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ON THE CHARACTERIZATION OF A SUBVARIETY OF SEMI-DE MORGAN ALGEBRAS

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ABSTRACT. In this note we characterize by a new set of axioms the largest subvariety of semi-De Morgan algebras with the congruence extension property.

1. Introduction

The equational class of semi-De Morgan algebras was introduced by Sankappanavar in [8]. It consists of bounded distributive lattices with an additional unary operation and it contains the variety of pseudocomplemented distributive lattices and $\mathcal{K}_{1,1}$, one of the subvarieties of Ockham algebras which includes De Morgan algebras.

In [4] Hobby developed a duality for semi-De Morgan algebras which he used to find the largest subvariety of semi-De Morgan algebras with the congruence extension property. This variety, which Hobby denoted by $\mathcal{C}$, contains both $\mathcal{K}_{1,1}$ and the equational class of demi-pseudocomplemented lattices, a generalization of pseudocomplemented lattices studied by Sankappanavar in [9] and [10].

The equations defining principal congruences as well as the subdirectly irreducibles of the variety $\mathcal{C}$ were determined by us in [6]; however, the two inequalities (α and β) that characterize this subvariety of semi-De Morgan are rather complicated. In fact Problem 2 in [4] is to find "nicer axioms for $\mathcal{C}$".

We solved this problem algebraically determining a new inequality (γ) such that $\mathcal{C}$ can be characterized by γ and β.

2. Preliminaries

We start by recalling some definitions and essential results from [8].

Definition (2.1). An algebra $L = \big(L, \vee, \wedge, ', 0, 1\big)$ is a *semi-De Morgan algebra* if the following five conditions hold $\big(a, b \in L\big)$:

(S1) $(L, \vee, \wedge, 0, 1)$ is a distributive lattice with 0, 1.

(S2) $0' \approx 1$ and $1' \approx 0$.

(S3) $\big(a \vee b\big)' \approx a' \wedge b'$.

(S4) $\big(a \wedge b\big)'' \approx a'' \wedge b''$.

(S5) $a''' \approx a'$.

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This equational class of algebras will be denoted by *SDMA*.

The following rules hold in *SDMA* and some of them are proved in [8] :

(S6) $(a \wedge b)' \approx (a'' \wedge b'')' \approx (a \wedge b'')'$.

(S7) $(a \wedge b)' \approx (a' \vee b')''$.

(S8) $(a \wedge b)'' \approx (a' \vee b')'$.

(S9) $a \leq b$ implies $b' \leq a'$.

(S10) $a \wedge (a \wedge b)' \geq a \wedge b'$.

(S11) $(a \vee b)'' \approx (a' \wedge b')' \approx (a'' \vee b'')'' \approx (a \vee b'')''$.

3. The variety $\mathcal{C}$

D. Hobby determined in [4] the largest subvariety of *SDMA* with the congruence extension property. He characterized this variety, which he denoted by $\mathcal{C}$, by the following inequalities:

$$(\alpha) \quad a' \vee b' \geq (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'$$

$$(\beta) \quad a' \vee (a' \wedge b \wedge b')' \geq (a \wedge b)'.$$

It is possible to obtain simpler inequalities characterizing $\mathcal{C}$. The search for these inequalities requires some rather nasty calculations so we consider several lemmas before we can reach our goal.

With this aim we will consider first the following identities:

$$(\alpha_1) \quad a' \vee b' = a' \vee b' \vee \Big((a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'\Big)$$

$$(\beta_1) \quad a' \vee (a' \wedge b \wedge b')' = (a \wedge b)' \vee (a' \wedge b \wedge b')'.$$

These identities are equivalent to α and β, respectively (note that $a \geq a \wedge b$ implies $a' \leq (a \wedge b)'$).

Now we can prove the following.

LEMMA (3.1). *Let $L \in \mathcal{C}$ and $a, d \in L$. Then the identity α_1 implies*

$$(\alpha_2) \qquad (a' \vee d') \wedge d'' = (a \wedge d)' \wedge d''.$$

Proof. By (S3), $(a' \wedge d'') \vee d' = (a \vee d')' \vee d'$. Replacing b by $a \vee d'$, a by d and c by d' in the identity α_1 and using commutativity, we obtain

$$\begin{aligned}
&(a \vee d')' \vee d' \\
&= (a \vee d')' \vee d' \vee \big(((a \vee d') \wedge d)' \wedge (d \wedge d')' \wedge ((a \vee d') \wedge d')' \wedge ((a \vee d') \wedge d'')'\big) \\
&= (a \vee d')' \vee d' \vee \Big(\big((a \vee d') \wedge d\big)' \wedge (d \wedge d')' \wedge d''\Big)
\end{aligned}$$

because $\big((a \vee d') \wedge d''\big)' = \big((a \vee d') \wedge d\big)'$ by (S6).

But $\big(d \wedge d'\big)' \geq \big((a \vee d') \wedge d\big)'$ since $d \wedge d' \leq \big(a \vee d'\big) \wedge d$; hence it follows from the previous equation that

$$\begin{aligned}
\big(a \vee d'\big)' \vee d' &= \big(a \vee d'\big)' \vee d' \vee \Big(\big(\big(a \vee d'\big) \wedge d\big)' \wedge d''\Big) \\
&= \big(a \vee d'\big)' \vee d' \vee \big(\big(\big(a \vee d'\big) \wedge d\big) \vee d'\big)' \qquad \text{(by S3)} \\
&= \big(a \vee d'\big)' \vee d' \vee \big(\big(a \vee d'\big) \wedge \big(d \vee d'\big)\big)' \qquad \text{(by distributivity)} \\
&= d' \vee \big((a \vee d') \wedge \big(d \vee d'\big)\big)' \ \text{(because } (a \vee d')' \leq \big(\big(a \vee d'\big) \wedge \big(d \vee d'\big)\big)' \\
&= d' \vee \big(\big(a \wedge d\big) \vee d'\big)' \qquad \text{(by distributivity)} \\
&= d' \vee \Big(\big(a \wedge d\big)' \wedge d''\Big).
\end{aligned}$$

Therefore we have $(a' \wedge d'') \vee d' = ((a \wedge d)' \wedge d'') \vee d'$.
Now, by distributivity, we obtain

$$(a' \vee d') \wedge (d'' \vee d') = ((a \wedge d)' \vee d') \wedge (d'' \vee d').$$

Since $(a \wedge d)' \geq d'$, it follows that

$$(a' \vee d') \wedge (d'' \vee d') = (a \wedge d)' \wedge (d'' \vee d')$$

and meeting the two members with d'', we have α_2. □

LEMMA (3.2). *Let $L \in SDMA$ and let $a, b, c \in L$. Then*

$$(\alpha_2) \qquad (a' \vee b') \wedge b'' = (a \wedge b)' \wedge b'' \quad \textit{and}$$

$$(\beta_1) \qquad a' \vee (a' \wedge b \wedge b')' = (a \wedge b)' \vee (a' \wedge b \wedge b')'$$

imply

$$(\delta) \qquad \big(a' \wedge (b \wedge (c \vee c'))'\big) \vee \big(b' \wedge (a \wedge c)'\big) = (a \wedge b)' \wedge (a \wedge c)' \wedge \big((b \wedge (c \vee c')\big)'.$$

Proof. Let us denote by A and B, respectively, the left and right sides of the identity δ. We are going to prove that β_1 and α_2 imply $A = B$ using the distributivity of L.

First we will verify that the joins of A and B with $(a' \wedge c' \wedge c'')'$ are equal:

$$\begin{aligned}
&A \vee (a' \wedge c' \wedge c'')' \\
&= \big(a' \wedge (b \wedge (c \vee c'))'\big) \vee \Big(\Big(\big(b' \vee (a' \wedge c' \wedge c'')'\big) \wedge \big((a \wedge c)' \vee (a' \wedge c' \wedge c'')'\big)\Big)\Big) \\
&\qquad \text{(by distributivity)} \\
&= \big(a' \wedge (b \wedge (c \vee c'))'\big) \vee \Big(\Big(\big(b' \vee (a' \wedge c' \wedge c'')'\big) \wedge \big(a' \vee (a' \wedge c' \wedge c'')'\big)\Big)\Big) \\
&\qquad \text{(by } \beta_1 \text{ and S6)} \\
&= \big(a' \wedge (b \wedge (c \vee c'))'\big) \vee \big(b' \wedge a'\big) \vee (a' \wedge c' \wedge c'')' \quad \text{(by distributivity)} \\
&= \big(a' \wedge (b \wedge (c \vee c'))'\big) \vee (a' \wedge c' \wedge c'')'
\end{aligned}$$

because, by S9, $(b \wedge (c \vee c'))' \geq b'$ and thus $a' \wedge (b \wedge (c \vee c'))' \geq a' \wedge b'$).

$$\begin{aligned}
&B \vee (a' \wedge c' \wedge c'')' \\
&= \Big(\big((a \wedge b)' \wedge (b \wedge (c \vee c'))'\big) \vee (a' \wedge c' \wedge c'')'\Big) \wedge \big((a \wedge c)' \vee (a' \wedge c' \wedge c'')'\big) \\
&\qquad \text{(by distributivity)}
\end{aligned}$$

$$= \Big(\big((a\wedge b)'\wedge(b\wedge(c\vee c'))'\big)\vee(a'\wedge c'\wedge c'')'\Big)\wedge\big(a'\vee(a'\wedge c'\wedge c'')'\big)$$
(by β_1 and S6)
$$= \big((a\wedge b)'\wedge(b\wedge(c\vee c'))'\wedge a'\big)\vee(a'\wedge c'\wedge c'')' \quad \text{(by distributivity)}$$
$$= \big(a'\wedge(b\wedge(c\vee c'))'\big)\vee(a'\wedge c'\wedge c'')' \quad \text{because, by S9,} \quad (a\wedge b)'\geq a'.$$

Thus we have proved that $A\vee(a'\wedge c'\wedge c'')' = B\vee(a'\wedge c'\wedge c'')'$.

Now we are going to see that the same is true with the meets. We will have to use identity α_2, so we must note that by S3, S11, S9 and S5,

$$(a'\wedge c'\wedge c'')' = (a\vee c\vee c')'' = (a\vee c''\vee c')'' \geq (a'\wedge c'\wedge c'')'' = (a\vee c\vee c')'.$$

Therefore, denoting by d the expression $a\vee c\vee c'$, we will have

$$(a'\wedge c'\wedge c'')' = d''\geq d'$$

and thus

$$A\wedge(a'\wedge c'\wedge c'')' = A\wedge d''$$
$$= \Big(\big(a'\wedge(b\wedge(c\vee c'))'\big)\vee(b'\wedge(a\wedge c)')\Big)\wedge d''$$
$$= \Big(\big(a\vee(b\wedge(c\vee c'))\big)'\vee(b'\wedge(a\wedge c)')\Big)\wedge d'' \quad \text{(by S3)}$$
$$= \Big(\big((a\vee b)\wedge(a\vee c\vee c')\big)'\wedge d''\Big)\vee\big((b'\wedge(a\wedge c)')\wedge d''\big)$$
(by distributivity)
$$= \Big(\big((a\vee b)\wedge d\big)'\wedge d''\Big)\vee\big((b'\wedge(a\wedge c)')\wedge d''\big)$$
(by the definition of d,)
$$= \Big(\big((a\vee b)'\vee d'\big)\wedge d''\Big)\vee\big((b'\wedge(a\wedge c)')\wedge d''\big) \quad \text{(by } \alpha_2)$$
$$= \Big(\big((a'\wedge b')\vee d'\big)\wedge d''\Big)\vee\big((b'\wedge(a\wedge c)')\wedge d''\big) \quad \text{(by S3)}$$
$$= \big((a'\wedge b')\vee d'\vee(b'\wedge(a\wedge c)'\big)\wedge d'' \quad \text{(by distributivity)}$$
$$= \big(d'\vee(b'\wedge(a\wedge c)'\big)\wedge d'' \quad \text{(because } (a\wedge c)'\geq a')$$
$$= \big(b'\wedge(a\wedge c)'\wedge d''\big)\vee(d'\wedge d'') \quad \text{(by distributivity)}$$
$$= \big(b'\wedge(a\wedge c)'\wedge d''\big)\vee d' \quad \text{(because } d''\geq d').$$

By a similar process,

$$B\wedge(a'\wedge c'\wedge c'')' = B\wedge d''$$
$$= (a\wedge b)'\wedge(b\wedge(c\vee c'))'\wedge d''\wedge(a\wedge c)' \quad \text{(by commutativity)}$$
$$= ((a\wedge b)\vee(b\wedge(c\vee c')))'\wedge d''\wedge(a\wedge c)' \quad \text{(by S3)}$$
$$= (b\wedge(a\vee c\vee c'))'\wedge d''\wedge(a\wedge c)' \quad \text{(by distributivity)}$$
$$= (b\wedge d)'\wedge d''\wedge(a\wedge c)' \quad \text{(by the definition of } d\ )$$
$$= (b'\vee d')\wedge d''\wedge(a\wedge c)' \qquad \text{(applying } \alpha_2)$$

$$
\begin{aligned}
&= ((b' \wedge d'') \vee (d' \wedge d'')) \wedge (a \wedge c)') \qquad \text{(by distributivity)}\\
&= ((b' \wedge d'') \vee d') \wedge (a \wedge c)') \qquad \text{(because } d'' \geq d' \text{)}\\
&= (b' \wedge d'' \wedge (a \wedge c)') \vee (d' \wedge (a \wedge c)') \qquad \text{(by distributivity)}\\
&= (b' \wedge d'' \wedge (a \wedge c)') \vee d'\\
&\qquad\qquad \text{(because, by S9, } d' = (a \vee c \vee c')' \leq (a \wedge c)').
\end{aligned}
$$

Therefore we have proved that

$$A \wedge (a' \wedge c' \wedge c'')' = B \wedge (a' \wedge c' \wedge c'')'.$$

By the characterization of $\theta_{latL}((a' \wedge c' \wedge c'')', (a' \wedge c' \wedge c'')')$ we conclude that $A = B$. □

LEMMA (3.3). *Let $L \in SDMA$ and let $a, b, c \in L$. Then the identity*

$$(\delta) \qquad \big(a' \wedge (b \wedge (c \vee c'))'\big) \vee \big(b' \wedge (a \wedge c)'\big) = (a \wedge b)' \wedge (a \wedge c)' \wedge \big((b \wedge (c \vee c')\big)'.$$

is equivalent to α.

Proof. First note that α is equivalent to

$$(a' \vee b') \wedge (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')' = (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'$$

and, by distributivity, this identity is equivalent to

$$
\begin{aligned}
&\big(a' \wedge (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'\,\big) \vee\\
&\qquad \big(b' \wedge (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'\,\big) =\\
&= (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'.
\end{aligned}
$$

By S9, it is known that a' is less than or equal to $(a \wedge b)'$ and to $(a \wedge c)'$ and that b' is also less than or equal to $(a \wedge b)'$, $(b \wedge c)'$ and $(b \wedge c')'$. Therefore the previous identity is equivalent to

$$\big(a' \wedge (b \wedge c)' \wedge (b \wedge c')'\big) \vee \big(b' \wedge (a \wedge c)'\big) = (a \wedge b)' \wedge (a \wedge c)' \wedge (b \wedge c)' \wedge (b \wedge c')'$$

and, by S3, also to

$$\big(a' \wedge ((b \wedge c) \vee (b \wedge c'))'\big) \vee \big(b' \wedge (a \wedge c)'\big) = (a \wedge b)' \wedge (a \wedge c)' \wedge \big((b \wedge c) \vee (b \wedge c')\big)'.$$

Finally, by the distributivity of L, we conclude that α is equivalent to δ.

□

From the previous lemmas we obtain the following:

PROPOSITION (3.4). *Let $L \in SDMA$. Then $L \in \mathcal{C}$ if and only if the identities*

$$(\alpha_2) \qquad (a' \vee b') \wedge b'' = (a \wedge b)' \wedge b''$$

and

$$(\beta_1) \qquad a' \vee (a' \wedge b \wedge b')' = (a \wedge b)' \vee (a' \wedge b \wedge b')'$$

hold.

Proof. We proved in Lemma 3.1 that the identity α_2 is a consequence of α_1 which is equivalent to α.

Conversely, by Lemma (3.2), (α_2 and β_1) imply (δ and β_1), and by Lemma (3.3), these are equivalent to α and β_1. □

It is now possible to characterize $\mathcal{C}$ by simpler axioms solving Problem 2 in Hobby [4]:

THEOREM (3.5). *The subvariety $\mathcal{C}$ of semi-De Morgan algebras can be characterized by inequalities γ and β:*

$$(\gamma) \qquad a' \vee b' \geq (a \wedge b)' \wedge b''$$

$$(\beta) \qquad a' \vee (a' \wedge b \wedge b')' \geq (a \wedge b)'$$

Proof. It is enough to prove that the identity α_2 of the previous lemma is equivalent to the inequality γ.

By α_2 we have

$$a' \vee b' \geq (a' \vee b') \wedge b'' = (a \wedge b)' \wedge b''.$$

Therefore α_2 implies γ.

On the other hand, from γ we know that

$$(a' \vee b') \wedge (a \wedge b)' \wedge b'' = (a \wedge b)' \wedge b''.$$

But $a' \leq (a \wedge b)'$ and $b' \leq (a \wedge b)'$ so that $a' \vee b' \leq (a \wedge b)'$ and therefore α_2 follows from γ. $\square$

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UNIFORMITY OF DISTRIBUTION MODULO 1 OF THE GEOMETRIC MEAN PRIME DIVISOR

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Abstract. We show that the fractional parts of $n^{1/\omega(n)}$, $n^{1/\Omega(n)}$ and the geometric mean of the distinct prime factors of n are uniformly distributed modulo 1 as n ranges over all the positive integers, where $\Omega(n)$ and $\omega(n)$ denote the number of distinct prime divisors of n counted with and without multiplicities. Note that $n^{1/\Omega(n)}$ is the geometric mean of *all prime divisors of* n taken with the corresponding multiplicities. The result complements a series of results of similar spirit obtained by various authors, while the method can be applied to several other arithmetic functions of similar structure.

1. Introduction

In [1], it is shown that the fractional part of the *arithmetic mean* of the prime factors of an integer n, that is, the function

$$f(n) = \frac{1}{\omega(n)} \sum_{p|n} p,$$

where $\omega(n)$ denotes the number of distinct prime divisors of n, is uniformly distributed in $[0, 1)$ as n ranges over all the positive integers. The same method can also be applied to the fractional part of the *arithmetic mean* of the prime factors of an integer n taken with the corresponding multiplicities, that is, to the function

$$F(n) = \frac{1}{\Omega(n)} \sum_{p^{\alpha_p}||n} \alpha_p p,$$

where $\Omega(n)$ denotes the number of distinct prime divisors of n counted with multiplicities.

This is in contrast with the main result from [3] where it is shown that the arithmetic mean of all the divisors of n, that is, the function

$$g(n) = \frac{1}{\tau(n)} \sum_{d|n} d,$$

is an integer for almost all positive integers n, where, as usual, $\tau(n)$ denotes the total number of positive divisors of n.

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The above result from [1] naturally leads to the question whether the fractional parts of the *geometric mean* of the prime factors of n

$$h(n) = \left(\prod_{p|n} p\right)^{1/\omega(n)}$$

are uniformly distributed in $[0, 1)$.

In this paper, we investigate the distribution modulo 1 of the function $h(n)$ and the closely related functions $n^{1/\omega(n)}$ and $n^{1/\Omega(n)}$. Note that all three functions coincide when n is square-free, and that $n^{1/\Omega(n)}$ has the natural interpretation of the geometric mean of the prime factors of n taken with the corresponding multiplicities.

We also recall that several more problems of a similar flavor have been treated previously in [1], [2], [3], [5], [6], [12], [13], [14], [15], [16], [17] (see also the references therein).

2. Notation and the Main Result

Recall that the *discrepancy* $D(\mathcal{A})$ of a sequence $\mathcal{A} = (a_n)_{n=1}^N$ of N (not necessarily distinct) real numbers is defined by the relation

$$D(\mathcal{A}) = \sup_{0 \le \gamma \le 1} \left| \frac{I(\mathcal{A}, \gamma)}{N} - \gamma \right|,$$

where $I(\mathcal{A}, \gamma)$ is the number of positive integers $n \le N$ such that $\{a_n\} < \gamma$.

We denote by $\delta(N)$, $\Delta(N)$ and $\nabla(N)$ the discrepancy of the sequences $\left(n^{1/\omega(n)}\right)_{n=1}^N$, $\left(n^{1/\Omega(n)}\right)_{n=1}^N$ and $(h(n))_{n=1}^N$, respectively.

THEOREM (2.1). *We have*

$$\delta(N) = (\log N)^{-1+o(1)}, \quad \Delta(N) = (\log N)^{-1+o(1)}, \quad \nabla(N) = (\log N)^{-1+o(1)}$$

as $N \to \infty$.

It is clear that the above result implies that the fractional parts $\{n^{1/\omega(n)}\}$, $\{n^{1/\Omega(n)}\}$ and $\{h(n)\}$ are all uniformly distributed in $[0, 1)$ as n ranges over all the positive integers.

3. Proof of the Main Result

(3.1) Preliminaries and the Scheme of the Proof. Since the proof of the upper bound on $\Delta(N)$ is completely analogous to the proof of the upper bound on $\delta(N)$ (and can be obtained from it by essentially making only typographical changes) we concentrate on the case of $\delta(N)$. We also indicate the tiny changes needed to deal with the case of the function $\nabla(N)$.

For a positive integer k we put $\log_k N$ for the kth-fold iterate of the natural logarithm function $\log N$. We assume that N is sufficiently large, in particular, large enough to make all the iterated logarithms well defined.

Also, given a set $\mathcal{A}$ we use $\pi(\mathcal{A})$ to denote the number of primes $p \in \mathcal{A}$. In particular, as usual, $\pi(x) = \pi(\{1, \dots, \lfloor x \rfloor\})$.

Let $P(n)$ denote the largest prime divisor of $n \ge 2$ and put $P(1) = 1$. As usual, we say that an integer $n \ge 1$ is *y-smooth* if $P(n) \le y$.

The proof follows the following steps:

- At the first step we remove integers $n \le N$ whose arithmetic structure is somewhat abnormal (for example, either n or $P(n)$ are small).
- For the remaining integers $n \le N$, we write $n = mP(n)$ and show that for every fixed m obtained in such a way, even the fractional parts of $(mp)^{1/(\omega(m)+1)}$ are already uniformly distributed when p runs through the set of prime values which $P(n)$ can take. Similar considerations apply to deal with $(mp)^{1/(\Omega(m)+1)}$ and $(h(mp))^{1/(\omega(m)+1)}$, respectively.

(3.2) The Exceptional Sets. We define the following sets $\mathcal{E}_i$, $i = 1, \dots, 7$, which are similar to those of [1] and estimated in the same way, although the choice of parameters is somewhat different. We show that total number of elements of these sets satisfies

$$\# \left(\bigcup_{i=1}^{7} \mathcal{E}_i \right) \ll \frac{N(\log_2 N)^2}{\log N}, \tag{3.2.1}$$

and thus they can be excluded from further considerations.

Let $\mathcal{E}_1$ denote the set of positive integers $n \le N/\log N$.

We choose $Q = N^{1/u}$, where

$$u = \frac{2\log_2 N}{\log_3 N},$$

and we denote by $\mathcal{E}_2$ the set of Q-smooth positive integers $n \le N$.

According to Corollary 1.3 of [9] (see also [4]), we have the bound

$$\#\mathcal{E}_2 = \psi(N, Q) \le Nu^{-u+o(u)} \ll \frac{N}{\log N},$$

where, as usual, $\psi(x, y) = \#\{n \le x : P(x) \le y\}$.

Next, we denote by $\mathcal{E}_3$ the set of the positive integers $n \le N$ not in $\mathcal{E}_2$ such that $P(n)^2 \mid n$. Clearly,

$$\#\mathcal{E}_3 \le \sum_{p>Q} \frac{N}{p^2} \ll \frac{N}{Q} \ll \frac{N}{\log N}.$$

Now let

$$K = \lfloor 4\log_2 N \rfloor,$$

and let $\mathcal{E}_4$ denote the set of positive integers $n \le N$ such that $\omega(n) > K$. Since $2^{\omega(n)} \le \tau(n)$ and

$$\sum_{n \le N} \tau(n) \sim N \log N$$

(see [7], Theorem 320), we get

$$\#\mathcal{E}_4 \le 2^{-K} N \log N \ll \frac{N}{\log N}.$$

Now let $n \le N$ be a positive integer not in $\cup_{i=1}^{4} \mathcal{E}_i$. This integer n has a unique representation of the form $n = mp$, where m is such that $m < N/Q$, and $p = P(n)$ is a prime number in the half-open interval $p \in \mathcal{L}(m)$, where

$$L(m) = \max\left\{Q, P(m), \frac{N}{m\log N}\right\} \qquad \text{and} \qquad \mathcal{L}(m) = (L(m),\ N/m].$$

Let $\mathcal{E}_5$ be the set of those n such that $L(m) = Q$. In this case, since $mQ \le mP(m) \le mP(n) = n \le N$, we have

$$\frac{N}{Q \log N} \le m \le \frac{N}{Q}.$$

When m is fixed, $p \le N/m$ can take at most $\pi(N/m)$ values. Thus, the number of elements $n \in \mathcal{E}_5$ is

$$\begin{aligned}
\#\mathcal{E}_5 &\ll \sum_{N/(Q\log N)\le m\le N/Q} \pi\left(\frac{N}{m}\right) \\
&\ll \sum_{N/(Q\log N)\le m\le N/Q} \frac{N}{m\log(N/m)} \\
&\ll \frac{N}{\log Q} \sum_{N/(Q\log N)\le m\le N/Q} \frac{1}{m} \\
&\ll \frac{Nu}{\log N}\left(\log\left(\frac{N}{Q}\right) - \log\left(\frac{N}{Q\log N}\right)\right) \\
&\ll \frac{Nu\log_2 N}{\log N} \ll \frac{N(\log_2 N)^2}{\log N}.
\end{aligned}$$

Let $\mathcal{E}_6$ be the set of those positive integers $n \le N$ which are not in $\cup_{i=1}^5 \mathcal{E}_i$ and such that $L(m) = P(m)$. In this case,

$$P(m) \ge \frac{N}{m\log N},$$

so we see immediately that $p = P(n) \le P(m)\log N$. Thus, $\mathcal{E}_6$ is contained in the set of all those positive integers $n \le N$ which are divisible by two primes $q < p$ such that $p \le q\log N$ and $p > Q$. In particular, $q \ge Q/\log N > Q^{1/2}$. Fix q and p. The number of such $n \le N$ is $O(N/pq)$. We recall the *Mertens formula* (see Theorem 427 in [7]), which asserts that the relation

$$\sum_{p\le x} \frac{1}{p} = \log_2 x + \alpha + O\left(\frac{1}{\log x}\right) \tag{3.2.2}$$

holds for all $x \ge 2$, where α is some absolute constant. Hence, we derive that for each q the total number $T_q(N)$ of such $n \le N$ with some prime p in the interval $(q, q\log N]$ can be estimated from (3.2.2) as

$$\begin{aligned}
T_q(N) &\ll \frac{N}{q}\sum_{q<p\le q\log N}\frac{1}{p} = \frac{N}{q}\left(\log_2(q\log N) - \log_2 q\right) + O\left(\frac{N}{q\log q}\right) \\
&= \frac{N}{q}\log\left(1 + \frac{\log_2 N}{\log q}\right) + O\left(\frac{N}{q\log q}\right) \ll \frac{N\log_2 N}{q\log q}.
\end{aligned}$$

Summing the above inequality over all $q > Q^{1/2}$, we get that

$$\begin{aligned}
\#\mathcal{E}_6 &\le \sum_{q>Q^{1/2}} T_q(N) \ll N\log_2 N \sum_{q>Q^{1/2}} \frac{1}{q\log q} \ll \frac{N\log_2 N}{\log Q} \\
&\ll \frac{Nu\log_2 N}{\log N} \ll \frac{N(\log_2 N)^2}{\log N}.
\end{aligned}$$

Finally, put $\rho(n)$ for the largest *square-full* divisor of n. Recall that a positive integer m is called square-full if $p^2 \mid m$ whenever $p \mid m$. For the purpose of the analysis of $h(n)$ only, we let $\mathcal{E}_7$ be the set of $n \le N$ such that $\rho(n) \ge (\log N)^2$. It is clear that an upper bound for the cardinality of $\mathcal{E}_7$ is

$$\#\mathcal{E}_7 \le \sum_{\substack{\rho \ge (\log N)^2 \\ \rho \text{ square-full}}} \frac{N}{\rho} \ll \frac{N}{\log N},$$

where we used the fact that

$$\sum_{\substack{\rho \ge x \\ \rho \ \text{square}-\text{full}}} \frac{1}{\rho} \ll \frac{1}{x^{1/2}},$$

which follows by partial summation from [11], Theorem 14.4.

Therefore, we have (3.2.1).

(3.3) The Remaining n.

3.3.1. *Bounds on* $\delta(N)$ *and* $\Delta(N)$. We only prove the claimed bound on $\delta(N)$ as the case of $\Delta(N)$ is completely analogous.

We assume that $n \notin \cup_{i=1}^{7} \mathcal{E}_i$ and that $n = mp$, where $p = P(n)$. Let $\mathcal{M}$ be the set of all acceptable values for m. For a given $m \in \mathcal{M}$, we have that

$$\mathcal{L}(m) = \left(\frac{N}{m \log N}, \frac{N}{m} \right].$$

Now, for $m \in \mathcal{M}$, we put

$$X_m = \lfloor (N/\log N)^{1/(\omega(m)+1)} \rfloor + 1 \qquad \text{and} \qquad Y_m = \lfloor N^{1/(\omega(m)+1)} \rfloor.$$

We let

$$\mathcal{R}(m) = \left(\frac{X_m^{\omega(m)+1}}{m}, \frac{Y_m^{\omega(m)+1}}{m} \right].$$

It is clear that $\mathcal{R}(m) \subset \mathcal{L}(m)$. Further,

$$\begin{aligned} \frac{N}{m} - \frac{Y_m^{\omega(m)+1}}{m} &= \frac{N}{m} \left(1 - \left(1 + O\left(\frac{1}{N^{1/(\omega(m)+1)}} \right) \right)^{\omega(m)+1} \right) \\ &= O\left(\frac{N\omega(m)}{mN^{1/(\omega(m)+1)}} \right) = O\left(\frac{N}{m(\log N)^2} \right) \end{aligned}$$

and similarly

$$\frac{X_m^{\omega(m)+1}}{m} - \frac{N}{m \log N} = O\left(\frac{N}{m(\log N)^2} \right), \tag{3.3.1.1}$$

which together with

$$\pi(\mathcal{L}(m)) = \pi\left(\frac{N}{m}\right) - \pi\left(\frac{N}{m\log N}\right) = \frac{N}{m\log(N/m)}\left(1 + O\left(\frac{1}{\log N}\right)\right)$$

shows that

$$\begin{aligned}\pi(\mathcal{R}(m)) &= \pi(\mathcal{L}(m)) + O\left(\left|\frac{N}{m} - \frac{Y_m^{\omega(m)+1}}{m}\right| + \left|\frac{N}{m\log N} - \frac{X_m^{\omega(m)+1}}{m}\right|\right)\\ &= \pi(\mathcal{L}(m)) + O\left(\frac{\pi(\mathcal{L}(m))}{\log N}\right).\end{aligned} \tag{3.3.1.2}$$

Let us fix $\gamma > 0$ and let $J_\gamma(N)$ be the number of $n \le N$ with $\{n^{1/\omega(n)}\} < \gamma$.

For each real $\gamma \in [0, 1)$ and positive integer $U \in [X_m, Y_m - 1]$, we put

$$Z_\gamma(m, U) = \frac{(U+\gamma)^{\omega(m)+1}}{m}$$

and define the set

$$\mathcal{R}_\gamma(m, U) = [Z_0(m, U), Z_\gamma(m, U)).$$

Now note that if $m = \mathcal{M}$ and $p \in \mathcal{L}(m)$, then for $n = pm$ we have $\omega(n) = \omega(m)+1$ and

$$n^{1/\omega(n)} \in [U, U+\gamma)$$

if and only if $p \in \mathcal{R}_\gamma(m, U)$.

Thus, it follows from (3.2.1) and (3.3.1.2) that

$$J_\gamma(N) = \sum_{m\in\mathcal{M}} \sum_{U=X_m}^{Y_m-1} \pi(\mathcal{R}_\gamma(m, U)) + O\left(\frac{N(\log_2 N)^2}{\log N}\right). \tag{3.3.1.3}$$

It is easy to see that it is enough to show that

$$J_\gamma(N) = \gamma N + O\left(\frac{N}{(\log N)^{1+o(1)}}\right) \tag{3.3.1.4}$$

as $N \to \infty$ uniformly for

$$\frac{1}{\log N} \le \gamma \le 1\,. \tag{3.3.1.5}$$

We have

$$\begin{aligned}Z_\gamma(m, U) - Z_0(m, U) &= Z_0(m, U)\left(\left(1+\frac{\gamma}{U}\right)^{\omega(m)+1} - 1\right)\\ &= \left(\gamma + O\left(\frac{\omega(m)}{U}\right)\right)\frac{Z_0(m, U)(\omega(m)+1)}{U},\end{aligned} \tag{3.3.1.6}$$

uniformly over all parameters (since $\omega(m) \le X_m^{1/2} \le U^{1/2}$ for $m \in \mathcal{M}$).

We now recall that, accordingly to Heath-Brown [8] and Huxley [10], we have

$$\pi(X+Y) - \pi(X) = \frac{Y}{\log X}\left(1 + O\left(\frac{(\log_2 X)^4}{\log X}\right)\right) \tag{3.3.1.7}$$

provided that $Y \ge X^{7/12}$.

Under the condition (3.3.1.5) and since $U \le Y_m = N^{o(1)}$ for $m \in \mathcal{M}$, we immediately see from (3.3.1.6) that

$$Z_\gamma(m, U) - Z_0(m, U) \ge Z_\gamma(m, U)^{7/12}.$$

Hence, the estimate (3.3.1.7) applies to $\pi(\mathcal{R}_\gamma(m,U))$. Remarking that

$$\frac{\omega(m)}{U} \le \frac{\omega(m)}{X_m} \ll \exp\left(-0.5\frac{\log N}{\log_2 N}\right) \tag{3.3.1.8}$$

and, by (3.3.1.1),

$$Z_\gamma(m,U) \ge Z_0(m,U) \ge \frac{U^{\omega(m)+1}}{m} \ge \frac{X_m^{\omega(m)+1}}{m} \gg \frac{N}{m\log N} \ge \frac{Q}{\log N} \gg Q^{1/2},$$

we deduce the bound

$$\begin{aligned}\pi(\mathcal{R}_\gamma(m,U)) &= \frac{Z_0(m,U)(\omega(m)+1)}{U}\left(\gamma + O\left(\frac{(\log_2 Q)^4}{\log Q}\right)\right)\\ &= \frac{Z_0(m,U)(\omega(m)+1)}{U}\left(\gamma + O\left(\frac{(\log_2 N)^5}{\log N}\right)\right). \qquad (3.3.1.9)\end{aligned}$$

Certainly, even much weaker results about primes in short intervals would suffice, but using (3.3.1.7) makes everything immediately obvious.

Substituting (3.3.1.9) in (3.3.1.3) leads us to the bound

$$\begin{aligned}J_\gamma(N) &= \left(\gamma + O\left(\frac{(\log_2 N)^5}{\log N}\right)\right)\sum_{m\in\mathcal{M}}\sum_{U=X_m}^{Y_m-1}\frac{Z_0(m,U)(\omega(m)+1)}{U}\\ &\qquad + O\left(\frac{N(\log_2 N)^2}{\log N}\right),\end{aligned}$$

which holds uniformly over all $\gamma \le 1$ under the condition (3.3.1.5). Using this formula with $\gamma = 1$ for which we obviously have $J_1(N) = N$, we see that

$$\sum_{m\in\mathcal{M}}\sum_{U=X_m}^{Y_m-1}\frac{Z_0(m,U)(\omega(m)+1)}{U} = N + O\left(\frac{N(\log_2 N)^5}{\log N}\right),$$

which concludes the proof of the upper bound.

Taking into account the contribution from the prime numbers, we see that for any $\gamma \ge 0$,

$$J_\gamma(N) \ge \pi(N) \gg \frac{N}{\log N},$$

which implies the lower bound and concludes the proof for $\delta(N)$.

As we have mentioned, the case of $\Delta(N)$ is entirely similar.

3.3.2. *Bound on* $\nabla(N)$. We let $r(n)$ be the product of all prime divisors of n, that is,

$$r(n) = \prod_{p|n} p = h(n)^{\omega(n)}.$$

To estimate $\nabla(N)$ one takes $m \in \mathcal{M}$, puts

$$\widetilde{X}_m = \left\lfloor \left(\frac{N}{(m/r(m))\log N}\right)^{1/(\omega(m)+1)} \right\rfloor + 1$$

and

$$\widetilde{Y}_m = \left\lfloor \left(\frac{N}{(m/r(m))}\right)^{1/(\omega(m)+1)} \right\rfloor,$$

and lets

$$\widetilde{\mathcal{R}}(m) = \left(\frac{\widetilde{X}_m^{\omega(m)+1}}{r(m)}, \frac{\widetilde{Y}_m^{\omega(m)+1}}{r(m)} \right].$$

As in the analysis of the previous case, one shows that

$$\pi(\widetilde{\mathcal{R}}(m)) = \pi(\mathcal{L}(m)) + O\left(\frac{\pi(\mathcal{L}(m))}{\log N}\right).$$

Thus, proceeding as in the previous analysis, we get that for all $\gamma \in (0, 1]$ the number of $n \le N$ such that $\{h(n)\} < \gamma$ is

$$\widetilde{J}_\gamma(N) = \sum_{m \in \mathcal{M}} \sum_{U=\widetilde{X}_m}^{\widetilde{Y}_m - 1} \pi(\widetilde{\mathcal{R}}_\gamma(m, U)) + O\left(\frac{N(\log_2 N)^2}{\log N}\right),$$

where $\widetilde{\mathcal{R}}_\gamma(m, U) = [\widetilde{Z}_0(m, U), \widetilde{Z}_\gamma(m, U))$ and

$$\widetilde{Z}_\gamma(m, U) = \frac{(U + \gamma)^{\omega(m)+1}}{r(m)}.$$

Since $m/r(m) \le \rho(m) = \rho(n) \le (\log N)^2$, the resulting intervals are still large enough to apply (3.3.1.7), and now an argument identical to the previous one finishes the proof of the upper bound.

For the lower bound, one uses again the contribution from the prime numbers. $\square$

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TRIALGEBRAS AND LEIBNIZ 3-ALGEBRAS

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ABSTRACT. We analyze the relationship between trialgebras (K-vector spaces equipped with three binary associative operations) and Leibniz 3-algebras (K-vector spaces equipped with a ternary bracket that verifies an identity which is a generalization of the Leibniz identity for Leibniz algebras) in a similar way as dialgebras are related to Leibniz algebras. The universal enveloping algebra $U_3L(\mathcal{L})$ of a Leibniz 3-algebras $\mathcal{L}$ is constructed and the equivalence between the categories of right $U_3L(\mathcal{L})$-modules and $\mathcal{L}$-representations is proved.

1. Introduction

It is well-known that there exists a functor $[-,-]$ from **Ass** (the category of associative algebras) to **Lie** (the category of Lie algebras) which endows an associative K-algebra A with a structure of Lie algebra by means of the bracket $[x, y] = x \cdot y - y \cdot x$ and that this functor is right adjoint to the universal enveloping algebra functor $U(-)$: **Lie** $\to$ **Ass**. In order to fix the notation, K denotes a fixed ground field throughout the paper.

A non skew-symmetric version of Lie algebras, called Leibniz algebras, was introduced by Loday [8, 9] as K-vector spaces equipped with a bilinear bracket which satisfies the Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

If we factorize a Leibniz algebra $\mathfrak{g}$ by the two-sided ideal spanned by the elements $[x, x], x \in \mathfrak{g}$, then we obtain the Lie algebra denoted by $\mathfrak{g}_{\mathrm{Lie}}$. The kernel of the canonical map $\mathfrak{g} \to \mathfrak{g}_{\mathrm{Lie}}$ is denoted by $\mathfrak{g}^{\mathrm{ann}}$. Thus we have defined a functor $(-)_{\mathrm{Lie}}$: **Leib** $\to$ **Lie** which is left adjoint to the inclusion functor from **Lie** to **Leib** which considers a Lie algebra as a Leibniz algebra.

Moreover Loday [10] introduced a type of algebras, called dialgebras, which are K-vector spaces endowed with two associative operations $\dashv$ and $\vdash$ (left and right) satisfying three axioms (see below). We denote by **Dias** the category corresponding to these objects. This kind of algebras is closely related to binary trees [10] and they play a similar role with respect to Leibniz algebras as associative algebras with respect to Lie algebras, that is, a dialgebra D can be functorially endowed with a structure of Leibniz algebra by means of the bracket $[x, y] = x \dashv y - y \vdash x$ and this functor is right adjoint to the universal enveloping dialgebra functor $Ud(-)$: **Leib** $\to$ **Dias** (see Proposition 1.9 in [10]).

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When Loday and Ronco were studying ternary planar trees [13], they found a type of algebras, called trialgebras, which are K-vector spaces equipped with three binary associative operations $\dashv$, $\perp$, and $\vdash$ (left, middle, and right) satisfying eight axioms (see below).

The goal of this paper (see section 2) is to endow trialgebras functorially with a structure of Leibniz 3-algebra (Leibniz n-algebras [2] are K-vector spaces endowed with an n-ary bracket satisfying the fundamental identity (2.4) below) by means of the bracket

$$[x, y, z] = x \dashv (y \perp z - z \perp y) - (y \perp z - z \perp y) \vdash x.$$

Moreover we construct the universal enveloping trialgebra functor $UT(-)$ from ${}_3\mathbf{Leib}$ (the category of Leibniz 3-algebras) to **Trias** (the category of trialgebras) which is left adjoint to the functor described previously. These adjoint functors are related with the adjoint functors $Ud(-) \dashv [-, -] : \mathbf{Leib} \to \mathbf{Dias}$ by means of the commutative diagram (2.12) below.

Section 3 is devoted to introducing the category ${}_3\mathbf{QLie}$ of QuasiLie 3-algebras as Leibniz 3-algebras for which the following identity holds:

$$[x, y, y] = 0$$

for all x, y. This kind of algebras plays a similar role with respect to Leibniz 3-algebras as Lie algebras with respect to Leibniz algebras. Concretely, we functorially endow an associative algebra with a structure of QuasiLie 3-algebras by mean of the bracket

$$[x, y, z] = x \cdot y \cdot z - x \cdot z \cdot y - y \cdot z \cdot x + z \cdot y \cdot x = x \cdot (y \cdot z - z \cdot y) - (y \cdot z - z \cdot y) \cdot x$$

We construct the universal Quasi-Lie 3-algebra functor $U_3(-)$ as the left adjoint to the functor $[-, -, -]$: $\mathbf{Ass} \to {}_3\mathbf{QLie}$ and, finally, we relate this pair of adjoint functors with the adjoint pair $UT(-) \dashv [-, -, -]$: ${}_3\mathbf{Leib} \to \mathbf{Trias}$ by means of the commutative diagram (3.4) below.

Finally, in section 4, we construct the universal enveloping algebra of a Leibniz 3-algebra and we prove that the category of representations of a Leibniz 3-algebra $\mathcal{L}$ is equivalent to the category of right-modules on the universal enveloping algebra $U_3L(\mathcal{L})$. Also we prove the typical properties of a universal enveloping algebra for $U_3L(\mathcal{L})$ (see Th. 1, p. 152 in [5]).

2. Trialgebras

Our goal in this section is to construct a pair of adjoint functors between the categories of Leibniz 3-algebras and trialgebras similar to the adjoint pair between the categories of Leibniz algebras and dialgebras. We start recalling a kind of algebras with three associative operations, called trialgebras, which were introduced by Loday and Ronco [13] when they studied ternary planar trees.

Definition (2.1). An associative trialgebra is a K-vector space A equipped with 3 binary associative operations: $\dashv, \perp, \vdash$: $A \otimes A \to A$ (called left, middle, and right, respectively), satisfying the following relations:

1. $(x \dashv y) \dashv z = x \dashv (y \vdash z) = x \dashv (y \perp z)$
2. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$

3. $(x \dashv y) \vdash z = x \vdash (y \vdash z) = (x \perp y) \vdash z$
4. $(x \perp y) \dashv z = x \perp (y \dashv z)$
5. $(x \dashv y) \perp z = x \perp (y \vdash z)$
6. $(x \vdash y) \perp z = x \vdash (y \perp z)$

A morphism between two associative trialgebras is a linear map which is compatible with the three operations. We denote by **Trias** the category of associative trialgebras.

Examples (2.2).

i) An associative K-algebra A endowed with the binary operations $x \dashv y = x \perp y = x \vdash y = x.y$, for all $x, y \in$ A. This operation defines a functor from **Ass** to **Trias** which has as left adjoint the functor $(-)_{\mathrm{Ass}}$: **Trias** $\to$ **Ass** which maps a trialgebra T to the associative algebra $\mathrm{T}_{\mathrm{Ass}}$, which is the factorization of T by the ideal (in sense of trialgebras) spanned by the elements of the form $x \vdash y - x \perp y; x \dashv y - x \perp y; x, y \in T$.

ii) Let A be an associative K-algebra. Take $T = \mathrm{A} \otimes \mathrm{A} \otimes \mathrm{A}$ and define the following operations:

$$a \otimes b \otimes c \dashv a' \otimes b' \otimes c' := a \otimes b \otimes ca'b'c'$$

$$a \otimes b \otimes c \vdash a' \otimes b' \otimes c' := abca' \otimes b' \otimes c'$$

$$a \otimes b \otimes c \perp a' \otimes b' \otimes c' := a \otimes bca'b' \otimes c'$$

Extending these formulas by linearity on $\mathrm{A} \otimes \mathrm{A} \otimes \mathrm{A}$ one obtains product applications $\dashv, \vdash, \perp$ which satisfy the trialgebra axioms.

iii) Opposite Trialgebra: The opposite trialgebra of a trialgebra $(T, \dashv, \vdash, \perp)$ is the trialgebra T^{op} with the same underlying vector space and product given by

$$x \dashv' y := y \vdash x; x \vdash' y := y \dashv x; x \perp' y := y \perp x$$

iv) For other examples we refer to [13].

In order to show the role which trialgebras play with respect to Leibniz 3-algebras we start recalling few well-known material.

Leibniz algebras [8, 9] are a non-skew symmetric version to Lie algebras, that is, they are K-vector spaces $\mathfrak{g}$ equipped with a bilinear bracket $[-,-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying the Leibniz relation $[x,[y,z]] = [[x,y],z] - [[x,z],y]$. We denote by **Lie** and **Leib** the categories of Lie algebras and Leibniz algebras, respectively.

A *dialgebra* [10] is a K-vector space equipped with two associative operations: $\dashv, \vdash$ (called left and right, respectively), satisfying the following relations:

1. $(x \dashv y) \dashv z = x \dashv (y \vdash z)$
2. $(x \vdash y) \dashv z = x \vdash (y \dashv z)$
3. $(x \dashv y) \vdash z = x \vdash (y \vdash z)$

A morphism between two associative dialgebras is a linear map which preserves the two operations. We will denote by **Dias** the category of associative dialgebras.

It is well-known that an associative algebra A can be endowed with a structure of dialgebra by means of the operations $x \dashv y = x \cdot y = x \vdash y$, for all $x, y \in$ A [10]. Conversely, for a dialgebra D, let $\mathrm{D}_{\mathrm{Ass}}$ be the factorization of D by the two sided ideal (in the sense of dialgebras) spanned by the elements of the form $x \dashv y - x \vdash y, \forall x, y \in$ D. It is clear that $\dashv = \vdash$ in $\mathrm{D}_{\mathrm{Ass}}$ and so $\mathrm{D}_{\mathrm{Ass}}$ is an associative algebra. The factorization map $\mathrm{D} \to \mathrm{D}_{\mathrm{Ass}}$ is universal among the applications from D to an associative algebra, that is, the associativization functor $(-)_{\mathrm{Ass}}\colon \mathbf{Dias} \to \mathbf{Ass}$ is left adjoint to the inclusion functor $\mathrm{inc}\colon \mathbf{Ass} \to \mathbf{Dias}$. We summarize this information in the following diagram:

(2.3)
$$\begin{tikzcd}
\mathbf{Lie} \arrow[d, shift right, "\mathbf{U(\text{-})}"'] \arrow[r, hook, shift right] & \mathbf{Leib} \arrow[l, shift right, "(-)_{\mathrm{Lie}}"'] \arrow[d, shift right, "\mathbf{Ud(\text{-})}"'] \\
\mathbf{Ass} \arrow[u, shift right, "\mathbf{[\text{-},\text{-}]}"'] \arrow[r, hook, shift right] & \mathbf{Dias} \arrow[l, shift right, "(-)_{\mathrm{Ass}}"'] \arrow[u, shift right, "\mathbf{[\text{-},\text{-}]}"']
\end{tikzcd}$$

From the diagonal composition of adjoint functors in diagram (2.3) one derives the isomorphism $Ud(\mathfrak{g})_{\mathrm{Ass}} \cong U(\mathfrak{g}_{\mathrm{Lie}})$ (see Lemma 4.8 in [11]).

On the other hand, a dialgebra D can be functorially endowed with a structure of trialgebra by means of one of the inclusion functors:

1. Taking the dialgebra operations $\dashv, \vdash$ and defining $x \perp y := x \dashv y$.
2. Taking the dialgebra operations $\dashv, \vdash$ and defining $x \perp y := x \vdash y$.

Conversely, for any trialgebra T, let $T_{1\mathrm{Dias}}$ be the factorization of T by the three-sided ideal I_1 spanned by the elements of the form $x \perp y - x \dashv y$, and let $T_{2\mathrm{Dias}}$ be the factorization of T by the three-sided ideal I_2 spanned by the elements of the form $x \perp y - x \vdash y$. It is clear that $x \perp y = x \dashv y$ in the first case and $x \perp y = x \vdash y$ in the second one. Therefore, $T_{i\mathrm{Dias}}, (i = 1, 2)$, is an associative dialgebra. Moreover, the factorization application $T \to T_{i\mathrm{Dias}}, (i = 1, 2)$ is universal for applications from T to any associative dialgebra, that is, the dialgebrization functor $(-)_{i\mathrm{Dias}}\colon \mathbf{Trias} \to \mathbf{Dias}$ is left adjoint to the inclusion functor $\mathrm{inc}_i\colon \mathbf{Dias} \to \mathbf{Trias}, (i = 1, 2)$.

Lie algebras and Leibniz algebras are a particular case $(n = 2)$ of Lie n-algebras and Leibniz n-algebras respectively [4, 14, 2]. We recall that a *Leibniz n-algebra* is a K-vector space $\mathcal{L}$ equipped with an n-linear operation $[-, \ldots, -]\colon \mathcal{L}^{\otimes n} \to \mathcal{L}$ satisfying the following fundamental identity:

(2.4)
$$[[x_1, x_2, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, y_2, \ldots, y_n], x_{i+1}, \ldots, x_n]$$

If the bracket is skew-symmetric, that means

$$[x_1, x_2, \ldots, x_n] = (-1)^{\epsilon(\sigma)}.[x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}]$$

for all $\sigma \in S_n$ (S_n is the symmetric group of n elements and the number $\epsilon(\sigma)$ is equal to 0 or 1 depending on the parity of the permutation σ), then we have a Lie n-algebra. A morphism of Leibniz n-algebras (Lie n-algebras) is a linear map which preserves the bracket. We denote by ${}_n\mathbf{Lie}$ and ${}_n\mathbf{Leib}$ the categories of Lie n-algebras and Leibniz n-algebras, respectively.

In this paper we concentrate on the case $n = 3$ in order to establish an adjunction between trialgebras and Leibniz 3-algebras similar to the adjunction between dialgebras and Leibniz algebras. To achieve this we need the following results:

PROPOSITION (2.5). *Let* A *be a trialgebra. Then* A *is a Leibniz 3-algebra with respect to the bracket*

$$[x, y, z] = x \dashv (y \perp z) - (y \perp z) \vdash x - x \dashv (z \perp y) + (z \perp y) \vdash x =$$
$$= x \dashv (y \perp z - z \perp y) - (y \perp z - z \perp y) \vdash x$$

for all $x, y, z \in \mathrm{A}$.

Proof. The proof is straightforward and we leave it to the reader. □

Also it is possible to establish other structures of Leibniz 3-algebras from a trialgebra. To do this we introduce the notion of noncommutative Leibniz-Poisson algebra. A cohomology theory of this kind of algebras was developed in [3].

Definition (2.6). A non-commutative Leibniz-Poisson algebra (in brief NLP-algebra) is a K-vector space P together with two bilinear operations

$$\cdot : P \times P \to P, (x, y) \mapsto x \cdot y$$
$$[-, -]: P \times P \to P, (x, y) \mapsto [x, y]$$

such that $(P, [-, -])$ is a Leibniz algebra, $(P, \cdot)$ is an associative algebra and the following identity holds:

$$[a \cdot b, c] = a \cdot [b, c] + [a, c] \cdot b$$

for all $a, b, c \in P$.

Examples (2.7).

i) Poisson algebras.

ii) Any Leibniz algebra is a NLP-algebra with trivial associative product $(a \cdot b = 0)$. On the other hand, any associative algebra is a NLP-algebra with the usual bracket $[a, b] = a \cdot b - b \cdot a$.

iii) Any associative dialgebra is a NLP-algebra with respect to the operations $a \cdot b = a \vdash b$; $[a, b] = a \dashv b - b \vdash a$.

iv) Any associative trialgebra is a NLP-algebra with respect to the operations $a \cdot b = a \perp b$; $[a, b] = a \dashv b - b \vdash a$.

v) If P_1 and P_2 are NLP-algebras, then the K-module $P_1 \otimes P_2$ endowed with the operations

$$(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = (a_1 \cdot b_1) \otimes (a_2 \cdot b_2)$$
$$[a_1 \otimes a_2, b_1 \otimes b_2] = [a_1, [b_1, b_2]] \otimes a_2 + a_1 \otimes [a_2, [b_1, b_2]]$$

is a NLP-algebra.

vi) For examples coming from Physics the reader is referred to [6].

LEMMA (2.8). *If* P *is a non-commutative Leibniz-Poisson algebra, then*

$$\{x, y, z\} := [x, y \cdot z]$$

defines a Leibniz 3-algebra structure on P.

Proof.

$$\begin{aligned}\{\{x,y,z\},a,b\} - \{\{x,a,b\},y,z\} - \{x,\{y,a,b\},z\} - \{x,y,\{z,a,b\}\} \\ &= [[x,y\cdot z],a\cdot b] - [[x,a\cdot b],y\cdot z] - [x,[y,a\cdot b]\cdot z] - [x,y\cdot[z,a\cdot b]] \\ &= [[x,y\cdot z],a\cdot b] - [[x,a\cdot b],y\cdot z] - [x,[y,a\cdot b]\cdot z + y\cdot[z,a\cdot b]] \\ &= [[x,y\cdot z],a\cdot b] - [[x,a\cdot b],y\cdot z] - [x,[y\cdot z,a\cdot b]]\end{aligned}$$

The last term vanishes thanks to the Leibniz identity. □

LEMMA (2.9). *If P is a non-commutative Leibniz-Poisson algebra , then P^{op} is also a non-commutative Leibniz-Poisson algebra. Here P^{op} has the same Leibniz algebra structure as P, but the associative algebra structure in P^{op} is*

$$x * y = y \cdot x$$

LEMMA (2.10). *If* A *is an associative trialgebra, then* $(\mathrm{A}, \perp, [-,-])$ *is a non-commutative Leibniz-Poisson algebra, where*

$$[a,b] = a \dashv b - b \vdash a\,.$$

COROLLARY (2.11). *Let* A *be a trialgebra, then* A *is a Leibniz 3-algebra in two ways:*

1. *with respect to the bracket:* $[x,y,z]_1 = x \dashv (y \perp z) - (y \perp z) \vdash x$*;*
2. *with respect to the bracket:* $[x,y,z]_2 = (z \perp y) \vdash x - x \dashv (z \perp y)$.

Proof. Apply Lemmas (2.8), (2.9) and (2.10). □

Let us observe that $[x,y,z]$ in Proposition (2.5) is equal to $[x,y,z]_1 + [x,y,z]_2$. We will use the structure in Proposition (2.5) since it is the structure which guarantees the commutativity of square (2.12) below.

Proposition (2.5) gives us a functor $[-,-,-]$: **Trias** $\to$ ${}_3$**Leib** which has as left adjoint the universal enveloping trialgebra functor **UT**$(-)$: ${}_3$**Leib** $\to$ **Trias**. This functor assigns to a Leibniz 3-algebra $\mathcal{L}$ the trialgebra $UT(\mathcal{L})$ defined by

$$UT(\mathcal{L}) = \frac{\oplus_{n\geq 1} K[P_n] \otimes \mathcal{L}^{\otimes n}}{I}$$

where $\oplus_{n\geq 1} K[P_n] \otimes \mathcal{L}^{\otimes n}$ is the free associative trialgebra over the underlying vector space $\mathcal{L}$ [13] and $I = \langle \{x \dashv (y \perp z) - (y \perp z) \vdash x - x \dashv (z \perp y) + (z \perp y) \vdash x - [x,y,z] \mid x,y,z \in \mathcal{L}\}\rangle$.

It is well-known that a Leibniz algebra $\mathfrak{g}$ can be endowed with a Leibniz 3-algebra structure by means of the operation $[x,y,z] = [x,[y,z]]$ [2], thus we obtain the following diagram which extends the diagram (2.3)

(2.12)
$$\begin{tikzcd} \mathbf{Leib} \arrow[r, "\omega"] \arrow[d, shift right, "\mathbf{Ud(\text{-})}"'] & {}_3\mathbf{Leib} \arrow[d, shift right, "\mathbf{UT(\text{-})}"'] \\ \mathbf{Dias} \arrow[u, shift right, "\mathbf{[\text{-},\text{-}]}"'] \arrow[r, hook, shift right, "\mathrm{inc}_i"'] & \mathbf{Trias} \arrow[u, shift right, "\mathbf{[\text{-},\text{-},\text{-}]}"'] \arrow[l, shift right, "(-)_{i\,\mathrm{Dias}}"'] \end{tikzcd}$$

It is easy to verify that a Leibniz 3-algebra coming from a dialgebra D via **Trias** is same as that via **Leib**; that is, diagram (2.12) is commutative in the following way: $\mathbf{Dias} \overset{[-,-]}{\to} \mathbf{Leib} \overset{\omega}{\to} {}_3\mathbf{Leib} \cong \mathbf{Dias} \overset{\mathrm{inc}_i}{\hookrightarrow} \mathbf{Trias} \overset{[-,-,-]}{\to} {}_3\mathbf{Leib}$.

3. QuasiLie 3-algebras

The goal of this section is to construct a diagram similar to diagram (2.3) in the category ${}_3\mathbf{Leib}$. In order to achieve our goal we need to introduce a new kind of ternary algebras.

Definition (3.1). A QuasiLie algebra of order 3 or QuasiLie 3-algebra is a Leibniz 3-algebra $\mathcal{L}$ for which the following identity holds:

$$[x, y, y] = 0$$

for all $x, y \in \mathcal{L}$.

Obviously, a homomorphism of QuasiLie 3-algebras is a linear map such that preserves the bracket. We denote by ${}_3\mathbf{QLie}$ the category of QuasiLie 3-algebras.

Examples (3.2). i) Lie 3-algebras [4, 14].

ii) Lie triple systems [7] are K-vector spaces equipped with a trilinear bracket which satisfies the identity (2.4) and, instead of skew-symmetry, satisfies the conditions

$$[x, y, z] + [z, x, y] + [y, z, x] = 0$$

and

$$[x, y, y] = 0$$

This is an example of QuasiLie 3-algebras which are not Lie 3-algebras.

iii) Let A be a K-associative algebra equipped with a K-linear map $D\colon \mathrm{A} \to \mathrm{A}$ satisfying

$$D(a \cdot Db) = Da \cdot Db = D(Da \cdot b)$$

for all $a, b \in \mathrm{A}$. If we define the bracket

$$[a, b, c] = a \cdot b \cdot Dc - a \cdot Dc \cdot b - b \cdot Dc \cdot a + Dc \cdot b \cdot a$$

then we have a Leibniz 3-algebra, when D is an endomorphism of algebras such that $D^2 = D$ or D is a derivation such that $D^2 = 0$, which is a QuasiLie 3-algebra, for instance, in particular case of $D = Id$.

iv) The particular case $D = Id$ in example iii) shows a K-associative algebra A endowed with a structure of Quasi-Lie 3-algebra by means of the bracket

$$[x, y, z] = x \cdot y \cdot z - x \cdot z \cdot y - y \cdot z \cdot x + z \cdot y \cdot x = x \cdot (y \cdot z - z \cdot y) - (y \cdot z - z \cdot y) \cdot x$$

$x, y, z \in$A. This QuasiLie 3-algebra is not a Lie 3-algebra.

If we factorize a Leibniz 3-algebra $\mathcal{L}$ by the three-sided ideal spanned by all brackets of the form $[x, y, y]$, $x, y \in \mathcal{L}$, then we obtain the QuasiLie 3-algebra denoted by $\mathcal{L}_{Q\,\mathrm{Lie}}$. The canonical morphism $\mathcal{L} \to \mathcal{L}_{Q\,\mathrm{Lie}}$ is universal for every morphism from $\mathcal{L}$ to a QuasiLie 3-algebra, that is, the functor $(-)_{Q\,\mathrm{Lie}}\colon {}_3\mathbf{Leib} \to {}_3\mathbf{QLie}$ is left adjoint to the inclusion functor. We denote the kernel of the canonical morphism $\mathcal{L} \to \mathcal{L}_{Q\,\mathrm{Lie}}$ by $\mathcal{L}^{Q\,\mathrm{ann}}$.

On the other hand, the functor $[-,-,-]\colon \mathbf{Ass} \to {}_3\mathbf{QLie}$ described in Example (3.2) iv) has as left adjoint the universal enveloping QuasiLie 3-algebra functor. The universal enveloping QuasiLie 3-algebra functor assigns to a QuasiLie 3-algebra $\mathcal{L}$ the factorization of the free QuasiLie 3-algebra $QL(\mathcal{L})$ by the ideal spanned by the elements of the form $[x,y,z] - x\cdot y\cdot z + x\cdot z\cdot y + y\cdot z\cdot x - z\cdot y\cdot x$, for all $x,y,z \in \mathcal{L}$.

PROPOSITION (3.3). *There exists the free QuasiLie 3-algebra over a set X.*

Proof. Following the same way as [15], we define a 3-magma $\mathcal{M}$ as a set together with a ternary operation $\omega\colon \mathcal{M}\times\mathcal{M}\times\mathcal{M}\to\mathcal{M}$, $(x,y,z)\mapsto\omega(x,y,z)$. For a set X define inductively the family of sets $X_n (n\geq 1)$ as follows:

i) $X_1 = X$, $X_2 = X\times X$,

ii) $X_n = \coprod_{p+q+r=n} X_p\times X_q\times X_r (n\geq 3)$ (= disjoint union).

Put $\mathcal{M}_X = \coprod_{n=1}^{\infty} X_n$ and define $\mathcal{M}_X\times\mathcal{M}_X\times\mathcal{M}_X\to\mathcal{M}_X$ by means of $X_p\times X_q\times X_r\to X_{p+q+r}\subset\mathcal{M}$.
$\mathcal{M}_X$ is the free 3-magma on X, since for every 3-magma $\mathcal{N}$ and every map $f\colon X\to\mathcal{N}$ there exists a unique homomorphism of 3-magma $F\colon\mathcal{M}_X\to\mathcal{N}$ which extends f.
Let A_X be the free K-3-algebra (K-vector space equipped with a ternary operation called product) associated to the free magma $\mathcal{M}_X$ and let I be the three-sided ideal of A_X spanned by the elements of the form $\omega(a,b,b)$ and

$$\begin{aligned} FI(a,b,c,d,e) = \omega(\omega(a,b,c),d,e) - \omega(\omega(a,d,e),b,c) - \omega(a,\omega(b,d,e),c)\\ - \omega(a,b,\omega(c,d,e)), \end{aligned}$$

then A_X/I is the free QuasiLie 3-algebra on X, denoted by QL(X).

After Proposition (3.3), the universal enveloping QuasiLie 3-algebra of $\mathcal{L}$ is

$$U_3(\mathcal{L}) = \frac{QL(\mathcal{L})}{\langle\{[x,y,z] - x\cdot y\cdot z + x\cdot z\cdot y + y\cdot z\cdot x - z\cdot y\cdot x\}\rangle}$$

since $\mathcal{L}\hookrightarrow QL(\mathcal{L})\twoheadrightarrow U_3(\mathcal{L})$ and every homomorphism of QuasiLie 3-algebras $f\colon\mathcal{L}\to[\Lambda]$ can be extended to a homomorphism $\overline{f}\colon QL(\mathcal{L})\to[\Lambda]$ by the universal property of the free QuasiLie 3-algebra. Plainly $\overline{f}$ vanishes on the ideal spanned by the elements $[x,y,z]-x\cdot y\cdot z+x\cdot z\cdot y+y\cdot z\cdot x-z\cdot y\cdot x$, so it induces a homomorphism of associative algebras $\varphi\colon U_3(\mathcal{L})\to\Lambda$ which extends f. Here $[\Lambda]$ denotes an associative algebra endowed with a QuasiLie 3-algebra structure given by Example (3.2) iv). $\square$

We can summarize this information in the following diagram of adjoint functors:

$$\begin{tikzcd} {}_3\mathbf{QLie} \arrow[r, hook, shift right] \arrow[d, shift right, "U_3(-)"'] & {}_3\mathbf{Leib} \arrow[l, shift right, "(-)_{Q\,\mathrm{Lie}}"'] \arrow[d, shift right, "\mathbf{UT(\text{-})}"'] \\ \mathbf{Ass} \arrow[u, shift right, "{[\text{-},\text{-},\text{-}]}"'] \arrow[r, shift right] & \mathbf{Trias} \arrow[u, shift right, "{[\text{-},\text{-},\text{-}]}"'] \arrow[l, shift right, "(-)_{\mathrm{Ass}}"'] \end{tikzcd} \tag{3.4}$$

PROPOSITION (3.5). *For a Leibniz 3-algebra $\mathcal{L}$ the following isomorphism holds:*

$$\mathbf{U_3}(\mathcal{L}_{Q\,\mathrm{Lie}}) \cong \mathbf{UT}(\mathcal{L})_{\mathrm{Ass}}\,.$$

Proof. The composition of adjunctions gives us an adjunction in the diagonal and apply that adjoint functors are unique up to isomorphism. □

4. Universal enveloping algebra

In the category **Lie** the universal enveloping algebra functor $U\colon \mathbf{Lie} \to \mathbf{Ass}$ has the following property: the category of representations over a Lie algebra $\mathfrak{g}$ is equivalent to the category of $U(\mathfrak{g})$-modules. Nevertheless, in **Leib** the same property does not hold in the sense that the composition of the adjoint functors from **Leib** to **Ass** in diagram (2.3) does not reproduce the analogous property. To solve this problem, in [12] Loday and Pirashvili constructed the functor $UL\colon \mathbf{Leib} \to \mathbf{Ass}$, defined by

$$UL(\mathfrak{g}) := T(\mathfrak{g}^l \oplus \mathfrak{g}^r)/I$$

where $\mathfrak{g}^l$ and $\mathfrak{g}^r$ are isomorphic copies of $\mathfrak{g}$ and I is an appropriate two-sided ideal. One verifies that a $\mathfrak{g}$-representation in **Leib** is equivalent to a $UL(\mathfrak{g})$-module. Moreover one verifies that $UL(\mathfrak{g}) \cong U(\mathfrak{g}_{\mathrm{Lie}}) \oplus (U(\mathfrak{g}_{\mathrm{Lie}}) \otimes \mathfrak{g})$ and the subalgebra of $UL(\mathfrak{g})$ generated by the elements r_x, $x \in \mathfrak{g}$, which are the isomorphic copies of x in $\mathfrak{g}^r$, is isomorphic to $U(\mathfrak{g}_{\mathrm{Lie}})$.

The goal of this section is to analyze this problem in the category ${}_3\mathbf{Leib}$.

Definition (4.1). [1]. A representation of a Leibniz n-algebra $\mathcal{L}$ is a K-vector space M endowed with n actions $[-,\overset{n}{\ldots},-]\colon \mathcal{L}^{\otimes i} \otimes \mathrm{M} \otimes \mathcal{L}^{\otimes n-i-1} \to \mathrm{M}$, $0 \le i \le n-1$, satisfying the following $2n-1$ axioms:

1. For $2 \le k \le n$

$$\rho_k([l_1,\ldots,l_n], l_{n+1},\ldots,l_{2n-2}) = \sum_{i=1}^{n} \rho_i(l_1,\ldots,\hat{l}_i,\ldots,l_n)\cdot\rho_k(l_i,l_{n+1},\ldots,l_{2n-2});$$

2. For $1 \le k \le n$

$$\begin{aligned}&[\rho_1(l_n,\ldots,l_{2n-2}),\rho_k(l_1,\ldots,l_{n-1})]\\ &\qquad\qquad= \sum_{i=1}^{n-1} \rho_k(l_1,\ldots,l_{i-1},[l_i,l_n,\ldots,l_{2n-2}],l_{i+1},\ldots,l_{n-1})\end{aligned}$$

the multilinear applications $\rho_i\colon \mathcal{L}^{\otimes n-1} \to \mathrm{End}_K(\mathrm{M})$ being defined by

$$\rho_i(l_1,\ldots,l_{n-1})(m) = [l_1,\ldots,l_{i-1},m,l_i,\ldots,l_{n-1}],\ (1 \le i \le n)$$

and the bracket on $\mathrm{End}_K(\mathrm{M})$ being the usual one for associative algebras.

In the particular case $n=3$, that is, $\mathcal{L}$ is a Leibniz 3-algebra, a representation M of $\mathcal{L}$ consists of three applications

$$[-,-,-]\colon \mathcal{L}\otimes\mathcal{L}\otimes\mathrm{M} \to \mathrm{M};\ [-,-,-]\colon \mathcal{L}\otimes\mathrm{M}\otimes\mathcal{L} \to \mathrm{M};\ [-,-,-]\colon \mathrm{M}\otimes\mathcal{L}\otimes\mathcal{L} \to \mathrm{M}$$

satisfying the following axioms:

1. $[[l_1,l_2,l_3],l_4,m] = [[l_1,l_4,m],l_2,l_3] + [l_1,[l_2,l_4,m],l_3] + [l_1,l_2,[l_3,l_4,m]]$
2. $[[l_1,l_2,l_3],m,l_4] = [[l_1,m,l_4],l_2,l_3] + [l_1,[l_2,m,l_4],l_3] + [l_1,l_2,[l_3,m,l_4]]$

3. $[[m, l_1, l_2], l_3, l_4] = [[m, l_3, l_4], l_1, l_2] + [m, [l_1, l_3, l_4], l_2] + [m, l_1, [l_2, l_3, l_4]]$
4. $[[l_1, m, l_2], l_3, l_4] = [[l_1, l_3, l_4], m, l_2] + [l_1, [m, l_3, l_4], l_2] + [l_1, m, [l_2, l_3, l_4]]$
5. $[[l_1, l_2, m], l_3, l_4] = [[l_1, l_3, l_4], l_2, m] + [l_1, [l_2, l_3, l_4], m] + [l_1, l_2, [m, l_3, l_4]]$

For a Leibniz 3-algebra $\mathcal{L}$, we consider three copies $(\mathcal{L}\otimes\mathcal{L})^l, (\mathcal{L}\otimes\mathcal{L})^m, (\mathcal{L}\otimes\mathcal{L})^r$ of the Leibniz algebra $(\mathcal{L} \otimes \mathcal{L})$ whose bracket is given by

$$[x \otimes y, a \otimes b] = [x, a, b] \otimes y + x \otimes [y, a, b]$$

We denote by $l_{x\otimes y}, m_{x\otimes y}, r_{x\otimes y}$ the elements of $(\mathcal{L} \otimes \mathcal{L})^l, (\mathcal{L} \otimes \mathcal{L})^m, (\mathcal{L} \otimes \mathcal{L})^r$ corresponding to $x \otimes y \in \mathcal{L}\otimes\mathcal{L}$. We consider the tensorial algebra $T((\mathcal{L}\otimes\mathcal{L})^l \oplus (\mathcal{L} \otimes \mathcal{L})^m \oplus (\mathcal{L} \otimes \mathcal{L})^r)$ and the following relations

i) $r_{([l_1,l_2,l_3]\otimes l_4)} - l_{(l_2\otimes l_3)}r_{(l_1\otimes l_4)} - m_{(l_1\otimes l_3)}r_{(l_2\otimes l_4)} - r_{(l_1\otimes l_2)}r_{(l_3\otimes l_4)}$
ii) $m_{([l_1,l_2,l_3]\otimes l_4)} - l_{(l_2\otimes l_3)}m_{(l_1\otimes l_4)} - m_{(l_1\otimes l_3)}m_{(l_2\otimes l_4)} - r_{(l_1\otimes l_2)}m_{(l_3\otimes l_4)}$
iii) $l_{(l_3\otimes l_4)}l_{(l_1\otimes l_2)} - l_{(l_1\otimes l_2)}l_{(l_3\otimes l_4)} - l_{([l_1,l_3,l_4]\otimes l_2)} - l_{(l_1\otimes [l_2,l_3,l_4])}$
iv) $l_{(l_3\otimes l_4)}m_{(l_1\otimes l_2)} - m_{([l_1,l_3,l_4]\otimes l_2)} - m_{(l_1\otimes l_2)}l_{(l_3\otimes l_4)} - m_{(l_1\otimes [l_2,l_3,l_4])}$
v) $l_{(l_3\otimes l_4)}r_{(l_1\otimes l_2)} - r_{([l_1,l_3,l_4]\otimes l_2)} - r_{(l_1\otimes [l_2,l_3,l_4])} - r_{(l_1\otimes l_2)}l_{(l_3\otimes l_4)}$

Let us observe that from the relations ii) and iv) we can deduce the following:

i') $m_{(l_1\otimes l_3)}m_{(l_2\otimes l_4)} + m_{(l_1\otimes l_4)}l_{(l_2\otimes l_3)} + r_{(l_1\otimes l_2)}m_{(l_3\otimes l_4)} + m_{(l_1\otimes [l_4,l_2,l_3])} = 0$

and from the relations i) and v) we can deduce the following:

ii') $m_{(l_1\otimes l_3)}r_{(l_2\otimes l_4)} + r_{(l_1\otimes l_4)}l_{(l_2\otimes l_3)} + r_{(l_1\otimes l_2)}r_{(l_3\otimes l_4)} + r_{(l_1\otimes [l_4,l_2,l_3])} = 0$

Definition (4.2). The universal enveloping algebra of the Leibniz 3-algebra $\mathcal{L}$ is the associative unitary algebra

$$U_3L(\mathcal{L}) := T((\mathcal{L} \otimes \mathcal{L})^l \oplus (\mathcal{L} \otimes \mathcal{L})^m \oplus (\mathcal{L} \otimes \mathcal{L})^r)/I$$

where I is the two-sided ideal corresponding to the relations i'), ii'), iii), iv), v).

THEOREM (4.3). *The category of representations of the Leibniz 3-algebra $\mathcal{L}$ is equivalent to the category of right modules on $U_3L(\mathcal{L})$.*

Proof. Let M be a representation of $\mathcal{L}$. We define a right action from $U_3L(\mathcal{L})$ on the K-vector space M as follows. Firstly $(\mathcal{L} \otimes \mathcal{L})^l, (\mathcal{L} \otimes \mathcal{L})^m, (\mathcal{L} \otimes \mathcal{L})^r$ act on M by

$$m \cdot l_{(x\otimes y)} = [m, x, y]; m \cdot m_{(x\otimes y)} = [x, m, y]; m \cdot r_{(x\otimes y)} = [x, y, m];$$

then we extend this actions to an action of $T((\mathcal{L} \otimes \mathcal{L})^l \oplus (\mathcal{L} \otimes \mathcal{L})^m \oplus (\mathcal{L} \otimes \mathcal{L})^r)$ by composition and linearity.

The axioms 1-5 of representation imply that the relations i'), ii'), iii)-v) act trivially. Thus M is endowed with a structure of $U_3L(\mathcal{L})$-module.

Conversely, we start with a $U_3L(\mathcal{L})$-module. The restriction of actions to $(\mathcal{L} \otimes \mathcal{L})^l, (\mathcal{L} \otimes \mathcal{L})^m, (\mathcal{L} \otimes \mathcal{L})^r$ provides three actions of $\mathcal{L} \otimes \mathcal{L}$ which make M a representation of $\mathcal{L} \otimes \mathcal{L}$.

Thanks to relation iii) we have that the subalgebra spanned by the elements $l_{x\otimes y}, x \otimes y \in \mathcal{L} \otimes \mathcal{L}$, is isomorphic to $U((\mathcal{L} \otimes \mathcal{L})_{Lie})$.

Let $d\colon (\mathcal{L} \otimes \mathcal{L})^l \oplus (\mathcal{L} \otimes \mathcal{L})^m \oplus (\mathcal{L} \otimes \mathcal{L})^r \to U((\mathcal{L} \otimes \mathcal{L})_{Lie})$ be the K-linear application defined by $d(l_{x\otimes y}) = -\overline{x \otimes y} = -x \otimes y + J$, where $J = \langle\{[x \otimes y, x \otimes y] \mid x \otimes y \in \mathcal{L} \otimes \mathcal{L}\}\rangle$, $d(m_{x\otimes y}) = 0, d(r_{x\otimes y}) = 0$. One extends d to an algebra homomorphism from $T((\mathcal{L} \otimes \mathcal{L})^l \oplus (\mathcal{L} \otimes \mathcal{L})^m \oplus (\mathcal{L} \otimes \mathcal{L})^r)$ to $U((\mathcal{L} \otimes \mathcal{L})_{Lie})$ which vanishes on the ideal spanned by the relations i)–v), hence d extends to an algebra homomorphism $d\colon U_3L(\mathcal{L}) \to U((\mathcal{L}\otimes\mathcal{L})_{Lie})$.

On the other hand, $s\colon \mathcal{L}\otimes\mathcal{L} \to U_3L(\mathcal{L})$, $s(x\otimes y) = -l_{x\otimes y}$ is a Leibniz algebra homomorphism which vanishes on $(\mathcal{L}\otimes\mathcal{L})^{\text{ann}}$ and hence it induces a Lie algebra homomorphism $s\colon (\mathcal{L}\otimes\mathcal{L})_{Lie} \to U_3L(\mathcal{L})$ which extends to an algebra homomorphism $s\colon U((\mathcal{L}\otimes\mathcal{L})_{Lie}) \to U_3L(\mathcal{L})$, $s(\overline{x\otimes y}) = -l_{x\otimes y}$. Moreover s is a section of d. Let H be the two-sided ideal of $U_3L(\mathcal{L})$ spanned by $m_{x\otimes y}, r_{x\otimes y}$, $x\otimes y \in \mathcal{L}\otimes\mathcal{L}$. It is clear that $H = \text{Ker } d$, so we have the following split exact sequence:

$$0 \longrightarrow H \longrightarrow U_3L(\mathcal{L}) \underset{d}{\overset{s}{\rightleftarrows}} U((\mathcal{L}\otimes\mathcal{L})_{\text{Lie}}) \longrightarrow 0.$$

□

Definition (4.4). Let be $\mathcal{L} \in {}_3\mathbf{Leib}$ and $A \in \mathbf{Ass}$. A trihomomorphism from $\mathcal{L}$ to A consists of a triple of K-linear maps $(\varphi, \psi, \phi) : \mathcal{L}\otimes\mathcal{L} \to A$ satisfying the following relations.

a) $\phi([l_1, l_2, l_3]\otimes l_4)) = \varphi(l_2\otimes l_3)\cdot\phi(l_1\otimes l_4)+\psi(l_1\otimes l_3)\cdot\phi(l_2\otimes l_4)+ \phi(l_1\otimes l_2)\cdot\phi(l_3\otimes l_4)$

b) $\psi([l_1, l_2, l_3]\otimes l_4) = \varphi(l_2\otimes l_3)\cdot\psi(l_1\otimes l_4)+\psi(l_1\otimes l_3)\cdot\psi(l_2\otimes l_4)+\phi(l_1\otimes l_2)\cdot\psi(l_3\otimes l_4)$

c) $\varphi[l_1\otimes l_2, l_3\otimes l_4] = \varphi(l_3\otimes l_4)\cdot\varphi(l_1\otimes l_2) - \varphi(l_1\otimes l_2)\cdot\varphi(l_3\otimes l_4)$

d) $\psi[l_1\otimes l_2, l_3\otimes l_4] = \varphi(l_3\otimes l_4)\cdot\psi(l_1\otimes l_2) - \psi(l_1\otimes l_2)\cdot\varphi(l_3\otimes l_4)$

e) $\phi[l_1\otimes l_2, l_3\otimes l_4] = \varphi(l_3\otimes l_4)\cdot\phi(l_1\otimes l_2) - \phi(l_1\otimes l_2)\cdot\varphi(l_3\otimes l_4)$

For a Leibniz 3-algebra $\mathcal{L}$ there exists a canonical trihomomorphism (l, m, r) from $\mathcal{L}$ to $U_3L(\mathcal{L})$ given by $l(x\otimes y) = l_{(x\otimes y)}$, $m(x\otimes y) = m_{(x\otimes y)}$, $r(x\otimes y) = r_{(x\otimes y)}$, for all $x\otimes y \in \mathcal{L}\otimes\mathcal{L}$.

PROPOSITION (4.5) (Universal Property). *The canonical trihomomorphism* $(l, m, r)\colon \mathcal{L}\otimes\mathcal{L} \to U_3L(\mathcal{L})$ *is universal for the trihomomorphisms of* $\mathcal{L}$, *that is,* $\text{Trihom}(\mathcal{L}, A) \cong \text{Ass}(U_3L(\mathcal{L}), A)$.

Let (φ, ψ, ϕ) be a trihomomorphism from $\mathcal{L}$ to A. We define a K-linear homomorphism $(\mathcal{L}\otimes\mathcal{L})^l \oplus (\mathcal{L}\otimes\mathcal{L})^m \oplus (\mathcal{L}\otimes\mathcal{L})^r \to A$ by $l_{(x\otimes y)} \mapsto \varphi(x\otimes y)$, $m_{(x\otimes y)} \mapsto \psi(x\otimes y)$, $r_{(x\otimes y)} \mapsto \phi(x\otimes y)$ which extends to $T((\mathcal{L}\otimes\mathcal{L})^l \oplus (\mathcal{L}\otimes\mathcal{L})^m \oplus (\mathcal{L}\otimes\mathcal{L})^r)$ and which vanishes on I, so it induces a homomorphism of associative algebras $U_3L(\mathcal{L}) \to A$.

Conversely, for a homomorphism of associative algebras $f\colon U_3L(\mathcal{L}) \to A$, the triple $(f\cdot l, f\cdot m, f\cdot r)$ is a trihomomorphism of $\mathcal{L}$. Moreover, both processes are inverses.

PROPOSITION (4.6). $U_3L(\mathcal{L})$ *is generated by the image* $(l, m, r)(\mathcal{L}\otimes\mathcal{L})$

Proof. Let B the subalgebra spanned by $l_{(x\otimes y)}, m_{x\otimes y}, r_{x\otimes y}, \forall x\otimes y \in \mathcal{L}\otimes\mathcal{L}$. (l, m, r) is a trihomomorphism from $\mathcal{L}$ to B, then Proposition (4.5) gives a unique homomorphism i such that $i\cdot(l, m, r) = (l, m, r)$. We can consider i as a homomorphism from $U_3L(\mathcal{L})$ to $U_3L(\mathcal{L})$, so Proposition (4.5) implies that $i = 1_{U_3L(\mathcal{L})}$. Hence $1_{U_3L(\mathcal{L})}(U_3L(\mathcal{L})) \subseteq B$, so $B = U_3L(\mathcal{L})$. □

LEMMA (4.7). *Let* (φ, ψ, ϕ) *be a trihomomorphism from a Leibniz 3-algebra* $\mathcal{L}_2$ *to a* K*-algebra* A *and let* $\alpha : \mathcal{L}_1 \to \mathcal{L}_2$ *be a homomorphism of Leibniz 3-algebras. Then* $(\varphi, \psi, \phi)\cdot(\alpha\otimes\alpha)$ *is a trihomomorphism from* $\mathcal{L}_1$ *to* A.

Proof. α induces the homomorphism of Leibniz algebras $\alpha \otimes \alpha \colon \mathcal{L}_1 \otimes \mathcal{L}_1 \to \mathcal{L}_2 \otimes \mathcal{L}_2$, $x \otimes y \mapsto \alpha(x) \otimes \alpha(y)$; hence it is a straightforward task to verify the properties of trihomomorphism for $(\varphi, \psi, \phi) \cdot (\alpha \otimes \alpha)$. □

PROPOSITION (4.8). *Let $\alpha \colon \mathcal{L}_1 \to \mathcal{L}_2$ be a homomorphism of Leibniz 3-algebras. There exists a unique homomorphism $\alpha' \colon U_3L(\mathcal{L}_1) \to U_3L(\mathcal{L}_2)$ such that $\alpha' \cdot (l_1, m_1, r_1) = (l_2, m_2, r_2) \cdot (\alpha \otimes \alpha)$.*

Proof. Apply Lemma (4.7) and Proposition (4.5). □

PROPOSITION (4.9). *Let B be a three-sided ideal of a Leibniz 3-algebra $\mathcal{L}$ and let R be the two-sided ideal of $U_3L(\mathcal{L})$ spanned by the elements $l_{x\otimes y}, m_{x\otimes y}, r_{x\otimes y}$ $\forall x \otimes y \in B \otimes B \oplus B \otimes \mathcal{L} \oplus \mathcal{L} \otimes B$. Then $U_3L(\mathcal{L}/B) \cong U_3L(\mathcal{L})/R$.*

Proof. We construct the well-defined trihomomorphism (φ, ψ, ϕ) from $\mathcal{L}/B$ to $U_3L(\mathcal{L})/R$ by $\varphi((x+B)\otimes(y+B)) = l_{x\otimes y} + R$; $\psi((x+B)\otimes(y+B)) = m_{x\otimes y} + R$; $\phi((x+B)\otimes(y+B)) = r_{x\otimes y} + R$. The Proposition (4.5) provides a unique homomorphism $\delta \colon U_3L(\mathcal{L}/B) \to U_3L(\mathcal{L})/R$.

Conversely, $\varphi'(x \otimes y) = l_{(x+B)\otimes(y+B)}$; $\psi'(x \otimes y) = m_{(x+B)\otimes(y+B)}$; $\phi'(x \otimes y) = r_{(x+B)\otimes(y+B)}$ define a trihomomorphism from $\mathcal{L}$ to $U_3L(\mathcal{L}/B)$, then Proposition (4.5) provides a unique homomorphism $\tau \colon U_3L(\mathcal{L}) \to U_3L(\mathcal{L}/B)$ which vanishes on R, so induces a homomorphism $\overline{\tau}$ from $U_3L(\mathcal{L})/R$ to $U_3L(\mathcal{L}/B)$.

Finally, it is easy to check that δ and $\overline{\tau}$ are inverses. □

PROPOSITION (4.10). *$U_3L(\mathcal{L})$ has a unique anti-homomorphism π such that $\pi \cdot (l, r, m) = (-l, -r, -m)$. Moreover, $\pi^2 = 1$.*

Proof. $(-l, -r, -m)$ is a trihomorphism; then Proposition (4.5) ends the proof. □

PROPOSITION (4.11). *There is a unique homomorphism $\delta \colon U_3L(\mathcal{L}) \to U_3L(\mathcal{L}) \wedge U_3L(\mathcal{L})$ such that $\delta \cdot (l, m, r) = (\varphi, \psi, \phi)$, where $\varphi(x\otimes y) = l_{x\otimes y} \wedge 1 + 1 \wedge l_{x\otimes y}$; $\psi(x\otimes y) = m_{x\otimes y} \wedge 1 + 1 \wedge m_{x\otimes y}$; $\phi(x \otimes y) = r_{x\otimes y} \wedge 1 + 1 \wedge r_{x\otimes y}$, $\forall x \otimes y \in \mathcal{L} \otimes \mathcal{L}$.*

Proof. The map (φ, ψ, ϕ) is a trihomomorphism from $\mathcal{L}$ to the associative algebra $U_3L(\mathcal{L}) \wedge U_3L(\mathcal{L})$. Proposition (4.5) ends the proof. □

PROPOSITION (4.12). *Let d be a derivation of a Leibniz 3-algebra $\mathcal{L}$. There exists a unique derivation D' of $U_3L(\mathcal{L})$ such that $D' \cdot (l, m, r) = (l, m, r) \cdot D$, where $D = d \otimes 1 + 1 \otimes d$.*

Proof. The derivation d induces a derivation of Leibniz algebras $D = d \otimes 1 + 1 \otimes d \colon \mathcal{L} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$. We consider the algebra $\mathcal{M}_2$ of 2×2 matrices with entries in $U_3L(\mathcal{L})$ and we define the K-linear maps φ, ψ, ϕ from $\mathcal{L} \otimes \mathcal{L}$ to $\mathcal{M}_2$ given by

$$\varphi \colon (x \otimes y) \mapsto \begin{pmatrix} l_{x\otimes y} & l_{d(x)\otimes y + x \otimes d(y)} \\ 0 & l_{x\otimes y} \end{pmatrix}$$

$$\psi \colon (x \otimes y) \mapsto \begin{pmatrix} m_{x\otimes y} & m_{d(x)\otimes y + x \otimes d(y)} \\ 0 & m_{x\otimes y} \end{pmatrix}$$

$$\phi \colon (x \otimes y) \mapsto \begin{pmatrix} r_{x\otimes y} & r_{d(x)\otimes y + x \otimes d(y)} \\ 0 & r_{x\otimes y} \end{pmatrix}$$

With a tedious verification we can see that (φ, ψ, ϕ) is a trihomomorphism from $\mathcal{L}$ to $\mathcal{M}_2$, so Proposition (4.5) gives us a homomorphism $\theta\colon U_3L(\mathcal{L}) \to \mathcal{M}_2$ such that $\theta \cdot (l, m, r) = (\varphi, \psi, \phi)$. Since $\theta(l_{x\otimes y}) = \begin{pmatrix} l_{x\otimes y} & l_{d(x)\otimes y + x\otimes d(y)} \\ 0 & l_{x\otimes y} \end{pmatrix}$, $\theta(m_{x\otimes y}) = \begin{pmatrix} m_{x\otimes y} & m_{d(x)\otimes y + x\otimes d(y)} \\ 0 & m_{x\otimes y} \end{pmatrix}$, $\theta(r_{x\otimes y}) = \begin{pmatrix} r_{x\otimes y} & r_{d(x)\otimes y + x\otimes d(y)} \\ 0 & r_{x\otimes y} \end{pmatrix}$ and $(l, m, r)(\mathcal{L} \otimes \mathcal{L})$ generates $U_3L(\mathcal{L})$, we have for any $X_{a\otimes b} \in U_3L(\mathcal{L})$ that $\theta(X_{a\otimes b}) = \begin{pmatrix} X_{a\otimes b} & X_{d(a)\otimes b + a\otimes d(b)} \\ 0 & X_{a\otimes b} \end{pmatrix}$. We write $D'(X_{a\otimes b}) = X_{d(a)\otimes b + a\otimes d(b)}$. From the calculations of trihomomorphism conditions for (φ, ψ, ϕ) we can deduce that D' is a derivation of $U_3L(\mathcal{L})$ considered as associative algebra in the usual way. Moreover $D' \cdot (l, m, r) = (l, m, r) \cdot D$. The uniqueness of D' follows from the fact that $(l, m, r)(\mathcal{L} \otimes \mathcal{L})$ generates $U_3L(\mathcal{L})$. □

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A METHOD TO INTEGRATE FILIFORM LIE ALGEBRAS

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ABSTRACT. Lie's Third Theorem states that given a Lie algebra $\mathfrak{g}$ of finite dimension, there exists a simply connected Lie group G whose associated Lie algebra is $\mathfrak{g}$. The classical proof of this result, which is not simple, is based on Ado's Theorem. According to it, every Lie algebra of finite dimension can be represented as a Lie subalgebra of the general linear group of matrices. We show in this paper a method to give a matrix representation of the simply connected group associated with a fixed nilpotent Lie algebra. Moreover, we give the representation of the Lie groups associated with filiform nilpotent Lie algebras whose derived is abelian.

Introduction

Lie's Third Theorem (and its converse, which is false in infinite dimension) establishes a unique correspondence between simply connected Lie groups and their associated Lie algebras. A direct proof of this result (see [4]), which is also known as Cartan's Theorem, uses a geometrical construction based on Maurer-Cartan constants and on the group of automorphisms.

Moreover, the customary proof in the literature is based on Ado's Theorem (see [7]), which states that given any Lie algebra $\mathfrak{g}$, there is a linear Lie algebra isomorphic to it. However, its proof cannot be called elementary.

Recently (see [10]), Tuynman has given an elementary proof of Lie's Third Theorem by using the correspondence between Lie subalgebras and Lie subgroups and the fact that, for a simply connected Lie group G, one has $H^2(G) = 0$. This proof generalizes, as the same author claims, the method to integrate nilpotent Lie algebras in a recursive way studied by A. Gray in [5].

In this paper we now improve Tuynman's work for a particular case, although far-reaching: a special class of nilpotent Lie algebras called filiform. To do this we give an explicit construction of a simply connected Lie group whose Lie algebra falls into a given filiform Lie algebra, say $\mathfrak{g}$. This is the main goal of the paper.

By using the group of automorphisms of filiform Lie algebras of a finite dimension n, in particular the unipotent automorphisms, we find a kind of basis with respect to which these automorphisms can be represented by triangular matrices. The dimension of the associated algebra, which is a subalgebra of Der $\mathfrak{g}$, is at most $2n - 3$.

Starting from the model filiform Lie algebra of each dimension (it is the most abelian of filiforms), its algebra of derivations contains as subalgebras those filiform Lie algebras such that $[\mathfrak{g}, \mathfrak{g}]$ is abelian. Note that one of them is the initial model algebra. These subalgebras determine, in a unique way, the corresponding subgroups from the initial group of automorphisms which

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constitute, as is claimed, a matrix representation of the simply connected Lie group corresponding to each of such algebras.

We also apply this method, as an example, to construct the Lie groups associated with complex filiform Lie algebras of dimension less or equal than 7.

We conclude this introduction by explaining the motivations for dealing with filiform Lie algebras. These algebras were introduced by M. Vergne in the late 60's of the past century [11], although before that, Blackburn had already studied the analogous class of finite p-groups and used the term *maximal class* for them, which is now also used for Lie algebras [3]. In fact, both terms *filiform* and *maximal class* are synonymous.

Vergne showed that within the variety of nilpotent Lie multiplications on a fixed vector space, non-filiforms can be relegated to small-dimensional components. Further, filiform Lie algebras are as the most structured as the least abelian within the nilpotent Lie algebras. In this sense, we can study them more easily than the set of nilpotent Lie algebras.

1. Definitions and notations

Let $\mathfrak{g} = (\mathbb{C}^n, [\,,\,])$ be a Lie algebra of dimension n with $[\,,\,]$ the associated law. We consider the *lower central series* of $\mathfrak{g}$ defined by $\mathcal{C}^1\mathfrak{g} = \mathfrak{g}$, $\mathcal{C}^i\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^{i-1}\mathfrak{g}]$. This was used by Ancochea and Goze to classify complex nilpotent Lie algebras of dimension 7 and complex filiform Lie algebras of dimension 8, since it is an invariant of these algebras, in the sense of not depending on the basis chosen (see [1]).

A Lie algebra $\mathfrak{g}$ of dimension n is *filiform* if $\dim \mathcal{C}^i\mathfrak{g} = n - i$ for $2 \leq i \leq n$. If $x \in \mathfrak{g}$ we denote by $ad(x)$ the adjoint mapping associated to x (i.e. the map $y \mapsto [x, y]$). As we already said, these algebras were introduced by Vergne in 1966 (see [11]). In the case of groups, the term filiform goes back at least as far as Ph. Hall in the 1930's and in the case of algebras, Vergne also used it in her paper, although in fact the term may appear in the works of Ph. Hall and Witt, also in the 1930's.

Let $\mathfrak{g}$ be a filiform Lie algebra of dimension n. Then there exists a basis $\mathcal{B} = \{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ such that $X_1 \in \mathfrak{g} \setminus \mathcal{C}^2\mathfrak{g}$, the matrix of $ad(X_1)$ with respect to $\mathcal{B}$ has a Jordan block of order $n-1$ and $\mathcal{C}^i\mathfrak{g}$ is the span of $\{X_{i+1}, \ldots, X_n\}$ with $2 \leq i \leq n-1$.

Note that the previous conditions involve $[X_1, X_h] = X_{h+1}$ for $2 \leq h \leq n-1$. Further, as $[X_2, X_n] = 0$ ($\{X_n\}$ is the center of $\mathfrak{g}$) and $[X_1, X_{n-1}] = X_n$, we can conclude that $[X_2, X_{n-1}] = \alpha X_n$ and thus the change of basis $X'_2 = X_2 - \alpha X_1$, $X'_k = X_k$ ($k \neq 2$) gives $[X'_2, X'_{n-1}] = 0$ and this does not change the remaining brackets. Such a basis $\mathcal{B}$ is called an *adapted basis*.

It is easy to deduce that, with respect to such a basis,

$$\begin{aligned}
\mathcal{C}^2(\mathfrak{g}) &\equiv \langle X_3, \ldots, X_n \rangle \\
\mathcal{C}^3(\mathfrak{g}) &\equiv \langle X_4, \ldots, X_n \rangle \\
&\vdots \\
\mathcal{C}^{n-1}(\mathfrak{g}) &\equiv \langle X_n \rangle \\
\mathcal{C}^n(\mathfrak{g}) &\equiv \{0\}
\end{aligned} \tag{1.1}$$

Moreover, expressions (1.1) supply, for each filiform Lie algebra $\mathfrak{g}$ of dimension n, a chain of ideals with successive quotients of dimension 1, which are also invariant as by automorphisms of Lie algebras as by derivations, that is:

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \dots \supseteq \mathfrak{g}_{n-1} \supseteq \mathfrak{g}_n = \{0\} \tag{1.2}$$

where $\mathfrak{g}_1 = Z_{\mathfrak{g}}(\mathcal{C}^{n-2}(\mathfrak{g}))$, $\mathfrak{g}_i = \mathcal{C}^i(\mathfrak{g})$ $(i \geq 2)$.

In this sense, a basis $\mathcal{B} = \{X_1, \dots, X_n\}$ of $\mathfrak{g}$ is adapted if and only if, up to order, the following holds:

$$X_i \in \mathfrak{g}_{i-1} \quad , \quad i = 1, \dots, n \tag{1.3}$$

A filiform Lie algebra is said to be a *model* Lie algebra if the unique nonzero brackets between the elements of an adapted basis are the following:

$$[X_1, X_h] = X_{h+1} \quad (h = 2, \dots, n-1).$$

It is immediate to check that there exists an unique model filiform Lie algebra for each dimension. It will denoted by $\mathcal{P}_n$.

Finally, if $\mathcal{C}^2(\mathfrak{g})$ is a abelian subalgebra of $\mathfrak{g}$, the nonzero brackets with respect to an adapted basis are the following (see [2]):

$$\begin{aligned} &[X_1, X_h] = X_{h+1} && (h = 2, \dots, n-1), \\ &[X_2, X_h] = \textstyle\sum_{l=1}^{n-h-1} \alpha_l X_{h+l+1} && (h = 3, \dots, n-2). \end{aligned} \tag{1.4}$$

2. The group of unipotent automorphisms of $\mathcal{P}_n$

The objective of this section is to parametrize the group of unipotent automorphisms of the model algebra of each dimension. To do this, we will prove first that with respect to an adapted basis, all automorphisms are represented by triangular matrices (it is general for every nilpotent Lie algebra).

Let $\mathcal{P}_n$ be the model filiform Lie algebra of dimension $n \geq 4$. Let $\mathcal{B} = \{X_1, \dots, X_n\}$ an adapted basis of $\mathcal{P}_n$, with respect to which the law of the algebra can be expressed by

$$[X_1, X_k] = X_{k+1} \quad ; \quad 2 \leq k \leq n-1$$

with the rest of brackets null, up to antisymmetry.

PROPOSITION (2.1). *Every automorphism of Lie algebras in $\mathcal{P}_n$ maps adapted bases to adapted bases. Moreover, for a given adapted basis, the matrices of all of automorphisms are (upper) triangular.*

Proof. The first assertion is a consequence of (1.2) and (1.3).

Let us fix an adapted basis $\mathcal{B} = \{X_1, \dots, X_n\}$ of $\mathcal{P}_n$, and let $\varphi : \mathcal{P}_n \to \mathcal{P}_n$ be an automorphism of Lie algebras. If we denote $Y_i = \varphi(X_i)$, we obtain a new adapted basis of $\mathcal{P}_n$ satisfying

$$[Y_1, Y_k] = Y_{k+1}, \quad 2 \leq k \leq n-1$$

and having the rest of brackets null, up to antisymmetry. The matrix corresponding to φ is triangular. Indeed, if we express Y_1 and Y_2 with respect to the basis $\mathcal{B}$,

$$Y_1 = \sum_{i=1}^{n} a_{1,i} X_i, \qquad Y_2 = \sum_{j=1}^{n} a_{2,j} X_j \, .$$

the rest of the basis vectors are determined by them:

$$Y_3 = [Y_1, Y_2] = \sum_{k=3}^{n} a_{3,k} X_k \, , \text{ where } \quad a_{3,k} = a_{1,1} a_{2,k-1} - a_{1,k-1} a_{2,1}$$

and by recurrence, we obtain ($3 \le h \le n$):
(2.2)

$$Y_h = [Y_1, Y_{h-1}] = \sum_{k=h}^{n} a_{h,k} X_k \, , \text{ where } \quad a_{h,k} = a_{1,1} a_{h-1,k-1} - a_{1,k-1} a_{k-1,1} \, .$$

Moreover, from $[Y_2, Y_{n-1}] = 0$ we deduce $a_{2,1} = 0$, which implies that the matrix of the basis change is triangular and gives the elements placed in the third and following rows as functions of those belonging to the first two rows,
(2.3)

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ 0 & 0 & a_{3,3} & \dots & a_{3,n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{n,n} \end{pmatrix} \qquad \begin{array}{l} a_{1,i}, \, a_{2,i} \in \mathbb{C} \\ \\ a_{h,k}\, , \text{ with } 3 \le h \le n, \text{ satisfying (2.2).} \end{array}$$

This construction allows us to parametrize the group of automorphisms of $\mathcal{P}_n$ by triangular matrices which have a Lie group structure. More specifically, the subgroup of unipotent automorphisms will be a simply connected Lie group of dimension $2n - 3$. It will contain the simply connected Lie group associated with $\mathcal{P}_n$ as a subgroup. $\square$

THEOREM (2.4). *The unipotent automorphisms of $\mathcal{P}_n$ give a $2n - 3$-dimensional Lie subgroup G of Aut($\mathcal{P}_n$) admitting a linear representation by upper triangular matrices.*

Proof. Let $\mathcal{B} = \{X_1, \dots, X_n\}$ be an adapted basis of $\mathcal{P}_n$, as in (2.1), and let $\varphi \colon \mathcal{P}_n \to \mathcal{P}_n$ be an unipotent automorphism of Lie algebras with $Y_i = \varphi(X_i)$.

If we choose the two first elements of the matrix as $a_{11} = a_{22} = 1$, that is,

$$\begin{aligned} Y_1 &= \; X_1 + a_{1,2} X_2 + a_{1,3} X_3 + \cdots + a_{1,n} X_n \\ Y_2 &= \; \phantom{X_1 + a_{1,2}} X_2 + a_{2,3} X_3 + \cdots + a_{2,n} X_n \, , \end{aligned}$$

the remaining diagonal elements satisfy, according to (2.2), that

$$a_{k,k} = 1 \, , \;\; k \ge 3$$

Moreover, due to $a_{i,1} = 0$ ($2 \le i \le n$), we have that

$$a_{h,k} = a_{h-1,k-1} \, , \;\; 3 \le h \le k \le n \, .$$

Therefore we have a group G formed by triangular matrices having diagonal elements equal 1, which represent all unipotent automorphisms of $\mathcal{P}_n$ with respect to an adapted basis. This allows us to parametrize G as a Lie subgroup of the general linear group,

(2.5)
$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & \dots & x_{n-3} & x_{n-2} & x_{n-1} & x_n \\ 0 & 1 & x_1 & x_{n+1} & \dots & x_{2n-6} & x_{2n-5} & x_{2n-4} & x_{2n-3} \\ 0 & 0 & 1 & x_1 & \ddots & \ddots & x_{2n-6} & x_{2n-5} & x_{2n-4} \\ 0 & 0 & 0 & 1 & \ddots & \ddots & \ddots & x_{2n-6} & x_{2n-5} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & x_1 & x_{n+1} & x_{n+2} \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & x_1 & x_{n+1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } (x_i) \in \mathbb{C}^{2n-3}.$$

□

3. The Lie algebra associated with G

The tangent space of G at the unit $I_n \in G$ is composed of all the matrices X for which we can find a differentiable curve $\varphi = \varphi(t)$ on G satisfying $\varphi(0) = I_n$ and $\varphi'(0) = X$. This space $\mathfrak{g}$ constitutes the Lie algebra associated with G and it is formed of the component at the origin of differentiable fields in G which are invariant under left translations of the group.

We will construct a basis of $\mathfrak{g}$ starting from a system of one-parameter groups of transformations $\{\varphi\}_{i=1,\dots,2n-3}$, which will define the paths of such left-invariant fields.

We will denote

$$\begin{array}{rcl} g : \mathbb{C}^{2n-3} & \to & G \\ (\overline{x}_i) & \to & g(\overline{x}_i) \end{array}$$

where $g(\overline{x}_i)$ is a generic point of G, as in (2.5).

We now define $2n-3$ differentiable curves in G, which will represent a system of one-parameter groups of transformations. Basically, these curves $\varphi_i\colon \mathbb{R} \to G$ will be parametrized by $x_i = t$ and the remaining coordinates equal 0, with the exception of suitable adjustments to make that φ_i into a group, that is, that $\varphi_i(t+s) = \varphi_i(t)\varphi_i(s)$. The first curve will be obtained in the following way:

$$\varphi_1 : \mathbb{R} \to G,\ t \mapsto g(\overline{x}_i) \quad \text{with} \quad \left\{ \begin{array}{lcl} x_1 & = & t \\ x_2 & = & 0 \\ \vdots & & \\ x_n & = & 0 \\ x_{n+k} & = & \frac{1}{(k+1)!} t^{k+1};\ 1 \le k \le n-3. \end{array} \right.$$

It is verified that $\varphi_1(t+s) = \varphi_1(t)\varphi_1(s)$ and $\varphi(0) = I_n$. The following $2n-4$ curves are defined by

$$\varphi_k\colon \mathbb{R} \to G,\ t \mapsto g(\overline{x}_i) \quad \text{with} \quad \left\{ \begin{array}{l} x_k = t \\ x_j = 0,\ j \ne k. \end{array} \right.$$

except in the case of n even. In this case, the one-parameter group $\varphi_{n+\frac{n}{2}-2}$ is parametrized by

$$\begin{cases} x_{n+\frac{n}{2}-2} & = & t \\ x_{2n-3} & = & \frac{1}{2}t^2 \\ x_j & = & 0\,, \text{ otherwise.} \end{cases}$$

For each k, the curves $g\varphi_k$ represent the paths of a differentiable field on G, which is invariant under left translations. Tangent vectors define at the origin (with respect to the canonical basis $\left\{e_i = \frac{\partial}{\partial x_i}\right\}_{i=1,\ldots,2n-3}$ of $\mathbb{C}^{2n-3}$), the following basis $\{X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n-3}\}$ of $\mathfrak{g}$:

$$\begin{array}{ll}
X_1 & = (1, 0, x_2, x_3, x_4, \ldots, x_{n-2}, x_{n-1}, x_1, x_{n+1}, x_{n+2}, \ldots, x_{2n-5}, x_{2n-4}) \\
X_2 & = (0, 1, 0, 0, 0, \ldots, 0, 0, 0, 0, 0, \ldots, 0, 0) \\
X_3 & = (0, 0, 1, 0, 0, \ldots, 0, 0, 0, 0, 0, \ldots, 0, 0) \\
\vdots & \\
X_n & = (0, 0, 0, 0, 0, \ldots, 0, \underbrace{1}_{n}, 0, 0, 0, \ldots, 0, 0) \\
X_{n+1} & = (0, 0, 0, x_2, x_3, \ldots, x_{n-3}, x_{n-2}, \underbrace{1}_{n+1}, x_1, x_{n+1}, \ldots, x_{2n-6}, x_{2n-5}) \\
X_{n+2} & = (0, 0, 0, 0, x_2, \ldots, x_{n-4}, x_{n-3}, 0, \underbrace{1}_{n+2}, x_1, \ldots, x_{2n-7}, x_{2n-6}) \\
\vdots & \\
X_{2n-4} & = (0, 0, 0, 0, 0, \ldots, x_2, x_3, 0, 0, 0, \ldots, 1, x_1) \\
X_{2n-3} & = (0, 0, 0, 0, 0, \ldots, 0, x_2, 0, 0, 0, \ldots, 0, 1)
\end{array}$$

In this way we obtain a Lie algebra $\mathfrak{g}$ of dimension $2n-3$ which is a subalgebra of $Der(\mathcal{P}_n)$, since the latter is the algebra associated with the group of automorphisms of $\mathcal{P}_n$ (see, for example, [8]). Note also that the algebra obtained is isomorphic to the nilradical of the derivation algebra of $\mathcal{P}_n$. With respect to this basis, the law of the algebra is the following:

$$[X_1, X_k] = -X_{k+1}\,, \quad 2 \le k \le n-1$$

$$[X_{n-h}, X_{n+k}] = X_{n-h+k+1}\,, \quad 2 \le h \le n-2\,, \quad 1 \le k \le n-3$$

Moreover, the basis so obtained allows us to check that $\mathfrak{g}$ has the following property, which is also satisfied by $\mathcal{P}_n$: the subalgebra $\langle X_2, \ldots, X_n \rangle$ is abelian. Indeed, we obtain the following result:

THEOREM (3.1). *Every filiform Lie algebra $\mathcal{L}$ of dimension n whose derived algebra $[\mathcal{L}, \mathcal{L}]$ is abelian, is a subalgebra of $\mathfrak{g}$. As a consequence, there exists a Lie subgroup of G whose associated algebra is $\mathcal{L}$.*

Proof. We consider, in the first instance, the case in which $\mathcal{L} = \mathcal{P}_n$. If $\{X_1, \ldots, X_{2n-3}\}$ is the basis of $\mathfrak{g}$ previously constructed, the subalgebra $\mathfrak{g}_n =$

$\langle X_1, \ldots, X_n \rangle$ satisfies

$$[X_1, X_2] = -X_3 \;;\; [X_1, X_3] = -X_4 \;;\; \ldots \;;\; [X_1, X_{n-1}] = -X_n \;;\; [X_1, X_n] = 0$$

$$[X_h, X_k] = 0 \qquad ;\; 2 \leq h < k \leq n$$

which implies $\mathfrak{g}_n = \mathcal{P}_n$.

On the other hand, as each Lie subalgebra of $\mathfrak{g}$ determines, in an unique way, a Lie subgroup of G, the existence of a subgroup of $G_n \subset G$ whose associated Lie algebra is $\mathcal{P}_n$ can be deduced. Moreover, $\mathfrak{g}_n = \langle X_1, \ldots, X_n \rangle$ determines in $\mathfrak{g}$ an involutive distribution. Therefore, G_n will be the n-dimensional connected integral subvariety containing every point of G, and it will be obtained as a solution of the system:

$$\{\omega_i = 0\}_{i=n+1,\ldots,2n-3}$$

where $\{\omega_i\}_i$ is the dual basis of $\{X_i\}_i$ in $\mathfrak{g}$.

If we also require

$$\{x_i = 0\}_{i=1,\ldots,2n-3}$$

as an initial condition, we will obtain the connected component of the unit. Recall that a representation of $\mathcal{P}_n$ (and of $\mathcal{Q}_n$) are explicitly described in [9].

Now let $\mathcal{L}$ be a filiform Lie algebra of dimension n whose derived Lie algebra $[\mathcal{L}, \mathcal{L}]$ is abelian. As in (1.4), the nonzero brackets are

$$[X_1, X_h] = X_{h+1} \qquad (h = 2, \ldots, n-1)$$

$$[X_2, X_h] = \textstyle\sum_{l=1}^{n-h-1} \alpha_l X_{h+l+1} \quad (h = 3, \ldots, n-2)$$

for some $\alpha_1, \ldots, \alpha_{n-4} \in \mathbb{C}$. After applying the change of basis

$$\begin{cases} Y_1 = X_1 \\ Y_2 = X_2 - \sum_{k=1}^{n-4} \alpha_k X_{n+k} \\ Y_i = X_i \;,\; i \geq 3 \end{cases}$$

the law of the algebra with respect to the basis $\{Y_i\}_{i=1,\ldots,2n-3}$ satisfies the following brackets:

$$[Y_1, Y_h] = Y_{h+1} \qquad 2 \leq h \leq n-1$$

$$[Y_2, Y_h] = \textstyle\sum_{l=1}^{n-h-1} \alpha_l Y_{h+l+1} \qquad 3 \leq h \leq n-2$$

$$[Y_h, Y_k] = 0 \;,\; 3 \leq h < k \leq n$$

and, as a consequence, $\langle Y_1, \ldots, Y_n \rangle \cong \mathcal{L}$. By a similar reasoning for G_n we deduce the existence of a simply connected Lie subgroup $G_{\mathcal{L}} \subset G$ corresponding to the solution of the system:

$$\begin{cases} \omega_i = 0 \,,\; i = n+1, \ldots, 2n-3 \\ \\ x_j(\overline{0}) = 0 \,,\; j = n+1, \ldots, 2n-3 \end{cases}$$

where $\{\omega_i\}$ is the dual basis of $\{Y_i\}$ in $\mathfrak{g}$.

4. Application to filiform Lie algebras of dimension less or equal than 7

In this section, we will give a list of filiform Lie algebras of dimension n, with $3 \leq n \leq 7$ (according to Goze and Ancoechea's classification given in [1], later corrected by Goze and Remm in [6]), whose derived Lie algebra is abelian (they make 13 out of a total of 17) and the simply connected associated Lie group. We will show the complete study of one of the cases, as an example.

EXAMPLE (4.1). *Let μ_7^2 be the complex filiform Lie algebra defined, with respect to a basis $\{Y_1, \ldots, Y_7\}$, by the nonzero brackets:*

$$\begin{cases} [Y_1, Y_k] = Y_{k+1}\,, \;\; k = 2, \ldots, 6 \\ [Y_2, Y_3] = Y_5 \\ [Y_2, Y_4] = Y_6 \\ [Y_2, Y_5] = Y_7 \end{cases}$$

Its derived Lie algebra is $\langle Y_2, \ldots, Y_6 \rangle$, which is, indeed, abelian. The corresponding values in (1.4) are $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_3 = 0$.

The group of unipotent automorphisms of the model Lie algebra $\mathcal{P}_7$ of dimension 7 is a Lie group G of dimension 11 whose representation with respect to an adapted basis is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0 & 1 & x_1 & x_8 & x_9 & x_{10} & x_{11} \\ 0 & 0 & 1 & x_1 & x_8 & x_9 & x_{10} \\ 0 & 0 & 0 & 1 & x_1 & x_8 & x_9 \\ 0 & 0 & 0 & 0 & 1 & x_1 & x_8 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \qquad (\overline{x}_i) \in \mathbb{C}^{11}$$

The associated Lie algebra $\mathfrak{g}$ is nilpotent of dimension 11, and a basis is $\{X_1, \ldots, X_{11}\}$. Coordinates of this basis with respect to the canonical basis of $\mathbb{C}^{11}$ are

$$\begin{aligned}
X_1 &= (1, 0, x_2, x_3, x_4, x_5, x_6, x_1, x_8, x_9, x_{10}) \\
X_2 &= (0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0) \\
X_3 &= (0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0) \\
X_4 &= (0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0) \\
X_5 &= (0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0) \\
X_6 &= (0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0) \\
X_7 &= (0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0) \\
X_8 &= (0, 0, 0, x_2, x_3, x_4, x_5, 1, x_1, x_8, x_9) \\
X_9 &= (0, 0, 0, 0, x_2, x_3, x_4, 0, 1, x_1, x_8) \\
X_{10} &= (0, 0, 0, 0, 0, x_2, x_3, 0, 0, 1, x_1) \\
X_{11} &= (0, 0, 0, 0, 0, 0, x_2, 0, 0, 0, 1)
\end{aligned}$$

The bracket products in $\mathfrak{g}$ are:

$$
\begin{array}{llll}
[X_1, X_{11}] = 0 & [X_2, X_{11}] = X_7 & & \\
[X_1, X_{10}] = 0 & [X_2, X_{10}] = X_6 & [X_3, X_{10}] = X_7 & \\
[X_1, X_9] = 0 & [X_2, X_9] = X_5 & [X_3, X_9] = X_6 & [X_4, X_9] = X_7 \\
[X_1, X_8] = 0 & [X_2, X_8] = X_4 & [X_3, X_8] = X_5 & [X_4, X_8] = X_6 \\
[X_5, X_8] = X_7 & & & \\
[X_1, X_7] = 0 & & & \\
[X_1, X_6] = -X_7 & & & \\
[X_1, X_5] = -X_6 & & & \\
[X_1, X_4] = -X_5 & & & \\
[X_1, X_3] = -X_4 & & & \\
[X_1, X_2] = -X_3 & & &
\end{array}
$$

The change of basis

$$
\begin{cases}
Y_1 = -X_1 \\
Y_2 = X_2 - X_8 \\
Y_i = X_i \ , \ \ i \neq 1, 2
\end{cases}
$$

allows us to deduce that the subalgebra $\mathfrak{g}_7 = \langle Y_1, \ldots, Y_7 \rangle$ of $\mathfrak{g}$ is isomorphic to μ_7^2. If $\{\omega_i\}$ denotes the dual basis of $\{Y_i\}$ in $\mathfrak{g}$, the connected integral subvariety of G of dimension 7 which corresponds to $\mathfrak{g}_7$ is the solution of the system $\{\omega_i = 0\}_{8 \leq i \leq 11}$ which contains the unit in G. The results of this integration is the following:

$$
\begin{aligned}
\omega_8 &= -x_1 \, \mathrm{d}x_1 - \mathrm{d}x_2 + \mathrm{d}x_8 \ \Rightarrow \ x_8 = \frac{1}{2} x_1^2 + x_2 \\
\omega_9 &= (x_1^2 - x_8) \, \mathrm{d}x_1 - x_1 \, \mathrm{d}x_8 + \mathrm{d}x_9 \\
&= \left(-\frac{1}{2} x_1^2 - x_2 \right) \, \mathrm{d}x_1 - x_1 \, \mathrm{d}x_2 + \mathrm{d}x_9 \ \Rightarrow \ x_9 = \frac{1}{6} x_1^3 + x_1 x_2 \\
\omega_{10} &= (-x_1^3 + 2x_1 x_8 - x_9) \, \mathrm{d}x_1 + (x_1^2 - x_8) \, \mathrm{d}x_8 - x_1 \, \mathrm{d}x_9 + \mathrm{d}x_{10} \\
&= \left(-\frac{1}{6} x_1^3 - x_1 x_2 \right) \, \mathrm{d}x_1 + \left(-\frac{1}{2} x_1^2 - x_2 \right) \, \mathrm{d}x_2 + \mathrm{d}x_{10} \\
&\qquad \Rightarrow \ x_{10} = \frac{1}{24} x_1^4 + \frac{1}{2} x_1^2 x_2 + \frac{1}{2} x_2^2 \\
\omega_{11} &= \left(x_1^4 - 3x_1^2 x_8 + 2x_1 x_9 - x_{10} + x_8^2 \right) \, \mathrm{d}x_1 + (-x_1^3 + 2x_1 x_8 - x_9) \, \mathrm{d}x_8 \\
&\qquad + (x_1^2 - x_8) \, \mathrm{d}x_9 - x_1 \, \mathrm{d}x_{10} + \mathrm{d}x_{11} = \\
&= \left(-\frac{1}{24} x_1^4 - \frac{1}{2} x_1^2 x_2 - \frac{1}{2} x_2^2 \right) \, \mathrm{d}x_1 + \left(-\frac{1}{6} x_1^3 - x_1 x_2 \right) \, \mathrm{d}x_2 + \mathrm{d}x_{11} \ \Rightarrow \\
&\Rightarrow x_{11} = \frac{1}{120} x_1^5 + \frac{1}{6} x_1^3 x_2 + \frac{1}{2} x_1 x_2^2
\end{aligned}
$$

and therefore, the simply connected Lie group associated with μ_7^2 is

$$\begin{pmatrix}
1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\
0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 & \frac{1}{6}x_1^3 + x_1x_2 & \frac{1}{24}x_1^4 + \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_2^2 & \frac{1}{120}x_1^5 + \frac{1}{6}x_1^3x_2 + \frac{1}{2}x_1x_2^2 \\
0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 & \frac{1}{6}x_1^3 + x_1x_2 & \frac{1}{24}x_1^4 + \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_2^2 \\
0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 & \frac{1}{6}x_1^3 + x_1x_2 \\
0 & 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 \\
0 & 0 & 0 & 0 & 0 & 1 & x_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

with $(\overline{x}_i) \in \mathbb{C}^7$. □

We finally show a table with the list of filiform Lie algebras of dimension less or equal than 7 (whose derived algebra is abelian) and the Lie group associated to each one.

Dimension 3.

Lie algebra $\boldsymbol{\mu_3^1}$ (model, Heisenberg's algebra). The law of this Lie algebra with respect to the basis $\{X_1, X_2, X_3\}$ is

$$[X_1, X_2] = X_3 \, .$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} .$$

Dimension 4.

Lie algebra $\boldsymbol{\mu_4^1}$ (model). The law of this Lie algebra with respect to the basis $\{X_1, X_2, X_3, X_4\}$ is

$$\begin{cases} [X_1, X_3] = X_4 \\ [X_1, X_2] = X_3 \, . \end{cases}$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 \\ 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Dimension 5.

Lie algebra $\boldsymbol{\mu_5^1}$. The law of this Lie algebra with respect to the basis $\{X_1, \ldots, X_5\}$ is

$$\begin{cases} [X_1, X_k] = X_{k+1} \, , \; k = 2, 3, 4 \\ [X_2, X_3] = X_5 \, . \end{cases}$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 & \frac{1}{6}x_1^3 + x_1x_2 \\ 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 \\ 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} ; \qquad (\overline{x}_i) \in \mathbb{C}^5$$

Lie algebra $\boldsymbol{\mu_5^2}$ *(model).* The law of this Lie algebra with respect to the basis $\{X_1, \dots, X_5\}$ is

$$[X_1, X_k] = X_{k+1}\ ,\ \ k = 2, 3, 4\,.$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 \\ 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 \\ 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \qquad (\overline{x}_i) \in \mathbb{C}^5\,.$$

Dimension 6.

Lie algebra $\boldsymbol{\mu_6^3}$. The law of this Lie algebra with respect to the basis $\{X_1, \dots, X_6\}$ is

$$\begin{cases} [X_1, X_k] = X_{k+1}\ ,\ \ k = 2, \dots, 5 \\ [X_2, X_4] = X_6 \\ [X_2, X_3] = X_5\,. \end{cases}$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 & \frac{1}{6}x_1^3 + x_1x_2 & \frac{1}{24}x_1^4 + \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_2^2 \\ 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 & \frac{1}{6}x_1^3 + x_1x_2 \\ 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 \\ 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \qquad (\overline{x}_i) \in \mathbb{C}^6\,.$$

Lie algebra $\boldsymbol{\mu_6^4}$. The law of this Lie algebra with respect to the basis $\{X_1, \dots, X_6\}$ is

$$\begin{cases} [X_1, X_k] = X_{k+1}\ ,\ \ k = 2, \dots, 5 \\ [X_2, X_3] = X_6\,. \end{cases}$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 + x_2 & \frac{1}{24}x_1^4 + x_1x_2 \\ 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 + x_2 \\ 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 \\ 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \qquad (\overline{x}_i) \in \mathbb{C}^6\,.$$

Lie algebra $\boldsymbol{\mu_6^5}$ *(model).* The law of this Lie algebra with respect to the basis $\{X_1, \dots, X_6\}$ is

$$[X_1, X_k] = X_{k+1}\ ,\ \ k = 2, \dots, 5\,.$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 & \frac{1}{24}x_1^4 \\ 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 \\ 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 \\ 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}; \qquad (\overline{x}_i) \in \mathbb{C}^6 .$$

Dimension 7.

Lie algebra μ_7^2. The law of this Lie algebra with respect to the basis $\{X_1, \dots, X_7\}$ is

$$\begin{cases} [X_1, X_k] = X_{k+1}\,, \ k = 2, \dots, 6 \\ [X_2, X_5] = X_7 \\ [X_2, X_4] = X_6 \\ [X_2, X_3] = X_5\,. \end{cases}$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 & \frac{1}{6}x_1^3 + x_1x_2 & \frac{1}{24}x_1^4 + \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_2^2 & \frac{1}{120}x_1^5 + \frac{1}{6}x_1^3x_2 + \frac{1}{2}x_1x_2^2 \\ 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 & \frac{1}{6}x_1^3 + x_1x_2 & \frac{1}{24}x_1^4 + \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_2^2 \\ 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 & \frac{1}{6}x_1^3 + x_1x_2 \\ 0 & 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $(\overline{x}_i) \in \mathbb{C}^7$.

Lie algebra μ_7^3. The law of this Lie algebra with respect to the basis $\{X_1, \dots, X_7\}$ is

$$\begin{cases} [X_1, X_k] = X_{k+1}\,, \ k = 2, \dots, 6 \\ [X_2, X_5] = X_7 \\ [X_2, X_4] = X_6 \\ [X_2, X_3] = X_5 + X_7\,. \end{cases}$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 & \frac{1}{6}x_1^3 + x_1x_2 & \frac{1}{24}x_1^4 + \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_2^2 + x_2 & \frac{1}{120}x_1^5 + \frac{1}{6}x_1^3x_2 + \frac{1}{2}x_1x_2^2 + x_1x_2 \\ 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 & \frac{1}{6}x_1^3 + x_1x_2 & \frac{1}{24}x_1^4 + \frac{1}{2}x_1^2x_2 + \frac{1}{2}x_2^2 + x_2 \\ 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 & \frac{1}{6}x_1^3 + x_1x_2 \\ 0 & 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 + x_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $(\overline{x}_i) \in \mathbb{C}^7$.

Lie algebra μ_7^5. The law of this Lie algebra with respect to the basis $\{X_1, \ldots, X_7\}$ is

$$\begin{cases} [X_1, X_k] = X_{k+1}\,, \ k = 2, \ldots, 6 \\ [X_2, X_4] = X_7 \\ [X_2, X_3] = X_6 + X_7\,. \end{cases}$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 + x_2 & \frac{1}{24}x_1^4 + x_1x_2 + x_2 & \frac{1}{120}x_1^5 + \frac{1}{2}x_1^2x_2 + x_1x_2 \\ 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 + x_2 & \frac{1}{24}x_1^4 + x_1x_2 + x_2 \\ 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 + x_2 \\ 0 & 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $(\overline{x}_i) \in \mathbb{C}^7$.

Lie algebra μ_7^6. The law of this Lie algebra with respect to the basis $\{X_1, \ldots, X_7\}$ is

$$\begin{cases} [X_1, X_k] = X_{k+1}\,, \ k = 2, \ldots, 6 \\ [X_2, X_4] = X_7 \\ [X_2, X_3] = X_6\,. \end{cases}$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 + x_2 & \frac{1}{24}x_1^4 + x_1x_2 & \frac{1}{120}x_1^5 + \frac{1}{2}x_1^2x_2 \\ 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 + x_2 & \frac{1}{24}x_1^4 + x_1x_2 \\ 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 + x_2 \\ 0 & 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $(\overline{x}_i) \in \mathbb{C}^7$.

Lie algebra μ_7^7. The law of this Lie algebra with respect to the basis $\{X_1, \ldots, X_7\}$ is

$$\begin{cases} [X_1, X_k] = X_{k+1}\,, \ k = 2, \ldots, 6 \\ [X_2, X_3] = X_7\,. \end{cases}$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 & \frac{1}{24}x_1^4 + x_2 & \frac{1}{120}x_1^5 + x_1x_2 \\ 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 & \frac{1}{24}x_1^4 + x_2 \\ 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 \\ 0 & 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with $(\overline{x}_i) \in \mathbb{C}^7$.

Lie algebra $\boldsymbol{\mu_7^8}$ *(model).* The law of this Lie algebra with respect to the basis $\{X_1, \ldots, X_7\}$ is

$$\{ \ [X_1, X_k] = X_{k+1}\ , \ k = 2, \ldots, 6\,.$$

Its associated Lie group is

$$\begin{pmatrix} 1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 & \frac{1}{24}x_1^4 & \frac{1}{120}x_1^5 \\ 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 & \frac{1}{24}x_1^4 \\ 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 & \frac{1}{6}x_1^3 \\ 0 & 0 & 0 & 0 & 1 & x_1 & \frac{1}{2}x_1^2 \\ 0 & 0 & 0 & 0 & 0 & 1 & x_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ with } (\overline{x}_i) \in \mathbb{C}^7.$$

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HÖLDER ESTIMATES FOR THE $\overline{\partial}$-EQUATION ON SURFACES WITH SIMPLE SINGULARITIES

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ABSTRACT. Let $\Sigma \subset \mathbb{C}^3$ be a 2-dimensional subvariety with an isolated simple (rational double point) singularity at the origin. The main objective of this paper is to solve the $\overline{\partial}$-equation on a neighbourhood of the origin in Σ, requiring a Hölder condition on the solution.

1. Introduction

Let $\Sigma \subset \mathbb{C}^3$ be a subvariety with an isolated singularity at the origin. Given a $\overline{\partial}$-closed $(0,1)$-differential form λ defined on Σ minus the origin, Gavosto and Fornæss proposed a general technique for solving the differential equation $\overline{\partial}g = \lambda$ on a neighbourhood of the origin in Σ. The calculations were done in the sense of distributions, and they required an extra Hölder condition on the solution g, see [2] and [3]. Their basic idea was to analyse Σ as a branched covering over $\mathbb{C}^2$, to solve the corresponding $\overline{\partial}$-equation on $\mathbb{C}^2$, and to *lift* the solution from $\mathbb{C}^2$ into Σ again. Gavosto and Fornæss completed all the calculations in the particular case when $\Sigma \subset \mathbb{C}^3$ is defined by the polynomial $x_1x_2 = x_3^2$, that is, when Σ is a surface with an isolated simple (rational double point) singularity of type A_2 at the origin, [1], p. 60.

Let X_N and Y_N be two subvarieties of $\mathbb{C}^3$ defined by the respective polynomials $x_1x_2 = x_3^N$ and $y_1^2y_3 + y_2^2 = y_3^{N+1}$, for any natural number $N \geq 2$. The surface X_N (respect. Y_N) has an isolated simple singularity of type A_{N-1} (respect. D_{N+2}) at the origin, see [1], p. 60. The main objective of this paper is to give an alternative and simplified solution to the equation $\overline{\partial}g = \lambda$ on both surfaces X_N and Y_N, with an extra Hölder condition on g. The central idea is to consider $\mathbb{C}^2$ as a branched covering over X_N and Y_N, instead of analysing X_N as a branched covering over $\mathbb{C}^2$. In the case of X_N, we use the natural branched N-covering $\pi_N : \mathbb{C}^2 \to X_N$ defined by $\pi_N(z_1, z_2) = (z_1^N, z_2^N, z_1z_2)$, in order to obtain the following theorem. We shall explain, at the end of the third section of this paper, why we use the covering π_N instead of a standard *blow up* mapping.

THEOREM (1.1). *Let $Ev(N)$ be the smallest even integer greater than or equal to N. Given an exponent $0 < \beta < 1/Ev(N)$ and an open ball $B_R \subset \mathbb{C}^2$ of radius $R > 0$ and centre in the origin, there exists a finite positive constant $C_1(R,\beta)$ such that: For every continuous $(0,1)$-differential form λ defined on the compact set $\pi_N(\overline{B_R}) \subset X_N$, and $\overline{\partial}$-closed on the interior $\pi_N(B_R)$, the equation $\overline{\partial}h = \lambda$ has*

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a continuous solution h on $\pi_N(B_R)$ which also satisfies the following Hölder estimate,

$$\|h\|_{\pi_N(B_R)} + \sup_{x,w\in\pi_N(B_R)} \frac{|h(x)-h(w)|}{\|x-w\|^\beta} \le C_1(R,\beta)\|\lambda\|_{\pi_N(B_R)}. \tag{1.2}$$

In the last section of this paper, we extend Theorem (1.1) to solve the $\overline{\partial}$-equation on the subvariety Y_N as well. The notation $\|h\|_S$ stands for the supremun of $|h|$ on the set S, and $\|x-w\|$ stands for the euclidean distance between x and w. Since $\|x-w\|$ is less than or equal to the distance between x and w measured *along* the surface X_N, we can assert that inequality (1.2) is indeed a Hölder estimate on X_N itself. Finally, all differentials are defined in terms of distributions. For example, the fact that the continuous $(0,1)$-differential form λ is $\overline{\partial}$-closed on $\pi_N(B_R)$ means that the integral

$$\int_{\pi_N(B_R)} \lambda \wedge \overline{\partial}\sigma = 0, \tag{1.3}$$

vanishes for every smooth $(2,0)$-differential form σ defined on $\pi_N(B_R)\setminus\{0\}$, such that both σ and $\overline{\partial}\sigma$ extend continuously to the origin, and these extensions have both compact support inside $\pi_N(B_R)$.

The proof of Theorem (1.1) is presented in the following two sections. The next section is devoted to introducing all the basic ideas for the particular case when $N=2$. Moreover, in the third section of this paper, we shall use these ideas for solving the $\overline{\partial}$-equation on X_N, in the extended case $N\ge 3$. Finally, in the last section of this paper, we extend Theorem (1.1) to solve the $\overline{\partial}$-equation on the subvariety Y_N as well.

2. Proof of Theorem (1.1), case $N=2$.

Consider the natural branched covering $\pi_2(z_1,z_2) = (z_1^2, z_2^2, z_1z_2)$ defined from $\mathbb{C}^2$ onto $X_2 := [x_1x_2 = x_3^2]$. It is easy to see that π_2 is a branched 2-covering, and that the origin is the only branch point of π_2, because the inverse image $\pi_2^{-1}(x)$ is a set of the form $\{\pm z\}$, for every $x\in X_2$. Further, define the antipodal automorphism $\phi(z)=-z$ which allows us to jump between the different *branches* of π_2. In particular, we have that $\phi^*\pi_2(z)=\pi_2(-z)=\pi_2(z)$.

We assert that the operators π_2^* and $\overline{\partial}$ commute. It is easy to see that π_2^* and $\overline{\partial}$ commute when $\overline{\partial}$ is a standard differential, for π_2 is holomorphic. However, calculations become more complicated when $\overline{\partial}$ is analysed in the sense of distributions. Let $B_R\subset\mathbb{C}^2$ be an open ball of radius $R>0$. We prove the commutativity of π_2^* and $\overline{\partial}$ for the particular case of a $\overline{\partial}$-closed $(0,1)$-differential form λ defined on $\pi_2(B_R)$; the proof with a general differential form follows exactly the same procedure. We have that $\overline{\partial}\lambda=0$ in the sense of equation (1.3), and we need to prove that $\overline{\partial}(\pi_2^*\lambda)$ is equal to $\pi_2^*(\overline{\partial}\lambda)=0$ in the sense of distributions, that is:

$$\int_{B_R} \pi_2^*\lambda \wedge \overline{\partial}v = 0, \tag{2.1}$$

for every smooth $(2,0)$-differential form v with compact support in B_R. The automorphism ϕ preserves the orientation of B_R, for it is analytic. Thus, after doing a simple change of variables, and recalling that $\phi^*\pi_2=\pi_2$, we have that the integral in equation (2.1) is equal to $\int_{B_R} \pi_2^*\lambda \wedge \overline{\partial}\phi^* v$. Moreover, since $v + \phi^* v$ is constant on the fibres of π_2 (it is invariant under the *pull back* ϕ^*) there exists a second differential form σ defined on $\pi_2(B_R)$ such that $v + \phi^* v$ is equal to $\pi_2^*\sigma$. Hence

$$\int_{B_R} \pi_2^*\lambda \wedge \overline{\partial} v = \int_{B_R} \pi_2^*\lambda \wedge \overline{\partial}\, \frac{v + \phi^* v}{2} = \int_{\pi_2(B_R)} \frac{\lambda \wedge \overline{\partial}\sigma}{2} = 0.$$

The equality to zero follows from equation (1.3), and so $\overline{\partial}(\pi_2^*\lambda) = 0$ on B_R, as we wanted to prove. Suppose now that the differential equation $\overline{\partial} g = \pi_2^*\lambda$ has a solution g on B_R. The sum $g + \phi^* g$ is also constant in the fibres of π_2 (it is invariant under the *pull back* ϕ^*), so there exists a continuous function f on B_R such that $\pi_2^* f$ is equal to $g + \phi^* g$. We assert that $\overline{\partial} f = 2\lambda$ on $\pi_2(B_R)$. This result follows automatically because

$$\pi_2^*\overline{\partial} f = \overline{\partial}(g + \phi^* g) = \pi_2^*\lambda + \phi^*\pi_2^*\lambda = \pi_2^*(2\lambda).$$

The previous equation requires that the operators ϕ^* and $\overline{\partial}$ commute as well in B_R, when $\overline{\partial}$ is seen as a distribution. This is an exercise based on the fact that $\int \phi^* \aleph = \int \aleph$, as we have indicated in the paragraph situated after equation (2.1), and because ϕ preserves the orientation of B_R. Suppose now that λ is also continuous on the compact set $\pi_2(\overline{B_R})$. Then we can apply Theorems 2.1.5 and 2.2.2 of [4] in order to get the following Hölder estimate.

THEOREM (2.2). *Given an exponent $0 < \delta < 1$ and an open ball $B_R \subset \mathbb{C}^2$ of radius $R > 0$ and centre at the origin, there exist two finite positive constants $C_2(R)$ and $C_3(R, \delta)$ such that: For every continuous $(0, 1)$-differential form λ defined on $\pi_2(\overline{B_R}) \subset X$, and $\overline{\partial}$-closed on the interior $\pi_2(B_R)$, the equation $\overline{\partial} g = \pi_2^*\lambda$ has a continuous solution g on B_R which also satisfies the following Hölder estimates,*

$$\|g\|_{B_R} + \sup_{z,\zeta \in B_R} \frac{|g(z) - g(\zeta)|}{\|z - \zeta\|^{1/2}} \leq C_2(R)\|\pi_2^*\lambda\|_{B_R}, \tag{2.3}$$

$$\textit{and} \quad \sup_{z,\zeta \in B_{R/2}} \frac{|g(z) - g(\zeta)|}{\|z - \zeta\|^{\delta}} \leq C_3(R, \delta)\|\pi_2^*\lambda\|_{B_R}. \tag{2.4}$$

Proof. Inequality (2.3) holds because of Theorem 2.2.2 in [4]. Further, recalling the proofs of Lemma 2.2.1 and Theorem 2.2.2, in [4], we have that inequality (2.4) holds whenever there exists a finite positive constant $C_4(R)$ such that

$$\sup_{z,\zeta \in B_{R/2}} \frac{|E(z) - E(\zeta)|}{\|z - \zeta\|} \leq C_4(R)\|\pi_2^*\lambda\|_{B_R}, \tag{2.5}$$

for every function $E(z)$ defined according to equation (2.2.7) of [4], p. 70. Let Υ be the closed interval which joins z and ζ inside the ball $B_{R/2}$. Then,

$$\begin{aligned} |E(z) - E(\zeta)| &\leq \int_0^1 \left| \frac{d}{dt} E(t\zeta + (1-t)z) \right| dt \\ &\leq \|z - \zeta\| \sup_{y \in \Upsilon} \sum_{k=1}^{2} \left| \frac{\partial E}{\partial y_k} \right| + \left| \frac{\partial E}{\partial \overline{y}_k} \right|. \end{aligned} \tag{2.6}$$

Finally, by equation (2.2.9) in [4], we know there exists a finite constant $C_4(R)$ such that all partial derivatives $\left|\frac{\partial E}{\partial y_k}\right|$ and $\left|\frac{\partial E}{\partial \overline{y}_k}\right|$ are less than or equal to $\frac{C_4(R)}{5}\|\pi_2^*\lambda\|_{B_R}$, for every $y \in B_{R/2}$ and each index $k = 1, 2$. Notice that $D = B_R$ in equations (2.2.7) and (2.2.9), but y lies inside the smaller ball $B_{R/2}$. Thus, equation (2.6) automatically implies that inequalities (2.5) and (2.4) holds, as we wanted. □

The problem is now reduced to estimating the distance $\|z - \zeta\|$ with respect to the projections $\|\pi_2(z) - \pi_2(\zeta)\|$.

LEMMA (2.7). *Given two points z and ζ in $\mathbb{C}^2$ such that $\|z - \zeta\|$ is less than or equal to $\|z + \zeta\|$, the following inequality holds.*

$$2\|\pi_2(z) - \pi_2(\zeta)\| \;\geq\; \|z - \zeta\| \max\{\|z\|, \|\zeta\|, \|z - \zeta\|\}.$$

Proof. We know that $2\|z\|$ and $2\|\zeta\|$ are both less than or equal to $\|z + \zeta\| + \|z - \zeta\|$. The given hypotheses indicates that $\|z - \zeta\| \leq \|z + \zeta\|$. Hence, the maximum of $\|z\|$, $\|\zeta\|$ and $\|z - \zeta\|$ is also less than or equal to $\|z + \zeta\|$. The desired result will follows after proving that $\|z - \zeta\| \cdot \|z + \zeta\|$ is less than or equal to $2\|\pi_2(z) - \pi_2(\zeta)\|$. Setting $P_1 = z_1 - \zeta_1$, $P_2 = z_2 - \zeta_2$, $Q_1 = z_1 + \zeta_1$ and $Q_2 = z_2 + \zeta_2$, allows us to write the following series of inequalities:

$$\begin{aligned}
& \|z - \zeta\|^2 \cdot \|z + \zeta\|^2 \;= \\
= \;& |P_1Q_1|^2 + |P_1Q_2|^2 + |P_2Q_1|^2 + |P_2Q_2|^2 \\
\leq \;& 4|P_1Q_1|^2 + 4|P_2Q_2|^2 + |P_1Q_2|^2 + |P_2Q_1|^2 - 2|P_1Q_1P_2Q_2| \\
\leq \;& 4|P_1Q_1|^2 + 4|P_2Q_2|^2 + |P_1Q_2 + P_2Q_1|^2 \\
= \;& 4\|\pi_2(z) - \pi_2(\zeta)\|^2.
\end{aligned}$$

□

We are now in position to prove Theorem (1.1) for the simplest case $N = 2$.

Proof. **(Theorem (1.1), case $N = 2$).** Suppose that $\lambda = \sum \lambda_k d\overline{x}_k$. Then,

$$\pi_2^*\lambda = [2\overline{z}_1\lambda_1(\pi_2) + \overline{z}_2\lambda_3(\pi_2)]d\overline{z}_1 + [2\overline{z}_2\lambda_2(\pi_2) + \overline{z}_1\lambda_3(\pi_2)]d\overline{z}_2. \tag{2.8}$$

We obviously have that $|z_k| < R$ for every point $z \in B_R$. Hence,

$$\|\pi_2^*\lambda\|_{B_R} \leq 3R\,\|\lambda\|_{\pi_2(B_R)}. \tag{2.9}$$

Let g be a continuous solution to the equation $\overline{\partial} g = \pi_2^*\lambda$ on B_R, and suppose that g satisfies the Hölder estimates given in equations (2.3) and (2.4) of Theorem (2.2). Recalling the analysis done in the paragraphs situated before Theorem (2.2), we know there exists a continuous function h defined on $\pi_2(B_R)$ such that $h \circ \pi_2$ is equal to $\frac{g+\phi^* g}{2}$. In particular, $\overline{\partial} h = \lambda$ on $\pi_2(B_R)$, and

$$\|h\|_{\pi_2(B_R)} = \frac{\|g + \phi^* g\|_{B_R}}{2} \;\leq \|g\|_{B_R}. \tag{2.10}$$

Note that $\beta < 1/2$ when $N = 2$. Given two points $x, w \in \pi_2(B_R)$, choose $z, \zeta \in B_R$ such that $x = \pi_2(z)$ and $w = \pi_2(\zeta)$. Since $\pi_2(\zeta) = \pi_2(-\zeta)$, we can even choose $\zeta \in B_R$ so that $\|z - \zeta\|$ is less than or equal to $\|z + \zeta\|$. If z and ζ are both inside the ball $B_{R/2}$, we may apply equation (2.4) of Theorem (2.2), and the inequality $2\|x-w\| \geq \|z-\zeta\|^2$ given in Lemma (2.7), in order to obtain the following equation for $0 < \beta < 1/2$

$$\begin{aligned} \frac{|h(x)-h(w)|}{2^{\beta}\|x-w\|^{\beta}} &\leq \frac{|g(z)-g(\zeta)|+|g(-z)-g(-\zeta)|}{2\,\|z-\zeta\|^{2\beta}} \\ &\leq C_3(R,2\beta)\|\pi_2^*\lambda\|_{B_R}. \end{aligned} \tag{2.11}$$

On the other hand, suppose, without lost of generality, that z is not inside the ball $B_{R/2}$; that is $\|z\| \geq \frac{R}{2}$. Lemma (2.7) implies then that $\|x-w\|$ is greater than or equal to $\frac{R}{4}\|z-\zeta\|$. Whence, equation (2.3) automatically implies the following,

$$\frac{|h(x)-h(w)|}{\|x-w\|^{1/2}} \leq \frac{2}{\sqrt{R}}\,C_2(R)\|\pi_2^*\lambda\|_{B_R}. \tag{2.12}$$

Finally, considering Theorem (2.2) and equations (2.9) to (2.12), we can deduce the existence of a bounded positive constant $C_1(R,\beta)$ such that equation (1.2) holds. □

We close this section with some observations about Theorem (1.1). Firstly, the procedure presented in this section yields a continuous solution h to the equation $\bar{\partial}h = \lambda$. Moreover, we are directly using the estimates given in [4], but we may use any integration kernel which produces estimates similar to those presented in equations (2.3) and (2.4) of Theorem (2.2).

On the other hand, the extension of Theorem (1.1) to considering a general subvariety Σ with an isolated singularity does not seem to be trivial. Theorem (1.1) requires the existence of a branched finite covering $\pi : W \to \Sigma$, where W is a *nice* non-singular manifold and the inverse image of the singular point is a singleton. It does not seem to be trivial to produce such a branched finite covering.

3. Proof of Theorem (1.1), case $N \geq 3$.

We analyse in this section the general case of the variety $X_N \subset \mathbb{C}^3$ defined by $x_1x_2 = x_3^N$, for any natural number $N \geq 3$. Surface X_N has an isolated simple singularity of type A_{N-1} at the origin, [1], p. 60. Define the automorphisms $\phi_k : \mathbb{C}^2 \to \mathbb{C}^2$ for each natural number k,

$$\phi_k(z_1,z_2) = (\rho_N^k z_1, \rho_N^{-k} z_2) \quad \text{where} \quad \rho_N = e^{2\pi i/N}. \tag{3.1}$$

Consider the natural branched covering $\pi_N(z_1,z_2) = (z_1^N, z_2^N, z_1z_2)$ defined from $\mathbb{C}^2$ onto X_N. It is easy to see that π_N is a branched N-covering, and that the origin is the only branch point of π_N, because the inverse image $\pi_N^{-1}(x)$ is a set of the form $\{\phi_k(z)\}_{1\leq k\leq N}$, for every $x \in X_N$. Thus, the automorphisms ϕ_k allow us to jump between the different *branches* of π_N. In particular, we have that $\pi_N = \phi_k^*\pi_N$ for every k. Besides, the operators π_N^* and $\bar{\partial}$ commute, the proof is based on the same ideas presented at the beginning of section two.

Given a $\bar{\partial}$-closed $(0,1)$-differential form λ defined on X_N, we obviously have that $\bar{\partial}(\pi_N^*\lambda) = 0$. Suppose the differential equation $\bar{\partial}g = \pi_N^*\lambda$ has a solution g in $\mathbb{C}^2$. The sum $\frac{1}{N}\sum_{k=1}^N \phi_k^* g$ is constant in the fibres of π_N (it is invariant under every *pull back* ϕ_j^*), so there exists a continuous function h on X_N such that $\pi_N^* h$ is equal to $\frac{1}{N}\sum \phi_k^* g$. We assert that $\bar\partial h = \lambda$ on X_N. This result follows automatically because

$$\pi_N^*\bar\partial h = \tfrac{1}{N}\sum \bar\partial\phi_k^* g = \tfrac{1}{N}\sum \phi_k^*\pi_N^*\lambda = \pi_N^*\lambda.$$

Let $B_R \subset \mathbb{C}^2$ be an open ball of radius $R > 0$. If λ is continuous on the compact set $\pi_N(\overline{B_R})$, then we can apply Theorem (2.2) in order to get a solution h which satisfies a Hölder estimate on the ball B_R. Obviously, as in the second section of this paper, the central part of the proof is an estimate of the distance $\|z-\zeta\|$ with respect to the projections $\|\pi_N(z)-\pi_N(\zeta)\|$. This estimate is done in the next lemma. Given two points z and ζ in $\mathbb{C}^2$, notation $\|z,\zeta\|_\infty$ stands for the maximum of $|z_1|$, $|z_2|$, $|\zeta_1|$ and $|\zeta_2|$. Moreover, $\|z\|_\infty := \|z, 0\|_\infty$ as well.

LEMMA (3.2). *Let $Ev(N)$ be the smallest even integer greater than or equal to N. Given two points z and ζ in $\mathbb{C}^2$ such that $\|z-\phi_k(\zeta)\|_\infty$ is greater than or equal to $\|z-\zeta\|_\infty$ for every automorphism ϕ_k defined in (3.1), the following inequality holds for δ equal to both N and $Ev(N)/2$.*

$$\|\pi_N(z)-\pi_N(\zeta)\| \geq \min\left\{\frac{\|z-\zeta\|_\infty^2}{12}, \frac{\|z,\zeta\|_\infty^{N-\delta}\|z-\zeta\|_\infty^\delta}{(8/3)^{N-\delta}\,2^\delta}\right\}. \tag{3.3}$$

Proof. Set $z=(a,b)$, so that $\pi_N(z)=(a^N, b^N, ab)$. Moreover, given $\zeta=(s,t)$, we can suppose without loss of generality that $|a-s| \geq |b-t|$, and so $\|z-\zeta\|_\infty = |a-s|$. We shall prove inequality (3.3) by considering three cases.

Case I. Whenever $|b| \geq |s| + \frac{|a-s|}{12}$, we have the inequality,

$$|ab-st| \geq |a-s|\cdot|b| - |b-t|\cdot|s| \geq \frac{|a-s|^2}{12}.$$

Finally, notice that $\|\pi_N(z)-\pi_N(\zeta)\| \geq |ab-st|$, so equation (3.3) holds in this particular case.

Case II. If $|b| \leq |s| + \frac{|a-s|}{12}$, and there exists a natural j such that $|a-\rho_N^j s|$ is less than or equal to $\frac{|a-s|}{2}$, we also have,

$$|a-s| \leq |a-\rho_N^j s| + 2|s| \leq \frac{|a-s|}{2} + 2|s|.$$

Consequently, $|s| \geq \frac{|a-s|}{4}$. On the other hand, we know that $\|z-\phi_j(\zeta)\|_\infty$ is equal to the maximum of $|a-\rho_N^j s|$ and $|\rho_N^j b - t|$. Recalling the hypotheses of Lemma (3.2) and this case (II), we have that $|a-\rho_N^j s| < |a-s|$, and that $\|z-\phi_j(\zeta)\|_\infty$ is greater than or equal to $\|z-\zeta\|_\infty = |a-s|$. Hence, both $|\rho_N^j b - t| \geq |a-s|$ and

$$\begin{aligned} |ab-st| &\geq |\rho_N^j b - t|\cdot|s| - |a-\rho_N^j s|\cdot|b| \\ &\geq \frac{|a-s|\cdot|s|}{2} - \frac{|a-s|^2}{24} \geq \frac{|a-s|^2}{12}. \end{aligned}$$

Notice that $|a-\rho_N^j s|\cdot|b|$ is less than or equal to $\frac{|a-s|\cdot|s|}{2} + \frac{|a-s|^2}{24}$ because of the hypotheses of this case (II). We may conclude that equation (3.3) holds in this particular case as well, after recalling that $\|\pi_N(z)-\pi_N(\zeta)\|$ is greater than or equal to $|ab-st|$.

Case III. If $|b| \leq |s| + \frac{|a-s|}{12}$, and $|a - \rho_N^k s| \geq \frac{|a-s|}{2}$ for every natural k, we automatically have the following inequality

$$|a^N - s^N| = \prod_{k=1}^{N} |a - \rho_N^k s| \geq \frac{|a-s|^N}{2^N}.$$

Finally, we know that $\|z-\zeta\|_\infty = |a-s|$, and that $\|\pi_N(z) - \pi_N(\zeta)\|$ is greater than or equal to $|a^N - s^N|$. The previous inequalities show that equation (3.3) holds for $\delta = N$. On the other hand, when $\delta = Ev(N)/2$, it is easy to deduce the existence of a subset J of $\{1, 2, \ldots, N\}$ composed of at least $N - \delta$ elements and which satisfies

$$|a - \rho_N^j s| \geq \max\left\{|a|, |s|, \frac{|a-s|}{\sqrt{2}}\right\} \quad \text{for each} \quad j \in J. \tag{3.4}$$

The set J can be built as follows. We may suppose, without lost of generality, that a is real and $a \geq 0$, for we only need to multiply both a and s by an appropriate complex number θ with $|\theta| = 1$. Thus, the set J is composed of all exponents $1 \leq j \leq N$ which satisfy $\Re(\rho_N^j s) \leq 0$. It is easy to see that $|a - \rho_N^j s|^2$ is greater than or equal to $|a|^2 + |s|^2$, for every $j \in J$. Moreover, J is composed of at least $N/2$ elements when N is even, and of at least $\frac{N-1}{2}$ elements when N is odd. Equation (3.4) follows automatically because $|a|^2$, $|s|^2$ and $\frac{|a-s|^2}{2}$ are all less than or equal to $|a|^2 + |s|^2$. The hypotheses of this case (III), and equation (3.4), directly imply that

$$2|a - \rho_N^j s| \geq |s| + \frac{|a-s|}{\sqrt{2}} \geq |b|, \quad \forall j \in J.$$

Moreover, since $\frac{5}{3} > \frac{13}{12}\sqrt{2}$, and we are supposing from the beginning of this proof that $|a - s| \geq |b - t|$, we may also deduce the following inequality,

$$\frac{8|a - \rho_N^j s|}{3} > |s| + \frac{|a-s|}{12} + |b - t| \geq |t|, \quad \forall j \in J.$$

Finally, considering all the results presented in previous paragraphs, equation (3.4) and the hypotheses of this case (III), we can deduce the desired result,

$$|a^N - s^N| = \prod_{k=1}^{N} |a - \rho_N^k s| \geq \frac{\|z, \zeta\|_\infty^{N-\delta} |a-s|^\delta}{(8/3)^{N-\delta}\, 2^\delta}$$

where $\delta = Ev(N)/2$, the norm $\|z - \zeta\|_\infty = |a - s|$ and $\|z, \zeta\|_\infty$ is the maximum of $|a|$, $|b|$, $|s|$ and $|t|$. We can conclude that equation (3.3) holds when δ is equal to N and $Ev(N)/2$. □

We are now in a position to complete the proof of Theorem (1.1). Notice that Lemma (3.2) automatically implies the following inequalities, whenever z and ζ lie inside the compact ball $\overline{B_R}$, and $\delta = N$,

$$\begin{aligned} \|\pi_N(z) - \pi_N(\zeta)\| \quad &\geq \quad \frac{\|z-\zeta\|_\infty^N}{2^N} \min\left\{\frac{1}{3\, R^{N-2}}, 1\right\} \\ &\geq \quad \frac{\|z-\zeta\|^N}{\left(\sqrt{8}\right)^N} \min\left\{\frac{1}{3\, R^{N-2}}, 1\right\}. \end{aligned}$$

Proof. **(Theorem (1.1), case $N \geq 3$).** We shall follow step by step the proof of Theorem (1.1), case $N = 2$, presented in section two; so we shall only indicate the main differences. Let g be a continuous solution to the equation $\overline{\partial} g = \pi^*\lambda$ on B_R which satisfies the Hölder estimates given in equations (2.3) and (2.4). Recalling the analysis done at the beginning of this section, we know there exists a continuous function h defined on $\pi(B_R)$ such that $h \circ \pi$ is equal to $\frac{1}{N}\sum \phi_k^* g$. In particular, $\overline{\partial} h = \lambda$ on $\pi(B_R)$, and $\|h\|_{\pi_N(B_R)}$ is less than or equal to $\|g\|_{B_R}$. Moreover, working as in equations (2.8) and (2.9), we may deduce the existence of a finite positive constant $C_5(R)$ such that $\|\pi_N^*\lambda\|_{B_R}$ is less than or equal to $C_5(R)\|\lambda\|_{\pi_N(B_R)}$.

Given two points $x, w \in \pi(B_R)$, choose $z, \zeta \in B_R$ such that $x = \pi_N(z)$ and $w = \pi_N(\zeta)$. Since $\pi_N(\zeta) = \pi_N(\phi_k(\zeta))$ for every automorphism ϕ_k defined in (3.1), we can even choose $\zeta \in B_R$ so that $\|z-\zeta\|_\infty$ is less than or equal to $\|z-\phi_k(\zeta)\|_\infty$ for every ϕ_k. A direct application of Lemma (3.2), with $\delta = N$, yields the existence of a finite positive constant $C_6(R)$ such that $\|x - w\|$ is greater than or equal to $C_6(R)\|z-\zeta\|^{1/\beta}$. Recall that $0 < \beta < 1/Ev(N)$ and $N \geq 3$. Thus, if z and ζ are both inside the ball $B_{R/2}$, we may apply equation (2.4) in order to deduce that $\frac{|h(x)-h(w)|}{\|x-w\|^\beta}$ is less than or equal to $\frac{C_3(R,Ev(N)\beta)}{C_6^\beta(R)}\|\pi^*\lambda\|_{B_R}$.

On the other hand, suppose, without lost of generality, that z is not inside the ball $B_{R/2}$. a direct application of Lemma (3.2), with $\delta = \frac{Ev(N)}{2}$, yields the existence of a finite positive constant $C_7(R)$ such that $\|x - w\|$ is greater than or equal to $C_7(R)\|z-\zeta\|^\delta$. Whence, equation (2.3) automatically implies that $\frac{|h(x)-h(w)|}{\|x-w\|^\beta}$ is less than or equal to $\frac{C_2(R)}{C_7^\beta(R)}\|\pi^*\lambda\|_{B_R}$ as well.

The analysis done in the previous paragraphs automatically implies the existence of a finite positive constant $C_1(R)$ such that equation (1.2) holds for every N greater than or equal to three. □

Finally, as we have already said at the end of section two, the proof of Theorem (1.1) works perfectly if we apply Theorem (2.2) of Henkin and Leiterer, or any other integration kernel which produces estimates similar to those posed in equations (2.3) and (2.4). For example, the hypotheses on λ can be relaxed in Theorem (1.1), to consider $(0,1)$-differential forms λ which are bounded and continuous on $\pi(\overline{B_R}) \setminus K$, for some compact set $K \subset B_R$ of zero-measure. Besides, the results presented in Theorem (1.1) hold as well, if we consider an arbitrary strictly pseudoconvex domain D, with smooth boundary and the origin in its interior, instead of the open ball B_R. In this case, the ball $B_{R/2}$ used in equation (2.4) of Theorem (2.2) would be a sufficiently small ball B_r whose closure is contained in the interior of D.

On the other hand, the work presented in this paper is strongly based on the existence of a branched finite covering π_N from $\mathbb{C}^2$ onto X_N, such that the inverse image of the singular point $\pi_N^{-1}(0) = \{0\}$ is a singleton. This property allows us to get the estimates presented in Lemmas (2.7) and (3.2), which are essential for this paper. It is obvious to consider a *blow-up* mapping $\eta : W \to X_N$ instead of the finite covering π_N. In any case, a blow-up is a 1-covering everywhere, except at the singular point 0. However, since the inverse image $\eta^{-1}(0)$ is not a singleton, and it is not even finite in general, we have strong problems for calculating a Hölder solution to the equation $\overline{\partial} h = \lambda$, unless we introduce stronger hypotheses. We finish this section by analysing the case of a blow-up.

Remark. Let Σ be a variety with an isolated singularity at $\sigma_0 \in \Sigma$, and $\eta : W \to \Sigma$ be a holomorphic *blow-up* of Σ at σ_0, such that W is a smooth manifold. Given a $\bar\partial$-closed $(0, 1)$-form λ defined on Σ minus σ_0, we automatically have that $\eta^*\lambda$ is also $\bar\partial$-closed on W minus $\eta^{-1}(\sigma_0)$. Thus, suppose there exists a continuous solution $g : W \to \mathbb{C}$ to the equation $\bar\partial g = \eta^*\lambda$. Since η is a blow-up, we automatically have that η^{-1} is well defined on $\Sigma \setminus \{\sigma_0\}$, and so λ is equal to $\bar\partial(g \circ \eta^{-1})$ there.

Define $h := g \circ \eta^{-1}$. Unless g is constant on the inverse fibre $\eta^{-1}(\sigma_0)$, the function h does not have a continuous extension to σ_0, and does not satisfy any Hölder condition in a neighbourhood of σ_0. Suppose there exists a pair of points a and b in $\eta^{-1}(\sigma_0)$ such that $g(a) \neq g(b)$. Besides, take $\{a_m\}$ and $\{b_m\}$ a pair of infinite sequences in $W \setminus \eta^{-1}(\sigma_0)$ which respectively converge to a and b. Notice that both $\eta(a_m)$ and $\eta(b_m)$ converge to the same point σ_0. However, $g(a_m)$ and $g(b_m)$ converge to different points, for $g(a) \neq g(b)$. Hence, given any metric Δ on Σ, which defines the topology, we have that

$$\limsup_m \frac{|h \circ \eta(a_m) - h \circ \eta(b_m)|}{\Delta[\eta(a_m), \eta(b_m)]^\beta} = \infty, \quad \forall \beta > 0.$$

That is, in order to introduce Hölder conditions on $h := g \circ \eta^{-1}$, it is essential that the solution to equation $\bar\partial g = \eta^*\lambda$ is constant on the inverse fibre of the singular point $\eta^{-1}(\sigma_0)$.

4. Surfaces with simple singularities of type D_{N+2}

We finish this paper by solving the $\bar\partial$-equation on a neighbourhood of the origin in the subvariety $Y_N \subset \mathbb{C}^3$, defined by the polynomial $y_1^2 y_3 + y_2^2 = y_3^{N+1}$. The surface Y_N has an isolated simple (rational double point) singularity of type D_{N+2} at the origin, for any natural number $N \geq 2$, [1], p. 60. We extend the results presented in Theorem (1.1), by introducing a branched 2-covering defined from $X_{2N} := [x_1 x_2 = x_3^{2N}]$ onto the surface Y_N. Consider the holomorphic mapping $\eta_2 : X_{2N} \to Y_N$, and the pair of matrices P and Q, given by the respective equations,

$$\eta_2(x_1, x_2, x_3) = \left(\frac{x_1 + x_2}{2}, x_3 \frac{x_1 - x_2}{2i}, x_3^2 \right), \tag{4.1}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ -i & i & 0 \\ 0 & 0 & 2 \end{bmatrix}. \tag{4.2}$$

It is easy to see that $\eta_2(x) = \eta_2(Px)$ for every $x \in X_{2N}$. Moreover, η_2 is a branched 2-covering, and the origin is the unique branch point, because the inverse image $\eta_2^{-1}(y)$ is a set of the form $\{x, Px\}$, for every $y \in Y_N$. For example, the inverse image of $(y_1, 0, 0)$ is composed of two points: $(2y_1, 0, 0)$ and $(0, 2y_1, 0)$. We have already defined a branched covering π_{2N} from $\mathbb{C}^2$ onto X_{2N}, so the composition $\eta_2 \circ \pi_{2N}$ is indeed a covering from $\mathbb{C}^2$ onto Y_N. The branch covering η_2 is a central part in the following result.

THEOREM (4.3). *Given an exponent $0 < \beta_2 < 1/(4N)$ and an open ball $B_R \subset \mathbb{C}^2$ of radius $R > 0$ and centre in the origin, define the open set $E_R := \eta_2(\pi_{2N}(B_R))$ in Y_N. There exists a finite positive constant $C_{11}(R, \beta_2)$ such that: for every continuous $(0,1)$-differential form $\aleph$ defined on the compact set $\overline{E_R} \subset Y_N$, and $\overline{\partial}$-closed on E_R, the equation $\overline{\partial} f = \aleph$ has a continuous solution f on E_R which also satisfies the following Hölder estimate*

$$\|f\|_{E_R} + \sup_{y,\xi\in E_R} \frac{|f(y) - f(\xi)|}{\|y - \xi\|^{\beta_2}} \leq C_{11}(R, \beta_2)\|\aleph\|_{E_R}. \tag{4.4}$$

The proof of this theorem follows exactly the same ideas and steps presented in the proof of Theorem (1.1), case $N = 2$, so we do not include it. Given a $(0,1)$-differential form $\aleph$ continuous on $\overline{E_R} \subset Y_N$, and $\overline{\partial}$-closed on E_R. We have that $\eta_2^*\aleph$ is also continuous on $\pi_{2N}(\overline{B_R})$, and $\overline{\partial}$-closed on $\pi_{2N}(B_R)$. Therefore, we can apply Theorem (1.1), in order to obtain a continuous solution h to the differential equation $\overline{\partial} h = \eta_2^*\aleph$, which also satisfies the Hölder conditions given in equation (1.2). There exists a continuous function f on E_R such that $\eta_2^* f$ is equal to $\frac{h+\psi^* h}{2}$, and so $\overline{\partial} f = \aleph$, as we wanted. Finally, inequality (4.4) follows from equation (1.2), after noticing that there exists a pair of finite positive constants $C_8(R)$ and $C_9(R)$ such that

$$\|f\|_{E_R} \leq \|h\|_{\pi_{2N}(B_R)}, \quad \|\eta_2^*\aleph\|_{\pi_{2N}(B_R)} \leq C_8(R)\|\aleph\|_{E_R},$$

and $\frac{|f(y)-f(\xi)|}{\|y-\xi\|^{\beta_2}}$ is also less than or equal to $C_9(R)\|\eta_2^*\aleph\|_{\pi_{2N}(B_R)}$ for every y and ξ in E_R. We obviously need an estimate of $\|x - w\|^2$, with respect to the projections $\|\eta_2(x) - \eta_2(w)\|$, in order to show that the inequality above holds. This estimate is presented in the following Lemma (4.5). In conclusion, the proof of Theorem (4.3) follows the same ideas and steps of the proof of Theorem (1.1), case $N = 2$, we only need to apply Theorem (1.1) instead of Theorem (2.2), and the following Lemma (4.5) instead of Lemma (2.7).

LEMMA (4.5). *Let x and w be two points in X_{2N} whose norms $\|x\|$ and $\|w\|$ are both less than or equal to a finite constant $\rho > 0$. If the distance $\|Q(w - Px)\|$ is greater than or equal to $\|Q(w - x)\|$, for the matrices P and Q defined in (4.2), then the following inequality holds.*

$$\begin{aligned} \|\eta_2(x) - \eta_2(w)\| &\geq C_{12}(\rho)\|x - w\|^2, \\ \textit{where} \quad C_{12}(\rho) &= \frac{1}{80} \min\left\{4, \frac{5}{3\rho}, \frac{1}{\rho^{2N-2}N}\right\}. \end{aligned} \tag{4.6}$$

Proof. Introducing the new variables $(a, b, c) := Qx$ and $(s, t, u) := Qw$, we have that $a^2 + b^2 = c^{2N}$ and $QPx = (a, -b, -c)$ for every $x \in X_{2N}$. Moreover,

$$\|\eta_2(x) - \eta_2(w)\|^2 = |a - s|^2 + |bc - tu|^2 + |c^2 - u^2|^2. \tag{4.7}$$

A main step in this proof is to shown that the following inequality holds,

$$\|\eta_2(x) - \eta_2(w)\| \geq 16C_{12}(\rho)\|\pi_2(b, c) - \pi_2(t, u)\|, \tag{4.8}$$

where $\pi_2(b, c) = (b^2, c^2, bc)$ was defined in the introduction of this paper, and $C_{12}(\rho)$ is given in equation (4.6) above. We know that $\|Q(w - Px)\|$ is greater than or equal to $\|Q(w - x)\|$, according to the hypotheses of this lemma, so it is easy to deduce that $\|t + b, u + c\|$ is also greater than or equal to $\|t - b, u - c\|$, because QPx is equal to $(a, -b, -c)$. Therefore, if equation (4.8) holds, a direct application of Lemma (2.7) yields

$$\|\eta_2(x) - \eta_2(w)\| \geq 8C_{12}(\rho)(|b-t|^2 + |c-u|^2). \tag{4.9}$$

On the other hand, we can easily calculate the following upper bound for $|a|$,

$$|a| \leq \|Qx\| \leq \|x\| \leq \rho. \tag{4.10}$$

A similar upper bound $|s| \leq \rho$ holds as well. Hence, recalling equation (4.7), we have that $\|\eta_2(x) - \eta_2(w)\|$ is greater than or equal to $|a-s| \geq \frac{|a-s|^2}{2\rho}$. Adding together the inequality presented in the previous statement and equation (4.9) yields the desired result, noting that $\frac{1}{2\rho} > 8C_{12}(\rho)$ and $2\|\xi\| \geq \|Q^{-1}\xi\|$ for $\xi \in \mathbb{C}^3$,

$$\begin{aligned} 2\|\eta_2(x) - \eta_2(w)\| &\geq 8C_{12}(\rho)\|Q(x-w)\|^2 \\ &\geq 2C_{12}(\rho)\|w-x\|^2. \end{aligned}$$

We may then conclude that inequality (4.6) holds, as we wanted. We only need to prove that equation (4.8) is always satisfied, in order to finish our calculations; and we will prove this by considering two complementary cases.

Case I Whenever $3|c^2-u^2|$ is greater than or equal to $\frac{|b^2-t^2|}{\rho^{2N-2}N}$, the following inequality holds,

$$|c^2-u^2|^2 \geq \frac{16\,|c^2-u^2|^2}{25} + \frac{|b^2-t^2|^2}{(5\rho^{2N-2}N)^2}.$$

Thus, in this particular case, inequality (4.8) follows directly from equation (4.7), because $\frac{1}{5\rho^{2N-2}N}$ and $4/5$ are both greater than or equal to $16C_{12}(\rho)$.

Case II Whenever $\frac{|b^2-t^2|}{\rho^{2N-2}N}$ is greater than or equal to $3|c^2-u^2|$, we proceed as follows. The absolute values $|a|$ and $|c|$ are both bounded by $\|Qx\| \leq \rho$, according to equation (4.10); the same upper bound can be calculated for $|s|$ and $|u|$. Whence, the following series of inequalities hold:

$$\begin{aligned} 2|b^2-t^2|/3 &\leq |b^2-t^2| - \rho^{2N-2}N|c^2-u^2| \\ &\leq |b^2-t^2| - |c^{2N}-u^{2N}| \\ &\leq |a^2-s^2| \leq 2\rho|a-s|. \end{aligned}$$

Recall that $a^2+b^2 = c^{2N}$ and that $(\xi^N - 1)$ is equal to the product of $(\xi - 1)$ times the sum $\sum_{k=0}^{N-1}\xi^k$. Inequality (4.8) follows then from equation (4.7), after noticing that $16C_{12}(\rho)|b^2-t^2|$ is less than or equal to $|a-s|$, and obviously, $16C_{12}(\rho)$ is also less than one. □

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A NOTE ON ASYMPTOTIC INTEGRATION OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. The paper is concerned with the asymptotic behavior of solutions to a second order nonlinear differential equation $u'' + f(t, u) = 0$. Using the Banach contraction principle, we establish global existence of solutions which satisfy $u(t) = At + o(t^{\nu})$ as $t \to +\infty$, where $A \in \mathbb{R}$ and $\nu \in (0, 1]$.

1. Introduction

Asymptotic behavior of solutions of nonlinear second order differential equations

$$u'' + f(t, u, u') = 0, \qquad t \geq t_0 \geq 1 \tag{1.1}$$

and

$$u'' + f(t, u) = 0, \qquad t \geq t_0 \geq 1 \tag{1.2}$$

has always been the subject of intensive research. Many papers published recently are concerned with existence of solutions to Eqs. (1.1) and (1.2) which behave at infinity like solutions of the simplest second order differential equation, $u'' = 0$, see, for instance, [1]-[9], [11]-[22]. A thorough study of the properties of such solutions, called *asymptotically linear* [5] or *linear-like* [16], is important, for instance, for the theory of oscillation of ordinary and functional differential equations, see the references in [9], as well as for the study of existence of positive solutions of elliptic problems in exterior domains, cf. [2] and [21]. We also note that this type of asymptotic behavior has been addressed recently by the authors in connection with Weyl's limit circle and limit point classification of differential operators in the theory of singular Sturm-Liouville problems [10].

Two particular types of behavior of asymptotically linear solutions of Eqs. (1.1) and (1.2) have been studied more extensively. Namely, Constantin [1], Rogovchenko and Rogovchenko [16], Yin [21] and Zhao [22] explored conditions which guarantee asymptotic representation

$$u(t) = At + o(t) \qquad \text{as } t \to +\infty, \tag{1.3}$$

whereas Lipovan [5], Mustafa [8] and the authors [9] established conditions for a more precise asymptotic development

$$u(t) = At + B + o(1) \qquad \text{as } t \to +\infty, \tag{1.4}$$

for some real constants A and B.

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Using a fixed point argument and a Wronskian-type representation similar to those exploited in [11], [12], the first author established recently in [8] existence of solutions of Eq. (1.2) which, for a given $\mu \in (0,1)$, have asymptotic representation

$$u(t) = At + o(t^{\mu}) \qquad \text{as } t \to +\infty. \tag{1.5}$$

As pointed out by Lipovan [5] and the authors [9], asymptotic formula (1.3) embraces large classes of solutions to Eq. (1.2), including those satisfying (1.4) or, in case this is not possible, solutions with the asymptotic representation (1.5).

It is known that Eq. (1.2) may possess solutions with the asymptotic development (1.4) in some situations where standard results on asymptotic integration guarantee only existence of solutions that behave at infinity as (1.3) or, at most, as (1.5), see the details in our paper [9, pp. 364-365]. Furthermore, a class of solutions with asymptotic representation (1.5) contains also solutions which satisfy (1.4). Therefore, in order to complete the study of solutions with asymptotic expansions (1.3)-(1.5) and understand completely relationship between all three classes, it is natural to explore existence of solutions of Eq. (1.2) that can be expressed in the form (1.5), but do not satisfy (1.4). The first attempt to answer this question has been made by the authors in [13]. To simplify the formulation of the result we adapt from the cited paper, we introduce two constants

$$\theta(n, t_0) := \int_{t_0}^{+\infty} s^n a(s)ds \qquad \text{and} \qquad \gamma := t_0^{\delta-(1+\varepsilon)c}\theta(m + (1+\varepsilon)c, t_0).$$

Application of [13, Theorem 2.2] to the celebrated Emden-Fowler equation

$$u'' + a(t)\,|u|^m \operatorname{sgn} u(t) = 0, \qquad t \geq 1, \qquad m \geq 1, \tag{1.6}$$

frequently encountered in applications, leads to the following proposition.

THEOREM (1.7). *Let $c \in (0,1)$, $\varepsilon \in (0, c^{-1} - 1)$, $\delta \in (c, (1+\varepsilon)c)$ and let $a(t)$ be a continuous, nonnegative function that does not vanish eventually. Assume also that*

(i) $\theta(m + (1+\varepsilon)c, 1) < +\infty$;
(ii) $\theta(m, t_0) < cm^{-1}$;
(iii) $\theta(m + (1+\varepsilon)c, t_0) < ct_0^{\delta}$.

Then, for every $A \in \left(0, 1 - \gamma(ct_0^{\delta})^{-1}\right)$, there exists a solution $u(t)$ of Eq. (1.6) with the asymptotic representation

$$u(t) = At + w(t) \qquad \text{as } t \to +\infty, \tag{1.8}$$

where $w(t) = o(t^{1-\delta})$ and, for all $t \geq t_0$,

$$A^m \left[\int_{t_0}^{t} s^{m+(1+\varepsilon)c} a(s)ds + t\int_{t}^{+\infty} s^{m+(1+\varepsilon)c-1} a(s)ds\right] \leq w(t) \leq \frac{\gamma}{(1+\varepsilon)c} t_0^{(1+\varepsilon)c-\delta} t^{1-(1+\varepsilon)c}.$$

It is clear that for the solution $u(t)$ whose existence is established in Theorem (1.7), one has

$$\liminf_{t\to+\infty} w(t) \geq B = A^m \gamma t_0^{(1+\varepsilon)c-\delta} > 0,$$

which, however, does not rule out the possibility that $u(t)$ has the asymptotic development (1.4).

In this note, using a modification of the Hale-Onuchic technique [4] that has been successfully applied by the first author [7] to investigate asymptotic behavior of solutions with prescribed decay of the first derivative, we establish, under rather general assumptions, existence of global solutions to Eq. (1.2) that satisfy (1.5) and can be written in the form (1.8), where

$$\lim_{t\to+\infty} w(t) = +\infty, \tag{1.9}$$

which, obviously, excludes for these solutions possibility of asymptotic representation (1.4).

2. Asymptotic behavior of solutions

THEOREM (2.1). *Let $A > 0$, $\nu \in [0, 1)$, $u_0 \in \mathbb{R}$ and α, $\beta \in C([t_0, +\infty); [0, +\infty))$ be two functions such that $\alpha(t) \le \beta(t)$ for all $t \ge t_0$ and $\beta(t) = o\left(t^{-\nu}\right)$ as $t \to +\infty$. Introduce the set D_{A,u_0} by*

$$D_{A,u_0} = \left\{u \in C^1([t_0, +\infty); \mathbb{R}) \mid \alpha(t) \le u'(t) - A \le \beta(t) \quad \text{for all } t \ge t_0, \quad u(t_0) = u_0\right\},$$

and assume that for all $t \ge t_0$ and $u \in D_{A,u_0}$,

$$\alpha(t) \le \int_t^{+\infty} f(s, u(s))ds \le \beta(t).$$

Suppose further that for all $t \ge t_0$ and any u_1, $u_2 \in D_{A,u_0}$,

$$|f(t, u_1(t)) - f(t, u_2(t))| \le \frac{k(t)}{t} |u_1(t) - u_2(t)|,$$

where a function $k \in C([t_0, +\infty); [0, +\infty))$ satisfies

$$\int_{t_0}^{+\infty} k(t)dt < 1 - \nu. \tag{2.2}$$

Then there exists a unique solution of the initial value problem

$$\begin{cases} u'' + f(t, u) = 0, & t \ge t_0 \ge 1, \\ u(t_0) = u_0, \end{cases}$$

defined on $[t_0, +\infty)$ such that

$$u(t) = At + o(t^{1-\nu}) \qquad \text{as } t \to +\infty,$$
$$\alpha(t) \le u'(t) - A \le \beta(t), \qquad t \ge t_0.$$

If, in particular,

$$\int_{t_0}^{+\infty} \alpha(t)dt = +\infty,$$

one has

$$\lim_{t\to+\infty} [u(t) - At] = +\infty.$$

Proof. Define the distance between the functions u_1 and u_2 in D_{A,u_0} by

$$d(u_1, u_2) = \sup_{t \geq t_0} \left[t^\nu \left| u_1'(t) - u_2'(t) \right| \right].$$

Then the metric space $E = (D_{A,u_0}, d)$ is complete. For $u \in D_{A,u_0}$ and $t \geq t_0$, introduce the operator $T : D_{A,u_0} \to C^1([t_0, +\infty); \mathbb{R})$ by the formula

$$(Tu)(t) = u_0 + A(t - t_0) + \int_{t_0}^{t} \int_{s}^{+\infty} f(\tau, u(\tau)) d\tau ds. \tag{2.3}$$

It is not hard to see that T is well-defined, that is, $TD_{A,u_0} \subseteq D_{A,u_0}$. Furthermore, we shall prove that T is a contraction in D_{A,u_0}. Let

$$\lambda = \frac{1}{1 - \nu} \int_{t_0}^{+\infty} k(s) ds.$$

It follows from the estimate

$$\begin{aligned}
|(Tu_1)'(t) - (Tu_2)'(t)| &\leq \int_{t}^{+\infty} \frac{k(\tau)}{\tau} \left| u_1(\tau) - u_2(\tau) \right| d\tau \\
&\leq \int_{t}^{+\infty} \frac{k(s)}{s} \int_{t_0}^{s} \left| u_1'(\tau) - u_2'(\tau) \right| d\tau ds \\
&\leq \left(\int_{t}^{+\infty} \frac{k(s)}{s} \int_{t_0}^{s} \frac{1}{\tau^\nu} d\tau ds \right) d(u_1, u_2) \\
&\leq t^{-\nu} \left(\frac{1}{1 - \nu} \int_{t_0}^{+\infty} k(s) ds \right) d(u_1, u_2) = t^{-\nu} \lambda d(u_1, u_2)
\end{aligned}$$

that, for $u_1, u_2 \in D_{A,u_0}$,

$$d(Tu_1, Tu_2) \leq \lambda d(u_1, u_2).$$

By virtue of (2.2), $\lambda \in (0, 1)$, and the existence of a solution follows now from the Banach contraction principle. Furthermore, for all $t \geq t_0$,

$$u(t) - At = u_0 - At_0 + \int_{t_0}^{t} \int_{s}^{+\infty} f(\tau, u(\tau)) d\tau ds \geq u_0 - At_0 + \int_{t_0}^{t} \alpha(s) ds,$$

which yields

$$\lim_{t \to +\infty} [u(t) - At] = \int_{t_0}^{+\infty} \alpha(t) dt = +\infty.$$

The proof is complete. □

Application of Theorem (2.1) to Emden-Fowler equation (1.6) leads to the proposition which complements results established in [3, 7, 13, 20]. In what follows, $C := t_0^{-(\nu+\varepsilon)} \theta(m + \nu + \varepsilon, t_0)$, where θ is defined as above.

COROLLARY (2.4). *Let $\nu \in [0, 1)$, $\varepsilon \in (0, 1 - \nu)$, and let $a(t)$ be a continuous, nonnegative function that does not vanish eventually. Assume that*

(a) $\theta(m + \nu + \varepsilon, 1) < +\infty$;

(b) $\theta(m, t_0) < m^{-1} (1 - \nu)$;

(c) $\theta(m + \nu + \varepsilon, t_0) < t_0^{\nu+\varepsilon}$.

Then, for every A, $0 < A < 1 - C$, there exists a solution $u(t)$ of Eq. (1.6) with the asymptotic representation (1.8), where $w(t) = o(t^{1-\nu})$ as $t \to +\infty$ and, for all $t \geq t_0$,

$$A^m \int_{t_0}^{t} \int_{s}^{+\infty} \tau^m a(\tau) d\tau ds \leq w(t) \leq \int_{t_0}^{t} \int_{s}^{+\infty} \tau^m a(\tau) d\tau ds.$$

In particular, $w(t)$ satisfies (1.9) provided that

(d) $\theta(m+1, 1) = +\infty$.

Proof. Let $u_0 = At_0$. For $t \geq t_0$, introduce the functions $\alpha(t)$ and $\beta(t)$ by

$$\alpha(t) = A^m \int_{t}^{+\infty} s^m a(s) ds \qquad \text{and} \qquad \beta(t) = \int_{t}^{+\infty} s^m a(s) ds.$$

Taking into account (2.3) and the fact that

$$\left(\frac{t}{t_0}\right)^{\nu+\varepsilon} \beta(t) \leq t_0^{-(\nu+\varepsilon)} \int_{t}^{+\infty} s^{m+\nu+\varepsilon} a(s) ds \leq C,$$

we deduce that for all $t \geq t_0$ and all $u \in D_{A, At_0}$,

$$\begin{aligned}
\alpha(t) \leq \int_{t}^{+\infty} a(s) \left(As + \int_{t_0}^{s} \alpha(\tau) d\tau\right)^m ds \leq \int_{t}^{+\infty} a(s) [u(s)]^m ds \\
= (Tu)'(t) - A \leq \int_{t}^{+\infty} a(s) \left(As + \int_{t_0}^{s} \beta(\tau) d\tau\right)^m ds \\
\leq \int_{t}^{+\infty} a(s) \left(As + \int_{t_0}^{s} C \left(\frac{t_0}{\tau}\right)^{\nu+\varepsilon} d\tau\right)^m ds \\
\leq \int_{t}^{+\infty} s^m a(s) (A+C)^m ds \leq \beta(t).
\end{aligned}$$

Furthermore, for any u_1, $u_2 \in D_{A, At_0}$ and for all $t \geq t_0$, one has

$$\begin{aligned}
|f(t, u_1(t)) - f(t, u_2(t))| = t^m a(t) \left| \left(\frac{u_1(t)}{t}\right)^m - \left(\frac{u_2(t)}{t}\right)^m \right| \\
\leq \frac{m t^m a(t)}{t} \sup_{s \geq t_0} \left[\left(\frac{1}{s}\left(As + \int_{t_0}^{s} \beta(\tau) d\tau\right)\right)^{m-1} \right] |u_1(t) - u_2(t)| \\
\leq \frac{k(t)}{t} |u_1(t) - u_2(t)|,
\end{aligned}$$

where $k(t) = m t^m a(t)$. The conclusion follows now from Theorem (2.1). □

We conclude the paper by noticing that it is not difficult to see that, given two positive constants $c_1 < c_2$, any continuous function $a(t)$ such that

$$c_1 t^{-m-2} \leq a(t) \leq c_2 t^{-m-2}, \qquad t \geq 1,$$

will satisfy conditions (a)-(d) of Corollary (2.4).

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POINCARÉ SERIES AND INSTABILITY OF EXPONENTIAL MAPS

P. MAKIENKO AND G. SIENRA

ABSTRACT. We relate the properties of the postsingular set for the exponential family regarding stability questions. We calculate the action of the Ruelle operator for the exponential family, and we prove that if the asymptotic (or singular) value is a summable point and its orbit satisfies certain topological conditions, the map is unstable. Hence there are no Beltrami differentials in the Julia set. We also show that if the Julia set is the whole sphere and the postsingular set is a compact set, then the singular value is summable and the map is unstable.

1. Introduction

If f is a transcendental entire map, we denote by f^n, $n \in \mathbb{N}$, the n-th iterate of f and write the Fatou set as $F(f) = \{z \in \mathbb{C}$; there is an open set U containing z in which $\{f^n\}$ is a normal family$\}$. The complement of $F(f)$ is called the Julia set $J(f)$. We say that f belongs to class S_q if the set of singularities of f^{-1} contains at most q points.

Two entire maps g and h are topologically equivalent if there exist homeomorphisms $\varphi, \psi \colon \mathbb{C} \to \mathbb{C}$ such that $\varphi \circ g = h \circ \psi$. Given a map f, let us denote by M_f the set of all entire maps topologically equivalent to f.

It is proved in [5] that M_f has the structure of a $(q+2)$-dimensional complex manifold. The Affine group acts on the space M_f and as it is shown in [5] the space $N_f = M_f/\{\textit{Affine group}\}$ is a q-dimensional complex orbifold.

A measurable field of tangent ellipses of bounded eccentricity determines a complex structure on the sphere. This ellipse field is recorded by a $(-1, 1)$-form $\mu(z)\frac{d\bar{z}}{dz}$ with $||\mu||_\infty < 1$, a *Beltrami differential*. If an entire map f is holomorphic in a complex structure defined by the Beltrami differential μ, then μ is the *invariant Beltrami differential*. Since the sphere admits a unique complex structure, there is a homeomorphism $\phi \colon \mathbb{C} \to \mathbb{C}$ such that μ is the pullback of the standard structure and the map $f_\phi = \phi \circ f \circ \phi^{-1}$ is an entire map.

The nonexistence of an invariant Beltrami differential (invariant line field) on the Julia set is related to the Fatou conjecture, see [9].

Now, let us consider the main hero of this paper – *The Exponential Family*: $E = \{f_\lambda(z) = \exp(\lambda z), \lambda \in \mathbb{C}^*\}$. Then $N_{f_1} \cong E$, where $f_1 = \exp(z)$. The map f_{λ_0} is *structurally stable* if for any λ close enough to λ_0 there exists a quasiconformal homeomorphism ϕ_λ, such that $f_\lambda = \phi_\lambda \circ f_{\lambda_0} \circ \phi_\lambda^{-1}$.

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Due to Mañé, P. Sad, D. Sullivan (see [10]) and A. Eremenko, M. Lyubich (see [5]) the following three items are equivalent for E:

Fatou conjecture
There are no invariant Beltrami differentials supported by the Julia set
If $J(f_\lambda) = \mathbb{C}$, then f_λ is structurally unstable.

In 1985 R. Devaney (see [2]) proved that $\exp(z)$ is structurally unstable. Afterwards, A. Douady and L. R. Goldberg (see [4]) showed that the maps $\lambda \exp(z)$, $\lambda \geq 1$ are topologically unstable. Zhuan Ye (see [13]) proved that f_λ is structurally unstable if $\lim_{n\to\infty} f_\lambda^n(0) = \infty$.

In this paper we follow the approach of papers [1], [6] and [7]–[8] (case of rational maps) and [3] (case of transcendental entire maps with algebraic singularities only) were we generalize the above mentioned results.

In holomorphic dynamics the stability of a map depends on the behavior of the *postsingular set*, denoted in this paper by $X_\lambda = \overline{\{\cup_{n\geq 1} f_\lambda^n(0)\}}$. In the case of the Exponential family we have only one asymptotic singularity, being a different situation than in [3]; however our tools can also be applied in this case to obtain concrete results.

Let us start with $f_\lambda \in E$ whose Julia set is equal to the plane. Then we have the following simple possibilities:

1). $\lim_{n\to\infty} |(f_\lambda^n)'(0)| = 0$,

2). there exists a subsequence $\{n_i\}$ such that $\lim_{i\to\infty} |(f_\lambda^{n_i})'(0)| = \infty$,

3). there exists a subsequence $\{n_i\}$ such that $\lim_{i\to\infty} |(f_\lambda^{n_i})'(0)| = M < \infty$ and $M \neq 0$.

We believe that the first case contains a contradiction, since in this situation the forward orbit of 0 should converge to an attractive cycle and hence $0 \notin J(f_\lambda)$. We show this conjecture under very strong additional conditions as an illustration that this conjecture is not completely false (see theorem 1).

As for the last two cases, the Fatou conjecture claims that f_λ is an unstable map.

Definition (1.1). Let $\lambda \in \mathbb{C}^*$, then the *Poincaré series* for f_λ is the following formal series

$$P_\lambda = 1 + \frac{1}{\lambda} \sum_{i=2}^{\infty} \frac{1}{(f_\lambda^{i-2})'(1)}.$$

Let

$$S_n = 1 + \frac{1}{\lambda} \sum_{i=2}^{n} \frac{1}{(f_\lambda^{i-2})'(1)}$$

be the partial sum of the Poincaré series P_λ. Thus we have the following theorem and proposition.

THEOREM (1.2). 1). *If there exists a sequence* $\{n_i\}$ *such that* $(f_\lambda^{n_i})'(1) \to \infty$ *and* $\lim_{i\to\infty} \sup |S_{n_i}| > 0$, *then* f_λ *is unstable.*

2). *If there exists a sequence* $\{n_i\}$ *such that* $\lim_{i\to\infty} (f_\lambda^{n_i})'(1) = c$, *where* $c \neq 0$ *is a constant and* $\lim_{i\to\infty} \sup |S_{n_i}| = \infty$, *then* f_λ *is unstable.*

3). *Let* $\lim_{n\to\infty} (f_\lambda^n)'(1) = 0$, *and suppose that one of the following conditions holds:*

$\lim_{n\to\infty} \sup \frac{|(f_\lambda^{n+1})'(1)|}{|(f_\lambda^{n})'(1)|} < \infty$, *or*
$\lim_{n\to\infty} \inf \frac{|(f_\lambda^{n+1})'(1)|}{|(f_\lambda^{n})'(1)|} > 0$.
Then $F(f_\lambda) \neq \emptyset$.

PROPOSITION (1.3). *There is no map* $f_\lambda \in E$, *with* $J(f_\lambda) = \mathbb{C}$ *satisfying that* $\lim_{n\to\infty} |f_\lambda^{n'}(1)| = C > 0$.

The next theorems discuss the best conditions on the Poincaré series and on the postsingular set for the map to be unstable.

Definition (1.4). A point $a \in \mathbb{C}$ is called "*summable*" if and only if the series

$$\sum_{i=0} \frac{1}{(f_\lambda^i)'(a)}$$

is absolutely convergent. Note that the point $z = 0$ is summable if and only if the Poincaré series P_λ is absolutely convergent.

Definition (1.5). Let $W \subset E$ be the subset of exponential maps f_λ with summable singular point $0 \in J(f_\lambda)$, satisfying one of the following conditions:

1). $m(X_\lambda) = 0$, where m is the Lebesgue measure.

2). The diameters of the components of $\mathbb{C}\backslash X_\lambda$ are uniformly bounded below away from zero.

THEOREM (1.6). *Let* $f_\lambda \in W$. *Then* f_λ *is an unstable map, and hence there is no invariant Beltrami differential on its Julia set.*

THEOREM (1.7). *Let* $f_\lambda \in E$, *with* $J(f_\lambda) = \mathbb{C}$. *Then*

1). *If* $0 \notin X_\lambda$ *(i.e.* 0 *is non-recurrent), there exists a subsequence* n_k *such that* $(f^{n_k})'(1) \to \infty$;

2). *If* X_λ *is bounded, the singular point* $z = 0$ *is summable for* f_λ *and* $m(X_\lambda) = 0$.

In section 2 we discuss and prove Theorem (1.7) and Proposition (1.3).
Finally, we have the following

COROLLARY. *If* X_λ *is bounded and* $J(f) = \mathbb{C}$, *then* f *is unstable.*

In section 3 we consider the basic definitions and properties of the Ruelle operator $R*_\lambda$ of f_λ and we study the potential of deformations and as a consequence we prove Theorem (1.2).

The rest of the paper is devoted to prove Theorem (1.6) for which we have the following strategy:

1). Assuming that f_λ is a stable map and that 0 is a summable point, we prove in Lemma (5.1) that $R_\lambda^*(\varphi(z)) = \varphi(z)$. Here
$\varphi(z) := \sum_{n\geq 0} \frac{1}{(f_\lambda^n)'(f_\lambda(1))} \gamma_{f_\lambda^n(f_\lambda(1))}(z)$ and $\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)}$, as in section 4.

2). In Propositon (5.7), we prove that if $\varphi \neq 0$ on $Y = \mathbb{C} - X_\lambda$ then f_λ is unstable

3). We prove in Proposition (5.9) that $\varphi \neq 0$ identically on Y if $f_\lambda \in W$.

The Corollary is proved at the end of this paper.

We would like to remark that Lemma (5.3), Corollary (4.3), Propositions (5.7) and (5.9) remain basically as in the paper [3]. We have included these results for reader's convenience.

2. Postsingular set and dynamics

Mañé has a result establishing expansion properties of rational maps on the compact subsets of their Julia sets, which are far away from the parabolic points and the ω-limit sets of recurrent critical points. Next we will consider this result for our map f_λ.

Remark (2.1). Note that if $f_\lambda^n(0) \to \infty$ then f_λ is summable. To see this, consider

$$|\frac{1}{(f_\lambda^{n+1})'(a)}|/|\frac{1}{(f_\lambda^n)'(a)}| = |\frac{1}{\lambda f_\lambda^n(a)}|$$

and choose $a = f_\lambda(1)$. Since the orbit of 0 tends to ∞ this fraction converges to zero, so the series $\sum \frac{1}{(f_\lambda^n)'(1)}$ converges absolutely.

(2.2) Proof of Theorem (1.7). The proof of the theorem follows exactly the proof in [12] by Shishikura and Tan Lei. For completeness we will state the lemmas used in the above mentioned paper, restricted to the situation of our case. Hence in order to prove our Theorem (1.7), we will follow their arguments.

Denote by $d(z, E)$ the Euclidean distance between a point $z \in \mathbb{C}$ and a closed subset $E \subset \mathbb{C}$. Let $d_Y(z, X)$ be the Poincaré distance on a hyperbolic surface Y between a point z and a closed subset $X \subset Y$ and $\mathrm{diam}_W(W')$ the diameter of W' with respect to the Poincaré metric of W.

LEMMA (2.2.1). ([12], lemma 2.1). *For any $0 \leq r \leq 1$, there exists a constant $C(1, r) \geq 0$ such that for any holomorphic proper map $g\colon V \to \mathbb{D}$ of degree 1, with V simply connected, each component of $g^{-1}(\overline{D_r(0)})$ has diameter $\leq C(1, r)$ with respect to the Poincaré metric on V. Moreover $\lim_{r\to 0} C(1, r) = 0$.*

Definition (2.2.2). $\mathbf{N_0}$: There exist $z_1, ..., z_{N_0-1} \in \mathbb{D}$ such that $\{\frac{2}{3} \leq |z| \leq 1\} \subset \bigcup_{i=1}^{N_0-1} D_{\frac{1}{3}}(z_i)$. Let $C_0 = N_0 C(1, \frac{2}{3})$.

The Julia set is $J(f_\lambda) = \mathbb{C}$. Hence we can choose a periodic point w so that the domain $\Omega = \mathbb{C} \setminus \{\text{forward orbit of the point } w\}$ satisfies $d_\Omega(0, X_\lambda) \geq 2C_0$.

LEMMA (2.2.3). ([12], lemma 2.3). *Let $U_0 = D_r(x)$ be a disc centered at $x \in X_\lambda$ with radius r so that $U_0 \subset \Omega$ and $\mathrm{diam}_\Omega(U_0) \leq C_0$. Then for every $n \geq 0$ the following is true:*

deg(n). *For every $D_s(z) \subset U_0$ with $0 \leq s \leq d(z, \partial U_0)/2$, and every connected component V' of $f_\lambda^{-n}(D_s(z))$, V' is simply connected and* $\deg(f_\lambda^n\colon V' \to D_s(z)){=}1$;

diam(n). *For every $D_r(w) \subset U_0$ with $0 \leq r \leq d(w, \partial U_0)/2$ and every connected component of V of $f_\lambda^{-n}(D_r(w))$,* $\mathrm{diam}_\Omega V \leq C_0$.

Now, we begin to prove Theorem (1.7). If ∞ is the unique point of accumulation of $\{\bigcup_n f_\lambda(0)\}$, then by Remark (2.1) above, the point $z = 0$ is a summable point and hence $\lim_{n\to\infty} |(f_\lambda^n)'(0)| = \infty$.

Now let $y \in X_\lambda$ be another point of accumulation of the orbit of $z = 0$. Let n_i be any subsequence such that $y = \lim_{i\to\infty} f_\lambda^{n_i}(1)$. Then we claim:

Claim $\lim_{i\to\infty}|(f_\lambda^{n_i})'(1)| = \infty$.

To prove the claim we repeat the arguments of Shishikura and Tan Lei. Let us assume that there exists a number $M < \infty$ and a sequence of natural numbers $\{n_j\} \subset \{n_i\}$ such that $|(f_\lambda^{n_j})'(1)| \leq M$. Then by Lemma (2.2.3) there exists an integer N and a number r such that components $W_j \subset f_\lambda^{-n_j}(D_r(y))$ containing the point $z = 1$ are simply connected and the respective restriction maps $f_\lambda^{n_j}\colon W_J \to D_r(y)$ are univalent for all $j \geq N$. Now let $B \subset \Omega$ be the hyperbolic ball of the radius C_0 centered at the point $z = 1$, then B is a precompact subset of Ω and hence has a bounded Euclidian diameter in $\mathbb{C}$. Besides, again by Lemma (2.2.3), the set $\{\cup_j W_j\} \subset B$.

Let $g_j\colon D \to W_j$ be the inverse maps then they form a normal family. Hence, after passing to a subsequence we can assume that g_j converge. Let g_∞ be a limit map, then $g_\infty \neq$ *constant* since the derivatives are $\geq \frac{1}{M}$ by hypothesis. Then there is a neighborhood U_0 of $z = 1$ such that $U_0 \subset g_j(D)$ for large j. Thus, $f_\lambda^{n_j}$ is normal in U_0, but there are many periodic expansive points in $U_0 \subset J(f_\lambda)$ and the derivative diverges, which is a contradiction. The claim and the first part of the theorem are done.

Finally, for the proof of the second part, we once more repeat the arguments of Shishikura and Tan Lei in [12]. First, assume that f_λ is not expansive on X_λ i.e. there are $n_k \to \infty$, $x_k \in X_\lambda$, such that $|(f^{n_k})'(x_k)| \leq 1$. Now using the compactness of X_λ and the arguments above, we obtain a contradiction. Therefore, expansivity immediately implies summability of the point $z = 1$.

Second, if $m(X_\lambda) \neq 0$ we follow the concepts of [9], page 44. Let x be a density point in X_λ and consider for each point in the orbit of x the disc $D_\delta(f^n(x)) = D_n$. It is clear that for all $z \in \mathbb{C}$ and $\delta \leq \frac{1}{2}d(0, X_\lambda)$, there exists a univalent branch $h\colon D_\delta(f(z)) \to \mathbb{C}$, such that $h(f(z)) = z$. Consecuently, since X_λ is bounded $0 \notin X_\lambda$. It follows that there exist univalent branches $g_n\colon D_\delta(f^n(x)) \to C_n$. Observe that by Koebe's principle, g_n has bounded nonlinearity. By part one above, $|(f^{n_k})'(x)| \to \infty$ for some subsequence n_k, and then $\operatorname{diam}(C_{n_k}) \to 0$.

Using that x is a density point, we have that

$$\lim_{n\to\infty} \frac{m(C_{n_k} \bigcap X_\lambda)}{m(C_{n_k})} \to 1,$$

then by the invariance of X_λ, we have that

$$\lim_{n\to\infty} \frac{m(D_{n_k} \bigcap X_\lambda)}{m(D_{n_k})} \to 1.$$

Boundness of X_λ implies that there exists a subsequence such that $D_k \to B$ in which the density of X_λ is equal to one. Then $B \subset X_\lambda$ a.e. and that implies that $f^n(B) = \mathbb{C}$, hence $X_\lambda = \mathbb{C}$ which is a contradiction with the hypothesis that X_λ is bounded. □

(2.3) Proof of Proposition (1.3).

Proof. We have $\lim_{n\to\infty}|\frac{1}{f_\lambda^{n+1'}(1)}/\frac{1}{f_\lambda^{n'}(1)}| = 1$. Since

$$\lim_{n\to\infty} \left|\frac{f_\lambda^{n'}(1)}{f_\lambda^{n+1'}(1)}\right| = \lim_{n\to\infty} \left|\frac{1}{\lambda f_\lambda^{n+1}(1)}\right|,$$

then $|f_\lambda^n(1)|$ is near $\frac{1}{|\lambda|}$ for all large values of n.

This implies that X_λ is bounded, hence compact and 0 is non-recurrent, by Theorem (1.7), and f_λ is summable. This is a contradiction of the hypothesis. □

3. Ruelle Operator: Definitions and Properties

For any $\lambda \in \mathbb{C}^*$ we define the following operators (compare with [7], [8], [6]).

Definition (3.1).

Ruelle operator (or push-forward operator)

$$R_\lambda^*(\varphi)(z) := \sum_{\xi_i} \varphi(\xi_i)\xi_i'^2 = \frac{1}{\lambda^2 z^2}\sum_{\xi_i}\varphi(\xi_i),$$

where the summation is taken over all branches ξ_i of f_λ^{-1}.

Modulus of the Ruelle operator $|R_\lambda^*|(\varphi)(z) = \frac{1}{|\lambda^2 z^2|}\sum_{\xi_i}\varphi(\xi_i)$.

Beltrami operator $B_\lambda(\varphi) = \varphi(f_\lambda)\frac{\overline{f_\lambda'}}{f_\lambda'}$.

Then we have the following simple lemma.

LEMMA (3.2). *For all* λ *;*

1). $R_\lambda^*: L_1(\mathbb{C}) \to L_1(\mathbb{C})$ *and* $\| R_\lambda^* \|_{L_1} \leq 1$,

2). $|R_\lambda^*|: L_1(\mathbb{C}) \to L_1(\mathbb{C})$, $\| \, |R_\lambda^*| \, \|_{L_1} \leq 1$, *and the fixed points of* $|R_\lambda^*|$ *define finite, complex-valued, invariant, and absolutely continuous measures on* $\mathbb{C}$.

3). $B_\lambda: L_\infty(\mathbb{C}) \to L_\infty(\mathbb{C})$ *is the dual operator to* R_λ^*, *and* $\| B_\lambda \|_{L_\infty} = 1$.

Proof. Immediately follows from the definitions. □

(3.3) Potential of Deformations. The open unit ball B of the space $Fix(B_\lambda) \subset L_\infty(\mathbb{C})$ of fixed points of B_λ is called the space of invariant Beltrami differentials for f_λ and describes all quasiconformal deformations of f_λ.

Let $\mu \in Fix(B_\lambda)$, then $t\mu \in B$ for all t such that $|t| < \frac{1}{\|\mu\|}$.

Let us denote by h_t their corresponding quasiconformal maps, normalized so that $h_t(0) = 0$ and $h_t(1) = 1$; then we have the following functional equation as explained in [1], [7], [8]:

$$F_\mu(f_\lambda(z)) - f_\lambda'(z)F_\mu(z) = G_\mu(z)$$

where $h_t \circ f_\lambda \circ h_t^{-1} = f_{\lambda(t)} \in M_{f_1}$ and $G_\mu(z) = \frac{\partial f_{\lambda(t)}}{\partial t}(z)|_{t=0} = z\exp(\lambda z)\lambda'(t)|_{t=0}$. The function

$$F_\mu(a) = \frac{\partial h_t}{\partial t}|_{t=0} = -\frac{a(a-1)}{\pi}\iint_{\mathbb{C}} \frac{\mu(z)}{z(z-1)(z-a)}$$

is called the *potential of the qc-deformations* generated by μ and $\bar{\partial}F_\mu = \mu$ in the sense of distributions, see [11]. $F_\mu(a)$ is a continuous function in $\mathbb{C}$ with $F_\mu(0) = 0$ and $F_\mu(1) = 0$.

LEMMA (3.3.1). *If* $F(f_\lambda) = \emptyset$, *then* $G_\mu = 0$ *if and only if* $\mu = 0$.

Proof. If $G_\mu = 0$, then $F_\mu(f_\lambda(z)) = f_\lambda'(z)F_\mu(z)$. Hence $F_\mu = 0$ on the set of repelling periodic points and hence $F_\mu = 0$ on the Julia set. Then $\mu = \bar\partial F_\mu = 0$. The lemma is finished. □

Then by an inductive argument we have that

$$F_\mu(f_\lambda^n(a)) = f_\lambda^{n'}(a)\left(F_\mu(a) + \sum_{i=1}^{n} \frac{G_\mu(f_\lambda^{i-1}(a))}{f_\lambda^{i'}(a)}\right).$$

from above $G_\mu(a) = \frac{af_\lambda'(a)c}{\lambda}$, where the constant $c = \lambda'(t)|_{t=0}$ and by Lemma (3.3.1) above $c \neq 0$.

$$F_\mu(f_\lambda^n(a)) = f_\lambda^{n'}(a)\left(F_\mu(a) + \frac{ac}{\lambda} + \frac{c}{\lambda^2}\sum_{i=2}^{n} \frac{1}{(f_\lambda^{i-2})'(a)}\right) \tag{3.4}$$

Now we are ready to prove Theorem (1.2).

(3.5) Proof of Theorem (1.2). We show first (3). By the assumption, we have either

$$\lim_{n\to\infty} \sup \frac{|(f_\lambda^{n+1})'(1)|}{|(f_\lambda^n)'(1)|} \le C < \infty$$

or

$$\lim_{n\to\infty} \inf \frac{|(f_\lambda^{n+1})'(1)|}{|(f_\lambda^n)'(1)|} \ge K > 0.$$

Since $\frac{|(f_\lambda^{n+1})'(0)|}{|(f_\lambda^n)'(0)|} = |\lambda f_\lambda^{n+1}(0)|$, either X_λ is a compact subset of the plane or $0 \notin X_\lambda$, respectively. Let us assume that $F(f_\lambda) = \emptyset$, then an application of Theorem (1.7) implies a contradiction with $\lim_{n\to\infty} |(f_\lambda^n)'(0)| = 0$. Hence we are done.

Now we show (1) and (2). Assume f_λ is stable.

From the equation (3.4) above, we have that

$$\frac{F_\mu(f_\lambda^n(a))}{(f_\lambda^n)'(a)} = F_\mu(a) + \frac{ac}{\lambda} + \frac{c}{\lambda^2}\sum_{i=2}^{n} \frac{1}{(f_\lambda^{i-2})'(a)} \tag{3.6}$$

From [11], Lemma 1 of chapter 4; we have the following inequality

$$|F_\mu(a)| \le M|a||\log|a||,$$

where M is a constant depending only on μ. Applying this estimate we obtain:

$$\frac{|F_\mu(f_\lambda^n(a))|}{|(f_\lambda^n)'(a)|} \le \frac{M|f_\lambda^n(a)||\log|f_\lambda^n(a)||}{|(f_\lambda^n)'(a)|}.$$

An easy calculation shows

$$\log|f_\lambda^n(a)| = |\lambda f_\lambda^{n-1}(a)|$$

and

$$(f_\lambda^n)'(a) = \lambda^2 f_\lambda^n(a) f_\lambda^{n-1}(a)(f_\lambda^{n-2})'(a).$$

So, $\frac{|F_\mu(f_\lambda^n(a))|}{|(f_\lambda^n)'(a)|} \le \frac{M}{\lambda(f_\lambda^{n-2})'(a)}$.

Now let n_j be the sequence from the assumptions of Theorem (1.2), items (1)-(2). Let the point $a = 1$. Since $F_\mu(1) = 0$, then from equation (3.6) we obtain the following equality:

$$\frac{F_\mu(f_\lambda^{n_j+2}(1))}{(f_\lambda^{n_j+2})'(1)} = \frac{c}{\lambda} + \frac{c}{\lambda^2}\sum_{i=2}^{n_j+2}\frac{1}{(f_\lambda^{i-2})'(1)} = \frac{c}{\lambda}\cdot S_{n_j}.$$

Therefore, this equation produces a contradiction in both cases with the hypothesis over S_{n_j}, consequently f_λ is unstable.

4. Calculation of the Ruelle Operator

In this section we calculate the action of the Ruelle operator on the family of rational functions $\gamma_a(z) = \frac{a(a-1)}{z(z-1)(z-a)}$ such that $a \neq 0, 1$. Let us recall that any rational integrable differential is a linear combination of such $\gamma_a(z)$.

Let $S = \mathbb{C}\backslash\{0, 1\}$ be the thrice punctured sphere.

PROPOSITION (4.1).

$$R_\lambda^*(\gamma_a(z)) = \frac{1}{(f_\lambda)'(a)}\gamma_{f_\lambda(a)}(z) - \frac{a}{(f_\lambda)'(1)}\gamma_{f_\lambda(1)}(z).$$

Proof. Let $h_a(z) = R_\lambda^*(\gamma_a)(z) - \frac{1}{(f_\lambda)'(a)}\gamma_{f_\lambda(a)}(z) + \frac{a}{(f_\lambda)'(1)}\gamma_{f_\lambda(1)}(z)$ be a function. Our aim is to show that $h_a(z)$ defines a holomorphic integrable function on the surface S, hence $h_a(z) = 0$ and we are done. By Lemma (3.2) the function $h_a(z)$ is integrable over the plane.

Now let us show that $h_a(z)$ is holomorphic by calculating the complex conjugate derivative in the sense of distributions. Let $\varphi \in C^\infty(S)$ be any differentiable function with compact support in S, and such that $\varphi = 0$ in $\mathbb{C} - S$.

By duality between the Ruelle operator and the Beltrami operator, we have

$$\begin{aligned}\iint_{\mathbb{C}} \varphi_{\bar z} h_a(z) &= \iint_{\mathbb{C}} B_\lambda(\varphi_{\bar z})\gamma_a(z) - \frac{1}{(f_\lambda)'(a)}\iint_{\mathbb{C}}\varphi_{\bar z}\gamma_{f_\lambda(a)}(z) + \frac{a}{(f_\lambda)'(1)}\iint_{\mathbb{C}}\varphi_{\bar z}\gamma_{f_\lambda(1)}(z)\\ &= \iint_{\mathbb{C}}\varphi_{\bar z}(f_\lambda)\frac{\overline{(f_\lambda)'(z)}}{(f_\lambda)'(z)}\gamma_a(z) - \frac{1}{(f_\lambda)'(a)}\iint_{\mathbb{C}}\varphi_{\bar z}\gamma_{f_\lambda(a)}(z)\\ &\qquad + \frac{a}{(f_\lambda)'(1)}\iint_{\mathbb{C}}\varphi_{\bar z}\gamma_{f_\lambda(1)}(z) = (*).\end{aligned}$$

On the other hand,

$$\begin{aligned}\iint_{\mathbb{C}}\varphi_{\bar z}(f_\lambda)\frac{\overline{(f_\lambda)'(z)}}{(f_\lambda)'(z)}\gamma_a(z) &= a(a-1)\iint_{\mathbb{C}}\frac{(\varphi\circ f_\lambda)_{\bar z}}{z(z-1)(z-a)(f_\lambda)'(z)}\\ &= (a-1)\iint_{\mathbb{C}}\frac{(\varphi\circ f_\lambda)_{\bar z}}{z(f_\lambda)'(z)} - a\iint_{\mathbb{C}}\frac{(\varphi\circ f_\lambda)_{\bar z}}{(z-1)(f_\lambda)'(z)}\\ &\qquad + \iint_{\mathbb{C}}\frac{(\varphi\circ f_\lambda)_{\bar z}}{(z-a)(f_\lambda)'(z)}.\end{aligned}$$

Since we have $\varphi(0) = 0$ and $\varphi(1) = 0$,

$$(a-1)\iint_{\mathbb{C}}\frac{(\varphi\circ f_\lambda)_{\bar z}}{z(f_\lambda)'(z)} = \frac{a-1}{f_\lambda'(0)}\varphi(f_\lambda(0)) = 0$$

$$\iint_{\mathbb{C}} \frac{(\varphi \circ f_\lambda)_{\overline{z}}}{(z-a)(f_\lambda)'(z)} = \frac{1}{(f_\lambda)'(a)}\varphi(f_\lambda(a))$$

Further,

$$\frac{1}{(f_\lambda)'(a)} \iint_{\mathbb{C}} \varphi_{\overline{z}}\gamma_{f_\lambda(a)}(z) = \frac{1}{(f_\lambda)'(a)}\varphi(f_\lambda(a))$$

hence, by cancelation, we obtain

$$(*) = \iint_{\mathbb{C}} \varphi_{\overline{z}} h_a(z) = 0.$$

By Weyl's Lemma $h_a(z)$ is a holomorphic function on S. Hence we are done. □

COROLLARY (4.2). *If $F(f_\lambda) = \emptyset$ and $\mu \neq 0 \in B$, then*

$$G_\mu(a) = \frac{af'_\lambda(a)}{f'_\lambda(1)} F_\mu(f_\lambda(1)).$$

Proof. Let $\mu \neq 0 \in B$ be invariant Beltrami differential for f_λ, then by Proposition (4.1), we have

$$\begin{aligned} -\pi F_\mu(a) &= \iint \gamma_a(z)\mu = \iint R^*_\lambda(\gamma_a(z))\mu \\ &= \frac{1}{f'_\lambda(a)}(-\pi)F_\mu(f_\lambda(a)) - \frac{a}{f'_\lambda(1)}(-\pi)F_\mu(f_\lambda(1)). \end{aligned}$$

Hence

$$F_\mu(a) = \frac{1}{f'_\lambda(a)} F_\mu(f_\lambda(a)) - \frac{a}{f'_\lambda(1)} F_\mu(f_\lambda(1)),$$

and

$$G_\mu(a) = F_\mu(f_\lambda(a)) - f'_\lambda(a)F_\mu(a) = \frac{af'_\lambda(a)}{f'_\lambda(1)} F_\mu(f_\lambda(1)).$$

□

Define the following series:

$$B(a) = \frac{1}{f'_\lambda(1)} \sum_{j=1}^{\infty} \frac{f_\lambda^{j-1}(a)}{(f_\lambda^{j-1})'(a)}.$$

We also have by direct calculation that

$$B(a) = \frac{1}{f'_\lambda(1)}(1 + \frac{1}{\lambda}\sum_{j=2}^{\infty} \frac{1}{(f_\lambda^{j-2})'(a)})$$

COROLLARY (4.3). *Let $\mu \neq 0 \in B$. Then for all $n > 0$ we have*

$$(*) \qquad (R^*_\lambda)^n(\gamma_a(z)) = \frac{1}{(f^n_\lambda)'(a)}\gamma_{f^n_\lambda(a)}(z) - \frac{f_\lambda^{n-1}(a)}{(f_\lambda^{n-1})'(a)f'_\lambda(1)}\gamma_{f_\lambda(1)}(z) -$$

$$- \frac{f_\lambda^{n-2}(a)}{(f_\lambda^{n-2})'(a)f'_\lambda(1)} R^*_\lambda(\gamma_{f_\lambda(1)}(z)) - \ldots - \frac{a}{f'_\lambda(1)}(R^*_\lambda)^{n-1}(\gamma_{f_\lambda(1)}(z)).$$

and

$$(**) \qquad F_\mu(a) = \frac{1}{f_\lambda^{n'}(a)} F_\mu(f^n_\lambda(a)) - B_n(a)F_\mu(f_\lambda(1)),$$

where $B_n(a)$ is the $n-th$ partial sum of the series $B(a)$ above.

Proof. By induction on the formula of Proposition (4.1) we obtain the equation for $(R_\lambda^*)^n(\gamma_a(z))$. An application of formula (3.6) in the proof of Theorem (1.2) together with the formula in Corollary (4.2) and second formula for $B(a)$, give us the required formula for $F_\mu(a)$. □

These two formulas are equivalent but we will use only the one for $F_\mu(a)$.

5. Proof of Theorem (1.6)

Assume that f_λ is a stable map. Then the summability of the singular value implies $F(f_\lambda) = \emptyset$, otherwise the critical point must tend to a periodic attracting point and so the Poincaré series is not convergent.

Let a be a summable point. Then the series $B(a)$ is absolutely convergent and by the arguments of Theorem (1.2), item (1), $\frac{1}{f_\lambda^{n\prime}(a)} F_\mu(f_\lambda^n(a)) \to 0$ as $n \to \infty$.

Then passing to the limit in the formula $(**)$ in Corollary (4.3), we have

$$F_\mu(a) = -B(a) F_\mu(f_\lambda(1)) .$$

Now set $a = f_\lambda(1)$, then

$$F_\mu(f_\lambda(1)) \left(1 + B(f_\lambda(1))\right) = 0.$$

and we have two possibilities:

1) $F_\mu(f_\lambda(1)) = 0$. Then by Corollary (4.2), $G_\mu = 0$ and by Lemma (3.3.1), $\mu = 0$ which contradicts the assumption above.

2) $B(f_\lambda(1)) = -1$.

Now we will finish Theorem (1.6) in 3 steps. Let φ be the following series,

$$\varphi(z) := \sum_{n \geq 0} \frac{1}{(f_\lambda^n)'(f_\lambda(1))} \gamma_{f_\lambda^n(f_\lambda(1))}(z) .$$

The summability of $z = 0$ implies

1) The function φ is an absolutely integrable function in $\mathbb{C}$.

2) Outside of X_λ, φ is holomorphic.

Indeed, again by Lemma 1, chapter 4 in [11] we have that for any $x \in \mathbb{C}$, $|\iint_{\mathbb{C}} \gamma_x(z)| \leq M|x|\,|\log(x)|$ being M a constant not depending on x. Then

$$||\varphi|| \leq \sum_{n \geq 0} \frac{1}{|(f_\lambda^n)'(f_\lambda(1))|} \left| \iint_{\mathbb{C}} \gamma_{f_\lambda^n(f_\lambda(1))}(z) \right| \leq M \sum_{n \geq 0} \frac{|(f_\lambda^n(1))|\,|\log((f_\lambda^n(1)))|}{|(f_\lambda^n)'(f_\lambda(1))|}$$

and from the calculations in Theorem (1.2), we have

$$M \sum_{n \geq 2} \left| \frac{(f_\lambda^n(1))}{(f_\lambda^n)'(f_\lambda(1))} \log((f_\lambda^n(1))) \right| = M \sum_{n \geq 2} \left| \frac{(f_\lambda^n(1)) \lambda (f_\lambda^{n-1}(1))}{\lambda^2 (f_\lambda^n(1))(f_\lambda^{n-1}(1))(f_\lambda^{n-2})'(f_\lambda(1))} \right|$$

$$||\varphi|| = \frac{M}{\lambda} \sum_{n \geq 2} \frac{1}{|(f_\lambda^{n-2})'(f_\lambda(1))|} + \frac{|\iint_{\mathbb{C}} \gamma_{(f_\lambda(1))}(z)|}{|f_\lambda(z)|} + \frac{|\iint_{\mathbb{C}} \gamma_{f_\lambda(f_\lambda(1))}(z)|}{|(f_\lambda)'(f_\lambda(1))|}$$

Summability of $f_\lambda(1)$ proves the first part of the assertion. For the second part, just observe that φ is uniformly approximated by the rational functions $\sum_{n=0}^{k} \frac{1}{(f_\lambda^n)'(f_\lambda(1))} \gamma_{f_\lambda^n(f_\lambda(1))}(z)$.

LEMMA (5.1). *Under assumption (2) above we have*

$$R_\lambda^*(\varphi(z)) = \varphi(z).$$

Proof. For any $n \geq 0$, by the formula of the Proposition (4.1), we have the following expression,

$$R_\lambda^*\left(\frac{1}{(f_\lambda^n)'(f_\lambda(1))}\gamma_{f_\lambda^n(f_\lambda(1))}(z)\right) =$$

$$\frac{1}{(f_\lambda^{n+1})'(f_\lambda(1))}\gamma_{f_\lambda^{n+1}(f_\lambda(1))}(z) - \frac{1}{f_\lambda'(1)}\gamma_{f_\lambda(1)}(z)\frac{f_\lambda^n(f_\lambda(1))}{(f_\lambda^n)'(f_\lambda(1))},$$

Then summation over all $n \geq 0$ gives

$$R_\lambda^*(\varphi) = R_\lambda^*\left(\sum_{n\geq 0}\frac{\gamma_{f_\lambda^n(f_\lambda(1))}(z)}{(f_\lambda^n)'(f_\lambda(1))}\right) =$$

$$= \sum_{n\geq 0}\frac{1}{(f_\lambda^{n+1})'(f_\lambda(1))}\gamma_{f_\lambda^{n+1}(f_\lambda(1))}(z) - \frac{1}{f_\lambda'(1)}\gamma_{f_\lambda(1)}(z)\sum_{n\geq 0}\frac{f_\lambda^n(f_\lambda(1))}{(f_\lambda^n)'(f_\lambda(1))} =$$

$$= \varphi(z) - \gamma_{f_\lambda(1)}(z) - \gamma_{f_\lambda(1)}(z)\left[B(f_\lambda(1))\right] = \varphi(z)$$

by hypothesis. □

LEMMA (5.2). *Under the assumption of Lemma (5.1) above the function $|\varphi|$ is a fixed point for the modulus of the Ruelle operator,*

$$|R_\lambda^*|(|\varphi|) = |\varphi|.$$

Proof. We recall that by definition, for every function φ

$$|R_\lambda^*|(|\varphi|) = \sum_{\zeta_i} |\varphi(\zeta_i)|\,|\zeta_i'|^2$$

where the summation is over all branches ζ_i of inverses of $f_\lambda(z) = e^{\lambda z}$.

By assumption

$$\| \varphi \| = \| R_\lambda^*(\varphi) \| = \iint_{\mathbb{C}} \left|\sum_{\zeta_i} \varphi(\zeta_i)(\zeta_i')^2\right|.$$

Now, define for each index i, $\alpha_i = \varphi(\zeta_i)(\zeta_i')^2$, $\beta_i = \sum_{j\neq i}\varphi(\zeta_j)(\zeta_j')^2 = \varphi - \alpha_i$. With this notation we have

$$\| \varphi \| = \| R_\lambda^*(\varphi) \| = \iint_{\mathbb{C}} \left|\sum_{\zeta_i} \varphi(\zeta_i)(\zeta_i')^2\right| dz \wedge d\bar{z} = \iint_{\mathbb{C}} |\sum_j \alpha_j| \leq$$

$$\leq \iint_{\mathbb{C}} |\alpha_i| + \iint_{\mathbb{C}} |\beta_i| \leq \sum \iint_{\mathbb{C}} |\alpha_j| = ||\varphi||$$

Hence

$$|\sum_j \alpha_j| = |\alpha_i| + |\beta_i| = \sum_j |\alpha_j|$$

That implies that

$$|\varphi| = |\sum_i \alpha_i| = \sum_i |\alpha_i| = \sum_{\zeta_i} |\varphi(\zeta_i)||\zeta_i'|^2 = |R_\lambda^*|(|\varphi|)\,.$$

$\square$

By Lemma (3.2), the measure $\sigma(A) = \iint_A |\varphi(z)|$ is a non-negative invariant absolutely continuous probability measure, where $A \subset \widehat{\mathbb{C}}$ is a measurable set. We have therefore completed the first step.

Let $Y = \mathbb{C} - X_\lambda$ be the complement to the postsingular set X_λ. In the second step we show that $\varphi = 0$ identically on Y.

In the notation of the lemmas above we have

LEMMA (5.3). *If $\alpha_j \neq 0$ identically on Y, then the function $k_j = \frac{\beta_j}{\alpha_j}$ is a non-negative constant on any component of Y.*

Proof. We have $|1 + \frac{\beta_j}{\alpha_j}| = 1 + \left|\frac{\beta_j}{\alpha_j}\right|$, then if $\frac{\beta_j}{\alpha_j} = \gamma_1^j + i\gamma_2^j$ we have

$$\left(1+(\gamma_1^j)\right)^2 + (\gamma_2^j)^2 = \left(1+\sqrt{(\gamma_1^j)^2+(\gamma_2^j)^2}\right)^2 = 1+(\gamma_1^j)^2+(\gamma_2^j)^2+2\sqrt{(\gamma_1^j)^2+(\gamma_2^j)^2}.$$

Hence $\gamma_2^j = 0$ and $\frac{\beta_j}{\alpha_j} = \gamma_1^j$ is a real-valued function but $\frac{\beta_j}{\alpha_j}$ is a meromorphic function. So $\gamma_1^j = k_j$ is constant on every connected component of Y and the condition $|1 + k_j| = 1 + |k_j|$ shows $k_j \geq 0$.

Definition (5.4). A measurable set $A \in \widehat{\mathbb{C}}$ is called back wandering if and only if $m(f^{-n}(A) \cap f^{-k}(A)) = 0$, for $k \neq n$.

Remark (5.5). If a set A is back wandering and μ is an invariant probability measure, then $\mu(A) = 0$.

COROLLARY (5.6). *If $\varphi \neq 0$ on Y, then (i) $m(X_\lambda) = 0$, where m is the Lebesgue measure and (ii) $\frac{\bar{\varphi}}{|\varphi|}$ defines an invariant Beltrami differential.*

Proof. (i) If $m(X_\lambda) > 0$, then $m(f_\lambda^{-1}(X_\lambda)) > 0$ so $m(f_\lambda^{-1}(X_\lambda) - X_\lambda) > 0$ since $f_\lambda^{-1}(X_\lambda) \neq X_\lambda$, $X_\lambda \neq \mathbb{C}$. Denote by $Z_1 = f_\lambda^{-1}(X_\lambda) - X_\lambda$. Then Z_1 is back wandering so $\varphi = 0$, on the orbit of Z_1, which is dense in $J(f_\lambda)$, hence $\varphi = 0$ in Y. Therefore $m(X_\lambda) = 0$.

(ii) By the notation and the proofs of Lemmas (5.1) and (5.2) we have $k_i(x) = \frac{\beta_i}{\alpha_i} = \frac{\varphi}{\alpha_i} - 1$so $\varphi(x) = (1 + k_i(x))\alpha_i = (1 + k_i(x))(\varphi(\zeta_i(x))(\zeta_i')^2(x)$. Hence,

$$\frac{\bar{\varphi}(x)}{|\varphi(x)|} = \frac{(1 + k_i(x))\bar{\varphi}(\zeta_i(x))(\bar{\zeta}_i')^2(x)}{(1 + k_i(x))|\varphi(\zeta_i(x)||(\zeta_i')^2(x)|},$$

and so for any branch ζ_i we have

$$\mu = \frac{\bar{\varphi}}{|\varphi|} = \frac{\bar{\varphi}(\zeta_i)\bar{\zeta}_i'}{|\varphi(\zeta_i)|\zeta_i'} = \mu(\zeta_i)\frac{\bar{\zeta}_i'}{\zeta_i}$$

as a result $\mu = \frac{\bar{\varphi}}{|\varphi|}$ is an invariant line field. Thus the corollary is proved. $\square$

Now we prove the main result of the second step.

PROPOSITION (5.7). *If $\varphi \neq 0$ on Y, then f_λ is unstable.*

Proof. Let us show first that $X_\lambda = \bigcup f_\lambda^i(1)$. We will use a McMullen argument as in [9]. By Corollary (4.3), $\mu = \frac{\bar{\varphi}}{|\varphi|}$ is an invariant Beltrami differential. That implies that φ is dual to μ and φ is defined by μ up to a constant. We will construct a meromorphic function ψ dual to μ and such that ψ has a finite number of poles on each ring $A_R = \{z :: \frac{1}{R} \leq |z| \leq R\}$ around the origin.

For that suppose that for $z \in \mathbb{C}$ there exists a branch g of a suitable f_λ^n, such that $g(U_z) \subset Y$ and U_z is a neighborhood of z. Then define $\psi(\zeta) = \varphi(g(\zeta))(g')^2(\zeta)$ for all $\zeta \in U_z$. Note that $\psi(\zeta)$ is dual to μ and has no poles in U_z.

By considering $R \to \infty$ we construct a meromorphic function ψ which is dual to μ. The poles of ψ form a discrete set accumulating $z = \infty$ and 0. Since φ is a dual to μ, then $\varphi = C \cdot \psi$, where C is a constant. Hence $X_\lambda = \bigcup f_\lambda^i(1)$ is a discrete closed set accumulating $z = \infty$ and 0, so Y is connected.

By Corollary (4.3) the functions k_i are globally defined constants on Y. Moreover, by the argument of Lemma (5.3) $\varphi(x) = (1 + k_i)(\varphi(\zeta_i(x))(\zeta_i')^2(x)$ for any $x \in \mathbb{C}$, thus $k_i = k_j$ for any i, j.

Therefore we have $\sum_i \frac{\varphi(x)}{1+k_i} = \sum_i \varphi(\zeta_i(x))(\zeta_i')^2(x) = \varphi(x)$. Since the first term of the equation is infinite, this can be only if $\varphi = 0$. □

To obtain a contradiction in step 3, we show that if $f_\lambda \in W$ is structurally stable, then $\varphi \neq 0$ identically on Y.

The following proposition is proved in [8] and [3].

PROPOSITION (5.8). *Let $a_i \in \mathbb{C}$ with $a_i \neq a_j$, for $i \neq j$ be points such that $Z = \overline{\bigcup_i a_i} \subset \mathbb{C}$ is a compact set. Let $b_i \neq 0$ be complex numbers such that the series $\sum b_i$ is absolutely convergent. Then the function $l(z) = \sum_i \frac{b_i}{z - a_i} \neq 0$ identically on $Y = \mathbb{C} \backslash Z$ in any of the following cases:*

(1) *the set Z has zero Lebesgue measure;*

(2) *the diameters of the components of $\mathbb{C} \backslash Z$ are uniformly bounded below away from zero.*

PROPOSITION (5.9). *Let f_λ be the exponential map and 0 a summable point. Then $\varphi(z) \neq 0$ identically on Y in any of the following cases:*

(1) *if $m(X_\lambda) = 0$, where m is the Lebesgue measure on $\mathbb{C}$;*

(2) *if the diameters of the components of Y are uniformly bounded below away from 0.*

Proof. Denote $d_\lambda = f_\lambda(1)$. We have two cases according to X_λ being bounded or not.

First assume that the set X_λ is bounded. Then by Proposition (5.8) we have that $\phi(z) = \frac{C_1}{z} + \frac{C_2}{z-1} + \sum \frac{1}{(f_\lambda^i)'(d_\lambda)(z - f_\lambda^i(d_\lambda))} = l(z) \neq 0$, which proves the proposition. □

Now let X_λ be unbounded. We want to reduce this situation to a bounded one, observe that under our assumptions $X_\lambda \neq \mathbb{C}$.

Let $y \in \mathbb{C}$ be a point such that the point $1-y \in Y$. Then the map $g(z) = \frac{yz}{z+y-1}$ maps X_λ onto a compact subset of $\mathbb{C}$. Let us consider the function $G(z) = \frac{1}{z}\sum_i \frac{(f_\lambda^i(d_\lambda)-1)}{(f_\lambda^i)'(d_\lambda)} - \frac{1}{z-1}\sum_i \frac{f_\lambda^i(d_\lambda)}{(f_\lambda^i)'(d_\lambda)} + \sum \frac{1}{(f_\lambda^i)'(d_\lambda)(z-g(f_\lambda^i(d_\lambda))}$. Then by Proposition (5.8) $G(z) \neq 0$ identically on $g(Y)$.

Now, we **Claim** that $G(g(z))g'(z) = \phi(z)$.

Proof of the claim. Let us define $C_1 = \sum_i \frac{(f_\lambda^i(d_\lambda)-1)}{(f_\lambda^i)'(d_\lambda)}$ and $C_2 = \sum_i \frac{f_\lambda^i(d_\lambda)}{(f_\lambda^i)'(d_\lambda)}$ then we have

$$\frac{C_1}{g(z)} = \frac{C_1(z+y-1)}{yz} \text{ and } \frac{C_2}{g(z)-1} = \frac{C_2(z+y-1)}{(y-1)(z-1)}$$

and for any n

$$\frac{1}{g(z)-g(f_\lambda^n(d_\lambda))} = \frac{(z+y-1)(f_\lambda^n(d_\lambda)+y-1)}{y(y-1)(z-f_\lambda^n(d_\lambda))} = \\ = \frac{1}{y(y-1)}\left(\frac{(z+y-1)^2}{z-f_\lambda^n(d_\lambda)} + 1 - y - z\right),$$

then

$$G(g(z)) = \frac{C_1(z+y-1)}{yz} - \frac{C_2(z+y-1)}{(y-1)(z-1)} + \sum \frac{1}{(f_\lambda^i)'(d_\lambda)(g(z)-g(f_\lambda^i(d_\lambda))} = \\ = \frac{1}{y(y-1)}\Bigg((1-y-z)\sum\frac{1}{(f_\lambda^i)'(d_\lambda)} + (z+y-1)^2\sum\frac{1}{(f_\lambda^i)'(d_\lambda)(z-f_\lambda^i(d_\lambda))} + \\ + \frac{C_1(z+y-1)}{yz} - \frac{C_2(z+y-1)}{(y-1)(z-1)}\Bigg) = *$$

and

$$* = \frac{1}{g'(z)}\left(\phi(z) + \frac{\sum_i \frac{f_\lambda^i(d_\lambda)-1}{(f_\lambda^i)'(d_\lambda)}}{z} - \frac{\sum_i \frac{f_\lambda^i(d_\lambda)}{(f_\lambda^i)'(d_\lambda)}}{z-1} + \frac{\sum \frac{1}{(f_\lambda^i)'(d_\lambda)}}{1-y-z}\right) + \\ + \frac{1}{g'(z)}\left(\frac{C_1(y-1)}{z(z+y-1)} - \frac{C_2 y}{(z-1)(z+y-1)}\right) = \\ = \frac{\phi(z)}{g'(z)}.$$

Hence $\phi(z) = 0$ identically on Y if and only if $G(z) = 0$ identically on $g(Y)$. So by Proposition (5.8) we complete the proof of this proposition.

Step 3 and Theorem (1.6) are finished.

Proof of the Corollary: Since X_λ is bounded, Theorem (1.7) part (2), implies that $z = 0$ is a summable point and $m(X_\lambda) = 0$. Then $f_\lambda \in W$ and this implies by Theorem (1.6) that f_λ is unstable. □

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A CHARACTERIZATION OF $C_k(X)$ FOR X NORMAL AND REALCOMPACT

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ABSTRACT. We present some internal conditions on a locally m-convex Φ-algebra A stated in terms of order and/or closed ideals of A. It turns out that a locally m-convex Φ-algebra satisfies these conditions if and only if it is l-isomorphic and homeomorphic to the locally m-convex Φ-algebra $C_k(X)$ for some realcompact normal space X. Here $C_k(X)$ is the set of all real-valued continuous functions on X endowed with the topology of compact convergence. One of the above mentioned internal conditions can be replaced by the requirement that A be a barreled space. We also prove that any Fréchet uniformly closed Φ-algebra satisfies the internal conditions in question.

Introduction

Throughout this paper, X will denote a completely regular and Hausdorff topological space, and $C(X)$ will be the Φ-algebra of all real-valued continuous functions on X with pointwise operations and order. Let $C_k(X)$ denote $C(X)$ endowed with the compact convergence topology. Recently we obtained a partial answer to the classical problem of characterizing $C_k(X)$. More precisely, we characterized $C_k(X)$ when X is a normal space (see [10]). Our aim in this article is to solve the same problem when X is a normal and realcompact space. But we should emphasize that the answer we obtain here is not derived as a particular case of that in [10]. Instead, our approach to the problem is based on the following result due to Feldman and Porter: A topological space X is realcompact if and only if the compact convergence topology of $C(X)$ coincides with the order topology [4]. It thus appears appropriate to seek a characterization in which the order plays a more important role in this case.

Let A be a uniformly closed Φ-algebra A endowed with a Hausdorff locally m-convex topology. The problem lies in looking for internal conditions on A which imply that A is l-isomorphic and homeomorphic to $C_k(X)$ for some normal and realcompact topological space X. Thus, if τ is such a topology on A then τ must be the order topology on A.

First we solve the algebraic part of the problem; i.e., we characterize those Φ-algebras that are l-isomorphic to some $C(X)$ with X normal and realcompact. Then, for these Φ-algebras, the problem turns into one of determining the order topology among all the Hausdorff locally m-convex topologies on those algebras.

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Before approaching the problem proper, we will make use of the solution given to the algebraic problem to prove the following result: A locally m-convex Φ-algebra A is l-isomorphic and homeomorphic to $C_k(X)$ for some hemicompact k-space X if and only if A is uniformly closed and Fréchet.

The article is organized into two sections. Section 1 sets out the results concerning topological algebras and Φ-algebras that we will need in the sequel. For easier reading, we shall give complete definitions of all the terms we will use, but not give any proofs. In Section 2 we prove the main results.

1. Preliminaries

In the sequel, every ring will be assumed to be commutative and to possess an identity, and every morphism of rings will carry the identity into the identity. We shall denote by $\mathbb{R}$-*algebra* (henceforth simply *algebra*) every ring A endowed with a morphism of rings $\mathbb{R} \to A$ (the *structural morphism* of the algebra) which must be injective and allow $\mathbb{R}$ to be identified with a subring of A; in particular 1 will denote indistinctly the identity of $\mathbb{R}$ and the identity of A. Given algebras A and B, a map $A \to B$ is a *morphism of algebras* if it is a morphism of rings that leaves $\mathbb{R}$ invariant.

Definition (1.1). A *topological algebra* is an algebra A endowed with a (not necessarily Hausdorff) topology for which A is a topological vector space, the product of A is continuous, and the map $a \mapsto a^{-1}$ (defined over the invertible elements) is continuous.

An important class of topological algebras consists of the *locally m-convex algebras*, i.e., those in which the topology may be defined by a family of m-seminorms (a seminorm q on an algebra A is an *m-seminorm* if $q(ab) \leq q(a)q(b)$ for all $a, b \in A$).

Example (1.2). Let us now consider the topological algebra that we are most interested in. For each compact subset K of X, one has the m-seminorm q_K on $C(X)$ defined by the equality $q_K(f) = \max\{|f(x)| : x \in K\}$ $(f \in C(X))$. The topology that the family $\{q_K : K \text{ compact subset of } X\}$ defines in $C(X)$ is known as the *topology of uniform convergence on compact sets* (in brief, *compact convergence topology*); $C(X)$ endowed with this topology will be denoted by $C_k(X)$.

Definitions (1.3). Let A be a topological algebra. We shall call the set of morphisms of algebras of A in $\mathbb{R}$ that are continuous the *topological spectrum* of A, and shall denote it by $\operatorname{Spec}_t A$. Each element $a \in A$ defines on $\operatorname{Spec}_t A$ the function $a : \operatorname{Spec}_t A \to \mathbb{R}$, $x \mapsto a(x) := x(a)$. The initial topology that these functions define on $\operatorname{Spec}_t A$ is known as the *Gelfand topology*. Except when otherwise specified, we shall assume that $\operatorname{Spec}_t A$ is endowed with this topology. Thus, it is clear that $\operatorname{Spec}_t A$ is a completely regular Hausdorff topological space (it may be that $\operatorname{Spec}_t A = \emptyset$).

Let us assume that $\operatorname{Spec}_t A \neq \emptyset$ and let $C(\operatorname{Spec}_t A)$ be the algebra of all real-valued continuous functions on $\operatorname{Spec}_t A$. There is a natural morphism of algebras $T : A \to C(\operatorname{Spec}_t A)$ known as the *spectral representation* of A. A is said to be *semisimple* when its spectral representation is injective.

A maximal ideal M of A is *real* if the residue class field A/M is $\mathbb{R}$. If $x\colon A \to \mathbb{R}$ is a morphism of algebras, then its kernel $\operatorname{Ker} x$ is a real maximal ideal of A, and x is continuous if and only if $\operatorname{Ker} x$ is closed. Hence there is a one-to-one correspondence between the points of $\operatorname{Spec}_t A$ and the set of all closed real maximal ideals of A. Clearly, A is semisimple if and only if the intersection of all its closed real maximal ideals is zero.

Definitions (1.4). Let A be a topological algebra. For every ideal I of A we have the closed set $(I)_0 := \{x \in \operatorname{Spec}_t A \,:\, a(x) = 0 \text{ for every } a \in I\}$ of $\operatorname{Spec}_t A$. We shall say that $(I)_0$ is the *zero set* of the ideal I. The *zero set* of an element $a \in A$ is the closed subset $(a)_0 := \{x \in \operatorname{Spec}_t A \,:\, a(x) = 0\}$ of $\operatorname{Spec}_t A$. It is clear that $(I)_0 = \bigcap_{a\in I}(a)_0$ for any ideal I of A. We shall say that A is *regular* if its elements separate points and closed sets of $\operatorname{Spec}_t A$ in the following sense: if $x \in \operatorname{Spec}_t A$ and F is a non-empty closed subset of $\operatorname{Spec}_t A$ such that $x \notin F$, then there exists $a \in A$ satisfying $a(F) = 0$ and $a(x) = 1$. It follows from the definition that A is regular if and only if $\{(a)_0 \,:\, a \in A\}$ is a basis of closed sets in $\operatorname{Spec}_t A$.

Example (1.5). Each $x \in X$ defines the continuous morphism of algebras $\delta_x\colon C_k(X) \to \mathbb{R}$, $\delta_x(f) := f(x)$, and so one has the natural map $i\colon X \to \operatorname{Spec}_t C_k(X)$, $i(x) := \delta_x$. On the one hand, $i\colon X \to i(X)$ is a homeomorphism because X is completely regular. On the other, if for each closed set C in X and each closed ideal J in $C_k(X)$ one denotes $J_C = \{f \in C(X) \,:\, f(C) = 0\}$, $(J)_0 = \{x \in X \,:\, f(x) = 0 \ \forall f \in J\}$, then the closed ideals in $C_k(X)$ are in one-to-one correspondence with the closed subsets of X (via $C \mapsto J_C$ and $J \mapsto (J)_0$, see [11]), and consequently the closed maximal ideals in $C_k(X)$ are in one-to-one correspondence with the points of X. Therefore, $X = \operatorname{Spec}_t C_k(X)$ and the spectral representation of $C_k(X)$ is an isomorphism. In particular $C_k(X)$ is regular and semisimple.

Definition (1.6). We shall say that a topological algebra A is *normal* if its elements separate disjoint closed sets of $\operatorname{Spec}_t A$ in the following sense: if F, G are disjoint non-empty closed sets of $\operatorname{Spec}_t A$, then there exists $a \in A$ such that $a(F) = 0$ and $a(G) = 1$. Clearly, if A is normal then A is regular.

According to Urysohn's Lemma, X is normal if and only if $C_k(X)$ is normal. Also, it is not difficult to show that $C_k(X)$ is normal if and only if in $C_k(X)$ there do not exist two closed ideals whose sum is dense and proper. One has the more general lemma:

LEMMA (1.7). ([10], Lemma 2.13 (ii)). *Let A be a regular topological algebra such that:* (i) *every non-dense ideal is contained in some closed real maximal ideal;* (ii) *there do not exist two closed ideals whose sum is dense and proper. Then A is normal.*

Next, let us describe the order structures that interest us.

Definitions (1.8). A *vector lattice* is a real vector space E endowed with an order relationship "$\leq$" with which it is a lattice (every non-empty finite subset has a supremum and an infimum), and is compatible with the vector structure (if $a, b \in E$ such that $a \leq b$, then $a + c \leq b + c$ for every $c \in E$, and $\lambda a \leq \lambda b$ for every $\lambda \in \mathbb{R}$, $\lambda \geq 0$). For $C(X)$ we shall always consider its usual order with which it is a vector lattice: the point-wise defined natural order.

Let E be a vector lattice. The set $E_+ = \{a \in E : a \geq 0\}$ is called the *positive cone* of E. As is usual, the supremum and infimum of a finite subset $\{a_1, \ldots, a_n\}$ of E will be denoted by $a_1 \vee \cdots \vee a_n$ and $a_1 \wedge \cdots \wedge a_n$, respectively. Given an element $a \in E$, its *positive part*, its *negative part*, and its *absolute value* are elements of E which are denoted by a^+, a^- and $|a|$, respectively, and are defined by the equalities $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, $|a| = a^+ \vee a^-$. Given $a, b \in E$, the *closed interval* of extremes a and b is the subset of E which is denoted $[a, b]$ and is defined by the equality $[a, b] := \{c \in E : a \leq c \leq b\}$. A subset C of E is said to be *solid* if $a \in C$ implies $\{b \in E : |b| \leq |a|\} \subseteq C$.

A map $T : E \to F$, where E and F are vector lattices, is a *morphism of vector lattices* if it is linear and is a morphism of lattices, i.e., if it is a linear map such that $T(a \vee b) = T(a) \vee T(b)$ and $T(a \wedge b) = T(a) \wedge T(b)$ for all $a, b \in E$.

Definition (1.9). Given a vector lattice E, the *order topology* in E, which we shall denote by τ_o, is defined as the finest locally convex topology for which all closed intervals of E are t-bounded (bounded in the topological sense, i.e., the closed intervals are absorbed by each 0-neighbourhood). The order topology on $\mathbb{R}$ is its usual topology.

It is easy to prove the following property: "Let $T : E \to F$ be a linear map between vector lattices. If T preserves the order, then T is continuous if E and F are endowed with their respective order topologies."

Definitions (1.10). An *l-algebra* is an algebra A endowed with an order relationship "$\leq$" with which it is a lattice and is compatible with the algebraic structure (if $a, b \in A$ such that $a \leq b$, then $a + c \leq b + c$ for all $c \in A$, $\lambda a \leq \lambda b$ for all $\lambda \in \mathbb{R}$, $\lambda \geq 0$, and $ac \leq bc$ for all $c \in A_+$). If A is an l-algebra, then in particular it is a vector lattice, so that the notions given for vector lattices in (1.8) above are valid in A.

Let A and B be l-algebras. A map $A \to B$ is said to be a *morphism of l-algebras* if it is a morphism of algebras and a morphism of lattices. A morphism of l-algebras is called an *l-isomorphism* if it is bijective. An ideal I of A is said to be an *l-ideal* if I is a solid set. A *maximal l-ideal* is a proper l-ideal that is not contained strictly in another proper l-ideal. $C(X)$ with its usual order is an l-algebra, and each closed ideal of $C_k(X)$ is an l-ideal, since for each closed subset C of X the ideal J_C is solid (Example (1.5)).

Remark (1.11). Let A be an l-algebra, I an ideal of A, and $\pi : A \to A/I$ the quotient morphism. Then $\pi(A_+)$ is the positive cone for an l-algebra structure on A/I for which π is a morphism of l-algebras if and only if I is an l-ideal.

Whenever we speak of the l-algebra A/I, we shall be assuming on A/I the above structure, and therefore that I is an l-ideal.

Definitions (1.12). Let A be an l-algebra. A is said to be *Archimedean* if for $a, b \in A$, $na \leq b$ for all $n \in \mathbb{N}$ implies $a \leq 0$. An element e of A is said to be a *weak order unit* if for $a \in A$, $a \wedge e = 0$ implies $a = 0$. An element e of A is said to be a *strong order unit* if $a \in A_+$ implies $a \leq ne$ for some non-negative integer n. A is said to be an *f-algebra* if for $a, b, c \in A$, $a \wedge b = 0$ and $c \geq 0$ imply $ca \wedge b = 0$. A *Φ-algebra* is an Archimedean f-algebra; equivalently, a

Φ-algebra is an Archimedean l-algebra in which 1 is a weak order unit (see [2], §9, Corollary 3). It is clear that $C(X)$ is a Φ-algebra.

Definitions (1.13). Let A be an f-algebra. Then it is known that A induces in $\mathbb{R}$ the usual order of $\mathbb{R}$. According to the above, given $\alpha, \beta \in \mathbb{R}$, one will have $\alpha \leq \beta$ in $\mathbb{R}$ if and only if $\alpha \leq \beta$ in A, so that we will make no distinction.

We shall say that an element $a \in A$ is *o-bounded* if there exists a non-negative integer n such that $|a| < n$. We shall denote the set of all the o-bounded elements of A by A^*. It is clear that A^*, with the order induced by the order of A, is an f-algebra. The f-algebra $C(X)^*$ is denoted by $C^*(X)$.

A sequence $(a_n)_n$ in A is said to be *Cauchy uniform* if for every real $\varepsilon > 0$ there exists a positive integer ν such that $|a_n - a_m| \leq \varepsilon$ for $n, m \geq \nu$. A sequence $(a_n)_n$ in A is said to be *uniformly convergent* to $a \in A$ if for each real $\varepsilon > 0$ there exists a positive integer ν such that $|a_n - a| \leq \varepsilon$ for $n \geq \nu$. It is easy to see that if $(a_n)_n$ is uniformly convergent to both a and b in A and if A is Archimedean then $a = b$. A subset S in A is said to be *uniformly closed* if each Cauchy uniform sequence in S is uniformly convergent in S. A subset S in A is said to be *uniformly dense* if for each element $a \in A$ there is a sequence in S that converges uniformly to a. It is known that $C(X)$ is uniformly closed.

We conclude this section with some results which we shall use later.

LEMMA (1.14). *Let A be a uniformly closed Φ-algebra. One has*

(i) *If $a \in A$ and $a \geq 1$ then a is invertible. As a consequence A is strictly real, i.e., $1 + a^2$ is invertible for all $a \in A$.*

(ii) *A is a Gelfand algebra; i.e., each prime ideal of A is contained in a unique maximal ideal.*

(iii) *If τ is a topology on A such that (A, τ) is a topological algebra, then (A, τ) is regular.*

(iv) *If B is another uniformly closed Φ-algebras, then every morphism of algebras $A \to B$ is a morphism of l-algebras. Consequently, every real maximal ideal of A is τ_o-closed.*

(v) *Let us consider A endowed with a Hausdorff locally m-convex topology. Every non-dense ideal of A is contained in some closed real maximal ideal. Consequently, every closed maximal ideal of A is real and $\operatorname{Spec}_t A \neq \emptyset$. Moreover, if A is complete, then an element $a \in A$ is invertible if and only if $a(x) \neq 0$ for all $x \in \operatorname{Spec}_t A$.*

Proof. See [9], Lemma 3.12, for (i), [10], Lemma 3.5, for (ii), [10], Lemma 3.7, for (iii), and [9], Lemma 3.15, for (iv). According to some results proved in [12] for complex algebras (whose proofs in the real case may be found in [16], Ejemplo II.1.6, Teorema II.3.10 and Teorema II.3.11), the property (v) holds in each locally m-convex, Hausdorff, and strictly real algebra, and by (i) this is the case here. □

2. The Main results

Definitions (2.1). Let A be an algebra and $\operatorname{Spec}_m A = \{$maximal ideals of $A\}$ the *maximal spectrum* of A. If for every ideal I of A one writes $[I]_0 := \{$maximal ideals of A that contain $I\}$, then the sets of the family $\{[I]_0 :$ I ideal of $A\}$ are the closed sets of a topology on $\operatorname{Spec}_m A$, known as the

Zariski topology. Under this topology $\operatorname{Spec}_m A$ is a compact topological space (not necessarily Hausdorff). One basis of closed sets for this topology is the collection $\{[a]_0 : a \in A\}$, where $[a]_0 := [(a)]_0$ and (a) is the principal ideal of A generated by a.

We shall call the set $\operatorname{Spec}_{\mathbb{R}} A := \{$morphisms of algebras of A into $\mathbb{R}\} = \{$real maximal ideals of $A\}$ the *real spectrum* of A. We shall say that A is *closed under inversion* if its invertible elements are just its non-null elements which do not belong to any real maximal ideal of A. We shall say that A is *real-semisimple* if the intersection of all the real maximal ideals of A is null.

It is clear that $C(X)$ is closed under inversion and real-semisimple.

Remarks (2.2). (i) When an algebra is real-semisimple, one implicitly assumes that its real spectrum is non-empty.

(ii) If A is a closed under inversion Φ-algebra, then it must be the case that $\operatorname{Spec}_{\mathbb{R}} A \neq \emptyset$. Indeed, if the real spectrum of A were empty, then every non-null element of A would be invertible and hence A would be a field. It is straightforward to see that this field would be totally ordered (see [2], p. 57), and as it is Archimedean, it would have to be a subfield of $\mathbb{R}$ (see [5], 0.21). Therefore $A = \mathbb{R}$ and $\operatorname{Spec}_{\mathbb{R}} A$ would be a point, which is absurd.

Recall that the topological space X it said to be *realcompact* if X satisfies one of the following equivalent properties: (i) every real maximal ideal of $C(X)$ is of the form J_x for some $x \in X$ (i.e., every real maximal ideal of $C_k(X)$ is closed; see Example (1.5)); (ii) X is homeomorphic to a closed subspace of a product of real lines.

Let A be a uniformly closed Φ-algebra with $\operatorname{Spec}_{\mathbb{R}} A \neq \emptyset$. Each element $a \in A$ defines on $\operatorname{Spec}_{\mathbb{R}} A$ the function $a\colon \operatorname{Spec}_{\mathbb{R}} A \to \mathbb{R}$, $x \mapsto a(x) := x(a)$. It is easy to see that the initial topology defined by these functions on $\operatorname{Spec}_{\mathbb{R}} A$ coincides with the Zariski topology induced by $\operatorname{Spec}_m A$. We shall denote by X_o the set $\operatorname{Spec}_{\mathbb{R}} A$ endowed with this topology. The natural map $A \to C(X_o)$ is a morphism of l-algebras (because it is a morphism of algebras); that A is real-semisimple means that this morphism is injective, and that A is closed under inversion means that, given $a \in A$, a is invertible if (and only if) $a(x) \neq 0$ for all $x \in \operatorname{Spec}_{\mathbb{R}} A$.

Considering $\operatorname{Spec}_{\mathbb{R}} A \subseteq \mathbb{R}^A = \{$maps of A into $\mathbb{R}\}$, one easily sees that the topology of X_o coincides with that induced by the product topology of $\mathbb{R}^A$, with X_o being a closed subspace of $\mathbb{R}^A$; i.e., X_o is realcompact.

Example (2.3). If $A = C(X)$, then from Example (1.5) it follows that X is realcompact if and only if $X = X_o$ (topological equality).

Let A be a uniformly closed Φ-algebra. When A is closed under inversion, in which case $\operatorname{Spec}_{\mathbb{R}} A \neq \emptyset$, it follows from an important result due to Buskes that the order topology on A is the initial topology induced by the morphism of algebras $A \to C_k(X_o)$ (see [3]; [9], Lemma 3.18 and Corollary 3.19). In particular, if X is realcompact then the order topology on $C(X)$ coincides with the compact convergence topology. One has:

PROPOSITION (2.4). *If A is a Φ-algebra that is uniformly closed and closed under inversion, then (A, τ_o) is a locally m-convex algebra such that*

(i) $\mathrm{Spec}_t(A, \tau_o) = X_o$ *and hence* $\mathrm{Spec}_t(A, \tau_o)$ *is realcompact;*
(ii) (A, τ_o) *is Hausdorff, regular and semisimple.*

Proof. From Buskes' result mentioned above it follows that (A, τ_o) is a locally m-convex algebra. The regularity of (A, τ_o) and the equality $\mathrm{Spec}_t(A, \tau_o) = X_o$ hold by (iii) and (iv) of Lemma (1.14), respectively. Since every uniformly closed and closed under inversion Φ-algebra is real-semisimple (see [9], Lemma 3.17, where *o-semisimple* means real-semisimple), from the equality $\mathrm{Spec}_t(A, \tau_o) = X_o$ it follows that (A, τ_o) is semisimple and Hausdorff. □

For the proof of the next theorem we will use a result of Tietze [18], namely: " Let E be a vector subspace of $C^*(X)$ that contains the constant functions. If E S^1-separates disjoint closed sets of X (i.e., for each pair of non-empty disjoint closed sets F and G of X, there exists $h \in E$ such that $0 \leq h \leq 1$, $h(F) = 0$ and $h(G) = 1$), then E is uniformly dense in $C^*(X)$".

THEOREM (2.5). *Let A be a uniformly closed Φ-algebra. A is l-isomorphic to $C(X)$ for some normal and realcompact topological space X if and only if*
(i) *A is closed under inversion;*
(ii) *(A, τ_o) is normal.*

Proof. Assume that A satisfies (i) and (ii). From Proposition (2.4) it follows that the spectral representation of (A, τ_o) is the injective morphism of l-algebras $A \to C(X_o)$, and that the condition "(A, τ_o) normal" means that A separates disjoint closed sets of X_o (in particular, the realcompact space X_o is normal). Identifying A with its image in $C(X_o)$, one has that A is a uniformly closed l-subalgebra of $C(X_o)$ that separates disjoint closed sets of X_o. Then A^* S^1-separates them, since, if for $a \in A$ one has $a(F) = 0$ and $a(G) = 1$ (F and G closed sets of X_o), the same is the case for $|a| \wedge 1 \in A^*$. From Tietze's result it follows that A^* is uniformly dense in $C^*(X_o)$, and as A^* is uniformly closed (since A is so) one concludes that $A^* = C^*(X_o)$. Now, if $f \in C(X_o)$, then $f_1 = 1/(f^+ + 1)$ and $f_2 = 1/(f^- + 1)$ are functions of A^* that do not vanish at any point of X_o, and such that $f = 1/f_1 - 1/f_2$. But, by hypothesis (i), $1/f_1, 1/f_2 \in A$, so that $f \in A$, and one concludes that $A = C(X_o)$; i.e., the spectral representation of (A, τ_o) is an isomorphism of l-algebras.

Now, let $A = C(X)$ with X normal and realcompact, in which case $X = X_o$ and $A \to C(X_o) = C(X)$ is an isomorphism. Hence $(A, \tau_o) = C_k(X)$ by Buskes' result, and therefore A satisfies (i) and (ii). □

We shall apply the preceding theorem to obtain the next result. We will take some notions from the theory of locally convex spaces: barreled, metrizable, and complete. Recall that the topological space X it said to be: (i) *hemicompact* if there exists a sequence $\{K_n\}_n$ of compact subsets of X such that every compact subset of X is contained in some K_n; (ii) a *k-space* if a subset of X is open if its intersection with each compact subset K of X is open in K. It is well-known that $C_k(X)$ is a Fréchet space (i.e., metrizable and complete) if and only if X is a hemicompact k-space (see [19]).

THEOREM (2.6). *Let A be an uniformly closed Φ-algebra. If τ is a locally m-convex topology on A, then (A, τ) is l-isomorphic and homeomorphic to $C_k(X)$ for some hemicompact k-space X if and only if (A, τ) is Fréchet.*

Proof. First, note that $\mathrm{Spec}_t(A,\tau) \neq \emptyset$ by Lemma (1.14)(v). From a well-known result of Michael [8] it follows that if B is a semisimple Fréchet algebra, then every morphism of algebras $A \to B$ is continuous (see [16], Teorema II.4.6, for the real case). As a consequence one obtains that $\mathrm{Spec}_t(A,\tau) = \mathrm{Spec}_{\mathbb{R}} A = \mathrm{Spec}_t(A,\tau_o)$, and hence from Lemma (1.14)(v) it follows that A is closed under inversion. It is also known that any regular and strictly real Fréchet algebra is normal (see [17]; [16], Teorema II.4.13, for the real case), then one has that (A,τ_o) is normal. We can apply Theorem (2.5) to deduce that the spectral representation $(A,\tau) \to C_k(X_o)$ is an l-isomorphism. Again, Michael's result shows that the composition $(A,\tau) \to C_k(X_o) \to C_k(K)$ is continuous for each compact subset K of X_o (where $C_k(X_o) \to C_k(K)$ is the restriction morphism), and thus we obtain that the spectral representation of (A,τ) is continuous. Finally, in order to prove the continuity of the inverse map $C_k(X_o) \to (A,\tau)$, note that $C_k(X_o)$ is barreled because X_o is realcompact (see [13]). Then, the proof follows from a generalization of the open mapping theorem: "a linear and continuous map from a Fréchet space onto a barreled space is open" (see [15]; [6], 4.1 and Proposition 3). □

Remark (2.7). The previous theorem is not a consequence or a particular case of Theorem (2.14) below.

LEMMA (2.8). *Let A be a uniformly closed Φ-algebra that is also a topological vector space. If the closed intervals of A are t-bounded, then every closed subset of A is uniformly closed.*

Proof. Let F be a closed subset of A. If $(a_n)_n$ is a uniform Cauchy sequence in F and $a \in A$ is the uniform limit of $(a_n)_n$, one will have to show that $a \in F$. Let V be a neighbourhood of 0 in A. On the one hand, there exists $\lambda \in \mathbb{R}_+$ such that $[-1,1] \subseteq \lambda V$, i.e., $[-\frac{1}{\lambda},\frac{1}{\lambda}] \subseteq V$; and on the other hand, there exists a non-negative integer m such that $a_n - a \in [-\frac{1}{\lambda},\frac{1}{\lambda}]$ for all $n \geq m$. Therefore $(a_n)_n$ converges to a, and one concludes that $a \in F$. □

LEMMA (2.9). *Let A be a uniformly closed Φ-algebra. The following are equivalent:*

(i) *every maximal ideal of A is real;*

(ii) *the unit element of A is a strong order unit.*

Proof. (i) ⇒ (ii) If A satisfies (i) then it is clear that $X_o = \mathrm{Spec}_m A$ (topological equality) and so X_o is compact. Since every maximal ideal of a uniformly closed Φ-algebra is an l-ideal (see [14], Theorem 3.7), and the intersection of all the maximal l-ideals is zero (see [7], Chapter II, Theorem 2.11), it follows that A is real-semisimple; i.e., A is l-isomorphic to a subalgebra of (bounded) functions in $C(X_o)$. Hence $A = A^*$, which is equivalent to saying that 1 is a strong order unit for A.

(ii) ⇒ (i) Let M be a maximal ideal of A. Since M is an l-ideal, one has that A/M is an f-field and thus a totally ordered field (see [2], p. 57). Moreover, A/M must be Archimedean since otherwise it would contain infinitely large elements, against that the unit of A/M is a strong order unit. It follows that $A/M = \mathbb{R}$. □

Definitions (2.10). Let A be a topological algebra. We shall say that A is a *Q-algebra* if the set of its invertible elements is open. We shall say that an ideal I of A is a *C-ideal*, if I is closed and every maximal ideal of A that contains I is real and closed.

Let A be a locally m-convex algebra and I an ideal of A. We shall endow the quotient A/I with the quotient topology, i.e., the final topology defined by the quotient morphism $A \to A/I$. Thus, A/I is also a locally m-convex algebra (see for instance [16], Teorema I.2.5), and there exists a one-to-one correspondence between the closed ideals of A containing I and the closed ideals of A/I.

When A is also a uniformly closed Φ-algebra, it is obvious that if I is a closed ideal such that A/I is a Q-algebra then I is a C-ideal (because in a Q-algebra every maximal ideal is closed). The converse is not true. Actually one has

LEMMA (2.11). *Let A be a locally m-convex and uniformly closed Φ-algebra. The spectral representation $A \to C_k(\mathrm{Spec}_t\, A)$ is continuous if and only if A/I is a Q-algebra for every C-ideal I of A.*

Proof. According to [10], Theorem 2.28, the lemma is true when A is a Gelfand regular locally m-convex algebra. As A is so by (ii) and (iii) of Lemma (1.14), the proof is complete. □

Remark (2.12). If A is a Gelfand regular topological algebra, then the C-ideals of A are in correspondence with the compact subsets of $\mathrm{Spec}_t\, A$ (see [10], Section 2, for the details). Thus it is reasonable that the C-ideals should play an essential role in a statement about the continuity of the spectral representation (such as Lemma (2.11)), since the topology of $C_k(\mathrm{Spec}_t\, A)$ is defined in terms of the compact subsets of $\mathrm{Spec}_t\, A$.

THEOREM (2.13). *Let A be a Φ-algebra that is uniformly closed and closed under inversion. If τ is a topology on A such that (A, τ) is a locally m-convex algebra, then $\tau = \tau_o$ if and only if*

(i) *each closed interval of A is τ-bounded;*

(ii) *every real maximal ideal of A is τ-closed;*

(iii) *if I is a τ-closed ideal of A such that the unit of A/I is a strong order unit, then A/I is a Q-algebra.*

Proof. We have that (A, τ_o) is a locally m-convex algebra (and thus regular) that satisfies condition (ii) (Proposition (2.4)) and condition (i). Let us see that τ_o satisfies (iii). Let I be a closed ideal of A. From Lemma (2.8) it follows that I is also uniformly closed, and so it is known that A/I is a uniformly closed Φ-algebra (see [14], Theorem 3.7 and Theorem 2.5). Assume that the unit of A/I is a strong order unit. According to the preceding lemma, this condition is equivalent to “every maximal ideal containing I is real”, i.e., to I being a C-ideal of A. That A/I is a Q-algebra follows from Lemma (2.11), because the spectral representation $(A, \tau_o) \to C_k(X_o)$ is continuous (by Buskes' result).

Conversely, assume that (A, τ) is a locally m-convex algebra satisfying conditions (i), (ii), and (iii). Then it is clear that $\tau \leq \tau_o$. A similar argument to the previous one allows one to prove that if I is a C-ideal of A then A/I is a Q-algebra, and hence from Lemma (2.11) one derives that the spectral representation $(A, \tau) \to C_k(\mathrm{Spec}_t\, A)$ is continuous. Since, according to condition (ii), the equality $\operatorname{Spec}_t A = X_o$ is satisfied, from Buskes' result it follows that τ_o is the initial topology associated with this spectral representation, so that it must be the case that $\tau_o \leq \tau$. $\square$

THEOREM (2.14). *Let A be a uniformly closed Φ-algebra endowed with a Hausdorff locally m-convex topology. A is l-isomorphic and homeomorphic to $C_k(X)$ for some normal and realcompact topological space X iff:*

(i) *there exist no principal ideals in A that are proper and dense;*

(ii) *each closed interval of A is t-bounded;*

(iii) *every real maximal ideal of A is closed;*

(iv) *if I is a closed ideal of A such that the unit of A/I is a strong order unit, then A/I is a Q-algebra.*

(v) *there do not exist two closed ideals in A whose sum is dense and proper.*

Proof. Let us see that these conditions are sufficient. We first prove that A is closed under inversion: if $a \in A$ such that $a(x) \neq 0$ for all $x \in \operatorname{Spec}_{\mathbb{R}} A$ ($\neq \emptyset$ by Lemma (1.14)(v)), then from Lemma (1.14)(v) it follows that the principal ideal (a) is dense. Condition (i) yields that (a) is the whole algebra A; i.e., a is an invertible element of A. Then we can apply Theorem (2.13) to deduce that the topology of A is the order topology. Also A is normal by condition (v) and Lemma (1.7). Finally, Theorem (2.5) shows that $A \to C_k(X_o)$ is an l-isomorphism, and therefore a homeomorphism.

Conversely, if X is realcompact then $C(X)$ is closed under inversion and the topology of $C_k(X)$ is the order topology. From Theorem (2.13) it follows that $C_k(X)$ satisfies conditions (ii), (iii), and (iv). Inverting the reasoning of the preceding paragraph, one has that $C_k(X)$ satisfies (i). Lastly, we have already said that condition (v) on $C_k(X)$ is equivalent to X being normal. $\square$

Remark (2.15). Condition (iv) in the preceding theorem can be replaced by the requirement that A be a barreled space. Indeed, on the one hand, as has already been pointed out, $C_k(X)$ is barreled when X is realcompact. On the other hand, if A is barreled, then it is easy to show that the spectral representation of A is continuous (see [1], (4.12-4)), and therefore, reasoning as in the proof of (2.13), one has that condition (iv) holds.

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ON TRANSVERSELY HOLOMORPHIC PRINCIPAL BUNDLES

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ABSTRACT. The notion of a transversely holomorphic structure on a foliated manifold is generalized. We define principal bundles, and their associated bundles, on these generalized transversely complex spaces. For any pair of holomorphic structures on a transversely differentiable principal bundle over such a space, we construct and study certain naturally associated elements of the Dolbeault cohomology of the foliated space. The construction of these cohomology classes are inspired by the construction of the Chern-Simons secondary invariants of flat vector bundles.

1. Introduction

Transversely holomorphic structures were introduced by Gómez-Mont in [GM80]. He defined such a structure on a topological space M to be an open covering of M by coordinate patches modeled after $\mathbb{R}^m \times \mathbb{C}^n$, with the property that the transition function between any two coordinate patches is of the form

$$\begin{aligned} \mathbb{R}^m \times \mathbb{C}^n &\to \mathbb{R}^m \times \mathbb{C}^n\,, \\ (x, y) &\mapsto (g(x, y), h(y))\,, \end{aligned} \tag{1.1}$$

where g is a C^∞ function and h is a holomorphic function. Gómez-Mont then showed that certain natural sheaves on such a foliated space M have finite-dimensional cohomologies, provided M is compact. He also studied deformations of such spaces. Deformations of transversely holomorphic structures were also studied by Girbau, Haefliger, and Sundararaman [GHS83], who introduced the notion of a versal deformation of a transversely holomorphic structure, and constructed it.

In this paper, we generalize the notion of transversely holomorphic structures as follows. We require that the space M in the above setting should have a covering by coordinate patches modeled after $X \times \mathbb{C}^n$, where X is an arbitrary fixed topological space, and that the transition function between any two coordinate patches is of the form $(x, y) \mapsto (g(x, y), h(y))$, where g is a continuous function and h is a holomorphic function. We shall call such a space a *transversely complex manifold*. We define principal bundles on such a space, and prove some properties of their characteristic classes.

Fix a complex Lie group G. Let P be a C^∞ principal G-bundle on a transversely complex manifold M. This means that, in terms of a coordinate chart on M, the principal bundle P is continuous in the X direction, and C^∞ in the $\mathbb{C}^n$ direction (the transverse direction). We shall define on P an analog of a Dolbeault operator; the Dolbeault operator on a smooth principal bundle

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over a complex manifold gives a complex structure on the principal bundle. The space $\mathrm{Dol}(P)$ of all Dolbeault operators on P is an affine space modeled after the space of all $\mathrm{ad}(P)$-valued transversely differential forms on M of Hodge type $(0, 1)$, where $\mathrm{ad}(P)$ is the adjoint vector bundle of P. For each pair of Dolbeault operators $\overline{\partial}_0$ and $\overline{\partial}_1$, we construct a sequence $\{D_k(\overline{\partial}_0, \overline{\partial}_1)\}_{k=0}^{\infty}$ of cohomology classes $D_k(\overline{\partial}_0, \overline{\partial}_1) \in \mathrm{H}^{2k+1}(M, \mathcal{O}_M)$, where $\mathcal{O}_M$ is the sheaf of transversely holomorphic functions on M. These cohomology classes are called the *secondary invariants* of P.

It may be pointed out that the transversely complex manifolds we define here are the 'transversal' analogue of the foliated spaces of Moore and Schochet [MS88]. The local models considered in [MS88] are $\mathbb{R}^m \times Y$, where Y is an arbitrary topological space, and the transition function between any two charts is of the form $\mathbb{R}^m \times Y \to \mathbb{R}^m \times Y'$ defined by $(x, y) \mapsto (g(x, y), h(y))$, where g is a C^∞ function and h is a continuous function. Another difference is that they study 'tangential' objects, i.e., objects along $\mathbb{R}^m$, whereas in this paper, we study 'transversal' objects, i.e., objects along Y.

Transversely complex manifolds are also related to laminations. For instance, a lamination by Riemann surfaces [Gh99] is a topological space locally homeomorphic to a model space of the type $\mathbb{D} \times T$, where $\mathbb{D}$ is the open unit disc in $\mathbb{C}$ and T is a topological space; the transition function between two such charts is assumed to be of the form $\mathbb{D} \times T \to \mathbb{D} \times T'$ defined by $(z, t) \mapsto (f(z, t), \gamma(t))$, where $f(z, t)$ is holomorphic in z and continuous in t, and $\gamma(t)$ is a continuous function of t (cf. [Gh99], Section 2, p. 50). The local models of a transversely complex manifold are also of the type $U \times T$, where U is open in $\mathbb{C}^n$ and T is a topological space. However, the transition function between two such charts is assumed to be of the form $U \times T \to U \times T'$ defined by $(z, t) \mapsto (f(z), \gamma(z, t))$, where $f(z)$ is holomorphic in z, and $\gamma(z, t)$ is continuous in z and t (Proposition (2.5)).

In Section (2), we establish the basic formalisms of transversely complex manifolds and principal bundles over them. We discuss, in Section (3), holomorphic structures on principal bundles over a transversely complex manifolds. In section (4), we develop the notion of secondary invariants of such principal bundles. In Section (5), we investigate the simplest case of these invariants.

2. Transversely complex manifolds and bundles

Fix a topological space X.

Definition (2.1). A *local model of dimension n* is a pair (X, U), where X is the above topological space, and U is an open subset of $X \times \mathbb{C}^n$. We shall often suppress the space X from the notation, and refer to U itself as a local model.

If (X, U) is a local model, and if x_0 is a point in X, we define $U_{(x_0,\cdot)} = \{y \in \mathbb{C}^n \mid (x_0, y) \in U\}$, and for any point y_0 of $\mathbb{C}^n$ we define $U_{(\cdot,y_0)} = \{x \in X \mid (x, y_0) \in U\}$. If $f : U \to T$ is a function from U to a set T, we define $f_{(x_0,\cdot)} : U_{(x_0,\cdot)} \to T$ by $f_{(x_0,\cdot)}(y) = f(x_0, y)$, and define $f_{(\cdot,y_0)} : U_{(\cdot,y_0)} \to T$ by $f_{(\cdot,y_0)}(x) = f(x, y_0)$.

We say that a continuous function $f\colon U \to \mathbb{C}$ on a local model (X, U) is *transversely holomorphic* if the functions $f_{(x_0,\cdot)}$ and $f_{(\cdot,y_0)}$ are holomorphic on $U_{(x_0,\cdot)}$ and $U_{(\cdot,y_0)}$, respectively, for all $(x_0, y_0) \in U \times \mathbb{C}^n$.

If V is an open subset of U, then (X, V) clearly is also a local model, so the above definition of transversely holomorphic functions applies to V also. We thus obtain the sheaf $\mathcal{O}_{(X,U)}$ of *transversely holomorphic* functions on U. The stalk of $\mathcal{O}_{(X,U)}$ at a point of U is the space of all transversely holomorphic functions defined around the point.

We shall, most of the time, abbreviate the notation $\mathcal{O}_{(X,U)}$ to $\mathcal{O}_U$. The stalk of $\mathcal{O}_{\mathbb{C}^n}$ (the sheaf of holomorphic functions on $\mathbb{C}^n$) at any point $y \in \mathbb{C}^n$ will be denoted by $\mathcal{O}_{\mathbb{C}^n,y}$.

Definition (2.2). If (X, U) and (X', U') are local models, then a continuous map $f\colon U \to U'$ is said to be a *transversely holomorphic map* if $f \circ u \in \mathcal{O}_U(f^{-1}(V))$ whenever V is an open subset of U', and $u \in \mathcal{O}_{U'}(V)$. We say that a continuous map $f\colon U \to U'$ is a *transversely biholomorphic map* if f is a homeomorphism and both the maps f and f^{-1} are transversely holomorphic.

Remark (2.3). We can define differentiable analogues of transversely holomorphic objects as follows. A *real local model* of dimension n is a pair (X, U), where X is a topological space, and U is an open subset of $X \times \mathbb{R}^n$. As before, we have the notion of a *transversely differentiable map*.

The following two Propositions (2.4) and (2.5) are easy consequences of the description of the topological inverse image given above, hence we omit their proofs.

PROPOSITION (2.4). *If (X, U) is a holomorphic local model, then the sheaf of transversely holomorphic functions $\mathcal{O}_U$ on U is canonically isomorphic to the topological inverse image* $\mathrm{pr}_2{}^{-1}\mathcal{O}_{\mathbb{C}^n}$, *where $\mathrm{pr}_2\colon U \to \mathbb{C}^n$ is the restriction of the second projection $X \times \mathbb{C}^n \to \mathbb{C}^n$, and $\mathcal{O}_{\mathbb{C}^n}$ is the sheaf of holomorphic functions on $\mathbb{C}^n$.*

PROPOSITION (2.5). *If (X, U) and (X', U') are local models, then a continuous map $f\colon U \to U'$ is transversely holomorphic if and only if each point $z_0 \in U$ has an open neighborhood $U_0 = U_1 \times U_2 \subset U$, with $U_1 \subset X$ and $U_2 \subset \mathbb{C}^n$, such that the restriction $f|_{U_0}$ is of the form*

$$(x, y) \longmapsto (g(x, y), h(y)), \tag{2.6}$$

where $g\colon U_0 \to X'$ is continuous, $h\colon U_2 \to \mathbb{C}^{n'}$ is holomorphic, and n' is the dimension of the local model (X', U').

Definition (2.7). Let M be a second-countable and metrizable topological space. A *transversely complex chart* of dimension n, or briefly a *chart*, on M is a homeomorphism $\phi\colon V \to U$, where V is an open subset of M and (X, U) is a local model. For notational convenience we shall sometimes denote such a chart by $\phi\colon V \to (X, U)$. Two charts $\phi\colon V \to (X, U)$ and $\phi'\colon V' \to (X', U')$ are said to be *compatible* if the transition function $\phi' \circ \phi^{-1}\colon \phi(V \cap V') \to \phi'(V \cap V')$ is a transversely biholomorphic map from the local model $(X, \phi(V \cap V'))$ to the local model $(X', \phi'(V \cap V'))$. An *atlas* of dimension n on M is a set of pairwise compatible charts $\phi_i\colon V_i \to (X_i, U_i)$ $(i \in I)$, whose domains V_i cover

M. A *transversely complex manifold* of dimension n is a second-countable and metrizable topological space, together with a maximal atlas of dimension n.

Remark (2.8). Every open subset of a transversely complex manifold is again a transversely complex manifold of the same dimension. If M and N are transversely complex manifolds, then their product $M \times N$ has a natural structure of a transversely complex manifold, whose dimension is the sum of the dimensions of M and N. Every complex manifold of dimension n is a transversely complex manifold of dimension n.

We will recall from [GM80] and [GHS83] a couple of examples of transversely complex manifolds; see [GM80], p. 164, Examples 1,2 and [GHS83], p. 131.

Example (2.9). Let M be a complex manifold of dimension $m + n$ equipped with a nonsingular holomorphic foliation $\mathcal{F}$ of complex dimension m. Then M has a natural structure of a transversely complex manifold of dimension n. Indeed, this follows immediately from the fact that the locally defined leaf–spaces for $\mathcal{F}$ have a natural complex structure. Nice applications of this fact that the locally defined leaf–spaces of a holomorphic foliation has a natural complex structure can be found in [LV97], [MV02].

Example (2.10). For the second example, let Z be a complex manifold of dimension n and X a topological space. Let Γ be a group acting freely and properly on $X \times Z$ through homeomorphisms of $X \times Z$. In other words, for any $\gamma \in \Gamma$, the bijection $X \times Z \longrightarrow X \times Z$ given by the action of γ is a homeomorphism. Assume that Γ acts on Z through biholomorphisms. So, for any $\gamma \in \Gamma$, the bijective map $Z \longrightarrow Z$ given by the action of γ on Z is a biholomorphism. The action of Γ on Z need not be free. Assume that the natural projection $X \times Z \to Z$ commutes with the actions of Γ on $X \times Z$ and Z. Then the quotient space $(X \times Z)/\Gamma$ has a natural structure of a transversely complex manifold of dimension n. For instance, X can be a Galois cover of another topological space X' and Γ the Galois group. For any holomorphic action of Γ on a complex manifold Z, the diagonal action of Γ on $X \times Z$ satisfies the above conditions. If we take Γ to be a closed subgroup of a Lie group X with Γ acting holomorphically on Z, then this also satisfies the above conditions.

A continuous function $f : M \to \mathbb{C}$ on a transversely complex manifold is called a *transversely holomorphic function* if, for every chart $\phi : V \to (X, U)$, the function $f \circ \phi^{-1} : U \to \mathbb{C}$ is a transversely holomorphic function on the local model (X, U). With this definition, we obtain the sheaf $\mathcal{O}_M$ of transversely holomorphic functions on M. A continuous map $f : M \to M'$ between transversely complex manifolds is said to be a *transversely holomorphic map* if $u \circ f \in \mathcal{O}_M(f^{-1}(V))$, whenever V is an open subset of M' and $u \in \mathcal{O}_{M'}(V)$.

Remark (2.11). Let M be a transversely complex manifold, and let N be a complex manifold. Therefore, the Cartesian product $M \times N$ is a transversely complex manifold. An explicit description of the transversely complex manifold structure on $M \times N$ is as follows. If $\phi : V \to U$ is a transversely complex chart, where (X, U) is a local model of of M dimension n, and if $\phi' : V' \to U'$ is a complex chart on N, where U' is an open subset of $\mathbb{C}^k$ and $k = \dim_{\mathbb{C}} N$, then

$$\phi \times \phi' : V \times V' \to U \times U' \tag{2.12}$$

is a transversely complex chart on $M \times N$, where we identify $\mathbb{C}^n \times \mathbb{C}^k$ with $\mathbb{C}^{n+k}$ in the obvious way, and consider $(X, U \times U')$ as a local model of dimension $n + k$. As ϕ and ϕ' vary over atlases on M and N, respectively, (2.12) defines a transversely complex structure on $M \times N$.

Consider the right action of G on $M \times G$, defined by $(x, g)h = (x, gh)$, where $x \in M$ and $g, h \in G$. Let

$$\lambda\colon M \times G \longrightarrow M \times G$$

be a G-equivariant transversely holomorphic map, with respect to the transversely complex structure on $M \times G$ as defined in Remark (2.11). Suppose that $\mathrm{pr}_1 \circ \lambda = \mathrm{pr}_1$, where pr_1 denotes the projection to the first factor (the projection to the second factor will be denoted by pr_2). Define two maps $\rho\colon M \times G \to G$ and $\sigma\colon M \to G$ by $\rho = \mathrm{pr}_2 \circ \lambda$ and $\sigma(x) = \rho(x, e)$, where e denotes the identity element of G.

PROPOSITION (2.13). *With the above notation, the map $\sigma\colon M \to G$ is transversely holomorphic. Indeed, every point $x_0 \in M$ has an open neighborhood U_0, and a chart $\phi\colon U_0 \to (X, V_0)$, such that*

1).$V_0 = V_1 \times V_2$, *where V_1 is a connected open subset of X, and V_2 is open in $\mathbb{C}^n$ with n being the dimension of M;*

2). $(\sigma \circ \phi^{-1})_{(\cdot,y)}\colon (V_0)_{(\cdot,y)} \to G$ *is a constant function for all $y \in \mathbb{C}^n$; and*

3). $(\sigma \circ \phi^{-1})_{(x,\cdot)}\colon (V_0)_{(x,\cdot)} \to G$ *is a holomorphic function for all $x \in X$.*

The proof is omitted, because it is a direct consequence of the definitions.

Definition (2.14). Let G be a complex Lie group. We say that a right action of G on a transversely complex manifold P is a *transversely holomorphic action* if the action map $P \times G \to P$ is transversely holomorphic, where the transversely complex structure on the product $P \times G$ is defined as in Remark (2.11). A *transversely holomorphic principal G-bundle* over a transversely complex manifold M is a transversely complex manifold P, together with a transversely holomorphic surjective map $\pi\colon P \to M$, and a transversely holomorphic right action of G on P, satisfying the usual local triviality condition. Namely, for every point x in M, there exist an open neighborhood U of x in M and a G-equivariant transversely biholomorphic map $\lambda\colon \pi^{-1}(U) \to U \times G$, where the action of G on $U \times G$ is defined by $(x, g)h = (x, gh)$, for all $x \in U$, and $g, h \in G$, such that $\mathrm{pr}_1 \circ \lambda = \pi$ on $\pi^{-1}(U)$.

Let $\pi\colon P \to M$ be a transversely holomorphic principal G-bundle over a transversely complex manifold M, and $\pi'\colon P' \to M$ a transversely holomorphic principal G'-bundles on M, where $G' \subset G$ is a complex Lie subgroup. A *transversely holomorphic morphism* from P' to P is a transversely holomorphic G'-equivariant map $f\colon P' \to P$ such that $\pi \circ f = \pi'$.

Let $\mathcal{O}_M(G)$ denote the sheaf of transversely holomorphic maps from a transversely complex manifold M to a complex Lie group G, where the complex manifold G is considered as a transversely complex manifold in the natural manner (see Remark (2.8)). Define the Čech cohomology set $\mathrm{H}^1(M, \mathcal{O}_M(G))$ in the usual manner.

Let $\pi\colon P \to M$ be a transversely holomorphic principal G-bundle over M. Let $\{U_i\}_{i\in I}$ be an open cover of M, such that for each $i \in I$, there exists a G-equivariant transversely biholomorphic map $\lambda_i\colon U_i \to U_i \times G$ as in Definition (2.14). For each pair $(i, j) \in I \times I$, define $\sigma_{ij}\colon U_i \cap U_j \longrightarrow G$ by $\sigma_{ij}(x) = \mathrm{pr}_2 \circ \lambda_i \circ \lambda_j^{-1}(x, e)$, where $e \in G$ is the identity element. By Proposition (2.13), these transition functions σ_{ij} are transversely holomorphic functions. Moreover, they satisfy the standard cocycle conditions, namely, $\sigma_{ij}\sigma_{jk} = \sigma_{ik}$ on $U_i \cap U_j \cap U_k$. Therefore, the family $\{U_i, \sigma_{ij}\}_{i,j\in I}$ defines an element $\theta(P)$ of $\mathrm{H}^1(M, \mathcal{O}_M(G))$. The cohomology class $\theta(P)$ depends only on the (transversely holomorphic) isomorphism class of the G-bundle P, and not on the choice of the family $\{U_i, \sigma_{ij}\}_{i,j\in I}$. The function which assigns to the isomorphism class of a G-bundle P, the cohomology class $\theta(P)$ is a bijection from the set of all isomorphism classes of transversely holomorphic principal G-bundles on M to the set $\mathrm{H}^1(M, \mathcal{O}_M(G))$.

A *transversely holomorphic vector bundle of rank r* on a transversely complex manifold M is a transversely complex manifold E, together with a transversely holomorphic surjective map $\pi\colon E \to M$, satisfying the following conditions:

1). For each point $x \in M$, the fiber $E_x = \pi^{-1}(x)$ is equipped with the structure of a $\mathbb{C}$-vector space of dimension r.

2). For every point $x \in M$, there exist an open neighborhood $U \subset M$ of x and a transversely biholomorphic map $\lambda\colon \pi^{-1}(U) \to U \times \mathbb{C}^r$ such that $\mathrm{pr}_1 \circ \lambda = \pi$ on $\pi^{-1}(U)$, where pr_1 is the projection to the first factor, and such that for each point $y \in U$, the induced map $\phi_y\colon E_y \to \{y\} \times \mathbb{C}^r \cong \mathbb{C}^r$ is an isomorphism of complex vector spaces.

Definition (2.15). Let $\pi\colon E \to M$ and $\pi'\colon E' \to M$ be transversely holomorphic vector bundles on a transversely complex manifold M. A *transversely holomorphic homomorphism* from E to E' is a transversely holomorphic map $f\colon E \to E'$ such that $\pi' \circ f = \pi$, and such that for each point $x \in M$, the induced map $f_x\colon E_x \to E'_x$ is linear. A *transversely holomorphic isomorphism* from E to E' is a transversely holomorphic homomorphism $f\colon E \to E'$ as above such that for each point $x \in M$, the induced map $f_x\colon E_x \to E'_x$ is a linear isomorphism. A *transversely holomorphic section* of a transversely holomorphic vector bundle E over M is a transversely holomorphic homomorphism to E from the trivial transversely holomorphic line bundle $M \times \mathbb{C}$ (the transversely complex structure of the product is defined as in Remark (2.11)). Equivalently, a transversely holomorphic section of E is a transversely holomorphic map $s\colon M \to E$ such that $\pi \circ s$ is the identity map of E.

Let M be a transversely complex manifold of dimension n. Let $\{(U_i, \phi_i)\}_{i\in I}$ be an atlas on M. For each $i \in I$, let $\sigma_i = \mathrm{pr}_2 \circ \phi_i\colon U_i \to \mathbb{C}^n$, and let $T_i = \sigma_i^* T\mathbb{C}^n$ be the pull-back of $T\mathbb{C}^n$, the holomorphic tangent bundle of $\mathbb{C}^n$, by σ_i. Then, the locally defined vector bundles T_i over M glue to form a transversely holomorphic vector bundle of rank n over M, which is called the *tangent bundle* of M, and is denoted by TM.

For any point $x \in M$, the fiber $(TM)_x$ of TM at x is canonically isomorphic to the vector space of all $\mathbb{C}$-derivations from $\mathcal{O}_{M,x}$ to $\mathbb{C}$. We can now, in the usual manner, define transversely holomorphic vector fields, differential forms, etc.

Set Ω^1_M to be sheaf of transversely holomorphic sections of $(TM)^*$, and more generally, for any $p \geq 0$, set Ω^p_M to be sheaf of transversely holomorphic sections of the exterior power $\Lambda^p(TM)^*$. Therefore, we have the sheaf of differential graded algebra $\Omega^1_M := \bigoplus_{p=0}^n \Omega^p_M$ of all transversely holomorphic differential forms, with the differential $\partial \colon \Omega^p_M \to \Omega^{p+1}_M$ (transversely holomorphic sections are defined in Definition (2.15)).

Remark (2.16). Using real local models (Remark (2.3)), we can define — imitating the definitions of transversely complex manifolds and transversely holomorphic bundles — transversely differentiable manifolds, transversely differentiable principal bundles, etc. All the assertions above carry over to this category, with "differentiable" replacing "holomorphic" and with "$\mathbb{R}$" replacing "$\mathbb{C}$" etc.

Remark (2.17). It can be shown, as done in [MS88], Proposition 2.8, that every transversely differentiable manifold M admits transversely differentiable partitions of unity. Therefore, the sheaf $\mathcal{A}^p_M$ of transversely differentiable p-forms on a transversely differentiable manifold M is a fine sheaf, for every integer p.

Example (2.18). If P is a holomorphic principal G-bundle on a complex manifold Z of dimension n, then the pull-back of P to $X \times Z$ is a transversely holomorphic principal G-bundle on the transversely complex manifold $X \times Z$ of dimension n.

Example (2.19). Let M be a complex manifold of dimension $m+n$ equipped with a nonsingular holomorphic foliation $\mathcal{F}$ of complex dimension m, and let $\pi : P \longrightarrow M$ be a holomorphic principal G-bundle over M. Let $\widetilde{\mathcal{F}}$ be a nonsingular holomorphic foliation on the total space of P of complex dimension n such that $\mathrm{d}\pi(\widetilde{\mathcal{F}}) = \pi^*\mathcal{F}$, where

$$\mathrm{d}\pi : T^{1,0}P \longrightarrow \pi^* T^{1,0}M$$

is the differential of the projection π. Further assume that the action of G on the principal G-bundle P preserves the subbundle $\widetilde{\mathcal{F}} \subset T^{1,0}P$. Then P has a natural structure of a transversely holomorphic principal G-bundle over the transversely complex manifold M.

Example (2.20). Take (X, Z, Γ) as in Example (2.9). Let P be a holomorphic principal G-bundle over Z. Assume that the G-bundle P is equipped with a lift of the action of Γ on Z. This means that Γ acts on the total space of P through biholomorphisms, that the actions of G and Γ on P commute, and that the bundle projection from P to Z is Γ-equivariant. Let $\widetilde{P} := p_Z^* P$ be the pullback of P to $X \times Z$ by the obvious projection $p_Z \colon X \times Z \to Z$. Then the quotient $\widetilde{P}/\Gamma$ is a principal G-bundle over $(X \times Z)/\Gamma$. This principal G-bundle has a structure of a transversely holomorphic principal G-bundle over the transversely complex manifold $(X \times Z)/\Gamma$.

3. Holomorphic structures on transversely differentiable bundles

We earlier defined the holomorphic tangent bundle of a transversely complex manifold. Note that the sheaf of sections of the holomorphic tangent bundle of a transversely complex manifold M is identified with the sheaf of derivations of transversely holomorphic functions on M. From now onwards, to distinguish the holomorphic tangent bundle from the real tangent bundle, we will denote the holomorphic tangent bundle of M by $T^{1,0}M$.

The (real) tangent bundle of a transversely differentiable manifold M is defined to be the sheaf of derivations of transversely differentiable real valued functions on M. The real tangent bundle of M will be denoted by TM. The conjugate bundle of $T^{1,0}M$ will be denoted by $\overline{T^{1,0}M}$. We have the decomposition $\Lambda(T_{\mathbb{C}}M) = \bigoplus_{p,q} \Lambda^{p,q}(TM)$, where

$$\Lambda^{p,q}(TM) = \Lambda^p(T^{1,0}M) \otimes \Lambda^q(\overline{T^{1,0}M}).$$

We will denote the spaces of transversely differentiable forms on a transversely complex manifold M by $\mathrm{A}^{p,q}(M)$ and $\mathrm{A}^p(M)$. So $\mathrm{A}^{p,q}(M)$ (respectively, $\mathrm{A}^p(M)$) is the space of all global sections of the transversely differentiable vector bundle $(\Lambda^{p,q}(TM))^*$ (respectively, $(\Lambda^p TM)^*$).

If E is a transversely differentiable vector bundle over M, we will denote by $\mathrm{A}^{p,q}(E)$ the space of all globally defined transversely differentiable homomorphisms from $\Lambda^{p,q}(TM)$ to E. Similarly, define $\mathrm{A}^p(E)$ to be the space of all globally defined transversely differentiable homomorphisms from $\Lambda^p TM$ to E.

Let G be a Lie group, and let $\mathfrak{g}$ denote its Lie algebra. Let $\pi\colon P \to M$ be a transversely differentiable principal G-bundle over M. Then we have the adjoint bundle

$$\mathrm{ad}(P) = (P \times \mathfrak{g})/G,$$

which is the vector bundle associated to P for the adjoint representation $\mathrm{ad}\colon G \to GL(\mathfrak{g})$; in the above quotient, the action of any $g \in G$ sends a point $(z, v) \in P \times \mathfrak{g}$ to $(zg, \mathrm{ad}(g^{-1}(v))$. Note that $\mathrm{ad}(P)$ is a transversely differentiable vector bundle over M.

There is a natural short exact sequence of transversely differentiable vector bundles

$$0 \to \mathrm{ad}(P) \to \pi_*^G TP \to TM \to 0, \tag{3.1}$$

where $\pi_*^G TP = TP/G$ is the vector bundle corresponding to the G-invariant direct image of TP; it is easy to see that $\pi_*^G TP$ is a transversely differentiable vector bundle over M. The construction and properties of this short exact sequence are analogous to its construction and properties in the usual case, i.e., in the category of differentiable principal bundles. See [At57] and [Kos86], Section 5.5 for the details. We shall refer to the above exact sequence as the *Atiyah sequence* of P.

Definition (3.2). Let $\pi\colon P \to M$ be a transversely differentiable principal G-bundle, where G is a Lie group. A *connection* on P is a transversely differentiable splitting

$$\gamma : \pi_*^G TP \longrightarrow \mathrm{ad}(P)$$

of the Atiyah exact sequence of P.

If γ and γ' are connections, then using (3.1) it follows that

$$\omega = \gamma - \gamma'$$

is a well defined transversely differentiable homomorphism from $\pi_*^G TP$ to $\mathrm{ad}(P)$. Moreover, $\omega(X) = 0$ for all $X \in \mathrm{ad}(P)$. Therefore, ω induces a transversely differentiable homomorphism from TM to $\mathrm{ad}(P)$, or in other words, ω is a transversely differentiable 1-form on M with values in $\mathrm{ad}(P)$. We will denote this form also by $\gamma - \gamma'$. The space of all connections on P is, thus, an affine space modeled after the vector space $\mathrm{A}^1(\mathrm{ad}(P))$.

Let G be a complex Lie group, and let $\pi\colon P \to M$ be a transversely differentiable principal G-bundle on a transversely complex manifold M. The space of $\mathrm{ad}(P)$-valued forms $\mathrm{A}^{p,q}(\mathrm{ad}(P))$ was defined above.

Definition (3.3). Let G be a complex Lie group, and let $\pi\colon P \to M$ be a transversely differentiable principal G-bundle on a transversely complex manifold M. Let us say that connections γ and γ' on P are *equivalent* if

$$\gamma - \gamma' \in \mathrm{A}^{1,0}(\mathrm{ad}(P))\,.$$

This defines an equivalence relation on the space of all connections on P. We will denote the equivalence class of γ by γ itself. An *almost holomorphic structure* on P is an equivalence class, with respect to the above relation, of connections on P.

The space of all almost holomorphic structures on P is an affine space modeled after the vector space $\mathrm{A}^{0,1}(\mathrm{ad}(P))$. Indeed, for any two almost holomorphic structures on P γ and γ', the difference $\gamma' - \gamma$ is an element of $\mathrm{A}^{0,1}(\mathrm{ad}(P))$, and conversely, for any $\theta \in \mathrm{A}^{0,1}(\mathrm{ad}(P))$ and any almost holomorphic structure γ on P

$$\gamma' := \gamma + \theta$$

is again an almost holomorphic structures on P.

Let $\pi\colon P \to M$ be a transversely differentiable principal G-bundle, where G is a complex Lie group. If γ is a connection on P, we will define its curvature, in analogy with the usual case, as follows. The transversely differentiable homomorphism of vector bundles on M, $\gamma\colon \pi_*^G TP \to \mathrm{ad}(P)$, gives rise to a transversely differentiable 1-form on P with values in $\mathfrak{g}$. We will denote this $\mathfrak{g}$-valued 1-form also by γ. By definition, the *curvature* is the 2-form on P with values in $\mathfrak{g}$, given by

$$\mathrm{K}(\gamma) \;=\; \mathrm{d}\,\gamma + [\gamma, \gamma]\,.$$

The form $\mathrm{K}(\gamma)$ descends to an $\mathrm{ad}(P)$-valued 2-form $\mathrm{K}(\gamma)' \in \mathrm{A}^2(\mathrm{ad}(P))$ on M. When there is no risk of confusion, we will denote the 2-form $\mathrm{K}(\gamma)'$ also by $\mathrm{K}(\gamma)$.

Definition (3.4). Let G be a complex Lie group, and let $\pi\colon P \to M$ be a transversely differentiable principal G-bundle on a transversely complex manifold M. Let γ be an almost holomorphic structure on P by a connection γ' on P. Consider its curvature $\mathrm{K}(\gamma') \in \mathrm{A}^2(\mathrm{ad}(P))$ of γ'. We say that the almost holomorphic structure γ is *integrable* if the $(0, 2)$-component, $\mathrm{K}(\gamma')^{0,2} \in \mathrm{A}^{0,2}(\mathrm{ad}(P))$, of $\mathrm{K}(\gamma')$ is zero. This condition on γ is independent of the choice of the representative γ' of the equivalence class of connections defined by γ. An integrable almost holomorphic structure is also known as a *holomorphic structure*.

Let M be a transversely differentiable manifold, and let E be a transversely differentiable real vector bundle on M of even rank. An *almost complex structure on* E is a transversely differentiable homomorphism of vector bundles, $J\colon E \to E$, such that

$$J^2 = -\mathbf{1}_E .$$

An *almost complex structure on* M is, by definition, an almost complex structure on the real tangent bundle TM. A *transversely differentiable almost complex manifold* is a transversely differentiable manifold together with a transversely differentiable almost complex structure on it.

On every transversely differentiable almost complex manifold (M, J), we have a decomposition $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$, where $T^{1,0}M$ is the $\sqrt{-1}$-eigenspace of $J \otimes \mathbf{1}_{\mathbb{C}}$, and $T^{0,1}M = \overline{T^{1,0}M}$. This induces a decomposition $\Gamma(TM \otimes \mathbb{C}) = \Gamma(T^{1,0}M) \oplus \Gamma(T^{0,1}M)$ of the vector space of transversely differentiable complex vector fields on M.

We say that a transversely differentiable almost complex structure $J\colon TM \to TM$ on a transversely differentiable manifold M is *integrable* if $\Gamma(\Lambda^{1,0}(TM))$ is a Lie subalgebra of $\Gamma(TM \otimes_{\mathbb{R}} \mathbb{C})$, where the Lie bracket of two transversely differentiable (complex) vector fields on M is defined in the same way as in the usual case. An integrable almost complex structure is also known as a *complex structure*.

Every transversely complex manifold M carries a natural transversely differentiable almost complex structure, which is integrable. Conversely, using the Newlander-Nirenberg Theorem [NN57], we can prove the following result.

PROPOSITION (3.5). *Let M be a transversely differentiable manifold, and let*

$$J : TM \longrightarrow TM$$

be a transversely differentiable almost complex structure on M. Suppose that J is integrable. Then, there exists a unique structure of a transversely complex manifold on M, that induces the almost complex structure J.

Let $\pi\colon P \to M$ be a transversely differentiable principal G-bundle, where G is a complex Lie group, and M is a transversely complex manifold. Let γ be a connection on P. Then γ induces a G-equivariant splitting of the short exact sequence of transversely differentiable G-vector bundles

$$0 \to \mathcal{O}_P \otimes_{\mathbb{R}} \mathfrak{g} \to TP \to \pi^*TM \to 0. \tag{3.6}$$

We will denote this G-equivariant splitting also by γ. Since G is a complex Lie group, the vector bundle $\mathcal{O}_P \otimes_{\mathbb{R}} \mathfrak{g}$ carries a transversely differentiable almost complex structure. Since M is a transversely complex manifold, the pull-back π^*TM also carries a transversely differentiable almost complex structure. Therefore, using the splitting γ of the Atiyah exact sequence (3.6), we can define a transversely differentiable almost complex structure $J_\gamma\colon TP \to TP$ on P.

Let $\pi\colon P \to M$ be a transversely differentiable principal G-bundle, where G is a complex Lie group, and M is a transversely complex manifold. Let γ be a connection on P. Then the transversely differentiable almost complex structure

$$J_\gamma : TP \longrightarrow TP$$

in the transversely differentiable real vector bundle TP is called the *almost complex structure induced by* γ.

We now have, as in [Kos86], Section 6.4, Propositions 2 and 3, the following result. The proof is an exact analogue of the proof in that reference.

PROPOSITION (3.7). *Let* $\pi\colon P \to M$ *be a transversely differentiable principal* G*-bundle, where* G *is a complex Lie group, and* M *is a transversely complex manifold. Let* γ *be a connection on* P*. Then, the following are true.*

1).*If* γ' *is a connection that is equivalent to* γ *in the sense of Definition (3.3), then the induced almost complex structures* $J_\gamma\colon TP \to TP$ *and* $J_{\gamma'}\colon TP \to TP$ *are equal.*

2). *The almost complex structure* J_γ *on* P *is a complex structure if and only if the almost holomorphic structure* γ *is a holomorphic structure in the sense of Definition (3.4).*

We thus see that almost holomorphic (respectively, holomorphic) structures on a transversely differentiable principal G-bundle P correspond bijectively to almost complex (respectively, complex) structures on P.

Let E be a transversely differentiable complex vector bundle over a transversely differentiable manifold M. A *connection* in E is a $\mathbb{C}$-linear map

$$\nabla:\ \mathrm{A}^0(E) \longrightarrow \mathrm{A}^1(E),$$

which satisfies the Leibniz identity:

$$\nabla(fs) = \mathrm{d} f \otimes s + f\nabla s \tag{3.8}$$

for all $f \in \mathrm{A}^0(\mathcal{O}_M)$ and $s \in \mathrm{A}^0(E)$.

Definition (3.9). Let E be a transversely differentiable complex vector bundle over a transversely complex manifold M. An *almost holomorphic structure* in E is a $\mathbb{C}$-linear map $D\colon \mathrm{A}^0(E) \to \mathrm{A}^{0,1}(E)$, which satisfies the Leibniz identity

$$D(fs) = \bar\partial f \otimes s + fDs \tag{3.10}$$

for all $f \in \mathrm{A}^0(\mathcal{O}_M)$ and $s \in \mathrm{A}^0(E)$. Given an almost holomorphic structure in E, we can extend it naturally to a $\mathbb{C}$-linear map $D^{(p,q)}\colon \mathrm{A}^{p,q}(E) \to \mathrm{A}^{p,q+1}(E)$, satisfying the generalized Leibniz identity

$$D^{(p+p',q+q')}(\omega \wedge \tau) = \bar\partial\omega \otimes \tau + \omega \wedge D\tau \tag{3.11}$$

for all $\omega \in \mathrm{A}^{p,q}(M)$ and $\tau \in \mathrm{A}^{p,q}(M)$. We will, as usual, drop the superscripts from $D^{p,q}$, and denote it by just D. We will say that the almost holomorphic structure D in E is *integrable* if $D^2 := D \circ D = 0$. A *holomorphic structure* in E is an integrable almost holomorphic structure.

4. Secondary invariants

Fix a nonnegative integer k. Let G be a complex Lie group with Lie algebra $\mathfrak{g}$, and let B be an G-invariant symmetric form on $\mathfrak{g}$ of degree $k+1$. In other words,

$$B \in (\mathrm{Sym}^{k+1}\mathfrak{g}^*)^G, \tag{4.1}$$

with G acting on $\mathrm{Sym}^{k+1}\mathfrak{g}^*$ through the adjoint action of G on $\mathfrak{g}$.

Let $\pi\colon P \to M$ be a transversely differentiable principal G-bundle over a transversely complex manifold M. Let $\rho\colon G \to GL(V)$ be a representation of G on a finite dimensional complex vector space V. Proceeding as in the usual case, one defines the associated transversely differentiable vector bundle $E_\rho = (P \times V)/G$. As usual, we identify sections of E_ρ with transversely differentiable functions from P to V satisfying an automorphy condition.

Note that the G-invariant $(k+1)$-form B on $\mathfrak{g}$ defines a smooth $(k+1)$-form

$$\mathrm{Sym}^{k+1}(\mathrm{ad}(P)) \longrightarrow \mathcal{A}_M^{\mathbb{C}}$$

on the adjoint bundle $\mathrm{ad}(P)$ of any smooth transversely differentiable principal P over M, where $\mathcal{A}_M^{\mathbb{C}}$ is the sheaf of transversely differentiable complex valued functions on M. For notational convenience, this form on $\mathrm{ad}(P)$ will also be denoted by B.

We say that a transversely differentiable vector field X on P is *projectable* if there exists a transversely differentiable vector field Y on M, such that $\mathrm{d}\,\pi_x(X_x) = Y_{\pi(x)}$ for all $x \in P$, where $\mathrm{d}\,\pi$ is the differential of the projection π from P to M.

PROPOSITION (4.2). *Let notation be as above. If γ is a connection on P, then there exists a unique connection ∇ in E_ρ such that, for every projectable vector field X on P, we have*

$$\nabla_X(\sigma) = X\sigma + (\rho(\gamma(X)))(\sigma) - \sigma \quad \textit{for all } \sigma \in \mathrm{A}^0(E_\rho). \tag{4.3}$$

Moreover, the $(0,1)$*-part, D, of the connection ∇ depends only on γ. The operator D is an almost holomorphic structure on E_ρ. It is integrable if and only if γ is integrable.*

The proof of this result is analogous to that of [Kos86], Section 5.6, Theorem 3.

Let γ_0 and γ_1 be two holomorphic structures on P, and let $\omega = \gamma_1 - \gamma_0$. Note that

$$\omega \in \mathrm{A}^{0,1}(\mathrm{ad}(P)).$$

Define

$$\gamma_t = \gamma_0 + t\omega$$

for all $t \in [0,1]$. Note that γ_t is an almost complex structures on P (almost complex structures form an affine space). The almost complex structure γ_t need not be integrable if $0 < t < 1$, so let

$$\Theta_t = \gamma_t \circ \gamma_t \in \mathrm{A}^{0,2}(\mathrm{ad}(P))$$

be the obstruction to the integrability of the Dolbeault operator γ_t on P, where $t \in [0,1]$.

Let $\overline{\partial}_t$ denote the almost holomorphic structure in $\mathrm{ad}(P)$ that is induced by γ_t following Proposition (4.2). Then, we note that the standard Bianchi identity says that $\overline{\partial}_t(\Theta_t) = 0$ for all t. Define

$$D_k(\gamma_0, \gamma_1) = \int_0^1 B(\omega \wedge \Theta_t^k)\,\mathrm{d}\,t, \tag{4.4}$$

where B is the earlier defined $(k+1)$-form on the adjoint vector bundle $\mathrm{ad}(P)$ (obtained from the G-invariant form on $\mathfrak{g}$). To explain the integral, note that

$B(\omega\wedge\Theta_t^k)$ is a transversely differentiable $(0, 2k+1)$-form on M for each $t \in [0\,, 1]$. So for each point $x \in M$, we have

$$\int_0^1 B(\omega \wedge \Theta_t^k)(x)\,\mathrm{d}\,t \;\in\; \Lambda^{2k+2}(T_x^{0,1}M)^* \,.$$

The integral $D_k(\gamma_0, \gamma_1)$ is a transversely differentiable $(0, 2k+1)$-form on M, and

$$D_k(\gamma_0, \gamma_1)(x) \;=\; \int_0^1 B(\omega \wedge \Theta_t^k)(x)\,\mathrm{d}\,t \tag{4.5}$$

for each point $x \in M$.

LEMMA (4.6). *The form $D_k(\gamma_0, \gamma_1)$ is $\overline{\partial}$-closed.*

Proof. For any $t \in [0\,1]$, consider the transversely differentiable $(0, 2k+2)$-form $B(\Theta_t^{k+1})$ on M. So the integral

$$\widehat{D}_k(\gamma_0, \gamma_1) \;:=\; \int_0^1 B(\Theta_t^{k+1})\,\mathrm{d}\,t$$

is a transversely differentiable $(0, 2k+2)$-form on M. Using the Bianchi identity it is straight-forward to check that

$$\overline{\partial} D_k(\gamma_0, \gamma_1) \;=\; \frac{1}{k+1}\widehat{D}_k(\gamma_0, \gamma_1)$$

(see [Ch95], p. 114, Lemma 3.1, for a very similar computation). This completes the proof of the lemma. □

Let $(\mathcal{A}_M^{\cdot,\cdot}, \mathrm{d})$ be the Dolbeault complex of transversely differentiable complex-valued forms on M; so $A_M^{p,q}$ is the space of all global sections of $\mathcal{A}_M^{p,q}$. The complex

$$0 \to \mathcal{O}_M \to \mathcal{A}_M^{0,0} \xrightarrow{\overline{\partial}} \mathcal{A}_M^{0,1} \xrightarrow{\overline{\partial}} \cdots \tag{4.7}$$

is an acyclic resolution (see Remark (2.17)) of $\mathcal{O}_M$, the sheaf of transversely holomorphic functions on M. Therefore, the $\overline{\partial}$-closed $(0, 2k+1)$-form $D_k(\gamma_0, \gamma_1)$ defines an element of $\mathrm{H}^{2k+1}(M, \mathcal{O}_M)$. We will denote this cohomology class by $D_k(\gamma_0, \gamma_1)$.

Example (4.8). Suppose that E is a transversely differentiable vector bundle of rank r over a transversely complex manifold M. By considering frames as usual, E gives rise to a transversely differentiable principal G-bundle on M, where $G = GL(r, \mathbb{C})$. There is a natural linear form $B \in (\mathfrak{g}^*)^G = (\mathrm{Sym}^1\, \mathfrak{g}^*)^G$, namely the trace form $B(g) = \mathrm{trace}(g)$. Using this linear form B in the above procedure, we get for every pair of holomorphic structures γ_0 and γ_1 on E, a cohomology class $D_0(\gamma_0, \gamma_1) \in \mathrm{H}^1(M, \mathcal{O}_M)$. In particular, if $L = \Lambda^r(E)$ is the determinant line bundle of E, we get a class $D_0(\gamma_0^L, \gamma_1^L) \in \mathrm{H}^1(M, \mathcal{O}_M)$, where γ_i^L is the holomorphic structure on L induced by the holomorphic structure γ_i on E. We use this notion in Lemma (5.1).

Let $\mathrm{Dol}(P)$ denote the set of almost holomorphic structures on the G-bundle P. We have seen that it is an affine space modeled after $\mathrm{A}^{0,1}(\mathrm{ad}(P))$. There is a natural topology on $\mathrm{A}^{0,1}(\mathrm{ad}(P))$ that makes it a Fréchet space. Therefore,

$\mathrm{Dol}(P)$ becomes a topological affine space in a natural way. Let $\mathrm{Hol}(P)$ denote the space of all holomorphic structures on P. So $\mathrm{Hol}(P)$ is a subspace of $\mathrm{Dol}(P)$.

THEOREM (4.9). *As in Lemma (4.6), take two holomorphic structures γ_0 and γ_1 on P. Let*

$$f : [0,1] \longrightarrow \mathrm{Dol}(P)$$

be a smooth map such that $f(0) = \gamma_0$ and $f(1) = \gamma_1$. Define

$$D_f(\gamma_0, \gamma_1) := \int_0^1 B(\omega \wedge (f(t) \circ f(t))^k)\, \mathrm{d}\, t \in \mathrm{A}^{0,2k+1}(M),$$

where ω as before is $\gamma_1 - \gamma_0$, and $f(t) \circ f(t)$ is the obstruction to integrability of the Dolbeault operator $f(t)$ on P. Then the form $D_f(\gamma_0, \gamma_1)$ is $\overline{\partial}$-closed, and the cohomology class in $H^{2k+1}(M, \mathcal{O}_M)$ represented by $D_f(\gamma_0, \gamma_1)$ coincides with the cohomology class $D_k(\gamma_0, \gamma_1)$ constructed earlier.

Proof. Let

$$F : [0,1] \times [0,1] \longrightarrow \mathrm{Dol}(P)$$

be the smooth map defined by

$$F(s,t) := (1-s)(\gamma_0 + t\omega) + sf(t).$$

Therefore, $F(s, 0) = \gamma_0$ and $F(s, 1) = \gamma_1$ for all $s \in [0, 1]$; similarly, we have $F(0,t) = \gamma_t$ and $F(1,t) = f(t)$.

Using this function F, there is a Dolbeault operator on the principal G-bundle q^*P over $M \times [0,1] \times [0,1]$, where

$$q\colon M \times [0,1] \times [0,1] \longrightarrow [0,1] \times [0,1]$$

is the natural projection. The Dolbeault operator on q^*P is uniquely determined by the following two conditions:

1. for any point $x \in M$, the Dolbeault operator on $(q^*P)|_{\{x\}\times[0,1]\times[0,1]}$ coincides with the one given by the natural trivialization of $(q^*P)|_{\{x\}\times[0,1]\times[0,1]}$ (since (q^*P is pull-back from M, any trivialization of the fiber P_x gives a trivialization of the G-bundle $(q^*P)|_{\{x\}\times[0,1]\times[0,1]}$), and
2. for any point $(s,t) \in [0,1] \times [0,1]$, the Dolbeault operator on $(q^*P)|_{M\times\{s\}\times\{t\}}$ coincides with Dolbeault operator $F(s,t)$ on P.

Let D denote the Dolbeault operator on q^*P constructed above. Let

$$D^2 \in \mathrm{A}^{0,2}(q^* \mathrm{ad}(P)) \tag{4.10}$$

be the obstruction to integrability of the Dolbeault operator D.

Consider the infinite dimensional vector space

$$\mathcal{V} := \mathrm{A}^{0,2k+1}(\mathrm{ad}(P)), \tag{4.11}$$

the space of all $\mathrm{ad}(P)$-valued transversely differentiable forms of type $(0, 2k+1)$ on M. There is a natural $\mathcal{V}$-valued $(0,1)$-form on $[0,1] \times [0,1]$ which will be constructed below.

Take any point $(s,t) \in [0,1] \times [0,1]$, and also take any point $x \in M$. Now we have a smooth function on $[0,1] \times [0,1]$ with values in $(T^{0,1}_x M)^* \otimes \mathrm{ad}(P)_x$ that sends any point $(s',t') \in [0,1] \times [0,1]$ to $(F(s',t') - F(s,t))(x)$; recall that

$$F(s',t') - F(s,t) \in \mathrm{A}^{0,1}(\mathrm{ad}(P)),$$

and hence $(F(s',t') - F(s,t))(x) \in (T^{0,1}_x M)^* \otimes \mathrm{ad}(P)_x$. This $(T^{0,1}_x M)^* \otimes \mathrm{ad}(P)_x$-valued smooth function on $[0,1] \times [0,1]$ will be denoted by $\phi_{s,t,x}$. Now take any tangent vector $v \in T^{0,1}_{(s,t)}([0,1] \times [0,1])$, and set

$$\psi(s,t,x)(v) := B(v(\phi_{s,t,x})(D^2(x,s,t))^k) \in (T^{0,2k+1}_x M)^*, \tag{4.12}$$

where $v(\phi_{s,t,x})$ is the derivation of the function $\phi_{s,t,x}$ in the direction v. Let ψ denote the $\mathcal{V}$-valued $(0,1)$-form on $[0,1] \times [0,1]$ which is defined as follows: for any point

$$(s,t) \in [0,1] \times [0,1]$$

and any tangent vector $v \in T^{0,1}_{(s,t)}([0,1] \times [0,1])$, define

$$\psi(v)(x) := \psi(s,t,x)(v), \tag{4.13}$$

where $\psi(s,t,x)(v)$ is constructed in (4.12).

Since $F(s,0) = \gamma_0$, $F(s,1) = \gamma_1$, $F(0,s) = \gamma_s$ and $F(1,s) = f(s)$ for all $s \in [0,1]$, we conclude that

$$\int_{\partial([0,1]\times[0,1])} \psi = D_f(\gamma_0,\gamma_1) - D_k(\gamma_0,\gamma_1), \tag{4.14}$$

where $D_f(\gamma_0,\gamma_1)$ is defined in the statement of the theorem and $D_k(\gamma_0,\gamma_1)$ is defined in (4.4), and $\partial([0,1] \times [0,1])$ denotes the oriented boundary of $[0,1] \times [0,1]$ (the boundary has the anti-clockwise orientation). Using Stokes' theorem, from (4.14) we conclude that

$$D_f(\gamma_0,\gamma_1) - D_k(\gamma_0,\gamma_1) = \int_{[0,1]\times[0,1]} \mathrm{d}\,\psi\,. \tag{4.15}$$

In view of (4.15), to prove the theorem it suffices to show that the $(0,2k+1)$-form $\int_{[0,1]\times[0,1]} \mathrm{d}\,\psi$ is $\bar\partial$-exact, or in other words, there is a form $\alpha \in \mathrm{A}^{0,2k}(M)$ such that

$$\bar\partial\alpha = \int_{[0,1]\times[0,1]} \mathrm{d}\,\psi\,.$$

Set

$$\mathcal{V}' := \mathrm{A}^{0,2k}(\mathrm{ad}(P)), \tag{4.16}$$

the space of all $\mathrm{ad}(P)$-valued transversely differentiable forms of type $(0,2k)$ on M. To prove that $\int_{[0,1]\times[0,1]} \mathrm{d}\,\psi$ is $\bar\partial$-exact we will construct below a $\mathcal{V}'$-valued $(0,2)$-form on $[0,1] \times [0,1]$.

Take any point $(s,t) \in [0,1] \times [0,1]$, and also take any point $x \in M$. We earlier constructed the $(T^{0,1}_x M)^* \otimes \mathrm{ad}(P)_x$-valued function $\phi_{s,t,x}$ on $[0,1] \times [0,1]$. Now take any two tangent vectors

$$v, w \in T^{0,1}_{(s,t)}([0,1] \times [0,1]),$$

and set

$$\beta(s,t,x)(v,w) := B(v(\phi_{s,t,x})w(\phi_{s,t,x})(D^2(x,s,t))^{k-1}) \in (T^{0,2k}_x M)^*,$$

where $w(\phi_{s,t,x})$, as in (4.12), is the derivation of the function $\phi_{s,t,x}$ in the direction w. Let β denote the $\mathcal{V}'$-valued $(0,2)$-form on $[0,1] \times [0,1]$ which is defined as follows: for any point $(s,t) \in [0,1] \times [0,1]$ and any ordered pair of tangent vectors $v, w \in T^{0,1}_{(s,t)}([0,1] \times [0,1])$, define

$$\beta(v,w)(x) := \beta(s,t,x)(v,w), \tag{4.17}$$

where $\beta(s,t,x)(v,w)$ is constructed above.

It is a straight-forward computation to see that the form ψ in (4.13) coincides with the Künneth component of type $(2k+1,1)$ of $B((D^2)^{k+1})$, where $D^2 \in \mathrm{A}^{0,2}(q^* \operatorname{ad}(P))$ is the obstruction of integrability in (4.10); by Künneth component of type $(2k+1,1)$ of a differential form of degree $2k+2$ on $M \times [0,1] \times [0,1]$ we mean the Künneth component which is a combination of a form of degree $2k+1$ on M and a form of degree one on $[0,1] \times [0,1]$.

Consider $\mathcal{V}$ and $\mathcal{V}'$ defined in (4.11) and (4.16) respectively. Let

$$\nu : \mathcal{V}' \longrightarrow \mathcal{V}$$

be the homomorphism defined by $\omega \longmapsto \bar{\partial}\omega$. Using ν, if θ is a $\mathcal{V}'$-valued differential form of degree c, then $\nu(\theta)$ is a $\mathcal{V}$-valued differential form of degree c in a natural way.

Using the Bianchi identity for the Dolbeault operator D (see (4.10)) it is a straight-forward computation that

$$\nu(\beta) = \frac{1}{k}\, \mathrm{d}\,\psi, \tag{4.18}$$

where β is defined in (4.17).

Since the image of the homomorphism ν is the space of $\bar{\partial}$-exact forms, from (4.18) it follows immediately that the integral $\int_{[0,1]\times[0,1]} \mathrm{d}\,\psi$ is a $\bar{\partial}$-exact form. Finally, from (4.15) we conclude that the form $D_f(\gamma_0, \gamma_1)$ is $\bar{\partial}$-closed (as $D_k(\gamma_0, \gamma_1)$ is $\bar{\partial}$-closed), and the cohomology classes represented by $D_f(\gamma_0, \gamma_1)$ and $D_k(\gamma_0, \gamma_1)$ coincide. This completes the proof of the theorem. $\square$

The above theorem has the following corollary:

COROLLARY (4.19). *Assume that $k \geq 1$. Let γ_1 and γ_2 be holomorphic structures on P, which lie in the same path component of* $\mathrm{Hol}(P)$. *Then the cohomology classes in $\mathrm{H}^{2k+1}(M, \mathcal{O}_M)$ represented by $D_k(\gamma_0, \gamma_1)$ and $D_k(\gamma_0, \gamma_2)$ coincide.*

By Theorem (4.9), to compute $D_k(\gamma_0, \gamma_2)$ we can use a path connecting γ_2 with γ_0 which is a composition of two segments of the following type: one segment connects γ_1 with γ_0 and the other segment lies in $\mathrm{Hol}(P)$ connecting γ_2 with γ_1. Since the integral vanishes identically on the second path, the cohomology classes represented by $D_k(\gamma_0, \gamma_1)$ and $D_k(\gamma_0, \gamma_2)$ coincide.

5. Some examples

In this section, we specialize the structure group G in Section (4) to $GL(r, \mathbb{C})$. Thus, the Lie algebra $\mathfrak{g}$ is the matrix algebra $M_r(\mathbb{C})$. We take the the symmetric form B in (4.1) to be the trace form, that sends any matrix $A \in M_r(\mathbb{C})$ to $\operatorname{trace}(A) \in \mathbb{C}$.

Let E be a transversely differentiable vector bundle of rank r over a transversely complex manifold M. Let $L = \det(E) := \bigwedge^r E$ be the determinant line bundle corresponding to E. Any holomorphic structure on E induces a holomorphic structure on L.

The following lemma shows that for $k = 0$, to compute the invariants in Theorem (4.9) for vector bundles, it is enough to compute it for line bundles.

LEMMA (5.1). *Take two holomorphic structures γ_0 and γ_1 on E. Let γ_0^L (respectively, γ_1^L) be the holomorphic structure on the determinant line bundle $L := \bigwedge^r E$ induced by γ_0 (respectively, γ_1). Let*

$$D_0(\gamma_0, \gamma_1) \in H^1(M, \mathcal{O}_M)$$

and

$$D_0(\gamma_0^L, \gamma_1^L) \in H^1(M, \mathcal{O}_M)$$

be the cohomology classes defined in Example (4.8). Then

$$D_0(\gamma_0, \gamma_1) = D_0(\gamma_0^L, \gamma_1^L).$$

Proof. Take any $\theta \in \mathrm{A}^{0,1}(\mathrm{End}(E))$. Let

$$\mathrm{trace}(\theta) \in \mathrm{A}^{0,1}(\mathcal{O}_M)$$

be the form obtained by taking the trace of θ. The almost holomorphic structures $\gamma_\theta := \gamma_0 + \theta$ on E has the following property: the almost complex structure on the determinant line bundle $L := \det(E)$ induced by γ_θ coincides with the almost complex structure $\gamma_0^L + \mathrm{trace}(\theta)$. Using this observation the lemma follows. □

In view of Lemma (5.1), we may restrict ourselves to line bundles to investigate the case of $k = 0$. In Lemma (4.6), set $G = GL(1, \mathbb{C}) = \mathbb{C}^*$, and also set B to be the natural identification of the Lie algebra of $\mathbb{C}^*$ with $\mathbb{C}$.

PROPOSITION (5.2). *Let $\theta \in H^1(M, \mathcal{O}_M)$ be any Dolbeault cohomology class. Let γ_0 be the trivial holomorphic structure on the trivial differentiable line bundle $M \times \mathbb{C}$ over M. There is a holomorphic structure γ on the trivial differentiable line bundle $M \times \mathbb{C}$ on M such that*

$$D_0(\gamma_0, \gamma_1) = \theta,$$

where $D_0(\gamma_0, \gamma_1)$ is the Dolbeault cohomology class constructed in Lemma (4.6).

Proof. Let $\widetilde{\theta} \in \mathrm{A}^{0,1}(\mathcal{O}_M)$ be a $\overline{\partial}$-closed $(0, 1)$-form representing the Dolbeault cohomology class θ. Let

$$\gamma_1 := \gamma_0 + \widetilde{\theta}$$

be the holomorphic structure on the trivial differentiable line bundle $M \times \mathbb{C}$ over M, where γ_0 is the trivial holomorphic structure. Note that as the form $\widetilde{\theta}$ is closed, the almost complex structure on $M \times \mathbb{C}$ given by γ_1 is integrable. Consider the path of holomorphic structures on L defined by $t \longmapsto \gamma_0 + t \cdot \widetilde{\theta}$. For this path, the form in (4.5) evidently coincides with $\widetilde{\theta}$. This completes the proof of the proposition. □

Remark (5.3). If M is a compact Kähler manifold with $H^1(M, \mathbb{Q}) \neq 0$, then we have $H^1(M, \mathcal{O}_M) \neq 0$. Therefore, if we take the transversely complex manifold $X \times M$, where X is any topological space and M a compact Kähler manifold with $H^1(M, \mathbb{Q}) \neq 0$, then for the transversely complex manifold $X \times M$ we have

$$H^1(X \times M, \mathcal{O}_M) \neq 0\,,$$

and furthermore, $H^1(M, \mathbb{Q})$ sits inside $H^1(X \times M, \mathcal{O}_M)$. Therefore, Proposition (5.2) provides examples where the invariant in Theorem (4.9) does not vanish.

We have a more precise formulation of the relationship between the holomorphic structures on line bundles and the invariant in Theorem (4.9).

LEMMA (5.4). *Let L be a transversely differentiable line bundle over a transversely complex manifold M. Take two holomorphic structures γ_0 and γ_1 on L. Let*

$$D_0(\gamma_0\,, \gamma_1) \in H^1(M, \mathcal{O}_M)$$

be the cohomology class defined ed in Example (4.8). Then the following two are equivalent:

1. The two holomorphic line bundles $(L\,, \gamma_0)$ and $(L\,, \gamma_1)$ are holomorphically isomorphic.

2. The Dolbeault cohomology class $D_0(\gamma_0\,, \gamma_1)$ is represented by the $(0\,, 1)$-part of a closed one-form $\theta \in \mathrm{A}^1(M \times \mathbb{C})$ such that the periods of θ are integers.

Proof. Let θ be a closed one-form as above with integral periods such that the Dolbeault cohomology class $D_0(\gamma_0\,, \gamma_1)$ is represented by the $(0\,, 1)$-part of θ. We will show that the two holomorphic line bundles $(L\,, \gamma_0)$ and $(L\,, \gamma_1)$ are holomorphically isomorphic.

Let f be the multi-valued function on M obtained by integrating θ along oriented paths starting from a base point in M. Since the periods of θ are integers,

$$g \,:=\, \exp(2\pi\sqrt{-1}f)$$

is a single-valued smooth function on M which is nowhere vanishing. Now from the construction of $D_0(\gamma_0\,, \gamma_1)$ it follows that the smooth automorphism of L given by the pointwise multiplication with the function g gives an isomorphism between the two holomorphic line bundles $(L\,, \gamma_0)$ and $(L\,, \gamma_1)$.

To prove the converse, assume that the two holomorphic line bundles $(L\,, \gamma_0)$ and $(L\,, \gamma_1)$ are isomorphic. Let g denote a nowhere zero smooth function on M such that the smooth automorphism of L given by the pointwise multiplication with the function g gives an isomorphism between the two holomorphic line bundles $(L\,, \gamma_0)$ and $(L\,, \gamma_1)$. Therefore, the $(0\,, 1)$-form

$$\theta \,:=\, \overline{\partial} \log g \,\in\, \mathrm{A}^{0,1}(\mathcal{O}_M)$$

is well-defined. Considering the path of holomorphic structures on L defined by $t \longmapsto \gamma_0 + t\theta$ we conclude that the Dolbeault cohomology class $D_0(\gamma_0\,, \gamma_1)$ is represented by the $(0\,, 1)$-part of a closed one-form with integral periods. This completes the proof of the lemma. □

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ALGEBRAIC TEST FOR THE HURWITZ STABILITY OF A GIVEN SEGMENT OF POLYNOMIALS

BALTAZAR AGUIRRE AND RODOLFO SUÁREZ

ABSTRACT. For the robust stability analysis of a linear system, due to the nonconvexity of the set of Hurwitz stable polynomials, it is important to have available computational methods to verify the stability of a convex combination of polynomials. In this paper, given two Hurwitz stable polynomials p_0 and p_1, a simple algebraic test (a matrix inequality) for the stability of the segment of polynomials determined by p_0 and p_1 is proposed. Based on this result the problem of estimating of the minimum left extreme is addressed.

1. Introduction

Motivated by the robustness analysis of systems with uncertain parameters, different approaches to study the stability of segments of polynomials have been proposed ([4], [5], [8], [9], [16]). The question is to find conditions on the stable polynomials $p_0(t)$ and $p_1(t)$ such that the segment of polynomials described by $p(t, \lambda) = \lambda p_0(t) + (1-\lambda)p_1(t)$ is stable for all $\lambda \in [0, 1]$. The first result where necessary and sufficient conditions were obtained was Bialas's Theorem which establishes that if p_0 is Hurwitz stable and $\deg(p_0) > \deg(p_1)$ then $p(t, \lambda)$ is Hurwitz stable for all $\lambda \in [0, 1]$ if and only if the matrix $H^{-1}(p_0)H(p_1)$ has no eigenvalues in $(-\infty, 0)$, where $H(p)$ is the Hurwitz matrix of the polynomial p (see [2], [4] and [11]). A different approach in terms of the frequency domain which is known as the Segment Lemma was established by Chapellat and Bhattacharyya (see [3] and [9]). In this lemma the stability of $p(t, \lambda)$ is equivalent to certain conditions that must be satisfied by the odd and even degree polynomials associated with the polynomials $p_0(t)$ and $p_1(t)$. On the other hand, a method to determine the stability of segments of complex polynomials was obtained by N. Bose and is known as Bose's Test [6].

Based on the above criteria, several algorithms have been developed to test efficiently the stability of segments of polynomials. The Segment Lemma has been used to develop an algorithm in [8]. In the same direction, more recently, in [14] there was obtained a procedure to check the Hurwitz stability of convex combinations of polynomials in a finite number of operations. Related to Bose's work [7], in [5] there is a test that can be used to determine the stability of segments of complex polynomials. Furthermore, in [16] there were obtained the well-known Rantzer conditions (see also [13]).

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Following the ideas exposed in [1] this work address the problem of obtaining simple algebraic conditions for checking the stability of a segment of polynomials. It is important to note that the approach proposed in this paper provides sufficient conditions used when $\deg p_0 = n$ and $\deg p_1 = n, n-1, n-2$ in contrast to the Segment Lemma where it is supposed that $\deg p_0 = \deg p_1$. As can be seen in [13], it is not necessary to study the cases when $\deg(p_1(t)) < n-2$.

Our approach for the case $\deg p_0 = \deg p_1$ is as follows: Given a Hurwitz stable polynomial $p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ which is the nominal polynomial, let $p_1(t) = c_1 t^n + c_2 t^{n-1} + \cdots + c_{n+1}$ be an arbitrary polynomial of degree n. Define the matrix $E_{(n,n)} \in \mathcal{M}_{(n+1)\times(n+1)}$ by

$$E_{(n,n)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -a_2 & a_1 & -1 & 0 & \dots & 0 & 0 \\ a_4 & -a_3 & a_2 & -a_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & -a_{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & a_n \end{pmatrix}. \tag{1.1}$$

If the polynomials $p_0(t)$ and $p_1(t)$ are Hurwitz stable and the vector $c = (c_1, c_2, \dots, c_{n+1})^T \succeq 0$ satisfies the system of linear inequalities

$$E_{(n,n)} c \gneqq 0, \tag{1.2}$$

then the convex combination $\lambda p_0(t) + (1-\lambda) p_1(t)$ is Hurwitz stable for every $\lambda \in [0, 1]$. Here the symbol $\succeq 0$ ($\preceq 0$) means that every component of a given vector is nonnegative (nonpositive) and the symbol $\gneqq 0$ means that every component of a given vector is nonnegative but there is at least one positive component.

A similar result can be obtained for the case $\deg(p_1(t)) = n-1$. In this case the matrix $E_{(n,n-1)} \in \mathcal{M}_{n\times n}$ is defined by

$$E_{(n,n-1)} = \begin{pmatrix} a_1 & -1 & 0 & 0 & \dots & 0 & 0 \\ -a_3 & a_2 & -a_1 & 1 & \dots & 0 & 0 \\ a_5 & -a_4 & a_3 & -a_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & -a_{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & a_n \end{pmatrix} \tag{1.3}$$

and the corresponding inequality is

$$E_{(n,n-1)} c \gneqq 0. \tag{1.4}$$

We also we study the situation when only one Hurwitz polynomial, say $p_0(t)$, is known and the problem is to find all possible $p_1(t)$ such that $\lambda p_0(t) + (1-\lambda) p_1(t)$ is Hurwitz for every $\lambda \in [0, 1]$.

Finally, we use the same approach to estimate the minimum left extreme of a stable segment, that is, given the Hurwitz stable polynomials $p_0(t)$ and $p_1(t)$ such that the vector of coefficients of p_1 satisfies (1.2) or (1.4) then we find a number $k_0 < 0$ such that $p_0(t) + k p_1(t)$ is Hurwitz stable for every $k > k_0$. The problem of calculating the minimum left extreme was solved by Bialas [4]. Although k_0 is only an estimate of $k_{\min}$, the novelty of our approach is that k_0 is obtained by a simple algebraic calculation. Contrary to stability of segments where a good deal of work has been reported about the minimum left extreme we can only mention Bialas' work, hence Section 5 might be interesting.

The paper is organized as follows: in Section 2 sufficient conditions assuring that a segment of polynomials consists of Hurwitz stable polynomials are given when it is known that the extremes $p_0(t)$ and $p_1(t)$ are Hurwitz stable. In Section 3 we compare our approach with other known sufficient conditions and two computational methods. In Section 4 we suppose that $p_0(t)$ is Hurwitz stable and we see that the matrix inequality (1.2) is a sufficient condition on the vector of coefficients of $p_1(t) = c_1t^n + c_2t^{n-1} + \cdots + c_{n+1}$ to establish that $[p_0, p_1]$ is a segment of Hurwitz polynomials and we characterize the solution set of (1.2). Finally, in Section 5 the minimum left extreme of a stable segment is estimated.

2. Hurwitz Stable segments

The aim of this section is to obtain conditions for the stability of segments of polynomials. The main results are based on the following lemma where sufficient conditions are given for a real polynomial to be Hurwitz stable.

LEMMA (2.1). *Let $F(t)$ and $f(t)$ be real polynomials of degree n, such that $f(t)$ has positive coefficients, $f(0) \neq 0$ and the roots of $F(t)$ are contained in $\mathbb{C}^+$. Consider the polynomial of degree $2n$ given by $F(t)f(t)$. If $F(i\omega)f(i\omega) \neq 0$ and $F(i\omega)f(i\omega)$ does not intersect $\mathcal{L}$ for all $\omega > 0$, where $\mathcal{L}$ is a straight line in the complex plane that passes through the origin, then all the roots of $f(t)$ are in $\mathbb{C}^-$.*

Proof. Suppose n is even (the odd case is analogous). Let $n = 2m$ and let $F(t)$, $f(t)$ be given by

$$F(t) = b_0t^{2m} + b_1t^{2m-1} + \cdots + b_{2m}, \quad f(t) = d_0t^{2m} + d_1t^{2m-1} + \cdots + d_{2m}\,.$$

Without loss of generality we may suppose that $b_0 > 0$, and then $b_{2m} > 0$ also since the roots of $F(t)$ are in $\mathbb{C}^+$. Let l and r be the number of roots of $F(t)f(t)$ contained in $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. Let $\theta(\omega)$ be the argument of $F(i\omega)f(i\omega)$. Denote by $\Delta_0^\infty\theta(\omega) = \theta(\infty) - \theta(0)$ the net change in the argument. Since $F(t)f(t)$ does not have roots on the imaginary axis we get that $\Delta_0^\infty\theta(\omega) = \frac{\pi}{2}(l - r)$ ([15], p. 406; [12], p. 174). The fact that $F(i\omega)f(i\omega)$ does not intersect $\mathcal{L}$ for $\omega > 0$ implies $|\Delta_0^\infty\theta(\omega)| \leq \pi$.

Now we will analyze $\theta(\omega) - \theta(0)$ when ω is large. First, we have that for large ω, $F(i\omega)f(i\omega) \approx b_0d_0\omega^{4m} - i[b_1d_0 + b_0d_1]\omega^{4m-1}$. Therefore $\operatorname{Re}\left[F(i\omega)f(i\omega)\right] > 0$ and $\frac{\operatorname{Im}[F(i\omega)f(i\omega)]}{\operatorname{Re}[F(i\omega)f(i\omega)]} \to 0$ when $\omega \to \infty$. Since $F(0)f(0) = b_{2m}d_{2m} > 0$ it follows that $\Delta_0^\infty\theta(\omega) = \theta(\infty) - \theta(0) = 2s\pi$, where s is an integer. Since $F(i\omega)f(i\omega)$ does not intersect $\mathcal{L}$ for $\omega > 0$ then $|\Delta_0^\infty\theta_1(\omega)| \leq \pi$, and therefore we get that $\Delta_0^\infty\theta(\omega) = 0$.

Consequently, the polynomial $F(t)f(t)$ has as many roots in $\mathbb{C}^-$ as in $\mathbb{C}^+$. Since such a polynomial has degree $2n$, there are n roots in $\mathbb{C}^+$. In fact the roots in $\mathbb{C}^+$ correspond to the roots of $F(t)$. Hence, the n roots in $\mathbb{C}^-$ correspond to the roots of $f(t)$, which means that $f(t)$ is Hurwitz stable. $\square$

Remark (2.2). Particular cases of Lemma (2.1) are the situations in which $\mathcal{L}$ is the real or the imaginary axis. When $\mathcal{L}$ is one of the axis, the associated matrices are easy to calculate. Our main results are based on these two cases.

In the following theorem we apply Lemma (2.1) when $\mathcal{L}$ is the imaginary axis.

THEOREM (2.3). *Consider the Hurwitz stable polynomials* $p_0(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ *and* $p_1(t) = c_1 t^n + c_2 t^{n-1} + \cdots + c_{n+1}$. *If* $c = (c_1, c_2, \ldots, c_{n+1})^T \succeq 0$ *is a solution to (1.2), then, for all* $\lambda \in [0,1]$, *the polynomial* $\lambda p_0(t) + (1-\lambda)p_1(t)$ *is Hurwitz stable.*

Proof. Suppose n is even (the odd case is analogous). Let $n = 2m$ and $\lambda \in [0,1]$. Let p, q, P, Q denote the polynomials

$$\begin{array}{rcl} p(L) & = & c_{2m+1} - c_{2m-1}L + c_{2m-3}L^2 + \ldots + (-1)^m c_1 L^m, \\ q(L) & = & c_{2m} - c_{2m-2}L + \ldots + (-1)^{m-1} c_2 L^{m-1}, \\ P(L) & = & a_{2m} - a_{2(m-1)}L + \ldots + (-1)^{m-1} a_2 L^{m-1} + (-1)^m L^m, \\ Q(L) & = & a_{2m-1} - a_{2m-3}L + \ldots + (-1)^{m-1} a_1 L^{m-1}. \end{array} \tag{2.4}$$

Then it holds that

$$\begin{array}{rcl} [\lambda p_0 + (1-\lambda)p_1](i\omega) & = & [\lambda P + (1-\lambda)p](\omega^2) + i\omega[\lambda Q + (1-\lambda)q](\omega^2), \\ p_0(i\omega) & = & P(\omega^2) + i\omega Q(\omega^2). \end{array}$$

Consider the polynomial $p_0(-t)\left[\lambda p_0(t) + (1-\lambda)p_1(t)\right]$. Thus we get

$$\begin{array}{rcl} p_0(-i\omega)[\lambda p_0 + (1-\lambda)p_1](i\omega) & = & P(\omega^2)\left[\lambda P(\omega^2) + (1-\lambda)p(\omega^2)\right] + \\ & & +\omega^2 Q(\omega^2)\left[\lambda Q(\omega^2) + (1-\lambda)q(\omega^2)\right] + \\ & & +i\omega(1-\lambda)\left[P(\omega^2)q(\omega^2) - Q(\omega^2)p(\omega^2)\right] \end{array}$$

That is,

$$\begin{array}{rcl} p_0(-i\omega)[\lambda p_0 + (1-\lambda)p_1](i\omega) & = & \lambda\left[P^2(\omega^2) + \omega^2 Q^2(\omega^2)\right] + \\ & & +(1-\lambda)\left[P(\omega^2)p(\omega^2) + \omega^2 Q(\omega^2)q(\omega^2)\right] + \\ & & +i\omega(1-\lambda)\left[P(\omega^2)q(\omega^2) - Q(\omega^2)p(\omega^2)\right]. \end{array} \tag{2.5}$$

Since $P(\omega^2)p(\omega^2) + \omega^2 Q(\omega^2)q(\omega^2) = \sum_{i=1}^{n+1}(E^i_{(n,n)}c)\omega^{2(n+1-i)}$ and the vector $c \succeq 0$ is a solution to the system of the linear inequalities (1.2), the polynomial $P(\omega^2)p(\omega^2) + \omega^2 Q(\omega^2)q(\omega^2)$ does not have positive roots. Consequently, for all $\omega > 0$, $p_0(-i\omega)[\lambda p_0 + (1-\lambda)p_1](i\omega)$ does not intersect the imaginary axis. Finally, since $p_0(-t)$ and $\lambda p_0(t) + (1-\lambda)p_1(t)$ satisfy the hypothesis of Lemma (2.1) we have that the polynomial $\lambda p_0(t) + (1-\lambda)p_1(t)$ is Hurwitz stable for all $\lambda \in [0,1]$. □

Remark (2.6). Theorem (2.12) can be extended to the case when $\deg p_1(t) = n-1$. To prove this result we need to redefine the polynomials $p(L)$ and $q(L)$ by

$$\begin{array}{rcl} p(L) & = & c_{2m} - c_{2(m-1)}L + \ldots + (-1)^{m-1} c_2 L^{m-1}, \\ q(L) & = & c_{2m-1} - c_{2m-3}L + \ldots + (-1)^{m-1} c_1 L^{m-1}, \end{array}$$

and the proof follows the same steps as the proof of Theorem (2.3).

On the other hand, using the same method, Theorem (2.3) cannot be extended to the case when $\deg(p_1(t)) = n-2$ since the corresponding matrix $E_{(n,n-2)}$ in $\mathcal{M}_{n\times(n-1)}$ is given by

$$E_{(n,n-2)} = \begin{pmatrix} -1 & 0 & 0 & 0 & \dots & 0 & 0 \\ a_2 & -a_1 & 1 & 0 & \dots & 0 & 0 \\ -a_4 & a_3 & -a_2 & a_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{n-1} & -a_{n-2} \\ 0 & 0 & 0 & 0 & \dots & 0 & a_n \end{pmatrix} \tag{2.7}$$

but the first inequality implies that $-c_1 \geq 0$ which is not satisfied since $c_1 > 0$.

Remark (2.8). In [1] we obtain a condition like (1.2) for the stability of rays of polynomials. There is an obvious relation between stable rays and stable segments of polynomials: if $p_0(t) + kg(t)$ is a Hurwitz stable polynomial for every $k \geq 0$ then $\left(\frac{1}{1+k}\right) p_0(t) + \left(\frac{k}{1+k}\right) g(t)$ is a Hurwitz stable polynomial for every $k \geq 0$, which means that the stability of the ray $p_0(t)+kg(t)$ is equivalent to the stability of the open segment $\left[p_0(t), g(t)\right)$. Observe that for g(t) Hurwitz stable we get the stability of the closed segment $[p_0(t), g(t)]$.

In the proof of Theorem (2.3), when we analyze the complex function $p_0(-i\omega)$ $[\lambda p_0 + (1-\lambda)p_1](i\omega)$ defined in (2.5), the straight line $\mathcal{L}$ was the imaginary axis. A different possibility is to consider $\mathcal{L}$ as the real axis. Such an analysis was done in [1] and the results were given in terms of a similar inequality $Dc \gneqq 0$ given by the following matrices:

$$D_{(n,n)} = \begin{pmatrix} a_1 & -1 & 0 & 0 & \dots & 0 & 0 \\ -a_3 & a_2 & -a_1 & 1 & \dots & 0 & 0 \\ a_5 & -a_4 & a_3 & -a_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & -a_{n-2} & a_{n-3} \\ 0 & 0 & 0 & 0 & \dots & a_n & -a_{n-1} \end{pmatrix} \tag{2.9}$$

for the case $\deg(p_1(t)) = n$, while

$$D_{(n,n-1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ -a_2 & a_1 & -1 & 0 & \dots & 0 & 0 \\ a_4 & -a_3 & a_2 & -a_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{n-2} & -a_{n-3} \\ 0 & 0 & 0 & 0 & \dots & -a_n & a_{n-1} \end{pmatrix} \tag{2.10}$$

for the case $\deg(p_1(t)) = n-1$, and for $\deg(p_1(t)) = n-2$, the matrix $D_{(n,n-2)} \in \mathcal{M}_{(n-1)x(n-1)}$ is

$$D_{(n,n-2)} = \begin{pmatrix} a_1 & -1 & 0 & 0 & \dots & 0 & 0 \\ -a_3 & a_2 & -a_1 & 1 & \dots & 0 & 0 \\ a_5 & -a_4 & a_3 & -a_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a_{n-2} & -a_{n-3} \\ 0 & 0 & 0 & 0 & \dots & -a_n & a_{n-1} \end{pmatrix}. \tag{2.11}$$

Rewriting the results in [1] for segments of polynomials instead of rays, we obtain the next result.

THEOREM (2.12). *Consider the Hurwitz stable polynomial* $p_0(t) = t^n + a_1t^{n-1} + \cdots + a_n$. *If* $p_1(t)$ *is a Hurwitz stable polynomial with* $\deg(p_1(t)) = n, n-1$, *or* $n-2$, *and its vector of coefficients* c *satisfies the system of linear inequalities*

$$Dc \gneqq 0 \tag{2.13}$$

where the matrix D *depending on* $\deg(p_1(t))$ *is one of the matrices* $D_{(n,n)}$, $D_{(n,n-1)}$ *or* $D_{(n,n-2)}$, *then the polynomial* $\lambda p_0(t) + (1-\lambda)p_1(t)$ *is Hurwitz stable for all* $\lambda \in [0, 1]$.

3. Comparison with other methods

In this section we present the qualities of our approach comparing it with other known methods.

(3.1) Comparison with the Method in Aguirre *et al.* There are segments of stable polynomials such that the stability can be verified using the approach introduced here, but it is not possible to check such stability with the test given in [1].

Example (3.1.1). Consider the Hurwitz stable polynomial $p_0(t) = t^3 + 2t^2 + t + 1$. The vector of coefficients c of the polynomial $p_1(t) = t^3 + 8t^2 + 13t + 1$ is a solution to the system of linear inequalities (1.2):

$$E_{(3,3)}c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 8 \\ 13 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}.$$

Consequently the segment $[p_0, p_1]$ is stable. However, c is not a solution to (2.13) since

$$D_{(3,3)}c = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 8 \\ 13 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -18 \\ 12 \end{pmatrix}.$$

(3.2) Comparison with the Rantzer-type conditions. Here we compare our approach with the known Rantzer-type conditions. Such conditions are explained in [13] and they are the following: Suppose that p_0 is a Hurwitz polynomial and p_1 is a semistable[1] polynomial. Then the ray of polynomials $p_0(t) + kp_1(t)$ consists of Hurwitz polynomials if one of the following four conditions holds:

(i) The difference $d = p_1 - p_0$ satisfies

$$\frac{\partial \arg(d(i\omega))}{\partial \omega} < 0, \quad \omega \in \{w > 0 / d(iw) \neq 0\}.$$

[1]A polynomial is semistable if the real parts of its roots are not positive.

(ii) Each of the polynomials p_0, p_1 has at least one root in the open left half-plane and

$$\frac{\partial \arg(d(i\omega))}{\partial \omega} < \left| \frac{\sin\left(2\arg[d(i\omega)]\right)}{2\omega} \right|, \qquad \omega \in \{w > 0/d(iw) \neq 0\}.$$

(iii) Each of the polynomials p_0, p_1 has at least one root in the open left half-plane and

$$\frac{\partial \arg(d(i\omega))}{\partial \omega} \leq 0, \qquad \omega \in \{w > 0/d(iw) \neq 0\}.$$

(iv) Each of the polynomials p_0, p_1 has at least two roots in the open left half-plane and

$$\frac{\partial \arg(d(i\omega))}{\partial \omega} \leq \left| \frac{\sin\left(2\arg[d(i\omega)]\right)}{2\omega} \right|, \qquad \omega \in \{w > 0/d(iw) \neq 0\}.$$

Although the Rantzer-type conditions offer four options to check the stability of segments of polynomials, they can not cover all the possibilities, as is illustrated by the following example.

Example (3.2.1). Consider the Hurwitz stable polynomial $p_0(t) = t^3 + 2t^2 + t + 1$. The vector of coefficients c of the polynomial $p_1(t) = t^3 + 7t^2 + 12t + 2$ is a solution to the system of linear inequalities (1.2):

$$E_{(3,3)}c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 7 \\ 12 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

Therefore the segment $[p_0, p_1]$ is stable. Furthermore, we will see that this example does not satisfy the Rantzer-type conditions (see [13], [16]).

For this example $p_0(t)$ and $p_1(t)$ are Hurwitz polynomials, and $d(t)$, $d(iw)$ and $\arg(d(i\omega))$ are given by $d(t) = (p_1 - p_0)(t) = 5t^2 + 11t + 1$, $d(i\omega) = 1 - 5\omega^2 + i11\omega$, $\arg(d(i\omega)) = \arctan\left(\dfrac{11\omega}{1-5\omega^2}\right)$.
It is not difficult to verify that $i) - iv)$ are not satisfied:

1) Since $\dfrac{\partial \arg(d(i\omega))}{\partial \omega} = \dfrac{11 + 55\omega^2}{\left(1-5\omega^2\right)^2 + 121\omega^2} > 0$ for all $\omega \in \{w > 0/d(i\omega) \neq 0\} = (0, \infty)$, $i)$ is not satisfied.

2) $\sin\left(2\arg[d(i\omega)]\right) = \dfrac{2\omega\left(11 - 55\omega^2\right)}{\left(1-5\omega^2\right)^2 + 121\omega^2}$, hence

$$\frac{\partial \arg(d(i\omega))}{\partial \omega} < \left| \frac{\sin\left(2\arg[d(i\omega)]\right)}{2\omega} \right|$$

is satisfied if and only if

$$\frac{11 + 55\omega^2}{\left(1-5\omega^2\right)^2 + 121\omega^2} < \frac{\left|11 - 55\omega^2\right|}{\left(1-5\omega^2\right)^2 + 121\omega^2}.$$

If $\omega = 1$ we have that $\frac{66}{137} < \frac{44}{137}$, which is a contradiction. Consequently *ii*) is not satisfied.

3) From the above inequalities it is immediate that *iii*) and *iv*) are not satisfied either.

Consequently, although the segment $\left[p_0(t), p_1(t)\right]$ consists of Hurwitz polynomials, it is not possible to verify that using the Rantzer-type conditions obtained in [13].

Remark (3.2.2). With respect to the method in [1] and the Rantzer-type conditions we believe that the main contribution of our new approach is that it can be applied to cases where the others do not succeed. However this does not mean that our approach subsumes the other methods and in a given segment our method could fail and some of the other methods could work.

(3.3) Comparison with the Algorithm of Hwang-Yang. Now we compare our approach with the computational method given in [14].

Example (3.3.1). Consider the polynomials $p_0(t) = t^5 + 6t^4 + 14\,t^3 + 16t^2 + 9t + 2$ and $p_1(t) = 2.16t^5 + 6.47t^4 + 8.58t^3 + 6.57t^2 + 3.38t + 1.08$. The polynomial $p_0(t)$ is Hurwitz stable and the vector of coefficients c of the polynomial $p_1(t)$ is a solution to the system of linear inequalities (1.2) since

$$E_{(5,5)}c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -14 & 6 & -1 & 0 & 0 & 0 \\ 9 & -16 & 14 & -6 & 1 & 0 \\ 0 & 2 & -9 & 16 & -14 & 6 \\ 0 & 0 & 0 & -2 & 9 & -16 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2.16 \\ 6.47 \\ 8.58 \\ 6.57 \\ 3.38 \\ 1.08 \end{pmatrix} = \begin{pmatrix} 2.16 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2.16 \end{pmatrix}.$$

Then we can conclude that the segment with extremes $p_0(t)$ and $p_1(t)$ consists of Hurwitz polynomials.

On the other hand, if we apply the approach given in [14] we begin with the following calculations:

$$\begin{aligned} p_0 + \lambda[p_1 - p_0] = {} & [1 + 1.16\lambda]t^5 + [6 + 0.47\lambda]t^4 + [14 - 5.42\lambda]t^3 \\ & + [16 - 9.43\lambda]t^2 + [9 - 5.62\lambda]t + 2 - 0.92\lambda, \end{aligned}$$

$$\begin{aligned}
a_{0,0}(\lambda) &= a_0(\lambda) = 2 - 0.92\lambda \\
a_{0,1}(\lambda) &= a_2(\lambda) = 16 - 9.43\lambda \\
a_{0,2}(\lambda) &= a_4(\lambda) = 6 + 0.47\lambda \\
a_{1,0}(\lambda) &= a_1(\lambda) = 9 - 5.62\lambda \\
a_{1,1}(\lambda) &= a_3(\lambda) = 14 - 5.42\lambda \\
a_{1,2}(\lambda) &= a_5(\lambda) = 1 + 1.16\lambda \\
a_{2,0}(\lambda) &= 116 - 151.07\lambda + 48.01\lambda^2 \\
a_{2,1}(\lambda) &= 52 - 30.89\lambda - 1.5742\lambda^2 \\
a_{3,0}(\lambda) &= 1156 - 2173.5\lambda + 1331.5\lambda^2 - 269.06\lambda^3 \\
a_{3,1}(\lambda) &= 116 - 16.51\lambda - 127.23\lambda^2 + 55.692\lambda^3 \\
a_{4,0}(\lambda) &= 5184 - 11128.5\lambda + 8745.6\lambda^2 - 3048.7\lambda^3 + 400.39\lambda^4
\end{aligned}$$

To finish the algorithm one must check that $a_{4,0}(\lambda) > 0$ for every $\lambda \in [0, 1]$.

Remark (3.3.2). In general, to apply the algorithm of Hwang-Yang one must calculate the all $a_{j,k}$'s. Next one must check whether $a_{n-1,0}(\lambda) > 0$ for $\lambda \in [0, 1]$. That is, this algorithm reduces the problem of determining the stability of a segment of polynomials to checking the positivity of a polynomial, which is usually verified by using Sturm sequences. But note that the number of calculations increases with the degree of the extremes of the segment of polynomials since the degree of $a_{n-1,0}(\lambda)$ is $n-1$.

(3.4) Comparison with the Algorithm of Bouguerra *et al.* Now we compare the calculations of our approach with the algorithm given in [8].

Example (3.4.1). Consider the Hurwitz polynomial $p_0(t) = t^6 + 7t^5 + 20t^4 + 30t^3 + 25t^2 + 11t + 2$ and let $p_1(t) = 15t^6 + 58t^5 + 100t^4 + 100t^3 + 65t^2 + 29t + 7.5$ be a polynomial. The vector of coefficients c of the polynomial $p_1(t)$ is a solution to the system of linear inequalities (1.2) since

$$E_{(6,6)}c = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -20 & 7 & -1 & 0 & 0 & 0 & 0 \\ 25 & -30 & 20 & -7 & 1 & 0 & 0 \\ -2 & 11 & -25 & 30 & -20 & 7 & -1 \\ 0 & 0 & 2 & -11 & 25 & -30 & 20 \\ 0 & 0 & 0 & 0 & -2 & 11 & -25 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 15 \\ 58 \\ 100 \\ 100 \\ 65 \\ 29 \\ 7.5 \end{pmatrix} = \begin{pmatrix} 15 \\ 6 \\ 0 \\ 3.5 \\ 5.0 \\ 1.5 \\ 15.0 \end{pmatrix}.$$

Consequently the segment $[p_0, p_1]$ is stable.

On the other hand, if we apply the algorithm posed in [8] first we have to determine the polynomials

$\widehat{a}(x) = 2 - 25x + 20x^2 - x^3$

$\widehat{b}(x) = 11 - 30x + 7x^2$

$\widehat{c}(x) = 7.5 - 65x + 100x^2 - 15x^3$

$\widehat{d}(x) = 29 - 100x + 58x^2$

Next we must make the following calculations:

1) Find the positive real roots of $\widehat{a}(x)$, $\widehat{b}(x)$, $\widehat{c}(x)$ and $\widehat{d}(x)$.

2) From these positive roots, one looks for intervals where both $\widehat{a}(x)\widehat{c}(x)$ and $\widehat{b}(x)\widehat{d}(x)$ are negative.

3) If such intervals exist, one needs to check for the existence of positive real roots of $\widehat{a}(x)\widehat{d}(x) - \widehat{b}(x)\widehat{c}(x) = 0$ inside these intervals. If $\widehat{a}(x)\widehat{d}(x) - \widehat{b}(x)\widehat{c}(x) = 0$ admits roots inside these intervals, then the segment is unstable, and stable otherwise.

Remark (3.4.2). In this algorithm one uses Sturm sequences as well. Observe that if the degrees of p_0 and p_1 are increased, the degrees of $\widehat{a}(x)$, $\widehat{b}(x)$, $\widehat{c}(x)$, $\widehat{d}(x)$, $\widehat{a}(x)\widehat{c}(x)$, $\widehat{b}(x)\widehat{d}(x)$ and $\widehat{a}(x)\widehat{d}(x) - \widehat{b}(x)\widehat{c}(x)$ are also increased.

Hence if the degrees of p_0 and p_1 are large, the application of 1), 2) and 3) requires costly effort.

Remark (3.4.3). It is natural that the algorithms of Hwang-Yang and Bouguerra require more work that our condition since they are based on necessary and sufficient conditions and consequently can check both situations: stable or unstable segments.

4. Stability of a segment when only a extreme is given

Now we study a different problem: a Hurwitz polynomial $p_0(t)$ is given and we ask whether there exist polynomials $p_1(t)$ such that $\left[p_0(t), p_1(t)\right]$ is a segment of Hurwitz polynomials.

Remark (4.1). Let $\mathcal{H}_n$ denote the set of Hurwitz stable polynomials of degree n. If the vector of coefficients of the polynomial $p_1(t) = c_1t^{n-1} + c_2t^{n-1} + \cdots + c_n$ is a solution to the system of linear inequalities $E_{(n,n-1)}c \succneqq 0$, then it can be proved that the segment of polynomials $[p_0(t), p_1(t))$ is Hurwitz stable. Observe that $p_1 \notin \mathcal{H}_n$ since $\deg(p_1) = n - 1$. However it is clear that $p_1(t)$ is on the boundary of $\mathcal{H}_n$.

Remark (4.2). Consider the Hurwitz polynomial $p_0(t) = t^n + a_1t^{n-1} + \cdots + a_n$. Let $E_{(n,n)}$ be the corresponding matrix defined by (1.1). If $p_1(t)$ is given by $p_1(t) = \sum_{i=1}^{n+1} c_it^{n+1-i}$ and the vector $c = (c_1, c_2, \ldots, c_{n+1})^T \succeq 0$ is a solution to the system of linear inequalities (1.2), then following a idea similar to that of Theorem (2.3) it can showed that $\left[p_0(t), p_1(t)\right]$ is a segment of Hurwitz polynomials. Contrary to Remark (4.1) we have in this case that $p_1 \in \mathcal{H}_n$ since $\deg(p_1) = n$. But the question is whether there exists any polynomial $p_1(t)$ that fulfills this property. In the following subsection we work on this problem. First we describe an example.

In the following example we present a segment of Hurwitz stable polynomials $\left[p_0(t), p_1(t)\right]$ such that the vector of coefficients of $p_1(t)$ does not satisfy the linear inequalities (1.2) and (2.13). This example proves that conditions (1.2) and (2.13) are only sufficient.

Example (4.3). Consider the Hurwitz stable polynomials $p_0(t) = t^3 + 2t^2 + t + 1$ and $p_1(t) = t^3 + \frac{5}{2}t^2 + 2t + \frac{19}{4}$. First, observe that the linear matrix inequality (1.2) is not satisfied:

$$E_{(3,3)}c = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{5}{2} \\ 2 \\ \frac{19}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -10 \\ \frac{19}{4} \end{pmatrix}.$$

On the other hand, the segment $\left[p_0(t), p_1(t)\right]$ is a segment of Hurwitz stable polynomials since the Routh-Hurwitz conditions $(\frac{5}{2} - \frac{1}{2}\lambda), (2 - \lambda), \frac{19}{4} - \frac{15}{4}\lambda > 0$ and $(\frac{5}{2} - \frac{1}{2}\lambda)(2-\lambda)t - \frac{19}{4} + \frac{15}{4}\lambda = \frac{1}{2}\lambda^2 + \frac{1}{4}\lambda + \frac{1}{4} > 0$ associated with the polynomial $\lambda p_0(t) + (1 - \lambda)p_1(t) = t^3 + (\frac{5}{2} - \frac{1}{2}\lambda)t^2 + (2 - \lambda)t + \frac{19}{4} - \frac{15}{4}\lambda$ are satisfied for all $\lambda \in [0, 1]$.

Furthermore, this example does not satisfy the condition (2.13) either, since

$$D_{(3,3)}c = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{5}{2} \\ 2 \\ \frac{19}{4} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{9}{4} \\ -\frac{11}{4} \end{pmatrix}.$$

This example and the Remark (4.2) illustrate the relevance of studying the problem of characterizing the solution set of (1.2), which will be addressed in the next subsection.

(4.1) Characterization of the set of solutions of (1.2). In this subsection, given the Hurwitz stable polynomial $p_0(t)$, we will find the polynomials $p_1(t)$ whose vector of coefficients satisfies the linear inequalities (1.2).

As was proved in [1] for the matrix $D_{(n,n-1)}$, it can be seen that the matrix $E_{(n,n)}$ is of monotone kind (*i.e.*, $E_{(n,n)}z \succeq 0$ implies $z \succeq 0$), which implies that it is invertible and $E^{-1}_{(n,n)} \succeq 0$, where $E^{-1}_{(n,n)} \succeq 0$ means that all its entries are nonnegative (see [10]). Denote by $V = \{z \in \mathbb{R}^{n+1}/\{0\} \mid z_i \geq 0, \forall i = 1, 2, \ldots, n+1\}$. The following result characterizes the solution set of (1.2).

THEOREM (4.1.1). *The set H of solutions of the system of linear inequalities (1.2) can be written as $H = E^{-1}_{(n,n)}V$.*

Proof. First we prove that $H \subseteq E^{-1}_{(n,n)}V$. Let $u \in H$, then $u \succeq 0$ and $E_{(n,n)}u \succeq 0$. Consequently, $u = E^{-1}_{(n,n)}E_{(n,n)}u$ with $E_{(n,n)}u \in V$. That is, $u \in E^{-1}_{(n,n)}V$.

Now, we prove $H \supseteq E^{-1}_{(n,n)}V$. Let $u \in E^{-1}_{(n,n)}V$, then $u = E^{-1}_{(n,n)}v$ with $v \succeq 0$ and $v \neq 0$. Hence, $E_{(n,n)}u = v \succeq 0$ and $E^{i}_{(n,n)}u > 0$ for some row $E^{i}_{(n,n)}$ with $1 \leq i \leq n$, that is, $u \in H$. $\square$

COROLLARY (4.1.2). *Let $p_0(t) = t^n + a_1t^{n-1} + \cdots + a_n$ be a Hurwitz stable polynomial. Let $E_{(n,n)}$ be the corresponding matrix defined by (1.1). If the vector $c = (c_1, c_2, \ldots, c_n + 1)^T \in E^{-1}_{(n,n)}V$, then $\left[p_0(t), p_1(t)\right]$ is a segment of Hurwitz polynomials, where the polynomial $p_1(t)$ is given by $p_1(t) = \sum_{i=1}^{n+1} c_i t^{n+1-i}$.*

Remark (4.1.3). Observe that the set of vectors that satisfies (1.2) is given by the polyhedral cone $\mathcal{C}$ generated by $w_1 = E^{-1}_{(n,n)}e_1, w_2 = E^{-1}_{(n,n)}e_2, \ldots, w_{n+1} = E^{-1}_{(n,n)}e_{n+1}$, where $e_1, e_2, \ldots, e_{n+1}$ are the canonical vectors in $\mathbb{R}^{n+1}$. Given the vector of coefficients $w_0 = (1, a_1, \ldots, a_n)$ of the Hurwitz stable polynomial $p_0(t)$, the vectors $w \in \mathcal{C}$ are vectors of coefficients of polynomials $p_1(t)$ such that $\left[p_0(t), p_1(t)\right]$ is a segment of Hurwitz polynomials.

Example (4.1.4). Consider the Hurwitz stable polynomial $p_0(t) = t^3 + 2t^2 + t + 1$. The matrices $E_{(3,3)}$ and $E^{-1}_{(3,3)}$ are given by

$$E_{(3,3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad E^{-1}_{(3,3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

From Theorem (4.1.1), the set of vectors that satisfy $E_{(3,3)}c \gneqq 0$ can be seen as the polyhedral cone $\mathcal{C}$ generated by

$$\{(1, 1, 1, 0)^T, (0, 1, 1, 0)^T, (0, 1, 2, 0)^T, (0, 2, 4, 1)^T\}.$$

5. The minimum left extreme

In this section, given the Hurwitz stable polynomials $p_0(t)$ and $p_1(t)$, we are concerned with the problem of estimating the minimum $k_{\min} < 0$ such that $p_0(t) + kp_1(t)$ is a Hurwitz stable polynomial $\forall k > k_{\min}$(see [4]). Using the results presented in the above sections, we will find a number $k_0 < 0$ such that $p_0(t) + kp_1(t)$ is Hurwitz stable for every $k > k_0$, if the vector of coefficients of p_1 satisfies (1.2) or (1.4). Here k_0 is an estimate of $k_{\min}$ ($k_0 \geq k_{\min}$) because we do not know if k_0 is the smallest number with this property. The problem of calculating the minimum left extreme $k_{\min}$ was solved by Bialas [4]. In our approach k_0 is obtained by an algebraic calculation.

Consider the polynomial

$$p(t,k) = p_0(t) + kp_1(t) \tag{5.1}$$

where $p_0(t) = t^n + a_1t^{n-1} + \cdots + a_n$ is the nominal polynomial. Assume $p_0(t)$ is a Hurwitz stable polynomial, and let $E_{(n,n)}$ be the corresponding matrix defined as in (1.1). If the vector of coefficients $c = (c_1, c_2, \ldots, c_{n+1})^T \succeq 0$ of the polynomial $p_1(t) = \sum_{i=1}^{n+1} c_i t^{n+1-i}$ is a solution to the system of linear inequalities (1.2), then $p_0(t) + kp_1(t)$ is a Hurwitz stable polynomial $\forall k \geq 0$. In [4] it was proved that

$$k_{\min} = \frac{1}{\lambda^-_{\min}[-H^{-1}(p_0)H(p_1)]} \tag{5.2}$$

where $H(p_0), H(p_1)$ are the Hurwitz matrices of p_0 and p_1 respectively and $\lambda^-_{\min}[-H^{-1}(p_0)H(p_1)]$ is the minimum negative eigenvalue of the matrix $-H^{-1}(p_0)H(p_1)$. Observe that numerically (5.2) is not easy to calculate because the calculation implies solving an nth-order eigenvalue problem. In what follows we give an algebraic procedure to obtain an estimate of $k_{\min}$.

Define the matrix

$$Z_{(n,n)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & a_1 & -2 & 0 & \ldots & 0 & 0 \\ 0 & 0 & a_2 & -2a_1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & a_{n-1} & -2a_{n-2} \\ 0 & 0 & 0 & 0 & \ldots & 0 & a_n \end{pmatrix} \tag{5.3}$$

and denote by $Z^i_{(n,n)}$ the i-th row of the matrix $Z_{(n.n)}$ and let $a = (1, a_1, \ldots, a_n)^T$.

THEOREM (5.4). *Let $p_0(t) = t^n + a_1t^{n-1} + \cdots + a_n$ be a Hurwitz stable polynomial. Let $E_{(n,n)}$ be the corresponding matrix defined by (1.1). If the vector $c = (c_1, c_2, \ldots, c_{n+1})^T \succeq 0$ is a solution to the system of linear inequalities (1.2) and each component of $E_{(n,n)}c$ is positive and the polynomial $p_1(t)$ is given by $p_1(t) = \sum_{i=1}^{n+1} c_i t^{n+1-i}$ then, $p_0(t) + kp_1(t)$ is a Hurwitz stable polynomial for all $k > k_0$, where*

$$k_0 = \max_{i=1,\ldots,n+1} \left(-\frac{Z^i_{(n,n)}a}{E^i_{(n,n)}c} \right).$$

Proof. In a similar way to the proof of Theorem (2.3) we get

$$\begin{aligned} p_0(-i\omega)[p_0 + kp_1](i\omega) &= \left[P^2(\omega^2) + \omega^2 Q^2(\omega^2)\right] + \\ &\quad + k\left[P(\omega^2)p(\omega^2) + \omega^2 Q(\omega^2)q(\omega^2)\right] + \\ &\quad + i\omega k\left[P(\omega^2)q(\omega^2) - Q(\omega^2)p(\omega^2)\right]. \end{aligned}$$

Note that the expression $P^2(\omega^2) + \omega^2 Q^2(\omega^2) + k\left[P(\omega^2)p(\omega^2) + \omega^2 Q(\omega^2)q(\omega^2)\right]$ can be rewritten as $\omega^{2(n+1)} + \sum_{i=1}^{n+1}(Z^i_{(n,n)}a + kE^i_{(n,n)}c)\omega^{2(n+1-i)}$. If $k > k_0$ then

$k > -\dfrac{Z^i_{(n,n)}a}{E^i_{(n,n)}c}$ $\forall i = 1, \ldots, n+1$. Since $E^i_{(n,n)}c > 0 \; \forall\, i = 1, \ldots, n+1$ it follows that $kE^i_{(n,n)}c > -Z^i_{(n,n)}a$ and then $Z^i_{(n,n)}a + kE^i_{(n,n)}c > 0 \; \forall i = 1, \ldots, n+1$. Consequently, for all $\omega > 0$, $p_0(-i\omega)[p_0 + kp_1](i\omega)$ does not intersect the imaginary axis, from which we have that $p_0(-t)$ and $p_0(t) + kp_1(t)$ satisfy the hypotheses of Lemma (2.1). This implies that the polynomial $p_0(t) + kp_1(t)$ is Hurwitz stable for all $k > k_0$, and the theorem is proved. □

Remark (5.5). The extension of Theorem (5.4) to the case where $\deg p_1(t) = n-1$ turns out to be as follows: $p_0(t) + kp_1(t)$ is Hurwitz stable for all $k > k_0$, if $k_0 = \max\limits_{i=1,\ldots,n,n} \left(-\dfrac{Z^i_{(n,n-1)}a}{E^i_{(n,n-1)}c} \right)$ and

$$(5.6) \qquad Z_{(n,n-1)} = \begin{pmatrix} a_1 & -2 & 0 & 0 & \ldots & 0 & 0 \\ 0 & a_2 & -2a_1 & 2 & \ldots & 0 & 0 \\ 0 & 0 & a_3 & -2a_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & a_{n-1} & -2a_{n-2} \\ 0 & 0 & 0 & 0 & \ldots & 0 & a_n \end{pmatrix}.$$

Remark (5.7). For the segment of polynomials $p(t, q) = p_0(t) + q[p_1(t) - p_0(t)]$ for $q \in [0, 1]$, it follows from Theorem (5.4) that $p(t, q)$ is Hurwitz stable for all $q \geq q_0$, where $q_0 = \dfrac{k_0}{1 + k_0}$. If $k_0 \leq -1$, it results that $p(t, q)$ is Hurwitz stable for all $k \in (-\infty, 1]$.

Next we present an example where $k_0 = k_{\min}$.

Example (5.8). Let $p_0(t) = t^3 + 7t^2 + 14t + 8$, $p_1(t) = t^2 + 4t + 6$. To calculate k_0, we first have that

$$\begin{aligned} Z_{(3,2)}a &= \begin{pmatrix} 7 & -2 & 0 \\ 0 & 14 & -14 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 7 \\ 14 \\ 8 \end{pmatrix} = \begin{pmatrix} 21 \\ 84 \\ 64 \end{pmatrix}, \\ E_{(3,2)}c &= \begin{pmatrix} 7 & -1 & 0 \\ -8 & 14 & -7 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 48 \end{pmatrix}. \end{aligned}$$

Then $k_0 = \max\left(-\frac{21}{3}, -\frac{84}{6}, -\frac{64}{48}\right) = -\frac{4}{3}$.

To calculate $k_{\min}$, we find that

$$H(p_0) = \begin{pmatrix} 7 & 8 & 0 \\ 1 & 14 & 0 \\ 0 & 7 & 8 \end{pmatrix}, H(p_1) = \begin{pmatrix} 1 & 6 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 6 \end{pmatrix}$$

and

$$H^{-1}(p_0)H(p_1) = \begin{pmatrix} \frac{7}{45} & \frac{26}{45} & 0 \\ -\frac{1}{90} & \frac{11}{45} & 0 \\ \frac{7}{720} & -\frac{4}{45} & \frac{3}{4} \end{pmatrix}, \sigma\left(-H^{-1}(p_0)H(p_1)\right) = \{-\tfrac{3}{4}, -\tfrac{1}{5} \pm \tfrac{1}{15}i\}.$$

Therefore $\lambda_{\min} = -\frac{3}{4}$, and thus $k_{\min} = -\frac{4}{3} = k_0$.

Example (5.9). For the polynomials $p_0(t) = t^3 + 7t^2 + 14t + 8$, $p_1(t) = 26t^2 + 137t + 90$ we obtain $k_0 > k_{\min}$. Defining the matrices $Z_{(3,2)}, E_{(3,2)}$ as in (5.6) and (1.3) respectively, we have

$$Z_{(3,2)}a = \begin{pmatrix} 21 \\ 84 \\ 64 \end{pmatrix}, \qquad E_{(3,2)}c = \begin{pmatrix} 45 \\ 1080 \\ 720 \end{pmatrix}.$$

From which $k_0 = \max\left(-\frac{21}{45}, -\frac{84}{1080}, -\frac{64}{720}\right) = -\frac{7}{90} = -0.07778$. On the other hand, given the Hurwitz matrices $H(p_0)$ and $H(p_1)$, $\sigma\left(-H^{-1}(p_0)H(p_1)\right) = \{-11.25, -4.1399, -9.5601\}$, and $\lambda_{\min} = -11.25$. Finally, $k_{\min} = -0.088889 < k_0 = -0.07778$.

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