{b} \log P(T_b < \infty) \leq -\inf_{\theta\in S(\mu)} w_\theta.$$

Proof of lower bound. Let $b, z > 0$ be arbitrarily fixed and let us set

$$n_z(b) = [\frac{b}{z}] + 1$$

where $[\cdot]$ denotes the integer part; then we have

$$P(T_b < \infty) \geq P(Z_{n_z(b)} > b)$$

since $\{T_b < \infty\} \supset \{Z_{n_z(b)} > b\}$ and we obtain

$$\frac{1}{b} \log P(T_b < \infty) \geq \frac{n_z(b)}{b} \frac{1}{n_z(b)} \log P(\frac{Z_{n_z(b)}}{n_z(b)} > \frac{b}{n_z(b)}) \geq$$

$$\frac{n_z(b)}{b} \frac{1}{n_z(b)} \log P(\frac{Z_{n_z(b)}}{n_z(b)} > z).$$

Thus, since $\lim_{b\to\infty} \frac{n_z(b)}{b} = \frac{1}{z}$ and $(\frac{Z_n}{n})$ satisfies the LDP with rate function λ in Proposition (3.1), we have

$$\liminf_{b\to\infty} \frac{1}{b} \log P(T_b < \infty) \geq -\frac{\lambda(z^+)}{z}$$

whence we obtain

$$\liminf_{b\to\infty}\frac{1}{b}\log P(T_b<\infty)\geq -\inf_{z>0}\frac{\lambda(z^+)}{z}.$$

Thus we complete the proof of the lower bound showing that

$$\inf_{z>0}\frac{\lambda(z^+)}{z}\leq \inf_{\theta\in S(\mu)} w_\theta. \tag{3.7}$$

In order to check (3.7) let $\theta\in S(\mu)$ be arbitrarily fixed; then we have

$$\inf_{z>0}\frac{\lambda(z^+)}{z}\leq \inf_{z>0}\frac{\lambda_\theta(z^+)}{z}=\inf_{z>0}\frac{\lambda_\theta(z)}{z}=w_\theta \tag{3.8}$$

indeed the inequality holds by construction, the first equality holds since λ_θ is convex and lower semicontinuous and the second equality follows from (2.2.2). In conclusion we obtain (3.7) by taking the infimum of w_θ with respect to θ in (3.8).

Proof of upper bound. Let $b>0$ be arbitrarily fixed. Then

$$P(T_b<\infty)=\int_{\mathbb{R}^E}P(T_b^{(\theta)}<\infty)\mu(d\theta)=\int_{S(\mu)}P(T_b^{(\theta)}<\infty)\mu(d\theta).$$

Now, for each fixed $\theta\in S(\mu)$, let us employ the densities $\ell^{(n)}_{\alpha,\theta}$ in (2.2.1) as in the proof of Theorem 3.1 in [8]. Then in general we have

$$P(T_b^{(\theta)}<\infty)=\mathbb{E}_{P^\theta_\alpha}[\ell^{(T_b^{(\theta)})}_{\alpha,\theta}1_{T_b^{(\theta)}<\infty}]$$

and choosing $\alpha=w_\theta$ we obtain

$$P(T_b^{(\theta)}<\infty)=\mathbb{E}_{P^\theta_{w_\theta}}[\ell^{(T_b^{(\theta)})}_{w_\theta,\theta}1_{T_b^{(\theta)}<\infty}]\leq$$

$$e^{-w_\theta b}\mathbb{E}_{P^\theta_{w_\theta}}[\frac{h^{(\theta)}_{J_0}(w_\theta)}{h^{(\theta)}_{J_{T_b^{(\theta)}}}(w_\theta)}1_{T_b^{(\theta)}<\infty}]\leq e^{-b\inf_{\theta\in S(\mu)}w_\theta}\sup_{\theta\in S(\mu)}\max_{x,y\in E}\frac{h^{(\theta)}_x(w_\theta)}{h^{(\theta)}_y(w_\theta)}$$

since we have $\sum_{k=1}^{T_b^{(\theta)}}\theta_{J_k}>b$ by construction and $\Lambda_\theta(w_\theta)=0$ by $(\mathbf{H})_\theta$. In conclusion

$$\limsup_{b\to\infty}\frac{1}{b}\log P(T_b<\infty)\leq -\inf_{\theta\in S(\mu)} w_\theta$$

follows from (3.6). □

We remark that (3.6) holds when $S(\mu)$ is a finite set (in such a case $S(\mu)$ is also a compact set) and in the proof of the upper bound we have $P^\theta_{w_\theta}(T_b^{(\theta)}<\infty)=1$ for each fixed $\theta\in S(\mu)$.

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ON SOME GENERALIZATIONS OF COMPACTNESS IN SPACES $C_p(X,2)$ AND $C_p(X,\mathbb{Z})$, ERRATUM

A. CONTRERAS-CARRETO AND A. TAMARIZ-MASCARÚA

Every topological space considered in this note is a Tychonoff space.

In [CT], it was proved that for a zero-dimensional normal space X, the space of all continuous functions defined on X and with integer values $C_p(X,\mathbb{Z})$ is σ-pseudocompact if and only if X is an Eberlein compact space. Afterwards, we gave an example, suggested by one of the referees of [CT], of a zero-dimensional space Y such that Y is not normal and $C_p(Y,\mathbb{Z})$ is σ-pseudocompact. The argumentations made there in order to prove the properties of Y contain some mistakes and they are not sufficiently clear. In the following, we rewrite some paragraphs which appear at the end of [CT] obtaining a correct and clearer demostration of the claims made about Y.

Recall that, for an $n \in \mathbb{N}$, $B \subset X$ is C_n-*embedded* in X if every continuous function $f : B \to \{0,...,n\}$ has a continuous extension to all X, where $\{0,...,n\}$ is the discrete space of cardinality $n+1$. It is not difficult to prove that $B \subset X$ is C_2-embedded in X if and only if B is C_n-embedded in X for every $n \in \mathbb{N}$.

There exists an infinite zero-dimensional pseudocompact space Y in which all countable subsets are closed and C_2-embedded in Y. Since Y is pseudocompact, $C_p(Y,\mathbb{Z}) \cong C_p(Y,\mathbb{N}) = \bigcup_{n\in\mathbb{N}} C_p(Y,n)$.

The space $C_p(Y,n)$ is a dense subset of n^Y; so, if $f : C_p(Y,n) \to \mathbb{R}$ is a continuous function, then, by Theorem 1 in [A1], there are a countable subset $B \subset Y$ and a continuous function $g : \pi_B(C_p(Y,n)) \to \mathbb{R}$ such that $f = g \circ \pi_B$, where $\pi_B : n^Y \to n^B$ is the canonical projection. Since B is a countable subset of Y, B is C_2-embedded in Y; so $\pi_B(C_p(Y,n)) = n^B$. Since n^B is compact, f has to be bounded. Thus, $C_p(Y,n)$ is pseudocompact for every $n \in \mathbb{N}$. Therefore, $C_p(Y,\mathbb{Z})$ is σ-pseudocompact.

Moreover, Y is not normal because normality plus pseudocompactness imply countable compactness, and every countably compact space contains some countable non-closed subset, contrary to one of the properties of Y.

Now, let us see how we can obtain such a space Y. We modify the example due to D.B. Shakhmatov of a pseudocompact space which has all its countable subsets closed and C^*-embedded (see [A2], Example I.2.5 and [S]). The example of Shakhmatov is connected but his proof can be carried out identically to find the space Y as a dense subspace of $2^{\mathfrak{c}}$. Indeed, let Σ be the Σ-product in $2^{\mathfrak{c}}$ with center at zero ($\Sigma = \{x \in 2^{\mathfrak{c}} : |\{\alpha \in \mathfrak{c} : \pi_\alpha(x) \neq 0\}| \leq \aleph_0\}$). Then $|\Sigma| = \mathfrak{c}$, so we can enumerate it as $\Sigma = \{g_\alpha : \alpha < \mathfrak{c}\}$ so that each $g \in \Sigma$ occurs $\mathfrak{c}$-many times in this enumeration. Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of all countably infinite subsets of $\mathfrak{c}$ such that each countably infinite $A \subset \mathfrak{c}$ also occurs $\mathfrak{c}$-many times in this enumeration. For each $\alpha < \mathfrak{c}$, we define a point $x_\alpha \in 2^{\mathfrak{c}}$ by the following conditions:

(i) $x_\alpha(\beta) = g_\alpha(\beta)$ for every $\beta \leq \alpha$,
(ii) $x_\alpha(\beta) = 0$ if $\beta > \alpha$ and $\alpha \notin A_\beta$,
(iii) $x_\alpha(\beta) = 1$ if $\beta > \alpha$ and $\alpha \in A_\beta$.

Then, the same reasoning as in Shakhmatov's paper [S] shows that $Y = \{x_\alpha : \alpha < \mathfrak{c}\}$ is a dense pseudocompact subspace of $2^{\mathfrak{c}}$ such that every countable subset of Y is closed and C_2-embedded in Y.

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