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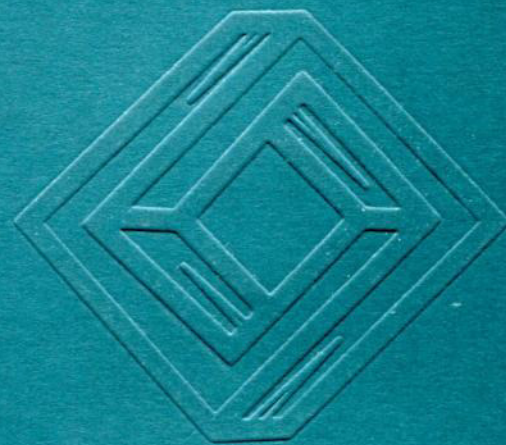
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Contenido

ARTÍCULOS DE INVESTIGACIÓN

A note on an additive problem with powers of a primitive root
V. C. García 1

Desigualdades y fórmulas asintóticas para sumas de potencias de primos
R. Jakimczuk 5

Structural invariants in crystallography and isometry of cyclotomic measures
G. Putinar 11

Branched coverings after Fox
J. M. Montesinos-Amilibia 19

On estimates for area functions of solutions to non-divergence parabolic equations
J. Rivera-Noriega 65

On the stochastic Aubry-Mather theory
R. Iturriaga and H. Sánchez-Morgado 91

Continúa/Continued on back cover

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MEXICANA



A NOTE ON AN ADDITIVE PROBLEM WITH POWERS OF A PRIMITIVE ROOT

VÍCTOR CUAUHTÉMOC GARCÍA

ABSTRACT. Let p be a prime number and g be a primitive root modulo p . We prove that any residue class modulo p is representable in the form $g^x - g^y \pmod{p}$ with $1 \leq x, y < 2^{5/4}p^{3/4}$.

1. Introduction

Let p be a prime number and g be a primitive root \pmod{p} . The problem of distribution of different arithmetical sequences \pmod{p} has been investigated in a number of papers, see for example [1–9]. M. Vâjaitu and A. Zaharescu [9] considered the problem of distribution of the set of differences

$$A := \{g^x - g^y \pmod{p} : 1 \leq x, y \leq N\}$$

where N is a positive integer, $N < p$. The problem, which according to [9] is due to A. Odlyzko, asks for which values of N the set A contains all residue system \pmod{p} . The conjecture is that one can take N to be as small as $p^{1/2+\varepsilon}$, for any positive ε and $p > p_0(\varepsilon) > 0$. From the result of Z. Rudnick and A. Zaharescu [8] it follows that one can take any integer $N > c_0 p^{3/4} \log p$, where $c_0 > 0$ is a suitable constant, in Odlyzko's problem. This result was improved to the range $N > cp^{3/4}$ with $c = 10$ by M. Z. Garaev and K. L. Kueh [5], and with $c = 4$ by S. V. Konyagin [6], p.99.

In this note we reduce the constant in this problem to $c = 2^{5/4}$.

THEOREM. *For any integer N , $N > 2^{5/4}p^{3/4}$, the set A contains the complete residue system modulo p .*

2. Proof of the Theorem

Our method is similar to the method of [5], where Vinogradov's estimate for the double exponential sum is used. Here we prove and use its version for the double sum of cosines.

LEMMA. *Let $(a, p) = 1$. Then for any complex numbers $\nu(i)$, $\rho(j)$, $0 \leq i, j \leq p-1$, the inequality*

$$\left| \sum_{n=0}^{p-1} \sum_{m=0}^{p-1} \nu(n)\rho(m) \cos 2\pi \frac{anm}{p} \right| \leq \sqrt{\frac{p}{2}} T(R_1 + R_2)$$

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holds, where

$$T = \sum_{n=0}^{p-1} |\nu(n)|^2, \quad R_1 = \sum_{m=0}^{p-1} |\rho(m)|^2, \quad R_2 = |\rho(0)|^2 + \sum_{m=1}^{p-1} \rho(m) \overline{\rho(p-m)}.$$

Proof. Let

$$S = \sum_{n=0}^{p-1} \sum_{m=0}^{p-1} \nu(n) \rho(m) \cos 2\pi \frac{anm}{p}.$$

According to the *Cauchy-Schwarz* inequality we have

$$\begin{aligned} |S|^2 &= \left| \sum_{n=0}^{p-1} \nu(n) \sum_{m=0}^{p-1} \rho(m) \cos 2\pi \frac{anm}{p} \right|^2 \leq \sum_{n=0}^{p-1} |\nu(n)|^2 \sum_{m=0}^{p-1} \left| \sum_{m=0}^{p-1} \rho(m) \cos 2\pi \frac{anm}{p} \right|^2 \\ &= T \sum_{n=0}^{p-1} \sum_{m_1=0}^{p-1} \sum_{m_2=0}^{p-1} \rho(m_1) \overline{\rho(m_2)} \cos 2\pi a \frac{nm_1}{p} \cos 2\pi a \frac{nm_2}{p} \\ &= \frac{1}{2} T \sum_{m_1=0}^{p-1} \sum_{m_2=0}^{p-1} \rho(m_1) \overline{\rho(m_2)} \left[\sum_{n=0}^{p-1} \cos 2\pi a \frac{n(m_1 - m_2)}{p} + \cos 2\pi a \frac{n(m_1 + m_2)}{p} \right]. \end{aligned}$$

Since

$$\sum_{n=0}^{p-1} \cos 2\pi a \frac{n(m_1 - m_2)}{p} = \begin{cases} p, & \text{if } m_1 = m_2, \\ 0, & \text{if } m_1 \neq m_2 \end{cases}$$

and

$$\sum_{n=0}^{p-1} \cos 2\pi a \frac{n(m_1 + m_2)}{p} = \begin{cases} p, & \text{if } m_1 + m_2 \equiv 0 \pmod{p}, \\ 0, & \text{if } m_1 + m_2 \not\equiv 0 \pmod{p} \end{cases}$$

then

$$\begin{aligned} \frac{1}{p} \sum_{m_1=0}^{p-1} \sum_{m_2=0}^{p-1} \rho(m_1) \overline{\rho(m_2)} \sum_{n=0}^{p-1} \cos 2\pi a \frac{n(m_1 - m_2)}{p} &= \sum_{m=0}^{p-1} |\rho(m)|^2 = R_1, \\ \frac{1}{p} \sum_{m_1=0}^{p-1} \sum_{m_2=0}^{p-1} \rho(m_1) \overline{\rho(m_2)} \sum_{n=0}^{p-1} \cos 2\pi a \frac{n(m_1 + m_2)}{p} &= |\rho(0)|^2 + \sum_{m=1}^{p-1} \rho(m) \overline{\rho(p-m)} = R_2. \end{aligned}$$

Hence

$$|S| \leq \sqrt{\frac{p}{2} T (R_1 + R_2)}.$$

The Lemma is proved. \square

Let M be a positive integer, $2M \leq p - 1$. Take in the Lemma

$$\nu(n) = \rho(n) = \begin{cases} 1, & \text{if } n \equiv g^{-x} \pmod{p} \text{ for some } 1 \leq x \leq M, \\ 0, & \text{otherwise} \end{cases}$$

Note that if $1 \leq x_1, x_2 \leq (p-1)/2$, then

$$g^{-x_1} + g^{-x_2} \not\equiv 0 \pmod{p}.$$

Therefore in this case

$$T = M, \quad R_1 = M, \quad R_2 = 0.$$

COROLLARY. Let $(a, p) = 1$, $2M \leq p - 1$. Then

$$\left| \sum_{z=1}^M \sum_{t=1}^M \cos 2\pi \frac{ag^{-z}g^{-t}}{p} \right| \leq \sqrt{\frac{p}{2}} M.$$

To prove the Theorem we let λ be any integer with $\lambda \not\equiv 0 \pmod{p}$, and denote

$$N_1 = [2^{1/4}p^{3/4}], \quad N_2 = [2^{-3/4}p^{3/4}].$$

Put $N = N_1 + 2N_2$. Obviously $N < 2^{5/4}p^{3/4}$. Further, let J denote the number of solutions of the congruence $g^x - g^y - \lambda g^{-z-t} \equiv 0 \pmod{p}$ in integers x, y, z, t subject to the condition

$$1 \leq x, y \leq N_1, \quad 1 \leq z, t \leq N_2.$$

If we prove that $J > 0$ for all λ then we are done. To this end we first note that if $N_1 > \frac{p-1}{2}$ then, by taking $z = t = 1$, we see that the set $\{g^x \pmod{p} : 1 \leq x \leq N_1\}$ as well as the set $\{g^y + \lambda g^{-2} \pmod{p} : 1 \leq y \leq N_1\}$ contains at least $\frac{p+1}{2}$ different values modulo p . Therefore, there are positive integers x, y , $1 \leq x, y \leq N_1$, such that

$$g^x \equiv g^y + \lambda g^{-2} \pmod{p},$$

and hence, $J > 0$ in this case.

Let now $N_1 \leq \frac{p-1}{2}$. In particular $p \geq 11$, and $2N_2 \leq p - 1$. We represent J by the mean of trigonometric sums:

$$pJ = \sum_{a=0}^{p-1} \sum_{x=1}^{N_1} \sum_{y=1}^{N_1} e^{2\pi i \frac{a(g^x - g^y)}{p}} \sum_{z=1}^{N_2} \sum_{t=1}^{N_2} e^{2\pi i \frac{-a\lambda g^{-z}g^{-t}}{p}}.$$

Note that

$$\sum_{x=1}^{N_1} \sum_{y=1}^{N_1} e^{2\pi i \frac{a(g^x - g^y)}{p}} = \left| \sum_{x=1}^{N_1} e^{2\pi i \frac{ag^x}{p}} \right|^2.$$

Therefore

$$pJ = \sum_{a=0}^{p-1} \left| \sum_{x=1}^{N_1} e^{2\pi i \frac{ag^x}{p}} \right|^2 \sum_{z=1}^{N_2} \sum_{t=1}^{N_2} e^{2\pi i \frac{-a\lambda g^{-z}g^{-t}}{p}}.$$

Since the left hand side is a real number, the imaginary part of the right hand side is equal to zero. Therefore, considering the real parts and separating the term corresponding to $a = 0$, we obtain

$$pJ = N_1^2 N_2^2 + \sum_{a=1}^{p-1} \left| \sum_{x=1}^{N_1} e^{2\pi i \frac{ag^x}{p}} \right|^2 \sum_{z=1}^{N_2} \sum_{t=1}^{N_2} \cos 2\pi \frac{a\lambda g^{-z}g^{-t}}{p}.$$

By the corollary to the Lemma (with $M = N_2 \leq (p-1)/2$),

$$pJ > N_1^2 N_2^2 - \sum_{a=1}^{p-1} \left| \sum_{x=1}^{N_1} e^{2\pi i \frac{ag^x}{p}} \right|^2 \sqrt{\frac{p}{2}} N_2.$$

Note that

$$\sum_{a=1}^{p-1} \left| \sum_{x=1}^{N_1} e^{2\pi i \frac{ax}{p}} \right|^2 = pN_1 - N_1^2.$$

Hence

$$pJ > N_1^2 N_2^2 - N_1 N_2 (p - N_1) \sqrt{\frac{p}{2}}.$$

Since $N_1 > 2^{1/4} p^{3/4} - 1$, $N_2 > 2^{-3/4} p^{3/4} - 1$ and $p \geq 11$, we have

$$\begin{aligned} \frac{pJ}{N_1 N_2} &> N_1 N_2 - (p - N_1) \sqrt{\frac{p}{2}} \\ &> (2^{1/4} p^{3/4} - 1)(2^{-3/4} p^{3/4} - 1) - 2^{-1/2} p^{3/2} + 2^{-1/2} p^{1/2} (2^{1/4} p^{3/4} - 1) \\ &= p^{1/2} 2^{-1/2} (2^{1/4} p^{1/4} (p^{1/2} - 3/2^{1/2}) - 1) + 1 > 0. \end{aligned}$$

The theorem is proved. \square

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INSTITUTO DE MATEMÁTICAS UNAM
CAMPUS MORELIA, AP. POSTAL 61-3 (XANGARI)
CP 58089, MORELIA, MICHOACÁN
MÉXICO
garci@matmor.unam.mx

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DESIGUALDADES Y FORMULAS ASINTÓTICAS PARA SUMAS DE POTENCIAS DE PRIMOS

RAFAEL JAKIMCZUK

ABSTRACT. We prove the inequality $\sum_{i=1}^n p_i^k > \frac{n^{k+1}}{k+1} \log^k n$, where $n \geq 1$ (p_i is the n -th prime number and k a positive integer). From Cipolla's formula we also obtain asymptotic formulae for $\sum_{i=1}^n p_i^k$.

RESUMEN. Se demuestra que la desigualdad estricta $\sum_{i=1}^n p_i^k > \frac{n^{k+1}}{k+1} \log^k n$ se satisface para todo $n \geq 1$ (donde p_i es el i -ésimo número primo y k un entero positivo). Se obtienen también fórmulas asintóticas para $\sum_{i=1}^n p_i^k$, partiendo de la fórmula asintótica de M. Cipolla para p_n .

1. Introducción

Fue probado por Salát y Znám [1] que para todo real $k \geq 0$ se cumple la siguiente fórmula asintótica (donde p_i denota el i -ésimo número primo)

$$(1.1) \quad \sum_{i=1}^n p_i^k \sim \frac{n^{k+1}}{k+1} \log^k n.$$

El principal resultado de este artículo es la demostración de la siguiente desigualdad estricta cuando k es un entero positivo

$$\sum_{i=1}^n p_i^k > \frac{n^{k+1}}{k+1} \log^k n.$$

Por otro lado, utilizando la fórmula de Cipolla, se obtienen también fórmulas similares para $\sum_{i=1}^n p_i^k$ cuando k es un entero positivo.

Estas fórmulas constituyen una demostración alternativa de (1.1), proporcionando una mejor aproximación cuando k es entero.

Recordemos que M. Cipolla [2] demostró el siguiente teorema para p_n .

TEOREMA DE CIPOLLA. *Existe una única sucesión $P_j(X)$ ($j \geq 1$) de polinomios con coeficientes racionales, tal que para cualquier entero no negativo m*

$$(1.2) \quad p_n = n \log n + n \log \log n - n + \sum_{j=1}^m \frac{(-1)^{j-1} n P_j(\log \log n)}{\log^j n} + o\left(\frac{n}{\log^m n}\right).$$

Los polinomios $P_j(X)$ tienen grado j y coeficiente principal $1/j$.

Si $m = 0$ el teorema de Cipolla se reduce a $p_n = n \log n + n \log \log n - n + o(n)$, lo cual tiene como consecuencia débil el teorema del número primo $p_n \sim n \log n$.

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2. La desigualdad estricta

Nosotros necesitamos el siguiente lema para demostrar la desigualdad principal de este trabajo.

LEMA (2.1). Si $n \geq \exp(\exp(1 + 1/k))$ se cumple

$$(2.2) \quad \left[\sum_{i=1}^n p_i^k - \frac{n^{k+1}}{k+1} \log^k n \right] > \left[\sum_{i=1}^{n-1} p_i^k - \frac{(n-1)^{k+1}}{k+1} \log^k(n-1) \right].$$

Demostración. Sea $f(x) = \frac{x^{k+1}}{k+1} \log^k x$, entonces $f'(x) = x^k \log^k x + \frac{k}{k+1} x^k \log^{k-1} x$. Por el teorema de Lagrange tenemos

$$(2.3) \quad f(n) - f(n-1) = f'(n-1+\varepsilon) < f'(n) \quad (0 < \varepsilon < 1)$$

y la desigualdad de Dusart [3] nos proporciona $p_n > n(\log n + \log \log n - 1)$ ($n \geq 2$).

Finalmente, $\log \log n \geq 1 + 1/k$ por las hipótesis dadas, así que $n(\log n + \log \log n - 1) \geq n(\log n + 1/k)$. Todo lo anterior implica

$$(2.4) \quad \begin{aligned} p_n^k &> n^k (\log n + \log \log n - 1)^k \geq n^k \left(\log n + \frac{1}{k} \right)^k \\ &= n^k \left(\sum_{i=0}^k \binom{k}{i} \log^{k-i} n \left(\frac{1}{k} \right)^i \right) \geq n^k \log^k n + n^k \log^{k-1} n. \end{aligned}$$

De las ecuaciones (2.3) y (2.4) obtenemos $p_n^k > \frac{n^{k+1}}{k+1} \log^k n - \frac{(n-1)^{k+1}}{k+1} \log^k(n-1)$ y ésta última desigualdad automáticamente implica (2.2), como queríamos. \square

COROLARIO. Si $k = 1$, la desigualdad se cumple para $n \geq 1619$, si $k = 2$ para $n \geq 89$, si $k = 3$ para $n \geq 45$, si $k = 4$ para $n \geq 33$, si $k = 5$ para $n \geq 28$, si $k = 6$ para $n \geq 25$, si $k = 7$ para $n \geq 24$, si $k = 8$ para $n \geq 22$, si $9 \leq k \leq 10$ para $n \geq 21$, si $11 \leq k \leq 12$ para $n \geq 20$, si $13 \leq k \leq 16$ para $n \geq 19$, si $17 \leq k \leq 24$ para $n \geq 18$, si $25 \leq k \leq 50$ para $n \geq 17$, si $k \geq 51$ para $n \geq 16$.

TEOREMA (2.5). Para todo $n \geq 1$ se cumple

$$(2.6) \quad \sum_{i=1}^n p_i^k > \frac{n^{k+1}}{k+1} \log^k n.$$

Demostración. Considerando la desigualdad de Rosser [4] $p_n > n \log n$, ($n \geq 1$), tendremos

$$(2.7) \quad 1 + \sum_{i=1}^{n-1} \left(\frac{p_i}{p_n} \right)^k > \frac{n^{k+1} \log^k n}{(k+1)p_n^k} \quad \text{cuando } k \geq n,$$

ya que el primer miembro es mayor que 1 y el segundo es menor que 1.

Multiplicando ambos miembros de (2.7) por p_n^k , obtenemos

$$(2.8) \quad \sum_{i=1}^n p_i^k > \frac{n^{k+1}}{k+1} \log^k n, \quad k \geq n.$$

Ahora bien, de acuerdo al Lema (2.1), también tenemos que la desigualdad (2.8) se satisface cuando $n \geq \exp(\exp(1 + 1/k))$.

Nótese que $k \geq \exp(\exp(1 + 1/k))$ cuando $k \geq 17$, ésto implica que (2.8) se satisface para todo $n \geq 1$ cuando $k \geq 17$. Finalmente, los casos restantes $k < \exp(\exp(1 + 1/k))$ con $1 \leq k \leq 16$, que no son cubiertos por (2.7) o el Lema (2.1), son una cantidad finita y pueden ser analizados con una computadora. Todo lo anterior implica que (2.8) se satisface para todos $k \geq 1$ y $n \geq 1$. \square

3. La formula asintótica para $\sum_{i=1}^n p_i$

TEOREMA (3.1). *Existe una única sucesión $Q_j(X)$ ($j \geq 1$) de polinomios con coeficientes racionales, tal que para cualquier entero no negativo m*

$$(3.2) \quad \sum_{i=1}^n p_i = \frac{n^2}{2} \log n + \frac{n^2}{2} \log \log n - \frac{3}{4}n^2 + \sum_{j=1}^m \frac{(-1)^{j-1} n^2 Q_j(\log \log n)}{\log^j n} + o\left(\frac{n^2}{\log^m n}\right).$$

Los polinomios $Q_j(X)$ son de grado j y coeficiente principal $\frac{1}{2^j}$. Si $m = 0$ la fórmula se reduce a $\sum_{i=1}^n p_i = \frac{n^2}{2} \log n + \frac{n^2}{2} \log \log n - \frac{3}{4}n^2 + o(n^2)$.

Demostración. En (1.2) tenemos que

$$o\left(\frac{n}{\log^m n}\right) = f(n) \frac{n}{\log^m n},$$

donde $\lim_{n \rightarrow \infty} f(n) = 0$.

Ahora consideremos la función $f(t)$ definida de la siguiente manera, si $t \in [n, n+1)$ entonces $f(t) = f(n) \frac{n}{\log^m n} \frac{\log^m t}{t}$. Por lo tanto $\lim_{t \rightarrow \infty} f(t) = 0$ y podemos escribir $f(t) \frac{t}{\log^m t} = o\left(\frac{t}{\log^m t}\right)$.

Consideremos por último la función

$$p_t = t \log t + t \log \log t - t + \sum_{j=1}^m \frac{(-1)^{j-1} t P_j(\log \log t)}{\log^j t} + o\left(\frac{t}{\log^m t}\right).$$

Observemos que $p_t - o\left(\frac{t}{\log^m t}\right)$ es estrictamente creciente desde algún valor t_0 en adelante.

De acuerdo con (1.2) obtenemos

$$(3.3) \quad \sum_{i=1}^n p_i = \int_2^n p_t dt + O(n \log n),$$

donde

$$\int_2^n p_t dt = \int_2^n (t \log t + t \log \log t - t) dt + \sum_{j=1}^m \int_2^n \frac{(-1)^{j-1} t P_j(\log \log t)}{\log^j t} dt + o\left(\int_2^n \frac{t}{\log^m t} dt\right).$$

Ahora bien

$$(3.4) \quad \int_2^n (t \log t + t \log \log t - t) dt = \frac{n^2}{2} \log n + \frac{n^2}{2} \log \log n - \frac{3}{4} n^2 + o(1) \\ - \frac{1}{2} \int_2^n \frac{t}{\log t} dt.$$

Si $l \leq j$, entonces

$$(3.5) \quad \int_2^n \frac{t(\log \log t)^l}{\log^j t} dt = \frac{n^2(\log \log n)^l}{2 \log^j n} + o(1) + \frac{j}{2} \int_2^n \frac{t(\log \log t)^l}{\log^{j+1} t} dt \\ - \frac{l}{2} \int_2^n \frac{t(\log \log t)^{l-1}}{\log^{j+1} t} dt.$$

Las ecuaciones (3.4) y (3.5) deben ser integradas sucesivamente hasta obtener integrales de la forma

$$\int_2^n \frac{t(\log \log t)^s}{\log^{m+1} t} dt = o\left(\frac{n^2}{\log^m n}\right).$$

Por otro lado, claramente

$$o\left(\int_2^n \frac{t}{\log^m t} dt\right) = o\left(\frac{n^2}{\log^m n}\right).$$

Por todo lo anterior, podemos deducir que (3.3) y (3.2) coinciden, como queríamos demostrar. \square

Se observa que el procedimiento por el cual demostramos este teorema es constructivo.

Ejemplo. En (1.2), $P_1(X) = X - 2$, de aquí resulta que (3.2) es, para $m = 1$,

$$\sum_{i=1}^n p_i = \frac{n^2}{2} \log n + \frac{n^2}{2} \log \log n - \frac{3}{4} n^2 + \frac{n^2(2 \log \log n - 5)}{4 \log n} + o\left(\frac{n^2}{\log n}\right).$$

4. La formula asintótica para $\sum_{i=1}^n p_i^k$

LEMA (4.1). *Existe una única sucesión $P_{k,j}(X)$ ($j \geq 1$) de polinomios con coeficientes racionales, tal que para cualquier entero no negativo m*

$$(4.2) \quad p_n^k = \sum c_i f_i(n) + \sum_{j=1}^m \frac{(-1)^{j-1} n^k P_{k,j}(\log \log n)}{\log^j n} + o\left(\frac{n^k}{\log^m n}\right).$$

Los polinomios $P_{k,j}(X)$ tienen grado $j + k - 1$ y coeficiente principal $1/\binom{j+k-1}{k}$. Las $f_i(n)$ son funciones de la forma $n^k \log^r n (\log \log n)^s$ donde $s \leq k$ y los c_i son ciertos coeficientes. Una de las funciones $f_i(n)$ es $n^k \log^k n$ con coeficiente 1. Las otras funciones $f_i(n)$ satisfacen $\lim_{n \rightarrow \infty} \frac{f_i(n)}{n^k \log^k n} = 0$. Si $m = 0$ la fórmula se reduce a

$$(4.3) \quad p_n^k = \sum c_i f_i(n) + o(n^k).$$

Demostración. La demostración se basa en calcular la k -ésima potencia de la fórmula de Cipolla (1.2), agrupando los términos necesarios bajo el símbolo de error, esto es:

$$(4.4) \quad \begin{aligned} p_n^k &= \left[n \log n + n \log \log n - n + \sum_{j=1}^{m+k-1} \frac{(-1)^{j-1} n P_j(\log \log n)}{\log^j n} + o\left(\frac{n}{\log^{m+k-1} n}\right) \right]^k \\ &= \sum c_i f_i(n) + \sum_{j=1}^m \frac{(-1)^{j-1} n^k P_{k,j}(\log \log n)}{\log^j n} + o\left(\frac{n^k}{\log^m n}\right). \end{aligned}$$

□

Ejemplo. Para $k = 2$ (4.3) se reduce a:

$$(4.5) \quad \begin{aligned} p_n^2 &= \left[n \log n + n \log \log n - n + n \frac{\log \log n - 2}{\log n} + o\left(\frac{n}{\log n}\right) \right]^2 \\ &= n^2 \log^2 n + 2n^2 \log n \log \log n - 2n^2 \log n + n^2 (\log \log n)^2 - 3n^2 + o(n^2). \end{aligned}$$

TEOREMA (4.6). *Existe una única sucesión $Q_{k,j}(X)$ ($j \geq 1$) de polinomios con coeficientes racionales, tal que para cualquier entero no negativo m*

$$(4.7) \quad \sum_{i=1}^n p_i^k = \sum d_i g_i(n) + \sum_{j=1}^m \frac{(-1)^{j-1} n^{k+1} Q_{k,j}(\log \log n)}{\log^j n} + o\left(\frac{n^{k+1}}{\log^m n}\right).$$

Los polinomios $Q_{k,j}(X)$ tienen grado $j + k - 1$ y coeficiente principal $\frac{1}{k+1} \frac{1}{\binom{j+k-1}{k}}$.

Las $g_i(n)$ son funciones de la forma $n^{k+1} \log^r n (\log \log n)^s$ y los d_i son ciertos coeficientes. Una de las funciones $g_i(n)$ es $n^{k+1} \log^k n$ con coeficiente $1/(k+1)$.

Las otras funciones $g_i(n)$ satisfacen $\lim_{n \rightarrow \infty} \frac{g_i(n)}{n^{k+1} \log^k n} = 0$. Si $m = 0$ la fórmula se reduce a

$$(4.8) \quad \sum_{i=1}^n p_i^k = \sum d_i g_i(n) + o(n^{k+1})$$

Demostración. Es análoga a la del teorema (3.1), ahora utilizando el lema (4.1). □

Ejemplo. Para $k = 2$, teniendo en cuenta (4.5), (4.8) se reduce a

$$\begin{aligned} \sum_{i=1}^n p_i^2 &= \frac{1}{3} n^3 \log^2 n + \frac{2}{3} n^3 \log n \log \log n - \frac{8}{9} n^3 \log n + \frac{1}{3} n^3 (\log \log n)^2 \\ &\quad - \frac{2}{9} n^3 \log \log n - \frac{25}{27} n^3 + o(n^3). \end{aligned}$$

Nota. (4.8) incluye como consecuencia débil a la fórmula de Salát y Znam (1.1) cuando k es un entero positivo.

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DIVISIÓN MATEMÁTICA
UNIVERSIDAD NACIONAL DE LUJÁN
BUENOS AIRES, ARGENTINA
jakimczu@mail.unlu.edu.ar

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STRUCTURAL INVARIANTS IN CRYSTALLOGRAPHY AND ISOMETRY OF CYCLOTOMIC MEASURES

GABRIELA PUTINAR

ABSTRACT. We prove that the higher-order structural invariants for the phase retrieval problem in crystallography form a complete set of invariants under isometry for measures with real coefficients supported at the n -th roots of unity. In particular the structural invariants of degree $\leq 2n$ suffice in order to distinguish between homometric non-isometric measures. For symmetrical measures with rational coefficients, we use a result of Grünbaum and Moore to show that in fact degrees at most 6 suffice to solve the phase retrieval problem in this setting.

1. Introduction

The phase retrieval problem in crystallography (cf. [2] for a general reference), asks for reconstructing a measure μ from its *homometry* data given by the absolute values

$$(1.1) \quad |\hat{\mu}(k)|, \quad k \in \mathbf{Z},$$

(where $\hat{\cdot}$ denotes the Fourier transform).

Since a measure μ is determined by its Fourier coefficients, what the homometry data does not include is information on the *phases* of the complex numbers $\hat{\mu}(k)$, $k \in \mathbf{Z}$.

That two homometric measures may correspond to two different crystals, was first realised in the 30's when studying X-ray diffraction patterns in crystals. The study of non-isometric homometric measures was initiated by L. Patterson with the simplest case of *cyclotomic measures*, which are supported at the n -th roots of the unity.

Two measures are *isometric* if they can be made to coincide by a rotation of angle a multiple of $2\pi/n$, or by a reflexion and conjugating the coefficients (and also by any combinations of these).

A set of invariants under isometry that is widely used (cf. [2], [4]) to distinguish between non-isometric homometric measures is that of the structural polynomials of degree three, which are real valued and defined by

$$(1.2) \quad A_{k,l}(\mu) = \mathbf{Re}(\hat{\mu}(k)\hat{\mu}(l)\overline{\hat{\mu}(k+l)}),$$

for $k, l \in \mathbf{Z}$. In [5] it is shown that the structural invariants distinguish between most uniformly distributed homometric sets (i.e., measures with coefficients 0 or 1) on S^1 .

In this paper we discuss higher-order structural invariants for cyclotomic measures with coefficients in an arbitrary field $K \subset \mathbf{C}$. In this setting, the

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structural invariants are polynomials in the variables (a_0, \dots, a_{n-1}) , where a_j is the mass at the j -th point of the cyclotomic measure. These invariants are constructed from the $(\hat{\mu}(k))_k$, as polynomial invariants under a natural action (defined in Section 2) of the dihedral group, which corresponds to isometry of measures.

In Section 3 we show that the structural invariants form a complete set of invariants. From this it follows that they suffice in order to distinguish between non-isometric measures with coefficients in K . Moreover, by Noether's lemma ([1]), degrees $\leq 2n$ suffice.

In [3], in the case of the action of the *cyclic* group of rotations on cyclotomic measures with rational coefficients, it is shown that the corresponding structural invariants of degree at most 6 suffice in order to distinguish between such measures, up to rotations. In Section 4 we show that this result can be extended to the case of the dihedral action, assuming that one of the measures is symmetrical (w.r.t. the Ox -axis).

Finally, let us add that in practice it is the *real* part of the cyclic invariants that one uses for computations, and for real measures this coincides with the dihedral structural invariants.

2. Dihedral group action on measures

A *cyclotomic* measure supported at the points $p_j = e^{2\pi i j/n}$, $j = 0, \dots, n-1$ is of the form

$$(2.1) \quad \mu = \sum_{j=0}^{n-1} a_j \delta_{p_j},$$

where δ_{p_j} denotes the Dirac distribution at p_j and the coefficients a_j are in a fixed field $K \subset \mathbf{C}$. We shall identify such a cyclotomic measure μ with its coefficients $(a_0, a_1, \dots, a_{n-1})$ and regard it as an element in K^n .

Two cyclotomic measures are *isometric* (or *\mathcal{D}_n -equivalent*) if they are equivalent under the action of the dihedral group \mathcal{D}_n of order $2n$, where \mathcal{D}_n is the (non-abelian) group generated by two transformations T and S , where T is the rotation of angle $2\pi/n$ and S is the reflexion in the Ox -axis. (For $K \subset \mathbf{R}$, this coincides with the standard notion of isometry, cf. Introduction).

Under the above identification of cyclotomic measures with points in K^n , the actions of S , respectively T become:

$$(2.2) \quad S(a_0, a_1, a_2, \dots, a_{n-1}) = (a_0, a_{n-1}, a_{n-2}, \dots, a_1),$$

$$(2.3) \quad T(a_0, a_1, \dots, a_{n-1}) = (a_1, a_2, \dots, a_{n-1}, a_0).$$

Let $\zeta = e^{-2\pi i/n}$. Note that in the cyclotomic case we have, for $k \in \mathbf{Z}$:

$$(2.4) \quad \hat{\mu}(k) = \int e^{-ik\theta} d\mu(\theta) = \sum_{j=0}^{n-1} a_j e^{-2\pi i j k/n} = \sum_{j=0}^{n-1} a_j \zeta^{kj},$$

and, since $\hat{\mu}(k) = \hat{\mu}(k+n)$, we may restrict the index k to $0 \leq k < n$.

Let us denote for convenience $L_k(\mu) = \hat{\mu}(k)$ and regard it as a (degree one) polynomial in the variables $(a_0, a_1, \dots, a_{n-1})$ and with coefficients in $K(\zeta)$. (Recall that $K(\zeta) = K[x]/(x^n - 1)$).

The above action of \mathcal{D}_n induces an action on the Fourier coefficients of cyclotomic measures, defined by

$$(2.5) \quad T(L_k) := L_k \circ T^{-1} = \zeta^k \cdot L_k, \text{ and } S(L_k) := L_k \circ S^{-1} = L_{-k}$$

for $0 \leq k < n$.

We can extend this action to polynomials of arbitrary degree by

$$(2.6) \quad (g(f))(a_0, \dots, a_{n-1}) := f(g^{-1}(a_0, \dots, a_{n-1})),$$

where f is a polynomial on K^n and $g \in \mathcal{D}_n$.

In general, if G is a finite group that acts by linear transformations on the vector space K^n , formula (2.6) defines an action of G on the ring \mathcal{S} of polynomials on K^n . In the case K an algebraically closed field (of zero characteristic), from basic facts in commutative algebra it follows ([1], p. 8 and the references there) that the ring \mathcal{S}^G of polynomials invariant under G separates the points in the space K^n/G of orbits under the action of G on K^n . Moreover, K^n/G is an affine variety which has for its "coordinate ring" the ring \mathcal{S}^G . We shall use these facts in the case $G = \mathcal{D}_n$ in Theorem (3.9) below. (For more background in invariant theory, the interested reader may consult [1], [6], [7].)

3. Structural invariants under the dihedral action.

In this section we describe a set of generators for the ring of invariants under \mathcal{D}_n .

For $l = 1$, let us define L_0 to be the only – up to a factor – structural invariant of degree one. For an arbitrary degree $l \geq 2$, and for any integers $0 \leq k_j \leq n - 1$, $j = 1, \dots, l - 1$, define the *structural invariant* (under the dihedral action) by

$$(3.1) \quad H_{k_1, \dots, k_{l-1}} = \frac{1}{2} (L_{k_1} \cdots L_{k_{l-1}} \cdot L_{-k_1 - \dots - k_{l-1}} \\ + L_{-k_1} \cdots L_{-k_{l-1}} \cdot L_{k_1 + \dots + k_{l-1}}).$$

Note that the structural invariants have coefficients in $\mathbf{K}(\zeta)$.

From formula (2.5) it is immediate that $H_{k_1, \dots, k_{l-1}}$ is indeed invariant under \mathcal{D}_n . Note that by permuting the indices k_1, \dots, k_{l-1} or by simultaneously changing their signs we obtain the same structural invariant.

For $K \subset \mathbf{R}$, the structural invariants are polynomials on K^n with *real* coefficients; in degree two, they coincide with the (squares of the) invariants for homometry (formula (1.1)); in degree three they are precisely the third-order (real) structural invariants (formula (1.2)), see [4], [5].

One can use the third-order invariants in order to prove the existence of homometric non-isometric cyclotomic measures with 0,1 coefficients.

Example (3.2) (Patterson). Let $n = 8$, $\alpha = e^{2\pi i/8}$, $\zeta = \alpha^{-1}$. Let the measures μ , ν be given by

$$\mu = \delta_1 + \delta_\alpha + \delta_{\alpha^3} + \delta_{\alpha^4} = (1, 1, 0, 1, 1, 0, 0, 0)$$

and

$$\nu = \delta_1 + \delta_{\alpha^3} + \delta_{\alpha^4} + \delta_{\alpha^5} = (1, 0, 0, 1, 1, 1, 0, 0).$$

It is easy to check (see e.g. [5]) that $H_k(\mu) = H_k(\nu)$ for all $0 \leq k < n$, but that $H_{1,1}(\mu) \neq H_{1,1}(\nu)$.

Our main result is the following.

THEOREM (3.3). *The structural invariants of degree l , $1 \leq l \leq 2n$, generate the $K(\zeta)$ -algebra of polynomial invariants under the action of \mathcal{D}_n on K^n .*

In fact, we also prove below that the structural invariants of a fixed degree are linear generators for the homogeneous invariants of that degree. For this purpose let us introduce, for integers $0 \leq \alpha_j \leq n-1$, $j = 1, \dots, l-1$, the *circular invariant*:

$$(3.4) \quad C_{\alpha_1, \dots, \alpha_{l-1}}(a_0, \dots, a_{n-1}) = \frac{1}{2} \left(\sum_{j=0}^{n-1} a_j a_{j+\alpha_1} \dots a_{j+\alpha_{l-1}} + \sum_{j=0}^{n-1} a_j a_{j-\alpha_1} \dots a_{j-\alpha_{l-1}} \right).$$

(Here and throughout we make the convention that indices $j + \alpha$ are taken modulo n , so that $0 \leq j + \alpha < n$.)

That $C_{\alpha_1, \dots, \alpha_{l-1}}$ is indeed invariant is clear by formulas (2.2) and (2.3).

The circular invariants can be expressed linearly in terms of the structural invariants. In fact, by computing explicitly we obtain:

$$(3.5) \quad H_{k_1, \dots, k_{l-1}} = \sum_{0 \leq \alpha_1, \dots, \alpha_{l-1} < n} \zeta^{k_1 \alpha_1 + \dots + k_{l-1} \alpha_{l-1}} \cdot C_{\alpha_1, \dots, \alpha_{l-1}}.$$

This shows that the relation between the structural invariants and the circular invariants in degree l (regarded as matrices indexed over k_1, \dots, k_{l-1} , when evaluated at a fixed point in K^n) is given by a matrix $A^{\otimes(l-1)}$, with A the Fourier transform matrix defined by

$$(3.6) \quad A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \zeta & \dots & \zeta^{n-1} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ 1 & \zeta^{n-1} & \dots & \zeta^{(n-1)^2} \end{pmatrix},$$

which is invertible.

Proof of Theorem (3.3). We show first that the circular invariants are linear generators for all the polynomial invariants under \mathcal{D}_n .

More generally [7], if G is a finite group acting linearly on K^n , the (Reynolds) operator

$$(3.7) \quad f \mapsto f^* := \frac{1}{|G|} \sum_{T \in G} Tf$$

acts K -linearly on polynomials on K^n and is such that $f = f^*$ if f is invariant under G .

Further, if $f = \sum \lambda_i m_i$ is an expression for the invariant f in terms of monomials, then $f = f^* = \sum \lambda_i m_i^*$. This shows that all polynomial invariants under G are linearly generated by the invariants of the form m^* , where m ranges over the monomials. And that the homogeneous invariants of a fixed degree l are linearly generated by the invariants m^* , where m is a monomial of degree l .

Returning to our case $G = \mathcal{D}_n$, it suffices to note that the invariants of the form m^* , with m a monomial, coincide (up to a constant factor) with the circular invariants.

Indeed, if a monomial $a_0 a_{\alpha_1} \dots a_{\alpha_{l-1}}$ of m^* is written as a product of distinct factors as

$$a_0 a_{\alpha_1} \dots a_{\alpha_{l-1}} = a_0^{p_0} \dots a_{n-1}^{p_{n-1}},$$

then for every $0 \leq j < n$ we have

$$a_j a_{j+\alpha_1} \dots a_{j+\alpha_{l-1}} = a_j^{p_0} \dots a_{j+n-1}^{p_{n-1}}.$$

Therefore the polynomial g defined by

$$(3.8) \quad g(a_0, \dots, a_{n-1}) = \sum_{j=0}^{n-1} a_j^{p_0} \dots a_{j+n-1}^{p_{n-1}} + \sum_{j=0}^{n-1} a_j^{p_0} \dots a_{j-n+1}^{p_{n-1}}$$

is (up to a constant factor) a circular invariant.

Thus we have a linear expression for every homogeneous invariant of degree l in terms of the circular invariants of degree l .

Further, since K is of zero characteristic, by Noether's lemma [1] there is a set of homogeneous generators of degree $\leq 2n$ for the algebra of invariants, and we can further express the invariants of this set in terms of the circular invariants of degree $\leq 2n$. Finally, that the structural invariants are also generating (in every degree $l \geq 1$) is clear by the (linear) relation (3.5) with the circular invariants. \square

Remark. For practical considerations, one needs to retrieve the information on the degree one invariant $L_0(a_0, \dots, a_{n-1}) = a_0 + \dots + a_{n-1}$ from the invariants of degree ≥ 2 . This is indeed possible, since if $L_0^3(\mu) = L_0^3(\nu)$, then $L_0(\mu) = L_0(\nu)$.

As a consequence of Theorem (3.3) and of the above remark, using standard facts from invariant theory, we have the following.

THEOREM (3.9). *The structural invariants (of degree l , $2 \leq l \leq 2n$) distinguish between non-isometric cyclotomic measures with coefficients in K .*

Proof. Let us assume first that $K = \mathbf{C}$. Let $G = \mathcal{D}_n$ and let f_1, \dots, f_p be generators for S^G (e.g. let these denote the circular invariants of degree $\leq 2n$, having coefficients in K). Then by Theorem (3.3) and the basic facts mentioned in the remark at the end of section 1, K^n/G is a subvariety of K^p , and its defining equations are the (finitely many) relations between f_1, \dots, f_p . Now if μ, ν are cyclotomic measures having the same structural invariants (or equivalently the same circular invariants), then their orbits under G are points in K^n/G of equal coordinates in K^p . It follows that μ and ν are \mathcal{D}_n -equivalent, therefore isometric.

Second, if we let $K \subset \mathbf{C}$ be an arbitrary subfield, then the action of G on K^n given by (2.2), (2.3) extends to \mathbf{C}^n . If $\mu, \nu \in K^n$ have the same structural invariants, then we regard them as elements in \mathbf{C}^n and apply the above considerations to obtain that μ, ν are in the same orbit under the G -action; therefore they are isometric. \square

Remark. It is easy to see by direct computations with the transformations $T^k S$ that for $n > 3$ the action of \mathcal{D}_n on K^n defined in Section 2 is *not* generated by reflexion, therefore for $n > 3$ the space of cyclotomic measures up to isometry is singular (cf. [1], [6]). In fact, the space of eigenvectors of $T^k S$ with corresponding eigenvalue $=1$ has dimension $\leq [\frac{n+1}{2}]$, which is $< n - 1$ for $n > 3$. (A reflexion is defined by requiring this dimension to be $n - 1$.)

By contrast, for $n \leq 3$, since $\mathcal{D}_n \cong \Sigma_n$, the group of permutations on n objects, and since the group Σ_n is generated by reflexion, the space of cyclotomic measures up to isometry is non-singular [6]. Equivalently, the coordinate ring $\mathcal{S}^{\mathcal{D}_n}$ has n algebraically independent generators.

In particular, we have $\mathcal{S}^{\mathcal{D}_3} = K[L_0, C_1, C_{1,2}]$, with $C_{1,2}$ a structural invariant that cannot be expressed as a polynomial (nor as a rational function) in terms of the homometry data (contained in $C_0 = C_1$ and L_0^2).

4. Structural invariants of degree ≤ 6

If, instead of the dihedral action on cyclotomic measures, one considers the action of the *cyclic* group \mathcal{C}_n generated by the rotation of angle $2\pi/n$ (see formula (2.2)), then one can define the invariants under this cyclic action:

$$(4.1) \quad P_{k_1, \dots, k_{l-1}} = L_{k_1} \cdots L_{k_{l-1}} \cdot L_{-k_1 - \dots - k_{l-1}},$$

which are related with the structural invariants defined in Section 2 by

$$(4.2) \quad H_{k_1, \dots, k_{l-1}} = \frac{1}{2}(P_{k_1, \dots, k_{l-1}} + P_{-k_1, \dots, -k_{l-1}}),$$

where k_j are integers such that $0 \leq k_j < n$, $j = 1, \dots, l-1$. Note that if $K \subset \mathbf{R}$,

$$(4.3) \quad H_{k_1, \dots, k_{l-1}} = \mathbf{Re} P_{k_1, \dots, k_{l-1}}.$$

For the invariants $P_{k_1, \dots, k_{l-1}}$ one can prove, as in Theorem (3.3), that they generate the algebra of invariants under the action of \mathcal{C}_n . Therefore they distinguish between real cyclotomic measures up to rotation, and by Noether's Lemma, degrees $\leq n$ suffice.

Example. Let us give an example of two measures μ, ν for which all the structural invariants coincide, but for which the cyclic invariants do not coincide. (This shows that the cyclic invariants distinguish more than the dihedral ones.) For this let $n \geq 3$, $\mu = (1, -1, 0, \dots, 0)$ and $\nu = S(\mu) = (1, 0, \dots, 0, -1)$. Then μ and ν are isometric, therefore all their structural invariants coincide, while direct computation shows $P_{1,1}(\mu) \neq P_{1,1}(\nu)$.

In the case $K = \mathbf{Q}$, A. Grünbaum and C. C. Moore proved that in fact degrees ≤ 6 suffice. Precisely they proved:

THEOREM (4.4) ([3]). *If two cyclotomic measures with rational coefficients $\mu = (a_0, \dots, a_{n-1})$, $\nu = (b_0, \dots, b_{n-1})$ are such that*

$$(4.5) \quad P_{k_1, \dots, k_{l-1}}(\mu) = P_{k_1, \dots, k_{l-1}}(\nu),$$

for $l \leq 6$ and arbitrary k_1, \dots, k_{l-1} such that $0 \leq k_j < n$, $j = 1, \dots, l-1$, then μ and ν differ by a rotation (of angle a multiple of $2\pi/n$).

For the structural invariants under the action of the dihedral group, we can use Theorem (4.4) to prove a similar result, in the case of symmetrical measures:

THEOREM (4.6). *Let μ, ν be cyclotomic measures with rational coefficients such that $\mu = S(\nu)$. If*

$$(4.7) \quad H_{k_1, \dots, k_{l-1}}(\mu) = H_{k_1, \dots, k_{l-1}}(\nu),$$

for degrees $l \leq 6$, then μ and ν are isometric.

Remark that the examples of [3] which show that for the cyclic invariants $l = 6$ is minimal with the above property, hold also for the structural invariants. Indeed, the pairs of measures in [3] which e.g. have the same third order invariants $(H_{k,l})_{k,l}$ but do not differ by a rotation, are also non-isometric. Similarly for pairs of measures with the same 4-th (and respectively 5-th) order invariants.

Proof. Let us fix the (multi-)index $\mathbf{k} = (k_1, \dots, k_{l-1})$. Then the sum and the product of $P_{\mathbf{k}}$ and $P_{-\mathbf{k}}$ are the same for μ and ν . Indeed, the sum is the same by the above formula relating $H_{\mathbf{k}}$ with $P_{\mathbf{k}}$ and by hypothesis, while for the product we have

$$(4.8) \quad \begin{aligned} P_{\mathbf{k}} \cdot P_{-\mathbf{k}} &= L_{k_1} \cdot L_{k_{l-1}} \cdot L_{-k_1 - \dots - k_{l-1}} \cdot L_{-k_1} \cdot \dots \cdot L_{-k_{l-1}} \cdot L_{k_1 + \dots + k_{l-1}} = \\ &= H_{k_1} \cdot \dots \cdot H_{k_{l-1}} \cdot H_{-k_1 - \dots - k_{l-1}}, \end{aligned}$$

which is a product of degree two \mathcal{D}_n -invariants, so by hypothesis it is the same for μ and ν .

Since from the sum and the product of the two conjugate complex numbers $P_{\mathbf{k}}$ and $P_{-\mathbf{k}}$ one can determine the numbers up to conjugation, and because we have assumed μ symmetrical (implying $P_{\mathbf{k}}$ real), we have that $P_{\mathbf{k}}(\mu) = P_{\mathbf{k}}(\nu)$.

To end the proof, let the index $\mathbf{k} = (k_1, \dots, k_{l-1})$ be arbitrary in the above formula for $l \leq 6$ and apply Theorem (4.4). Then μ and ν differ by a rotation, in particular they are isometric. \square

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA AT SANTA BARBARA
CALIFORNIA USA
gputinar@att.net

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BRANCHED COVERINGS AFTER FOX

Dedicated to the memory of Sylvia de Neymet de Christ.

JOSÉ MARÍA MONTESINOS-AMILIBIA

ABSTRACT. Branched coverings, folded coverings, and branched folded covering are all particular cases of the spreads introduced by Fox in a celebrated article which is freshly presented here together with Fox's approach to the ideal (or Freudenthal) compactification. A topological definition of "branched folded covering" is given. To that end, the new concept of "singular covering," slightly more general than Fox's definition of branched covering, is introduced and studied.

1. Introduction

The origins of Branched Covering Theory are closely related to the history of Riemann Surfaces. Given a multivalued function f , Riemann constructed a new domain Σ (the Riemann surface), a map p from this domain onto the original one, and a new univalued function g defined on Σ , such that $g = p^{-1} \circ f$. This process was called "uniformization" of f (see, for instance, [8]). At the beginning, the three objects were inseparable. But soon they gave rise to two theories. The concept of (abstract) Riemann surface Σ was defined and isolated from the functions f , p and g . And the functions p and g were the prototype of branched covering: the pair (Σ, p) being constructed by "cuts" performed in the original domain and the function g by lifting f . The definition by Weyl [32] of abstract Riemann surface soon gave rise to a complete theory of topological manifolds. However the theory of branched coverings lacked a firm topological basis, despite the generalizations to other dimensions. These generalizations forced themselves when it was discovered that the singularities of algebraic curves were essentially cones over classical tame knots (see, for instance [19]).

Branched coverings soon became an important instrument in the study of knots and manifolds (see, for instance [25]). "Folded coverings" were investigated by Tucker [29] and they were put to good use to place the concept of orbifold, rediscovered by Thurston [30], on solid ground (compare [16]).

At the time when Ralph H. Fox, professor of the University of Princeton in the United States, wrote his famous article [9] it was unknown whether all topological (metrizable) manifolds were polyhedra and whether there existed two polyhedra, underlying the same topological manifold, without isomorphic subdivisions. The positive solution of these two problems (at least in dimension 3) was the key to establish the topological nature of certain knot invariants that were defined starting from a polyhedron (combinatorial invariants).

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The definition process of these invariants consists in constructing branched coverings (überlagerungen, revêtements) over these knots, defined in a canonical way, and then obtaining the ordinary invariants of these covering manifolds: homology groups, linking form, etc. The difficulty of knowing whether these invariants were topological resided in that the construction of the branched coverings depended essentially on the polyhedral structure of the base space.

The part of the total space that covers what is outside of the singular set is an unbranched covering (the “associated” unbranched covering) and it was very well known already that this part depends only on the topology and not on the triangulation. It remained to be discovered whether the total space only depends on the topology of the pair formed by the base and the singular set.

The problem was centered on how to complete the part of the total space that covers what is outside of the singular set, building topologically and in a unique way the part of the total space lying over the singular set, removing all additional structure that is not topological. This involved a formidable challenge. Indeed, the singular set that until then was habitually an ordinary knot (tame), can now be a wild knot. And, how to define the cover on the knot if it doesn’t have a tubular neighborhood?

Fox dealt with all these problems by generalizing the concept (until then, non-topological) of branched covering. He introduced a new type of map called a *spread* to which he could apply a completion procedure. In this way he developed a topological theory of branched coverings. His theory possesses all the prospective advantages: it is valid for very general spaces (locally connected and T_1); the singular set is very general (basically it must have codimension 2); and the branched covering depends exclusively on the associated unbranched covering and on the topological pair (base, singular set).

A measure of the precision of the definitions of Fox and of the import of his theorems it is that the Theory of the Ideal Compactification of Freudenthal (see [11] and [5]) is a particular case of the Theory of Fox that also shows an elegant relationship with the one point compactification of Alexandroff (see [9] and [20]).

Fox developed all these ideas in lectures that he delivered in Mexico. There he found a favorable atmosphere, influenced by the conferences of Lefschetz. The late Sylvia de Neymet was among those in attendance. She had written a Masters Thesis under the direction of Lefschetz on branched coverings (see [26], [27], [28].)

Several authors have generalized more or less successfully these ideas of Fox. First, we must mention Michael’s work ([18], [17]) who develops his ideas from a footnote of [9] with masterful ability. Other works that take these ideas in a wider context are those mentioned before [26], [27], [28]. Finally it is indispensable to mention the work of Hunt [12] who used the Theory of Uniformities to develop the Theory of Fox.

Contemporary to the work of Fox was that of Church and Hemmingsen (see [3] and the bibliography mentioned there). This work can be read in conjunction with the work of Fox. It contains many important results on open maps. The work of Cernavskii [2] and Väisälä [31] characterize branched coverings

between compact manifolds. All this bears fruit in the important results of Edmonds and Berstein (see [1] and the bibliography mentioned there).

Moise and Bing later demonstrated (see [23]) that all metrizable 3-manifolds are polyhedra and that two such underlying polyhedra to homeomorphic 3-manifolds possess isomorphic subdivisions, solving in one stroke the problem of the topological nature of the combinatorial invariants of 3-manifolds and knots. However, the Theory of Fox still retains its importance.

We have already mentioned above the application of the Theory to the Ideal Compactification of Freudenthal and the results of Edmonds-Berstein and Edmonds. To this we can add an elegant way to build the desingularization of an algebraic curve. The theory can be applied also to construct branched coverings over sets more complicated than tame knots. Recently we have dedicated several papers to these investigations. We have generalized, for example, the theorem obtained independently by Hilden and Montesinos, proving that all open 3-manifolds are branched coverings of S^3 minus a tame set homeomorphic to the space of ends of the 3-manifold ([21], see also [22]).

In the course of these last investigations it was found necessary to ascertain the accuracy of the statements in [9] relative to the concept of branched covering and Fox's approach to the ideal (or Freudenthal) compactification. The result of that critical reading and the necessity to extend the theory to encompass folded and branched folded coverings was the original reason for writing the present expository article.

We have given a topological definition of "branched folded covering", introducing first the new concept of "singular covering", slightly more general than Fox's definition of branched covering. The paper is intended to be applied to the theory of (not necessarily locally flat) knots.

Besides the use of inverse limits to clarify the completion process, there are some new results. Among them, we offer sufficient conditions to ensure that a singular covering is surjective or has discrete fiber. There is also a surprising example of a branched covering with no discrete fiber (see [20]).

We have divided this exposition in various sections. The first seven sections cover the definition of spread, a number of sufficient conditions for a map to be a spread, which are essentially new, and the Theorem of existence and uniqueness of the completion of a spread. Sections 8 covers Fox's approach to the ideal (or Freudenthal) compactification as the completion of a particular spread. Sections 9 to 11 are devoted to the theory of branched coverings. Here, the concept of singular covering allows the common study of branched coverings and of branched folded coverings.

I dedicate this paper to the memory of Silvia de Neymet de Christ. She was interested in this topic all her life, and contributed to its understanding with some interesting papers ([26], [27], [28]).

2. Spreads: definition and first properties

All maps between topological spaces will be assumed continuous. The reference to topological matters will be the book of Kelley [13]. A map $g: Y \rightarrow Z$ between T_1 -spaces is a *spread* iff the connected components of inverse images of open sets of Z constitute a base for the topology of Y . That is, given a point

$y \in Y$ and an open neighborhood (“nbd” for short) U of y in Y there exists an open set W in Z such that the connected component of $g^{-1}(W)$ containing y (the y -component of $g^{-1}(W)$ from now on) is an open set between y and U .

THEOREM (2.1). *In the spread $g : Y \rightarrow Z$ the space Y is locally connected.*

Proof. By definition Y has a base of connected sets. Therefore, it is locally connected ([13], Chapter 1, S). \square

We remark that the space Z is assumed to be locally connected in the original paper [9], but this is not necessary (compare [12]).

PROPOSITION (2.2). *If $g : Y \rightarrow Z$ is a spread and Z_1 is an open subset of Z , then*

$$g \upharpoonright g^{-1}(Z_1) : g^{-1}(Z_1) \rightarrow Z_1$$

is a spread.

Example (2.3). Let Y be a T_1 -space. Then the identity map $1_Y : Y \rightarrow Y$ is a spread iff Y is locally connected.

Example (2.4). Let $g : Y \rightarrow Z$ be a spread and let Z be a subspace of some T_1 -space Z' . Then the natural map $i \circ g : Y \rightarrow Z'$ is a spread, where i is the inclusion.

Example (2.5). A spread which is surjective but Z is not locally connected.

The map $g : Y \rightarrow Z$, where

$$Y = \{n\}_{n=0}^{\infty} \subset \mathbb{R}; \quad Z = \{1/n\}_{n=1}^{\infty} \cup \{0\} \subset \mathbb{R};$$

and g is defined by $g(0) = 0$ and $g(n) = 1/n$, for $n > 0$, is a spread such that Y and Z are metrizable; Z is not locally connected and g is onto. Since g fails to preserve local connectedness g is not an identification.

Example (2.6). A spread which is surjective, g is neither open nor closed and Y and Z are metrizable, connected and locally connected.

The map $g : Y \rightarrow Z$, where

$$Y = \{z \in \mathbb{C} : |z| = 1\} - \{i\}; \quad Z = \mathbb{R}; \quad g(z) = \operatorname{Re}(z).$$

is a spread such that Y and Z are metrizable, connected and locally connected, but g is neither open nor closed. (One can modify the example to make g onto: take $g : \mathbb{R} \rightarrow \mathbb{S}^1$ defined by $g(x) = e^{i|x|}$.)

A map $g : Y \rightarrow Z$ is *light* iff $g^{-1}(z)$ is totally disconnected, for all $z \in Z$.

We want to prove that every spread is light. We first recall some definitions (compare [7], p. 24). If $x \in X$, the x -component is the connected component of X containing x ; it is the union of all connected sets containing x . The x -quasicomponent is the intersection of all clopen (=closed and open) sets containing x . The x -component is contained in the x -quasicomponent. A space X is *totally disconnected* iff for all $x \in X$ the x -component is $\{x\}$. A space X is *0-dimensional* iff it has a base of clopen sets. *Every 0-dimensional and T_1 -space X is totally disconnected.* In fact, each point has a nbd base formed by clopen sets. Hence the intersection of its members is $\{x\}$, since X is T_1 . (In such a space even the x -quasicomponents are singletons.)

THEOREM (2.7). *Let $g: Y \rightarrow Z$ be a spread. Let $y_1 \neq y_2$ be two points in the same fiber, that is,*

$$g(y_1) = g(y_2) = z \in Z.$$

Then there is an open nbd W of z in Z such that the y_1 -component of $g^{-1}(W)$ and the y_2 -component of $g^{-1}(W)$ fail to intersect. Therefore, there are nbds U and V of y_1 and y_2 in Y , respectively, such that

$$U \cap V = \emptyset.$$

In other words, the fibers of g have the Hausdorff property in Y .

Proof. In fact, since Y is T_1 , there is an open set U of y_1 in Y such that $y_2 \notin U$. We may assume that U belongs to the preferred base of Y , that is, there is an open nbd W of $g(y_1) = z$ in Z such that U is the y_1 -component of $g^{-1}(W)$. Let V be the y_2 -component of $g^{-1}(W)$. Then

$$U \cap V = \emptyset,$$

since the components of $g^{-1}(W)$ partition $g^{-1}(W)$ (by open sets). This completes the proof. \square

COROLLARY (2.8). *If in the spread $g: Y \rightarrow Z$ the space Z is T_2 , then Y is also T_2 .*

COROLLARY (2.9). *The fibers of any spread are totally disconnected, 0-dimensional and T_3 . In particular, every spread is light.*

Proof. Note that in the proof of Theorem (2.7), the set U is a clopen set in $g^{-1}(W)$. Thus, $g^{-1}(z)$ has a base of clopen sets of the form $U \cap g^{-1}(z)$. Hence $g^{-1}(z)$ is 0-dimensional and regular. Being $g^{-1}(z)$ a T_1 -space (even T_2), $g^{-1}(z)$ is totally disconnected. \square

The following example shows that the necessary conditions of Theorem (2.1) and Corollary (2.9) for a map to be a spread are not sufficient.

Example (2.10). A closed, open, onto map with discrete finite fibers which is not a spread.

Take the disjoint union of two copies Z and Z' of some compact Hausdorff n -manifold, and a finite subset $F_1 \subset Z$, and identify each point of $Z \setminus F_1$ with the corresponding point in Z' . The result is a compact n -manifold Y which is T_1 but not T_2 . Define

$$g: Y \rightarrow Z$$

in the obvious way. The fiber $g^{-1}(z)$ is discrete and g is closed, open and onto, but g is not a spread because Z is T_2 while Y is not (Corollary (2.8)).

There are spreads with no discrete fibers.

Example (2.11). Two spreads with no discrete fibers.

In the spread of Figure 1 $g^{-1}(0)$ has a limit point. Figure 2 depicts a spread in which $g^{-1}(1)$ is a Cantor set.

There are spreads with no locally compact fibers.

Example (2.12). A spread with no locally compact fibers.

The points of the Cantor set of Figure 2 can be parametrized by an infinite sequence of symbols 0 (read “up”) and 1 (read “down”) recording the way to proceed in each successive fork. Points corresponding to sequences that become constantly 0 or 1 from some index onwards, define a dense and countable subset of the Cantor set. If in Example (2.11) we delete this subset from the fiber $g^{-1}(1)$ the resulting spread will have a 0-dimensional fiber which is not locally compact.

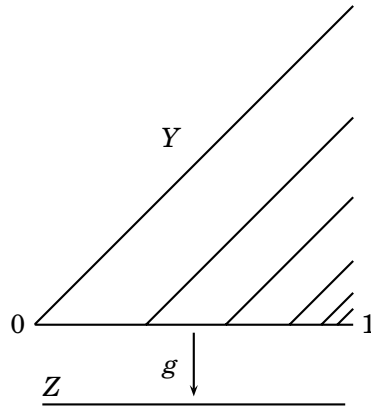


Figure 1. $g^{-1}(0)$ has a limit point.

LEMMA (2.13). *If in the spread $g: Y \rightarrow Z$, Z is regular, then Y is also regular.*

Proof. Set $g(y) = z$. Let V be a basic nbd of y in Y , that is, there is an open set W in Z such that V is the y -component of $g^{-1}(W)$. Let W_1 be an open nbd of z in Z such that $\overline{W_1} \subset W$ (Z is regular), and let V_1 be the y -component of $g^{-1}(W_1)$. Then

$$V_1 \subset V \cap g^{-1}(W_1).$$

Hence

$$\overline{V_1} \subset \overline{V} \cap \overline{g^{-1}(W_1)} = \overline{V} \cap g^{-1}(\overline{W_1}) \subset \overline{V} \cap g^{-1}(W) = V,$$

since V is closed in $g^{-1}(W)$. This completes the proof. \square

The following result is not used in this paper.

PROPOSITION (2.14) (John Isbell, see [17], p. 632). *If in the spread $g: Y \rightarrow Z$, Z is completely regular, then Y is also completely regular.*

Proof. Let $y \in Y$ and let V be a basic nbd of y in Y , that is, there is an open nbd W of $z = g(y)$ in Z and V is the y -component of $g^{-1}(W)$. Let

$$f: Z \rightarrow [0, 1]$$

be a map which is 0 at z and 1 outside W (by definition of completely regular space). Let

$$\tilde{f}: Y \rightarrow [0, 1]$$

be

$$\tilde{f}(v) = f \circ g(v),$$

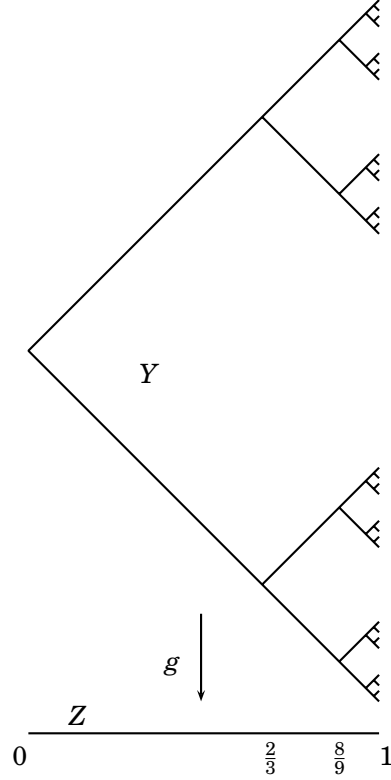


Figure 2. $g^{-1}(1)$ is the Cantor set.

for $v \in \bar{V}$ and let $\tilde{f}(v) = 1$, for $v \in Y \setminus V$. The map will be continuous if it is well defined in the intersection

$$\partial V = Y \setminus V \cap \bar{V},$$

because $Y \setminus V$ and \bar{V} are closed sets whose union is Y . Now, since V is closed in $g^{-1}(W)$,

$$g^{-1}(W) \cap \bar{V} = V.$$

Therefore

$$\partial V \cap g^{-1}(W) = \emptyset.$$

Thus, $g(\partial V) \subset Z \setminus W$. Then, for $v \in \partial V$, $f \circ g(v) = 1$. Hence, \tilde{f} is well defined and is continuous. Moreover

$$\tilde{f}(y) = f \circ g(y) = f(z) = 0.$$

Thus, Y is completely regular. \square

3. Discussion of spreads in case Y is not locally connected

In [9], section 8, R. H. Fox generalized the concept of spread when Y is not necessarily locally connected as follows. A map $g: Y \rightarrow Z$ is a *generalized-spread* if Y and Z are T_1 -spaces and if the clopen subsets of inverse images of open subsets of Z form a base for Y .

THEOREM (3.1). *If Y is locally connected, a map $g: Y \rightarrow Z$ is a generalized-spread iff it is a spread.*

Proof. Assume g is a generalized-spread. Let U be an open nbd of $y \in Y$. There exist an open set W in Z and a clopen subset G of $g^{-1}(W)$ such that

$$y \in G \subset U.$$

Let Q_y be the y -quasicomponent of $g^{-1}(W)$. Then

$$y \in Q_y \subset G \subset U.$$

Let C_y be the y -component of $g^{-1}(W)$. Then $C_y \subset Q_y$. Thus

$$y \in C_y \subset U$$

and C_y is open in $g^{-1}(W)$ (therefore in Y) because Y is locally connected. Thus the connected components of inverse images of open sets of Z form a base of the topology of Y , and therefore g is a spread. Conversely, assume g is a spread. Let U be an open nbd of $y \in Y$. There exists an open set W in Z such that $y \in C_y \subset U$, where C_y is the y -component of $g^{-1}(W)$. Then C_y is a clopen subset of $g^{-1}(W)$ and therefore g is a generalized-spread. \square

Example (3.2). Let Y be any 0-dimensional T_1 -space and let Z be a one-point space. Let $g: Y \rightarrow Z$ be the constant map. Then g is a generalized-spread. If Y is not locally connected, g is not a spread. In fact, Y is a spread iff Y is a discrete space.

Remark (3.3). Theorem (2.7) is false for generalized-spreads. Take in Example (3.2) the space Y to be a T_1 -but not T_2 -space. For instance, in the disjoint union of two copies of the rationals identify any two corresponding rationals, except for the “numbers” zero. This is why we will restrict ourselves in this paper to the locally connected case. For the general case see [17].

4. Five useful criteria

A topological space is locally compact iff every point has a base of compact nbds. (This definition differs from the one used in [13].) Given a subset A of a topological space X we denote by ∂A the frontier of A in X . That is, ∂A is the adherence of A in X minus the interior of A in X .

A useful result to decide if a certain light $g: Y \rightarrow Z$ between T_1 -spaces is a spread is the next Theorem (4.5) (compare [3], Lemma 1.4, Väisälä [31], Lemma 5.1, [17], p. 633 and [18]).

We need four lemmas.

The following lemma is well known (see, for instance, [14], XII.1.34 (b)).

LEMMA (4.1). *In a compact T_2 -space the connected components coincide with the quasicomponents.*

Proof. Let x be an element of a compact T_2 -space X . Let

$$\{A_i\}_{i \in I}$$

be the collection of clopen sets containing x . Their intersection K is the quasicomponent of X containing x . Thus, K is closed in X . Since the connected component of X containing x is contained in K , we only need to prove that K

is connected. Assume the contrary holds. Then there are disjoint closed sets F_1, F_2 in K (therefore in X) whose union is K . Assume $x \in F_1$. Since X is normal we can separate F_1, F_2 by disjoint open sets U_1, U_2 containing F_1, F_2 respectively. The union of the open sets

$$\{X \setminus A_i\}_{i \in I}$$

is

$$X \setminus \bigcap_{i \in I} A_i = X \setminus K.$$

Therefore the open sets $\{X \setminus A_i\}_{i \in I}$ cover the compact set

$$X \setminus (U_1 \cup U_2).$$

Therefore

$$X \setminus (U_1 \cup U_2)$$

is covered by a finite set $\{X \setminus A_j\}_{j=1}^k$. Hence,

$$X \setminus (U_1 \cup U_2) \subset X \setminus \bigcap_{j=1}^k A_j.$$

That is,

$$U_1 \cup U_2 \supset \bigcap_{j=1}^k A_j.$$

Now, the set $\bigcap_{j=1}^k A_j$ is closed and open in X and contains x . Therefore,

$$U_1 \cap \left(\bigcap_{j=1}^k A_j \right)$$

is open and contains x . But

$$U_1 \cap \left(\bigcap_{j=1}^k A_j \right) = (X \setminus U_2) \cap \left(\bigcap_{j=1}^k A_j \right)$$

is also closed. Then, by definition of K

$$K \subset U_1 \cap \left(\bigcap_{j=1}^k A_j \right).$$

From $F_2 \subset K$, the contradiction $F_2 \subset U_1$ follows. This completes the proof. \square

LEMMA (4.2) ([7], Theorem 1.4.5). *Every locally compact, totally disconnected T_2 -space is 0-dimensional.*

Proof. To show that a locally compact, totally disconnected T_2 -space X is 0-dimensional we take an open nbd V of $x \in X$ and try to find a clopen set W between x and V . There is an open nbd U of x such that \bar{U} is compact and contained in V . Then $R = \bar{U}$ is compact, totally disconnected and T_2 . By Lemma (4.1) the connected components of R coincide with its quasicomponents.

Let $\{A_i\}_{i \in I}$ the collection of clopen sets of R containing x . Their intersection $\bigcap_{i \in I} A_i$ is $\{x\}$. Consider the compact set ∂U . The collection of closed sets

$$\{A_i \cap \partial U\}_{i \in I}$$

have intersection

$$\bigcap_{i \in I} (A_i \cap \partial U) = \{x\} \cap \partial U = \emptyset.$$

It follows that the collection

$$\{A_i \cap \partial U\}_{i \in I}$$

of closed sets of the compact set ∂U fails to have the finite intersection property. Then, there is a finite subcollection $\{A_j\}_{j=1}^k$ such that

$$W = \bigcap_{j=1}^k A_j$$

does not intersect ∂U . The set W is clopen in $R = \bar{U}$. Thus

$$x \in W \subset \bar{U}$$

and W is closed in X . But W is open in $R = \bar{U}$ and is contained in U because

$$W \cap \partial U = \emptyset.$$

Therefore W is open in U and therefore in X . Then W is a clopen set in X between x and V . This completes the proof. \square

LEMMA (4.3). *Let Y be a T_2 -space and R a compact, totally disconnected, closed subset of Y . Then, for each open nbd U of every y in R there is an open set V in Y such that*

- (i) $y \in V \subset U$;
- (ii) $\partial V \cap R = \emptyset$.

Proof. Assume Y is T_2 and R is a compact and totally disconnected, closed subset of Y . Then, R is locally compact, totally disconnected and T_2 . Therefore R is 0-dimensional by Lemma (4.2). Let U be an open nbd of $y \in R$ in Y . Then $U \cap R$ is a nbd of $y \in R$ in R . Since R is 0-dimensional, there is a clopen subset U_0 of R such that

$$y \in U_0 \subset R \cap U.$$

Since R is compact, the closed sets U_0 and $R \setminus U_0$ are both compact. Since the space Y is T_2 , we can separate the two disjoint compact sets U_0 and $R \setminus U_0$ by open sets. Thus, there is an open set W of Y containing U_0 and whose adherence \bar{W} does not intersect $(R \setminus U_0)$. Set $V = W \cap U$. Then V is open in Y , contains U_0 and $\bar{V} \subset \bar{W}$. But this implies that

$$(\bar{V} - V) \cap R = \emptyset.$$

In fact,

$$(\bar{V} - V) \cap R = (\bar{V} - V) \cap [(R \setminus U_0) \cup U_0] = (\bar{V} - V) \cap U_0,$$

because

$$\bar{V} \cap (R \setminus U_0) \subset \bar{W} \cap (R \setminus U_0) = \emptyset.$$

But,

$$(\bar{V} - V) \cap U_0 = (\bar{V} \cap U_0) \setminus (V \cap U_0) = (\bar{V} \cap U_0) \setminus U_0 = \emptyset.$$

Therefore, $y \in V \subset U$ and $\partial V \cap R = \emptyset$, as we wanted to prove. \square

LEMMA (4.4). *Let Y be a space and R a closed subset of Y . Assume one of the following two conditions is satisfied:*

- (1) Y is T_3 and R is locally compact and totally disconnected;
- (2) Y is T_4 and R is 0-dimensional.

Then for each open nbd U of every y in R there is an open set V in Y such that

- (i) $y \in V \subset \bar{V} \subset U$, and
- (ii) $\partial V \cap R = \emptyset$.

Proof. Assume Y is T_3 and R is locally compact and totally disconnected. Since R is 0-dimensional (Lemma (4.2)) and locally compact, there is a clopen compact subspace U_0 of R such that

$$y \in U_0 \subset R \cap U.$$

Consider the disjoint sets U_0 and

$$(Y - U) \cup (R - U_0).$$

Here U_0 is compact, and

$$(Y - U) \cup (R - U_0)$$

is closed in Y because $R - U_0$ is closed in R . The space Y is T_3 . Then, there is an open set V of Y containing U_0 and whose adherence \bar{V} does not intersect

$$(Y - U) \cup (R - U_0).$$

But this implies that $y \in V$, that $\bar{V} \subset U$, and that

$$(\bar{V} - V) \cap R = \emptyset.$$

This last holds because

$$\bar{V} \cap (R - U_0) = \emptyset,$$

and

$$\bar{V} \cap U_0 = V \cap U_0 = U_0.$$

If Y is T_4 and R is 0-dimensional it is not needed that U_0 be compact. In fact, the closed sets U_0 and

$$(Y - U) \cup (R - U_0)$$

can be separated by open sets and the conclusion follows equally. \square

THEOREM (4.5). *Let Y be a locally connected T_2 -space, let Z be T_1 , and let $g: Y \rightarrow Z$ be a light map. Assume one of the following five conditions is satisfied:*

1. Y is locally compact and Z is T_2 .
2. Y is locally compact and g is closed.
3. Y is T_4 and g is closed with 0-dimensional fibers.
4. Y is T_3 and g is closed with locally compact fibers.
5. g is a proper map, that is, g is closed with compact fibers (Michael [17], [18])

Then g is a spread.

Proof. Let U be an open nbd of y in Y . We want to find an open nbd W of $z = g(y)$ in Z such that the y -component G of $g^{-1}(W)$ is contained in U . Assume first that Y is locally compact. Since any locally compact T_2 -space is regular, Y is T_3 . We may well assume that \bar{U} is compact. To find W , note that $g^{-1}(z)$ is a totally disconnected closed subset of a locally compact and T_3 space Y . Then $g^{-1}(z)$ is totally disconnected and locally compact. By Lemma (4.4) there is an open nbd V of $y \in g^{-1}(z)$ in Y such that \bar{V} is contained in U . Then \bar{V} is compact. Moreover, ∂V is closed in Y and compact and fails to intersect $g^{-1}(z)$. Then, if g is closed (case 2) $g(\partial V)$ is closed in Z . And if Z is T_2 (case 1) the set $g(\partial V)$, being compact, is also closed in Z . Hence

$$W := Z \setminus g(\partial V)$$

is an open nbd of z in Z not intersecting $g(\partial V)$. If we can show that the y -component G of $g^{-1}(W)$ is contained in V we will have finished the proof. But $G \cap V \neq \emptyset$ and $G \cap \partial V = \emptyset$, and if we assume

$$G \cap (Y - V) \neq \emptyset,$$

then

$$G = (G \cap V) \cup (G \cap (Y - \bar{V}))$$

would be the union of two disjoint non-empty open sets. This is not the case because G is connected. Therefore the assumption

$$G \cap (Y - V) \neq \emptyset$$

is false. Hence

$$G \subset V \subset U,$$

as we wanted.

In case 3, $g^{-1}(z)$ is closed and 0-dimensional and Y is T_4 ; while in case 4, $g^{-1}(z)$ is closed, totally disconnected and locally compact, and Y is T_3 . In both cases, Lemma (4.4) provides an open nbd V of $y \in g^{-1}(z)$ in Y such that $\bar{V} \subset U$ and

$$\partial V \cap g^{-1}(z) = \emptyset.$$

In case 5, $g^{-1}(z)$ is closed, totally disconnected and compact, and Y is T_2 . Lemma (4.3) provides an open nbd V of $y \in g^{-1}(z)$ in Y such that $V \subset U$ and

$$\partial V \cap g^{-1}(z) = \emptyset.$$

In all these three cases g is assumed to be closed. Then $g(\partial V)$ is closed in Z and the proof continues as before. This ends the proof of the theorem. \square

Remark (4.6). The spreads given in Theorem (4.5), Condition 1 (Y is locally compact and Z is T_2) enjoy the property that every point of Y has a base of open nbds U such that the restriction

$$g|_U : U \rightarrow g(U)$$

of g to any of its members is closed.

Proof. Let $y \in Y$. Let G be the y -component of any open subset W of Z containing $g(y)$. By the proof of the previous theorem, we can select W in such a way that \bar{G} is compact. We claim that

$$g|_G : G \rightarrow g(G)$$

is closed. Because if F is a closed subset of G , then $F = G \cap \overline{F}$ with \overline{F} compact. Then

$$g(F) = g(G) \cap g(\overline{F})$$

is closed because $g(\overline{F})$ is compact in a T_2 space, and hence, closed. The only point that needs checking is that

$$g(G) \cap g(\overline{F}) \subset g(F).$$

This will follow if we prove that

$$g(G) \cap g(\overline{F} \setminus F) = \emptyset.$$

Let $z = g(u) = g(v)$, for $u \in \overline{F} \setminus F$, and $v \in G$. Then $u \in g^{-1}(z) \subset g^{-1}(W)$. Since $u \in \overline{F} \setminus F$ then $u \in \overline{G} \setminus G$ because F is closed in G . But G is closed in $g^{-1}(W)$, so $(\overline{G} \setminus G) \cap g^{-1}(W) = \emptyset$. This contradicts the fact that $u \in (\overline{G} \setminus G) \cap g^{-1}(W)$. \square

Theorem (4.5) is especially neat when Y and Z are topological manifolds T_2 :

COROLLARY (4.7). *If Y and Z are topological manifolds T_2 , the map $g : Y \rightarrow Z$ is a spread iff it is light.*

A particular case of the following Corollary was proved by Fox ([9], note on p. 245).

COROLLARY (4.8). *Let Y be T_2 , locally connected, and compact. Let Z be T_1 . If $g : Y \rightarrow Z$ is light and closed (or g is light and Z is T_2), then g is a spread.*

Proof. It follows from case 2 of Theorem (4.5), because Y , being compact and T_2 , is locally compact. (If Z is T_2 , g is closed.) \square

If in Corollary (4.8) the condition T_2 for Y is relaxed to condition T_1 , the result is false (Example (2.10)).

Case 1 of Theorem (4.5) applies to Examples (2.5), (2.6), and (2.11). Case 3 applies to Example (2.12) and Case 4 to Example (4.12) below.

The next set of examples show that the conditions in the above Theorem are very difficult to relax.

Example (4.9) (J. Milnor [9], note in p. 245). Y is locally connected, T_4 , but not locally compact; Z is T_4 ; g is light; g is not closed, and has 0-dimensional (but not locally compact) fibers. The map g is not a spread.

The space Y is the union of the lines with equation

$$y = ax + b$$

in the xy -plane, in which a and b are rational numbers; Z is the x -axis and g is the orthogonal projection. The space Y is locally (path-) connected, metrizable separable, hence T_4 , but not locally compact. The space Z is T_4 (and metrizable separable). The map g is not closed. The map g is light (its fibers are 0-dimensional but not locally compact). However g is not a spread because the inverse image of any open interval of Z is connected.

Example (4.10) (Compare [15], Exercise 8.4 and [30]). Y is locally connected, T_4 , and locally compact; Z is T_1 but not T_2 ; g is light; g is not closed, and has 0-dimensional and locally compact fibers. The map g is not a spread.

The space Y is the real projective plane minus a point, obtained by identifying $(x, 0)$ with $(0, x^{-1})$, for every $x > 0$, in the subspace

$$\{(x, y) \in \mathbb{R}^2 - \{(0, 0)\} : x \geq 0, y \geq 0\}.$$

Let G be the group of homeomorphisms of Y generated by

$$(x, y) \mapsto (2x, y/2).$$

(The action of G in Y has discrete orbits, but is not discontinuous.) Let Z be the space obtained by identifying each orbit of G in Y to a point. The space Z is T_1 but not T_2 . The natural map $g : Y \rightarrow Z$ is light (with discrete fibers, hence 0-dimensional and locally compact fibers), but it is not a spread. In fact, for each $x > 0$, the preimage of a nbd of $g(x, 0)$ is connected. Note that the map g is not closed (if

$$E = \{1\} \times [0, 1],$$

the set

$$g^{-1}g(E) = G(E)$$

is not closed, because the y -axis lies in its adherence).

Example (4.11). Y is locally connected, T_2 , but not T_3 ; Z is T_2 ; g is light; g is closed, and has 0-dimensional and locally compact (but not compact) fibers. The map g is not a spread.

In

$$Y = \{(x, y) \in \mathbb{R}^2 : y \geq 0\},$$

with the usual topology, change the topology on the boundary ∂Y as follows. Let $p \in \partial Y$ and let U be a nbd of p in Y . Then

$$(U \setminus \partial Y) \cup \{p\}$$

is a new nbd of p . The space Y is called the plane of Moore. It is T_2 , but not T_3 . Therefore it is not metrizable nor locally compact. It is separable and satisfies the first axiom of countability but not the second. Let Z be the quotient of Y obtained by collapsing ∂Y to one point. The space Z is T_2 . The natural map $g : Y \rightarrow Z$ is closed and light (its fibers are either isolated points or the discrete set ∂Y ; therefore they are totally disconnected and locally compact). However g is not a spread because if W is an open nbd of ∂Y in Z , there is a component of $g^{-1}(W)$ containing ∂Y .

Example (4.12). Y is locally connected, T_3 (but not T_4), and not locally compact; Z is T_2 ; g is light; g is closed, with locally compact (but not compact) fibers. Precisely case 4 of Theorem (4.5) applies, and the map g is a spread.

In

$$Y = \{(x, y) \in \mathbb{R}^2 : y \geq 0\},$$

with the usual topology, change the topology in the boundary ∂Y as follows. Let $p \in \partial Y$ and let U be an open disc tangent to ∂Y at p . Then $U \cup \{p\}$, for all U , form a basis of open nbds of p . The nbds defined analogously but using closed disks form a basis of closed nbds of p . These closed nbds are not compact. Thus the space Y is T_3 and separable but not locally compact. (Y is known as the space of Niemytzki.) The set ∂Y has the discrete topology and is uncountable. Therefore Y is not normal. Hence it is not metrizable. It satisfies the first

axiom of countability but not the second. Let Z be the quotient of Y obtained by collapsing ∂Y to one point. The space Z is T_2 . The natural map $g : Y \rightarrow Z$ is light (its fibers are either isolated points or the discrete set ∂Y ; therefore they are 0-dimensional and locally compact). The map g is closed because if C is closed in Y the set $g^{-1}g(C)$ coincides with C (if $C \cap A = \emptyset$) or is the closed set $C \cup A$. By Theorem (4.5) the map g is a spread. It is interesting to see this directly. Consider two discs F and U tangent to ∂Y at p . The disc F is closed and the disc U is open and contained in F . Then $F \cup \{p\}$ is a closed nbd of p and $U \cup \{p\}$ is an open nbd of p . The set

$$V := [Y \setminus (F \cup \{p\})] \cup [U \cup \{p\}]$$

is an open nbd of ∂Y , has two connected components and is saturated for g . Therefore $g(V)$ is an open nbd of the point $g(\partial Y)$ in Z , such that one connected component of $g^{-1}g(V)$ is $U \cup \{p\}$.

Example (4.13). Y is locally connected, T_3 (but not T_4), and not locally compact; Z is T_4 ; g is light; g is not closed, and has discrete fibers. Moreover, any two points of the fiber $g^{-1}(z)$ are separated by components of $g^{-1}(W)$ for some open nbd W of z , and for every $z \in Z$. However, g is not a spread.

Let $g : Y \rightarrow Z$ of Figure 2, but modifying the topology of Y as follows. For each point F of the Cantor set C of Figure 2 there is an arc $\alpha_F : [0, 1] \rightarrow Y$ such that $\alpha_F(0)$ is the vertex of Y of valence two and $\alpha_F(1) = F$. The open arc $\alpha_F((t, 1))$ crosses successively the nodes n_1, n_2, \dots of Y . Each node n_i is the foot of a segment σ_i of Y not lying in $\alpha_F((t, 1))$. We may assume that Y is so constructed that $\text{diam}(\sigma_{i+1}) = \text{diam}(\sigma_i)/3$. Let $U_F(t, \epsilon)$ be the union of $\alpha_F((t, 1))$ with the open ball $B(n_i, \epsilon/3^i)$ in Y , for all i . The collection of all $U_F(t, \epsilon)$ for all $t \in (0, 1)$, and all $\epsilon < \text{diam}(\sigma_1)$ is, by definition, our nbd base of F . With this topology, C is discrete. The space Y is T_3 , locally connected, separable and satisfies the first axiom of countability but not the second. Moreover, Y is neither compact nor locally compact (the infinite set $E = \bigcup_{i=1}^{\infty} \{y \in \sigma_i : d(n_i, y) = \epsilon/3^i\}$ is closed and discrete). The map g is not closed because $g(E)$ is not closed in Z . The preimages of open, connected nbds of $g(C)$ contain finitely many components. Therefore g cannot be a spread. However, there are enough of these components to separate any two points in $g^{-1}g(C)$.

We end our list of examples with the following subtle one, contributed by E. Outerelo.

Example (4.14) (of Outerelo [24]). Y is locally connected, T_4 ; Z is T_4 ; g is light and closed, but there is a fiber which fails to be 0-dimensional (and, “*a fortiori*”, locally compact). The map g is not a spread.

The construction is based in a famous example by Erdős of a non-0-dimensional space X which is totally disconnected. The space X is a subset of Hilbert space H , which by definition is the set of all sequences $\{x_n\}_{n \in \mathbb{N}}$ of real numbers such that

$$\sum_{n=1}^{\infty} x_n^2 < +\infty.$$

The space H is separable and metrizable. In fact, it has the structure of a normed real vector space. The scalar product in H is given by

$$\langle \{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \rangle = \sum_{n=1}^{\infty} x_n y_n.$$

The associated norm is

$$\|\{x_n\}_{n \in \mathbb{N}}\| = \sqrt{\sum_{n=1}^{\infty} x_n^2}.$$

The metric is given, therefore, by

$$d(\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}.$$

The subset D of H formed by all sequences of H with rational coordinates having only a finite number of non-zero elements, is countable and dense in H . Thus H is separable. Moreover, being a normed, real vector space, it is locally connected because the open balls are connected sets.

Consider now the product space $H \times \mathbb{R}^+$, where $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$. Let Y be the subset of $H \times \mathbb{R}^+$ obtained by deleting from $H \times \{0\}$ all elements $(\{x_n\}_{n \in \mathbb{N}}, 0)$ having at least one coordinate x_i irrational. Set $X = Y \cap H \times \{0\}$. This is a closed subset of Y . Let $p : Y \rightarrow Z$ be the canonical projection from Y onto the quotient space $Z = Y/X$. Then p is a closed map.

The space Y , being a subspace of the metric space $H \times \mathbb{R}^+$, is metrizable. Therefore Y is T_4 , and it is easily seen that the quotient space Z is also T_4 . Moreover Y is easily seen to be locally connected. Now, the space of Erdős X is totally disconnected but not 0-dimensional (see, for instance, [7]). It cannot be locally compact by Lemma (4.2). Therefore $p : Y \rightarrow Z$ is closed and light, and the spaces Y and Z are locally connected T_4 spaces, but p is not a spread because it has a non 0-dimensional fiber (and a non locally compact fiber).

If in Case 1 of Theorem (4.5) the condition locally compact and T_2 for Y is relaxed to the condition T_3 the theorem is false (Example (4.9)). If in this same Case 1 the condition of Z being T_2 is relaxed to condition T_1 , the theorem is false (Example (4.10)). If in Case 2 of Theorem (4.5) the condition of Y to be T_2 is relaxed to condition T_1 , the conclusion of the theorem is false (Example (2.10)). If in Case 3 of Theorem (4.5) the condition “0-dimensional fibers” of g is deleted, the conclusion of the theorem is false (Example (4.14)). If in Case 4 of Theorem (4.5) the condition of Y being T_3 is relaxed to condition T_2 , the conclusion of the theorem is false (Example (4.11)). If in Case 4 of Theorem (4.5) the condition of g being closed is deleted, the conclusion of the theorem is false (Examples (4.10) and (4.13)). If in Case 4 of Theorem (4.5) the condition “locally compact fibers” of g is deleted, the conclusion of the theorem is false (Example (4.14)).

5. Complete spreads

Let $g : Y \rightarrow Z$ be a spread and let z be a point in Z . Denote by $\mathcal{E}(z)$ the set of open nbds of z in Z . If W_1 and $W_2 \in \mathcal{E}(z)$ and if we define $W_2 \geq W_1$ iff $W_2 \subset W_1$

(“ W_2 follows W_1 ”) then $(\mathcal{E}(z), \geq)$ is a directed set. Associated with this directed set $\mathcal{E}(z)$, there is an inverse system of topological spaces and continuous maps defined as follows. If $W \in \mathcal{E}(z)$, the space Y_W is by definition the topological space obtained by identifying each connected component of $g^{-1}(W)$ to one point (it is a discrete space). (If $y_z(W)$ denotes a connected component of $g^{-1}(W)$, the same notation will serve to denote the corresponding point in Y_W .) For $W_2 \geq W_1$ define the bonding map

$$j_{W_1 W_2} : Y_{W_2} \rightarrow Y_{W_1}$$

as the map induced by the canonical inclusion

$$g^{-1}(W_2) \subset g^{-1}(W_1).$$

Then

$$\{Y_W; j_{W_1 W_2}; \mathcal{E}(z)\}$$

is an inverse system whose limit will be denoted by Y_z . The elements of Y_z , classically called *threads over z* , are members $y_z = \{y_z(W)\}$ of the infinite cartesian product

$$\prod_{W \in \mathcal{E}(z)} Y_W$$

such that

$$j_{W_1 W_2}(y_z(W_2)) = y_z(W_1),$$

for every $W_2 \geq W_1$. That is, such that

$$y_z(W_2) \subset y_z(W_1) \text{ if } W_2 \subset W_1.$$

The restriction to Y_z of the canonical projection

$$\prod_{W \in \mathcal{E}(z)} Y_W \rightarrow Y_W$$

will be denoted by p_W . Let U be a fixed component of $g^{-1}(W)$. Let U_W be the union of threads

$$y_z = \{y_z(W)\}$$

of Y_z over z whose W -coordinate is U ; that is, such that $y_z(W) = U$. It is well known [6], Proposition 2.5.5, that a base for the topology of Y_z is

$$\{U_W : W \in \mathcal{E}(z), U \text{ component of } g^{-1}(W)\}.$$

Note that a thread is perfectly determined by the coordinates $y_z(V)$ of the members V of a base of the system of nbds of z . Because if $W \in \mathcal{E}(z)$ and V belongs to the base and is contained in W then $y_z(W)$ is the component of $g^{-1}(W)$ containing $y_z(V)$.

PROPOSITION (5.1). *If*

$$y_z = \{y_z(W)\} \in Y_z$$

is a thread over z , the intersection

$$\bigcap_{W \in \mathcal{E}(z)} y_z(W)$$

consists at most of one point (in $g^{-1}(z)$).

Proof. If the intersection is not empty it is a subset of the fiber $g^{-1}(z)$. But the points of the fiber are separated by components of $g^{-1}(W)$ for some W (see Theorem (2.7)). \square

PROPOSITION (5.2) (Compare [13], 4.5 Lemma of embedding). *The function*

$$\lambda_z : g^{-1}(z) \rightarrow Y_z$$

defined by

$$\lambda_z(y) = \{yW\},$$

where yW is the y -component of $g^{-1}(W)$, is continuous, open onto its image, and injective.

Proof. By Proposition (5.1)

$$\bigcap_{W \in \mathcal{E}(z)} (yW) = \{y\}.$$

Thus λ_z is injective. Let us see that λ_z is continuous. Let U_W be a member of the base of Y_z . Then

$$y \in U \cap g^{-1}(z) \Leftrightarrow yW = U \Leftrightarrow \lambda_z(y) \in U_W.$$

That is

$$\lambda_z^{-1}(U_W) = U \cap g^{-1}(z)$$

which is open in $g^{-1}(z)$. Thus λ_z is continuous. Moreover λ_z is open onto its image because

$$\lambda_z(U \cap g^{-1}(z)) = \lambda_z \lambda_z^{-1}(U_W) = U_W \cap \lambda_z(g^{-1}(z)).$$

□

Definition (5.3) ([9]). The spread g is **complete** iff, for every thread

$$y_z = \{y_z(W)\}$$

over z , and for every z , the intersection

$$\bigcap_{W \in \mathcal{E}(z)} y_z(W)$$

is non-empty (and consists of just one point).

COROLLARY (5.4). *The spread g is complete iff*

$$\lambda_z : g^{-1}(z) \rightarrow Y_z$$

is a homeomorphism for every $z \in Z$.

The spread of Example (2.6) is not complete. The spreads of Figures 1 and 2 are complete.

In some cases it is possible to visualize the inverse system

$$\{Y_W; j_{W_1 W_2}; \mathcal{E}(z)\}.$$

This is specially so when Z is first countable and the sets Y_W are finite. We then construct a tree whose vertices are the points in the union of all Y_W , where W runs over a countable nbd base of z , and whose edges connect points with their images under j . For instance, these trees for the fiber over $\{0\}$ in the spreads of Figures 1 and 2 coincide with the spaces Y in the Figures. In these examples, the fiber over $\{0\}$ is the end space of the tree in its Freudenthal compactification (see Section 8 below). When each branch of the tree becomes eventually linear, then the space Y_z is discrete.

Example (5.5). A complete spread $g : Y \rightarrow Z$ with Y and Z compact, with a point z_0 in Z whose fiber $g^{-1}(z_0)$ is finite, but for each open connected nbd W of z_0 in Z its inverse image $g^{-1}(W)$ has more than $\#g^{-1}(z_0)$ components.

In Figure 1 consider a straight line L passing through $\{1\}$ and inclined to the left. Delete from Y the points situated to the right of L . The restriction of g to this subset of Y is the required spread.

The following test for completeness is false if we only assume that Y is locally compact (Example (2.6), or take the inclusion $(0, 1) \subset [0, 1]$).

THEOREM (5.6). *Let $g : Y \rightarrow Z$ be a spread where Z is regular and Y is compact. Then g is complete.*

Proof. Let $y_z : \mathcal{E}(z) \rightarrow \mathcal{U}$ be a thread of g over $z \in Z$. We will show that

$$\bigcap_{W \in \mathcal{E}(z)} y_z(W)$$

is non-empty. For each open nbd V of z in Z let $y_z(\bar{V})$ be the component of $g^{-1}(\bar{V})$ containing $y_z(V)$. If $V_1 \subset V_2$ then

$$y_z(\bar{V}_1) \subset y_z(\bar{V}_2).$$

The set $y_z(\bar{V})$ is closed in $g^{-1}(\bar{V})$ hence in Y . The system of closed sets

$$\{y_z(\bar{V}) : V \in \mathcal{E}(z)\}$$

has the finite intersection property. In fact, let

$$V_1, \dots, V_n$$

be open nbds of z in Z . Let V be their intersection. Then

$$y_z(\bar{V}) \subset \bigcap_{i=1}^n y_z(\bar{V}_i).$$

Since Y is compact, the intersection

$$\bigcap \{y_z(\bar{V}) : V \in \mathcal{E}(z)\}$$

is non-empty. Since Z is regular, there is an open nbd V of z in Z such that

$$\bar{V} \subset W.$$

Then

$$y_z(V) \subset y_z(\bar{V}) \subset y_z(W).$$

Therefore

$$\bigcap \{y_z(\bar{V}) : V \in \mathcal{E}(z)\} = \bigcap \{y_z(W) : W \in \mathcal{E}(z)\}.$$

Thus, the intersection

$$\bigcap \{y_z(W) : W \in \mathcal{E}(z)\}$$

is non-empty. This completes the proof. \square

Example (5.7) (Outerelo). A non-complete spread $g : Y \rightarrow Z$ with Y compact (Z fails to be regular).

The space Y is $[0, 1] \times [0, 1]$. This is a compact, connected, locally connected, T_2 space. The space Z is obtained by adding to Y an ideal point ∞ . A base of nbds of ∞ is formed by all the sets containing ∞ and intersecting Y in the complement of a finite subset. The space Z is T_1 but fails to be regular. By Examples (2.3) and (2.4) the canonical inclusion $g : Y \rightarrow Z$ is a spread. However g is not complete. In fact, define a thread y_∞ of g over $\infty \in Z$ by associating to each basic nbd W of ∞ its intersection with Y (which is connected). This is a thread, but $\bigcap_{W \in \mathcal{E}(\infty)} (W \setminus \{\infty\}) = \emptyset$.

6. Existence of Fox's completion

In Fox's terminology X is **locally connected in** Y iff Y has a base whose members intersect X in connected sets.

Definition (6.1) ([9]). A spread $g : Y \rightarrow Z$ is a **completion** of a spread

$$f : X \rightarrow Z$$

if (i) X is a subspace dense and locally connected in Y ; and (ii) g is complete and extends f .

The spread $g : Y \rightarrow Z$ where

$$Y = \{z \in \mathbb{C} : |z| = 1\}; Z = [-1, 1]; g(z) = \operatorname{Re}(z)$$

is not a completion of

$$f = g|_{(Y - \{\pm 1\})},$$

because $Y - \{\pm 1\}$ is not locally connected in Y . In fact, the completion of f , given by the next theorem, has a disconnected domain (two copies of Z).

THEOREM (6.2) (see [9]). *Every spread has a completion.*

Proof. Given the spread $f : X \rightarrow Z$ we will construct a completion $g : Y \rightarrow Z$ in various steps.

(a) *Construction of the set Y .* By definition Y will be the set of threads y_z of f over z , for every $z \in Z$. We define

$$g : Y \rightarrow Z$$

by $g(y) = z$.

(b) *Topology of Y .* Let W be an open set in Z and let U be any component of $f^{-1}(W)$. Define

$$U/W = \{y_z : z \in W, y_z(W) = U\}$$

as the union of threads of f over points of W whose W -coordinate $y_z(W)$ is U .

We need to show that the collection of sets U/W where W runs over all open sets in Z , and U runs over all components of $f^{-1}(W)$, is a base of a topology for Y . Since it is obvious that

$$Y = \bigcup_{W,U} U/W,$$

it remains to see that for every

$$y \in U_1/W_1 \cap U_2/W_2$$

there is U_3/W_3 such that

$$y \in U_3/W_3 \subset U_1/W_1 \cap U_2/W_2.$$

Take

$$W_3 = W_1 \cap W_2.$$

The W_3 -coordinate $y(W_3)$ of the thread y is some component U_3 of $f^{-1}(W_3)$.

By the definition of thread

$$U_3 = y(W_1 \cap W_2) \subset y(W_1) \cap y(W_2).$$

In other words:

(*) $y(W_i) = U_i$ is the U_3 -component of $f^{-1}(W_i)$, $i = 1, 2$.

Let $\hat{y} \in U_3/W_3$. Then $\hat{y}(W_3) = U_3$. By the definition of thread

$$U_3 = \hat{y}(W_3) = \hat{y}(W_1 \cap W_2) \subset \hat{y}(W_1) \cap \hat{y}(W_2).$$

In other words:

(**) $\hat{y}(W_i)$ is the U_3 -component of $f^{-1}(W_i)$, $i = 1, 2$.

Comparing (*) with (**) we see that

$$\hat{y}(W_i) = U_i, \quad i = 1, 2.$$

That is $\hat{y} \in U_i/W_i$, $i = 1, 2$. Therefore

$$U_3/W_3 \subset U_1/W_1 \cap U_2/W_2,$$

as we wanted to prove.

(c) Y is T_1 . A base of nbds of $y \in Y$ is the system of U_i/W_i such that $y(W_i) = U_i$ and $g(y) \in W_i$. A member in the intersection of all U_i/W_i is a thread over

$$\bigcap W_i = g(y),$$

whose W_i -coordinate is U_i . But this is precisely the definition of y . Therefore y is the only thread in the intersection

$$\bigcap U_i/W_i.$$

Hence Y is T_1 .

(d) *The imbedding of X in Y .* Every member x of X defines a member \hat{x} of Y , namely, the thread of f over $f(x) \in Z$ whose W -coordinate (where W is a nbd of $f(x)$) is the x -component xW of $f^{-1}(W)$. Thus

$$\hat{x}(W) = xW.$$

But X is T_1 and the sets xW form a nbd base of x . Therefore

$$\bigcap xW = \{x\}.$$

If $\hat{x}_1 = \hat{x}_2$ then

$$\{x_1\} = \bigcap x_1W = \bigcap \hat{x}_1(W) = \bigcap \hat{x}_2(W) = \bigcap x_2W = \{x_2\}.$$

Thus $x \rightarrow \hat{x}$ is injective, and we identify x with its image \hat{x} in Y . But

$$(U/W) \cap X = U,$$

because

$$\hat{x}(W) = U = xW$$

iff $x \in U$. Since the sets U/W form a base of Y , and the sets U form a base of X it follows that X is a topological subspace of Y . Moreover, U being connected, the formula

$$(U/W) \cap X = U$$

implies that X is dense and locally connected in Y .

(e) g is a spread. We will presently see that for every U/W we have

$$\overline{U} \supset U/W \supset U.$$

But U is connected. Therefore, every set, such as U/W , sitting between U and \overline{U} will be connected. To see that every point $y_z \in \overline{U/W}$ is in \overline{U} consider a basic nbd U_i/W_i of y_z . Thus $y_z(W_i) = U_i$ and $g(y_z) \in W_i$. Since y_z belongs to $\overline{U/W}$, the intersection

$$(U_i/W_i) \cap (U/W)$$

is non-empty, and there exists some basic non-empty $U_j/(W_i \cap W)$ such that

$$U_j/(W_i \cap W) \subset (U_i/W_i) \cap (U/W).$$

Intersecting this relation with X we see that the non-empty set U_j lies in $(U_i/W_i) \cap U$. In other words, if the basic nbd U_i/W_i of y_z cuts U/W , it cuts also U . Hence $\overline{U/W}$ lies in \overline{U} , as we wanted to prove.

On the other hand $g^{-1}(W)$ is the union of open sets U_i/W , where the sets U_i are the components of $f^{-1}(W)$. Thus g is continuous.

But the open sets U_i/W are disjoint. Therefore the open sets U_i/W are closed in $g^{-1}(W)$, and being connected, they are the components of $g^{-1}(W)$. Then g is a spread.

(f) g is a complete spread. Let us consider a fixed thread \hat{y} of g over $z \in Z$. This thread assigns to each open nbd W of z a component U/W of $g^{-1}(W)$, in such a way that

$$U_1/W_1 \subset U_2/W_2$$

if $W_1 \subset W_2$. Then

$$U_1 = (U_1/W_1) \cap X \subset (U_2/W_2) \cap X = U_2.$$

This shows that there is a thread y of f over z whose W_i -coordinate is U_i . (This y is in fact a thread because if $W_1 \subset W_2$ then $U_1 \subset U_2$.) Then y is, by definition, a point of Y lying in the intersection of the sets U_i/W_i . Therefore g is complete because the intersections of the coordinates U_i/W_i of the thread \hat{y} is non-empty. \square

PROPOSITION (6.3). *Let the spread $g : Y \rightarrow Z$ be a completion of the spread $f : X \rightarrow Z$. Then*

$$f(X) \subset g(Y) \subset \overline{f(X)}.$$

Proof. $f(X) \subset g(Y) = g(\overline{X}) \subset \overline{g(X)} = \overline{f(X)}$. \square

The following example (6.4) shows that $g(Y)$ is not necessarily closed in Z . Therefore, for this example, the inclusion $g(Y) \subset \overline{f(X)}$ is proper.

Example (6.4). The spread $g : Y \rightarrow Z$ such that

$$Y = \{n\}_{n=1}^{\infty},$$

Z is the real line, and

$$g(n) = 1/n,$$

is a complete spread but $g(Y)$ is not closed in Z (and g is not surjective).

PROPOSITION (6.5). *Let the spread $g : Y \rightarrow Z$ be a completion of the spread $f : X \rightarrow Z$. If the spaces X and Z have countable bases, then Y also has a countable base.*

Proof. Since each component of $f^{-1}(W)$ contains a member of the countable base of X it follows that the components of $f^{-1}(W)$ are countable. Then the base of Y formed by the sets U/W when the sets W run over a countable base of Z is countable. \square

7. Uniqueness of Fox's completion

We need the following

LEMMA (7.1) (see [9]). *If X is dense and locally connected in Y , the intersection of X with any open connected set of Y is connected.*

Proof. Let V be an open, connected subset of Y . Then $U = V \cap X$ is non-empty because X is dense in Y . Suppose that $U = V \cap X$ is not connected. Then, there are non-empty disjoint open subsets A_1 and A_2 of X , such that

$$A_1 \cup A_2 = U.$$

We will construct non-empty disjoint open subsets B_1 and B_2 of Y , such that $B_1 \cup B_2 = V$. Let $y \in V$ be an arbitrary point. Using the fact that X is locally connected in Y find a nbd $N(y)$ of y in V , whose intersection with X , denoted by $M(y)$, is connected. Since

$$M(y) \subset U = A_1 \cup A_2,$$

the set $M(y)$ will be contained either in A_1 or A_2 . In the first case we define y to be a member of B_1 ; in the second, it will define a member of B_2 . Clearly

(i) $A_i \subset B_i$, $i = 1, 2$;

(ii) B_i is open:

$$y \in B_i \implies N(y) \subset B_i;$$

(iii) $B_1 \cap B_2 = \emptyset$;

(iv) $B_1 \cup B_2 = V$.

Therefore V is not connected. This contradiction completes the proof. \square

Let

$$g_i : Y_i \rightarrow Z_i$$

be completions of the spreads

$$f_i : X_i \rightarrow Z_i, i = 1, 2.$$

Let

$$h : X_1 \rightarrow X_2$$

and

$$l : Z_1 \rightarrow Z_2$$

be maps such that

$$f_2 \circ h = l \circ f_1$$

(h covers l).

THEOREM (7.2) (compare [9]). *The map h covering l extends uniquely to a map $k : Y_1 \rightarrow Y_2$ covering l .*

Proof. (a) *Definition of k .* Let

$$y_1 \in Y_1;$$

$$z_2 := l \circ g_1(y_1);$$

$$W_2 \in \mathcal{E}(z_2);$$

$$W_1 := l^{-1}(W_2);$$

and let

$$V_1 := y_1(g_1^{-1}W_1),$$

denote the y_1 -component of $g_1^{-1}W_1$. By Lemma (7.1)

$$U_1 := V_1 \cap X_1$$

is connected, and open in X_1 . Then

$$h(U_1) \subset X_2$$

is connected and it is contained in $g_2^{-1}W_2$. Then there is a component V_2 of

$$g_2^{-1}W_2$$

containing $h(U_1)$. Thus we have constructed a thread t_2 of g_2 over z_2 whose W_2 -coordinate, for $W_2 \in \mathcal{E}(z_2)$, is

$$V_2 \subset g_2^{-1}W_2.$$

And, since g_2 is complete, the intersection of the coordinates of such a thread consists of one point, namely $y_2 := k(y_1)$. This defines k as a function.

(b) k extends h and covers l . This can be easily checked.

(c) k is continuous. (Notation as in part (a)). Since the coordinates of the thread t_2 form a nbd base of the nbd system of $y_2 = k(y_1)$, we only need to prove that

$$k(V_1) \subset V_2$$

because V_1 is a nbd of y_1 . Let $y'_1 \in V_1$. To obtain $k(y'_1)$ we select a nbd of

$$z := l \circ g_1(y'_1).$$

But since $z \in W_2$, we select just $W_2 \in \mathcal{E}(z)$. Consider

$$V := y'_1(g_1^{-1}W_1).$$

Since $y'_1 \in V_1$, the component V coincides with V_1 . Thus, the thread t of g_2 over z defining $k(y'_1)$ has, as W_2 -coordinate, the component V_2 . Then $k(y'_1) \in V_2$, as we wanted to prove.

(d) *k is unique.* Assume not. There exists \tilde{k} , different from k , covering l and extending h . There is some $y \in Y_1$ for which

$$k(y) \neq \tilde{k}(y).$$

Since both $k(y), \tilde{k}(y)$ belong to $g_2^{-1}(z)$ where

$$z = l \circ g_1(y),$$

there are disjoint open nbds U, V of $k(y), \tilde{k}(y)$ in Y_2 , respectively (property T_2 in the fibers of a spread: Theorem (2.7)). Since X_1 is dense in Y_1 , there is a net R in X_1 converging to y . Since \tilde{k} and k are both continuous, the net

$$k(R) = \tilde{k}(R)$$

in Y_2 converges to $k(y)$ and to $\tilde{k}(y)$ simultaneously. This is not possible, because the nbds U, V of $k(y), \tilde{k}(y)$, respectively, are disjoint, and the same net cannot be eventually in two disjoint sets. This concludes the proof of the theorem. \square

As a Corollary, we obtain the theorem of uniqueness of the completion of a spread.

Note (7.3). Fox's proof of his Unicity Theorem contains an error detected and corrected by F. González Acuña (see [12], p.149). Fox's proof is valid for T_2 spaces, but can be repaired easily observing that points in the same fiber can be separated by disjoint neighborhoods. This is the path followed here instead of the more arduous one used in [12].

COROLLARY (7.4). *Two completions of the same spread are topologically equivalent.*

Proof. Apply Theorem (7.2) to two completions

$$g_i : Y_i \rightarrow Z$$

of the spread

$$f : X \rightarrow Z, \quad i = 1, 2;$$

and let

$$1_X : X \rightarrow X$$

and

$$1_Z : Z \rightarrow Z$$

denote the identity maps. We obtain maps

$$k_1 : Y_1 \rightarrow Y_2$$

and

$$k_2 : Y_2 \rightarrow Y_1,$$

covering 1_Z and extending 1_X . Now

$$k_2 \circ k_1 : Y_1 \rightarrow Y_1$$

and

$$1_{Y_1} : Y_1 \rightarrow Y_1$$

coincide by uniqueness. Also $k_1 \circ k_2 = 1_{Y_2}$. Hence k_1 is a homeomorphism that restricted to X is the identity, and such that $g_2 \circ k_1 = g_1$. This completes the proof. \square

LEMMA (7.5). *Let A be a subspace of X and let $Y \subset X$ be an open subset. Then $Y \cap A$ is locally connected in Y if A is locally connected in X . Moreover $Y \cap A$ is dense in Y if A is dense in X .*

Proof. If there is a base of X whose members when intersected with A give connected sets, take the members of that base lying in Y . Their intersection with A coincides with their intersection with $Y \cap A$ and is therefore connected. If A is dense in X , every open set in Y is open in X . Hence it cuts A , and a fortiori $Y \cap A$. \square

PROPOSITION (7.6). *Let*

$$g : Y \rightarrow Z$$

be a complete spread. Assume $Y \subset Y'$ is dense and locally connected in Y' . Let

$$g' : Y' \rightarrow Z$$

be a spread extending g . Then $Y = Y'$ and $g = g'$.

Proof. If g' is complete then $g' = g$ because both g and g' are completions of g . If g' is not complete, take a completion of it and proceed as before. \square

Definition (7.7). Let $g : Y \rightarrow Z$ be a spread. An automorphism of g is a homeomorphism $\alpha : Y \rightarrow Y$ such that $g \circ \alpha = g$. The group of all automorphisms of g is denoted by $\text{Aut}(g)$.

COROLLARY (7.8). *Let Z_1 be an open subset of Z and let*

$$j : Z_1 \rightarrow Z$$

be the canonical inclusion. Let

$$f : X \rightarrow Z_1$$

be a complete surjective spread. If $g : Y \rightarrow Z$ is the completion of the spread

$$j \circ f : X \rightarrow Z,$$

then the restriction to X of any automorphism of Y is an automorphism of X . The map so defined between $\text{Aut}(g)$ and $\text{Aut}(f)$ is an isomorphism.

Proof. Let $\alpha : Y \rightarrow Y$ be an automorphism of Y . We have to see that $\alpha(X) = X$. Now,

$$g \mid g^{-1}(Z_1) : g^{-1}(Z_1) \rightarrow Z_1$$

is a spread because Z_1 is an open subset of Z . The set $g^{-1}(Z_1)$ is an open subset of Y , and X is a subset of Y which is dense and locally connected in Y , then (Lemma (7.5)) X is dense and locally connected in $g^{-1}(Z_1)$. By Proposition (7.6), $X = g^{-1}(Z_1)$. Then, $\alpha(X) = X$. \square

8. The ideal compactification

The Theory so far developed serves, as proved by Fox, to ascertain the existence and uniqueness of a compactification introduced by Freudenthal [11] and called the ideal or Freudenthal compactification. This compactification adds one point for each end of the topological space. It is very useful in the study of topological manifolds and desingularization of algebraic curves. However, despite this fact, it is hard to find good expositions in the literature. The reference [5] is an exception. We will follow closely Fox [9].

The following theorem will be used to show that the completion of some particular spread to be constructed later has compact domain. I am indebted to E. Outerelo for explaining to me how to avoid the condition T_2 for $\overline{f(X)}$ in the next theorem.

We first prove an auxiliary result.

LEMMA (8.1). *Let*

$$f : X \rightarrow Z$$

be a spread and assume Z satisfies the first axiom of countability. Let

$$g : Y \rightarrow Z$$

be the completion of f . If Z has a base such that for each W in the base, $f^{-1}(W)$ has a finite number of components, then every sequence $\{y_i\}_{i=1}^{\infty}$ in Y covering a convergent sequence $\{g(y_i)\}_{i=1}^{\infty}$ to $z_0 \in Z$, has a convergent subsequence to a point covering z_0 .

Proof. By hypothesis

$$\{z_i\}_{i=1}^{\infty} = \{g(y_i)\}_{i=1}^{\infty} \rightarrow z_0 \in Z$$

Since Z satisfies the first axiom of countability, the hypothesis implies that there is a nbd base $\{W_i\}_{i=1}^{\infty}$ of z_0 such that

$$(i) \ W_{i+1} \subset W_i;$$

and

$$(ii) \ f^{-1}(W_i)$$

has a finite number of components, for $i \geq 1$. We now construct a sequence by induction,

$$\mathbb{N} \supset J_1 \supset \cdots \supset J_i \supset J_{i+1} \supset \cdots$$

(where J_i is infinite for all i) and a second sequence

$$U_1 \supset \cdots \supset U_i \supset U_{i+1} \supset \cdots$$

(where U_i is a component of $f^{-1}(W_i)$) such that

$$y_n(W_i) = U_i, \text{ for all } n \in J_i.$$

Let us construct J_1 and U_1 . Since $z_n \rightarrow z_0$, there is $k_1 \in \mathbb{N}$ such that

$$\{z_n : n \geq k_1\} \subset W_1.$$

Then $y_n(W_1)$ is a component of $f^{-1}(W_1)$, for all $n \geq k_1$. But $f^{-1}(W_1)$ has a finite number of components. Therefore there is a component U_1 of $f^{-1}(W_1)$ and an infinite subset

$$J_1 \subset \{n : n \geq k_1\}$$

such that $y_n(W_1) = U_1$, for all $n \in J_1$. Assume J_i and U_i have been constructed. Take $k_{i+1} \in \mathbb{N}$ such that

$$\{z_n : n \geq k_{i+1}\} \subset W_{i+1}.$$

By induction, for all $n \in J_i$ with $n \geq k_{i+1}$, the set $y_n(W_{i+1})$ is contained in $y_n(W_i) = U_i$ and is a component of $f^{-1}(W_{i+1})$. Since $W_{i+1} \subset W_i$ the set $f^{-1}(W_{i+1}) \cap U_i$ is a (finite) union of components of $f^{-1}(W_{i+1})$. But the set

$$\{n : n \in J_i, n \geq k_{i+1}\}$$

is infinite. Therefore there is a component U_{i+1} of $f^{-1}(W_{i+1})$ and an infinite subset $J_{i+1} \subset J_i$ such that $y_n(W_{i+1}) = U_{i+1} \subset U_i$, for all $n \in J_{i+1}$. This ends the inductive construction. Therefore, we have a thread y_{z_0} of f over z_0 whose W_i -coordinate is U_i . The point y_{z_0} belongs to Y , and we claim that there is a subsequence of y_i converging to y_{z_0} . Take

$$n_1 < \cdots < n_i < n_{i+1} < \cdots$$

with $n_i \in J_i$, for $i \geq 1$. Then the subsequence $\{y_{n_i}\}$ of $\{y_i\}_{i=1}^\infty$ converges to y_{z_0} , because

$$\{U_i/W_i : i \geq 1\}$$

is a nbd base of y_{z_0} in Y , and given k , we have

$$\{y_{n_i} : i \geq k\} \subset \{y_n : n \in J_k\} \subset U_k/W_k$$

because $y_n(W_k) = U_k$, for all $n \in J_k$. This completes the proof. \square

COROLLARY (8.2). *Let*

$$f : X \rightarrow Z$$

be a spread and assume Z satisfies the first axiom of countability. Let

$$g : Y \rightarrow Z$$

be the completion of f . If Z has a base such that for each member W of it, $f^{-1}(W)$ has a finite number of components, then g is closed and $g(Y) = \overline{f(X)}$. Moreover, if both X and Z have a countable base then g is proper; that is, g is closed with compact fibers.

Proof. Let $A \subset Y$ be a closed set. Let z_0 be a point in $\overline{g(A)}$. Since Z satisfies the first axiom of countability, there is a sequence of the form $\{z_i\}_{i=1}^\infty = \{g(y_i)\}_{i=1}^\infty$, with $\{y_i\}_{i=1}^\infty$ in A , converging to z_0 (see, for instance, [14], VI.1.5). By Lemma (8.1), the sequence $\{y_i\}_{i=1}^\infty$ in Y has a convergent subsequence to some y_0 covering z_0 . Because A is closed, the point y_0 belongs to A . Therefore $g(y_0) = z_0 \in g(A)$. Thus, $g(A)$ is closed, and g is closed.

Since $f(X) \subset g(Y) \subset \overline{f(X)}$ (Proposition (6.3)) and $g(Y)$ is closed, $g(Y) = \overline{f(X)}$ follows. This completes the proof of the first part of the Corollary.

Now, assume that both X and Z have a countable base. Let K be a compact subspace of Z and $\{y_i\}_{i=1}^\infty$ a sequence contained in $g^{-1}(K)$. Then, the sequence $\{z_i\}_{i=1}^\infty = \{g(y_i)\}_{i=1}^\infty$ has a subsequence converging to a point z_0 in K . By Lemma (8.1), the sequence $\{y_i\}_{i=1}^\infty$ in $g^{-1}(K)$ has a subsequence convergent to some y_0 covering $z_0 \in K$. By Proposition (6.5), Y has a countable base and, therefore, $g^{-1}(K)$ is compact (by [14], VIII.1.87, for instance). Thus g has compact fibers and is closed; that is, g is proper. \square

THEOREM (8.3). *Let*

$$f : X \rightarrow Z$$

be a spread and assume both X and Z have a countable base. Let

$$g : Y \rightarrow Z$$

be the completion of f . If (i) Z has a base such that, for each W in the base, $f^{-1}(W)$ has a finite number of components, and (ii) $\overline{f(X)}$ is compact, then Y is compact and g is proper with image $\overline{f(X)}$.

Proof. This is an immediate consequence of the last Corollary. \square

We are ready to prove the Theorem of existence and uniqueness of the *ideal* (or *Freudenthal*) *compactification*.

THEOREM (8.4). *Let X be a T_3 -space, with countable base (that is, a metrizable separable space). Assume that X is connected, locally connected, locally compact but not compact. Then, there exists a compact space $Y \supset X$, unique up to homeomorphism, with the same properties as X and such that (i) X is dense, open and locally connected in Y ; and (ii) the set of ends*

$$\mathcal{E}(X) := Y \setminus X$$

is totally disconnected (compact and 0-dimensional).

Proof. (a) *Existence.* The space X is T_2 because it is metrizable. Moreover, since X is locally compact, the one point compactification

$$Z = X \cup \{z_0\}$$

is T_2 . The space X , being metrizable with countable base and locally compact, can be covered by an increasing sequence of compact sets, whose complements form a nbd base of $\{z_0\}$. Thus Z is metrizable because it is compact, T_2 and has countable base. Thus we see that both X and Z share the same properties. The inclusion

$$j : X \rightarrow Z$$

is a spread because Z is T_1 and X is locally connected. Let

$$g : Y \rightarrow Z$$

be its completion. Then $X \subset Y$ is dense, locally connected in Y and open (since it is the inverse image of the open set X of Z). Moreover Y is T_1 , locally connected, regular (since Z is regular: Lemma (2.13)), and with countable base (Proposition (6.5)). The space Y is connected because it is the adherence of the connected space X . To see that Y is compact we will use Theorem (8.3). Therefore, we will prove that $z_0 \in Z$ possesses a base of open nbds such that, if W belongs to it, the number of connected components of

$$j^{-1}(W) = W \cap X = W \setminus \{z_0\}$$

is finite. (It helps to think of $\{z_0\}$ as the point $\{0\}$ in Figures 1 and 2.)

Let W_1 be an open nbd of z_0 such that $W_1 \neq Z$. Now Z is regular. Then there is an open nbd W_2 of z_0 such that $\overline{W_2} \subset W_1$. The frontier

$$B = \overline{W_2} \setminus W_2$$

of W_2 is non-empty and closed (Z is connected) in the compact space Z . Therefore B is compact. Now

$$B \subset \overline{W_2} \subset W_1$$

and $z_0 \notin B$. Then

$$B \subset W_1 \setminus \{z_0\} \subset X.$$

We want to prove that B intersects only a finite number of connected components of $W_1 \setminus \{z_0\}$. Assume the contrary. Then there is a sequence $\{x_i\}_{i=1}^{\infty}$ in B whose points belong to different connected components of $W_1 \setminus \{z_0\}$. Now B is compact. Then $\{x_i\}_{i=1}^{\infty}$ possesses a subsequence converging to some $x \in B$. This limit point x cannot lie in any connected component of $W_1 \setminus \{z_0\}$. This is clear, because the connected components of $W_1 \setminus \{z_0\}$ are open sets in X (locally connected) and if one of them contains x it must also contain infinitely many points of $\{x_i\}_{i=1}^{\infty}$, when, as a matter of definitions, it only contains one. But then $x \notin W_1 \setminus \{z_0\}$, contradicting the fact that

$$x \in B \subset W_1 \setminus \{z_0\}.$$

Let then

$$U_1, U_2, \dots, U_n$$

be the connected components of $W_1 \setminus \{z_0\}$ intersecting B .

On the other hand, let us see that W_2 fails to contain connected components U of $W_1 \setminus \{z_0\}$. Assume the opposite. Then $U \subset W_2 \subset \overline{W_2}$. Since U is closed in $W_1 \setminus \{z_0\}$ there is a closed set F in X such that

$$(W_1 \setminus \{z_0\}) \cap F = U.$$

But $U \subset \overline{W_2}$. Hence

$$U = \overline{W_2} \cap U = \overline{W_2} \cap (W_1 \setminus \{z_0\}) \cap F = \overline{W_2} \cap X \cap F$$

because $\overline{W_2} \subset W_1$. Therefore U is closed in X . But U is non-empty and open in X . Therefore X is not connected. This contradicts the hypothesis.

From these two results we conclude that the connected components of $W_1 \setminus \{z_0\}$ intersecting $\overline{W_2}$ are precisely those intersecting its frontier B , that is

$$U_1, U_2, \dots, U_n.$$

Now $\overline{W_2} \setminus \{z_0\}$ is the union of its connected components, and these connected components are contained in connected components of $W_1 \setminus \{z_0\}$ (intersecting $\overline{W_2}$). We deduce that

$$\overline{W_2} \setminus \{z_0\} \subset U_1 \cup U_2 \cup \dots \cup U_n.$$

Hence

$$\overline{W_2} \subset \{z_0\} \cup U_1 \cup U_2 \cup \dots \cup U_n.$$

Set

$$W = \{z_0\} \cup U_1 \cup U_2 \cup \dots \cup U_n.$$

Since

$$W = \{z_0\} \cup U_1 \cup U_2 \cup \dots \cup U_n = W_2 \cup U_1 \cup U_2 \cup \dots \cup U_n,$$

W is a union of open sets in Z . In fact, U_i (being open in $W_1 \setminus \{z_0\}$) is open in X , hence in Z . Thus W is an open nbd of z_0 in Z whose intersection with X has a finite number of connected components

$$U_1, U_2, \dots, U_n.$$

This completes the proof of the existence of the ideal compactification of X .

(b) *Uniqueness.* Consider now a compact space $Y \supset X$, with the same properties as X and such that

- (i) X is dense, open and locally connected in Y , and
- (ii) the set of ends

$$\mathcal{E}(X) := Y \setminus X$$

is totally disconnected (compact and 0-dimensional).

Let $Z = X \cup \{z_0\}$ be the one-point compactification of X . Let $g : Y \rightarrow Z$ the extension of the identity map in X defined by

$$g(\mathcal{E}(X)) = z_0.$$

The application g is continuous. In fact, let W be an open nbd of z_0 in Z . Then $Z \setminus W$ is a compact set K . Then

$$Y \setminus g^{-1}(W) = g^{-1}(Z \setminus W) = K$$

is closed in Y . Hence, $g^{-1}(W)$ is open in Y . Therefore g is continuous. Moreover, g is a spread because it satisfies condition 1 of Theorem (4.5). In fact, Y is locally connected and locally compact T_2 , while Z is T_2 and g is light ($g^{-1}(z_0)$ is totally disconnected). Moreover, g is a complete spread by theorem (5.6). Therefore Y is unique, up to homeomorphism, by the theorem of uniqueness of the completion of a spread (Corollary (7.4)). This concludes the proof. \square

The following theorem is very useful.

THEOREM (8.5). *Let X and Z_1 be T_3 -spaces, with countable base (that is, they are metrizable and separable). Assume also that they are connected, locally connected, locally compact, but not compact. Let*

$$f : X \rightarrow Z_1$$

be a surjective complete spread. Let Z be the ideal compactification of Z_1 and let

$$j : Z_1 \rightarrow Z$$

be the canonical inclusion. If

$$g : Y \rightarrow Z$$

denotes the completion of the spread

$$j \circ f : X \rightarrow Z,$$

then Y is the ideal compactification of X if Z has a base such that, for each member W of it, $f^{-1}(W)$ has a finite number of components.

Proof. By Theorem (8.3), Y is compact. By Proposition (6.3), g is surjective. Now, Z_1 is an open subset of Z and $f : X \rightarrow Z_1$ is a surjective and complete spread. Then

$$E_X := Y \setminus X$$

equals $g^{-1}(E_{Z_1})$, where

$$E_{Z_1} = Z \setminus Z_1.$$

It remains to see that E_X is totally disconnected. But let C be a connected subset of E_X . Then, the set $g(C)$ is a connected subset of the totally disconnected space E_{Z_1} . Therefore $g(C)$ is a singleton $\{z\}$. But then, C is a connected subset of the totally disconnected space $g^{-1}(z)$. Therefore C is also a singleton. This concludes the proof. \square

9. Singular Coverings

Let $g : Y \rightarrow Z$ be a (continuous) map. An open nbd W of $z \in Z$ is called *elementary* if $z \in g(Y)$ and g maps each (connected) component of $g^{-1}(W)$ homeomorphically onto W . A point of Z that admits an elementary nbd is an *ordinary point*. A non-ordinary point is *singular*. The set of ordinary points of Z is an open subset Z_o of Z . Its complement Z_s is the closed set of singular points. A *multi-covering* is a map $g : Y \rightarrow Z$, where Z is connected and $Z_o = Z$ (g is surjective). A multi-covering $g : Y \rightarrow Z$ with Y connected, is an (ordinary) *covering*. A covering with Y locally connected is open. Therefore it is an identification. This implies that Z is also locally connected and that the components of the inverse images of open connected sets of Z form a base of the topology of Y . Moreover, in this case $g : Y \rightarrow Z$ is *discrete*; that is, for all $z \in Z$ the fiber $g^{-1}(z)$, as subspace of Y , possesses the discrete topology.

LEMMA (9.1). *Let $g : Y \rightarrow Z$ be a multi-covering with Y locally connected. Let W be an open connected set of Z , and let C be any connected component of $g^{-1}(W)$. Then $g(C) = W$. Moreover*

$$g|_C : C \rightarrow W$$

is a covering.

Proof. $g(C)$ is open connected and contained in W . Let

$$z \in \overline{g(C)} \cap W.$$

Let V be an elementary nbd of z contained in W . Since

$$V \cap g(C) \neq \emptyset,$$

there exists $x \in C$ such that $g(x) \in V$. Let D be the x -component of $g^{-1}(V)$. Then $V \subset W$ implies $D \subset C$. Since V is elementary:

$$g(D) = V \subset g(C).$$

Hence $z \in V$ lies in $g(C)$. Thus $g(C)$ is clopen in W . Since W is connected, $g(C) = W$. This proves the first part.

To see that

$$h = g|_C : C \rightarrow W$$

is a covering, take an elementary nbd (with respect to g) $V \subset W$ of a point $z \in W$. Let D be any component of $g^{-1}(V)$. Since $V \subset W$, D is contained in some component of $g^{-1}(W)$. In other words D is a component of $h^{-1}(V)$ iff D is a component of $g^{-1}(V)$ and $D \subset C$. Then

$$h|_D = (g|_C)|_D = g|_D$$

is a homeomorphism onto V . Thus V is an elementary nbd (with respect to h) of the point $z \in W$. \square

In the theory of orbifolds, the “branched coverings” of Fox [9] and the “folded coverings” and “branched folded coverings” of Tucker [29] appear in a natural way (compare [16]). Next we propose a unified topological theory.

Definition (9.2). A *singular covering* is a complete spread $g : Y \rightarrow Z$ with Y connected, such that Z_o is connected, dense and locally connected in Z , and $g^{-1}(Z_o)$ is dense in Y . The restriction

$$g \mid g^{-1}(Z_o) : g^{-1}(Z_o) \rightarrow Z_o$$

is then a multi-covering, called the multi-covering associated to g . The *degree* of the multi-covering associated to g is, by definition, the degree of g .

Example (9.3). A complete, surjective spread which fails to be a singular covering only because Z_o is not locally connected in Z .

Proof. $g : \mathbb{R} \rightarrow \mathbb{S}^1$ defined by $g(x) = e^{i|x|}$. \square

Example (9.4). An example of complete spread $g : Y \rightarrow Z$ with Y connected and Z_o connected, dense and locally connected in Z , with $g^{-1}(Z_o)$ not dense in Y (and connected).

Proof. Put

$$\begin{aligned} Z &= \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\} \\ Y &= Z \cup \{(x, y) \in \mathbb{R}^2 : x \leq 0, y = 0\} \end{aligned}$$

and define $g : Y \rightarrow Z$ by

$$g(x, y) = (|x|, |y|).$$

\square

LEMMA (9.5). *Let A be a subspace of X which is connected, dense, and locally connected in X . Let*

$$A \subset B \subset X.$$

Then B is connected, dense, and locally connected in X and, moreover, X is locally connected.

Proof. Since

$$X = \overline{A} \subset \overline{B} \subset X,$$

B is dense. Since

$$A \subset B \subset \overline{A}$$

and A is connected, B is connected. Since A is locally connected in X , there is a base of X such that if W is one member of it, $W \cap A$ is connected. Then, because

$$W \cap A \subset W \cap B \subset W \subset \overline{W \cap A},$$

we deduce that $W \cap B$ is connected (hence B is locally connected in X) and that W is connected (hence X is locally connected). \square

PROPOSITION (9.6). *If $g : Y \rightarrow Z$ is a singular covering, then Z is connected and locally connected.*

Proof. Since $\overline{Z_o} = Z$ and Z_o is connected, Z is also connected. Lemma (9.5) implies that Z is locally connected because Z_o is connected, dense and locally connected in Z . \square

PROPOSITION (9.7). *Let $g : Y \rightarrow Z$ be a singular covering. Let W be an open connected subset of Z , and let C be any component of $g^{-1}(W)$. Then*

$$C \cap g^{-1}(Z_o)$$

is a union of components of

$$g^{-1}(W \cap Z_o)$$

and g restricted to

$$C \cap g^{-1}(Z_o)$$

is a multi-covering onto $W \cap Z_o$.

Proof. Since g is a spread, C is open in Y . Since $g^{-1}(Z_o)$ is dense in Y , C intersects $g^{-1}(Z_o)$. Then

$$C \cap g^{-1}(Z_o)$$

is contained in the union D of the components D_i of

$$g^{-1}(W \cap Z_o)$$

such that

$$D_i \cap C \neq \emptyset.$$

Let us see that

$$C \cap g^{-1}(Z_o) = D.$$

In fact,

$$D_i \subset g^{-1}(W)$$

is a connected set intersecting the component C of $g^{-1}(W)$. Hence $D_i \subset C$. Hence $D \subset C$. Now, $g|_{g^{-1}(Z_o)} : g^{-1}(Z_o) \rightarrow Z_o$ is a multi-covering and $W \cap Z_o$ is a connected open subset of Z_o (Lemma (7.1)). By Lemma (9.1), g restricted to

$$C \cap g^{-1}(Z_o)$$

is a multi-covering onto $W \cap Z_o$. \square

THEOREM (9.8). *Let $g : Y \rightarrow Z$ be a singular covering. Let W be an open connected subset of Z . Let C be any component of $g^{-1}(W)$. Then*

$$g|_C : C \rightarrow W$$

is a singular covering.

Proof. Since $g : Y \rightarrow Z$ is a complete spread, so is

$$g|_{g^{-1}(W)} : g^{-1}(W) \rightarrow W,$$

and

$$h = g|_C : C \rightarrow W.$$

Let W_o be the subset of W of ordinary points with respect to the map h . By Proposition (9.7), $C \cap g^{-1}(Z_o)$ is a union of components of $g^{-1}(W \cap Z_o)$ and

$$g|_{C \cap g^{-1}(Z_o)} : C \cap g^{-1}(Z_o) \rightarrow W \cap Z_o$$

is a multi-covering onto $W \cap Z_o$. Hence W_o contains $W \cap Z_o$. To prove that h is a singular covering it is enough to show (Lemma (9.5)) that $W \cap Z_o$ is connected, dense and locally connected in W and that $h^{-1}(W \cap Z_o)$ is dense in C (because if this is true, the same properties will be shared by W_o and $h^{-1}(W_o)$). But, since W is connected, $W \cap Z_o$ is connected (Lemma (7.1)). Since W is open in Z and Z_o is dense and locally connected in Z , then $W \cap Z_o$ is dense and locally connected in W (Lemma (7.5)). Using this same Lemma we see that

$$h^{-1}(W \cap Z_o) = C \cap g^{-1}(Z_o)$$

is dense in C because C is open in Y and $g^{-1}(Z_o)$ is dense in Y . \square

THEOREM (9.9). *A singular covering $g : Y \rightarrow Z$ is surjective if Z satisfies the first axiom of countability.*

Proof. Recall that if $g : Y \rightarrow Z$ is a spread and z is a point of Z , $\mathcal{E}(z)$ denotes the system of open nbds of z . If $W_1, W_2 \in \mathcal{E}(z)$ and we define that $W_2 \geq W_1$ iff $W_2 \subset W_1$, then $(\mathcal{E}(z), \geq)$ is a directed set. Since Z is locally connected and satisfies the first axiom of countability, there exists a base

$$\mathcal{B}(z) = \{V_i\}_{i=1}^{\infty}$$

of open and connected nbds of z such that $V_{i+1} \geq V_i$ for all i . This means that the base $\mathcal{B}(z)$ is a cofinal subset in $(\mathcal{E}(z), \geq)$. Recall that we have defined an inverse system of topological spaces and maps as follows. If $W \in \mathcal{E}(z)$ the space Y_W is, by definition, the topological (discrete) space obtained from $g^{-1}(W)$ by identifying each of its component to a point. For $W_2 \geq W_1$ define

$$j_{W_1 W_2} : Y_{W_2} \rightarrow Y_{W_1}$$

as the map induced by the canonical inclusion $g^{-1}(W_2) \subset g^{-1}(W_1)$. Then

$$\{Y_W; j_{W_1 W_2}; \mathcal{E}(z)\}$$

is an inverse system whose limit Y_z is homeomorphic to $g^{-1}(z)$ (Corollary (5.4)). Since the base $\mathcal{B}(z)$ is cofinal in $(\mathcal{E}(z), \geq)$ the limit Y_z is homeomorphic to the inverse limit of the inverse sequence

$$\{Y_W; j_{W_1 W_2}; \mathcal{B}(z)\}$$

(see, for instance, [6], Corollary 2.5.11). Let us see that the map

$$j_{W_1 W_2} : Y_{W_2} \rightarrow Y_{W_1}$$

is surjective. If C_1 is a component of $g^{-1}(W_1)$, then (Proposition (9.7)) $C_1 \cap g^{-1}(Z_o)$ is a union of components of $g^{-1}(W_1 \cap Z_o)$, where $W_1 \cap Z_o$ is connected (Lema (7.1)). Let D be any of them. Then

$$g(C_1) \supset g(D) = W_1 \cap Z_o \supset W_2 \cap Z_o.$$

Let $y \in C_1$ be such that $g(y) \in W_2 \cap Z_o$. Let C_2 be the y -component of $g^{-1}(W_2)$. Then $C_2 \subset C_1$. Thus

$$j_{W_1 W_2} : Y_{W_2} \rightarrow Y_{W_1}$$

is surjective. This implies that the inverse limit Y_z of the inverse sequence

$$\{Y_W; j_{W_1 W_2}; \mathcal{B}(z)\},$$

which is homeomorphic to $g^{-1}(z)$, is non-empty ([6], Exercise 2.5.A). Thus g is surjective. \square

Example (9.10) (Outerelo). A surjective singular covering $g : Y \rightarrow Z$ in which Z fails to satisfy the first axiom of countability.

$Y = Z = [0, 1]^{\mathbb{R}}$ and $g : Y \rightarrow Z$ is the identity map. Here $[0, 1]^{\mathbb{R}}$ is compact, regular, connected, locally connected but fails to satisfy the first axiom of countability. By Theorem (5.6), g is a complete spread.

Problem (9.11). Is every singular covering surjective?

COROLLARY (9.12) (A. Costa [4], Teorema 2.1). *A singular covering $g : Y \rightarrow Z$ is an open, surjective map if Z satisfies the first axiom of countability.*

Remark (9.13). Compare this with Example (9.3): this corollary is false if in the definition of singular covering it is not assumed that Z_o is locally connected in Z .

Proof. Let $g : Y \rightarrow Z$ be a singular covering. By Proposition (9.6), Z is locally connected and g is a spread. Then the components of inverse images of open *connected* subsets of Z form a base of the topology of Y . Let W be an open connected subset of Z and let C be any component of $g^{-1}(W)$. Then

$$g|_C : C \rightarrow W$$

is a singular covering, where W satisfies the first axiom of countability. Hence, $g|_C$ is surjective; that is, $g(C) = W$. Thus g is open. \square

In a singular covering $g : Y \rightarrow Z$, the set $Z_s = Z \setminus Z_o$ will be called the *singular locus* and will be said that g is a *covering singular onto Z_s* . Let z_s be a point of Z_s . Take $y \in g^{-1}(z_s)$. The *branching index $b(y)$* of y is the infimum of the degrees of the multi-coverings associated with the singular coverings

$$g|_C : C \rightarrow W,$$

where C is the y -component of $g^{-1}(W)$, for every open, connected nbd W of z_s . If all these degrees are infinite, we will say that the branching index is infinite.

There are examples of singular coverings in which a particular fiber possesses a limit point (and therefore is not discrete). The examples are in fact branched coverings (see Example (10.6) in the next section). However, we have:

THEOREM (9.14). *If in a singular covering $g : Y \rightarrow Z$ the branching indexes are all finite, then g is discrete.*

Proof. Fibers lying over points of Z_o are evidently discrete. Take $y \in Y$ such that $g(y) = z \in Z_s$. Let W be an open, connected nbd of z and let C be the y -component of $g^{-1}(W)$. Since by hypothesis, the branching index of y is finite, we may assume that the singular covering

$$g|_C : C \rightarrow W$$

has finite degree m . If y is an accumulation point of $g^{-1}(z)$, then $g^{-1}(z) \cap C$ possesses infinitely many points. Let

$$y_1, \dots, y_{m+1}$$

be $m + 1$ different points of $g^{-1}(z) \cap C$. By Theorem (2.7), there is an open, connected nbd W_1 of z in W such that the y_i -components

$$V_i, i = 1, \dots, m + 1,$$

of $g^{-1}(W_1)$ are all different (and disjoint). By Proposition (9.7),

$$g(V_i \cap g^{-1}(W_o)) = W_1 \cap W_o,$$

for $i = 1, \dots, m + 1$. Since W_o is dense in W , there exists $z_o \in W_1 \cap W_o$. This point has a preimage in each of the $m + 1$ disjoint sets V_i . This is impossible because z_o has exactly m preimages in C . This concludes the proof. \square

THEOREM (9.15). *Let $g : Y \rightarrow Z$ be a singular covering with Y locally compact. Then all the branching indexes are finite and therefore all fibers are discrete. If Y is compact the fibers are finite.*

Proof. Assume Y locally compact. Then, for every point $y \in Y$ and every nbd U of y there is an open connected nbd W of $z = g(y)$ in Z such that the y -component C of $g^{-1}(W)$ is contained in a compact subset K of U . (If Y is compact take $W = Z$ and $C = Y$.) If $z \in Z_o$ the discrete set $C \cap g^{-1}(z_o)$ lies in the compact set K . Hence it is finite. If $z \in Z_s$, for every $z_o \in W \cap Z_o$, the (non-empty) set $C \cap g^{-1}(z_o)$ is finite. Otherwise, there is an accumulation point q in K and then $g(q) = z_o$ is not an ordinary point, contrary to hypothesis. Then the branching index of y is finite. This ends the proof. \square

10. Branched coverings

Part of this section appears in [20].

Definition (10.1). A branched covering is a singular covering $g : Y \rightarrow Z$ such that $g^{-1}(Z_o)$ is locally connected in Y .

A branched covering is surjective and open if Z satisfies the first axiom of countability (Corollary (9.12)). We do not know of any nonsurjective branched covering.

Definition (10.2). An unbranched covering is a spread $f : X \rightarrow Z$ such that

- (i) Z_o is connected, dense and locally connected in Z ; and
- (ii) $X = f^{-1}(Z_o)$ is connected.

Then the covering $\tilde{f} : X \rightarrow Z_o$, defined by $\tilde{f}(x) = f(x)$, is called the *associated covering* of f . Since f is a spread, X is locally connected. Then \tilde{f} is open. Since Z_o is open in Z , f is also open.

THEOREM (10.3). *Let $g : Y \rightarrow Z$ be a branched covering. Then*

$$g \upharpoonright g^{-1}(Z_o) : g^{-1}(Z_o) \rightarrow Z$$

is an unbranched covering whose associated covering is

$$g \upharpoonright g^{-1}(Z_o) : g^{-1}(Z_o) \rightarrow Z_o.$$

Moreover g is the (unique) completion of

$$g \upharpoonright g^{-1}(Z_o) : g^{-1}(Z_o) \rightarrow Z.$$

Thus, g is determined by the unbranched covering

$$g | g^{-1}(Z_o) : g^{-1}(Z_o) \rightarrow Z;$$

or by the associated covering

$$g | g^{-1}(Z_o) : g^{-1}(Z_o) \rightarrow Z_o$$

and the inclusion $Z_o \subset Z$.

Proof. By Lemma (7.1) applied to the connected set Y , $g^{-1}(Z_o)$ is connected. Then

$$g | g^{-1}(Z_o) : g^{-1}(Z_o) \rightarrow Z$$

is an unbranched covering. Since $g^{-1}(Z_o)$ is connected, dense and locally connected in Y , then g is the completion of

$$g | g^{-1}(Z_o) : g^{-1}(Z_o) \rightarrow Z.$$

□

A useful method for constructing branched coverings is to complete unbranched coverings. In this process the set of ordinary points might increase (the completion might even be a covering).

THEOREM (10.4). *Let $g : Y \rightarrow Z$ be a branched covering. Let W be an open, connected subset of Z , and let C be any component of $g^{-1}(W)$. Then*

$$g | C : C \rightarrow W$$

is a branched covering whose image contains $W \cap Z_o$.

Proof. By Theorem (9.8) we know that

$$h = g | C : C \rightarrow W$$

is a singular covering. It remains to show that $h^{-1}(W_o)$ is locally connected in C . By Lemma (9.5), it is enough to show that the subspace $h^{-1}(W \cap Z_o)$ of $h^{-1}(W_o)$ is locally connected in C . Since C is open in Y ,

$$h^{-1}(W \cap Z_o) = C \cap g^{-1}(Z_o),$$

and $g^{-1}(Z_o)$ is locally connected in Y , it follows from Lemma (7.5) that $C \cap g^{-1}(Z_o)$ is locally connected in Y . □

In a branched covering $g : Y \rightarrow Z$, the set $Z_s = Z - Z_o$ is called the *branching set* and we say that g is a *covering branched over Z_s* . The subset of $g^{-1}(Z_s)$ of points with branching index 1 is called the *pseudobranching cover*, and the subset of $g^{-1}(Z_s)$ of points with branching index ≥ 1 is called the *branching cover*.

We have seen that if Y is compact the fibre is finite, and that if Y is locally compact, the branching indexes are all finite and the fibers are discrete.

Note however that even if the fiber is discrete the branching indexes might be infinite (for instance, a fiber which is just one point with branching index infinite).

There is an example of a branched covering having a non-discrete fiber. To capture the topology of the fiber in this example it is convenient to determine

it from its complement in Y . This is accomplished in the next theorem (which is trivially false for general singular coverings).

THEOREM (10.5). *Let $g : Y \rightarrow Z$ be a branched covering. Let $z_s \in Z_s$ and set $X = Y \setminus g^{-1}(z_s)$ and*

$$f = g \mid X : X \rightarrow Z.$$

Then $g^{-1}(z_s)$ is homeomorphic to the inverse limit X_{z_s} of the inverse system

$$\{X_W; h_{W_1 W_2}; \mathcal{E}(z_s)\}$$

where X_W is the discrete space obtained from $f^{-1}(W)$ by identifying its components to points, for every $W \in \mathcal{E}(z_s)$; and

$$h_{W_1 W_2} : X_{W_2} \rightarrow X_{W_1}$$

is induced by the canonical inclusion $f^{-1}(W_2) \subset f^{-1}(W_1)$, for every $W_2 \geq W_1$.

Proof. Since $g^{-1}(Z_o) \subset X \subset Y$, Lemma (9.5) implies that X is connected, dense and locally connected in Y . Then, by Lemma (7.1), if C is any component of $g^{-1}(W)$, then

$$C \cap X = C \cap f^{-1}(W)$$

is connected. Since C is open in Y and X is dense in Y , then $C \cap X \neq \emptyset$. Moreover, $C \cap X$ is a component of $f^{-1}(W)$. In fact, if

$$C \cap X \subset D \subset f^{-1}(W)$$

with D connected, we have $D \cap C \neq \emptyset$ and $D \subset g^{-1}(W)$. But C is a component of $g^{-1}(W)$, then necessarily $D \subset C$. Hence

$$D = D \cap X \subset C \cap X.$$

Thus, the map

$$k_W : Y_W \rightarrow X_W; k_W(C) = C \cap X$$

is well defined and sends the component C of $g^{-1}(W)$ to the component $C \cap X$ of $f^{-1}(W)$. This map is bijective. In fact, if $D \subset f^{-1}(W)$ is a component of $f^{-1}(W)$, there must be a component C of $g^{-1}(W)$ such that $D \subset C$. Then

$$k_W(C) = C \cap X$$

is a component of $f^{-1}(W)$ containing the component D of $f^{-1}(W)$. Thus

$$k_W(C) = C \cap X = D.$$

Hence k_W is surjective. If C_1 and C_2 are components of $g^{-1}(W)$ such that $k_W(C_1) = k_W(C_2)$, then

$$C_1 \cap X = C_2 \cap X \neq \emptyset,$$

which implies that $C_1 \cap C_2 \neq \emptyset$. Hence $C_1 = C_2$. Therefore, $k_W : Y_W \rightarrow X_W$ is a homeomorphism. Now it is evident that the limits of the inverse systems

$$\{X_W; h_{W_1 W_2}; \mathcal{E}(z_s)\}$$

and

$$\{Y_W; j_{W_1 W_2}; \mathcal{E}(z_s)\}$$

are homeomorphic. That is, X_{z_s} is homeomorphic to Y_{z_s} which, by Corollary (5.4), is homeomorphic to $g^{-1}(z_s)$. \square

We apply this theorem to the following example.

Example (10.6). A branched covering possessing a non-discrete compact fiber.

In the 3-sphere $S^3 = \mathbb{R}^3 + \infty$, take a circle C_1 of radius 1 in the xy -plane. Consider a translation h sending C_1 to a circle C_2 of radius 1 in the xy -plane, such that $C_1 \cap C_2 = \emptyset$. Define

$$C_{i+1} = h^i(C_1).$$

Denote the center of C_i by p_i . Let B_i be the 3-ball of center at p_i and radius 1. Set

$$S = \bigcup_{i=1}^{\infty} C_i.$$

The group $\pi_1(\mathbb{R}^3 \setminus S)$ is free of infinite rank and it is freely generated by the meridians μ_i of the circles C_i . Define a transitive representation

$$\omega : \pi_1(\mathbb{R}^3 \setminus S) \rightarrow \Sigma,$$

where Σ is the group of bijections of the numbers $\{0, 1, 2, \dots\}$, by $\omega(\mu_i) = (0i)$, for all $i \geq 1$.

Let

$$\tilde{f} : \tilde{X} \rightarrow \mathbb{R}^3 \setminus S$$

be the covering whose monodromy is ω (see, for instance [15]) that is, \tilde{f}_i maps $\pi_1(\tilde{X})$ (injectively) onto $\omega^{-1}(E_0)$, where E_0 is the subgroup of Σ stabilizing 0. The composition of

$$\tilde{f} : \tilde{X} \rightarrow \mathbb{R}^3 \setminus S$$

with the canonical inclusion $\mathbb{R}^3 \setminus S \subset S^3$ is an unbranched covering $f : \tilde{X} \rightarrow S^3$. Its unique completion

$$g : Y \rightarrow S^3$$

is a branched covering.

We are going to apply Theorem (10.5) to show that the fiber $g^{-1}(\infty)$ is compact but not discrete (it has a limit point). That is, we will obtain $g^{-1}(\infty)$ from

$$f = g \mid X : X \rightarrow S^3$$

where $X = Y - g^{-1}(\infty)$.

We claim that X is $\mathbb{R}^3 \setminus \bigcup_{i=1}^{\infty} \{p_i\}$, and we presently construct $f : X \rightarrow S^3$. Set

$$H := \mathbb{R}^3 \setminus \bigcup_{i=1}^{\infty} \text{Int}(B_i),$$

and let $\pi : H \rightarrow \mathbb{R}^3$ be the surjection identifying points of ∂B_i by reflection in the xy -plane, for all $i \geq 1$.

Set

$$H_i := \mathbb{R}^3 \setminus \text{Int}(B_i),$$

and let $\pi_i : H_i \rightarrow \mathbb{R}^3$ be the surjection identifying points of ∂B_i by reflection in the xy -plane.

Define

$$f : X \rightarrow \mathbb{R}^3 \subset S^3$$

as follows. The map f restricted to

$$X \setminus \bigcup_{i=1}^{\infty} \text{Int}(B_i) = H$$

coincides with π . The restriction of f to $(B_i \setminus \{p_i\})$ is the composition of the reflection in the xy -plane with the inversion through ∂B_i , with π_i . It is easy to see that this map is the required branched covering f .

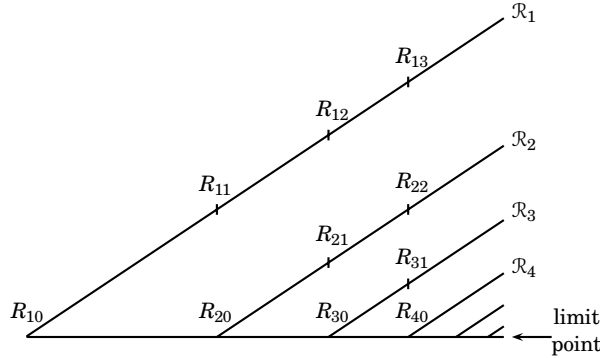
Assume that p_i is $(4i, 0, 0)$. Take the sequence $\{D_i : i \geq 0\}$ of 3-balls D_i with center at $(0, 0, 0)$ and radius $2 + 4i$. Then the fiber $g^{-1}(\infty)$ is homeomorphic to the inverse limit of the inverse sequence

$$\{X_{W_i}; h_{W_i W_{i+1}}; \mathcal{B}(\infty)\},$$

where $W_i = S^3 \setminus D_i$ and $\mathcal{B}(\infty)$ is the base $\{W_i\}_{i=0}^{\infty}$ of the system of open nbds of $\infty \in S^3$. To understand this limit we construct a graph as follows. For each $i \geq 0$ take the discrete space $X_i (= X_{W_i})$ with as many points as

$$f^{-1}(W_i) = X \setminus f^{-1}(D_i)$$

has components.



$X \setminus f^{-1}(D_1)$ lying in $\text{Int}(B_1)$) and R_{20} (the unbounded component). Thus R_{10} forks into two new vertices R_{20} and R_{11} .

Take D_2 . This 3-ball contains $B_1 \cup B_2$ in its interior and the remaining 3-balls B_i , $i \geq 3$, in its exterior. Then $f^{-1}(D_2)$ is the union of $D_2 \setminus \text{Int}(B_1 \cup B_2)$ with the inverse of $D_2 \setminus \text{Int}(B_1)$ with respect to ∂B_1 , plus the inverse of $D_2 \setminus \text{Int}(B_2)$ with respect to ∂B_2 (a twice punctured ball) plus the inverses of D_2 with respect to ∂B_i for all $i \geq 3$, and these are 3-balls in $\text{Int}(B_i) \setminus \{p_i\}$, $i \geq 3$. Thus $X \setminus f^{-1}(D_2)$ has three components R_{12} , R_{21} and R_{30} (the two bounded component and the unbounded component). We see that R_{12} , is contained in the interior of R_{11} . The other two components are contained in the interior of R_{20} . Thus X_2 has three points R_{12} , R_{21} and R_{30} . The vertex R_{20} forks into two new vertices R_{21} and R_{30} ; the vertex R_{11} extends linearly to R_{12} . Continuing in this way we obtain the tree of Figure 3. Therefore $g^{-1}(\infty)$ has a limit point.

The space Y is the result of adding to $X = \mathbb{R}^3 \setminus \bigcup_{i=1}^{\infty} \{p_i\}$ the compact fiber $g^{-1}(\infty)$. This fiber is $(\bigcup_{i=1}^{\infty} \{p_i\}) \cup \{\infty\}$ and is the union of a discrete part (filling up the punctures of $X = \mathbb{R}^3 \setminus \bigcup_{i=1}^{\infty} \{p_i\}$; the pseudobranching over ∞) and an ideal point ∞ (the branching over ∞ , with infinite branching index) which possesses a peculiar nbd system. The space $Y \setminus \{\infty\}$ is homeomorphic to \mathbb{R}^3 , but, by Theorem (9.15), Y is neither compact nor locally compact. Hence Y , though as a set it coincides with S^3 , as a topological space it is not even a manifold. Note that if $x \in \mathbb{R}^3$, the fiber $g^{-1}(x)$ has limit point $\{\infty\}$ in the ordinary topology of S^3 , but not in the topology of Y . This is in fact a remarkable example.

The restriction of $g : Y \rightarrow S^3$ to the preimage of the yz -plane (plus $\{\infty\}$) offers a 2-dimensional example with the same properties.

This example can be modified to obtain an uncountable number of branched coverings

$$g : \mathbb{R}^n \cup \{\infty\} \rightarrow S^n,$$

with $n = 3, 4$, branched over a wild knot and whose fiber over ∞ is not discrete. For instance the branching set in $S^3 = \mathbb{R}^3 + \{\infty\}$ can be taken to be the infinite connected sum $(K_1 \# K_2 \# K_3 \# \dots) \cup \{\infty\}$ where K_i is the trefoil knot 3_1 (in the Reidemeister tables), or its mirror image, lying in B_i , for $i \geq 1$. Taking 3_1 or its mirror image at random in the above connected sum one gets uncountably many different knots with the same properties.

Problem (10.7). Is there a branched covering possessing a fiber homeomorphic to a Cantor set? If, in fact, it does exist, what would be the smallest degree?

Problem (10.8). Is there a branched covering possessing a non locally compact fiber?

In the case that Y and Z are T_2 topological manifolds there is a useful criterium to decide if a given map $g : Y \rightarrow Z$ is a branched covering, due to Cernavskii and Väisälä independently [2], [31]. We offer this result without proof.

THEOREM (10.9) ([1]; see [31]). *Let M and N be compact, connected, T_2 topological manifolds without boundary (closed n -manifolds). Let $f : M \rightarrow N$ be an open and discrete map. Then f is a branched covering of finite degree.*

Sketch of the proof. By Theorem (4.5), f is a spread; and it is complete by Theorem (5.6). Let B_f denote the closed subset of points of M in which f fails to be a local homeomorphism. Then B_f has dimension $\leq n - 2$ (see [31], Theorem 5.4). By [3], Lemma 2.1, $f(B_f)$ and B_f have equal dimension. Then $f(B_f)$ fails to separate N both globally and locally. Since f is closed, $f(B_f)$ and $f^{-1}f(B_f)$ are closed sets. By [3], Lemma 2.1 again, $f(B_f)$ and $f^{-1}f(B_f)$ have the same dimension. Hence $f^{-1}f(B_f)$ fails to separate M both globally and locally. Moreover, $N \setminus f(B_f)$ is dense in N and

$$M \setminus f^{-1}f(B_f)$$

is dense in M . By [31], Theorem 5.5, the degree of f is finite. Therefore, the restriction

$$f | M \setminus f^{-1}f(B_f): M \setminus f^{-1}f(B_f) \rightarrow N \setminus f(B_f)$$

of f to $M \setminus f^{-1}f(B_f)$ is a complete, surjective spread of finite degree and a local homeomorphism. Then it is possible to see that $f | M \setminus f^{-1}f(B_f)$ is a covering. This finally proves that f is a branched covering of finite degree. \square

Here it is an open question (see [3], p. 536):

Problem (10.10). Is there a branched covering between 3-manifolds with a (totally disconnected) 0-dimensional branching set?

11. Branched foldings

An important example of singular covering, which is a topological version of the “folded coverings” and “branched folded coverings” of Tucker [29], can be obtained as follows. Let Z be a T_2 , connected topological manifold (not necessarily closed). In $Z \times [-1, 1]$ define an equivalence relation as follows. The point (z_1, t_1) is equivalent to (z_2, t_2) iff $z_1 = z_2$ lies in ∂Z and $t_1 = -t_2$. The quotient space is denoted by \widehat{Z} . There is a natural projection

$$p: Z \times [-1, 1] \rightarrow \widehat{Z}.$$

We identify Z with $p(Z \times \{0\})$. Thus Z is a subspace of \widehat{Z} .

Then $\widehat{Z} - \partial Z$ is connected, dense, and locally connected in \widehat{Z} . Let

$$\widehat{g}: \widehat{Y} \rightarrow \widehat{Z}$$

be a covering branched over ∂Z . This branched covering is determined by its monodromy,

$$\omega: \pi_1(\widehat{Z} - \partial Z) \rightarrow \Sigma_F$$

which is a transitive representation into the group Σ_F of bijections of the standard fiber F of the covering associated to \widehat{g} . The group $\pi_1(\widehat{Z} - \partial Z)$ is obtained from $\pi_1(Z)$ by adding a new generator c for each component C of ∂Z , and imposing that c commutes with all the elements of the image of $\pi_1(C)$ in $\pi_1(Z)$. The new generators will be called *boundary generators*.

The restriction of

$$\widehat{g}: \widehat{Y} \rightarrow \widehat{Z}$$

to $Y := \widehat{g}^{-1}(Z)$ is a singular covering $g: Y \rightarrow Z$ with singular locus contained in ∂Z . This singular covering $g: Y \rightarrow Z$ will be called the *multi-folding over Z determined by the monodromy ω* .

If the image under ω of any boundary generator has order 1 or 2, all the branching indexes of g divide 2. This is the case that appears in the definition of orbifold and we call $g : Y \rightarrow Z$ a *folding over Z determined by the monodromy*

$$\omega : \bar{\pi}_1(Z) \rightarrow \Sigma_F$$

where $\bar{\pi}_1(Z)$ is the quotient of the group $\pi_1(\widehat{Z} - \partial Z)$ by the subgroup normally generated by the squares of the boundary generators of $\pi_1(\widehat{Z} - \partial Z)$. We call $\bar{\pi}_1(Z)$ the *mirror fundamental group* of Z . The idea is that Z , besides being an ordinary manifold, has the structure of “manifold with mirror boundary” (a concept well known in orbifold theory). A folding in this “manifold with mirror boundary” realm is the analogue of a covering in the “ordinary manifold” realm.

An *unbranched folding* is a map $f : X \rightarrow Z$ such that (i) $f : X \rightarrow f(X)$ is a folding; and (ii) $f(X)$ is connected dense and locally connected in Z . Let Σ be $Z \setminus f(X)$. The (unique) completion $g : Y \rightarrow Z$ of the unbranched folding $f : X \rightarrow Z$ will be called a *branched folding* (compare [16]) with branching set Σ . The map g is a singular covering.

The branched folding $g : Y \rightarrow Z$ is uniquely determined by the monodromy $\omega : \bar{\pi}_1(Z \setminus \Sigma) \rightarrow \Sigma_F$ of the folding $f : X \rightarrow Z \setminus \Sigma$ and the canonical inclusion $Z \setminus \Sigma \subset Z$.

Example (11.1). Let Z be a closed 2-simplex. Let $\Sigma \subset Z$ be the sets of vertexes of Z . Then

$$\bar{\pi}_1(Z \setminus \Sigma) = | A, B, C : A^2 = B^2 = C^2 = 1 |$$

where a, b, c are the boundary generators. Let T be the group

$$T = | a, b, c : a^2 = b^2 = c^2 = 1; (ab)^4 = (bc)^3 = (ca)^2 = 1 |.$$

This is the full group of symmetry of the octahedron: it has order 48. Let

$$\rho : \bar{\pi}_1(Z \setminus \Sigma) \rightarrow T$$

be the homomorphism such that

$$\rho(A) = a, \rho(B) = b, \rho(C) = c.$$

The branched folding associated to the kernel of ρ is $g : Y \rightarrow Z$, where Y is the 2-sphere and g is the quotient of Y under the action of T . The branching set is Σ . The degree of g is 48. The monodromy $\omega : \bar{\pi}_1(Z \setminus \Sigma) \rightarrow \Sigma_{48}$ is the composition of ρ with the canonical faithful representation of T in Σ_{48} .

Problem (11.2). Let M and N be n -manifolds T_2 , compact, and connected. Assume ∂N is non-empty. Let

$$f : (M, \partial M) \rightarrow (N, \partial N)$$

be an open and discrete map. Is f a branched folding?

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FACULTAD DE MATEMÁTICAS
UNIVERSIDAD COMPLUTENSE
28040 MADRID
montesin@mat.ucm.es

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**ON ESTIMATES FOR AREA FUNCTIONS OF SOLUTIONS TO
NON-DIVERGENCE PARABOLIC EQUATIONS**

JORGE RIVERA-NORIEGA

ABSTRACT. We prove local equivalences of the L^p -norms over the lateral boundary of a non-cylindrical domain, of certain area functions and the non-tangential maximal function, of solutions of parabolic equations in non-divergence form. As an application, we indicate how to extend the result in [12] on the approximation over Lipschitz domains of harmonic functions by functions of bounded variation, to solutions of parabolic equations in non-divergence form, with some consequences.

1. Introduction and statement of the main results

We consider an operator as

$$(1.1) \quad Lu = \sum_{i,j=1}^n a_{i,j}(X, t) \partial_i \partial_j u(X, t) - \partial_t u(X, t),$$

where $(X, t) \in \mathbb{R}^n \times \mathbb{R}$, and $(a_{i,j})$ is a symmetric matrix of real-valued functions that satisfies the ellipticity condition

$$(1.2) \quad \lambda |\xi|^2 \leq \sum_{i,j} a_{i,j}(X, t) \xi_j \xi_i \leq |\xi|^2 / \lambda, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n,$$

$$(X, t) \in \mathbb{R}^n \times \mathbb{R}.$$

Here and in the future we use the notation $\partial_i \partial_j u = \frac{\partial^2 u}{\partial x_i \partial x_j}$, and $\partial_t u = \frac{\partial u}{\partial t}$. We make the qualitative assumption that the coefficients $a_{i,j}(X, t)$ are smooth. We note however, that in [5] it is proved that one may obtain some basic regularity results and solvability of the Cauchy-Dirichlet problem for coefficients $a_{i,j}(X, t)$ in the class VMO of functions of *vanishing mean oscillation*. This space of functions may be regarded as the closure with respect to the BMO norm of the space of functions which are uniformly continuous (see [46]). Whenever we have a result where VMO coefficients may be taken, we will try to indicate so. In our parabolic environment, the BMO norm of a locally integrable function f is defined as

$$\|f\|_* = \sup_{\mathbf{C}} \frac{1}{|\mathbf{C}|} \int_{\mathbf{C}} |f(Y) - f_{\mathbf{C}}| dY,$$

where $f_{\mathbf{C}} = \frac{1}{|\mathbf{C}|} \int_{\mathbf{C}} f(Y) dY$, and \mathbf{C} is a cylinder as described in the next section.

Throughout the paper we adopt a standard convention for constants in inequalities that we explain now. Constants in different inequalities may

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change, and we may denote them with the same letter. Whenever $A \leq kB$ with k depending at most on the ellipticity of L , the so called *Lipschitz character* of the domain Ω in which we work, and n , we write $A \lesssim B$. The notation $A \approx B$ indicates that the quantity A/B is bounded above and below by constants in the sense just explained.

In this work we prove the local comparability of L^p norms on the *lateral boundary* S of a time-varying domain $\Omega \subset \mathbb{R}^{n+1}$ of the first and second order area functions with the nontangential maximal function of solutions to the parabolic equation. Precise definitions of all of these objects is written below or postponed for the next section. This local comparability is obtained by adapting techniques of [17] and [14]. Results of this type have interest on their own in a variety of situations (see e.g. [8], [22], [9], [13], [14], [36], [28], [2], [7], [39], [51], [17], [37], [15], [50]). In this note we apply the estimates to generalize a result of [12]. In fact such a generalization is more or less immediate from [43], [44]. The issue is then to obtain some properties of solutions and adjoint solutions of the equations considered, by adapting the techniques of the corresponding elliptic results. For the properties not proved here we provide the pertinent reference.

To give a more precise statement of our theorems we need to introduce several definitions. The *parabolic measure* associated to L is the unique probability Borel measure $\omega^{(X,t)(\cdot)} \equiv \omega(X, t; \cdot)$ defined on S that represents the solution u_f of

$$(1.3) \quad \begin{cases} Lu_f = 0 & \text{on } \Omega, \\ u_f|_{\partial_p \Omega} = -f, \end{cases}$$

whenever this solution exists (even in the Perron-Wiener-Brelot sense) in the following sense

$$u_f(X, t) = \int_S f(Y, s) d\omega(X, t; Y, s).$$

Here $f : \partial_p \Omega \rightarrow \mathbb{R}$ is a continuous function supported in S . Observe that our definition of the parabolic measure implies that it is a measure supported in S , and not in the entire *parabolic boundary* $\partial_p \Omega$.

Harnack's inequality for parabolic equations has a time lag (cf. [31, 35]) and so we can choose $(\Xi, \mathcal{J}) \in \mathbb{R}^n \times \mathbb{R}_+$ so that we have $\omega(X, t; \cdot) \lesssim \omega(\Xi, \mathcal{J}; \cdot)$ for every $(X, t) \in \Omega$. Let $\omega(\cdot)$ denote the measure $\omega(\Xi, \mathcal{J}; \cdot)$. By the fundamental results of [45], we know that ω is a doubling measure, in the sense that if $(Q, s) \in S$ and $\Delta_r \equiv \Delta_r(Q, s) = S \cap \{(X, t) \in \mathbb{R}^n \times \mathbb{R} : |X - Q| + |t - s|^{1/2} < r\}$ is a surface ball in S of radius r and such that $\Delta_{2r}(Q, s) \subset S$, then $\omega^{(X,t)}(\Delta_{2r}) \lesssim \omega^{(X,t)}(\Delta_r)$ with certain conditions relating (Y, s) and (X, t) , due to the time lag (see (2.24) below). A doubling Borel measure μ defined in S belongs to the class $A_\infty(\omega)$ if there exist constants $C, \theta > 0$ such that for every $\Delta \subset S$ and any Borel set $E \subseteq \Delta$ we have

$$(1.4) \quad \frac{\mu(E)}{\mu(\Delta)} \leq C \left(\frac{\omega(E)}{\omega(\Delta)} \right)^\theta.$$

It is well known that this defines an equivalence relation (see [11] and [37] for the parabolic statement). Also, $\mu \in A_\infty(d\omega)$ if and only if, letting $k = d\omega/d\mu$,

the following estimate holds for some $q > 1$ and for every surface ball $\Delta \subset S$ as defined above

$$(1.5) \quad \left(\frac{1}{\mu(\Delta)} \int_{\Delta} k^q d\mu \right)^{1/q} \leq \frac{C}{\mu(\Delta)} \int_{\Delta} k d\mu$$

with constant C independent of Δ . This property in turn is connected with the solvability of an L^p Dirichlet problem, $1/p + 1/q = 1$, for the corresponding parabolic equation (see e.g. [37]).

Let $\mathbf{C}(X, t)$ denote a cylinder of the form $B(X, t) \times (t - d(X, t)/2, t + d(X, t)/2)$, where $B(X, t) = \{Y \in \mathbb{R}^n : |Y - X| < d(X, t)\}$ and $d(X, t) = \text{dist}(X, t; S)$. The *parabolic distance* between points of $\mathbb{R}^n \times \mathbb{R}$ (and by extension the parabolic distance between sets) is defined as $\text{dist}(X, t; Y, s) = |X - Y| + |t - s|^{1/2}$. For $(Y, s) \in S$ the *parabolic cones* of aperture α are

$$\Gamma_{\alpha}(Y, s) = \{(X, t) \in \Omega : d(X, t; Y, s) < (1 + \alpha)d(X, t; S)\}.$$

Defining the *parabolic balls* as $\mathbf{Q}_r(Y, s) = \{(X, t) \in \mathbb{R}^{n+1} : |Y - X| + |s - t|^{1/2} < r\}$, we also have the *truncated cones* as $\Gamma_{\alpha}^r(Y, s) = \Gamma_{\alpha}(Y, s) \cap \mathbf{Q}_r(Y, s)$.

For any function $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ set $\bar{F}(B) = \int_B F(Y, s) dY ds$ for any Borel set $B \subset \mathbb{R}^{n+1}$ and $F(B', s) = \int_{B'} F(Y, s) dY$ for any Borel set $B' \subset \mathbb{R}^n$. Define the *nontangential maximal function* as

$$N_{\alpha}F(Y, s) = \sup_{(X, t) \in \Gamma_{\alpha}(Y, s)} |F(X, t)|, \text{ for } (Y, s) \in S.$$

Following [17] we define the *first order area function* by

$$S_{\alpha}F(Y, s) = \left(\int_{\Gamma_{\alpha}(Y, s)} |\nabla_X F(X, t)|^2 \frac{d(X, t)^2}{G(\mathbf{C}(X, t))} G(X, t) dX dt \right)^{1/2} \text{ for } (Y, s) \in S,$$

where G is the Green's function at a fixed pole of the operator L in a cylinder containing Ω (cf. [18]; see section 3 for details). When F is a solution of the heat equation or a divergence form equation, $S_{\alpha}u$ is equivalent to the definition given in [7] and [37] respectively (cf. [19]). Finally, based on a definition from [40], the *second order area function* is

$$A_{\alpha}F(Y, s) = \left(\int_{\Gamma_{\alpha}(Y, s)} [|\nabla_X^2 F(X, t)|^2 + |\partial_t F(X, t)|^2] \frac{d(X, t)^4}{G(\mathbf{C}(X, t))} G(X, t) dX dt \right)^{1/2}$$

for $(Y, s) \in S$. Here $\nabla_X^2 F$ denotes the vector of second derivatives of F in the X variable only. In the future we omit this subindex from the symbols ∇ and ∇^2 . There is a similar definition of the truncated operators $N^r u$, $S^r u$, $A^r u$ when truncated cones replace the regular cones.

THEOREM (1.6). *Let u be a solution to $Lu = 0$ in Ω , where the coefficients of L are in VMO. Let $\alpha, \beta > 0$, and let μ be a measure in S , the lateral boundary of Ω , such that $\mu \in A_{\infty}(\omega)$. Let $(Q, s) \in S$ and $\Delta \equiv \Delta_r(Q, s)$ be any surface ball such that $\Delta_{2r}(Q, s) \equiv 2\Delta \subset S$ and $\bigcup_{P \in \Delta} \Gamma_{\alpha+\beta}(P) \subset \Omega$. Then for each $0 < p < \infty$ there exists $C_1 > 0$ (depending only on the ellipticity of L , n and p) and $C_2 > 0$*

(depending also on r) such that

$$(1.7) \quad \int_{\Delta} [A_{\beta}^r u]^p d\mu \leq \int_{\Delta} [S_{\beta}^r u]^p d\mu \leq C_1 \int_{2\Delta} [N_{\alpha}^{2r} u]^p d\mu$$

$$(1.8) \quad \int_{\Delta} [N_{\alpha}^r u]^p d\mu \leq C_2 \int_{2\Delta} [S_{\beta}^{2r} u]^p d\mu + C_2 |u(Y_r(\mathbf{Q}, s))|^p \mu(\Delta)$$

where $Y_r(\mathbf{Q}, s)$ is roughly a point in Ω with distance to (\mathbf{Q}, s) proportional to r .

A theorem of this type was proved in the parabolic setting for solutions of the heat equation [7], and for divergence form operators [37]. It is clear from these articles that one needs only to prove certain local distributional inequalities relating the operators involved. We include sketches of the proofs of these distributional inequalities, to point out where the adaptations were required.

As applications of this Theorem we explain an extension of a result of Dahlberg's [12] and some of its consequences. Since cylinders given locally by continuous functions with mixed Hölder property (the so called $\text{Lip}(1,1/2)$ cylinders) are not smooth enough for the purposes of solving an L^p Dirichlet problem for our parabolic operators (see the counterexamples of [29]), we introduce in the next section a class of slightly smoother domains, that we call here the *parabolic Lipschitz cylinders*, and that were used in [25, 26]. In the next theorems we use parabolic Lipschitz cylinders to avoid the above mentioned counterexamples.

THEOREM (1.9). *Let L be an operator as (1.1) with VMO coefficients and let u be a bounded solution to $Lu = 0$ in a parabolic Lipschitz cylinder $\Omega \subset \mathbb{R}^{n+1}$, whose lateral boundary S has surface measure σ . Then for every $\varepsilon > 0$ there exists a function $\varphi : \Omega \rightarrow \mathbb{R}$ such that*

- (i) $\|u - \varphi\|_{L^{\infty}(\Omega)} \leq \varepsilon$;
- (ii) For every C_0^{∞} function $w \geq 0$ defined on S , any surface cube $\Delta \subset S$ of radius r , such that $2\Delta \subset S$, and with an appropriate aperture $\beta > \alpha$,

$$(1.10) \quad \int_{\Delta} [B_{\alpha}^r \varphi]^2 w d\sigma \lesssim \varepsilon \int_{2\Delta} M w d\sigma + \frac{1}{\varepsilon} \int_{2\Delta} [S_{\beta}^r u]^2 w d\sigma,$$

where Mw denotes the Hardy-Littlewood maximal function of w with respect to σ , and where

$$B_{\alpha}^r(u)(\mathbf{Q}, s) = \int_{\Gamma_{\alpha}^r(\mathbf{Q}, s)} |\nabla u(X, t)| \delta(X, t; \mathbf{Q}, s)^{-n-1} dX dt,$$

The proof uses the estimates of Theorem (1.6), and may be obtained by combining the arguments of [43] and [44] with the ideas for the parabolic adaptation of [42, Proposition 4.2]. These articles contains adaptations of the original proof of B. Dahlberg [12]: by choosing w as an appropriate bump function, one recovers the result of Dahlberg's.

For the construction of φ as well as for the verification of the A_{∞} property, we look at local surface balls, and so in the next theorem we can work in domains above the graph of a *parabolic Lipschitz function*. These domains are called *parabolic Lipschitz domains* (see details in the next section).

THEOREM (1.11). *Let Ω be a parabolic Lipschitz domain whose lateral boundary S has surface measure σ and let L be an operator as (1.1) with VMO coefficients. Consider an arbitrary parabolic Lipschitz domain $\Omega' \subset \Omega$ and a bounded solution u of $Lu = 0$ on Ω . For any surface cube $\mathcal{Q}_0 \subset \partial\Omega$ let $\Psi(\mathcal{Q}_0)$ denote the Carleson region above \mathcal{Q}_0 (this is roughly the intersection of Ω and a parabolic ball centered at a point in $\partial\Omega$; see the next section). Suppose that for any $\Delta \subset \partial\Omega' \cap \Psi(\mathcal{Q}_0)$ of radius $r > 0$ and such that $\tilde{\Delta} = 2\Delta \subset \Psi(\mathcal{Q}_0)$ we have*

$$\begin{aligned} \int_{\Delta} (N_{\alpha}^r u)^2 d\sigma &\leq C \int_{\tilde{\Delta}} (A_{\beta}^r u)^2 d\sigma + K(\mathcal{Q}_0) \\ \int_{\Delta} (A_{\alpha}^r u)^2 d\sigma &\leq C \int_{\tilde{\Delta}} (N_{\beta}^r u)^2 d\sigma, \end{aligned}$$

where $K(\mathcal{Q}_0) = |\mathcal{Q}_0| \sup_{\Psi(\mathcal{Q}_0)} |u|$. Then $d\omega \in A_{\infty}(d\sigma)$.

This is based on a technique in [30], and in fact is a consequence of Theorem (1.9) and ideas from the parabolic divergence case in [42].

Another consequence of Theorem (1.9), as explained in [38], is the next theorem.

THEOREM (1.12). *Under the assumptions and with the notation of Theorem (1.9), assume that $\|S_{\alpha}u\|_{L^{\infty}} < \infty$ and that the non-tangential limit $f(P) = \lim_{\substack{(X,t) \rightarrow P \\ (X,t) \in \Gamma(P)}} u(X,t)$ exists for σ -almost every $P \in S$. Then for any surface cube $\Delta \subset S$ we have*

$$(1.13) \quad \frac{1}{|\Delta|} \int_{\Delta} \exp\left(\frac{c_1 |f - f_{\Delta}|^2}{\|S_{\alpha}u\|_{L^{\infty}}^2}\right) d\sigma < c_2$$

with constants independent of Δ , and where $f_{\Delta} = \frac{1}{|\Delta|} \int_{\Delta} f d\sigma$.

Estimate (1.13) is the analogue of the estimate defining the *exponential square class* of [10], where the same estimate is proved for harmonic functions (see also [48]). As observed in [3], (1.13) implies an improvement in the constants of the distributional inequality (3.12) that implies (1.8) (we refer the reader to section 3 for details):

With the notations of Proposition (3.11), if $0 < \beta < \alpha$ then there exist constants $a_1, a_2 > 0$ (depending on α, β, n) such that for any $M > 1, \lambda_0 > 0$, one has

$$(1.14) \quad \begin{aligned} &|\{P \in \Delta : N_{\beta}^r u(P) > M\lambda_0, S_{\alpha}^r u(P) < \lambda_0\}| \\ &\leq a_1 \exp(-a_2 M^2) |\{P \in \Delta : N_{\beta}^r u(P) > \lambda_0\}| \end{aligned}$$

In the next section we obtain some preliminary results, and review the background material. Section 3 contains the proof of the distributional inequalities that yield theorem (1.6), which after the results of section 2, are really parabolic adaptations of the elliptic arguments.

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2. Notations and preliminary results

In the space \mathbb{R}^{n+1} the parabolic norm of a point $(X, t) \in \mathbb{R}^n \times \mathbb{R}$ is defined as $\|X, t\| = |X| + |t|^{1/2}$. We denote the corresponding distance function by $\text{dist}(X, t; Y, s) = \|X - Y, t - s\|$. To stress that x_0 is the variable that depends on (x, t) , we may sometimes write points in \mathbb{R}^{n+1} as $(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$. The metric introduced above will be used whenever the coordinates include the t variable. For an arbitrary domain $\Omega \subset \mathbb{R}^{n+1}$ we define its *parabolic boundary* $\partial_p \Omega$ as the set $\partial_p \Omega = \{(Y, s) \in \partial \Omega : Q_r(Y, s) \setminus \Omega \neq \emptyset\}$. Here, the *backward cylinder* $Q_r(Y, s)$ is defined by

$$(2.1) \quad Q_r(Y, s) = \{(X, t) \in \mathbb{R}^{n+1} : |X - Y| < r, s - r^2 < t \leq s\}.$$

The Euclidian balls in \mathbb{R}^n (with a fixed s) are denoted as $B_r(Y, s) = \{(X, s) \in \mathbb{R}^n \times \{s\} : |Y - X| < r\}$, and when $s = 0$ we will simply write $B_r(Y)$. The *cylinders* in \mathbb{R}^{n+1} are defined as $\mathbf{C}_r(Y, s) = B_r(Y, s) \times (s - r^2, s + r^2)$. We may omit the center or the radius of \mathbf{C} when they are not important in the discussion.

Throughout this work we will fix constants $M > 0$, $R > 0$, and they may be referred to as *Lipschitz character* of a $\text{Lip}(1, 1/2)$ cylinder $D \subset \mathbb{R}^{n+1}$, as defined below (we follow [7]). Let $\mathcal{Z} = \{(x_0, x, t) \in \mathbb{R}^{n+1} : |x_0| < 2nMR, |x| < R, t \in \mathbb{R}\}$, and $\varphi : \mathbb{R}^n \rightarrow (-MR, MR)$. We say that (\mathcal{Z}, φ) is a *coordinate cylinder* for a domain $D \subset \mathbb{R}^{n+1}$ if

- (i) $2\mathcal{Z} \cap \partial D = \{(x_0, x, t) : x_0 = \varphi(x, t)\} \cap 2\mathcal{Z}$,
- (ii) $2\mathcal{Z} \cap D = \{(x_0, x, t) : x_0 > \varphi(x, t)\} \cap 2\mathcal{Z}$,

where $2\mathcal{Z}$ denotes the concentric double of \mathcal{Z} . $D \subset \mathbb{R}^{n+1}$ is called an *infinite Lip(1, 1/2) cylinder* with parameters (M, R) , or simply *Lip(1, 1/2) cylinder*, if there is a covering of ∂D by coordinate cylinders $\{(\mathcal{Z}_i, \varphi_i) : i = 1, 2, \dots, N\}$ such that $|\varphi(x, t) - \varphi(y, s)| \leq M\|x - y, t - s\|$. In this case we write $\varphi \in \text{Lip}(1, 1/2)$. The coordinate systems used to define \mathcal{Z}_i are allowed to differ by a rigid motion in the X variable only. We will use the notation $\tilde{S} = \partial D$ for the boundary of an infinite $\text{Lip}(1, 1/2)$ cylinder D . For a fixed $T > R > 0$, and when there is no risk to confuse the domain Ω , we will set $\Omega = D \cap \{0 < t < T\}$, $S = \tilde{S} \cap \{0 < t < T\}$. This last set will be called the *lateral boundary* of Ω . By extension, we denote by $S(\mathbf{C}_r(Q, s)) = \partial B_r(Q) \times [s - r^2, s + r^2]$ the *lateral boundary* of a cylinder.

Let $\text{diam}(\Omega) = \sup_{t \in \mathbb{R}} \text{diam} \Omega(t)$ where $\Omega(t) = \{(Y, s) \in \Omega : s = t\}$ is the level surface of Ω at the level t . We will assume throughout this work that $\text{diam}(\Omega) < \infty$ and that there exists $\Xi \in \mathbb{R}^n$ such that $(\Xi, t) \in \Omega$ for all $t \in \mathbb{R}$.

At this point we can define the domains in which one can solve L^p Dirichlet problems for the heat equation, and that are used in the statement of Theorems (1.9) and (1.11). Define the operator I_p by setting $(I_p f)^\wedge(\xi, \tau) = \|\xi, \tau\|^{-1} \hat{f}(\xi)$, $(\xi, \tau) \in \mathbb{R}^n$. Here both $\hat{\phi}$ and ϕ^\wedge represent the Fourier transform of the object ϕ . The *parabolic Sobolev-BMO space* $I_p(BMO)$ is defined as the space of distributions $I_p(BMO) = \{\psi : \psi = I_p(a), a \in BMO\}$. We say that $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a *parabolic Lipschitz function* with constant N if $|\varphi(x, t) - \varphi(y, t)| \leq N|x - y|$, uniformly for $t \in \mathbb{R}$, and $\varphi \in I_p(BMO)$ with $\|I_p^{-1} \varphi\|_* \leq N$. We abbreviate this

by witting $\varphi \in \Lambda_p(N)$. It can be proved (see e.g. [24]) that parabolic Lipschitz functions are $\text{Lip}(1,1/2)$.

For $\varphi \in \Lambda_p(N)$, the domain $\Omega(\varphi) = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : \varphi(x, t) < x_0\}$ is called a *parabolic Lipschitz domain*. For parabolic Lipschitz domains one has the absolute continuity of its surface measure with respect to caloric measure [26, 33, 42]. More recently [27] it was proved that certain parabolic locally flat domains may be approximated by domains whose boundary is a parabolic Lipschitz function. If the local functions φ defining a $\text{Lip}(1,1/2)$ cylinder are in $\Lambda_p(M)$, we say that Ω is a *parabolic Lipschitz cylinder* (of character M, R).

Given a point $(Y, s) = (y_0, y, s) \in S$ define $\bar{A}_r(Y, s) = (\varphi(y, s) + Mnr, y, s + 2r^2)$, $\underline{A}_r(Y, s) = (\varphi(y, s) + Mnr, y, s - 2r^2)$ where φ defines locally the boundary of Ω around the the point $(Y, s) \in S$. Define the *parabolic cones* of aperture α as $\Gamma_\alpha(Y, s) = \{(X, t) \in \Omega : \|X - Y, t - s\| < (1 + \alpha)d(X, t; \partial\Omega)\}$. There are also the truncated versions of these non-tangential approach regions defined as $\Gamma_\alpha^r(Y, s) = \Gamma_\alpha(Y, s) \cap \mathbf{Q}_r(Y, s)$. Here $\mathbf{Q}_r(Y, s)$ denotes the *parabolic ball* $\mathbf{Q}_r(Y, s) = \{(X, t) \in \mathbb{R}^{n+1} : |Y - X| + |s - t|^{1/2} < r\}$. Similarly the surface cubes are defined by $\Delta_r(Y, s) = \mathbf{Q}_r(Y, s) \cap \partial\Omega$ and the *Carleson regions* $\Psi_r(y_0, y, s) = \{(x_0, x, t) \in \Omega : |x_0 - y_0| < Mr, |x - y| < r, |t - s| < r^2\}$ (this is roughly the portion of a parabolic ball centered at a point in $\partial\Omega$ which is contained in Ω).

For L an operator as in (1.1) with smooth coefficients, a solution for the *adjoint operator* L^* (or *adjoint solution*) over Ω , is a smooth function v for which

$$L^*v = \sum_{i,j=1}^n \partial_i \partial_j [a_{i,j}(X, t)v(X, t)] + \partial_t u(X, t) = 0.$$

If the coefficients are in VMO, an adjoint solution is a locally integrable function v that satisfies $\int L\varphi v = 0$ for every $\varphi \in C_0^\infty(\Omega)$ vanishing in $\partial_p\Omega$. From now on, operators L will be taken as (1.1), and we denote by $P(\text{VMO})$ the class of operators as (1.1) with VMO coefficients.

For $D \subset \mathbb{R}^{n+1}$ an open domain we write $w \in W_p^{2,1}(D)$ if it belongs to the closure of $C^\infty(D)$ with respect to the norm

$$\|w\|_{W_p^{2,1}(D)} = \|\partial_t w\|_{L^p(D)} + \sum_{i,j}^n \|\partial_{i,j} w\|_{L^p(D)} + \sum_i^n \|\partial_i w\|_{L^p(D)}.$$

The closure of the set of $C^\infty(D)$ functions vanishing in $\partial_p D$ with respect to the $W_p^{2,1}(D)$ norm is denoted by $W_p^0(D)$. There is a similar definition when the Lebesgue measure is substituted by any measure of the form $w(X, t)dXd t$ for a positive bounded function $w : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, with the obvious change of notation.

We will gradually introduce some basic properties of solutions to $Lu = 0$ with the corresponding reference. We also introduce some objects and concepts needed throughout the paper.

Aleksandrov-Tso's maximum principle [49]

Let $\mathbf{C}_r \subset \Omega$ is a cylinder of radius r . Suppose that $v \in W_{n+1}^{2,1}(\Omega) \cap C(\bar{\Omega})$ is a function bounded from above, with $Lv \geq f$ in \mathbf{C}_r . Then there exists a constant

$C = C(r, n, \lambda) > 0$ such that

$$(2.2) \quad \sup_{\mathbf{C}_r} v \leq \sup_{\partial_p \mathbf{C}_r} v^+ + Cr^{n/(n+1)} \|f\|_{L^{n+1}(\mathbf{C}_r)}$$

where $v^+(P) = \max\{0, v(P)\}$

Interior regularity [32]

Suppose that $L \in P(\text{VMO})$. For $1 < p < \infty$ there exists a constant $C = C(n, p, \lambda)$ and r_0 such that for $r < r_0$, $\mathbf{C}_{2r} \subset \Omega$ and u in $W_p^0(\Omega)$ one has

$$(2.3) \quad \|u\|_{W_p^{2,1}(\mathbf{C}_r)} \leq C(\|Lu\|_{L^p(\mathbf{C}_{2r})} + \|u\|_{L^p(\mathbf{C}_{2r})}).$$

Interior Harnack's Inequality [31]

Let u be a nonnegative solution of $Lu = 0$ in the bounded region D . Let D' be any subregion of D such that $d(D', \partial D) = \delta > 0$. Then there is a constant $C > 0$ such that for all $(X, t), (Y, s) \in D'$ with $t - s \geq \delta$ we have

$$(2.4) \quad u(Y, s) \leq Cu(X, t).$$

We define the *Green's function* associated to an operator L with smooth coefficients on Ω by

$$g(X, t; Y, s) = \mathcal{G}(X - Y, t - s) - \int_{\partial_p \Omega} \mathcal{G}(Z - Y, \tau - s) d\omega(X, t; Z, \tau)$$

where \mathcal{G} is the fundamental solution for L (cf. [32]). The connection between parabolic measure and Green's function is given by the following identity:

$$(2.5) \quad v(X, t) = \int_{\partial_p \Omega} v(Y, s) d\omega(X, t; Y, s) - \int_{\Omega} g(X, t; Y, s) Lv(Y, s) dY ds$$

for every $(X, t) \in \Omega$, and every v smooth in Ω . Also, the solution of the Cauchy L^p -Dirichlet problem, for $f \in L^p(\Omega)$,

$$(2.6) \quad \begin{cases} Lu = f & \text{on } \Omega \\ u|_{\partial_p \Omega_T} = 0 \end{cases}$$

is represented by

$$(2.7) \quad u(X, t) = \int_{\Omega} g(X, t; Z, \tau) f(Z, \tau) dZ d\tau,$$

Thus $g(X, t; \cdot) \in L^{(n+1)/n}$ and by (2.2)

$$\left(\int_{\Omega} g(X, t; Y, s)^{\frac{n+1}{n}} dY ds \right)^{\frac{n}{n+1}} \leq CT^{\frac{n}{n+1}}.$$

Similarly, if g_r denotes the Green's function in a cylinder \mathbf{C}_r then

$$(2.8) \quad \left(\int_{\mathbf{C}_r} g_r(X, t; Y, s)^{\frac{n+1}{n}} dY ds \right)^{\frac{n}{n+1}} \leq Cr^{\frac{n}{n+1}}.$$

When $L \in P(\text{VMO})$ we can follow the usual considerations (cf. [21] and [4]) based on the solvability of (2.6) for $p > n$, Tso's maximum principle and Riesz representation theorem to define the Green's function of L . Moreover we can prove that (2.7) still holds, and that (2.5) and (2.8) remain valid.

The following properties (2.9)-(2.15) hold when $L \in P(\text{VMO})$. Recall that $R > 0$ was the size of the coordinate cylinders defining Ω (see page 70).

Hölder continuity

Let u be a solution to $Lu = 0$ in the cylinder $\mathbf{C}_r(X, t)$. Then for $(X_0, t_0), (X_1, t_1) \in \mathbf{C}_{r/2}(X, t)$,

$$(2.9) \quad |u(X_0, t_0) - u(X_1, t_1)| \leq C \left(\frac{\|X_0 - X_1, t_0 - t_1\|}{r} \right)^\theta \sup_{\mathbf{C}_r(X, t)} |u|.$$

Boundary Harnack Inequality or Carleson type estimate

Let $(Y, s) \in S$, $s < T$ and u be a nonnegative solution to $Lu = 0$ in Ω . Suppose that u vanishes in $\Delta_r(Y, s)$, with $0 < r < R$, and that $\Delta_{2r}(Y, s) \subset S$. Then

$$(2.10) \quad \sup_{\Psi_r(Y, s)} u \leq C u(\bar{A}_r(Y, s)).$$

Boundary Hölder continuity

If u is a nonnegative solution to $Lu = 0$ which vanishes continuously in $\Delta_{r_0}(Y, s)$, for some $(Y, s) \in S_T$ and $\Delta_{2r_0}(Y, s) \subset S_T$, then there exist constants C and α depending on ellipticity and dimension n such that

$$(2.11) \quad \sup_{\Psi_r(Y, s)} u \leq \left(\frac{r}{r_0} \right)^\alpha \sup_{\Psi_{r_0}(Y, s)} u$$

for $0 < r < r_0$.

Inequality (2.9) is a consequence of Harnack's principle (2.4). Estimates (2.10) and (2.11) may be proved by adapting the arguments of E. Fabes, M. Safonov and Y. Yuan (see [45] and [20]) to our non-cylindrical setting. For one way to perform this type of adaptation see [37].

With all these estimates one may prove with a standard argument the following related result (see e.g. [37], Lemma 2.2).

Backward Carleson type estimate

Let $(Y, s) \in S$, $s < T$ and u be a nonnegative solution to $Lu = 0$ in Ω . Suppose that u vanishes in $\Delta_r(Y, s)$, with $0 < r < R$, and that $\Delta_{2r}(Y, s) \subset S$. Then

$$(2.12) \quad \sup_{\Psi_r(Y, s)} u \geq C u(\underline{A}_r(Y, s)).$$

First order Caccioppoli type inequality

Let u be a solution to $Lu = 0$, and let v be a non-negative adjoint solution on $\mathbf{C}_{3r}(X_0, t_0) \subset \Omega$. Then for any constant β ,

$$(2.13) \quad \int_{\mathbf{C}_r(X_0, t_0)} |\nabla u(X, t)|^2 v(X, t) dX dt \leq \frac{C}{r^2} \int_{\mathbf{C}_{2r}(X_0, t_0)} |u - \beta|^2 v dX dt.$$

To prove this last inequality one may follow [17], Lemma 3 with the necessary and easy adaptations to the parabolic setting.

Elliptic-type Harnack's inequality

Let u be a nonnegative solution of $Lu = 0$ on Ω , vanishing continuously on

$\Delta_r \subset S$. Let $0 < \delta < 1$ and $\eta > 1$ be constants such that $\text{diam } \Omega + r/\delta \leq \eta$. Then

$$(2.14) \quad \sup_{\Omega_r^\delta} u \leq C \inf_{\Omega_r^\delta} u,$$

where the constant depends also on η and δ , and

$$\Omega_r^\delta = \{(X, t) \in \Omega : d(X; S) > \delta, \text{ uniformly on } t, \delta^2 < t < r^2 - \delta^2\}.$$

Comparison theorem for nonnegative solutions

Let $(Y, s) \in S$ and $r > 0$ small enough. Assume that u and v are two nonnegative solutions of $Lu = 0$ in Ω vanishing continuously in $\Delta_{2r}(Y, s) \subset S$. Then

$$(2.15) \quad \frac{u(X, t)}{Cv(X, t)} \leq \frac{u(\bar{A}_r(Y, s))}{v(\underline{A}_r(Y, s))} \leq \frac{u(X, t)}{Cv(X, t)}$$

for every $(X, t) \in \Psi_r(Y, s)$. The constant C depends only on ellipticity, dimension and the Lipschitz character of Ω .

For the proof of (2.14) and (2.15) we may adapt again the proofs of [20] to our non-cylindrical domains.

Remark. As observed in [16], estimates (2.10), (2.11), (2.14) and (2.15) may be obtained for normalized adjoint solutions (see below) following the techniques from [20]. This is a consequence of the estimates for the fundamental solution of L obtained in [16].

Following [23], p. 747, we say that a function $w: \Omega \rightarrow \mathbb{R}^+$ is a $B_p(\Omega, \mu)$ weight, $1 < p < \infty$, or $w \in B_p(\Omega, \mu)$ (μ is a Borel measure on Ω), if for every cylinder \mathbf{C} as defined above, such that $2\mathbf{C} \subset \Omega$, the following inequality holds:

$$(2.16) \quad \left(\frac{1}{|\mathbf{C}|} \int_{\mathbf{C}} w^p d\mu \right)^{1/p} \lesssim \frac{1}{|\mathbf{C}|} \int_{\mathbf{C}} w d\mu.$$

Similarly we say that $w \in A_p(\Omega, \mu)$ if

$$(2.17) \quad \left(\frac{1}{|\mathbf{C}|} \int_{\mathbf{C}} w d\mu \right) \left(\frac{1}{|\mathbf{C}|} \int_{\mathbf{C}} w^{-\frac{1}{p-1}} d\mu \right)^{p-1} \leq C.$$

Whenever μ is the Lebesgue measure in Ω we will omit both Ω and μ from the notation.

We now sketch the proof of certain weighted *a priori* estimate which will be used in the proof of a second order Caccioppoli type inequality.

PROPOSITION (2.18) (Interior weighted $\mathbf{W}_p^{2,1}$ regularity). *Suppose $L \in P(\text{VMO})$. Let $w \in A_p(\mathbf{C}_{2r})$ for some $1 < p < \infty$ and let $L^p(\mathbf{C}_{2r}, w)$ denote the L^p space on \mathbf{C}_{2r} with measure $w(X, t)dXd t$. Then there exists a constant $C = C(n, p, \lambda)$ and r_0 such that for $r < r_0$, $\mathbf{C}_{2r} \subset \Omega$ and u in $\mathbf{W}_p^0(\Omega)$ one has*

$$(2.19) \quad \|\partial_{i,j}u\|_{L^p(\mathbf{C}_r, w)} + \|\partial_t u\|_{L^p(\mathbf{C}_r, w)} \leq C \|Lu\|_{L^p(\mathbf{C}_{2r}, w)}.$$

Proof. The proof is an adaptation of [5], Theorem 4.1, using the weighted inequalities for singular integrals and commutators of singular integrals with multiplication by *BMO* functions [6], Theorem 3.1. For completeness we

include a sketch of the proof. Let $\mathcal{G}_{(X,t)}(Y, s)$ be the fundamental solution in \mathbb{R}^{n+1} of the frozen coefficients operator

$$L_{(X,t)}u = \sum_{i,j} a_{i,j}(X, t)\partial_{i,j}u - \partial_t u$$

(cf. [32]). One may prove that $\mathcal{G}_{(X,t)}(Y, s) = 0$ for $s > t$ and

$$(2.20) \quad \sup_{\Sigma} |\nabla_Y \mathcal{G}_{(X,t)}(Y, s)| \leq C,$$

where C depends only on ellipticity of L and $\Sigma = \{|X|^2 + t^2 = 1\}$ with surface measure σ . Let n_j be the j component of the vector n normal to the Lipschitz domain $\Omega(s)$, assuming $Q = (Y, s)$. Define

$$\begin{aligned} \alpha_{i,j}(P) &= \int_{\Sigma_n} \partial_i \mathcal{G}_P(Q) n_j d\sigma(Q), \\ \mathcal{K}_{i,j}f(P) &= \lim_{\epsilon \rightarrow 0} \int_{\|P-Q\| > \epsilon} \partial_{i,j} \mathcal{G}_P(P-Q) f(Q) dQ, \\ \mathcal{C}_{i,j}[a, f](P) &= \lim_{\epsilon \rightarrow 0} \int_{\|P-Q\| > \epsilon} \partial_{i,j} \mathcal{G}_P(P-Q) [a(Q) - a(P)] f(Q) dQ \\ &= \mathcal{K}_{i,j}(af)(P) - a(P) \mathcal{K}_{i,j}f(P) \end{aligned}$$

for f a suitable smooth function and $a \in BMO$. Then we have [5], Theorem 1.4, the following interior representation formula, for any function u sufficiently smooth:

$$(2.21) \quad u_{i,j}(X, t) = \sum_{k,l}^n \mathcal{C}_{i,j}[a_{k,l}, u_{h,k}](X, t) + \mathcal{K}_{i,j}(Lu)(X, t) + \alpha_{i,j} \cdot Lu(X, t).$$

By (2.20) above, the last term is uniformly bounded by $Lu(X, t)$. On the other hand, as observed in [6] we have

$$\|\mathcal{K}_{i,j}f\|_{L^p(\mathbf{C}_r, \omega)} \leq C \|f\|_{L^p(\mathbf{C}_r, \omega)} \quad \text{and} \quad \|\mathcal{C}_{i,j}[a, f]\|_{L^p(\mathbf{C}_r, \omega)} \leq C \|a\|_* \|f\|_{L^p(\mathbf{C}_r, \omega)}.$$

This already implies the desired result. \square

We state some useful properties of parabolic measure, which are consequences of the previous properties for solutions to $Lu = 0$ when $L \in \mathbf{P}(\text{VMO})$.

Normalizing lemma [45]

Let $(Q, s) \in \mathcal{S}$, $r > 0$, and $(X, t) \in \Omega$ such that $\|X - Q; t - s\| \approx r \approx d(X, t; \mathcal{S})$. Then

$$(2.22) \quad \omega(X, t; \Delta_r(Q, s)) \approx 1.$$

Scaling relation for parabolic measures [45]

There exists r_0 depending on the Lipschitz character of Ω such that if $(Q, s) \in \mathcal{S}$, $0 < 5r < r_0$, $s < r/2$, $F \subset \Delta \equiv \Delta_s(Q, s)$ a Borel set, and $(X, t) \in \Omega$ such that $\|X - Q; t - s\| \approx r \approx d(X, t; \mathcal{S}_T)$, then

$$(2.23) \quad \frac{\omega(F)}{\omega(\Delta_s(Q, s))} \approx \frac{\omega(X, t; F)}{\omega(X, t; \Delta_s(Q, s))}.$$

Doubling property of parabolic measure [45]

Let $(Y, s) \in S$ be fixed. Then, given $k > 1, \eta > 1$, for $0 < r < \eta r_0/4$ and $(X, t) \in \Omega$ satisfying $|X - Y| \leq k\sqrt{t - s}$, $4r \leq \sqrt{t - s} \leq \eta r_0$ we have

$$(2.24) \quad \omega(X, t; \Delta_{2r}(Y, s)) \leq C\omega(X, t; \Delta_r(Y, s)).$$

Doubling property of adjoint solutions

Let $v \in L^1_{loc}(\Omega)$ be a nonnegative adjoint solution on Ω . Then

$$(2.25) \quad \int_{\mathbf{C}_r} v(X, t) dXd t \approx \int_{\mathbf{C}_{r/2}} v(X, t) dXd t$$

for every cylinder $\mathbf{C}_r \subset \mathbb{R}^{n+1}$ of radius r such that $\mathbf{C}_{2r} \subset \Omega$.

Remark. The proof of this property is actually a consequence of [16], Theorem 1.3. We may take backward cylinders Q instead of the cylinders \mathbf{C} , as observed in [16]. This will be used in the proof of weighted Poincaré inequality (2.33) below. See also the proof given in [1].

The proofs of the following properties are adaptations of the corresponding elliptic results from [21] and [4], and we include the proofs for completeness.

PROPOSITION (2.26). *Suppose $L \in P(VMO)$. Let $1 < p < \infty$ and let $v \in L^p_{loc}(\Omega)$ be a nonnegative adjoint solution in Ω . Then $v \in B_p(\Omega)$.*

Proof. We must prove that

$$\left(\frac{1}{|\mathbf{C}_r|} \int_{\mathbf{C}_r} v^p(X, t) dXd t \right)^{1/p} \leq \frac{C}{|\mathbf{C}_r|} \int_{\mathbf{C}_r} v(X, t) dXd t$$

for any cylinder \mathbf{C}_r of radius $r > 0$ such that $\mathbf{C}_{2r} \subset \Omega$. Given $f \in C_0^\infty(\mathbf{C}_{2r})$, with $\|f\|_{L^q(\mathbf{C}_{2r})} \leq 1$, $1/p + 1/q = 1$, let $u \in W_q^{2,1}(\mathbf{C}_{2r})$ be the solution to $Lu = f$ in \mathbf{C}_{2r} with $u|_{\partial_p \mathbf{C}_{2r}} = 0$ and $\|u\|_{W_q^{2,1}(\mathbf{C}_{2r})} \leq \|f\|_{L^q(\mathbf{C}_{2r})}$ (cf. [34], Chapter VII).

If $1 < p < (n+2)/(n+1)$ then $q > n+2$ and we can follow the proof of either [21] or [4]. If $p > (n+2)/(n+1)$ let q_0 such that $\frac{1}{q_0} = \frac{1}{p} - \frac{2}{n+2}$ and q_1 such that $\frac{1}{q_1} = \frac{1}{p} - \frac{1}{n+2}$. Since $q_0 > q_1 > q$ we can apply [32], p. 80 to prove

$$\begin{aligned} \|u\|_{L^{q_0}(\mathbf{C}_{2r})} &\lesssim \frac{r^2}{r^{(n+2)(1/q-1/q_0)}} \|u\|_{W_q^{2,1}(\mathbf{C}_{2r})} \\ \|\partial_j u\|_{L^{q_1}(\mathbf{C}_{2r})} &\lesssim \frac{r}{r^{(n+2)(1/q-1/q_1)}} \|u\|_{W_q^{2,1}(\mathbf{C}_{2r})} \end{aligned}$$

Let $\psi \in C_0^\infty(\mathbf{C}_{2r})$ be a nonnegative function on Ω_T satisfying $\psi \equiv 1$ on \mathbf{C}_r and $|\nabla_X^2 \psi(X, t)|, |\partial_t \psi(X, t)| \lesssim 1/r^2, |\nabla_X \psi(X, t)| \lesssim 1/r$. Since

$$L(\psi u) = uL\psi + f\psi + 2\langle A\nabla\psi, \nabla u \rangle,$$

we have $L(\psi u) \in L^{q_0}(\mathbf{C}_{2r})$, and since there exists $v \in W_{q_0}^0(\mathbf{C}_{2r})$ such that $Lv = L(\psi u)$ in \mathbf{C}_{2r} (cf. [32], p. 323), we have $\psi u = v \in W_{q_0}^{2,1}(\mathbf{C}_{2r})$ and hence $u \in W_{q_0}^{2,1}(\mathbf{C}_{2r})$. Iterating this argument we obtain $u \in W_{q'}^{2,1}(\mathbf{C}_{2r})$ for $q' > n+2$, where we may apply again the ideas from [21] or [4]. \square

PROPOSITION (2.27). *If G denotes the Green's function of L in the cylinder \mathbf{C}_3 (where \mathbf{C}_r denotes the cylinder of radius r centered at the origin), then*

$$(2.28) \quad \int_{\mathbf{C}_1} G(X, t; Y, s) dY ds \approx 1$$

for $(X, t) \in \mathbf{C}_2$.

Proof. Let φ be the function $\varphi(X, t) = [(1 + \delta)^2 - |X|^2]^2$ and define

$$h(X, t) = \int_{\partial_p \mathbf{C}_{1+\delta}} G(X, t; Y, s) dY ds - \kappa \varphi(X, t).$$

If κ is chosen small enough, then $Lh \leq 0$ on $\mathbf{C}_{1+\delta}$ and $h \geq 0$ on $\partial_p \mathbf{C}_{1+\delta}$. Hence by the maximum principle, $h \geq 0$ on $\mathbf{C}_{1+\delta}$; i.e.,

$$\int_{\partial_p \mathbf{C}_{1+\delta}} G(X, t; Y, s) dY ds \geq \kappa \varphi(X, t).$$

An iteration argument and Harnack's principle (2.14) applied to $G(X, t; Y, s)$ allow us to conclude the proof of the lower bound. The upper bound follows from (2.2). \square

We say that a function w is a *normalized adjoint solution* for L in Ω with respect to an adjoint solution \tilde{w} (or when the context is clear we will call it simply NAS), if w is smooth and $\tilde{w}w$ is an adjoint solution for L in Ω (cf. [4]).

In the fundamental work [16] it was observed how to obtain a Harnack's principle as (2.4) for NAS. Accordingly, one can obtain a boundary Harnack's inequality, a backward Carleson type inequality, boundary Hölder continuity and an elliptic type Harnack's inequality for NAS with the techniques of [20, 45].

In what follows $N\Omega$ will denote the cylinder $\{(Y, s) \in \mathbb{R}^n \times \mathbb{R} : |Y| < N \text{diam } \Omega, s \in (0, NT)\}$. Also $S(N\Omega)$ will be the lateral boundary of $N\Omega$. Let $G(Y, s) = G(\Xi, 10T; Y, s)$ denote the Green's function for L in 20Ω with pole $(\Xi, 10T)$. As before let $G(F) = \int_F G(Z, \tau) dZ d\tau$ for $F \subset \mathbb{R}^{n+1}$. Let $\tilde{g}(X, t; Ys) = g(X, t; Ys)/G(Y, s)$, where g denotes the Green's function of L in Ω . Recall that $\mathbf{C}(X, t) = \mathbf{C}_{d(X,t)/2}(X, t)$.

PROPOSITION (2.29) (Second order Caccioppoli type inequality). *Let $L \in P(\text{VMO})$ and let u be a solution to $Lu = 0$ in $\mathbf{C}_{3r}(X_0, t_0)$, and $v \in L^1_{loc}(\Omega)$ a nonnegative adjoint solution in $\mathbf{C}_{3r}(X_0, t_0) \subset \Omega$. Then for $0 < 2r < R$ and for any constant β ,*

$$(2.30) \quad \begin{aligned} & \int_{\mathbf{C}_r(X_0, t_0)} [|\nabla^2 u(X, t)|^2 + |\partial_t u(X, t)|^2] v(X, t) dX dt \\ & \lesssim \frac{C}{r^2} \int_{\mathbf{C}_{2r}(X_0, t_0)} |u(X, t) - \beta|^2 v(X, t) dX dt \\ & \quad + \frac{C}{r^4} \int_{\mathbf{C}_{2r}(X_0, t_0)} |\nabla u(X, t)|^2 v(X, t) dX dt. \end{aligned}$$

Proof. We sketch the proof following the arguments of [41]. One proves first the result for $v(X, t) = G(X, t)$ and then for $w = v/G$ one may apply elliptic type Harnack's principle for NAS. Take $\varphi \in C_0^\infty(\mathbf{C}_{2r}(X_0, t_0))$ such that $\varphi \equiv 1$

in $\mathbf{C}_r(X_0, t_0)$ and $|\nabla^i \varphi| \leq M/r^i$, $i = 1, 2, 3$, $|\partial_t \varphi| \leq M/r^2$. By (2.19) applied to $(u - \beta)\varphi(X, t)$, and using Proposition (2.26),

$$\int_{\mathbf{C}_{2r}(X_0, t_0)} [|\nabla^2 u(X, t)|^2 + |\partial_t u(X, t)|^2] G(X, t) dX dt \leq \int_{\mathbf{C}_{2r}(X_0, t_0)} (L[(u(X, t) - \beta)\varphi])^2 G(X, t) dX dt.$$

The right hand side of this equation may be rearranged to see that it is bounded by

$$(2.31) \quad \int (|u(X, t) - \beta|^2 |\nabla^2 \varphi(X, t)|^2 + |\nabla u(X, t) \cdot \nabla \varphi(X, t)|^2) G(X, t) dX dt.$$

From here the proposition is immediate. \square

Now we prove a weighted version of a Poincaré type inequality. We adapt the argument of [17], Lemma 9 to the parabolic equation, and for completeness we include the entire proof.

PROPOSITION (2.32) (Weighted Poincaré inequality). *Let $L \in P(VMO)$ and let u be a solution to $Lu = 0$ in $\mathbf{Q}_{4r} \equiv \mathbf{Q}_{4r}(X_0, t_0) \subset \Omega$. Set $\mathbf{Q}_\tau \equiv \mathbf{Q}_\tau(X_0, t_0)$ (backward cylinder of radius τ) for any $\tau > 0$. Then*

$$(2.33) \quad \sup_{(Y, s) \in \mathbf{Q}_\tau} |u(Y, s) - u(X_0, t_0)|^2 \leq \frac{r^2}{G(\mathbf{Q}_{2r})} \int_{\mathbf{Q}_{2r}} |\nabla u(X, t)|^2 G(X, t) dX dt.$$

Remark. In the statement and proof of this result it is convenient to use backward cylinders \mathbf{Q} . However in the applications, we will use the doubling property of $G(X, t)$ to obtain the result for cylinders \mathbf{C} .

Proof. Throughout the proof ω_τ represents the parabolic measure of L in \mathbf{Q}_τ , with pole at $(X_0, t_0 + \tau^2)$, and g_τ represents the Green's function for L in \mathbf{Q}_τ . To avoid cumbersome subindices we let $\mathbf{Q}_k = \mathbf{Q}_{2r/2^k}$, $g_k = g_{2r/2^k}$ and $\omega_k = \omega_{2r/2^k}$, where k is an integer.

Since $L(u^2) = 2\langle A\nabla u, \nabla u \rangle$, using (2.5) we obtain

$$(2.34) \quad \begin{aligned} & \int_{\partial_p \mathbf{Q}_{2r}} |u(\mathbf{Q}, z) - u(X_0, t_0)|^2 d\omega_{2r}(X_0, t_0) \\ &= \iint_{\mathbf{Q}_{2r}} g_{2r}(X_0, t_0; X, t) 2\langle A\nabla u(X, t), \nabla u(X, t) \rangle dX dt \end{aligned}$$

and by the same token, for every $0 < \tau \leq 2r$,

$$\iint_{\mathbf{Q}_\tau} g_{2r}(X_0, t_0; X, t) 2\langle A\nabla u(X, t), \nabla u(X, t) \rangle dX dt \leq \sup_{(X, t) \in \mathbf{Q}_\tau} |u(X, t) - u(X_0, t_0)|^2.$$

Therefore, for k large to be chosen

$$(2.35) \quad \iint_{\mathbf{Q}_k} g_k(X_0, t_0; X, t) 2\langle A\nabla u(X, t), \nabla u(X, t) \rangle dX dt \leq \sup_{(X, t) \in \mathbf{Q}_k} |u(X, t) - u(X_0, t_0)|^2.$$

Hence, the right hand side of (2.34) is bounded by

$$\iint_{\mathbf{Q}_k} (g_{2r}(X_0, t_0; X, t) - g_k(X_0, t_0; X, t)) 2\langle A\nabla u(X, t), \nabla u(X, t) \rangle dX dt$$

$$\begin{aligned}
 & + \iint_{Q_{2r} \setminus Q_k} g_{2r}(X_0, t_0; X, t) 2 \langle A \nabla u(X, t), \nabla u(X, t) \rangle dX dt \\
 & + \sup_{(X, t) \in Q_k} |u(X, t) - u(X_0, t_0)|^2 \equiv I + II + III.
 \end{aligned}$$

Observe that the estimate (2.9) implies

$$III \leq \theta^k \sup_{(X, t) \in Q_{2r}} |u(X, t) - u(X_0, t_0)|^2$$

for some $k > 3$ and that by Harnack's inequality (2.4), for every $(X, t) \in Q_k$ the measure $\omega(X, t)$ is absolute continuous with respect to $\omega(X_0, t_0)$. Thus for $(X, t) \in Q_k$

$$u(X, t) - u(X_0, t_0) = \int_{\partial_p Q_{2r}} (u(Q, z) - u(X_0, t_0)) d\omega_{2r}(X, t),$$

and since the Radon-Nikodým derivative $d\omega(X, t)/d\omega(X_0, t_0)$ is essentially bounded, then

$$\sup_{(X, t) \in Q_k} |u(X, t) - u(X_0, t_0)|^2 \leq C \int_{\partial_p Q_{2r}} |u(Q, z) - u(X_0, t_0)|^2 d\omega_{2r}(X_0, t_0).$$

If k is now chosen large enough,

$$III \leq \frac{1}{2} \int_{\partial_p Q_{2r}} |u(Q, z) - u(X_0, t_0)|^2 d\omega_{2r}(X_0, t_0)$$

and we can hide this term in the left hand side of (2.34).

Next observe that $\tilde{g}_{2r}(X_0, t_0; \cdot) - \tilde{g}_k(X_0, t_0; \cdot)$ is a NAS for L in Q_k . Hence, its maximum is attained on $\partial_p Q_k$ where actually

$$\tilde{g}_{2r}(X_0, t_0; \cdot) - \tilde{g}_k(X_0, t_0; \cdot) = \tilde{g}_{2r}(X_0, t_0; \cdot).$$

Applying the elliptic type Harnack's principle for NAS, since $\tilde{g}_{2r}(X_0, t_0; \cdot)$ vanishes in $\partial_p Q_{2r}$, for $(X, t) \in \partial_p Q_k$,

$$\tilde{g}_{2r}(X_0, t_0; X, t) \leq \frac{C}{G(Q_{2r})} \iint_{Q_{2r}} g_{2r}(X_0, t_0; X, t) dX dt$$

by the doubling property of adjoint solutions (2.25). Applying (2.8) and Cauchy's inequality,

$$\frac{1}{G(Q_{2r})} \iint_{Q_{2r}} g_{2r}(X_0, t_0; X, t) dX dt \leq \frac{Cr^{\frac{n}{n+1}} r^{\frac{n+2}{n+1}}}{G(Q_{2r})} \approx \frac{r^2}{G(Q_{2r})}.$$

Divide and multiply the integrand of I by $G(X, t)$ to obtain the desired bound for term I .

Similarly, $\tilde{g}_{2r}(X_0, t_0; \cdot)$ attains its maximum over $Q_{2r}(X_0, t_0) \setminus Q_k(X_0, t_0)$ at some point on $\partial_p Q_k(X_0, t_0)$ and so we can get the desired bound for term II as we did for term I . To finish the proof observe that

$$\sup_{(X, t) \in Q_r} |u(X, t) - u(X_0, t_0)|^2 \leq C \int_{\partial_p Q_{2r}} |u(Q, z) - u(X_0, t_0)|^2 d\omega_{2r}(X_0, t_0).$$

□

The following result will be useful at several stages. It substitutes a well known property of Green's function for divergence form operators (see [19], Theorem 1.4). We adapt the proof of [17], Lemma 2.

PROPOSITION (2.36). *There exists r_0 depending on the Lipschitz character of Ω , such that for every $(\mathbf{Q}_0, s_0) \in \mathcal{S}$, $r < r_0$ and $(Y, s) \in \beta(\mathbf{Q}_0, s_0) = \mathbf{C}_{d(\bar{\mathcal{A}}_r(\mathbf{Q}_0, s_0))/2}(\bar{\mathcal{A}}_r(\mathbf{Q}_0, s_0))$ and $(X, t) \in \Omega \cap \{(Z, \tau) \in \Omega : s_0 + 4r^2 < \tau < T\}$ we have*

$$(2.37) \quad \tilde{g}(X, t; Y, s) \frac{G(\mathbf{C}(Y, s))}{d(Y, s)^2} \approx \omega(X, t; \Delta_r(\mathbf{Q}_0, s_0))$$

whenever $\Delta_{2r}(\mathbf{Q}_0, s_0) \subset \mathcal{S}$.

Proof. Let $\varphi \in C_0^\infty(\mathbf{Q}_r(\mathbf{Q}_0, s_0))$ with $\varphi \equiv 1$ on $\mathbf{Q}_{r/2}(\mathbf{Q}_0, s_0)$ and $|\nabla\varphi| \leq 1/r$, $|\nabla^2\varphi|, |\partial_t\varphi| \leq 1/r^2$. By the doubling property (2.24) of parabolic measure and (2.5), and by the choice of (X, t) ,

$$\begin{aligned} \omega(X, t; \Delta_r(\mathbf{Q}_0, s_0)) &\leq \int_{\mathcal{S}} \varphi d\omega(X, t) = \int_{\Omega} g(X, t; Y, s) L\varphi(Z, \tau) dZd\tau \\ &\leq \frac{1}{r^2} \int_{\text{suppt}(\varphi)} \tilde{g}(X, t; Z, \tau) G(Z, \tau) dZd\tau. \end{aligned}$$

Applying a Carleson type estimate and the elliptic type Harnack's inequality for NAS, for $(Y, s) \in \beta(\mathbf{Q}_0, s_0)$, the last expression is bounded by

$$\frac{1}{r^2} \tilde{g}(X, t; Y, s) \int_{\mathbf{Q}_r(\mathbf{Q}_0, s_0)} G(Z, \tau) dZd\tau \leq C \frac{1}{r^2} \tilde{g}(X, t; Y, s) \int_{\mathbf{Q}_{3r}(\mathbf{Q}_0, s_0)} G(Z, \tau) dZd\tau.$$

Observe that $\mathbf{Q}_{3r}(\mathbf{Q}_0, s_0) \subset \mathbf{C}_\rho(\mathbf{Q}_1, s_1)$ for some $\rho \approx r \approx d(Y, s)$ and $(\mathbf{Q}_1, s_1) \in \partial\mathbf{Q}_{3r} \cap \mathcal{S}$. So by the doubling property of adjoint solutions (2.25) and (2.14), the last expression is dominated by

$$\frac{1}{d(Y, s)^2} \tilde{g}(X, t; Y, s) \int_{\mathbf{C}_\rho} G(Z, \tau) dZd\tau \approx \frac{1}{d(Y, s)^2} \tilde{g}(X, t; Y, s) G(\mathbf{C}(Y, s)).$$

To prove the opposite inequality, define

$$u(X, t) = \frac{1}{r^2} \int_{\Psi_{r/8}(\mathbf{Q}_0, s_0)} g(X, t; Z, \tau) dZd\tau,$$

and observe that by the doubling property of G and (2.14),

$$\begin{aligned} u(X, t) &\leq \frac{C}{r^2} \int_{\Psi_{r/8}(\mathbf{Q}_0, s_0)} \tilde{g}(X, t; Z, \tau) G(Z, \tau) dZd\tau \\ &\leq \frac{C}{r^2} G(\Psi_{r/8}(\mathbf{Q}_0, s_0)) \tilde{g}(X, t; \bar{\mathcal{A}}_{r/4}(\mathbf{Q}_0, s_0)) \\ &\leq \frac{C}{r^2} G(\tilde{\mathcal{C}}) \tilde{g}(X, t; Y, s) \end{aligned}$$

where $\tilde{\mathcal{C}} = \mathbf{C}_{r/8}(\bar{\mathcal{A}}_{r/4}(\mathbf{Q}_0, s_0))$ and $(Y, s) \in \tilde{\mathcal{C}}$. Once again applying (2.14),

$$u(X, t) \leq \frac{C}{r^2} \int_{\tilde{\mathcal{C}}} \tilde{g}(X, t; Z, \tau) G(Z, \tau) dZd\tau \leq c_1,$$

by Tso's maximum principle, for $(X, t) \in \Omega$ with $s_0 + 4r^2 < t < T$. On the other hand for $(X_0, t_0) \in \beta(Q_0, s_0)$ and $(X, t) \in \Omega$ with $s_0 + 4r^2 < t < T$,

$$\omega(X, t; \Delta_r(Q_0, s_0)) \geq \omega(X_0, t_0; \Delta_r(Q_0, s_0)) \geq c_2.$$

Hence for (X, t) as above

$$\omega(X, t; \Delta_r(Q_0, s_0)) \geq \frac{c_3}{r^2} \int_{\Psi_{r/8}(Q_0, s_0)} g(X, t; Z, \tau) dZ d\tau.$$

Applying (2.12) and (2.14)

$$\begin{aligned} \omega(X, t; \Delta_r(Q_0, s_0)) &\geq \frac{C}{r^2} \tilde{g}(X, t; \underline{A}_r(Q_0, s_0)) \int_{\Psi_{r/8}(Q_0, s_0)} G(Z, \tau) dZ d\tau \\ &\geq \frac{C}{r^2} \tilde{g}(X, t; Y, s) \int_{\mathbf{C}_{r/8}(Q_0, s_0)} G(Z, \tau) dZ d\tau \end{aligned}$$

for $(Y, s) \in \beta(Q_0, s_0)$. Now we can apply (2.25) to estimate from below the last expression by

$$\frac{C \tilde{g}(X, t; Y, s)}{d(Y, s)^2} \int_{\mathbf{C}_{r/8}(Q_0, s_0)} G(Z, \tau) dZ d\tau$$

as desired. \square

3. Distributional inequalities

Having developed the machinery of last section, the distributional inequalities of this section become parabolic adaptations of the results of [17] and [14]. We sketch the proofs for completeness.

The first result of this section is a pointwise comparison in which the second area function is dominated by the first order area function, and that implies the first part of (1.7). It was proved in [40] for elliptic operators, and the proof may be easily adapted using second order Caccioppoli inequality (2.30) and weighted Poincare inequality (2.33).

PROPOSITION (3.1). *Let u be a solution to $Lu = 0$ in Ω . Then there exists $c > 1$ such that for any $\alpha > 0$ and any $(Q, s) \in S$,*

$$(3.2) \quad A_\alpha u(Q, s) \lesssim S_{c\alpha} u(Q, s).$$

Following the strategy of [51] and [17], for the proof of the next good- λ inequality we will focus on proving that $\|H\Lambda\|_1 + \|H\Lambda\|_* \leq c_1$ for an appropriate function $H\Lambda$.

PROPOSITION (3.3). *Let u be a solution to $Lu = 0$ in Ω , and let $0 < \alpha < \beta$ and $\mu \in A_\infty(\omega)$. Let $\Delta \subset S$ be any surface ball of radius $r > 0$ and with $2\Delta \subset S$. Then, for $\gamma > 0$ suitable chosen, and $t > 0$ the following holds*

$$\mu\{P \in \Delta : S_\alpha^r u(P) > \gamma t, N_\beta^r u(P) \leq t\} \leq C e^{-\gamma^2 c} \mu\{P \in \Delta : S_\alpha^r u(P) > t\}.$$

Let us recall first the following result of [51], which we state here adapted to our parabolic setting.

PROPOSITION (3.4). For $P, Q \in S$ and $X \in \Omega$, let $l_{P,Q}(X) = \min\{\|X - P\|, \|X - Q\|\}$ and $\Delta(Q, P, X) = \Delta_{l_{P,Q}(X)}(X_{Q,P})$; here $X_{Q,P}$ is either P or Q satisfying $\|X - X_{Q,P}\| = l_{P,Q}(X)$. Let $H(X, Q)$ be a continuous function on $\Omega \times S$ which satisfies the following conditions:

(i) There exists a positive constant C such that

$$\sup_{X \in \Omega} \int_S |H(X, Q)| d\omega \leq C,$$

(ii) For all $Q, P \in S$ and $X \in \Omega$ with $l_{P,Q}(X) > 2\|Q - P\|$

$$|H(X, Q) - H(X, P)| \leq \frac{C}{\omega(\Delta(Q, P, X))} \left(\frac{\|Q - P\|}{l_{P,Q}(X)} \right).$$

Suppose that Λ is a Carleson measure with respect to ω ; that is,

$$(3.5) \quad \Lambda(\Psi_r(Y, s)) \leq C\omega(\Delta_r(Y, s)).$$

Then the function

$$H\Lambda(Q) = \int_{\Omega} H(X, Q) d\Lambda(X),$$

is in BMO with $\|H\Lambda\|_1 + \|H\Lambda\|_* \leq C\|\Lambda\|$, where $\|\Lambda\|$ denotes the smallest constant for which (3.5) holds.

We will define a suitable Λ and $H(X, Q)$ to which we may apply this proposition and then we will use an auxiliary functional to conclude our distributional inequality.

To define the measure Λ , let

$$(3.6) \quad E = \{(Y, s) \in S : N_{\beta}u(Y, s) \leq 1\},$$

and define

$$(3.7) \quad \Lambda(F) = \int_{\Gamma(E) \cap F} g(\Xi, 3T/2; X, t) |\nabla u(X, t)|^2 dXd t,$$

where F is any Borel set of \mathbb{R}^{n+1} and $\Gamma(E) = \bigcup_{Q \in E} \Gamma(Q) \cap \Omega$ is the sawtooth region of E . Also, one can define ([34], p. 71) the *regularized distance function* $d^*(X, t)$ satisfying the following three properties: (A) $d(X, t) \approx d^*(X, t)$, (B) $|\nabla d^*(X, t)| \leq C$, (C) $|\nabla^2 d^*(X, t)|, |\partial_t d^*(X, t)| \leq C/d$. For $\varepsilon > 0$ set $\Omega_*^\varepsilon = \{P \in \Omega : d^*(P) > \varepsilon\}$, $\Omega^\varepsilon = \{P \in \Omega : d(P) > \varepsilon\}$

LEMMA (3.8). Λ is a Carleson measure with respect to ω on S whose norm $\|\Lambda\|$ is bounded by a constant which is independent of u .

The proof of this result is similar to that of [17], Lemma 5 using results of the previous section. We now define for $(X, t) \in \Omega$ and $(Y, s) \in S$

$$H(X, t; Y, s) = \varphi(X, t) \psi \left(\frac{\|X - Y; t - s\|}{d(X, t)} \right) \frac{d(X, t)^2}{G(\mathbf{C}(X, t))} \frac{G(X, t)}{\tilde{g}(\Xi, 3T/2; X, t)}$$

where $\varphi, \psi \in C_0^\infty(\mathbb{R}^{n+1})$ with $0 \leq \varphi, \psi \leq 1$, $\varphi = 1$ on $\Omega \setminus \Omega^{r_0/10}$, $\varphi = 0$ on $\Omega^{r_0/5}$; $\psi = 0$ for $\|X, t\| > 1 + 2\alpha$.

Adapting the arguments from [17], using results of the previous section, one can prove that H satisfies conditions (i) and (ii) of Proposition (3.4) and hence $H\Lambda$ is a function in $BMO(d\omega) \cap L^1(d\omega)$ with $\|H\Lambda\|_* + \|H\Lambda\|_{L^1} \leq C$.

Let $\Delta \subset S$ be as in the statement of the proposition. One may conclude as in [51] that there are constants $C, c > 0$ such that

$$(3.9) \quad \mu\{Q \in \Delta : H\Lambda(Q) > t\} \leq Ce^{-tc} \mu\{Q \in \Delta : H\Lambda(Q) > 1\}.$$

Define now for any function $F : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ and $(Y, s) \in S$

$$\tilde{S}_\alpha^r F(Y, s) = \left(\int_{\Gamma_\alpha(Y, s)} \frac{d(X, t)^2}{G(C(X, t))} |\nabla F(X, t)|^2 G(X, t) \varphi(X, t) dXd t \right)^{1/2}.$$

LEMMA (3.10). *For $\gamma > 1$ large enough we have*

$$\{P \in \Delta : S_\alpha^r u(P) > \gamma, N_\beta^r u(P) \leq 1\} \subset \{P \in \Delta : \tilde{S}_\alpha^r u(P) > \gamma/2, N_\beta^r u(P) \leq 1\}.$$

The proof is similar to that of [17], p. 284.

Proof of Proposition (3.3). Take $\mu \in A_\infty(\omega)$. Without loss of generality, we may take $t = 1$. Observe that for $(Y, s) \in E \cap \Delta$, $\tilde{S}_\alpha u(Y, s)^2 \leq H\Lambda(Y, s)$ and that $H\Lambda(Y, s) \leq \tilde{S}_{2\alpha} u(Y, s)^2$ for every $(Y, s) \in \Delta$. Applying (3.9) and this remark,

$$\begin{aligned} \mu\{P \in \Delta : S_\alpha^r u(P) > \gamma, N_\beta^r u(P) \leq 1\} &\leq \mu\{P \in \Delta : \tilde{S}_\alpha^r u(P) > \gamma/2\} \\ &\leq \mu\{P \in \Delta : H\Lambda(P) > \gamma^2/4\} \\ &\leq Ce^{-\gamma^2} \mu\{P \in \Delta : S_{2\alpha}^{2r} u(P) > 1\} \end{aligned}$$

□

Before stating our second good- λ inequality, we recall that if r is small enough, and if $F \subset \Delta_r(Q_0, s_0)$ is a closed set, then following [7] or [37], one may prove that there exists a point $(X_0, t_0) \in \Gamma_{(\alpha+\beta)/2}(F)$ with $d(X_0, t_0; S) \approx r$, and so that $t_0 > t$ for every $(X, t) \in \Psi_{2r}(Q_0, s_0)$.

Recall also the definition of the *Hardy-Littlewood maximal function* (on the lateral boundary) of a function f with respect to a measure $\mu \in A_\infty(\omega)$

$$M_\mu(f)(Q, s) = \sup_{(Q, s) \in \Delta} \frac{1}{\mu(\Delta)} \int_\Delta f d\mu,$$

for $(Q, s) \in S$.

PROPOSITION (3.11). *Suppose u is a solution to $Lu = 0$ in Ω , $\Delta \equiv \Delta_r(P_0)$ is a surface ball, $0 < \alpha < \beta$, $\mu \in A_\infty(\omega)$. Assume that for some $\lambda_0 > 0$, $N_\alpha u(P_1) \leq \lambda_0$ for some $P_1 \in S$ with $d(P_1; \Delta) \approx r$. Then given constants $\alpha, \gamma > 1$ there exist $\epsilon, \delta > 0$ (depending only on the A_∞ property of μ , the Lipschitz character of Ω , the ellipticity of L , α , β and n) such that*

$$(3.12) \quad \alpha\mu\{P \in \Delta : N_\alpha^r u(P) \geq \gamma\lambda_0, S_\beta^r u(P) \leq \epsilon\lambda_0, M_\mu(\chi_{G_{\epsilon\lambda_0}}) < \delta\} \leq C(\gamma)\mu(\Delta)$$

where $G_{\lambda_0} = \{P \in \Delta : S_\beta^r u(P) > \lambda_0\}$, and where $C(\gamma)$ is a constant depending on γ and the A_∞ property of μ .

Proof. Once again we adopt ideas from the elliptic case [14], Lemma 2, [17], Lemma 8. Set $E = \{P \in \Delta : N_\alpha^r u(P) \geq \gamma\lambda_0, S_\beta u(P) \leq \epsilon\lambda_0, M_\mu(\chi_{G_\epsilon}) \leq \delta\lambda_0\}$. Let $W = \Gamma_{(\alpha+\beta)/2}(E) \cap \Omega$, the sawtooth region above the set E . Define for $(Q, s) \in E$,

$$\tilde{N}u(Q, s) = \sup_{(X, t) \in \Gamma_\alpha(Q, s) \cap W} |u(X, t) - u(X_0, t_0)|.$$

Observe that $\tilde{N}u \leq CN_\alpha u$ on $\partial\Omega$. In particular, \tilde{N} satisfies a weak-(2, 2) boundedness property by Hardy-Littlewood's maximal theorem [47], p. 13.

The following lemma follows from the doubling property of ω , (2.22) and (2.23), as in the ‘‘Main Lemma’’ of [14]. For parabolic operators (divergence and nondivergence form) one can follow the arguments of [7], Lemma 2.10.

LEMMA (3.13). *Let ν be the parabolic measure of W with pole at (X_0, t_0) , and let $F \subset 2\Delta \equiv \Delta_{2r}(P_0)$ a Borel set. Define*

$$(3.14) \quad \tilde{\nu}(F) = \nu(E \cap F) + \sum_j \frac{\omega(F \cap I_j)}{\omega(I_j \cap S)} \nu(\tilde{I}_j \cap (\partial W \cap \Omega)),$$

where $\{I_j\}$ is a Whitney decomposition of $2\Delta \setminus E$, and \tilde{I}_j is the projection of I_j on W . Then there exists $\theta, C > 0$ such that

$$\frac{\omega(F)}{\omega(\Delta')} \leq C \left(\frac{\tilde{\nu}(F)}{\tilde{\nu}(\Delta')} \right)^\theta,$$

where $\Delta' \subset \Delta$ any surface ball.

The projection function used above is a function mapping any point $P \in \Omega$ to the point $\tilde{P} \in W$ exactly above P in the x_0 direction. Observe that in particular one has $d(I_j; \tilde{I}_j) \approx d(I_j; E) \approx \text{diam} I_j \equiv r_j$. The ‘‘Main Lemma’’ mentioned above states that, with the notation of the previous lemma, we actually have

$$(3.15) \quad \left(\frac{\omega(F)}{\omega(\Delta)} \right)^\theta \leq C \nu(F).$$

On the other hand, one may prove, using (2.33), (see [14], p. 104), that $E \subset \{P : \tilde{N}(P) \geq \gamma\lambda_0\}$ (here we use that $Nu(P_1) \leq \lambda_0$). To prove (3.12), since $\mu \in A_\infty(\omega)$, we claim that it suffices to prove that if $\xi = \gamma\lambda_0$, then

$$(3.16) \quad \omega(\{(Q, s) \in \Delta : \tilde{N}u(Q, s) > \xi\}) \leq C\xi^{-2\theta} \omega(\Delta).$$

Proof of (3.16). Recall the measure $\tilde{\nu}$ defined in (3.14). Let $H_\xi = \{(Q, s) \in \Delta : \tilde{N}u(Q, s) > \xi\}$. Then, by Chebyshev's inequality

$$\tilde{\nu}(H_\xi) \leq \frac{1}{\xi^2} \int_E (\tilde{N}u)^2 d\nu + \sum_j \frac{\omega(H_\xi \cap I_j)}{\omega(I_j \cap S)} \nu(\tilde{I}_j \cap (\partial W \cap \Omega)).$$

Observe that, by Lemma (3.13), it is enough to prove

$$(3.17) \quad \tilde{\nu}(H_\xi) \leq C\xi^{-2}$$

for $\xi \gg 1$. We prove (3.17) in three steps.

Step 1. We will prove first that $\omega(H_\xi \cap I_j) \approx \omega(I_j \cap S)$.

For $Q \in H_\xi \cap I_j$ one has $d(Q, E) \approx \text{diam} I_j$; also, if $P \in E$ satisfies $d(Q; P) \approx \text{diam} I_j$ then there exists $\mathbf{X} \in \Gamma_\alpha(Q) \cap \Gamma_{\frac{\alpha+\beta}{2}}(P)$ with $d(\mathbf{X}; S(\Omega)) \approx \text{diam} I_j$, and such that $u(\mathbf{X}) - u(X_0, t_0) > \xi$, where (X_0, t_0) is the point from the construction of the domain W . So one may chose β as a large multiple of α , and for a constant ρ one has $\tilde{N}u > \xi$ on the set $\Delta_{\rho \text{diam} I_j}(Q)$. The doubling property of ω implies the first claim.

Step 2. Next we prove that

$$\tilde{\nu}(H_\xi)\xi^2 \leq C \int_W g_0(X_0, t_0; Y, s) |\nabla u(Y, s)|^2 dY ds,$$

where g_0 is the Green's function of L in W . By our choice of \mathbf{X} we have $|u(\mathbf{X}) - u(X_0, t_0)| > \xi$. For $(Z, \tau) \in \tilde{I}_j \cap (\partial W \cap \Omega)$, by the weighted Poincaré's inequality (2.33),

$$|u(Z, \tau) - u(\mathbf{X})| \leq (S_\alpha u(P))^2 < \epsilon^2 t^2.$$

Hence, for $\epsilon > 0$ small $|u(Z, \tau) - u(X_0, t_0)| > C$. Therefore

$$\tilde{\nu}(H_\xi)\xi^2 \leq C \int_E (\tilde{N}u)^2 d\nu + \sum_j \int_{\tilde{I}_j} |u(Q) - u(X_0, t_0)|^2 d\nu.$$

Since there is only a finite overlapping, we may use the Hardy-Littlewood's maximal theorem to obtain

$$\tilde{\nu}(H_\xi)\xi^2 \leq C \int_W |u(Q) - u(X_0, t_0)|^2 d\nu.$$

By equation (2.5)

$$\begin{aligned} \tilde{\nu}(H_\xi)\xi^2 &\leq 2 \int_W g_0(X_0, t_0; Y, s) \langle A \nabla u, \nabla u \rangle dY ds \\ &\lesssim \int_W g_0(X_0, t_0; Y, s) |\nabla u(Y, s)|^2 dY ds. \end{aligned}$$

Step 3. We finally prove that

$$\int_W g_0(X_0, t_0; Y, s) |\nabla u(Y, s)|^2 dY ds \leq C.$$

Set $W^r = \{X \in W : d(X, S) \leq rr\}$. We split the integral over W in the part away from the boundary and the one close to the boundary.

Observe that applying (2.5)

$$\begin{aligned} \int_{W \setminus W^r} g_0(X_0, t_0; Y, s) |\nabla u|^2 dY ds &\leq C \int_{W \setminus W^r} g_0(X_0, t_0; Y, s) \langle A \nabla u, \nabla u \rangle dY ds \\ &\leq C \int_{W \setminus W^r} g_0(X_0, t_0; Y, s) (Lu^2) dY ds \\ &\leq C \sup_{(X, t) \in W \setminus W^{r/2}} |u(X, t) - u(X_0, t_0)|^2. \end{aligned}$$

Now, for $(X, t) \in W \setminus W^{r/2}$ one may find a finite sequence of points $\{(X_i, t_i)\} \subset W \setminus W^{r/2}$ with $\|X_i - X_{i-1}, t_i - t_{i-1}\| \leq N_0$ and cylinders $\mathbf{C}_i \subset \Gamma_{(\alpha+\beta)/2}(P_i)$ for some $P_i \in E$, $i = 1, 2, \dots, N_1$, $X_1 = (X, t)$, $X_{N_1} = (X_0, t_0)$, and N_0, N_1 depending on the Lipschitz character of Ω and on the apertures α and β . Also, in $W \setminus W^{r/2}$ one has by (2.28) that $G(\mathbf{C}_i) \equiv 1$. Therefore, using the weighted Poincaré's inequality (2.33) and that $S_\beta u(P) \leq \epsilon \lambda_0$ for $P \in E$ we have $|u(X_j) - u(X_{j+1})|^2 \lesssim \epsilon \lambda_0$ and hence $|u(X, t) - u(X_0, t_0)|^2 \leq C$ for $\epsilon > 0$ small, and with constant depending only on α, β and on the Lipschitz character of Ω .

To handle the part close to the boundary observe that

$$C \geq \int_{S \setminus G_{\lambda_0}} S_{\beta}^r u(Q) d\omega(X_0, t_0; Q) \geq \int_W |\nabla u(Y, s)|^2 G(Y, s) \frac{d(Y, s)^2}{G(\mathbf{C}(Y, s))} \psi(Y, s) dY ds$$

where $\psi(Y, s) = \omega(X_0, t_0; \{P \in S \setminus G_{\lambda_0} : (Y, s) \in \Gamma_{\beta}^r(P)\})$, and G_{λ_0} is as in the statement of the theorem.

Observe also that for $Y \in W^r$ there exists $\tilde{Y} \in E \subset S \setminus G_{\lambda_0}$ such that $\|Y - \tilde{Y}\| \approx d(Y)$ and $Y \in \Gamma_{\beta}(\tilde{Y})$. Hence, setting $\tilde{\Delta} \equiv \Delta_{\gamma d(Y)}(\tilde{Y})$ we have $\tilde{\Delta} \cap S \setminus G_{\lambda_0} \subset \{P \in S \setminus G_{\lambda_0} : Y \in \Gamma_{\beta}^r(P)\}$ and consequently by (2.37)

$$\begin{aligned} \psi(Y, s) &\geq \omega(X_0, t_0; \tilde{\Delta} \cap S \setminus G_{\lambda_0}) = \frac{\omega(X_0, t_0; \tilde{\Delta} \cap S \setminus G_{\lambda_0})}{\omega(X_0, t_0; \tilde{\Delta})} \omega(X_0, t_0; \tilde{\Delta}) \\ &\approx \frac{\omega(\tilde{\Delta} \cap S \setminus G_{\lambda_0})}{\omega(\tilde{\Delta})} \omega(X_0, t_0; \tilde{\Delta}) \\ &\approx \left(\frac{\omega(\tilde{\Delta} \cap S \setminus G_{\lambda_0})}{\omega(\tilde{\Delta})} \right) \left(\frac{\tilde{g}(X_0, t_0; Y, s) G(\mathbf{C}(Y, s))}{d^2(Y, s)} \right) \end{aligned}$$

By the definition of E and by the A_{∞} property of μ

$$\psi(Y, s) \geq \frac{\tilde{g}(X_0, t_0; Y, s) G(\mathbf{C}(Y, s))}{d^2(Y, s)}.$$

Thus

$$\int_{W^r} g_0(X_0, t_0; Y, s) |\nabla u(Y, s)|^2 dY ds \leq \int_{W^r} g(X_0, t_0; Y, s) |\nabla u(Y, s)|^2 dY ds \leq C$$

which completes the proof of (3.17). \square

From the local distribution inequality (3.12) we may obtain the following inequality, which is a precise statement of inequality (1.8): for $p > 0$ and for Δ a surface ball of radius $r > 0$ and center $(q_0, q, s) \in S$,

$$(3.18) \quad \int_{\Delta} (N_{\alpha}^r u)^p d\mu \lesssim \int_{2\Delta} (S_{\beta}^r u)^p d\mu + \mu(\Delta) |u(\bar{Q})|$$

where $\bar{Q} = (q_0 + r, q, s)$. We will use the notation \bar{P} whenever one lifts P in the x_0 direction by a magnitude of the radius of the ball in question. To prove (3.18), we follow [7]. Define $E_{\lambda_0} = \{P \in \Delta : N_{\alpha}^r u > \lambda_0\}$ and let $I_j \equiv I_{r_j}(P_j) \subset \Delta$ be a covering of E_{λ_0} by of surface balls with finite overlapping and satisfying that $d(I_j; \Delta \setminus E_{\lambda_0}) \lesssim r_j$. Define

$$I_j^p = \{(p_0, p, s) \in \Omega : (p, s) \in I_j, \psi(p, s) + \rho < p_0 < \psi(p, s) + Mr_j\}$$

where we assumed that ψ defines S in a coordinate cylinder containing Δ . Then we claim that

$$(3.19) \quad \sup_{I_j^p} |u(X, t)| \leq |u(\bar{P}_j)| + c_0 \left(\frac{1}{\mu(I_j)} \int_{I_j} (S_{\beta}^r u)^p d\mu \right)^{1/p}.$$

Indeed, let $P \in I_j$ and $(X, t) \in \Gamma_{\beta/2}^r(P) \cap I_j^p$. Applying (2.33) and (2.25) as in the proof of the proposition, $|u(X, t) - u(\bar{P}_j)| \lesssim S_{\beta}^r(P)$. Since for any $(X, t) \in I_j^p$

the set $\{P : (X, t) \in \Gamma_{\beta/2}^r(P)\}$ contains a surface ball I_j^+ of diameter at least ρ , then by (2.24)

$$\mu(I_j)|u(X, t)|^p \lesssim |u(\bar{P})|^p \mu(I_j) + \int_{I_j} (S_\beta^r u)^p d\mu$$

and (3.19) is proved. Let

$$\lambda_1 = |u(\bar{A}_r(Q))| + k \left(\frac{1}{\mu(\Delta)} \int_{2\Delta} (S_\beta^r u)^p d\mu \right)^{1/p}.$$

Choosing $k \gg c_0$ will guarantee that the hypothesis of Proposition (3.11) are satisfied for each of the sets I_j , whenever $\lambda_0 > \lambda_1$ by Harnack's inequality. Then one may follow the arguments from [7], p. 582, to prove (3.18).

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DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
 1409 W. GREEN ST
 URBANA IL 61801
 rnoriega@math.uiuc.edu

FACULTAD DE CIENCIAS
 UNIVERSIDAD AUTÓNOMA DEL ESTADO DE MORELOS
 AV. UNIVERSIDAD 1001
 COL. CHAMILPA
 CUERNAVACA MOR. CP 62210
 MÉXICO
 rnoriega@servm.fc.uaem.mx

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ON THE STOCHASTIC AUBRY-MATHER THEORY

RENATO ITURRIAGA AND HÉCTOR SÁNCHEZ-MORGADO

ABSTRACT. In this paper we prove the differentiability of the stochastic analogues of Mather's functions α and β , introduced implicitly by D. Gomes [G]. We also prove that the solution to the viscous Hamilton Jacobi equation associated to α is differentiable in the parameter.

1. Introduction

In the work of Gomes [G] a stochastic analogue of Aubry Mather theory was developed. More precisely let \mathbb{T}^d be the d dimensional torus, $L : \mathbb{T}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a strictly convex, C^∞ , superlinear Lagrangian. The problem is to find a probability measure $\mu(x, v)$ on $\mathbb{T}^d \times \mathbb{R}^d$ that minimizes the average action

$$A(\mu) = \int L(x, v) d\mu$$

among the probability measures that satisfy

$$(1.1) \quad \int (d\phi(x)v + \varepsilon \Delta \phi(x)) d\mu = 0$$

for all C^2 functions ϕ . Measures satisfying (1.1) are called stochastic for the following reason. Consider a controlled Markov diffusion

$$dX = vdt + \sigma dW$$

where $W(t)$ is Brownian motion, and the measure μ_T is given by

$$\int_{\mathbb{T}^d} \phi d\mu_T = \frac{1}{T} E \left(\int_0^T \phi(X(t), v(t)) dt \right).$$

If μ is a weak limit of a sequence μ_{T_n} , $T_n \rightarrow \infty$, using Dynkin's formula we have that μ satisfies (1.1) with $2\varepsilon = \sigma^2$.

For $\varepsilon = 0$, the nonstochastic case, (1.1) is the so-called holonomic condition and the previous fact is the analogous of the fact that weak limits of measures supported on liftings of closed curves are holonomic. According to results of Mather [Ma] and Mañé [M], holonomic minimizing measures are invariant.

Gomes proves that there is only one minimizing measure (Theorem 2). As in the nonstochastic case the theory is enhanced if we define a homology of

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rotation map. Let \mathcal{M} be the space of Borel probability measures on $\mathbb{T}^d \times \mathbb{R}^d$ endowed with the weak topology and define the space

$$\mathcal{N}_\varepsilon = \text{cl}\{\mu \in \mathcal{M} : \forall \phi \in C^2(\mathbb{T}^d) \int (\varepsilon \Delta \phi(x) + d\phi(x)v) d\mu = 0\}.$$

Define $\rho : \mathcal{N}_\varepsilon \rightarrow H_1(M, \mathbb{R})$ by

$$(1.2) \quad \langle \rho(\mu), [\omega] \rangle = \int (\varepsilon \delta \omega(x) + \omega(x)v) d\mu$$

where δ is the adjoint of d for the flat metric, so that $\Delta \phi = \delta d\phi$.

It is an immediate consequence of the fact that μ belongs to \mathcal{N}_ε that the integral in (1.2) does not depend on the representative of the cohomology class $[\omega]$. Thus, according to the Poincaré duality we can think of $\rho(\mu)$ as a homology class.

The function ρ is onto, see Lemma (2.1) below. Now we can define β_ε and α_ε , the stochastic analogues to Mather's functions β and α . See [Ma1].

$$(1.3) \quad \beta_\varepsilon(h) = \inf_{\mu \in \mathcal{N}_\varepsilon, \rho(\mu)=h} \int L(x, v) d\mu,$$

$$(1.4) \quad \alpha_\varepsilon(\omega) = - \inf_{\mu \in \mathcal{N}_\varepsilon} \int L(x, v) - \omega d\mu.$$

For each function the infimum is achieved at a unique measure. This is an easy consequence of Theorem 2 in [G]. Due to the convexity of L it is easy to see that the functions $\alpha_\varepsilon, \beta_\varepsilon$ are convex and dual of one another.

For $\varepsilon = 0$, Mather's function β is related to the stable norm for a Riemannian metric g . Recall that for h in $H_1(M, \mathbb{R})$ the stable norm is defined as

$$\|h\| = \inf \left\{ \sum |r_i| l(\sigma_i) : \sum r_i \sigma_i \text{ is a 1-cycle representing } h \right\},$$

where l is the length associated to the metric g . In this setting the stable norm is the square root of β .

Mather's function α coincides with the so called Mañé critical value [CIPP] and the effective Hamiltonian $\bar{H}(P)$ [EG]; that is, the unique number such that the equation

$$H(x, d\phi + P) = \bar{H}(P)$$

has a viscosity solution.

In the case of integrable systems, $\bar{H}(P)$ is a differentiable function which is the value of the Hamiltonian in action-angle variables.

According to Theorem 4 in [G], the function α_ε is characterized by

$$(1.5) \quad \alpha_\varepsilon(P) = \min_{h \in C^2} \max_x H(x, Dh(x) + P) + \varepsilon \Delta h(x).$$

Moreover by Theorems 5, 6, 7 in [G], the value $\alpha_\varepsilon(P)$ is the only number such that

$$(1.6) \quad H(x, D\varphi(x) + P) + \varepsilon \Delta \varphi(x) = \alpha_\varepsilon(P)$$

has C^2 solutions $\varphi : \mathbb{T}^d \rightarrow \mathbb{R}$. Moreover, the solution is C^∞ by elliptic regularity and unique besides constants.

Mather's functions α and β are in general not differentiable. It is known since [Ma2] that for time periodic Lagrangians on the circle the function β

is differentiable at irrational homologies and generically not differentiable at rational homologies. See also [Ba], [Au]. For higher dimensions, to give conditions for the differentiability is still an active research area. See [BIK], [Os], [Mas], [AB].

The first goal of this work is to prove the following.

THEOREM (1.7). *For $\varepsilon > 0$ the functions α_ε and β_ε are C^∞ .*

Varying P , equation (1.6) defines a family of PDE's having smooth solutions, and the second goal of the paper is to prove that the solution varies smoothly with respect to P .

THEOREM (1.8). *If we normalize the solutions $\phi(\cdot, P)$ of (1.6) by fixing its value, say 0 at a point x_0 , then $\phi \in C^\infty(\mathbb{T}^d \times \mathbb{R}^d)$.*

Gomes [G] proved several properties under the assumption that ϕ or α_ε are differentiable. Most notably he proved the stochastic analogue of the fact that, at points ω of differentiability of the Mather α function, ω -minimizing orbits have asymptotic homology $h = D\alpha(\omega)$. More precisely he proved that

$$\lim_{t \rightarrow \infty} E\left(\frac{X(t)}{t}\right) = D\alpha_\varepsilon(P)$$

where $X(t)$ is the stochastic process given by

$$dX = \partial_p H(X, d\phi(X, P))dt + \sqrt{2\varepsilon}dW, \quad X(0) = x.$$

To prove the differentiability of the functions β_ε and α_ε we will actually prove that they are strictly convex. Therefore the dual functions, meaning respectively α_ε and β_ε , are differentiable.

The strict convexity of β_ε follows from the uniqueness of a density satisfying a stationary Focker-Plank equation. The strict convexity of α_ε follows from Lemma (2.7), interesting by itself, which claims that subsolutions of (2.5) are actually solutions.

Lemma (2.7) is the counterpart of a result of Fathi [F], saying that any critical subsolution is a solution in the Aubry set. Indeed, for $\varepsilon = 0$ the Aubry set contains the projection of the support of minimizing measures, and by theorem 9 in [G], the projection of minimizing measures for $\varepsilon > 0$ is the whole manifold.

We prove the smoothness results in Theorems (1.7) and (1.8) by induction, applying the implicit function theorem on Sobolev spaces together with some well known facts of elliptic operators.

We thank J. Ize for suggesting the use of the implicit function theorem.

2. Differentiability of α_ε and β_ε

From now on we assume ε positive.

LEMMA (2.1). *The map ρ is onto.*

Proof. The idea of the proof of this lemma is to adapt the proof of the same lemma for $\varepsilon = 0$. Since \mathcal{N}_ε is convex, and ρ takes convex combinations to convex combinations, it is enough to prove that $me_i \in \rho(\mathcal{N}_\varepsilon)$ for any $m \in \mathbb{Z}$ and any of the standard generators e_i of $H_1(\mathbb{T}^d, \mathbb{Z}) \cong \mathbb{Z}^d$.

In the nonstochastic case one takes a multiple of the lift of a closed curve with homology e_i as a support of a measure. For $\varepsilon > 0$ we follow this idea, taking into account that the associated diffusion process concentrates densities according to the stationary Fokker-Planck equation.

Consider $f: \mathbb{T}^{d-1} \rightarrow \mathbb{R}$ with a unique maximum and let

$$V = (D_1 f, \dots, D_{d-1} f, m), \quad k = \int_{\mathbb{T}^{d-1}} \exp(f/\varepsilon).$$

Then $\theta(x) = \frac{1}{k} \exp(f(x_1, \dots, x_{d-1})/\varepsilon)$ is a solution to the Fokker-Planck equation

$$(2.2) \quad \varepsilon \Delta \theta - \operatorname{div}(\theta V) = 0.$$

Consider the measure $\mu \in \mathcal{N}_\varepsilon$ defined by

$$(2.3) \quad \int \psi(x, v) d\mu = \int_{\mathbb{T}^d} \psi(x, V(x)) \theta(x) dx.$$

Recalling (1.2), we have

$$\langle \rho(\mu), [\omega] \rangle = \int_{\mathbb{T}^d} \omega(x) (-\varepsilon d\theta(x) + \theta(x)V(x)) dx = m \int_{\mathbb{T}^d} \omega_d(x) \theta(x) dx.$$

Taking the representative given by $\omega(x)v = Pv$ with $P \in \mathbb{R}^d$, we identify $[\omega]$ with P and have

$$\langle \rho(\mu), P \rangle = mP_d$$

so $\rho(\mu) = me_d$. Similarly for the other e_i 's. \square

Next we have

PROPOSITION (2.4). *The function β_ε is strictly convex.*

Proof. Suppose that there are $h_1, h_2 \in H_1(M, \mathbb{R})$ different such that

$$\beta_\varepsilon(th_1 + (1-t)h_2) = t\beta_\varepsilon(h_1) + (1-t)\beta_\varepsilon(h_2), \quad t \in [0, 1].$$

Let μ_1 and μ_2 be minimizing measures with homologies h_1 and h_2 . In particular $\mu_1 \neq \mu_2$.

Consider a supporting hyperplane of the epigraph of β_ε , defined by $P \in \mathbb{R}^d \cong H^1(M, \mathbb{R})$, containing the segment S from h_1 to h_2 . Therefore

$$\beta_\varepsilon(h_1) - \langle h_1, P \rangle \leq \beta_\varepsilon(h) - \langle h, P \rangle$$

and the equality holds for $h \in S$. Thus, for any $h \in S$

$$\alpha_\varepsilon(P) = \langle h, P \rangle - \beta_\varepsilon(h).$$

By Theorem 8 in [G], this implies that any minimizing $\mu \in \mathcal{N}_\varepsilon$ with $\rho(\mu) \in S$ is supported on the graph of $V(x) = \partial_p H(x, D\phi(x) + P)$ for any solution ϕ to (1.6) and its density θ is a solution to (2.2), then it is unique and μ is given by (2.3). In particular $\mu_1 = \mu_2$, giving a contradiction. \square

A function f_ε is said to be a critical subsolution of

$$(2.5) \quad H(x, D\phi(x)) + \varepsilon \Delta \phi(x) = c(\varepsilon) := \alpha_\varepsilon(0)$$

if

$$(2.6) \quad H(x, Df_\varepsilon(x)) + \varepsilon \Delta f_\varepsilon(x) \leq c(\varepsilon).$$

LEMMA (2.7). *Any critical subsolution f_ε of (2.5) is in fact a solution.*

Proof. Let B be the set of points where equality in (2.6) holds. Consider the Hamiltonian

$$\mathbb{H}(x, p) = H(x, p + Df_\varepsilon(x)) + \varepsilon \Delta f_\varepsilon(x)$$

with Lagrangian

$$\mathbb{L}(x, v) = L(x, v) - Df_\varepsilon(x)v - \varepsilon \Delta f_\varepsilon(x).$$

Write the solution ϕ_ε of (2.5) as $\phi_\varepsilon = \psi_\varepsilon + f_\varepsilon$. It follows that ψ_ε is a solution of

$$(2.8) \quad \mathbb{H}(x, D\psi_\varepsilon(x)) + \varepsilon \Delta \psi_\varepsilon(x) = c(\varepsilon).$$

Suppose $D\psi_\varepsilon(z) = 0$, then

$$\varepsilon \Delta \psi_\varepsilon(z) = c(\varepsilon) - \mathbb{H}(z, 0) \geq 0.$$

Therefore, if z is a local maximum of ψ_ε , necessarily $\Delta \psi_\varepsilon(z) = 0$ and $\mathbb{H}(z, 0) = c(\varepsilon)$ and so $z \in B$.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space endowed with a Brownian motion $W(t) : \Omega \rightarrow \mathbb{T}^d$ on the flat d -torus. We denote by \mathbb{E} the expectation with respect to the probability measure \mathbb{P} . For $z \in B$, Lax's formula ([FS], Theorem IV 11.1) corresponding to (2.8) gives

$$(2.9) \quad \psi_\varepsilon(z) = \mathbb{E} \left(\psi_\varepsilon(X_\varepsilon(T)) - \int_0^T \mathbb{L}(X_\varepsilon(s), u_\varepsilon(X_\varepsilon(s))) ds - c(\varepsilon)T \right),$$

where $u_\varepsilon(x) = \partial_p H(x, D\phi_\varepsilon(x))$ is a maximal control and X_ε is the solution of the stochastic differential equation

$$(2.10) \quad \begin{cases} dX_\varepsilon(t) &= u_\varepsilon(X_\varepsilon(t))dt + \sqrt{2\varepsilon} dW(t), \\ X_\varepsilon(0) &= z. \end{cases}$$

For a local maximum z of ψ_ε we have

$$\mathbb{E}(\psi_\varepsilon(X_\varepsilon(T))) \leq \psi_\varepsilon(z),$$

$$\mathbb{L}(x, v) + c(\varepsilon) \geq \mathbb{L}(x, v) + \mathbb{H}(x, 0) \geq 0.$$

Thus $\psi_\varepsilon(z) = \mathbb{E}(\psi_\varepsilon(X_\varepsilon(T)))$ and

$$\mathbb{P}(\{\omega : \psi_\varepsilon(X_\varepsilon(T, \omega)) \neq \psi_\varepsilon(z)\}) = 0.$$

This implies that ψ_ε is a constant, therefore the subsolution f_ε is in fact a solution. \square

PROPOSITION (2.11). *The function α_ε is strictly convex.*

Proof. Assume that there exist P and Q such that for any λ in $[0, 1]$ we have $\alpha_\varepsilon(\lambda P + (1 - \lambda)Q) = \lambda \alpha_\varepsilon(P) + (1 - \lambda) \alpha_\varepsilon(Q)$.

Let f and g be solutions respectively of

$$H(x, df + P) + \varepsilon \Delta f = \alpha_\varepsilon(P),$$

$$H(x, dg + Q) + \varepsilon \Delta g = \alpha_\varepsilon(Q).$$

Defining $h = \lambda f + (1 - \lambda)g$ we have

$$\begin{aligned}
dh + \lambda P + (1 - \lambda)Q &= \lambda(df + P) + (1 - \lambda)(dg + Q), \\
H(x, dh + \lambda P + (1 - \lambda)Q) + \varepsilon \triangle h &\leq \lambda(H(x, df + P) + \varepsilon \triangle f) \\
&\quad + (1 - \lambda)(H(x, dg + Q) + \varepsilon \triangle g) \\
(2.12) \qquad \qquad \qquad &= \lambda\alpha_\varepsilon(P) + (1 - \lambda)\alpha_\varepsilon(Q) \\
&= \alpha_\varepsilon(\lambda P + (1 - \lambda)Q).
\end{aligned}$$

Therefore h is a subsolution of

$$H(x, du + \lambda P + (1 - \lambda)Q) + \varepsilon \triangle u = \alpha_\varepsilon(\lambda P + (1 - \lambda)Q).$$

By Lemma (2.7) we have that h is in fact a solution and so inequality (2.12) is an equality. Since H is assumed to be strictly convex, it follows that $df + P = dg + Q$ at all points, so $P - Q = d(f - g)$ is an exact differential and then $P = Q$. \square

LEMMA (2.13). *The functions $\alpha_\varepsilon, \beta_\varepsilon$ are C^1 .*

Proof. According to Theorem 26.3 on page 253 of the book [R], the dual of a strictly convex function is differentiable. Moreover, a differentiable convex function is always C^1 (Corollary 25.5.1, page 246 of the same book). \square

3. Smoothness

Fix $x_0 \in \mathbb{T}^d$ and let $\phi(x, P)$ be the solution to (1.6) with $\phi(x_0, P) = 0$. We know that $\phi(x, P)$ is C^∞ in x and we are going to prove that it is also C^∞ in P . Letting $H^k(\mathbb{T}^d, \mathbb{R}^m)$ be the usual Sobolev space we have that $\phi(\cdot, P)$ belongs to $H^k(\mathbb{T}^d, \mathbb{R})$ for all P and k . Define

$$\begin{aligned}
V(x, P) &= \partial_p H(x, \partial_x \phi(x, P) + P), \\
M_P &= \varepsilon \triangle + V(x, P) : H^k(\mathbb{T}^d, \mathbb{R}) \rightarrow H^{k-2}(\mathbb{T}^d, \mathbb{R}),
\end{aligned}$$

where the vector field $V(\cdot, P)$ acts as a first order differential operator.

We will use the following facts from the theory of elliptic operators. See for example Chapter 5 of [T].

1. A regularity result that says that if f is in H^{k-2} and $u \in H^1$ is a solution to $M_P u = f$, then u is in H^k .
2. $M_P : H^1 \rightarrow H^{-1}$ is Fredholm of index zero.
3. $\ker M_P$ is the set of constant functions.

LEMMA (3.1). *We have that $\phi \in C^1(\mathbb{T}^d \times \mathbb{R}^d)$ and $\partial_P \phi(\cdot, P) \in C^\infty(\mathbb{T}^d)$.*

Proof. Let $Q \in \mathbb{R}^d$, let $g \in H^{k-2} - M_Q(H^k)$, $2k > d + 2$, and define the map

$$\begin{aligned}
(3.2) \qquad \qquad \qquad F : \mathbb{R}^d \times H^k(\mathbb{T}^d, \mathbb{R}) &\rightarrow H^{k-2}(\mathbb{T}^d, \mathbb{R}) \\
F(P, \varphi) &= H \circ (I, d\varphi + P) + \varepsilon \triangle \varphi - \varphi(x_0)g - \alpha_\varepsilon(P)
\end{aligned}$$

where I is the identity map on \mathbb{T}^d .

Since $\phi(\cdot, P)$ is the solution to (2.5) with $\phi(x_0, P) = 0$, we have $F(P, \phi(\cdot, P)) = 0$. To prove the differentiability of ϕ with respect to P we use the implicit function theorem.

The map F is C^1 by the following facts:

- The map α_ε is C^1 ;
- Sobolev's inequality

$$(3.3) \quad \|\varphi\|_{C^{k-\lfloor \frac{d}{2} \rfloor - 1, \frac{1}{2}}} \leq C(d, k) \|\varphi\|_{H^k};$$

- If $G : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is C^∞ , then the map

$$\bar{G} : H^k(\mathbb{T}^d, \mathbb{R}^n) \rightarrow H^k(\mathbb{T}^d, \mathbb{R}^m), \quad \bar{G}(\Psi) = G \circ \Psi$$

is C^∞ and $D\bar{G} = \bar{D}G$.

The partial derivatives of F are

$$\begin{aligned} D_1 F(P, \varphi) &= \partial_p H \circ (I, d\varphi + P) - d\alpha_\varepsilon, \\ D_2 F(P, \varphi)\xi &= \partial_p H \circ (I, D\varphi + P) \cdot d\xi + \varepsilon \triangle \xi - \xi(x_0)g. \end{aligned}$$

Since $D_2 F(Q, \phi(\cdot, Q))\xi = M_Q \xi - \xi(x_0)g$, it is invertible. By the implicit function theorem there is a neighbourhood U of Q such that the map

$$\psi : U \rightarrow H^k(\mathbb{T}^d, \mathbb{R}), \quad P \mapsto \phi(\cdot, P)$$

is C^1 and moreover $\Psi(x, P)h = d\psi(P) \cdot h(x)$ satisfies

$$(3.4) \quad M_P \Psi(x, P)h + V(x, P) \cdot h - (\Psi(x_0, P)h)g(x) - d\alpha_\varepsilon(P)h = 0.$$

For $P \in U$ fixed write

$$\phi(x, P+h) = \phi(x, P) + \Psi(x, P)h + \Phi_h(x)|h|$$

so we have that $\lim_{h \rightarrow 0} \|\Phi_h\|_{H^k} = 0$. By (3.3)

$$\lim_{h \rightarrow 0} \|\Phi_h\|_{C^{k-\lfloor \frac{d}{2} \rfloor - 1, \frac{1}{2}}} = 0.$$

Thus, $\phi \in C^1(\mathbb{T}^d \times U)$ with

$$\partial_P \phi(x, P) = \Psi(x, P)$$

and so $\Psi(x_0, P) = 0$. Letting $\varphi_j = \partial_{P_j} \phi$, we have that φ_j is a solution of the affine PDE

$$(3.5) \quad M_P(\varphi_j) + V_j(x, P) = \partial_{P_j} \alpha(P)$$

and therefore $\varphi_j \in C^\infty(\mathbb{T}^d)$. \square

Let $\theta(x, P)$ be the solution to the Fokker-Planck equation (2.2) with $\int \theta dx = 1$. Multiply each equation (3.5) by θ and integrate to obtain

$$(3.6) \quad \int \theta(x, P)V(x, P)dx = d\alpha_\varepsilon(P).$$

We now prove by induction that α_ε and ϕ are smooth.

LEMMA (3.7). *For any $r \in \mathbb{N}$, $\alpha_\varepsilon \in C^r(\mathbb{R}^d)$, $\phi \in C^r(\mathbb{T}^d \times \mathbb{R}^d)$ and $\partial_P^r \phi(\cdot, P) \in C^\infty(\mathbb{T}^d)$.*

Proof. Suppose the assertion of the Lemma holds for $r \in \mathbb{N}$. Thus $V \in C^r(\mathbb{T}^d \times \mathbb{R}^d)$. By the theory of elliptic operators quoted above, the adjoint $\mathbf{FP}_P : H^k \rightarrow H^{k-2}$ of M_P given by

$$(3.8) \quad \mathbf{FP}_P \varphi = \varepsilon \triangle \varphi - \operatorname{div}(\varphi V(x, P))$$

satisfies properties (1) and (2) of M_P and $\dim \ker \mathbf{FP}_P = 1$ as well. For $Q \in \mathbb{R}^d$ let $g \in H^{k-2} - \mathbf{FP}_Q(H^k)$ and define

$$J : \mathbb{R}^d \times H^k \rightarrow H^{k-2}, \quad J(P, \varphi) = \mathbf{FP}_P \varphi + \left(\int \varphi - 1 \right) g$$

so that $J(P, \theta(\cdot, P)) = 0$. The map J is C^r (affine in φ) with

$$D_2 J(P, \varphi) = \mathbf{FP}_P + g \int.$$

By construction, any $D_2 J(Q, \varphi)$ is invertible. By the implicit function theorem, $\theta \in C^r(\mathbb{T}^d \times U)$ for U a neighbourhood of Q , but Q is arbitrary. By (3.6), $\alpha_\varepsilon \in C^{r+1}(\mathbb{R}^d)$.

The map F defined in (3.2) is now C^{r+1} and so, by the implicit function theorem $\phi \in C^{r+1}(\mathbb{T}^d \times \mathbb{R}^d)$ and $\partial^{r+1} \phi(\cdot, P) \in C^\infty(\mathbb{T}^d)$. \square

COROLLARY (3.9). *The function β_ε is smooth.*

Since α_ε is strictly convex and smooth, the map $d\alpha_\varepsilon$ has a smooth inverse γ_ε . Then $\beta_\varepsilon(h) = h\gamma_\varepsilon(h) - \alpha_\varepsilon(\gamma_\varepsilon(h))$ is smooth. \square

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R. ITURRIAGA
 CIMAT
 A.P. 402,
 GUANAJUATO. GTO, 3600
 MÉXICO
 renato@cimat.mx

H. SÁNCHEZ-MORGADO
 INSTITUTO DE MATEMÁTICAS, UNAM
 CD. UNIVERSITARIA
 MÉXICO, D.F. 04510
 MÉXICO
 hector@matem.unam.mx

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AN ATOMIC DECOMPOSITION FOR THE BERGMAN SPACE OF TEMPERATURE FUNCTIONS ON A CYLINDER

MARCOS LÓPEZ-GARCÍA

ABSTRACT. For $1 \leq p < \infty$, we define the weighted Bergman space $b_\beta^p(S_T)$ as consisting of the temperature functions on the cylinder $S_T = \mathbb{S}^1 \times (0, T)$ that belong to $L^p(\Omega_T, t^\beta dx dt)$, where $\Omega_T = (0, 2) \times (0, T)$. For $\alpha > \beta > -1$ we construct a family of bounded projections $P_\alpha: L^1(\Omega_T, t^\beta dx dt) \rightarrow b_\beta^1(S_T)$. We use this to get an atomic decomposition of the Bergman space $b^p(S_T) = b_0^p(S_T)$ for all $p \geq 1$.

1. Introduction

For \mathbb{D} the open unit disk in the complex plane \mathbb{C} , the classic Bergman space $L_\alpha^p(\mathbb{D})$ is the subspace of holomorphic functions $f: \mathbb{D} \rightarrow \mathbb{C}$ such that $f \in L^p(\mathbb{D})$.

The theory of Bergman spaces has a long history. It goes back to the work of S. Bergman (see [3]), who gave the first treatment of $L_\alpha^2(\mathbb{D})$. Today there are rich theories describing the Bergman spaces in various domains and their operators. Two of the most important classes of operators in Bergman space theory are the Toeplitz and Hankel operators, which are defined in terms of the Bergman projection P . This theory was mainly developed in the late 1980's. For a very nice exposition of the $L^p(\mathbb{D})$ -theory of Bergman spaces, operators defined of them, and further historical references, we refer to Axler's survey paper [1], Zhu's book [10], and a more modern approach in [2], [6].

For $\Omega \subset \mathbb{R}_+^2$ an open set, let

$$H(\Omega) = \left\{ u \in C^2(\Omega) : \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \text{ on } \Omega \right\}.$$

We shall call the elements of $H(\Omega)$ temperature functions on Ω .

For $T > 0$ finite, we define the linear space of temperature functions on the cylinder S_T as

$$H(S_T) = \{u \in H(\mathbb{R} \times (0, T)) : u(x, t) = u(x + 2, t)\}.$$

Roughly speaking, a function in $H(S_T)$ determines a temperature distribution on a circular ring of fine wire with radius $1/\pi$, up to a certain time T .

For $p \geq 1$ we define the weighted Bergman-type spaces $b_\beta^p(S_T)$ as consisting of the temperature functions on the cylinder S_T that belong to $L^p(\Omega_T, t^\beta dx dt)$, where $\Omega_T = (0, 2) \times (0, T)$, $\beta > -1$.

The following result was proved in [7].

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LEMMA (1.1). *Let $R = (a, b) \times (c, d) \subset \mathbb{R}_+^2$ be such that $d - c = (b - a)^2$. If $u \in H(R) \cap C(\overline{R})$ and (x_0, t_0) is the midpoint of the upper boundary of R , then*

$$|u(x_0, t_0)|^p \leq \frac{C_p}{|R|} \iint_R |u(x, t)|^p dx dt,$$

where $|R|$ is the area of R and C_p is a constant that depends only on $p > 0$.

From this lemma, it was proved in [8] that $b_\beta^p(S_T)$ is a Banach space. In particular, $b^2(S_T) = b_0^2(S_T)$ is a Hilbert space and each point evaluation is a bounded linear functional on $b^2(S_T)$. Therefore, the Riesz representation theorem implies the existence of a reproducing kernel $N(z, w)$ for $b^2(S_T)$. The orthogonal projection $P: L^2(\Omega_T) \rightarrow b^2(S_T)$ is called the Bergman projection and can be written as the integral operator

$$Pu(z) = \int_{\Omega_T} N(z, w)u(w)dw,$$

for $u \in L^2(\Omega_T)$. The function $N(z, w)$ is called the Bergman reproducing kernel of $b^2(S_T)$.

We define the incomplete gamma function γ as

$$\gamma(\alpha, z) = \int_0^z t^{\alpha-1} e^{-t} dt \quad \text{if } \alpha, z > 0.$$

For $\alpha > -1$ let $c_{m,\alpha} = \frac{2^\alpha \pi^{2(1+\alpha)} m^{2(1+\alpha)}}{\gamma(1+\alpha, 2\pi^2 m^2 T)}$ for $m \in \mathbb{Z}^*$, $c_{0,\alpha} = \frac{1+\alpha}{2T^{1+\alpha}}$, and

$$N_\alpha(z, w) = \sum_{m \in \mathbb{Z}} c_{m,\alpha} e^{-\pi^2 m^2 (t+\tau) + \pi m i(x-y)}.$$

In [8] we constructed a family of bounded projections P_α of $L^p(\Omega_T, t^\beta dx dt)$ onto $b_\beta^p(S_T)$ as follows.

THEOREM (1.2). *Let $\alpha, \beta > -1$. If $p > \max\left(1, \frac{1+\beta}{1+\alpha}\right)$, the operator $P_\alpha: L_\beta^p(\Omega_T) \rightarrow b_\beta^p(S_T)$ given by*

$$P_\alpha u(z) = \int_{\Omega_T} N_\alpha(z, w) u(w) \tau^\alpha dw, \quad z \in \Omega_T$$

is a continuous projection onto $b_\beta^p(S_T)$.

In this paper we solve the case $p = 1$; i.e.,

THEOREM (1.3). *Let $\alpha > \beta > -1$. The operator $P_\alpha: L_\beta^1(\Omega_T) \rightarrow b_\beta^1(S_T)$ given by*

$$P_\alpha u(z) = \int_{\Omega_T} N_\alpha(z, w) u(w) \tau^\alpha dw, \quad z \in \Omega_T$$

is a continuous projection onto $b_\beta^1(S_T)$.

Remark (1.4). In particular, if $\alpha > \beta = 0$ we have that P_α is a bounded projection from $L^1(\Omega_T)$ onto $b^1(S_T)$.

In [5], Coifman and Rochberg gave an atomic decomposition for the Bergman spaces of holomorphic and harmonic functions defined on bounded symmetric domains and on the n -dimensional ball respectively. In this paper we get an atomic decomposition of the space $b^p(S_T)$, for every $p \geq 1$. The main result is that every function in the Bergman space $b^p(S_T)$ can be written as a sum of terms that depend on the reproducing kernel N . Our proof of the atomic decomposition is based on the methods and techniques in [5] and [10], and it involves estimating integral operators whose kernel is a power of the Bergman kernel.

Let us denote by p' the conjugate exponent of $p > 1$. The main results are the following ones:

THEOREM (1.5). *Let $p > 1$. There exists a sequence $\{w_m = (y_m, \tau_m)\} \subset \Omega_T$ and a constant $C_p > 0$ with the following properties:*

(1) *For any $\{a_m\} \in \ell^p$, the function*

$$u(z) = \sum_{m=0}^{\infty} a_m \tau_m^{\frac{3}{2p'}} N(z, w_m)$$

is in $b^p(S_T)$ with $\|u\|_{b^p(S_T)} \leq C_p \|\{a_m\}\|_{\ell^p}$.

(2) *Given $u \in b^p(S_T)$ there exists a sequence $\{a_m\} \in \ell^p$ such that*

$$u(z) = \sum_{m=0}^{\infty} a_m \tau_m^{\frac{3}{2p'}} N(z, w_m)$$

and $\|\{a_m\}\|_{\ell^p} \leq C_p \|u\|_{b^p(S_T)}$.

Remark (1.6). Part (2) of this theorem can be regarded as the discrete analogue of the integral reproducing formula for functions in $b^p(S_T)$.

Using the reproducing formula $P_\alpha u = u$ for $u \in b^1(S_T)$, $\alpha > 0$, we have the following

THEOREM (1.7). *Let $\alpha > 0$. There exists a sequence $\{w_m = (y_m, \tau_m)\} \subset \Omega_T$ and a constant $C_\alpha > 0$ with the following properties:*

(1) *For any $\{a_m\} \in \ell^1$, the function*

$$u(z) = \sum_{m=0}^{\infty} a_m \tau_m^\alpha N_\alpha(z, w_m)$$

is in $b^1(S_T)$ with $\|u\|_{b^1(S_T)} \leq C_\alpha \|\{a_m\}\|_{\ell^1}$;

(2) *If $u \in b^1(S_T)$, then there exists a sequence $\{a_m\} \in \ell^1$ such that*

$$u(z) = \sum_{m=0}^{\infty} a_m \tau_m^\alpha N_\alpha(z, w_m)$$

and $\|\{a_m\}\|_{\ell^1} \leq C_\alpha \|u\|_{b^1(S_T)}$.

Remark (1.8). With minor modifications and some computations, we can obtain an atomic decomposition for the space $b_\beta^p(S_T)$, $1 \leq p < \infty$, $\beta > -1$. It is enough to redefine the functions T_δ , R_δ in the proof of Theorem (1.5) (setting $|\hat{D}_m|_\beta$

instead of $|\hat{D}_m|$, use the fact that t^β is comparable with τ_m^β if $z = (x, t) \in \hat{D}_m$, and $|\hat{D}_m|_\beta = \tau_m^\beta |\hat{D}_m|$, where

$$|\hat{D}_m|_\beta = \int_{\hat{D}_m} t^\beta dx dt.$$

In any case, the condition $p > (1 + \beta)/(1 + \alpha)$ is needed.

This paper is organized as follows: In Section 2 we give some definitions and preliminary results. Section 3 is devoted to the construction of the atomic decomposition for the Bergman spaces. The geometric construction needed for the atomic decomposition can be found in the Appendix.

2. Notation and preliminary results

Throughout this paper we will write $z = (x, t)$, $w = (y, \tau)$, $dz = dx dt$, $dw = dy d\tau$, $\mathbb{S}^1 = \{e^{i\theta} : \theta \in [0, 2]\}$ and $\mathbb{Z}^* = \{n \in \mathbb{Z} : n \neq 0\}$. For $u \in L^1(\mathbb{R})$ we define its Fourier transform as

$$(\mathcal{F}u)(s) = \int_{-\infty}^{\infty} u(x)e^{-2\pi ixs} dx.$$

$K(x, t)$ will denote the Gauss-Weierstrass kernel. For $t > 0$, let

$$\theta(x, t) = \sum_{n \in \mathbb{Z}} K(x + 2n, t) = \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{-\pi^2 n^2 t + \pi n i x} \quad (\text{see [9]}).$$

In [4] it was proved that there is a constant $C > 0$ such that

$$\theta(x, t) \leq C(1 + t)K(x, t), \quad \text{if } -1 < x < 1.$$

In particular for $0 < t \leq T < \infty$ there is a constant $C_T > 0$ such that

$$(2.1) \quad \theta(x, t) \leq C_T K(x, t), \quad \text{if } -1 < x < 1.$$

We define the linear space of continuous functions on the cylinder $\overline{S_T}$ as

$$C(\overline{S_T}) = \{u \in C(\mathbb{R} \times [0, T]) : u(x, t) = u(x + 2, t)\}.$$

Definition (2.2). Let $1 \leq p < \infty$ and $-1 < \beta$; we define the weighted Bergman space $b_\beta^p(S_T)$ of temperature functions on the cylinder S_T as

$$b_\beta^p(S_T) = \left\{ u \in H(S_T) : \int_{\Omega_T} |u(z)|^p t^\beta dz < \infty \right\}.$$

We equip $b_\beta^p(S_T)$ with the norm

$$\|u\|_{b_\beta^p(S_T)} = \left(\int_{\Omega_T} |u(z)|^p t^\beta dz \right)^{\frac{1}{p}}.$$

Notice that if $u \in b_\beta^p(S_T)$ then $u|_{\Omega_T} \in L_\beta^p(\Omega_T)$. Hence $b_\beta^p(S_T) \subset L_\beta^p(\Omega_T)$.

If

$$(2.3) \quad u_n(x, t) = e^{-\pi^2 n^2 t + \pi n i x},$$

then $u_n \in b_\beta^p(S_T)$ for $n \in \mathbb{Z}$, $\beta > -1$.

Remark (2.4). In [8], it was proved that the subspace $H(S_T) \cap C(\overline{S_T})$ is dense in $b_\beta^p(S_T)$, $\beta > -1$. Likewise, the linear space generated by $(u_n)_{n \in \mathbb{Z}}$ is dense in $b_\beta^p(S_T)$, $\beta > -1$.

Next, let us consider a family of kernels N_α and their corresponding integral operators P_α .

Definition (2.5). Given $\alpha > -1$, we define $N_\alpha : \mathbb{R}_+^2 \times \mathbb{R}_+^2 \rightarrow \mathbb{C}$ as

$$(2.6) \quad N_\alpha(z, w) = \sum_{m \in \mathbb{Z}} c_{m,\alpha} e^{-\pi^2 m^2(t+\tau) + \pi m i(x-y)},$$

where $c_{m,\alpha} = \frac{2^\alpha \pi^{2(1+\alpha)} m^{2(1+\alpha)}}{\gamma(1+\alpha, 2\pi^2 m^2 T)}$ for $m \in \mathbb{Z}^*$ and $c_{0,\alpha} = \frac{1+\alpha}{2T^{1+\alpha}}$.

Since $0 < C_{\alpha,T} = \gamma(1+\alpha, 2\pi^2 T) \leq \gamma(1+\alpha, 2\pi^2 m^2 T)$ for $m \in \mathbb{Z}^*$, it follows that $c_{m,\alpha} = O(m^{2\alpha+2})$.

For $x > 0$ and $\sigma \geq 0$ we have $e^{-x} \leq C'_\sigma x^{-\sigma}$. Hence the series defining N_α converges absolutely and uniformly on $\Omega' \times \mathbb{R}_+^2$ provided $\Omega' \subset \mathbb{R}_+^2$ is compact. Furthermore, the function N_α is bounded on $\Omega' \times \mathbb{R}_+^2$. So $N_\alpha \in C^\infty(\mathbb{R}_+^2 \times \mathbb{R}_+^2)$ and $N_\alpha(\cdot, w) \in H(S_T)$ for all $w \in \mathbb{R}_+^2$. Since $c_{m,\alpha} = c_{-m,\alpha}$ the function N_α is real-valued and symmetric.

Definition (2.7). For $\alpha > -1$ let P_α be the integral operator given by

$$P_\alpha u(z) = \int_{\Omega_T} N_\alpha(z, w) u(w) \tau^\alpha dw, \quad z \in \Omega_T.$$

This integral is well-defined for $u \in C_c^\infty(\Omega_T)$.

Remark (2.8). When $\alpha = 0$, N_α is the reproducing kernel for $b^2(S_T)$ and P_α is the Bergman projection P (see [8]).

It is easy to see that

$$(2.9) \quad P_\alpha \left(e^{-\pi^2 n^2 t + \pi n i x} \right) = e^{-\pi^2 n^2 t + \pi n i x}$$

for all $n \in \mathbb{Z}$; therefore P_α is a projection on the linear space generated by $\{e^{-\pi^2 n^2 t + \pi n i x}\}$.

We want to determine the continuity of P_α on $L_\beta^1(\Omega_T)$. In order to do so, we analyze the following operator

$$T_\alpha u(z) = \int_{\Omega_T} \Theta_\alpha(z, w) u(w) \tau^\alpha dw, \quad z \in \Omega_T$$

where

$$\Theta_\alpha(z, w) = \theta_\alpha(x-y, t+\tau) = \frac{1}{2} \pi^{2\alpha+2} \sum_{m \in \mathbb{Z}} m^{2\alpha+2} e^{-\pi^2 m^2(t+\tau) + \pi m i(x-y)}.$$

Clearly, the series defining Θ_α has the same properties of convergence as N_α , and $\Theta_\alpha(\cdot, w) \in H(S_T)$ for all $w \in \mathbb{R}_+^2$.

Remark (2.10). If $\alpha \in \mathbb{N}$, then $\theta_\alpha(x, t) = (-1)^{1+\alpha} \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}} \theta(x, t)$.

Let $K_\alpha(x, t)$ given by

$$\begin{aligned} K_\alpha(x, t) &= \frac{1}{2} \mathcal{F}^{-1} \left(\pi^{2(1+\alpha)} \varsigma^{2(1+\alpha)} e^{-\pi^2 \varsigma^2 t} \right) \left(\frac{x}{2} \right) \\ &= \frac{1}{\sqrt{\pi t^{1+\alpha}}} K(x, t) \int_{-\infty}^{\infty} \left(\sigma + i \frac{x}{2\sqrt{t}} \right)^{2(1+\alpha)} e^{-\sigma^2} d\sigma, \end{aligned}$$

where \mathcal{F}^{-1} is the inverse Fourier transform with respect to the variable ς . From the dominated convergence theorem we can see that $K_\alpha(x, t) \in C(\mathbb{R}_+^2)$. Furthermore,

$$(2.11) \quad |K_\alpha(x, t)| \leq \frac{C_\alpha}{t^{1+\alpha}} (K(x, t) + K(x, 2t)) \quad \text{for all } (x, t) \in \mathbb{R}_+^2.$$

Remark (2.12). If $\alpha \in \mathbb{N}$, then $K_\alpha(x, t) = (-1)^{1+\alpha} \frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}} K(x, t)$.

We have an alternate expression for the function θ_α in terms of the function K_α , (see [8])

$$(2.13) \quad \theta_\alpha(x, t) = \sum_{m \in \mathbb{Z}} K_\alpha(x + 2m, t).$$

The next result is the key to reach our goals.

LEMMA (2.14). $\int_{\Omega_T} |\Theta_\alpha(z, w)| t^\beta dz \leq C_{\alpha, \beta} \tau^{\beta-\alpha}$ provided $\alpha > \beta > -1$.

Proof. From (2.11) and (2.13) we have that

$$|\theta_\alpha(x, t)| \leq \frac{C_\alpha}{t^{1+\alpha}} (\theta(x, t) + \theta(x, 2t)).$$

From (2.1) we get

$$\begin{aligned} |\theta_\alpha(x - y, t + \tau)| &\leq \frac{C_\alpha}{(t + \tau)^{1+\alpha}} (K(x - y + 2, t + \tau) + K(x - y, t + \tau) \\ &\quad + K(x - y - 2, t + \tau) \\ &\quad + K(x - y + 2, 2(t + \tau)) \\ &\quad + K(x - y, 2(t + \tau)) \\ &\quad + K(x - y - 2, 2(t + \tau))). \end{aligned}$$

For $\lambda > 0$, let $h_{y, \tau, \lambda}(x, t) = K(x - y \pm 2, \lambda(t + \tau))$ we have

$$\begin{aligned} \int_0^T \frac{t^\beta}{(t + \tau)^{1+\alpha}} \int_0^2 h_{y, \tau, \lambda}(x, t) dx dt &\leq \int_0^T \frac{t^\beta}{(t + \tau)^{1+\alpha}} \int_{-\infty}^{\infty} K(x, \lambda(t + \tau)) dx dt \\ &= \int_0^T \frac{t^\beta dt}{(t + \tau)^{1+\alpha}} \leq \tau^{\beta-\alpha} \int_0^{\infty} \frac{s^\beta ds}{(s + 1)^{1+\alpha}} \end{aligned}$$

where $C_{\alpha, \beta} = \int_0^{\infty} \frac{s^\beta ds}{(s+1)^{1+\alpha}} < \infty$ whenever $\alpha > \beta > -1$. \square

In fact, the lemma above and the Schur's test imply that $T_\alpha : L_\beta^p(\Omega_T) \rightarrow b_\beta^p(S_T)$ is bounded if $p > \max\left(1, \frac{1+\beta}{1+\alpha}\right)$ (see [8]). We show that the same conclusion holds for $p = 1$.

PROPOSITION (2.15). *Let $\alpha > \beta > -1$. The operator $T_\alpha : L_\beta^1(\Omega_T) \rightarrow b_\beta^1(S_T)$ given by*

$$T_\alpha u(z) = \int_{\Omega_T} \Theta_\alpha(z, w) u(w) \tau^\alpha \bar{d}w, \quad z \in \Omega_T$$

is bounded.

Proof. Apply Fubini's theorem and the previous lemma. \square

Now we define an isomorphism \mathcal{M}_α connecting the operators T_α and P_α .

LEMMA (2.16). *Let (λ_n) be a bounded sequence such that $\lambda_n = C_1 + O(|n|^{-\epsilon})$, $\epsilon > 1$ and $0 < C_2 \leq |\lambda_n|$ for all $n \in \mathbb{Z}$. Then (λ_n) induces an isomorphism \mathcal{M} of $b_\beta^1(S_T)$ onto itself.*

Proof. If $(\mu_n) \in \ell^1(\mathbb{Z})$ then

$$(2.17) \quad \left\| \sum \mu_n \widehat{f}(n) e^{\pi n i} \right\|_{L^1(\mathbb{S}^1)} \leq \sum |\mu_n| |\widehat{f}(n)| \leq \|f\|_{L^1(\mathbb{S}^1)} \|(\mu_n)\|_{\ell^1(\mathbb{Z})}$$

for all trigonometric polynomials f .

It is enough to define the operator \mathcal{M} on the sequence $(e^{-\pi^2 n^2 t + \pi n i x})$ (see Remark (2.4)), and extend \mathcal{M} by linearity

$$\mathcal{M}(e^{-\pi^2 n^2 t + \pi n i x}) = \lambda_n e^{-\pi^2 n^2 t + \pi n i x}.$$

From (2.17), we obtain

$$\left\| \mathcal{M} \left(\sum_{|n| \leq N} a_n e^{-\pi^2 n^2 t + \pi n i x} \right) \right\|_{L^1(\mathbb{S}^1)} \leq C_\epsilon \left\| \sum_{|n| \leq N} a_n e^{-\pi^2 n^2 t + \pi n i x} \right\|_{L^1(\mathbb{S}^1)},$$

where $C_\epsilon = |C_1| + C \left\| (|n|^{-\epsilon}) \right\|_{\ell^1(\mathbb{Z})}$.

Using Tonelli's theorem, we have

$$\left\| \mathcal{M} \left(\sum_{|n| \leq N} a_n e^{-\pi^2 n^2 t + \pi n i x} \right) \right\|_{b_\beta^1(S_T)} \leq C_\epsilon \left\| \sum_{|n| \leq N} a_n e^{-\pi^2 n^2 t + \pi n i x} \right\|_{b_\beta^1(S_T)}.$$

On the other hand, the hypothesis on (λ_n) implies that $\lambda_n^{-1} = C_1^{-1} + O(|n|^{-\epsilon})$. Therefore \mathcal{M}^{-1} is bounded on $b_\beta^1(S_T)$. \square

Now we are ready to prove Theorem (1.3).

Proof. Since $N_\alpha(\cdot, w) \in H(S_T)$, it follows that $P_\alpha u \in H(S_T)$ for $u \in C_c^\infty(\Omega_T)$.

Let $\lambda_{n,\alpha} = 2^{-\alpha-1}\gamma(1+\alpha, 2\pi^2 n^2 T)$ for $n \in \mathbb{Z}^*$ and $\lambda_{0,\alpha} = \frac{2T^{1+\alpha}}{1+\alpha}$. Then

$$0 < c_\alpha = 2^{-\alpha-1} \int_0^{2\pi^2 T} t^\alpha e^{-t} dt \leq \lambda_{n,\alpha} \leq 2^{-\alpha-1} \int_0^\infty t^\alpha e^{-t} dt = C_\alpha$$

for $n \in \mathbb{Z}^*$. Thus, $0 < c_\alpha \leq \lambda_{n,\alpha} \leq C_\alpha$ for $n \in \mathbb{Z}$.

Using the fact that $e^{-t} \leq C'_\sigma t^{-\sigma}$ for $t, \sigma > 0$, it follows that

$$\int_{2\pi^2 n^2 T}^\infty t^\alpha e^{-t} dt \leq C'_\sigma \int_{2\pi^2 n^2 T}^\infty t^{\alpha-\sigma} dt = \frac{C_{\sigma,T}}{\sigma-\alpha-1} n^{2(\alpha-\sigma+1)}$$

provided $\alpha - \sigma < -1$. Letting $\sigma = \alpha + 1 + \frac{\epsilon}{2}$ we have

$$(2.18) \quad \lambda_{n,\alpha} = C_\alpha + O(|n|^{-\epsilon}) \quad \text{for all } \epsilon > 0.$$

Let \mathcal{M}_α be the isomorphism induced by the sequence $(\lambda_{n,\alpha})$ on $b_\beta^1(S_T)$. That is,

$$\mathcal{M}_\alpha \left(\sum a_n e^{-\pi^2 n^2 t + \pi n i x} \right) = \sum \lambda_{n,\alpha} a_n e^{-\pi^2 n^2 t + \pi n i x}.$$

In particular, by Lemma (2.14) we have

$$(2.19) \quad \begin{aligned} \mathcal{M}_\alpha N_\alpha(z, w) &= 1 + \Theta_\alpha(z, w), \quad \text{if } \alpha > \beta > -1, \\ \mathcal{M}_\alpha \left(\frac{\partial^n N_\alpha}{\partial y^n}(z, w) \right) &= (-i)^n \Theta_{\alpha+\frac{n}{2}}(z, w), \quad \text{if } \alpha + \frac{n}{2} > \beta > -1, \\ \mathcal{M}_\alpha \left(\frac{\partial^n N_\alpha}{\partial \tau^n}(z, w) \right) &= (-1)^n \Theta_{\alpha+n}(z, w), \quad \text{if } \alpha + n > \beta > -1, \end{aligned}$$

where \mathcal{M}_α is acting on the variable z .

The dominated convergence theorem together with the fact that the series defining $N_\alpha(z, w)$ and $\Theta_\alpha(z, w)$ converge uniformly on the set $\{z\} \times \Omega_T$ give

$$(\mathcal{M}_\alpha \circ P_\alpha)u(z) = (\mathbf{1}_\alpha + T_\alpha)u(z) \quad \text{for all } u \in C_c^\infty(\Omega_T),$$

where $\mathbf{1}_\alpha u(z) = \int_{\Omega_T} u(w) \tau^\alpha dw$ is a bounded operator on $L_\beta^1(\Omega_T)$. Hence P_α is continuous on $L_\beta^1(\Omega_T)$. \square

Remark (2.20). Similarly, it was proved in [8] that $\mathcal{M}_\alpha \circ P_\alpha = \mathbf{1}_\alpha + T_\alpha$ on $L_\beta^p(\Omega_T)$.

Thus, if $p > \max\left(1, \frac{1+\beta}{1+\alpha}\right)$ the operator $P_\alpha : L_\beta^p(\Omega_T) \rightarrow b_\beta^p(S_T)$ is a continuous projection onto $b_\beta^p(S_T)$. In particular, the Bergman projection is a bounded operator on $L_\beta^p(\Omega_T)$ for all $p > 1 + \beta$. Using this fact we have obtained that $b_\beta^p(S_T)^* = b_\beta^{p'}(S_T)$ under the integral pairing:

$$\langle u, v \rangle = \int_{\Omega_T} u(z) \overline{v(z)} t^\beta dx dt.$$

Finally, from the continuity of the adjoint operator $P_\alpha^* : b_\beta^p(S_T) \rightarrow L^p(\Omega_T)$ given by

$$P_\alpha^* u(z) = t^\alpha \int_{\Omega_T} N_\alpha(z, w) u(w) dw,$$

it was proved that there are constants C_n, C_d such that

$$(2.21) \quad \begin{aligned} \left\| t^{\frac{n}{2}} \frac{\partial^n u}{\partial x^n} \right\|_{L^p(\Omega_T)} &\leq C_n \|u\|_{b^p(S_T)}, \\ \left\| t^d \frac{\partial^d u}{\partial t^d} \right\|_{L^p(\Omega_T)} &\leq C_d \|u\|_{b^p(S_T)}, \end{aligned}$$

for all $u \in b^p(S_T)$, $p > 1$.

Remark (2.22). We claim that (2.21) holds for $p = 1$. By Remark (1.4),

$$u(z) = \int_{\Omega_T} N_\alpha(z, w) u(w) \tau^\alpha dw, \quad \forall z \in \Omega_T, \quad u \in b^1(S_T),$$

provided $\alpha > 0$. Differentiating under the integral sign leads to

$$\frac{\partial^n u}{\partial x^n}(z) = \int_{\Omega_T} \frac{\partial^n N_\alpha}{\partial x^n}(z, w) u(w) \tau^\alpha dw.$$

From (2.19) and $\frac{\partial^n N_\alpha}{\partial x^n} = (-1)^n \frac{\partial^n N_\alpha}{\partial y^n}$ we have

$$\int_0^{2\pi} \left| \frac{\partial^n N_\alpha}{\partial x^n}(z, w) \right| dx \leq C \int_0^{2\pi} |\Theta_{\alpha+\frac{n}{2}}(z, w)| dx,$$

multiplying this inequality by $t^{n/2}$, integrating on $(0, T)$ and using Tonelli's theorem we obtain

$$\int_{\Omega_T} t^{\frac{n}{2}} \left| \frac{\partial^n N_\alpha}{\partial x^n}(z, w) \right| dz \leq C \int_{\Omega_T} t^{\frac{n}{2}} |\Theta_{\alpha+\frac{n}{2}}(z, w)| dz.$$

The result follows from Fubini's theorem, the last inequality and Lemma (2.14).

3. An atomic decomposition for $b^p(S_T)$.

In the Appendix we construct some families $\{D_m\}$, $\{\widehat{D}_m\}$, and $\{\widetilde{D}_m\}$ of rectangles on the cylinder S_T satisfying the following properties:

1. $\{D_m\}$ is a cover of S_T and $|D_m \cap D_n| = 0$ whenever $m \neq n$.
2. $\text{height}(\widehat{D}_m) = \{\text{base}(\widehat{D}_m)\}^2$.
3. $D_m \subset \widehat{D}_m$, and any point on the cylinder S_T is at most in 2 rectangles of the collection $\{\widehat{D}_m\}$.
4. Any point on the cylinder S_T is in at most 6 rectangles of the collection $\{\widetilde{D}_m\}$.
5. $w_m = (y_m, \tau_m)$ is the midpoint of the upper boundary of the rectangles D_m , \widehat{D}_m and \widetilde{D}_m . Also,

$$(3.1) \quad \left| \widehat{D}_m \right| = (1 - \delta)^{\frac{3}{2}} \tau_m^{\frac{3}{2}},$$

where $0 < \delta < 1$ is a constant to be chosen later.

Starting from this construction we have the following results which will be proved in the Appendix.

LEMMA (3.2). *Let $p \geq 1$. There exists a constant $C_p > 0$ such that*

$$\sum_{m=0}^{\infty} |\widehat{D}_m| |u(w_m)|^p \leq C_p \|u\|_{b^p(S_T)}^p,$$

for all $u \in b^p(S_T)$.

LEMMA (3.3). *Let $p \geq 1$ and $0 < \delta < 1$. There exists a constant $C_{p,\delta} > 0$ such that*

$$\sum_{m=0}^{\infty} \int_{D_m} |u(z) - u(w_m)|^p dz \leq C_{p,\delta} \|u\|_{b^p(S_T)}^p,$$

for all $u \in b^p(S_T)$. Furthermore, $C_{p,\delta} \rightarrow 0$ as $\delta \rightarrow 1$.

For $p > 1$ we define the operators $R_\delta : b^p(S_T) \rightarrow b^p(S_T)$, $T_\delta : \ell^p \rightarrow b^p(S_T)$ as

$$\begin{aligned} (R_\delta u)(z) &= \sum_{m=0}^{\infty} |D_m| u(w_m) N(z, w_m), \\ T_\delta(\{a_m\})(z) &= \sum_{m=0}^{\infty} a_m |\widehat{D}_m|^{\frac{1}{p'}} N(z, w_m). \end{aligned}$$

Clearly, the operator R_δ can be regarded as the discrete analogue of the integral reproducing formula $Pu = u$ for $u \in b^p(S_T)$, $p > 1$.

LEMMA (3.4). *The operator R_δ is bounded on $b^p(S_T)$ and the operator T_δ is bounded on ℓ^p for all $p > 1$.*

Proof. Let $u \in C(\overline{S_T}) \cap H(S_T)$. Let us denote with $\langle \cdot, \cdot \rangle$ the usual duality. Using the reproducing formula $Pu = u$ we have that

$$\langle T_\delta(\{a_m\}), u \rangle = \sum_{m=0}^{\infty} a_m |\widehat{D}_m|^{\frac{1}{p'}} \overline{u(w_m)}.$$

Using Hölder's inequality and Lemma (3.2), we get

$$\begin{aligned} |\langle T_\delta(\{a_m\}), u \rangle| &\leq \|\{a_m\}\|_{\ell^p} \left\| \left\{ |\widehat{D}_m|^{\frac{1}{p'}} \overline{u(w_m)} \right\} \right\|_{\ell^{p'}} \\ &\leq C_{p'} \|\{a_m\}\|_{\ell^p} \|u\|_{b^{p'}(S_T)}. \end{aligned}$$

Since $b^{p'}(S_T)^* = b^p(S_T)$, it follows that the operator $T_\delta : \ell^p \rightarrow b^p(S_T)$ is bounded.

To prove that the operator R_δ is bounded, let us consider $u \in b^p(S_T)$, and we define

$$a_m = \frac{|D_m| u(w_m)}{|\widehat{D}_m|^{\frac{1}{p'}}}, \quad m \in \mathbb{N}.$$

Since $D_m \subset \widehat{D}_m$, Lemma (3.2) implies that the mapping $u \mapsto \{a_m\}$ is continuous from $b^p(S_T)$ to ℓ^p .

We note that

$$(R_\delta u)(z) = T_\delta(\{a_m\})(z).$$

Hence the boundedness of the operator R_δ follows from the boundedness of the operator T_δ . \square

The following lemma is essential for proving the main result.

LEMMA (3.5). *Let $p > 1$. There exists $\delta' < 1$ such that if $\delta \in (\delta', 1)$, then R_δ is invertible on $b^p(S_T)$.*

Proof. Let $A_\delta = I - R_\delta$ where I is the identity operator. It suffices to show that $\|A_\delta\| < 1$. Let us denote by $\langle \cdot, \cdot \rangle$ the usual duality. Fix $u, v \in C(\overline{S_T}) \cap H(S_T)$; using the reproducing formula we have

$$\begin{aligned} \langle A_\delta u, v \rangle &= \int_{\Omega_T} u(z) \overline{v(z)} dz - \sum_{m=0}^{\infty} |D_m| u(w_m) \overline{v(w_m)} \\ &= \sum_{m=0}^{\infty} \int_{D_m} (u(z) \overline{v(z)} - u(w_m) \overline{v(w_m)}) dz \\ &= \sum_{m=0}^{\infty} \int_{D_m} u(z) (\overline{v(z)} - \overline{v(w_m)}) dz + \sum_{m=0}^{\infty} \int_{D_m} \overline{v(w_m)} (u(z) - u(w_m)) dz \\ &= I_1 + I_2. \end{aligned}$$

Using Hölder's inequality twice and applying Lemma (3.3), we obtain

$$\begin{aligned} |I_1| &\leq \left[\sum_{m=0}^{\infty} \int_{D_m} |u(z)|^p dz \right]^{\frac{1}{p}} \left[\sum_{m=0}^{\infty} \int_{D_m} |v(z) - v(w_m)|^{p'} dz \right]^{\frac{1}{p'}} \\ &\leq C_p C_{p', \delta}^{\frac{1}{p'}} \|u\|_{b^p(S_T)} \|v\|_{b^{p'}(S_T)}. \end{aligned}$$

Similarly, applying Hölder's inequality twice, we get

$$|I_2| \leq \left[\sum_{m=0}^{\infty} |v(w_m)|^{p'} |D_m| \right]^{\frac{1}{p'}} \left[\sum_{m=0}^{\infty} \int_{D_m} |u(z) - u(w_m)|^p dz \right]^{\frac{1}{p}}.$$

Since $D_m \subset \widehat{D}_m$, using Lemma (3.2) we can see that

$$\sum_{m=0}^{\infty} |v(w_m)|^{p'} |D_m| \leq C_{p'} \|v\|_{b^{p'}(S_T)}^{p'}.$$

Lemma (3.3) gives us

$$|I_2| \leq C_{p'} C_{p, \delta}^{\frac{1}{p}} \|u\|_{b^p(S_T)} \|v\|_{b^{p'}(S_T)}.$$

Assuming $C_p \geq C_{p'}$ and combining the inequalities just obtained, we have

$$|\langle A_\delta u, v \rangle| \leq C_p \left(C_{p, \delta}^{\frac{1}{p}} + C_{p', \delta}^{\frac{1}{p'}} \right) \|u\|_{b^p(S_T)} \|v\|_{b^{p'}(S_T)}.$$

Therefore $\|A_\delta\| = \|I - R_\delta\| \leq C_p \left(C_{p,\delta}^{\frac{1}{p}} + C_{p',\delta}^{\frac{1}{p'}} \right)$. Since $C_{p,\delta} \rightarrow 0$ as $\delta \rightarrow 1$ for all $p \geq 1$, the result follows. \square

Now, we prove the main theorem.

Proof of Theorem (1.5). Let $\{a_m\} \in \ell^p$, we define the function

$$u(z) = \sum_{m=0}^{\infty} a_m \tau_m^{\frac{3}{2p'}} N(z, w_m).$$

By (3.1) we have

$$(1 - \delta)^{\frac{3}{2p'}} u(z) = T_\delta(\{a_m\})(z).$$

Then (1) follows from Lemma (3.4):

$$\|u\|_{b^p(S_T)} = \tilde{C}_{p,\delta} \|T_\delta(\{a_m\})\|_{b^p(S_T)} \leq \tilde{C}_{p,\delta} \|\{a_m\}\|_{\ell^p}.$$

To prove (2) we choose $\delta \in (\delta', 1)$, with δ' given by Lemma (3.5). Then the operator $R_\delta : b^p(S_T) \rightarrow b^p(S_T)$ is invertible. Thus, given $u \in b^p(S_T)$ there exists $v \in b^p(S_T)$ such that

$$u(z) = R_\delta v(z) = \sum_{m=0}^{\infty} |D_m| v(w_m) N(z, w_m).$$

Let $\{a_m\} = \left\{ \tau_m^{-\frac{3}{2p'}} |D_m| v(w_m) \right\}$. Using the fact that $D_m \subset \widehat{D}_m$ together with (3.1) we get

$$|a_m| \leq (1 - \delta)^{\frac{3}{2}} \tau_m^{\frac{3}{2p}} |v(w_m)| = (1 - \delta)^{\frac{3}{2p'}} \left| \widehat{D}_m \right|^{\frac{1}{p}} |v(w_m)|.$$

From Lemma (3.2) we have $\{a_m\} \in \ell^p$ and also

$$\|\{a_m\}\|_{\ell^p} \leq \tilde{C}_{p,\delta} \|v\|_{b^p(S_T)}.$$

Therefore, $\|\{a_m\}\|_{\ell^p} \leq \tilde{C}_{p,\delta} \|R_\delta^{-1}\| \|u\|_{b^p(S_T)}$. \square

COROLLARY (3.6). *Let δ' be given by Lemma (3.5). If $\delta \in (\delta', 1)$, the operator $T_\delta : \ell^p \rightarrow b^p(S_T)$ is surjective.*

Proof. Let $u \in b^p(S_T)$, by Lemma (3.5) there exists $v \in b^p(S_T)$ such that $u(z) = R_\delta v(z)$. As before, we have that $\{a_m\} = \left\{ \tau_m^{-\frac{3}{2p'}} |D_m| v(w_m) \right\} \in \ell^p$, and $u(z) = (1 - \delta)^{-\frac{3}{2p'}} T_\delta(\{a_m\})$. \square

The treatment of the case $p = 1$ is slightly different. First, we need the following result which will be proved in the Appendix.

LEMMA (3.7). *Let $\alpha > 0$. There is a constant $C_{\delta,\alpha} > 0$ such that*

$$\int_{\Omega_T} |N_\alpha(z, w_m) \tau_m^\alpha - N_\alpha(z, w) \tau^\alpha| dz \leq C_{\delta,\alpha}, \quad \text{for all } w \in D_m.$$

Furthermore, $C_{\delta, \alpha} \rightarrow 0$ as $\delta \rightarrow 1$.

Proof of Theorem (1.7). Let us consider the operator $F : \ell^1 \rightarrow b^1(S_T)$ defined by

$$F(\{a_m\})(z) = \sum_{m=0}^{\infty} a_m \tau_m^\alpha N_\alpha(z, w_m).$$

We claim that F is bounded and surjective. By Lemma (2.14) and (2.19) with $\beta = 0$, there exists a constant \tilde{C}_α such that $\int_{\Omega_T} \tau_m^\alpha |N_\alpha(z, w_m)| dz \leq \tilde{C}_\alpha$ for all $m \in \mathbb{N}$, so the boundedness of F follows. To prove the surjectivity of the operator F , we introduce the operator $G : b^1(S_T) \rightarrow b^1(S_T)$ given by

$$Gu(z) = \sum_{m=0}^{\infty} N_\alpha(z, w_m) u(w_m) \tau_m^\alpha |D_m|.$$

The operator G can be regarded as the discrete analogous of the integral reproducing formula $u = P_\alpha u$ for $u \in b^1(S_T)$. If $u \in b^1(S_T)$ then by Lemma (3.2) we have $\{|D_m|u(w_m)\} \in \ell^1$. Thus the surjectivity of F will be proved if we can show that G is onto. Actually, we prove that $\|I - G\| < 1$ and therefore G is invertible.

Given $u \in b^1(S_T)$, from Remark (1.4) we have

$$u(z) = P_\alpha u(z) = \int_{\Omega_T} N_\alpha(z, w) u(w) \tau^\alpha dw.$$

It follows that

$$\begin{aligned} Gu(z) - u(z) &= \sum_{m=0}^{\infty} \int_{D_m} (N_\alpha(z, w_m) u(w_m) \tau_m^\alpha - N_\alpha(z, w) u(w) \tau^\alpha) dw \\ &= \sum_{m=0}^{\infty} \int_{D_m} N_\alpha(z, w_m) (u(w_m) - u(w)) \tau_m^\alpha dw \\ &\quad + \sum_{m=0}^{\infty} \int_{D_m} u(w) (N_\alpha(z, w_m) \tau_m^\alpha - N_\alpha(z, w) \tau^\alpha) dw. \end{aligned}$$

Therefore,

$$\begin{aligned} |Gu(z) - u(z)| &\leq \sum_{m=0}^{\infty} |N_\alpha(z, w_m)| \tau_m^\alpha \int_{D_m} |u(w_m) - u(w)| dw \\ &\quad + \sum_{m=0}^{\infty} \int_{D_m} |N_\alpha(z, w_m) \tau_m^\alpha - N_\alpha(z, w) \tau^\alpha| |u(w)| dw \\ &= I_1(z) + I_2(z). \end{aligned}$$

Lemma (3.3) implies that there is a constant C'_δ such that

$$\sum_{m=0}^{\infty} \int_{D_m} |u(w_m) - u(w)| dw \leq C'_\delta \|u\|_{b^1(S_T)}.$$

Since $\int_{\Omega_T} |N_\alpha(z, w_m)| \tau_m^\alpha dz \leq \tilde{C}_\alpha$ for $m \in \mathbb{N}$, we have $\int_{\Omega_T} I_1(z) dz \leq \tilde{C}_\alpha C'_\delta \|u\|_{b^1(S_T)}$.
By Lemma (3.7), there is a constant $C_{\delta, \alpha}$ such that

$$\begin{aligned} \int_{\Omega_T} I_2(z) dz &\leq \sum_{m=0}^{\infty} \int_{D_m} \left(\int_{\Omega_T} |N_\alpha(z, w_m) \tau_m^\alpha - N_\alpha(z, w) \tau^\alpha| dz \right) |u(w)| dw \\ &\leq C_{\delta, \alpha} \sum_{m=0}^{\infty} \int |u(w)| dw = C_{\delta, \alpha} \|u\|_{b^1(S_T)}. \end{aligned}$$

So, $\|I - G\| \leq \tilde{C}_\alpha C'_\delta + C_{\delta, \alpha}$. Since $C'_\delta, C_{\delta, \alpha} \rightarrow 0$ as $\delta \rightarrow 1$ then G is invertible. \square

4. Appendix

Let $0 < \delta < 1$ be fixed. Let $b_n = T^{\frac{1}{2}} (1 - \delta)^{\frac{1}{2}} \delta^{\frac{n}{2}}$ and let k_n be the greatest integer which is less or equal to $2b_n^{-1}$.

For every $n \in \mathbb{N}$, and $i = 0, \dots, k_n - 1$ we define

$$\begin{aligned} w_{n,i} &= (y_{n,i}, \tau_n) = \left(\left(i + \frac{1}{2} \right) b_n, T\delta^n \right), \\ w_{n,k_n} &= (y_{n,k_n}, \tau_n) = \left(\frac{k_n b_n + 2}{2}, T\delta^n \right), \\ D_{n,i} &= [ib_n, (i+1)b_n] \times [T\delta^{n+1}, T\delta^n], \\ D_{n,k_n} &= [k_n b_n, 2] \times [T\delta^{n+1}, T\delta^n], \\ \hat{D}_{n,i} &= D_{n,i}, \\ \hat{D}_{n,k_n} &= \left[\frac{(k_n - 1)b_n + 2}{2}, \frac{(k_n + 1)b_n + 2}{2} \right] \times [T\delta^{n+1}, T\delta^n], \\ \tilde{D}_{n,i} &= \left[ib_n - \frac{1}{2}b_{n+1}, (i+1)b_n + \frac{1}{2}b_{n+1} \right] \times [T\delta^{n+2}, T\delta^n], \\ \tilde{D}_{n,k_n} &= \left[k_n b_n - \frac{1}{2}b_{n+1}, 2 + \frac{1}{2}b_{n+1} \right] \times [T\delta^{n+2}, T\delta^n]. \end{aligned}$$

For every $n \in \mathbb{N}$, we have:

1. $w_{n,i}$ is the midpoint of the upper boundary of the rectangles $D_{n,i}$, $\hat{D}_{n,i}$ and $\tilde{D}_{n,i}$ for $i = 0, \dots, k_n$.
2. The rectangles $D_{n,i}$ are ‘‘almost disjoint’’; they can intersect only on vertical line segments.
3. Each rectangle $\hat{D}_{n,i}$ satisfies the geometric condition of Lemma (1.1).
4. From the definition of k_n it follows that $D_{n,k_n} \subset \hat{D}_{n,k_n}$. So, $D_{n,i} \subset \hat{D}_{n,i}$ for $i = 0, \dots, k_n$. Furthermore,

$$(4.1) \quad \left| \hat{D}_{n,i} \right| = b_n (T\delta^n - T\delta^{n+1}) = T^{\frac{3}{2}} (1 - \delta)^{\frac{3}{2}} \delta^{\frac{3n}{2}} = (1 - \delta)^{\frac{3}{2}} \tau_n^{\frac{3}{2}},$$

for $i = 0, \dots, k_n$.

5. Since $b_{n+1} < b_n$ it follows that $b_n < \text{base}(\tilde{D}_{n,i}) < 2b_n$ for $i = 0, \dots, k_n - 1$.

From now on we will suppose that all rectangles $D_{n,i}$, $\widehat{D}_{n,i}$ and $\widetilde{D}_{n,i}$ are on the cylinder S_T . Hence

$$(4.2) \quad \bigcup_{i=0}^{k_n} D_{n,i} = \mathbb{S}^1 \times [T\delta^{n+1}, T\delta^n] = G_n, \quad \text{and}$$

$$(4.3) \quad \bigcup_{i=0}^{k_n} \widetilde{D}_{n,i} = \mathbb{S}^1 \times [T\delta^{n+2}, T\delta^n] = F_n.$$

Notice that $\text{int}(G_i) \cap \text{int}(G_j) = \emptyset$ whenever $i \neq j$ and $\text{int}(F_i) \cap \text{int}(F_j) = \emptyset$ whenever $|i - j| \geq 2$.

Since $D_{n,i} \subset \widehat{D}_{n,i}$ for $i = 0, \dots, k_n$ we have

$$(4.4) \quad \bigcup_{i=0}^{k_n} \widehat{D}_{n,i} = \mathbb{S}^1 \times [T\delta^{n+1}, T\delta^n].$$

With this construction, we have the following results.

LEMMA (4.5). *Except for a set of measure zero, any point on the cylinder S_T is in at most 2 rectangles of the collection $\{\widehat{D}_{n,i} : n \in \mathbb{N}, i = 0, \dots, k_n\}$.*

Proof. Since $\widehat{D}_{n,0} = D_{n,0} = [0, b_n] \times [T\delta^{n+1}, T\delta^n]$ and $\widehat{D}_{n,i} = (b_n, 0) + \widehat{D}_{n,i-1}$, it follows that $\widehat{D}_{n,i-1}$ and $\widehat{D}_{n,i}$ intersect on vertical line segments for $i = 1, \dots, k_n - 1$.

On the other hand, since $\text{base}(\widehat{D}_{n,k_n}) = b_n$ and

$$d(w_{n,k_n}, w_{n,0}), d(w_{n,k_n}, w_{n,k_n-1}) \leq b_n$$

it follows that \widehat{D}_{n,k_n} intersects at most the two rectangles \widehat{D}_{n,k_n-1} and $\widehat{D}_{n,0}$.

If $z \in \text{int}(G_n)$, from (4.4) and the above we have that z belongs to at most 2 rectangles of the collection $\{\widehat{D}_{n,i} : i = 0, \dots, k_n\}$. \square

LEMMA (4.6). *Except for a set of measure zero, any point on the cylinder S_T is in at most 6 rectangles of the collection $\{\widetilde{D}_{n,i} : n \in \mathbb{N}, i = 0, \dots, k_n\}$.*

Proof. Since $\text{base}(\widetilde{D}_{n,i}) < 2b_n$, $w_{n,i}$ is the midpoint of the upper lid of $\widetilde{D}_{n,i}$, and $w_{n,i} = (b_n, 0) + w_{n,i-1}$ for $i = 0, \dots, k_n - 1$, $\widetilde{D}_{n,i}$ intersects at most half the area of the rectangles $\widetilde{D}_{n,i-1}$ and $\widetilde{D}_{n,i+1}$ for $i = 0, \dots, k_n - 2$ (letting $\widetilde{D}_{n,-1} = \widetilde{D}_{n,k_n-1}$). Also \widetilde{D}_{n,k_n-1} intersects at most half the area of the rectangles \widetilde{D}_{n,k_n-2} and $\widetilde{D}_{n,0}$.

Therefore if $z \in \bigcup_{i=0}^{k_n-1} \widetilde{D}_{n,i}$ then z belongs to at most 2 rectangles of the collection $\{\widetilde{D}_{n,i} : i = 0, \dots, k_n - 1\}$. Finally, if $w \in F_n$, from (4.3) and the above we have that w belongs to at most 3 rectangles of the collection $\{\widetilde{D}_{n,i} : i = 0, \dots, k_n\}$.

If $z \in \text{int}(G_n)$ then $z \in \text{int}(F_{n-1}) \cap \text{int}(F_n)$, therefore z belongs to at most 6 rectangles of the collection $\{\widetilde{D}_{n,i} : n \in \mathbb{N}, i = 0, \dots, k_n\}$. \square

Now, we prove the technical lemmas.

LEMMA (4.7). *Let $p \geq 1$. There exists a constant $C_p > 0$ such that*

$$\sum_{n=0}^{\infty} \sum_{i=0}^{k_n} \left| \widehat{D}_{n,i} \right| |u(w_{n,i})|^p \leq C_p \|u\|_{b^p(S_T)}^p$$

for all $u \in b^p(S_T)$.

Proof. The point $w_{n,i}$ and the rectangle $\widehat{D}_{n,i}$ satisfy the geometric condition of Lemma (1.1). Thus, there exists a constant C_p such that

$$\sum_{n=0}^{\infty} \sum_{i=0}^{k_n} \left| \widehat{D}_{n,i} \right| |u(w_{n,i})|^p \leq C_p \sum_{n=0}^{\infty} \sum_{i=0}^{k_n} \int_{\widehat{D}_{n,i}} |u(z)|^p dz \leq 2C_p \|u\|_{b^p(S_T)}^p,$$

for $u \in b^p(S_T)$ and $p \geq 1$. The last inequality follows from Lemma (4.5). \square

LEMMA (4.8). *Let $p \geq 1$. There exists a constant $C_p > 0$ such that*

$$\sup \{ |u(z)|^p : z \in D_{n,i} \} \leq \frac{C_p}{b_{n+1}^3} \int_{\widetilde{D}_{n,i}} |u(w)|^p dw$$

for $u \in H(S_T)$, $n \in \mathbb{N}$, $i = 0, \dots, k_n$.

Proof. Let $z = (x, t)$ be any point in $D_{n,i}$. Let us consider the rectangle

$$R_z = \left[x - \frac{1}{2}b_{n+1}, x + \frac{1}{2}b_{n+1} \right] \times [t - b_{n+1}^2, t].$$

So, $R_z \subset \widetilde{D}_{n,i}$ for $i = 0, \dots, k_n$.

The point z and the rectangle R_z satisfy the geometric condition of Lemma (1.1), therefore there exists a constant $C_p > 0$ such that

$$|u(z)|^p \leq \frac{C_p}{|R_z|} \int_{R_z} |u(w)|^p dw \leq \frac{C_p}{b_{n+1}^3} \int_{\widetilde{D}_{n,i}} |u(w)|^p dw$$

for $u \in H(S_T)$. \square

LEMMA (4.9). *Let $p \geq 1$ and $0 < \delta < 1$. There exists a constant $C_{p,\delta} > 0$ such that*

$$\sum_{n=0}^{\infty} \sum_{i=0}^{k_n} \int_{D_{n,i}} |u(z) - u(w_{n,i})|^p dz \leq C_{p,\delta} \|u\|_{b^p(S_T)}^p,$$

for $u \in b^p(S_T)$. Furthermore, $C_{p,\delta} \rightarrow 0$ as $\delta \uparrow 1$.

Proof. Let $u \in b^p(S_T)$. If $z \in D_{n,i}$, by the Mean Value Theorem we have that

$$|u(z) - u(w_{n,i})| \leq b_n \left| \frac{\partial u}{\partial y}(\xi_z) \right| + b_n^2 \left| \frac{\partial u}{\partial \tau}(\xi_z) \right|$$

where ξ_z is a point on the line segment joining the points z and $w_{n,i}$. The inequality still holds if $i = k_n$ since, by construction, the base of D_{n,k_n} is less than or equal to b_n .

Applying Lemma (4.8) to $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial \tau}$ we obtain

$$|u(z) - u(w_{n,i})|^p \leq C_p \left\{ \frac{b_n^p}{b_{n+1}^3} \int_{\tilde{D}_{n,i}} \left| \frac{\partial u}{\partial y}(w) \right|^p dw + \frac{b_n^{2p}}{b_{n+1}^3} \int_{\tilde{D}_{n,i}} \left| \frac{\partial u}{\partial \tau}(w) \right|^p dw \right\}.$$

On the other hand, if $w = (y, \tau) \in \tilde{D}_{n,i}$ then $\tau \geq T\delta^{n+2}$. So,

$$\begin{aligned} \int_{D_{n,i}} |u(z) - u(w_{n,i})|^p dz &\leq C_p \left\{ \frac{b_n^p |D_{n,i}|}{T^{\frac{p}{2}} \delta^{(n+2)\frac{p}{2}} b_{n+1}^3} \int_{\tilde{D}_{n,i}} \left| \tau^{\frac{1}{2}} \frac{\partial u}{\partial y}(w) \right|^p dw \right. \\ &\quad \left. + \frac{b_n^{2p} |D_{n,i}|}{T^p \delta^{(n+2)p} b_{n+1}^3} \int_{\tilde{D}_{n,i}} \left| \tau \frac{\partial u}{\partial \tau}(w) \right|^p dw \right\}. \end{aligned}$$

Since $|D_{n,i}| \leq |\widehat{D}_{n,i}| = b_n^3$, we get

$$\int_{D_{n,i}} |u(z) - u(w_{n,i})|^p dz \leq C_{p,\delta} \left(\int_{\tilde{D}_{n,i}} \left| \tau^{\frac{1}{2}} \frac{\partial u}{\partial y}(w) \right|^p dw + \int_{\tilde{D}_{n,i}} \left| \tau \frac{\partial u}{\partial \tau}(w) \right|^p dw \right)$$

where

$$C_{p,\delta} = \delta^{-\frac{3}{2}} C_p \max \left\{ \delta^{-p} (1-\delta)^{\frac{p}{2}}, \delta^{-2p} (1-\delta)^p \right\}.$$

Therefore $C_{p,\delta} \rightarrow 0$ as $\delta \rightarrow 1$ for all $p \geq 1$.

Finally, by Lemma (4.6), (2.21) and Remark (2.22) we get

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{i=0}^{k_n} \int_{D_{n,i}} |u(z) - u(w_{n,i})|^p dz &\leq 6C_{p,\delta} \left\| t^{\frac{1}{2}} \frac{\partial u}{\partial x} \right\|_{L^p(\Omega_T)}^p + 6C_{p,\delta} \left\| t \frac{\partial u}{\partial t} \right\|_{L^p(\Omega_T)}^p \\ &\leq C_{p,\delta} \|u\|_{b^p(S_T)}^p. \end{aligned}$$

□

LEMMA (4.10). *Let $\alpha > 0$. There exists a constant $C_{\delta,\alpha} > 0$ such that*

$$\int_{\Omega_T} |N_\alpha(z, w_{n,i}) \tau_n^\alpha - N_\alpha(z, w) \tau^\alpha| dz \leq C_{\delta,\alpha}, \quad \text{for } w \in D_{n,i}.$$

Furthermore, $C_{\delta,\alpha} \rightarrow 0$ as $\delta \uparrow 1$.

Proof. Let $h_z(w) = N_\alpha(z, w) \tau^\alpha$. So,

$$|h_z(w) - h_z(w_{n,i})| \leq b_n \left| \frac{\partial h_z}{\partial y}(\xi_w) \right| + b_n^2 \left| \frac{\partial h_z}{\partial \tau}(\xi_w) \right|,$$

where $\xi_w = (y^*, \tau^*)$ is a point in the line segment joining the points w and $w_{n,i}$. Therefore $T\delta^{n+1} \leq \tau^* \leq \tau_n = T\delta^n$.

Applying Lemma (4.8) to $N_\alpha(z, \cdot)$, $\frac{\partial N_\alpha}{\partial y}(z, \cdot)$, and $\frac{\partial N_\alpha}{\partial \tau}(z, \cdot)$ we have

$$C_1^{-1} \int_{\Omega_T} |h_z(w) - h_z(w_{n,i})| dz \leq \frac{b_n (T\delta^n)^\alpha}{b_{n+1}^3} \int_{\tilde{D}_{n,i}} \int_{\Omega_T} \left| \frac{\partial N_\alpha}{\partial y}(z, w) \right| dz dw$$

$$+ \frac{\alpha b_n^2 (T\delta^n)^{\alpha-1}}{A_\delta^{-1} b_{n+1}^3} \int_{\tilde{D}_{n,i}} \int_{\Omega_T} |N_\alpha(z, w)| dz dw + \frac{b_n^2 (T\delta^n)^\alpha}{b_{n+1}^3} \int_{\tilde{D}_{n,i}} \int_{\Omega_T} \left| \frac{\partial N_\alpha}{\partial \tau}(z, w) \right| dz dw,$$

where $A_\delta = 1$ if $\alpha \geq 1$ and $A_\delta = \delta^{\alpha-1}$ otherwise. From (2.19) (with $\beta = 0$) and Lemma (2.14) we get

$$\begin{aligned} \int_{\Omega_T} |N_\alpha(z, w)| dz &\leq \tilde{C}_\alpha \left(|\Omega_T| + \int_{\Omega_T} |\Theta_\alpha(z, w)| dz \right) \leq \tilde{C}_\alpha (|\Omega_T| + \tau^{-\alpha}), \\ \int_{\Omega_T} \left| \frac{\partial N_\alpha}{\partial y}(z, w) \right| dz &\leq \tilde{C}_\alpha \int_{\Omega_T} |\Theta_{\alpha+\frac{1}{2}}(z, w)| dz \leq \tilde{C}_\alpha \tau^{-\alpha-\frac{1}{2}}, \\ \int_{\Omega_T} \left| \frac{\partial N_\alpha}{\partial \tau}(z, w) \right| dz &\leq \tilde{C}_\alpha \int_{\Omega_T} |\Theta_{\alpha+1}(z, w)| dz \leq \tilde{C}_\alpha \tau^{-\alpha-1}. \end{aligned}$$

Since $\tau \geq T\delta^{n+2}$ if $w = (y, \tau) \in \tilde{D}_{n,i}$, and $|\tilde{D}_{n,i}| \leq (T\delta^n - T\delta^{n+2})(b_n + b_{n+1}) \leq 4b_n^3$, we have

$$\int_{\Omega_T} |h_z(w) - h_z(w_{n,i})| dz \leq C_1 \tilde{C}_\alpha C_{\alpha,\delta},$$

where

$$C_{\alpha,\delta} = 4\delta^{-2\alpha-\frac{3}{2}} \max \left\{ \delta^{-1}(1-\delta)^{\frac{1}{2}}, \frac{\alpha T^{\alpha+1} \delta^{2\alpha}(1-\delta)}{A_\delta^{-1}}, \frac{\alpha(1-\delta)}{A_\delta^{-1}}, \delta^{-2}(1-\delta) \right\}.$$

□

Finally, we arrange the elements of the set $\{w_{n,i} : n \in \mathbb{N}, i = 0, \dots, k_n\}$ in a sequence $\{w_m\}$. Likewise, we set

$$\begin{aligned} \{D_{n,i} : n \in \mathbb{N}, i = 0, \dots, k_n\} &= \{D_m\}, \\ \{\hat{D}_{n,i} : n \in \mathbb{N}, i = 0, \dots, k_n\} &= \{\hat{D}_m\}, \\ \{\tilde{D}_{n,i} : n \in \mathbb{N}, i = 0, \dots, k_n\} &= \{\tilde{D}_m\}, \end{aligned}$$

in such a way that $w_m = (y_m, \tau_m)$ is the midpoint of the upper boundary of the rectangles D_m , \hat{D}_m , and \tilde{D}_m . Therefore (3.1) follows from (4.1), lemma (3.2) from lemma (4.7), and so on.

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INSTITUTO DE MATEMÁTICAS, UNAM
CD. UNIVERSITARIA
MÉXICO, D.F. 04510
MÉXICO
flopez@matem.unam.mx

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ABSOLUTE HYPERSPACES AND HYPERSPACES THAT ARE ABSOLUTE CONES AND ABSOLUTE SUSPENSIONS

In memory of Professor J. J. Charatonik

SAM B. NADLER, JR.

ABSTRACT. The notion of absolute hyperspaces, for several types of hyperspaces, is introduced; the notion is analogous to J. de Groot's notion of absolute cones. It is determined which continua are absolute hyperspaces and which hyperspaces of certain types are absolute cones and absolute suspensions.

1. Introduction

All spaces are metric. A *compactum* is a nonempty compact space, and a *continuum* is a connected compactum.

We use $X \approx Y$ to denote that the spaces X and Y are homeomorphic; if $A_i \subset X$ and $B_i \subset Y$, $i \in \{1, \dots, n\}$, then $(X, A_1, \dots, A_n) \approx (Y, B_1, \dots, B_n)$ means that there is a homeomorphism of X onto Y that takes A_i onto B_i for all i . When A_i or B_i is a point, we write the point without set brackets.

The notions of cones, suspensions and hyperspaces (with the Hausdorff metric) are well known and not defined here (see [9], [17],[18]). We denote the cone over a space Y by $\text{Cone}(Y)$ and its vertex by v_Y , the suspension over Y by $\text{Sus}(Y)$ and its vertices by v_Y^- and v_Y^+ , the hyperspace of subcontinua (or of subcompacta) of a continuum Y by $C(Y)$ (2^Y , respectively), and the n -fold hyperspace of a continuum Y by $C_n(Y)$ ($= \{A \in 2^Y : A \text{ has at most } n \text{ components}\}$, $n < \infty$; note that $C_1(Y) = C(Y)$). Interest in n -fold hyperspaces when $n > 1$ is fairly recent (e.g., [5], [6], [7], [11], [12], [13], [14]).

The following two notions were introduced by de Groot [2]. A continuum X is called an *absolute suspension* provided that for each two (different) points $p, q \in X$, there is a compactum $Y_{(p,q)}$ such that

$$(X, p, q) \approx (\text{Sus}(Y_{(p,q)}), v_{Y_{(p,q)}}^-, v_{Y_{(p,q)}}^+).$$

A continuum X is called an *absolute cone* provided that for each point $p \in X$, there is a compactum Y_p such that

$$(X, p) \approx (\text{Cone}(Y_p), v_{Y_p}).$$

De Groot [2] asked whether every finite-dimensional absolute suspension is a sphere; Szymański [20] answered this question affirmatively in dimensions ≤ 3 , but the question remains unanswered in higher dimensions. W. Mitchell [16]

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gave a partial answer by showing that finite-dimensional absolute suspensions are regular generalized manifolds that are homotopy equivalent to spheres.

De Groot also asked if every finite-dimensional absolute cone is an n -cell. No work seems to have been done on this question (recently, I have shown that the answer is affirmative in dimensions 1 and 2). Regarding absolute open cones, see [16].

In analogy with the vertex of a cone as the highest point of the cone, Y is the highest point of $C_n(Y)$ and of 2^Y with respect to the partial order \subset . Thus, the following notions of absolute hyperspaces are natural analogues of absolute cones: Let X be a continuum; then

X is an *absolute hyperspace of continua* or of *compacta* provided that for each point $p \in X$, there is a continuum Y_p such that $(X, p) \approx (C(Y_p), Y_p)$ or, respectively, $(X, p) \approx (2^{Y_p}, Y_p)$;

X is an *absolute n -fold hyperspace* provided that for each point $p \in X$, there is a continuum Y_p such that $(C_n(Y_p), Y_p)$.

It is reasonable to extend the definitions just given to any hyperspace that has a largest element with respect to the partial order \subset ; we do not consider this extension here.

Motivated by de Groot's questions, several questions can be asked about hyperspaces: What continua are absolute hyperspaces (of the three types above)? What hyperspaces $C(X)$, 2^X and $C_n(X)$ are absolute cones or absolute suspensions?

In this paper we focus on the questions about hyperspaces and provide complete answers to most of them. In section 3, we determine the continua that are absolute hyperspaces for all three types of hyperspaces we defined. In section 4, we determine the hyperspaces $C(X)$ and 2^X that are absolute cones. In section 5, we determine the hyperspaces $C_n(X)$ and 2^X that are absolute suspensions. The question of when $C_n(X)$, $n > 1$, is an absolute cone remains open (more models for n -fold hyperspaces than are now known are probably needed). The reason we can determine when $C_n(X)$ is an absolute suspension but not when it is an absolute cone for $n > 1$ is that absolute suspensions are homogeneous—see the proof of Theorem (5.1).

In connection with our theorems about hyperspaces that are absolute cones, we note that the question of when hyperspaces are cones has been studied extensively (see [9], pp. 59–63 and 424–431 and, more recently, [4], [8], [13], [15] and the references therein). On the other hand, in connection with our theorems concerning hyperspaces that are absolute suspensions, the only definitive result for when hyperspaces are suspensions is the theorem in [19], which says that the only finite-dimensional continuum X such that $C(X) \approx \text{Sus}(X)$ is the arc.

Regarding definitions we have given, note that Y_p is only required to be a compactum in de Groot's definitions of absolute suspension and absolute cone, whereas Y_p is required to be a continuum in our definitions of the three types of absolute hyperspaces. No generality would be achieved for our results by only requiring Y_p to be a compactum in our definitions of absolute hyperspaces; this is because if Y is a nonconnected compactum, then v_Y separates $\text{Cone}(Y)$ and $\{v_Y^-, v_Y^+\}$ separates $\text{Sus}(Y)$, whereas, for a continuum X , no finite set separates

$C(X)$, 2^X or $C_n(X)$ (which is easy to prove using order arcs defined in [9], p. 110; a stronger result is in [17], p. 224, 2.15).

Further Notation and Terminology: I denotes the unit interval $[0, 1]$, and an *arc* is a space $\approx I$; I^n ($n \in \{1, 2, \dots, \infty\}$) denotes the Cartesian product of n copies of I , an n -cell is a space $\approx I^n$ for $n < \infty$ and a *Hilbert cube* is a space $\approx I^\infty$; S^1 denotes the unit circle in the Euclidean plane \mathbb{R}^2 , and a *simple closed curve* is a space $\approx S^1$; \dim stands for topological dimension and \dim_p denotes dimension at a point p [3]; ∂X denotes the manifold boundary of a manifold X .

A *Peano continuum* is a locally connected continuum. A *finite graph* is a compact connected polyhedron of dimension ≤ 1 . A *simple n -od*, $n < \infty$, is the cone over an n -point discrete space (called a *simple triod* when $n = 3$). A *free arc in a space X* is an arc A such that $A - \partial A$ is open in X . The term *nondegenerate* refers to a space that contains more than one point.

2. Preliminary Results

We will use a number of basic results several times. We state the results below. We provide references for results that are known (references are to books whenever possible, not to original sources) and we give proofs for results not in the literature. It should be kept in mind that results for $C_n(X)$ for all n include $C(X)$ since $C(X) = C_1(X)$.

(2.1). *If a continuum Z is an absolute cone, an absolute suspension, an absolute n -fold hyperspace for some n or an absolute hyperspace of compacta, then Z is a Peano continuum.*

Proof. Cones and suspensions are locally connected at their vertices. Also, for any continuum Y , $C_n(Y)$ and 2^Y are locally connected at Y ([9], p. 122, 15.5 and, for $C_n(Y)$ with $n \geq 2$, repeat the proof of [9], p. 122, 15.5 using [9], p. 125, 15.11). Therefore, it follows easily from the definitions of absolute cones, etc. that Z is locally connected at each of its points. \square

(2.2). $C(I) \approx I^2$ and $\partial C(I) = \{[a, b] : a = 0 \text{ or } b = 1 \text{ or } a = b\}$. [9], p. 33, 5.1.

(2.3). $C(S^1) \approx I^2$ and $\partial C(S^1) = \{\{x\} : x \in S^1\}$. [9], p. 35, 5.2.

(2.4). *Let Z be a nondegenerate Peano continuum. Then $2^Z \approx I^\infty$ and, if Z contains no free arc, $C_n(Z) \approx I^\infty$ all n .* [9], p. 89, 11.3 and [12], p. 250, 7.1.

(2.5). *The Hilbert cube is an absolute n -fold hyperspace for all n , an absolute hyperspace of compacta, an absolute cone and an absolute suspension.*

Proof. By (2.4), $C_n(I^\infty) \approx I^\infty$ (all n) and $2^{I^\infty} \approx I^\infty$; also, I^∞ is homogeneous [21], p. 254, 6.1.6. Hence, for any point $p \in I^\infty$, $(I^\infty, p) \approx (C_n(I^\infty), I^\infty)$ and $(I^\infty, p) \approx (2^{I^\infty}, I^\infty)$. This proves the first two parts of (2.5).

Since $I^\infty \approx \prod_{i=1}^\infty [0, 2^{-i}]$, $\text{Cone}(I^\infty)$ and $\text{Sus}(I^\infty)$ may be considered to be compact convex infinite-dimensional subsets of the Hilbert space ℓ_2 ; thus, $\text{Cone}(I^\infty) \approx I^\infty$ and $\text{Sus}(I^\infty) \approx I^\infty$ by Keller's Theorem [21], p. 378, 8.2.4. Therefore, for any point $p \in I^\infty$, $(I^\infty, p) \approx (\text{Cone}(I^\infty), v_{I^\infty})$ by homogeneity [21], p. 254, 6.1.6; also, for any two distinct points $p, q \in I^\infty$, $(I^\infty, p, q) \approx$

($\text{Sus}(I^\infty), v_{I^\infty}^-, v_{I^\infty}^+$) by Anderson's Homogeneity Theorem [9], p. 93, 11.9.1, which applies since a 2-point set in I^∞ is a Z -set in I^∞ . \square

(2.6). For a continuum Z , any one of Z , $C_n(Z)$ for some n , and 2^Z is a Peano continuum if and only if any one of the other two is a Peano continuum. [17], p. 134, 1.92 and [12], p. 140, 3.2.

Our next result follows from [1], Corollaries 2 and 4, pp. 390-391 for the case of 2^X and $C(X)$ and from proofs in [1] for the case of $C_n(X)$, $n \geq 2$. For completeness, we include a short, direct proof.

(2.7). If Z is a continuum such that $C_n(Z)$ (some n) or 2^Z is locally connected at each singleton $\{p\}$, then Z is a Peano continuum.

Proof. Fix $p \in Z$. Let $\epsilon > 0$. Then $\{p\}$ has a continuum neighborhood A_ϵ in $C_n(Z)$ or 2^Z of diameter $< \epsilon$. Hence, $\bigcup A_\epsilon$ is a continuum [17], p. 102, 1.49, $\bigcup A_\epsilon$ is a neighborhood of p in Z (since $\{z\} \in A_\epsilon$ for all z sufficiently close to p), and $\bigcup A_\epsilon$ has diameter $< 2\epsilon$ (if $z \in \bigcup A_\epsilon$, then $z \in A_z \in A_\epsilon$ and thus, since the Hausdorff distance between A_z and $\{p\}$ is $< \epsilon$, $d(z, p) < \epsilon$). This proves that Z is a Peano continuum (cf. [18], p. 84, 5.22(b)). \square

(2.8). If Z is a Peano continuum, then $\dim(C_n(Z)) < \infty$ for a given n if and only if Z is a finite graph. [11], p. 270, 5.1.

(2.9). If Z is a Peano continuum that is not a finite graph, then $\dim_Z(C_n(Z)) = \infty$ for each n .

Proof. Since $C(Z) \subset C_n(Z)$ for each n , it suffices to prove the result for $C(Z)$.

Fix an integer $k \geq 1$. Then, since Z is not a finite graph, there are k distinct points p_1, p_2, \dots, p_k in Z such that the set $A_k = \{p_1, p_2, \dots, p_k\}$ does not separate Z [18], p. 152, 9.24. Hence, there are mutually disjoint connected open neighborhoods U_i of p_i such that $Z - \bigcup_{i=1}^k U_i = K$ is continuum (apply [18], 8.45, p. 137 to the quotient space Z/A_k , which identifies the points of A_k to a single point denoted by p , to obtain a connected open neighborhood U of p with nonempty connected complement in Z/A_k ; then the neighborhoods U_i are the components of $\pi^{-1}(U)$, $\pi =$ the quotient map).

For each $i = 1, 2, \dots, k$, \overline{U}_i is a continuum containing $\overline{U}_i - U_i$; hence (segment refers to a segment in the sense of Kelley [9], p. 128), for each i , there is a segment $\sigma_i : I \rightarrow 2^{\overline{U}_i}$ with $\sigma_i(0) = \overline{U}_i - U_i$ and $\sigma_i(1) = \overline{U}_i$ [9], p. 130, 16.7. Define $h : I^k \rightarrow C(Z)$ by

$$h((t_1, t_2, \dots, t_k)) = K \cup \left[\bigcup_{i=1}^k \sigma_i([0, t_i]) \right].$$

It is easy to check that h is an embedding and that $Z = h((1, 1, \dots, 1))$.

We have proved that the point Z of $C(Z)$ belongs to a k -cell in $C(Z)$ for all k . It follows that $\dim_Z(C(Z)) = \infty$ (use the definition of dimension at a point and [3], p. 48, Corollary 2). \square

Remark (2.9). is a weak form of 71.8 in [9], p. 343, and 71.8 was referenced to be in [10] (with a slightly different title). However, 71.8 is not in [10], so I have included 2.9 (and its proof), which is sufficient for our purposes here.

(2.10) *If X is a nondegenerate continuum such that $C_n(X)$ is homogeneous for a given n , then $C_n(X)$ is a Hilbert cube.*

Proof. Fix n . We prove that X is a Peano continuum that contains no free arc; then $C_n(X)$ is a Hilbert cube by (2.4).

By [9], pp. 122 and 125, 15.5 and 15.11, $C_n(X)$ is locally connected at X ; thus, since $C_n(X)$ is homogeneous, $C_n(X)$ is Peano continuum. Hence, X is a Peano continuum by (2.6).

Suppose by way of contradiction that X contains a free arc F . Then some points of $C_n(F)$ have open neighborhoods in $C_n(X)$ homeomorphic to \mathbb{R}^{2n} (e.g., a point in $C_n(F - \partial F)$ with n nondegenerate components [14], p. 264, 4.2); on the other hand, some points of $C_n(F)$ do not have such neighborhoods (e.g., a singleton $\{y_0\}$ does not have such a neighborhood by the proof of in [11], p. 271, Theorem 5.4 or, alternatively, since $\{y_0\}$ is a Z -set in each of its neighborhoods). This contradicts our assumption that $C_n(X)$ is homogeneous. Therefore, X contains no free arc. \square

(2.11). *A nondegenerate Peano continuum that contains no simple triod is an arc or a simple closed curve.* [18], p. 135, 8.40(b).

(2.12). *The cone over a simple triod is not embeddable in a 2-cell.*

Proof. This follows from Brouwer's Invariance of Domain Theorem [3], p. 95, VI 9, since the cone over a simple triod contains a 2-cell B and a sequence of points not in B that converges to a point of $B - \partial B$. \square

3. Continua that are absolute hyperspaces

We determine the continua that are absolute hyperspaces of continua, absolute hyperspaces of compacta, and absolute n -fold hyperspaces.

THEOREM (3.1). *A nondegenerate continuum X is an absolute hyperspace of continua if and only if X is a 2-cell or a Hilbert cube.*

Proof. A 2-cell T is an absolute hyperspace of continua by (2.2) and (2.3) (if $p \in \partial T$, then $(T, p) \approx (C(I), I)$ by (2.2) and if $p \in T - \partial T$, then $(T, p) \approx (C(S^1), S^1)$ by (2.3)).

A Hilbert cube is an absolute hyperspace of continua by (2.5).

To prove the other direction of the theorem, assume that X is an absolute hyperspace of continua. For each point $p \in X$, let Y_p be a continuum such that

$$(X, p) \approx (C(Y_p), Y_p).$$

By (2.1), X is a Peano continuum. Hence, each $C(Y_p)$ is a Peano continuum. Thus, by (2.6), each Y_p is a Peano continuum. Also, since X is nondegenerate (by assumption), each Y_p is a nondegenerate. We take two cases.

Case 1: $\dim(X) < \infty$. Then $\dim(C(Y_p)) < \infty$ for all $p \in X$; thus, since each Y_p is a nondegenerate Peano continuum, each Y_p is a nondegenerate finite graph by (2.8). We prove that X is a 2-cell.

Suppose by way of contradiction that X is not a 2-cell. Then no Y_p is an arc or a simple closed curve by (2.2) and (2.3). Also, the hyperspaces $C(Y_p)$, $p \in X$, are mutually homeomorphic (since they are all homeomorphic to X). Thus,

all the finite graphs Y_p are homeomorphic to one another [17], p. 32, 0.59; we denote all these finite graphs by Y . Then

$$(X, p) \approx (C(Y), Y) \text{ for all } p \in X.$$

Therefore, X has the same dimension at every point. Hence, $C(Y)$ has the same dimension at every point. However, since Y is a nondegenerate finite graph that is not an arc or a simple closed curve, $C(Y)$ does not have the same dimension at every point (for $y_1 \in Y$ that is the vertex of a simple triod in Y (2.11), $\dim_{\{y_1\}}(C(Y)) \geq 3$ [9], p. 41, 5.4, and for y_2 in the interior in Y of a free arc F , $\dim_{\{y_2\}}(C(Y)) = 2$ since $C(F)$ is a neighborhood of $\{y_2\}$ in $C(Y)$ and $C(F)$ is a 2-cell by (2.2)). Hence, we have a contradiction. Therefore, X is a 2-cell.

Case 2: $\dim(X) = \infty$. Then, for each $p \in X$, $\dim(C(Y_p)) = \infty$, hence Y_p is not a finite graph by (2.8) and, thus, $\dim_{Y_p}(C(Y_p)) = \infty$ by (2.9). Therefore, since $(X, p) \approx (C(Y_p), Y_p)$ for each $p \in X$, we have proved that X is infinite-dimensional at every point. Now, fix Y_{p_0} . Then, since $X \approx C(Y_{p_0})$, $C(Y_{p_0})$ is infinite-dimensional at every point. Hence, Y_{p_0} does not contain a free arc (otherwise, $C(Y_{p_0})$ would contain a 2-cell neighborhood by (2.2)). Thus, $C(Y_{p_0})$ is a Hilbert cube by (2.4). Therefore, X is a Hilbert cube. \square

THEOREM (3.2). *A nondegenerate continuum X is an absolute hyperspace of compacta if and only if X is a Hilbert cube.*

Proof. A Hilbert cube is an absolute hyperspace of compacta by (2.5).

Conversely, assume that the nondegenerate continuum X is an absolute hyperspace of compacta. Then, in particular, $X \approx 2^Y$ for some nondegenerate continuum Y . Furthermore, since X is a Peano continuum by (2.1), Y is a Peano continuum by (2.6). Thus, since Y is nondegenerate, 2^Y is a Hilbert cube by (2.4). Therefore, X is a Hilbert cube. \square

In Theorem (3.1) we determined the absolute 1-fold hyperspaces. We now determine the absolute n -fold hyperspaces for $n \geq 2$.

THEOREM (3.3). *A nondegenerate continuum X is an absolute n -fold hyperspace for some $n \geq 2$ if and only if X is a Hilbert cube.*

Proof. A Hilbert cube is an absolute n -fold hyperspace by (2.5).

To prove the other direction of the theorem, assume that X is an absolute n -fold hyperspace for some fixed $n \geq 2$. For each point $p \in X$, let Y_p be a continuum such that

$$(X, p) \approx (C_n(Y_p), Y_p).$$

By (2.1) and (2.6) (as in the proof of Theorem (3.1)), X and Y_p are nondegenerate Peano continua.

We first prove that

$$(*) \quad \dim(X) = \infty.$$

\square

Proof of ().* Suppose by way of contradiction that $\dim(X) < \infty$. Then $\dim(C_n(Y_p)) < \infty$ for all $p \in X$. Thus, each Y_p is a nondegenerate finite graph by (2.8). Also, all the hyperspaces $C_n(Y_p)$ are homeomorphic to one another

(since they are all homeomorphic to X). Therefore, since $n \geq 2$, all the finite graphs Y_p are homeomorphic to one another ([5] and [6]). We denote all these finite graphs by Y . Then

$$(X, p) \approx (C_n(Y), Y) \text{ for all } p \in X.$$

Therefore, for any $p_1, p_2 \in X$, there are homeomorphisms h_1 and h_2 of X onto $C_n(Y)$ such that $h_i(p_i) = Y$ for each i . Hence, $h_2^{-1} \circ h_1$ is a homeomorphism of X onto X such that $h_2^{-1} \circ h_1(p_1) = p_2$. This proves that X is homogeneous. Thus, $C_n(Y)$ is homogeneous. Hence, by (2.10), $C_n(Y)$ is a Hilbert cube. Therefore, X is a Hilbert cube, which contradicts our assumption that $\dim(X) < \infty$. This proves (*).

In view of (*), we can conclude that X is a Hilbert cube by repeating the argument for Case 2 in the proof of Theorem (3.1) (use (2.8) instead of (2.2) to obtain that $C_n(Y_{p_0})$ would contain a finite-dimensional neighborhood if Y_{p_0} had a free arc). □

4. Hyperspaces that are absolute cones

We determine the hyperspaces $C(X)$ and 2^X that are absolute cones.

THEOREM (4.1). *A hyperspace $C(X)$, X a continuum, is an absolute cone if and only if $C(X)$ is a 2-cell or a Hilbert cube (i.e., X is an arc, a simple closed curve or a nondegenerate Peano continuum with no free arc).*

Moreover, if X is a continuum such that for each singleton $\{p\} \in C(X)$ there is a compactum $Y_{\{p\}}$ such that

$$(C(X), \{p\}) \approx (\text{Cone}(Y_{\{p\}}), v_{Y_{\{p\}}}),$$

then $C(X)$ is a 2-cell or a Hilbert cube (and conversely).

Proof. A 2-cell T is an absolute cone since $(T, p) \approx (\text{Cone}(I), v_I)$ when $p \in \partial T$ and $(T, p) \approx (\text{Cone}(S^1), v_{S^1})$ when $p \notin \partial T$.

A Hilbert cube is an absolute cone by (2.5).

Conversely, assume $Y_{\{p\}}$ is as in the second part of the theorem for each $p \in X$.

We first show that X and each $Y_{\{p\}}$ are nondegenerate Peano continua. Since each $\text{Cone}(Y_{\{p\}})$ is locally connected at $v_{Y_{\{p\}}}$, $C(X)$ is locally connected at each singleton $\{p\}$. Hence, by (2.7), X is a Peano continuum; also, X is nondegenerate since $C(X)$, being a cone, is nondegenerate. Since X is a Peano continuum, $C(X)$ is a Peano continuum by (2.6); thus, each $\text{Cone}(Y_{\{p\}})$ is a Peano continuum. Hence, each $Y_{\{p\}}$ is a Peano continuum. Finally, each $Y_{\{p\}}$ is nondegenerate: otherwise, $C(X)$ would be an arc and, thus, would have separating points, in contradiction to [17], p. 195, 1.204.1.

We complete the proof of our theorem by considering two situations—when X contains a free arc and when X does not contain a free arc.

Assume that X contains a free arc F , and let p be a point in the interior of F in X . Then $C(F)$ is a 2-cell (by (2.2)) and $C(F)$ is a neighborhood of $\{p\}$ in $C(X)$; hence, $v_{Y_{\{p\}}}$ has a 2-cell neighborhood N in $\text{Cone}(Y_{\{p\}})$. Therefore, $Y_{\{p\}}$ does not contain a simple triod T (otherwise, N would contain a cone over a simple triod in $\text{Cone}(T)$, which is impossible by (2.12)). Thus, since $Y_{\{p\}}$ is a

nondegenerate Peano continuum, $Y_{\{p\}}$ is an arc or a simple closed curve by (2.11). Therefore, $\text{Cone}(Y_{\{p\}})$, hence $C(X)$, is a 2-cell.

Finally, assume that X does not contain a free arc. Then, since X is a nondegenerate Peano continuum, $C(X)$ is a Hilbert cube by (2.4). \square

THEOREM (4.2). *A hyperspace 2^X , X a continuum, is an absolute cone if and only if 2^X is a Hilbert cube (i.e., X is a nondegenerate Peano continuum).*

Moreover, if X is a continuum such that for each singleton $\{p\} \in 2^X$ there is a compactum $Y_{\{p\}}$ such that

$$(2^X, \{p\}) \approx (\text{Cone}(Y_{\{p\}}), v_{Y_{\{p\}}}),$$

then 2^X is a Hilbert cube (and conversely).

Proof. A Hilbert cube is an absolute cone by (2.5).

Conversely, assume that 2^X satisfies the condition involving singletons in the second part of our theorem. Then, using (2.7) as we did in the proof Theorem (4.1), we see that X is a nondegenerate Peano continuum. Therefore, 2^X is a Hilbert cube by (2.4). This is equivalent to X being a nondegenerate Peano continuum by (2.4) and (2.6). \square

5. Hyperspaces that are absolute suspensions

We determine the hyperspaces $C_n(X)$ and 2^X that are absolute suspensions. In contrast to Theorem (4.1), the first part of Theorem (5.1) shows that the only time $C(X)$ is absolute suspension is when $C(X)$ is a Hilbert cube.

THEOREM (5.1). *Let X be a continuum. Then*

- (1) *for a given n , $C_n(X)$ is an absolute suspension if and only if $C_n(X)$ is a Hilbert cube (i.e., X is a nondegenerate Peano continuum with no free arc) and*
- (2) *2^X is an absolute suspension if and only if 2^X is a Hilbert cube (i.e., X is a nondegenerate Peano continuum).*

Proof. A Hilbert cube is an absolute suspension by (2.5).

To prove the converse, first note that any suspension admits a self-homeomorphism that interchanges the vertices of the suspension. It follows from this that (as noted in [16], p. 242, Lemma 1)

(#) absolute suspensions are homogeneous.

Now, assume that $C_n(X)$ is an absolute suspension for some $n \geq 1$. Then, by (#), $C_n(X)$ is homogeneous. Therefore, $C_n(X)$ is a Hilbert cube by (2.10) (when $n = 1$, this also follows from (#) and [17], p. 564, 17.2). This is equivalent to X being a nondegenerate Peano continuum with no free arc by (2.4), (2.6) and (2.8).

Finally, assume that 2^X is an absolute suspension. Then, by (#), 2^X is homogeneous. Therefore, 2^X is a Hilbert cube by [17], p. 565, 17.3. This is equivalent to X being a nondegenerate Peano continuum by (2.4) and (2.6). \square

In relation to Theorem (5.1), all continua X whose hyperspaces $C(X)$ and 2^X are suspensions have not been determined. This problem and related problems were posed in [17], pp. 337–338 and are as yet not answered. In addition, it is not known when $C_n(X)$ is a suspension for $n > 1$. We note that R. Schori has

shown that $C_2(I) \approx I^4$ (a proof is in [5], p. 349); thus, $C_2(I) \approx \text{Sus}(I^3)$, which is the only finite-dimensional example relating $C_n(X)$, $n > 1$, to suspensions that we have at present.

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DEPARTMENT OF MATHEMATICS
WEST VIRGINIA UNIVERSITY
MORGANTOWN, WV 26506–6310
U.S.A.
nadler@mat.wvu.edu

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LIMIT PROPERTIES OF SPACES OF SOME MAPPINGS BETWEEN CONTINUA

JANUSZ J. CHARATONIK*

ABSTRACT. A mapping is said to be confluent over locally connected continua if for each locally connected subcontinuum Q of the range each component of its preimage is mapped onto Q . For mappings of compact metric spaces this class is a very natural generalization of locally confluent mappings. Limit properties of these and related mappings are studied in the paper.

1. Introduction

The study of limit properties of spaces of (special kinds of) mappings is an old and important area of interest for mathematicians. For a class Λ of mappings the two basic questions in the area are: (1) When is Λ closed under uniform convergence? (2) When is Λ topologically complete? (see [18], p. 562). The reader is referred to the Introduction of [18] for motivation of this subject. Let us only mention here some important papers devoted to these problems.

The limit properties of the class of monotone mappings were studied by Whyburn [21] (compare also [20], Theorem 3.1, p. 174), Kuratowski [10], Kuratowski and Lacher [11], of non-alternating mappings by McAuley [17]; and of confluent and related mappings by Maćkowiak [14] and Nadler [18]. The reader is referred to articles [4] and [16] for a survey of these and related topics. The limit properties for various classes of mappings were investigated, and the results were collected by Maćkowiak in [15], Chapter 5, Section E, p. 41; Table III, p. 48 and 49. The aim of this article is a continuation of this study for some larger classes of mappings between continua.

Special attention is paid to mappings which are *confluent over locally connected continua*, that is, to mappings such that for each locally connected subcontinuum of the range each component of its preimage is mapped onto the subcontinuum. For mappings of compact spaces this class is a very natural generalization of locally confluent mappings. The concept of confluence over a subcontinuum of the range was exploited in several places of continuum theory and was proved to be a useful tool to investigate various properties of continua. For example H. Cook [3] and D. R. Read [19] (see also [12]) proved that a continuum is hereditarily indecomposable if and only if every mapping of a continuum onto it is confluent. An essential part of the proof of that result was a characterization of terminal continua saying that a subcontinuum Q of a

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continuum Y is terminal if and only if each map of a continuum onto Y is confluent over Q , see [6], Theorem 1, p. 84. W. T. Ingram has used the concept of mappings confluent over each unicoherent proper subcontinuum to characterize continua having hereditarily the property that each proper subcontinuum with nonempty interior is unicoherent, see [6], Theorem 5, p. 89. Mappings which are confluent over locally connected continua were used to characterize the path lifting property for light surjections between compact metric spaces in [7], Corollary 2.3, p. 364 and to characterize dendrites in [2], Corollary 17, p. 1216.

It is shown in the paper that the classes of mappings confluent over locally connected continua do not have the limit property, but they have the weak limit property (i.e., under an additional assumption that the range space is locally connected). The former assertion is presented in Example (3.1) where a sequence of homeomorphisms on a continuum is shown that converges to a mapping which is not confluent over locally connected continua. The latter one is Corollary (4.3) which says that the set of mappings $f : X \rightarrow Y$ between continua X and Y such that f are confluent over locally connected continua is closed in the space Y^X (with the compact-open topology) provided that Y has the property of Kelley and the arc approximation property.

2. Preliminaries

All spaces considered in this paper are assumed to be metric and all mappings are continuous. A *continuum* means a compact connected space.

A surjective mapping $f : X \rightarrow Y$ between topological spaces is said to be:

- *confluent* provided that for each continuum $Q \subset Y$ each component of $f^{-1}(Q)$ is mapped under f onto Q , see [1];
- *semi-confluent* provided that for each continuum $Q \subset Y$ and for every two components C_1 and C_2 of $f^{-1}(Q)$ we have either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$, see [13].

We refer the reader to [15] for definitions of, and relations between, concepts not defined here. In particular, the reader is referred to [15], Chapters 3 and 4, p. 12-27; and Table II, p. 28 for relations between these and other known classes of mappings like monotone, open, locally confluent etc.

We say that a class \mathfrak{M} of mappings is *defined by the subcontinua of the range* provided that it is defined by a property $\mathcal{P}(Q)$ which is satisfied by the inverse image of an arbitrary subcontinuum Q of the range space (see [15], p. 12). For example, classes of monotone, confluent, semi-confluent, weakly confluent or joining mappings are such, while homeomorphisms, open or atomic mappings are not (the reader is referred to [15], p. 12–14 for the definitions of these concepts). The reader can observe that the definition of the class of mappings which are confluent over locally connected continua (as defined in the Introduction) is constructed in this way.

Definition (2.1). Let \mathfrak{M} be a class of mappings defined by the subcontinua of the range, and let \mathcal{C} be a class of continua. A surjective mapping $f : X \rightarrow Y$ between continua is said to be:

- \mathfrak{M} *over* \mathcal{C} provided that for each subcontinuum $Q \subset Y$ with $Q \in \mathcal{C}$ the property $\mathcal{P}(Q)$ is satisfied by the inverse image $f^{-1}(Q)$;

– \mathfrak{M} with respect to \mathcal{C} provided that for each subcontinuum $Q \subset Y$ with $Q \in \mathcal{C}$ the partial mapping $f|_{f^{-1}(Q)} : f^{-1}(Q) \rightarrow Q$ is in \mathfrak{M} .

Remark (2.2). The condition formulated in the second part of Definition (2.1) is stronger than that of the first part. Indeed, if the class \mathfrak{M} is defined by the subcontinua of the range and if $\mathcal{P}(Q)$ is the property required in the definition, then the second condition requires that the partial mapping $f|_{f^{-1}(Q)} : f^{-1}(Q) \rightarrow Q$ is in \mathfrak{M} , that is, for each subcontinuum $Q' \subset Q$ the property $\mathcal{P}(Q')$ holds (only continuum Q has to be in \mathcal{C} , while its subcontinua Q' need not be in \mathcal{C} ; this means that the second condition demands \mathcal{P} hereditarily: for Q and for all its subcontinua, independently if they are or are not in \mathcal{C}). The first one demands \mathcal{P} only for those continua Q which belong to \mathcal{C} . No condition is required for subcontinua of Q .

A property \mathcal{P} of subcontinua of a continuum Y is said to be *hereditary* provided that if $\mathcal{P}(Q)$ holds for a subcontinuum Q of Y , then $\mathcal{P}(Q')$ holds for each subcontinuum Q' of Q .

Remark (2.2) implies the following statement.

Statement (2.3). Let the class \mathfrak{M} be defined by the subcontinua of the range and let \mathcal{C} be a class of continua. Then

$$(2.3.1) \quad (\mathfrak{M} \text{ with respect to } \mathcal{C}) \subset (\mathfrak{M} \text{ over } \mathcal{C}).$$

If the property \mathcal{P} that defines \mathfrak{M} is hereditary, then the inclusion in (2.3.1) can be replaced by the equality.

Applying Definition (2.1) to the classes \mathfrak{M} of confluent and of semi-confluent mappings and to the class \mathcal{C} of locally connected continua we get the following proposition.

PROPOSITION (2.4). Let $f : X \rightarrow Y$ be a surjective mapping between continua X and Y . Then the following implication holds.

(2.4.1) If f is confluent (semi-confluent) with respect to locally connected continua, then f is confluent (semi-confluent) over locally connected continua.

The following result is known, see [2], Proposition 8, p. 1213.

PROPOSITION (2.5). Each locally confluent mapping between compact metric spaces is confluent over locally connected continua.

As it can be seen from the proof, metric arguments are used to show this proposition. Thus the following question seems to be natural.

Question (2.6). Is the implication in Proposition (2.5) true in the general case, i.e., for mappings between Hausdorff, not necessarily metric, spaces?

Let topological spaces X and Y be given, and denote by Y^X the space of all mappings $f : X \rightarrow Y$ with the compact-open topology. A class \mathfrak{M} of mappings is said to have the *(weak) limit property* provided that for every two spaces X and Y (for each space X and for each locally connected space Y) the set of all surjective mappings $f : X \rightarrow Y$ belonging to \mathfrak{M} is closed in the space Y^X .

3. The example

To see that the class of mappings that are confluent over locally connected continua does not have the limit property consider the following example.

Example (3.1). *There exist a metric continuum X and a sequence of homeomorphisms $f_n : X \rightarrow X$ such that the limit mapping $f = \lim f_n : X \rightarrow X$ is not semi-confluent over locally connected continua.*

Proof. In the Cartesian coordinates in the plane put:

$$p = (0, 0), \quad c = (1, 0), \quad q = (2, 0), \quad a_n = (1, \frac{1}{n}), \quad b_n = (1, -\frac{1}{n}) \quad \text{for each } n \in \mathbb{N}.$$

Let uv denote the straight line segment from u to v ; define

$$X_1 = pc \cup \bigcup \{pa_n : n \in \mathbb{N}\}, \quad X_2 = qc \cup \bigcup \{qb_n : n \in \mathbb{N}\},$$

and

$$X = X_1 \cup X_2.$$

Thus X_1 and X_2 are well known harmonic fans. In [21], Example, p. 465 a sequence of homeomorphisms $\{h_n\}$ on a harmonic fan is constructed so that the sequence converges uniformly to a surjective non-monotone mapping h on the harmonic fan. For a fixed $n \in \mathbb{N}$ define $f_n : X \rightarrow X$ so that the partial mappings $f_n|X_1 : X_1 \rightarrow X_1$ and $f_n|X_2 : X_2 \rightarrow X_2$ are just the mentioned mappings h_n . Thus each f_n is a homeomorphism of X onto itself. The limit mapping $f : X \rightarrow X$ is such that the partial mappings $f|X_1$ and $f|X_2$ are the limit mapping h taken from the above quoted Example in [21].

To see that f is not semi-confluent over locally connected continua, let $b_1 = (\frac{1}{2}, 0)$ and $b_2 = (\frac{3}{2}, 0)$ and $Q = b_1b_2 \subset pq$. Then the components A and B of $f^{-1}(Q)$ containing the points a_1 and b_1 , respectively, are such that $f(A) = b_1c$ and $f(B) = cb_2$, so neither of the images is contained in the other one, and thereby f is not semi-confluent over the locally connected continuum Q . The proof is complete. \square

Thus the following statement is a consequence of Example (3.1).

Statement (3.2). *The classes of mappings confluent (or semi-confluent) over locally connected continua do not have the limit property, even in the realm of mappings between arcwise connected continua in the plane.*

However, we will show in the next section that the class of confluent mappings have the weak limit property.

4. The weak limit property

Observe that the (range) continuum in Example (3.1) is not only non-locally connected, but moreover, it does not have the property of Kelley. Recall that a continuum Y is said to have *the property of Kelley* (see e.g. [5], p. 167) provided that for each $\varepsilon > 0$ there is a $\delta > 0$ such that if $a, b \in Y$ with $d(a, b) < \delta$ and A is a subcontinuum of Y with $a \in A$, then there exists a subcontinuum B of Y with $b \in B$ and $\text{dist}(A, B) < \varepsilon$ (where d stands for the metric in Y and dist means the Hausdorff distance, see [8], §21, VII, p. 214). In [18], Theorem 3.1, p. 570 Nadler has shown that if Y has the property of Kelley, then the set of all surjective and confluent mappings from X onto Y is a closed subset

of Y^X . Since each locally connected continuum has the property of Kelley, [5], Example 20.4, p. 167, it is tempting to prove a similar result for the class of surjective mappings which are confluent over locally connected continua. As we will see in Example (4.6), property of Kelly is not enough for proving such a result, however a weaker version can be obtained if we assume a stronger hypothesis on Y (see Corollary (4.3)).

For compact spaces X and Y define

$$\Phi_c = \{f \in Y^X : f \text{ is surjective and confluent over locally connected continua}\},$$

and

$$\Phi_{sc} = \{f \in Y^X : f \text{ is surjective and semi-confluent over locally connected continua}\}.$$

Definition (4.1). A metric continuum Y is said to have the *arc approximation property* provided that for each subcontinuum B of Y and for each point $p \in B$ there exists a sequence of arcwise connected subcontinua B_n of Y such that $p \in B_n$ for each $n \in \mathbb{N}$ and $B = \text{Lim } B_n$.

Notice that in Definition (4.1) it is possible to change arcwise connected continua for finite trees. Observe that continua having the property of Kelley do not necessarily have the arc approximation property. For example, the pseudo-arc, being a hereditarily indecomposable continuum, has the property of Kelley, [5], Example 20.6, p. 168, but none of its nondegenerate subcontinua is approximated by arcwise connected ones.

THEOREM (4.2). *Let X and Y be continua. Suppose that Y has the arc approximation property. Then $\Phi_c = \{f \in Y^X : f \text{ is surjective and confluent}\}$.*

Proof. Let $f \in \Phi_c$. We have to show that f is confluent. Let K be a subcontinuum of Y and let C be a component of $f^{-1}(K)$. We show that $f(C) = K$. Note that

$$(4.2.1) \quad f(C) \subset K$$

Fix a point $p \in C$. Then $f(p) \in K$. Since Y has the arc approximation property, there exist a sequence B_n of finite trees in Y such that $\text{Lim } B_n = K$ and $f(p) \in B_n$ for each $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ let C_n be the component of $f^{-1}(B_n)$ such that

$$(4.2.2) \quad p \in C_n.$$

Then by confluence of f over locally connected continua we conclude that

$$(4.2.3) \quad f(C_n) = B_n \text{ for each } n \in \mathbb{N}.$$

Without loss of generality (by going to a subsequence if necessary) we may assume that the sequence $\{C_n : n \in \mathbb{N}\}$ converges and put

$$(4.2.4) \quad C_0 = \text{Lim } C_n.$$

By (4.2.2) we get

$$(4.2.5) \quad p \in C_0.$$

Further, (4.2.4) and the continuity of f imply

$$(4.2.6) \quad f(C_0) = \text{Lim } f(C_n).$$

Therefore, by (4.2.3), $f(C_0) = \text{Lim } B_n$. Hence

$$(4.2.7) \quad f(C_0) = K,$$

which implies

$$(4.2.8) \quad C_0 \subset f^{-1}(K).$$

It follows from (4.2.4) by [9], §47, II, Theorem 6, p. 171 that C_0 is connected. Therefore, since C is a component of $f^{-1}(K)$, it follows from (4.2.5) and (4.2.8) that $C_0 \subset C$. Hence by (4.2.7) it follows that $f(C) = K$, which finishes the proof. \square

COROLLARY (4.3). *Let X and Y be continua. Suppose that Y has the arc approximation property and Y has the property of Kelley. Then Φ_c is a closed subspace of Y^X .*

Since locally connected continua have the property of Kelley (see [5], Example 20.4, p. 167) and since locally connected continua have the arc approximation property (which can be easily seen), hence the following corollary is a direct consequence of Theorem (4.2).

COROLLARY (4.4). *If a continuum Y is locally connected, then Φ_c is a closed subspace of Y^X .*

It can be seen that the proof of Theorem (4.2) cannot be applied for Φ_{sc} instead of Φ_c , so the following question arises naturally.

Question (4.5). Is Corollary (4.3) true for the space Φ_{sc} ?

The following example shows that the hypothesis that Y has the arc approximation property cannot be replaced by the weaker condition that each subcontinuum of Y is limit of locally connected subcontinua of Y .

Example (4.6). There exist a metric continuum X and a sequence of mappings $f_n : X \rightarrow X$ such that the limit mapping $f = \text{Lim } f_n : X \rightarrow X$ is not confluent over locally connected continua and

(4.6.1) each f_n is confluent over locally connected continua,

(4.6.2) X has the property of Kelley,

(4.6.3) for each subcontinuum B of X there exists a sequence of locally connected subcontinua B_r of X such that $B = \text{Lim } B_r$.

Proof. For each $k \in \mathbb{N}$, let $W_k = \{\frac{1}{k}\} \times [-\frac{1}{k}, \frac{1}{k}]$, W_k is considered as a subspace of the Euclidean plane \mathbb{R}^2 . Let S_k be a subspace of the convex hull in \mathbb{R}^2 of the set $W_k \cup W_{k+1}$ such that S_k is homeomorphic to the real line \mathbb{R} , for each point $p \in S_k$, p separates S_k in two components U and V and the sets $\{p\} \cup U \cup W_{k+1}$ and $\{p\} \cup V \cup W_k$ are homeomorphic to the $\sin(\frac{1}{x})$ -curve. Let

$$Y = \{(0, 0)\} \cup (\cup \{W_k \cup S_k : k \in \mathbb{N}\}).$$

Note that Y is a continuum with the property of Kelley.

For each $m \in \mathbb{N}$, let $Y_m = \{(x + 1, \frac{1}{2^{m-1}} + \frac{y}{2^{m+2}}) \in \mathbb{R}^2 : (x, y) \in Y\}$. Notice that each Y_m is homeomorphic to Y , Y_1, Y_2, \dots are pairwise disjoint and $\text{lim } Y_m$

is the convex segment joining the points $(1, 0)$ and $(2, 0)$. Let J_m be the convex segment joining the points $(0, 0)$ and $(1, \frac{1}{2^{m-1}})$ in \mathbb{R}^2 . Let J be the convex segment joining the points $(0, 0)$ and $(2, 0)$ in \mathbb{R}^2 . Define

$$Z = J \cup \left(\bigcup \{J_m \cup Y_m : m \in \mathbb{N}\} \right).$$

Clearly, Z is a continuum with the property of Kelley.

For each $m \in \mathbb{N}$, consider a sequence of arcs $\{L_r^{(m)}\}_{r=1}^{\infty}$ with the following properties

$$(4.6.4) \quad L_r^{(m)} \cap L_s^{(m)} = \{(0, 0)\}, \text{ if } r \neq s;$$

$$(4.6.5) \quad L_r^{(m)} \cap (J \cup J_m \cup Y_m) = \{(0, 0)\};$$

$$(4.6.6) \quad \text{Lim}_{r \rightarrow \infty} L_r^{(m)} = J_m \cup Y_m;$$

(4.6.7) the continuum $X_m = J_m \cup Y_m \cup (\bigcup \{L_r^{(m)} : r \in \mathbb{N}\})$ has the property of Kelley and each one of its subcontinua is the limit of a sequence of some of its locally connected subcontinua;

$$(4.6.8) \quad \text{if } m_1 \neq m_2, \text{ then } X_{m_1} \cap X_{m_2} = \{(0, 0)\};$$

$$(4.6.9) \quad \text{Lim } X_m = J;$$

$$(4.6.10) \quad L_1^{(1)} \text{ is a convex segment.}$$

Thus the continuum X defined as

$$X = J \cup (\bigcup \{X_m : m \in \mathbb{N}\})$$

satisfies properties (4.6.2) and (4.6.3).

Consider an onto mapping $g : J_1 \cup Y_1 \cup (\bigcup \{L_r^{(1)} : r \in \{2, 3, \dots\}\}) \rightarrow X_1$ such that $g|(J_1 \cup Y_1)$ is the identity map on $J_1 \cup Y_1$ and g maps $L_r^{(1)}$ homeomorphically onto $L_{r-1}^{(1)}$.

For each $n \in \mathbb{N}$, let $f_n : X \rightarrow X$ be given by the following properties:

$$(4.6.11) \quad f_n|(J \cup (\bigcup \{X_m : m \in \{2, 3, \dots\}\})) \text{ is the identity map on the set } J \cup (\bigcup \{X_m : m \in \{2, 3, \dots\}\});$$

$$(4.6.12) \quad f_n|(J_1 \cup Y_1 \cup (\bigcup \{L_r^{(m)} : r \in \{2, 3, \dots\}\})) = g;$$

$$(4.6.13) \quad f_n \text{ sends } L_1^{(1)} \text{ linearly onto } J_n.$$

It is easy to check that each f_n is confluent over locally connected continua and the sequence f_n converges uniformly to the mapping $f : X \rightarrow X$ which satisfies the corresponding properties to (4.6.11), (4.6.12) and

$$(4.6.14) \quad f \text{ sends } L_1^{(1)} \text{ linearly onto the convex segment which joins the points } (0, 0) \text{ and } (1, 0).$$

Let B be the convex segment which joins the points $(1, 0)$ and $(2, 0)$. Then the component C of $f^{-1}(B)$ contained in $L_1^{(1)}$ is a one-point set and $f(C) = \{(1, 0)\} \neq B$. Therefore, f is not confluent over locally connected continua. \square

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INSTITUTO DE MATEMÁTICAS, UNAM
 CIUDAD UNIVERSITARIA
 MÉXICO, D. F. 04510
 MÉXICO
 jjc@matem.unam.mx

MATHEMATICAL INSTITUTE
 UNIVERSITY OF WROCLAW
 PL. GRUNWALDZKI 2/4, 50-384
 WROCLAW
 POLAND

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HEREDITARILY INDECOMPOSABLE CONTINUA WITH FINITELY MANY CONTINUOUS SURJECTIONS

ELŻBIETA POL

ABSTRACT. For every positive integer n we construct a one-dimensional hereditarily indecomposable continuum X_n which has exactly n continuous surjections onto itself. Moreover, we construct a collection of cardinality 2^{\aleph_0} consisting of continua of this type, such that no two different elements of this family are comparable either by continuous mappings or by embeddings. We show also that for every $m \in N \cup \{\infty\}$ there exists a family $\{Z_s : s \in S\}$ of cardinality 2^{\aleph_0} consisting of hereditarily indecomposable m -dimensional continua such that if $f : Z_s \rightarrow Z_t$ is a continuous surjection then $s = t$ and f is the identity. The essential role in our constructions is played by a continuum constructed by H. Cook.

1. Introduction

Our terminology follows [6] and [9]. The symbol $X \stackrel{\text{top}}{\cong} Y$ means that X is homeomorphic to Y . By the *dimension* we understand the covering dimension \dim and by a *continuum* we mean a compact connected space (we assume that all our spaces are metrizable and separable). A continuum X is *hereditarily indecomposable*, abbreviated HI, if for any two intersecting subcontinua K, L of X , either $K \subset L$ or $L \subset K$. A subcontinuum K of a continuum X is *terminal*, if every subcontinuum of X which intersects both K and its complement must contain K . A continuous mapping from a continuum X onto Y is called *atomic*, if every fiber of f is a terminal subcontinuum of X .

We say that continua X and Y are *comparable by continuous mappings* (respectively, *by embeddings*), if there is a continuous mapping from one of these continua onto the other (respectively, if there is an embedding of one of these continua into the other). In [5] H. Cook constructed an HI continuum H such that no two different non-trivial subcontinua of X are comparable by continuous mappings. He also constructed, for every $n \in N$, a continuum H_n such that there exist n , and only n , continuous mappings of H_n onto H_n , each of them being a homeomorphism. The continuum H_n is decomposable and admits an atomic mapping onto a simple closed curve.

In this note we show that for every positive integer n there exists an HI one-dimensional continuum X_n which has exactly n continuous mappings onto itself, each of them being a homeomorphism. The continuum X_n admits an atomic mapping onto the pseudo-arc.

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The construction of X_n is taken from the paper [15], where we have given examples of HI continua (of arbitrary given dimension), whose groups of autohomeomorphisms are cyclic groups of order n , for every given $n \in N$. The proof that every surjection of X_n onto X_n is a homeomorphism needs a more subtle reasoning than the one in [15] and uses the fact that every mapping of a continuum onto an HI continuum is confluent.

We will show also that there exists a collection of cardinality 2^{\aleph_0} consisting of continua of this type, such that no two different elements of this family are comparable either by continuous mappings or by embeddings.

In section 4 we will prove that for every $m \in N \cup \{\infty\}$ there exists an m -dimensional HI continuum Z_m such that the identity is the only continuous surjection of Z_m onto itself. As we will see, such a continuum can be obtained by “replacing” one point of the Cook continuum H by an HI m -dimensional Cantor manifold. Moreover, we will construct a family of 2^{\aleph_0} continua of this type, such that no two elements of this family are comparable either by continuous mappings or by embeddings.

2. Preliminaries

The first HI continuum, now called the pseudo-arc, was constructed by B. Knaster [7] in 1922. The *pseudo-arc*, which will be denoted by P , is an HI one-dimensional chainable continuum (unique, up to a homeomorphism —see [3]); and is the only (up to homeomorphism) non-degenerate, homogeneous, chainable continuum, see [4]. Recall that a continuum X is *chainable* if for every $\epsilon > 0$ it can be mapped onto the interval by a mapping with fibers of diameter $< \epsilon$. We will often use the fact that every non-trivial subcontinuum of P is homeomorphic to P (cf. [9], §48, X or [11]). Our construction uses also a theorem of W. Lewis [10] stating that for each $n \in N$ there exists an embedding of the pseudo-arc P in the plane such that the restriction r of a period n rotation of the plane around $(0, 0)$ to P is a homeomorphism of P onto P of period n .

The *composant* of a point x in a continuum X is the union of all proper subcontinua of X containing x . If X is a non-trivial HI continuum, then X has 2^{\aleph_0} different composants, which are pairwise disjoint and are connected F_σ -subsets of X , both dense and co-dense in X (see [9], §48, VI).

LEMMA (2.1) (H. Cook [5]). *There exists a one-dimensional HI continuum H such that for any two different nontrivial subcontinua of H , there is no continuous mapping from one onto another.*

In particular, the identity is the only surjection of H onto H .

Note that T. Maćkowiak proved in [14], (4.1) that if a continuum X has the property that no two nondegenerate disjoint subcontinua of X are comparable by continuous mappings, then the dimension of X is not greater than 2 (it seems to be an open question whether there exists a 2-dimensional continuum of this kind). As was observed by J. Krzempek [8], Proposition 6.4, every HI continuum X with this property must be 1-dimensional.

There exist, however, HI continua X of arbitrarily large dimension such that the identity is the only continuous surjection from X onto itself. Examples of such continua will be given in Section 4.

Recall that a mapping $f : X \rightarrow Y$ from a space X onto Y is *confluent* if for each subcontinuum Q of Y each component of $f^{-1}(Q)$ is mapped onto the whole Q under f . As proved by Cook in [5], Th. 4, each mapping of a continuum onto a hereditarily indecomposable continuum is confluent.

As observed by Cook in the note after Theorem 12 in [5], each non-degenerate subcontinuum of the continuum H from Lemma (2.1) (denoted by M_1 in [5]) contains a continuum which can be mapped onto a poly-adic solenoid and no plane continuum can be mapped onto a poly-adic solenoid. Since every mapping of a continuum into H is confluent, this immediately implies that H contains only degenerate continuous images of plane continua (see also [19], where this fact is explicitly stated). In particular,

LEMMA (2.2). *If $f : P \rightarrow H$ is a continuous mapping of the pseudo-arc P into Cook's continuum H from Lemma (2.1), then f is a constant mapping.*

Remark (2.3). *Let us note that Lemma (2.2) follows also from the following fact:*

(*) *Every weakly chainable hereditarily indecomposable continuum X is continuously homogeneous; i.e., for every $x, y \in X$ there is a continuous surjection $p : X \rightarrow X$ such that $p(x) = y$.*

Indeed, a continuum X is *weakly chainable* if it is the image of the pseudoarc P under some continuous mapping $f : P \rightarrow X$. Since X is HI, by a theorem of D. Bellamy (see [1]) there exists a continuous mapping $g : X \rightarrow P$ of X onto the pseudo-arc P . To prove that X is continuously homogeneous, take two points x and y of X . We will find a continuous surjection $p : X \rightarrow X$ such that $p(x) = y$. Take $y' \in P$ such that $f(y') = y$. Since P is homogeneous, there is a homeomorphism h of P onto P such that $h(g(x)) = y'$. Then $f(h(g(x))) = f(y') = y$, so we can take $f \circ h \circ g$ as p .

Let us note that if X is an HI continuum which is homogeneous with respect to continuous mappings then $\dim X \leq 1$ (see [14]).

Note also that it is not known if the non-degenerate confluent image of the pseudo-arc is chainable (equivalently, if it is homeomorphic to the pseudo-arc); cf. [11], Question 4.17.

3. An HI continuum with exactly n continuous surjections onto itself

We will show the existence of the spaces X_n described in the Introduction.

THEOREM (3.1). *For every positive integer n there exists a hereditarily indecomposable continuum X_n of dimension 1 which has exactly n continuous surjections onto itself. Moreover,*

- 1) X_n admits an atomic mapping onto the pseudo-arc P , and
- 2) each of the surjections of X_n onto itself is a homeomorphism and they form a cyclic group of order n .

Proof. We will show that for a fixed integer $n \in \mathbb{N}$, the 1-dimensional HI continuum $X_{n,1}$ constructed in [15] has the required properties. We have proved there that the group of autohomeomorphisms of $X_{n,1}$ is the cyclic group of order n . Below we will show, using a more subtle reasoning than the one in [15],

that every continuous surjection of X_{n1} onto itself is equal to one of these autohomeomorphisms.

Let us recall now the most important properties of X_{n1} without repeating its construction.

Let us take a pseudo-arc $P \subset R^2$ and a homeomorphism $r : P \rightarrow P$ of period n , which is the restriction of the period n rotation of the plane around $(0, 0)$ (see [10]). Let $P_0 = \{(x_1, x_2) \in P : x_1 = \lambda \cos \alpha \text{ and } x_2 = \lambda \sin \alpha \text{ for some } 0 < \lambda < \infty \text{ and } 0 < \alpha < \frac{2\pi}{n}\}$, and $P_k = r^k(P_0)$ for $k = 0, 1, \dots, n-1$. Let $A_0 = \{a_1, a_2, \dots\}$ be a countable dense subset of P_0 such that a_i and a_j are in the same component of P if and only if $i = j$. Put $A_k = r^k(A_0)$ for $k = 0, 1, \dots, n-1$ and let $A = \bigcup_{k=0}^{n-1} A_k$. The set A is dense in P . Finally, let K_1, K_2, \dots be a sequence of disjoint non-trivial subcontinua of Cook's continuum H from Lemma (2.1).

Given such P, r, A and $\{K_i\}_{i=1}^\infty$ we have constructed in [15] an HI continuum X_{n1} (which will be denoted in the sequel by X_n), an atomic mapping $\tilde{p} : X_n \rightarrow P$ onto P and a homeomorphism $\tilde{r} : X_n \rightarrow X_n$ of period n such that

$$(3.2) \quad \tilde{p} \circ \tilde{r} = r \circ \tilde{p} \text{ (and consequently, } \tilde{p} \circ \tilde{r}^k = r^k \circ \tilde{p} \text{ for } k = 0, 1, \dots, n-1),$$

$$(3.3) \quad \tilde{p} | \tilde{p}^{-1}(P \setminus A) \text{ is a homeomorphism of } \tilde{p}^{-1}(P \setminus A) \text{ onto } P \setminus A,$$

(3.4) for each $i \in N$ and each $k = 0, 1, \dots, n-1$, the set $\tilde{p}^{-1}(r^k(a_i))$, which is equal to $\tilde{r}^k(\tilde{p}^{-1}(a_i))$, is homeomorphic to K_i ,

$$(3.5) \quad \tilde{p}^{-1}(P \setminus A) \text{ is dense in } X_n.$$

Note that from the upper-semi-continuity of \tilde{p}^{-1} it follows that if a sequence $\{b_i\} \subset A$ converges to $b \in P \setminus A$, then $\tilde{p}^{-1}(b) = \text{Lim } \tilde{p}^{-1}(b_i)$. Indeed, $\tilde{p}^{-1}(b)$ is a one-point set and $\text{Ls } \tilde{p}^{-1}(b_i) \subset \tilde{p}^{-1}(b)$ (see [9]). In particular,

(3.6) if a sequence $\{b_i\} \subset A$ converges to $b \in P \setminus A$, and if $x_i \in \tilde{p}^{-1}(b_i)$ for every $i \in N$, then the sequence $\{x_i\}$ converges to the point $\tilde{p}^{-1}(b)$.

Roughly speaking, for every $i \in N$ we have "replaced" points $a_i, r(a_i), \dots, r^{n-1}(a_i)$ of the pseudo-arc P by continua $\tilde{p}^{-1}(a_i), \tilde{p}^{-1}(r(a_i)), \dots, \tilde{p}^{-1}(r^{n-1}(a_i))$, which are all copies of K_i , so that for every $k = 0, 1, \dots, n-1$, $\tilde{r}^k | \tilde{p}^{-1}(a_i) : \tilde{p}^{-1}(a_i) \rightarrow \tilde{p}^{-1}(r^k(a_i))$ is a (unique) homeomorphism between two copies of K_i . At the same time, $P \setminus A$ is "replaced" by its topological copy $\tilde{p}^{-1}(P \setminus A)$.

Note that every component of P contains at most n points of A . Since \tilde{p} is an atomic mapping, every component of X_n is equal to $\tilde{p}^{-1}(L)$ for some component L of P (see [17], Lemma 2.8), so every component of X_n contains at most n copies of the continua from the family $\{K_1, K_2, \dots\}$.

We will show that every continuous mapping $f : X_n \rightarrow X_n$ is equal to \tilde{r}^k for some $k = 0, 1, \dots, n-1$, so it is an autohomeomorphism.

First we will prove that

(3.7) for every $a \in A$ there is a non-trivial subcontinuum C of $\tilde{p}^{-1}(a)$ and $l \in \{0, 1, \dots, n-1\}$ such that $f(\tilde{r}^l(C)) = C$.

Fix $a \in A$. Then $\tilde{p}^{-1}(a)$ is homeomorphic to some continuum $K_i \subset H$. Denote $\tilde{p}^{-1}(a)$ by \tilde{K}_i . Since f is confluent, then there exists a (non-trivial) subcontinuum T of X_n such that $f(T) = \tilde{K}_i$. Since T is a proper subcontinuum of X_n , then T is contained in some component of X_n . By (3.3), each proper subcontinuum of X_n disjoint with $\tilde{p}^{-1}(A)$ is embeddable into P and thus is homeomorphic to P ; therefore, by Lemma (2.2), it cannot be mapped onto \tilde{K}_i .

Thus T intersects at least one (and at most n) of the continua of the form $\tilde{p}^{-1}(b)$, for $b \in A$. Denote these continua by $\tilde{p}^{-1}(b_1), \dots, \tilde{p}^{-1}(b_k)$, where $b_i \in A$ and $k \leq n$. There exists $j_0 \in \{0, 1, \dots, k\}$ such that $f(T \cap \tilde{p}^{-1}(b_{j_0}))$ is non-degenerate.

Indeed, in the opposite case one would find a non-trivial subcontinuum K of \tilde{K}_i disjoint from $\bigcup_{j=1}^k f(T \cap \tilde{p}^{-1}(b_j))$ and a non-trivial subcontinuum K' of T such that $f(K') = K$. Since $K' \cap \bigcup_{j=1}^k (T \cap \tilde{p}^{-1}(b_j)) = \emptyset$, then $K' \subset \tilde{p}^{-1}(P \setminus A)$ and K' is homeomorphic to P . Since K embeds in Cook's continuum H , we get a contradiction with Lemma (2.2).

Since either $T \subset \tilde{p}^{-1}(b_{j_0})$ or $T \supset \tilde{p}^{-1}(b_{j_0})$, $T \cap \tilde{p}^{-1}(b_{j_0})$ is a non-trivial subcontinuum of $\tilde{p}^{-1}(b_{j_0})$ and $\tilde{p}^{-1}(b_{j_0}) \stackrel{\text{top}}{=} K_j$ for some $j \in N$. Also, $C = f(T \cap \tilde{p}^{-1}(b_{j_0}))$ is a subcontinuum of the continuum $\tilde{K}_i \stackrel{\text{top}}{=} K_i$. It follows that $j = i$ and $b_{j_0} = r^l(a_i)$ for some $l \in \{0, 1, \dots, n-1\}$, because C and $T \cap \tilde{p}^{-1}(b_{j_0})$ must be topological copies of the same subcontinuum of the continuum $K_i \subset H$. It follows also that $T \cap \tilde{p}^{-1}(b_{j_0}) = \tilde{r}^l(C)$ and thus $C = f(\tilde{r}^l(C))$. This completes the proof of (3.7).

Observe that $f \circ \tilde{r}^l$ is the identity on C in (3.7). By choosing $x(a)$ to be a point of C , we get the result that

(3.8) for every $a \in A$ there is a point $x(a) \in \tilde{p}^{-1}(A)$ such that $f(\tilde{r}^l(x(a))) = x(a)$ for some $l \in \{0, 1, \dots, n-1\}$.

Let us note that

(3.9) the set $Y = \{x(a) : a \in A\}$ is dense in X_n .

Indeed, let $x \in \tilde{p}^{-1}(P \setminus A)$. Since A is dense in P there exists a sequence $\{a_j\} \subset A$ such that $a_j \rightarrow \tilde{p}(x)$. For every $j \in N$ let $x(a_j) \in \tilde{p}^{-1}(a_j)$ be as in (3.8). Then $x(a_j) \rightarrow x = \tilde{p}^{-1}(\tilde{p}(x))$ by (3.6), so $x \in \text{cl} Y$. It follows that $\tilde{p}^{-1}(P \setminus A) \subset \text{cl} Y$. Since $\tilde{p}^{-1}(P \setminus A)$ is dense in X_n , $X_n = \text{cl} Y$, which finishes the proof of (3.9).

We will show that

(3.10) for every $x \in X_n$ there is $l \in \{0, 1, \dots, n-1\}$ such that $f(\tilde{r}^l(x)) = x$.

Indeed, by (3.9), there exists a sequence $\{x(a_j)\} \subset Y$ converging to x . Replacing $\{x(a_j)\}$ by its subsequence, we can assume that there exists $l \in \{0, 1, \dots, n-1\}$ such that $f(\tilde{r}^l(x(a_j))) = x(a_j)$ for every j . Then $f(\tilde{r}^l(x(a_j))) \rightarrow f(\tilde{r}^l(x))$ and $\{x(a_j)\} \rightarrow x$, so $f(\tilde{r}^l(x)) = x$. It follows that

(3.11) for every $x \neq (0, 0)$ there exists exactly one $k \in \{0, 1, \dots, n-1\}$ such that $f(x) = \tilde{r}^k(x)$.

Indeed, the set $Y(x) = \bigcup_{i=1}^{\infty} \tilde{r}^i(x)$ has n elements and every point of $Y(x)$ is the image of a point from $Y(x)$, so $f \upharpoonright Y(x) : Y(x) \rightarrow Y(x)$ is one-to-one.

For every $k \in \{0, 1, \dots, n-1\}$ let $X_n(k) = \{x \in X_n : f(x) = \tilde{r}^k(x)\}$. It is easy to see that every $X_n(k)$ is closed in X_n . Since $r((0, 0)) = (0, 0)$, we have $\tilde{r}\tilde{p}^{-1}((0, 0)) = \tilde{p}^{-1}((0, 0))$ and $\tilde{p}^{-1}((0, 0)) \in X_n(k)$ for every $k \in \{0, 1, \dots, n-1\}$. Thus $X_n(k) \cap X_n(l) = \tilde{p}^{-1}((0, 0))$ for $k \neq l, k, l \in \{1, 2, \dots, n-1\}$.

Every $X_n(k)$ is a continuum. Indeed, if $X_n(k)$ is the union of two disjoint closed subsets F_1 and F_2 , with F_2 containing the point $\tilde{p}^{-1}((0, 0))$, then X_n is the union of two disjoint closed subsets F_1 and $F_2 \cup \bigcup_{l \neq k} X_n(l)$. Thus $F_1 = \emptyset$, since X_n is a continuum.

Since X_n is HI, we have $X_n(k) = X_n$ for some k . Thus $f = \tilde{r}^k$ for some $k \in \{1, 2, \dots, n-1\}$.

This finishes the proof of the fact that the set of all continuous mappings from X_n onto X_n is equal to the set $\{\tilde{r}^0, \tilde{r}^1, \dots, \tilde{r}^{n-1}\}$ and is the cyclic group of order n . \square

THEOREM (3.2). *There exists a family $\{X_n^s : s \in S\}$, where S is a set of cardinality 2^{\aleph_0} , of topologically different HI one-dimensional continua such that every X_n^s has exactly n continuous surjections onto itself and admits an atomic mapping p_s onto the pseudo-arc P . Moreover, no two different continua from this family are comparable either by continuous mappings or by embeddings.*

Proof. Let S be a set of cardinality 2^{\aleph_0} and H be Cook's continuum from Lemma (2.1). For every $s \in S$ choose a sequence $\{K_1^s, K_2^s, \dots\}$ of non-trivial subcontinua of H in such a way that $K_i^s \cap K_j^t = \emptyset$ if $s \neq t$ or $i \neq j$. If we replace in the proof of Theorem (3.1) a sequence K_1, K_2, \dots by a sequence $\{K_1^s, K_2^s, \dots\}$, then we obtain an HI continuum X_n^s with exactly n continuous surjections onto itself, which admits an atomic mapping $p_s : X_n^s \rightarrow P$. We will show that the family $\{X_n^s : s \in S\}$ satisfies the required conditions.

From the construction it follows that X_n^s is the union of a set homeomorphic to a subset of P and of countably many continua from the family \mathcal{K}_s which contains exactly n copies of every continuum K_i^s , for every $i \in N$.

Suppose that $f : X_n^s \rightarrow X_n^t$ is a continuous surjection. Suppose that $t \neq s$. Then X_n^t contains a subcontinuum \tilde{K}_1^t , homeomorphic to K_1^t . Since f is confluent, there is a proper subcontinuum $T \subset X_n^s$ such that $f(T) = \tilde{K}_1^t$. By Lemma (2.2), T is not homeomorphic to P . It is not contained topologically in any element K_j^s of \mathcal{K}_s , since no nontrivial subcontinuum of K_j^s is comparable to \tilde{K}_1^t . Thus T must contain at least one and at most n continua from the family \mathcal{K}_s , say $\tilde{K}_{i_1}^s, \dots, \tilde{K}_{i_p}^s$, where $\tilde{K}_{i_j}^s \stackrel{\text{top}}{=} K_{i_j}^s$ (since every component of X_n^s contains at most n continua from this family). For some $k \in \{1, \dots, p\}$, $f(\tilde{K}_{i_k}^s)$ is a non-trivial subcontinuum of \tilde{K}_1^t (otherwise, some subcontinuum homeomorphic to P would be mapped onto a nontrivial subcontinuum of \tilde{K}_1^t). But this is impossible, because $\tilde{K}_{i_k}^s$ and \tilde{K}_1^t are homeomorphic to different non-trivial subcontinua of H . This contradiction shows that $s = t$.

We will show now that if $s \neq t$ then X_n^s does not embed in X_n^t . Suppose on the contrary that h is such an embedding. Let $\tilde{K}_1^s \subset X_n^s$ be an element of \mathcal{K}_s homeomorphic to K_1^s . Then \tilde{K}_1^s does not embed in P so $h(\tilde{K}_1^s)$ must intersect some element $\tilde{K}_i^t \in \mathcal{K}_t$ homeomorphic to K_i^t . Thus $h(\tilde{K}_1^s) \subset \tilde{K}_i^t$ or $h(\tilde{K}_1^s) \supset \tilde{K}_i^t$. But this is impossible because these continua are homeomorphic to disjoint subcontinua of the Cook continuum H . \square

4. Higher-dimensional HI continua with only one continuous surjection

The first HI m -dimensional continua, for arbitrary $m \in \{2, 3, \dots, \infty\}$, were constructed by R. H. Bing [2]. Recently M. Reńska [18] has constructed, for every $m = 2, 3, \dots, \infty$, an m -dimensional HI Cantor manifold Y_m such that every autohomeomorphism $h : Y_m \rightarrow Y_m$ is the identity. We shall show that

for every such m there exists an m -dimensional HI continuum which admits exactly one continuous surjection onto itself (i.e., which is rigid with respect to continuous mappings). Moreover, we shall show that there exists a family of cardinality 2^{\aleph_0} consisting of such continua which are pairwise not comparable by continuous mappings (respectively, by embeddings).

It would be interesting to construct such continua which are m -dimensional Cantor manifolds for $m \geq 2$.

We will apply a construction of a pseudosuspension, described in the following theorem of Maćkowiak, which is the special case of Theorem 1.14 from [12] (cf. [13], Th. 15).

LEMMA (4.1) (T. Maćkowiak). *Suppose that X and K are two continua and a is a point of X . Then there exists a continuum $M(X, K, a)$ (called a pseudosuspension of K over X at a point a) which admits an atomic mapping $p : M(X, K, a) \rightarrow X$ onto X such that*

- (i) $p \upharpoonright p^{-1}(X \setminus \{a\}) : p^{-1}(X \setminus \{a\}) \rightarrow X \setminus \{a\}$ is a homeomorphism,
- (ii) the set $p^{-1}(X \setminus \{a\})$ is dense in $L(X, K, a)$,
- (iii) $p^{-1}(a)$ is homeomorphic to K .

THEOREM (4.2). *For every natural number $m \geq 2$ there exists a family $\{Z_s : s \in S\}$, where $|S| = 2^{\aleph_0}$, of hereditarily indecomposable m -dimensional continua such that*

- 1) if $f : Z_s \rightarrow Z_t$ is a continuous surjection for $s, t \in S$ then $s = t$ and f is the identity,
- 2) if $f : Z_s \rightarrow Z_t$ is an embedding for $s, t \in S$ then $s = t$ and f is the identity.

Proof. Fix $m \in \{2, 3, \dots, \infty\}$ and a set S of cardinality 2^{\aleph_0} . As proved in [16], there exists a family $\{M_s : s \in S\}$ of m -dimensional HI Cantor manifolds such that

- (4.3) if $h : M_s \rightarrow M_t$ is an embedding, then $s = t$ and h is the identity.

Let H be Cook's continuum described in Lemma (2.1) and let $\{a_s : s \in S\}$ be a family of points of H such that $a_s \neq a_t$ for $s \neq t$.

For $s \in S$, let $Z_s = M(H, M_s, a_s)$ be a pseudosuspension of M_s over H at a_s and $p_s : Z_s \rightarrow H$ be the natural projection.

Thus $p_s^{-1}(a_s)$ is homeomorphic to M_s and $p_s \upharpoonright p_s^{-1}(H \setminus \{a_s\})$ is a homeomorphism onto $H \setminus \{a_s\}$. Since Z_s is the union of a 1-dimensional open subset $p_s^{-1}(H \setminus \{a_s\})$ and an m -dimensional space $p_s^{-1}(a_s)$, we see Z_s is m -dimensional.

Since p_s is an atomic mapping and H and M_s are HI, we have that Z_s is HI as the preimage of an HI continuum under an atomic mapping with HI fibers (see [12], Proposition 11, (ii)).

We will show that if $s, t \in S$ and $f : Z_s \rightarrow Z_t$ is a continuous surjection, then $s = t$ and f is the identity.

Note that every component of Z_s (resp., Z_t) is the preimage under p_s (resp., p_t) of a component in H (see [17], Lemma 2.8). Let K_0 be a component of H not containing a_s or a_t . Then $p_s^{-1}(K_0)$ is a component of Z_s homeomorphic to K_0 . Since $p_s^{-1}(K_0)$ is dense in Z_s , then $f(p_s^{-1}(K_0))$ is dense in Z_t , hence there exists $x_0 \in K_0$ such that $f(p_s^{-1}(x_0)) \notin p_t^{-1}(a_s) \cup p_t^{-1}(a_t)$. Let $C \subset Z_t$ be a non-trivial continuum containing $f(p_s^{-1}(x_0))$ and disjoint from $p_t^{-1}(a_s) \cup p_t^{-1}(a_t)$. Let C' be a component of $f^{-1}(C)$ containing $p_s^{-1}(x_0)$. Since f is confluent, $f(C') = C$.

Since C' and C are comparable, they are homeomorphic to the same continuum $C_0 \subset K_0 \subset H$ and $C' = p_s^{-1}(C_0)$, $C = p_t^{-1}(C_0)$; moreover, for every $x \in C_0$

$$(4.4) \quad f(p_s^{-1}(x)) = p_t^{-1}(x).$$

Suppose now that x is an arbitrary point of $K_0 \setminus C_0$ and let D_0 be any continuum in K_0 joining x_0 and x .

Let C be a component of the point $p_s^{-1}(x_0)$ in the set $f^{-1}(p_t^{-1}(D_0))$. Since f is confluent, $f(C) = p_t^{-1}(D_0)$. Thus C and D_0 are comparable, so $C = p_s^{-1}(D_0)$ and (4.4) holds for every $y \in D_0$; in particular, for $y = x$. It follows that (4.4) holds for every $x \in K_0$. Since K_0 is dense in H , $f(p_s^{-1}(x)) \subset p_t^{-1}(x)$ for every $x \in H$ and since f is “onto”, $f(p_s^{-1}(x)) = p_t^{-1}(x)$ for every $x \in H$.

Suppose now that $s \neq t$. Then $p_s^{-1}(a_t)$ is degenerate while $p_t^{-1}(a_t)$ is a copy of M_t , so $f(p_s^{-1}(a_t)) \neq p_t^{-1}(a_t)$, contrary to (4.4). Thus $s = t$ and $f \mid p_s^{-1}(H \setminus (\{a_s\} \cup \{a_t\}))$ is the identity. Since $p_s^{-1}(H \setminus (\{a_s\} \cup \{a_t\}))$ is dense in Z_s , we have that f is the identity on the entire Z_s . This ends the proof of 1).

It remains to show 2). Let $h : Z_s \rightarrow Z_t$ be an embedding. Then $\tilde{M}_s = p_s^{-1}(a_s)$ is an m -dimensional Cantor manifold homeomorphic to M_s . Since $p_t^{-1}(H \setminus \{a_t\})$ is 1-dimensional and open in Z_t then $h(\tilde{M}_s)$ must be contained in $\tilde{M}_t = p_t^{-1}(a_t)$. By (4.3), this is possible only if $s = t$ and $h \mid \tilde{M}_s : \tilde{M}_s \rightarrow \tilde{M}_s$ is the identity. Thus $h \mid Z_s \setminus \tilde{M}_s \rightarrow Z_s \setminus \tilde{M}_s$ is an embedding. But $Z_s \setminus \tilde{M}_s$ is homeomorphic to the subspace $H \setminus \{a_s\}$ of the Cook continuum H , so h is the identity on every non-trivial subcontinuum of this space. This implies that h is the identity also on $Z_s \setminus \tilde{M}_s$ and ends the proof of 2). \square

Remark (4.3). Let us note that the property (4.3) of the family $\{M_s : s \in S\}$ is needed to achieve the condition 2). If we want to obtain a family $\{Z_s : s \in S\}$ satisfying only the condition 1), we can take as M_s any m -dimensional HI Cantor manifold (for every $s \in S$).

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF WARSAW
BANACHA 2
02-097 WARSAW
POLAND
pol@mimuw.edu.pl

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THE ABEL-JACOBI THEORY FOR PRODUCT LAMINATIONS BY RIEMANN SURFACES

EMIGDIO MARTÍNEZ-OJEDA

ABSTRACT. We introduce a version of the Abel-Jacobi theory for product laminations by Riemann surfaces.

Introduction

The theory of laminations by Riemann surfaces is starting to play a central role in mathematics as shown by the applications of Y. Minsky and D. Gabai to the topology of three manifolds [9], [4], as well as the applications of D. Sullivan to solve the Feigenbaum conjecture [12]. The paper *Laminations par surfaces de Riemann* [7], is a wonderful survey that gives an account of laminations by Riemann surfaces.

On the other hand, the Abel-Jacobi theory is one of the jewels of mathematics that has interested many of the greatest mathematicians of the nineteenth century, such as Riemann, Weierstrass, Abel, and Jacobi [2], [3], [5].

We recall that the Abel-Jacobi theory starts with the Jacobian variety of a compact Riemann surface X_g of genus $g \geq 1$; which is a pair $(\text{Jac}(X_g), A)$, where $\text{Jac}(X_g) = \frac{H^0(X_g, \mathcal{O}^{(1,0)})^*}{H_1(X_g, \mathbb{Z})} \cong \mathbb{C}^g / \mathbb{Z}^{2g}$ is an Abelian variety (a complex torus) and $A: X_g \rightarrow \text{Jac}(X_g)$ is a holomorphic embedding called the Abel-Jacobi map. Using the divisor group $\text{Div}(X_g)$ and the Jacobi morphism $\varphi: \text{Div}(X_g) \rightarrow \text{Jac}(X_g)$ ones can state the celebrated Abel's theorem. This theorem claims that two divisors $D, D' \in \text{Div}_k(X_g)$ of degree k satisfy $\varphi(D) = \varphi(D')$ if and only if D is linearly equivalent to D' .

The aim of this paper is to extend the beautiful Abel-Jacobi theory to a particular case of compact laminations by Riemann surfaces, namely, *product laminations by Riemann surfaces* which are defined as $\mathcal{L}_g = X_g \times K$, where K is any compact totally disconnected topological Hausdorff space. It is important to note that if K is reduced to a single point, we obtain the classical Abel-Jacobi theory for compact Riemann surfaces.

This paper is organized in the following way. In the first section we shall give the basic definitions of laminations by Riemann surfaces. In the second section we shall start with the de Rham and Dolbeault tangential cohomology, and tangential homology for \mathcal{L}_g in order to prove an analogue to Serre duality

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theorem for product laminations by Riemann surfaces, as well as to define *the Jacobian*

$$\text{Jac}(\mathcal{L}_g) = \frac{H_T^0(\mathcal{L}_g, \mathcal{O}_T^{(1,0)})}{H_1(\mathcal{L}_g, \mathbb{Z})_T}$$

which results in a topological Abelian group. In the third section we shall define the Abel-Jacobi map $\widehat{A} : \mathcal{L}_g \rightarrow \text{Jac}(\mathcal{L}_g)$ which is a tangentially holomorphic embedding.

The notions of divisor, of tangential degree, of group of divisors as well as the principal divisors in product laminations by Riemann surfaces will be clarified at beginning of the fourth section. We shall show that a divisor $D(t) \in \text{Div}(\mathcal{L}_g)$ is a principal divisor if and only if it corresponds to the divisor of some tangential meromorphic function in \mathcal{L}_g . Then we will have a linear equivalence between divisors. The Picard group will be defined in terms of classes of divisors under linear equivalence as well as in terms of holomorphic line bundles. A canonical correspondence between the Jacobian and another subgroup of the Picard group will be given. All of these are described in order to state an analogous theorem to the celebrated Abel theorem for product laminations by Riemann surfaces in the fifth section.

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1. Fundamentals of Laminations by Riemann Surfaces

Let M be a separable, locally compact, metrizable space. We say that M has a *structure of lamination atlas by Riemann surfaces* if there is a cover of M by open sets U_α (called *distinguished open sets*) and homeomorphisms $h_\alpha : U_\alpha \rightarrow \mathbb{D} \times K_\alpha$, where \mathbb{D} is the unit open disk in \mathbb{C} , and, K_α is a topological Hausdorff space, such that the transition maps $h_\alpha \circ h_\beta^{-1}$ are of the form

$$h_{\alpha\beta}(z, t) = (\phi_{\alpha\beta}(z, t), \gamma_{\alpha\beta}(t)),$$

where each $\phi_{\alpha\beta} : \mathbb{D} \times K_\beta \rightarrow \mathbb{D}$ is holomorphic in the z -variable, and, all its partial differentials with respect to the z -variable are continuous functions of all the variables. Thus a separable, locally compact, metrizable space endowed with a structure of lamination atlas by Riemann surfaces will be a *lamination by Riemann surfaces* (compare with [7]).

A *holomorphic vector bundle* $\pi : E \rightarrow M$ of rank p over a lamination by Riemann surfaces M is defined by a space E which has the structure of a $(p+2)$ -foliated space which is compatible with the local tangentially holomorphic product structure of the bundle, and $\pi : E \rightarrow M$ is a tangentially holomorphic map of laminations. Any example of holomorphic line bundle over Riemann surfaces can be extended to a holomorphic line bundle over product laminations by Riemann surfaces. Examples of this are the tangent line bundle $F\mathcal{L}_g$ and the canonical line bundle $F^*\mathcal{L}_g$, which extends the tangent and canonical bundle of Riemann surfaces (see [10] p. 43). A *tangentially smooth (holomorphic) section*

of E is a tangentially smooth (holomorphic) map of laminations $s : M \rightarrow E$ such that $\pi \circ s = 1_M$.

2. The Jacobian

The de Rham tangential cohomology. For each $i = 0, 1, 2$, let $\Omega_T^i(\mathcal{L}_g)$ be the locally convex topological vector space of tangentially smooth i -differential forms in \mathcal{L}_g . The topology is given by the union of the \mathbb{C}^k -topology for all $k \in \mathbb{N}$. Let $d_T : \Omega_T^i(\mathcal{L}_g) \rightarrow \Omega_T^{i+1}(\mathcal{L}_g)$ be the tangential exterior derivative operator. It is well-known (see [10] pp. 70) that there exists the de Rham exact sequence

$$(2.1) \quad 0 \longrightarrow \mathbb{C}_T \longrightarrow \mathcal{C}_T^\infty \longrightarrow \Omega_T^1 \xrightarrow{d_T} \Omega_T^2 \longrightarrow 0,$$

and the i -th de Rham tangential cohomology group of \mathcal{L}_g is given by

$$H^i(\mathcal{L}_g; \mathbb{C})_T = \frac{\text{Ker } d_T : \Omega_T^i(\mathcal{L}_g) \rightarrow \Omega_T^{i+1}(\mathcal{L}_g)}{\text{Im } d_T : \Omega_T^{i-1}(\mathcal{L}_g) \rightarrow \Omega_T^i(\mathcal{L}_g)}.$$

Let $\Omega^i(X_g)$ be the locally convex topological vector space of smooth i -differential forms in X_g , and $\mathcal{C}^0(K, \Omega^i(X_g))$ be the topological vector space of continuous maps. We remark that a canonical correspondence between $\Omega_T^i(\mathcal{L}_g)$ and $\mathcal{C}^0(K, \Omega^i(X_g))$ exists. The following result relates the de Rham tangential cohomology groups of \mathcal{L}_g and the de Rham cohomology groups of X_g . The proof of this result follows from the definitions.

PROPOSITION (2.2). *Let $\mathcal{C}^0(K, H_{DR}^i(X_g))$ be the group of continuous maps from K to the i -th de Rham cohomology group of X_g . Then $H^i(\mathcal{L}_g; \mathbb{C})_T$ is canonically isomorphic to $\mathcal{C}^0(K, H_{DR}^i(X_g))$.*

The Dolbeault tangential cohomology. The tangential exterior derivative d_T can be written as the sum

$$(2.3) \quad d_T = \partial_T + \bar{\partial}_T,$$

where ∂_T (resp. $\bar{\partial}_T$) is the tangential ∂ -operator (resp. $\bar{\partial}$ -operator) in \mathcal{L}_g .

Let us choose some coordinate mapping $h_\alpha = (z_\alpha, t_\alpha)$ in $U(p)$, an open neighbourhood point $p \in \mathcal{L}_g$. Let $V = z_\alpha(U(p)) \subset \mathbb{C} \times K$. We will assume that $\mathbb{D} \times K \subset V$.

LEMMA (2.4). *Let $g \in \mathcal{C}_T^\infty(V)$ and let D be a connected open subset of \mathbb{D} such that $\bar{D} \times K$ is a compact subset of V . Then a function $f \in \mathcal{C}_T^\infty(V)$ such that $\bar{\partial}_T f(z, t) = g(z, t)$ whenever $(z, t) \in D \times K$ exists.*

Proof. It is not hard to prove a ‘‘Stokes theorem for product lamination by Riemann surfaces’’ assuming a probability measure on K , and applying the classical Stokes theorem to X_g . With this theorem in hand, the proof of this lemma is completely analogous to the proof of the classical one (see [5] lemma 6). \square

Definition (2.5). The Dolbeault tangential cohomology group of \mathcal{L}_g (of type (p, q)) is the group $H_{\bar{\partial}_T}^{(p,q)}(\mathcal{L}_g)$ given by

$$(2.6) \quad H_{\bar{\partial}_T}^{(p,q)}(\mathcal{L}_g) = \frac{\text{Ker } \bar{\partial}_T : \Omega_T^{(p,q)}(\mathcal{L}_g) \longrightarrow \Omega_T^{(p,q+1)}(\mathcal{L}_g)}{\text{Im } \bar{\partial}_T : \Omega_T^{(p,q-1)}(\mathcal{L}_g) \longrightarrow \Omega_T^{(p,q)}(\mathcal{L}_g)}.$$

Remark (2.7). From the previous definition it follows that

$$\begin{aligned} H_{\bar{\partial}_T}^{(0,0)}(\mathcal{L}_g) &= \mathcal{O}_T(\mathcal{L}_g) = \mathcal{C}^0(K, \mathcal{O}_{X_g}) = \mathcal{C}^0(K, H_{\bar{\partial}}^{(0,0)}(X_g)), \\ H_{\bar{\partial}_T}^{(1,0)}(\mathcal{L}_g) &= \mathcal{O}_T^{(1,0)}(\mathcal{L}_g) = \mathcal{C}^0(K, \mathcal{O}_{X_g}^{(1,0)}) = \mathcal{C}^0(K, H_{\bar{\partial}}^{(1,0)}(X_g)). \end{aligned}$$

Lemma (2.4) implies the following proposition.

PROPOSITION (2.8) (Dolbeault sequence). *The sequence*

$$(2.9) \quad 0 \longrightarrow \mathcal{O}_T \longrightarrow \Omega_T^{(0,0)} \xrightarrow{\bar{\partial}_T} \Omega_T^{(0,1)} \longrightarrow 0$$

is an exact sequence of sheaves.

We remark that \mathcal{L}_g can be seen as a fibration $X_g \rightarrow \mathcal{L}_g \rightarrow K$. Let us consider the vector bundle $\pi : E \rightarrow K$ given by $E_t = \pi^{-1}(t) = H_{\bar{\partial}_T}^*(X_g)$. Then we have the following result.

THEOREM (2.10) (The Dolbeault Theorem for product laminations by Riemann surfaces). *The Dolbeault tangential cohomology group $H_{\bar{\partial}_T}^{(p,q)}(\mathcal{L}_g)$ is canonically isomorphic to the group of continuous maps from K to the Dolbeault cohomology group $H_{\bar{\partial}}^{(p,q)}(X_g)$.*

Serre's Duality. Let $\overline{F^*\mathcal{L}_g}$ be the anti-canonical line bundle of \mathcal{L}_g and L be a holomorphic line bundle over \mathcal{L}_g . Let U be any distinguished open set in \mathcal{L}_g such that L has a tangentially holomorphic trivialization s . Let $f \in \mathcal{C}_T^\infty(U, L)$; then $(\bar{\partial}_T f \cdot s) \in \mathcal{C}_T^\infty(U, L \otimes \overline{F^*\mathcal{L}_g})$. It is independent of the tangentially holomorphic trivialization chosen. It is not hard to show that $\bar{\partial}_T : \mathcal{C}_T^\infty(U, L) \rightarrow \mathcal{C}_T^\infty(U, L \otimes \overline{F^*\mathcal{L}_g})$ defines a unique first order tangential operator which acts on the smooth sections of L with values in the smooth sections of $L \otimes \overline{F^*\mathcal{L}_g}$. It will be called *the tangential $\bar{\partial}$ -operator with coefficients in L* and will be denoted by $\bar{\partial}_L$.

We remark that $\bar{\partial}_L s' = 0$ if and only if s' is a tangentially holomorphic section of L in U .

PROPOSITION (2.11) (The Dolbeault-Serre sequence). *The sequence*

$$(2.12) \quad 0 \longrightarrow \mathcal{O}_T(L) \longrightarrow \Omega_T^{(0,0)}(L) \xrightarrow{\bar{\partial}_L} \Omega_T^{(0,1)}(L) \longrightarrow 0$$

is an exact sequence of sheaves.

Proof. Consider any distinguished open set $U \subset \mathcal{L}_g$. Then the sequence given in (2.12) restricted to U is reduced to the Dolbeault sequence given in (2.9). Thus we obtain the desired result. \square

The kernel of the $\bar{\partial}_L$ -operator, $H^0(\mathcal{L}_g; L)_T$, will be called *the zero Dolbeault tangential cohomology group of L* ; the cokernel of the $\bar{\partial}_L$ -operator, $H^1(\mathcal{L}_g; L)_T$, will be called *the first Dolbeault tangential cohomology group of L* .

Serre's Duality theorem is one of the most important results for the classical theory of compact Riemann surfaces. It is also valid for product laminations by Riemann surfaces.

THEOREM (2.13) (Serre's Duality for product laminations by Riemann surfaces). *Let L be a holomorphic line bundle over \mathcal{L}_g . Then a canonical isomorphism*

$$H^1(\mathcal{L}_g; L)_T \cong H^0(\mathcal{L}_g; L^* \otimes \overline{F^* \mathcal{L}_g})^*$$

exists.

Before we prove this theorem we need to consider the following. It is well-known ([10] p. 86) that elements in the strong dual of $\Omega_T^i(\mathcal{L}_g)$, i.e., $c \in \Omega_T^i(\mathcal{L}_g) = \text{Hom}_{\text{cont}}(\Omega_T^i(\mathcal{L}_g), \mathbb{C})$, are *i -currents* in de Rham's sense [1]. For convenience of the reader we shall recall the notion of *0-currents* (or *distributions*) in \mathcal{L}_g . Let $\{U_\alpha, h_\alpha\}$ be a finite coordinate covering for \mathcal{L}_g and $h_{\alpha\beta}$ be the transition functions. A *0-current* (or *distribution*) in $h_\alpha(U_\alpha) \cong \mathbb{D} \times K$ is a linear mapping $c : \mathcal{C}_T^\infty(h_\alpha(U_\alpha)) \rightarrow \mathbb{C}$ such that for every compact subset $\Omega \subset U_\alpha$ there are constants $M \in \mathbb{R}^+$ and $n \in \mathbb{N}$ with the property that

$$(2.14) \quad |c(f)| \leq M \sum_{I=i+j \leq n} \text{Sup}_{z \in \Omega} \left| \frac{\partial^{I=i+j} f}{\partial x^i \partial y^j} \right|$$

where $\text{supp}(f) \subset \Omega$. The set of all 0-currents in $h_\alpha(U_\alpha)$ is a linear space which is denoted by $\mathcal{K}_{h_\alpha(U_\alpha)}$.

If $c \in \mathcal{K}_{h_\alpha(U_\alpha)}$, then *the tangential derivatives* of c are defined by $\frac{\partial c}{\partial x}(f) = -c(\frac{\partial f}{\partial x})$ and $\frac{\partial c}{\partial y}(f) = -c(\frac{\partial f}{\partial y})$ for any $f \in \mathcal{C}_T^\infty(h_\alpha(U_\alpha))$. These derivatives are 0-currents. Thus, the tangential partial differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ can be applied to 0-currents.

We note that the tangentially holomorphic maps can be considered as embedded in the space of 0-currents by associating to a tangentially holomorphic map h the 0-current $c_h(f) = \int_{h_\alpha(U_\alpha)} fh$ for any $f \in \mathcal{C}_T^\infty(h_\alpha(U_\alpha))$.

The proof of the following proposition is an adaptation of the classical one when a continuous variation in the transversal K is considered.

PROPOSITION (2.15). *If c is a 0-current on a subset $h_\alpha(U_\alpha) \subset \mathbb{C} \times K$ such that $\frac{\partial c}{\partial \bar{z}} = 0$, then c is a tangentially holomorphic function in $h_\alpha(U_\alpha)$.*

A *0-current* (or *distribution*) c on the coordinate covering $\{U_\alpha, h_\alpha\}$ is defined to be a collection $\{c_\alpha\}$ of 0-currents on the various subsets $h_\alpha(U_\alpha)$ such that for each non-empty intersection $U_\alpha \cap U_\beta \subset \mathcal{L}_g$, we have $h_{\alpha\beta}^*(c_\alpha) = c_\beta$. Here $(h_{\alpha\beta}^* c_\alpha) = c_\alpha[\frac{(f \circ h_{\alpha\beta}^{-1})}{J_{h_{\alpha\beta}}}]$ is a linear map, where $c_\alpha \in \mathcal{K}_{h_\alpha(U_\alpha)}$, $f \in \mathcal{C}_T^\infty(h_\alpha(U_\alpha))$, and $J_{h_{\alpha\beta}}$ is the tangential Jacobian determinant of $h_{\alpha\beta}$. Two 0-currents c and c' on coordinate coverings $\{U_\alpha, h_\alpha\}$ and $\{U'_\alpha, h'_\alpha\}$ are called *equivalent* if they define a 0-current on the union of those coordinate coverings. A *0-current on the product lamination by Riemann surfaces \mathcal{L}_g* is an equivalence class of 0-currents on coordinate coverings of \mathcal{L}_g . *The sheaf \mathcal{K} of germs of 0-currents on*

\mathcal{L}_g is then a well-defined sheaf of Abelian groups. The global sections of the sheaf \mathcal{K} are precisely the 0-currents on \mathcal{L}_g . For any holomorphic line bundle L in \mathcal{L}_g , the corresponding sheaf $\mathcal{K}(L)$ of *germs of distributional cross-sections of the line bundle L* can be constructed in a similar manner to the classical construction of the germs of smooth cross-sections of a holomorphic line bundle over X_g . To simplify the notation, we shall write $\mathcal{K}^{(1,0)}(L) = \mathcal{K}(F^*\mathcal{L}_g \otimes L)$, $\mathcal{K}^{(0,1)}(L) = \mathcal{K}(\overline{F^*\mathcal{L}_g} \otimes L)$ and $\mathcal{K}^{(1,1)}(L) = \mathcal{K}(F^*\mathcal{L}_g \otimes \overline{F^*\mathcal{L}_g} \otimes L)$.

Let us suppose that $\{U_\alpha, h_\alpha\}$ is a finite coordinate covering of \mathcal{L}_g with the property that the mapping h_α can be extended to a \mathcal{C}_T^∞ homeomorphism of an open neighbourhood of \bar{U}_α into $\mathbb{C} \times K$. For any integer $n \geq 0$ and any element $f = \{f_\alpha\} \in \Gamma(\mathcal{L}_g, \mathcal{C}_T^\infty(L))$ set

$$(2.16) \quad P_n(f) = \sum_\alpha \sum_{I=i+j \leq n} \sup_{(z_\alpha, t_\alpha) \in h_\alpha(U_\alpha)} \left| \frac{\partial^{I=i+j} f_\alpha}{\partial x_\alpha^i \partial y_\alpha^j}(z_\alpha, t_\alpha) \right|.$$

It is clear that the functions P_n thus defined on $\Gamma(\mathcal{L}_g, \mathcal{C}_T^\infty(L))$ are norms. Give $\Gamma(\mathcal{L}_g, \mathcal{C}_T^\infty(L))$ the topology defined by this family of norms. In this way it becomes a topological vector space.

LEMMA (2.17). *On any product lamination by Riemann surfaces \mathcal{L}_g the space $\Gamma(\mathcal{L}_g, \mathcal{K}^{(1,1)}(L^{-1}))$ is equal to the space of continuous linear functionals on the topological vector space $\Gamma(\mathcal{L}_g, \mathcal{C}_T^\infty(L))$.*

Proof. If $T = \{T_\alpha\} \in \Gamma(\mathcal{L}_g, \mathcal{K}^{(1,1)}(L^{-1}))$ then $T(\varphi) = \sum_\alpha T_\alpha(r_\alpha \varphi)$ determines a continuous linear functional on $\Gamma(\mathcal{L}_g, \mathcal{C}_T^\infty(L))$ for any partition of unity $\{r_\alpha\}$. Conversely, if T is a continuous linear functional on $\Gamma(\mathcal{L}_g, \mathcal{C}_T^\infty(L))$, take $T_\alpha(f) = T(f)$ for the continuous extension of any $f \in \mathcal{C}_T^\infty(U_\alpha)$. It is possible to have a well-defined $\{T_\alpha\} \in \Gamma(\mathcal{L}_g, \mathcal{K}^{(1,1)}(L^{-1}))$ in such a way that for any partition of unity $\{r_\alpha\}$ the equality $T(f) = \sum_\alpha T_\alpha(r_\alpha f)$ holds. So T is identified with the section $\{T_\alpha\} \in \Gamma(\mathcal{L}_g, \mathcal{K}^{(1,1)}(L^{-1}))$. \square

Let us consider the complex vector space $A = \Gamma(\mathcal{L}_g, \Omega_T^{(0,0)}(L))$ as well as $B = \Gamma(\mathcal{L}_g, \Omega_T^{(0,1)}(L))$, and the homomorphism $\bar{\partial}_L : A \rightarrow B$. It follows from the definition of the first Dolbeault tangential cohomology group of L that $H^1(\mathcal{L}_g, \mathcal{O}_T(L))_T \cong B/\bar{\partial}_L(A)$. Consider both of the spaces A and B with the norms P_n given by (2.16). These norms make A and B into Fréchet spaces. It follows that $\bar{\partial}_L : A \rightarrow B$ is a continuous map in terms of this topology. It can be proved that $\bar{\partial}_L(A) \subset B$ is a closed subspace. Then $B/\bar{\partial}_L(A)$ is a Fréchet space. Thus any continuous linear functional on the quotient $B/\bar{\partial}_L(A)$ leads to a continuous linear functional on B which vanishes on the subspace $\bar{\partial}_L(A)$.

Let A^* and B^* be the dual spaces to A and B respectively and let $\bar{\partial}_L^* : B^* \rightarrow A^*$ be the dual homomorphism to $\bar{\partial}_L$. We remark that the kernel $\Omega^* \subset B^*$ of this homomorphism is the dual space to $B/\bar{\partial}_L(A)$.

Now we are in position to prove Serre's Duality theorem.

Proof. It follows from lemma (2.17) that $A^* \cong \Gamma(\mathcal{L}_g, \mathcal{K}^{(1,1)}(L^{-1}))$ and $B^* \cong \Gamma(\mathcal{L}_g, \mathcal{K}^{(1,0)}(L^{-1}))$. From the definition of derivatives applied to 0-currents we have that $\bar{\partial}^* = -\bar{\partial}$. Now $B/\bar{\partial}_L(A) \cong H^1(\mathcal{L}_g, \mathcal{O}(L))_T$ is therefore dual to the kernel of the map $-\bar{\partial} : \Gamma(\mathcal{L}_g, \mathcal{K}^{(1,0)}(L^{-1})) \rightarrow \Gamma(\mathcal{L}_g, \mathcal{K}^{(1,1)}(L^{-1}))$; however, by

proposition (2.15) this kernel is precisely $\Gamma(\mathcal{L}_g, \mathcal{O}^{(1,0)}(L^{-1}))=H^0(\mathcal{L}_g, \mathcal{O}^{(1,0)}(L^{-1}))$, and thus the proof is completed. \square

The tangential homology. We will consider continuous families of 1-cycles (boundaries) in X_g parametrized by K in such a way we can identify a 1-cycle (boundary) in \mathcal{L}_g with it. So let us consider $\mathcal{C}^0(K, H_1(X_g, \mathbb{C}))$, the Banach space of continuous maps from K to the first de Rham homology group of X_g as the first tangentially homology group of \mathcal{L}_g with coefficients in \mathbb{C} .

Remark (2.18). If we consider the natural inclusion $H_1(X_g, \mathbb{Z})$ into $H_1(X_g, \mathbb{C})$, then, the set $\mathcal{C}^0(K, H_1(X_g, \mathbb{Z}))$ will be a discrete subgroup of $\mathcal{C}^0(K, H_1(X_g, \mathbb{C}))$. We define $H_1(\mathcal{L}_g; \mathbb{Z})_T = \mathcal{C}^0(K, H_1(X_g, \mathbb{Z}))$.

The Jacobian. Let $EC^0(K, \mathbb{C}^g)$ be the set of step \mathbb{C}^g -valued functions defined in K . Let $\langle \mathcal{C}^0(K, \mathbb{Z}^{2g}) \rangle_{\mathbb{R}}$ be the set generated by $\mathcal{C}^0(K, \mathbb{Z}^{2g})$ over the real numbers.

LEMMA (2.19). *The topological closure of $EC^0(K, \mathbb{C}^g)$ is equal to $\mathcal{C}^0(K, \mathbb{C}^g)$.*

Proof. Let $f \in \mathcal{C}^0(K, \mathbb{C}^g)$, then f is uniformly continuous. Take $\epsilon > 0$ such that $|f(x) - f(y)| < \epsilon$ if $d_K(x, y) < \delta$. Since K is compact and totally disconnected there exists a finite covering $\{U_j\}_{j=1}^n$ of K such that all its elements are disjoint and have diameter less than δ . Let us consider one point $x_j \in U_j$ for each j and define $f_j \in EC^0(K, \mathbb{C}^g)$ as $f_j(t) = f(x_j)\chi_{U_j}$. Further, we should consider $f_\delta \in EC^0(K, \mathbb{C}^g)$ defined by $f_\delta = \sum f_j$. By the previous construction and considerations we have that for all $x, y \in K$ such that $d_K(x, y) < \delta$, the inequality $|f(x) - f_\delta(y)| < \epsilon$ holds, so we obtain the desired result. \square

Remark (2.20). We remark that the set of step \mathbb{C}^g -valued functions defined on K is a subset of the set generated by $\mathcal{C}^0(K, \mathbb{Z}^{2g})$ over the real numbers.

The previous lemma and remark together with the fact that $\langle \mathcal{C}^0(K, \mathbb{Z}^{2g}) \rangle_{\mathbb{R}} \subset \mathcal{C}^0(K, \mathbb{C}^g)$ imply the following result.

PROPOSITION (2.21). *The topological closure of $\langle \mathcal{C}^0(K, \mathbb{Z}^{2g}) \rangle_{\mathbb{R}}$ is equal to $\mathcal{C}^0(K, \mathbb{C}^g)$.*

Using remark (2.18) and the previous proposition we have the following result.

LEMMA (2.22). *The quotient*

$$(2.23) \quad \frac{\mathcal{C}^0(K, H^0(X_g, \mathcal{O}_{X_g}^{(1,0)*})}{\mathcal{C}^0(K, H_1(X_g, \mathbb{Z}))}.$$

is a well-defined Hausdorff topological space.

Analogous to the classical theory of curves [3], [5] we make the definition of the Jacobian of the product lamination by Riemann surfaces.

Definition (2.24). The Jacobian of the product lamination by Riemann surfaces \mathcal{L}_g is given by

$$(2.25) \quad \text{Jac}(\mathcal{L}_g) := \frac{\mathcal{C}^0(K, H^0(X_g, \mathcal{O}_{X_g}^{(1,0)*}))}{\mathcal{C}^0(K, H_1(X_g, \mathbb{Z}))}.$$

We remark that the quotient $\mathcal{C}^0(K, \mathbb{C}^g)/\mathcal{C}^0(K, \mathbb{Z}^{2g})$ is a complex Abelian topological group of infinite dimension associated to the product lamination by Riemann surfaces. In a naive way, $\text{Jac}(\mathcal{L}_g)$ is like a “complex torus of infinite dimension”.

PROPOSITION (2.26). *There exists a natural isomorphism between $\mathcal{C}^0(K, \mathbb{C}^g/\mathbb{Z}^{2g})$ and $\mathcal{C}^0(K, \mathbb{C}^g)/\mathcal{C}^0(K, \mathbb{Z}^{2g})$.*

Proof. Let us consider the map $F : \mathcal{C}^0(K, \mathbb{C}^g)/\mathcal{C}^0(K, \mathbb{Z}^{2g}) \longrightarrow \mathcal{C}^0(K, \mathbb{C}^g/\mathbb{Z}^{2g})$ defined by $F : [f] \mapsto f \bmod \mathbb{Z}^{2g}$. This gives the isomorphism. \square

As a consequence of the isomorphism $\text{Jac}(X_g) \cong \mathbb{C}^g/\mathbb{Z}^{2g}$ and the natural isomorphism given by the previous proposition we obtain another characterization of the Jacobian of product laminations by Riemann surfaces.

COROLLARY (2.27). *There exists a natural isomorphism between the Jacobian of the product lamination by Riemann surfaces \mathcal{L}_g and the abelian topological group of continuous maps from K to the Jacobian of the compact Riemann surface X_g .*

3. The Abel-Jacobi Map

Let $Z : K \longrightarrow X_g$ be any continuous function. Let us consider its graph $\text{graph}(Z)$; it will be called *the base points (with respect to Z) in \mathcal{L}_g* .

For each $t \in K$, let us denote

$$A_{Z(t)}(z) = \int_{Z(t)}^z \omega := \left(\int_{Z(t)}^z \omega_1, \int_{Z(t)}^z \omega_2, \dots, \int_{Z(t)}^z \omega_g \right)$$

which is the classical Abel-Jacobi map from the compact Riemann surface $X_g \times \{t\} (\cong X_g)$ with base points $Z(t)$ to its Jacobian.

The Abel-Jacobi map is a cornerstone in the theory of compact Riemann surfaces. The following result shows that this map can also be defined for product laminations by Riemann surfaces.

LEMMA (3.1). *The map $\widehat{A} : (\mathcal{L}_g, Z) \longrightarrow \text{Jac}(\mathcal{L}_g)$ given by*

$$(3.2) \quad (z, t, Z(t)) \mapsto A_{Z(t)}(z) = \int_{Z(t)}^z \omega,$$

is a well-defined continuous map.

The continuous map \widehat{A} will be called *the Abel-Jacobi map from the product lamination by Riemann surfaces \mathcal{L}_g (with base points Z) to its Jacobian*.

Remark (3.3). Let $p_0 = (z_0, t_0)$ be a point in \mathcal{L}_g . By corollary (2.27) the Abel-Jacobi map \widehat{A} (with base points Z) associates the map

$$(3.4) \quad f(t) = \begin{cases} \int_{Z(t_0)}^{z_0} \omega & \text{if } t = t_0, \\ 0 & \text{otherwise} \end{cases}$$

to the point p_0 , where $\int_{Z(t_0)}^{z_0} \omega \in \text{Jac}(X_g \times \{t_0\}) (\cong \text{Jac}(X_g))$ and 0 is the zero element of the group $\text{Jac}(X_g)$.

By the previous considerations and remark we have the following result.

THEOREM (3.5). *The Abel-Jacobi map is a tangentially holomorphic embedding from the product lamination by Riemann surfaces (with base points Z) into its Jacobian.*

4. The group of divisors and the Picard group

Divisors, degree of divisors and the group of divisors. Let us consider closed subsets C_i in \mathcal{L}_g such that their restrictions to each leaf $X_g \times \{t\}$ are divisors. A finite formal sum $D(t) = \sum_{i=1}^n C_i$ in \mathcal{L}_g will be called a *divisor in the product lamination by Riemann surfaces* \mathcal{L}_g . The set of all divisors in \mathcal{L}_g will be denoted by $\text{Div}(\mathcal{L}_g)$. *The tangential degree of $D(t)$* is the continuous function $\text{deg}_T : K \rightarrow \mathbb{Z}$ such that for each $t_0 \in K$, $\text{deg}_T(D(t_0))$ is the degree of the divisor in the leaf $X_g \times \{t_0\}$.

Since K is a compact and totally disconnected set, we have the following result.

LEMMA (4.1). *For any $D(t) \in \text{Div}(\mathcal{L}_g)$ the tangential degree of $D(t)$ is a locally constant function. Moreover, it is an additive function.*

By this lemma we have that for any divisor in \mathcal{L}_g , its tangential degree must be constant in open sets along the K set. Let $d \in \mathbb{Z}$. The set of divisors with constant tangential degree d in \mathcal{L}_g will be denoted by $\text{Div}_d(\mathcal{L}_g)$.

Remark (4.2). Using the fact that a divisor with constant tangential degree d in \mathcal{L}_g is equivalent to having a continuous family of divisors of degree d in X_g parametrized by K , we have that the set of divisors with constant tangential degree d in \mathcal{L}_g can be canonically identified with the set of continuous maps from K to the set of divisors of degree d in X_g .

The *sum* in $\text{Div}(\mathcal{L}_g)$ is given by the sum of the divisors, leaf by leaf, in a continuous way.

From the definitions of divisor and sum of divisors in \mathcal{L}_g , we have the following result.

PROPOSITION (4.3). *The group of divisors in the product lamination by Riemann surfaces \mathcal{L}_g is canonically isomorphic to the group of continuous maps from K to the group of divisors in the compact Riemann surface X_g .*

Principal divisors and tangentially meromorphic functions. A divisor $D(t)$ will be called a *principal divisor in \mathcal{L}_g* if its restriction to each leaf $X_g \times \{t\}$ is a principal divisor. The Abelian subgroup of all principal divisors in \mathcal{L}_g will be denoted by $\text{Div } P(\mathcal{L}_g)$.

Analogous to the previous proposition we have the following result.

PROPOSITION (4.4). *The group of principal divisors in the product lamination by Riemann surfaces \mathcal{L}_g is canonically isomorphic to the group of continuous maps from K to the group of principal divisors in the compact Riemann surface X_g .*

Let $\mathcal{M}_T(\mathcal{L}_g)$ be the set of tangentially meromorphic functions in the product lamination by Riemann surfaces \mathcal{L}_g with the compact-open topology.

We note that $\mathcal{M}_T(\mathcal{L}_g)$ can be canonically identified with the set of continuous maps from K to the space of meromorphic functions in the compact Riemann surface X_g .

The following result is analogous to the classical Rouché's theorem for product laminations by Riemann surfaces.

LEMMA (4.5). *Let f be a tangentially meromorphic function defined on the product lamination by Riemann surfaces \mathcal{L}_g . Suppose that f has a zero (a pole) in $p_0 = (z_0, t_0)$ of order n . Then, there exists an open set $U \in K$ such that for every $t \in U$, $f(z_0, t)$ has a zero (a pole) of order n .*

It is easy to see that if $f \in \mathcal{M}_T(\mathcal{L}_g)$ and L is any leaf in \mathcal{L}_g , then the set $(f^{-1}(0) \cap L)$ (respectively $(f^{-1}(\infty) \cap L)$) has only a finite number of elements. Let us denote by $(f)_0 = \{(z, t) \in \mathcal{L}_g : f_t(z) = 0\}$ and $(f)_\infty = \{(z, t) \in \mathcal{L}_g : f_t(z) = \infty\}$ the zero divisor and the polar divisor of f , respectively. Let $(f) = (f)_0 - (f)_\infty$.

PROPOSITION (4.6). *A divisor $D(t)$ in the product lamination by Riemann surfaces \mathcal{L}_g is a principal divisor if and only if there exists $f \in \mathcal{M}_T(\mathcal{L}_g)$ such that $(f) = D(t)$.*

Proof. Let us consider the family $\{f_t\}_{t \in K}$ of meromorphic functions such that $(f_t) = D(t)$ for each $t \in K$. It is a continuous family with respect to t . \square

If D and D' are two divisors in \mathcal{L}_g we will say that they are *linearly equivalent*, $D \sim D'$, if and only if there is an $f \in \mathcal{M}_T(\mathcal{L}_g)$ such that $D - D' = (f)$.

The Picard group. In virtue of the previous section we have that the quotient

$$(4.7) \quad \text{Pic}(\mathcal{L}_g) := \frac{\text{Div}(\mathcal{L}_g)}{\text{Div } P(\mathcal{L}_g)}$$

is a well-defined abelian group. It will be called *the Picard group of the product lamination by Riemann surfaces \mathcal{L}_g* .

PROPOSITION (4.8). *The Picard group of the product lamination by Riemann surfaces \mathcal{L}_g can be canonically identified with the group of continuous maps from K to the Picard group of the compact Riemann surface X_g .*

Proof. From propositions (4.3) and (4.4), we have that $\text{Pic}(\mathcal{L}_g) = \frac{\mathcal{C}^0(K, \text{Div}(X_g))}{\mathcal{C}^0(K, \text{Div } P(X_g))}$.
 The map $F : \frac{\mathcal{C}^0(K, \text{Div}(X_g))}{\mathcal{C}^0(K, \text{Div } P(X_g))} \longrightarrow \mathcal{C}^0(K, \frac{\text{Div}(X_g)}{\text{Div } P(X_g)})$ given by

$$[f] = f + \mathcal{C}^0(K, \text{Div } P(X_g)) \mapsto f(t) \bmod \text{Div } P(X_g)$$

gives the identification. □

COROLLARY (4.9). *There is a canonical identification between the Picard group of the product lamination by Riemann surfaces \mathcal{L}_g and*

$$\mathcal{C}^0 \left(K, \bigsqcup_{d \in \mathbb{Z}} \frac{\text{Div}_d(X_g)}{\text{Div } P(X_g)} \right).$$

Now we shall give a description of $\text{Pic}(\mathcal{L}_g)$ in terms of holomorphic line bundles over product laminations by Riemann surfaces.

Let L be a holomorphic line bundle over \mathcal{L}_g . *The tangential Chern class of L* is the continuous function $c_{1T} : K \longrightarrow \mathbb{Z}$ such that for each $t \in K$, $c_{1T}(t)$ is the Chern class of the holomorphic line bundle $L(t)$ on the leaf $X_g \times \{t\}$. Since K is a compact and totally disconnected set, we have the following result.

LEMMA (4.10). *The tangential Chern class of L is a locally constant function. Moreover, it is an additive function.*

Let $d \in \mathbb{Z}$. The set of holomorphic line bundles over \mathcal{L}_g with constant tangential Chern class d will be denoted by $\text{Pic}_d(\mathcal{L}_g)$.

We remark that the canonical holomorphic line bundle $F^*\mathcal{L}_g$ over \mathcal{L}_g has constant tangential Chern class $2g - 2$.

Remark (4.11). The set of holomorphic line bundles over \mathcal{L}_g with constant tangential Chern class d can be canonically identified with the set of continuous maps from K to the set of holomorphic line bundles over X_g with Chern class d .

PROPOSITION (4.12). *There exists a canonical correspondence between*

$$\text{Pic}_0(\mathcal{L}_g) \quad \text{and} \quad \text{Pic}_{2g-2}(\mathcal{L}_g).$$

Proof. For any $L \in \text{Pic}_0(\mathcal{L}_g)$ consider the tensor product with the canonical line bundle over \mathcal{L}_g . This gives the desired correspondence. □

Since $\text{Pic}_0(X_g)$ is isomorphic to $\text{Jac}(X_g)$ for any compact Riemann surface X_g , then, we have that $\mathcal{C}^0(K, \text{Pic}_0(X_g))$ is isomorphic to $\mathcal{C}^0(K, \text{Jac}(X_g))$. This note together with theorem (2.27) and remark (4.11) give us the following result.

THEOREM (4.13). *The Jacobian of the product lamination by Riemann surfaces is isomorphic to the group of line bundles over \mathcal{L}_g with constant tangential Chern class zero.*

5. Abel's Theorem

The W_k set in $\text{Jac}(\mathcal{L}_g)$. Let us consider the map $\widehat{\varphi} : \text{Div}(\mathcal{L}_g) \longrightarrow \text{Jac}(\mathcal{L}_g)$ given by $\widehat{\varphi}(D(t)) = (\varphi \circ D)(t)$, where φ is the classical Jacobi morphism. It will be called *the Jacobi morphism in the product lamination by Riemann surfaces \mathcal{L}_g* .

The image of the product lamination by Riemann surfaces \mathcal{L}_g under the Abel-Jacobi map \widehat{A} will be called the $W_1(\mathcal{L}_g)$ subset in $\text{Jac}(\mathcal{L}_g)$.

Let us consider $\text{Div}_k^+(\mathcal{L}_g)$, the set of positive divisors in \mathcal{L}_g with constant tangential degree k . The image of $\text{Div}_k^+(\mathcal{L}_g)$ under the Jacobi morphism $\widehat{\varphi}$ will be called the $W_k(\mathcal{L}_g)$ subset in $\text{Jac}(\mathcal{L}_g)$. Using the arguments given in remark (3.3) we have that the $W_k(\mathcal{L}_g)$ set can be canonically identified with the set of continuous maps from K to the classical $W_k(X_g)$ sub-variety of $\text{Jac}(X_g)$ (see [6]).

It is well-known that for any $k \in \mathbb{N}$ such that $k \geq g > 0$, the equality $W_k(X_g) = W_g(X_g)$ holds (see [6]). Then, we have $C^0(K, W_k(X_g)) = C^0(K, W_g(X_g))$. This equality implies the following result.

THEOREM (5.1). *For any $k \in \mathbb{N}^+$ such that $k \geq g$ the equality $W_k(\mathcal{L}_g) = W_g(\mathcal{L}_g)$ holds for any product lamination by Riemann surfaces \mathcal{L}_g .*

COROLLARY (5.2) (The inversion theorem for product lamination by Riemann surfaces). *The $W_g(\mathcal{L}_g)$ set can be canonically identified with the Jacobian of the product lamination by Riemann surfaces \mathcal{L}_g .*

It follows from corollary (2.27) that the restriction of the Jacobi morphism $\widehat{\varphi}$ to the group $\text{Div}_k^+(\mathcal{L}_g)$ can be canonically identified with $C^0(K, X_g^{(k)})$, where $X_g^{(k)}$ is the k -symmetric product of X_g .

Abel's theorem is a cornerstone on the theory of compact Riemann surfaces. From the definition of linear equivalence of divisors and the previous discussion, it is valid for product laminations by Riemann surfaces as well.

THEOREM (5.3) (Abel's theorem). *For divisors $D, D' \in \text{Div}_k(\mathcal{L}_g)$ we have*

$$\widehat{\varphi}(D) = \widehat{\varphi}(D') \text{ if and only if } D \sim D'.$$

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INSTITUTO DE MATEMÁTICAS, UNAM
 UNIDAD CUERNAVACA
 AV. UNIVERSIDAD S/N
 CHAMILPA
 CUERNAVACA, MORELOS
 MÉXICO
 ojeda@math.unam.mx; emigdio@matcuer.unam.mx

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