Boletín de la

SOCIEDAD MATEMÁTICA MEXICANA

Contenido

Tercera Serie

Volumen 13
Número 1
Abril de 2007

ARTÍCULOS DE INVESTIGACIÓN

Continúa/Continued on back cover

SOCIEDAD
MATEMATICA
MEXICANA

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

ON THE VALUE SET OF $n!m!$ MODULO A LARGE PRIME

VICTOR C. GARCÍA

ABSTRACT. We prove that for a large prime number p

$$\#\{n!m! \pmod p \ : \ 1 \le n, m \le p\} \ge \left(\tfrac{41}{48} + o(1)\right) p.$$

This improves previously known results from Chen and Dai [1] and Garaev, Luca, and Shparlinski [5].

1. Introduction

The problem of distribution of factorials modulo a prime number p has been a topic of much investigation, see, for example, the recent papers [1]–[7], [10] and references therein. In [8], **F11**, it is conjectured that about p/e of the residue classes modulo p are missed by the sequence $n!$. If this conjecture were true, the sequence $n!$ modulo p should assume about $(1-1/e)p$ distinct values, see [2] for some results of this spirit. This in turn would imply the representability of every residue class modulo p as a product of two factorials. Unconditionally, in [5] it was shown that

$$\#\{n!m! \pmod p \ : \ 1 \le n, m \le p\} \ge \tfrac{5}{8}p + O(p^{1/2}\log^2 p),$$

which has been improved in [1] to

$$\#\{n!m! \pmod p \ : \ 1 \le n, m \le p\} \ge \tfrac{3}{4}p + O(p^{1/2}\log^2 p).$$

In the present paper, using hybrid character sum estimates, we improve this further to the following result.

THEOREM (1.1). *The following bound holds:*

$$\#\{n!m! \pmod p \ : \ 1 \le n, m \le p\} \ge \tfrac{41}{48}p + O(p^{1/2}\log^3 p).$$

2. Proof

Let

$$\mathcal{E} = \{n!m! \pmod p \ : \ 1 \le n, m \le p\}.$$

The starting point, as in [1, 2, 5], is to employ the congruence

$$(2x-1)! \cdot (p-2x)! \equiv 1 \pmod p, \tag{2.1}$$

which holds for any positive integer $x \le p_1$, where $p_1 = (p-1)/2$.

Let

$$\mathcal{E}_1 = \{2, 4, \dots, 2p_1\}.$$

Let $\mathcal{E}_2$ be the set of positive odd integers less than p and having the form

$$(2x-1)^* \pmod p, \quad 1 \le x \le p_1.$$

2000 *Mathematics Subject Classification*: 11A07, 11B65, 11L40.

Keywords and phrases: congruences modulo a prime, distribution of factorials.

Here a^* is defined from $aa^* \equiv 1 \pmod p$.

Let $\mathcal{E}_3$ be the set of positive odd integers less than p which can be represented in the form $(2z)^* \pmod p$, for some $1 \le z \le p_1$, and at the same time in the form

$$(2x)^*(2x+1)^* \pmod p, \quad 1 \le x \le p_1 - 1.$$

Next, we define $\mathcal{E}_4$ be the set of positive odd integers less than p which can be represented in the form $(2z)^* \pmod p$ for some $1 \le z \le p_1$ and at the same time in the form

$$(2x-1)^*(2x)^*(2x+1)^* \pmod p$$

for some $1 \le x \le p_1 - 1$ satisfying the conditions

$$\left(\frac{4(2x-1)(2x)(2x+1)+1}{p}\right) = -1, \qquad \left(\frac{1-3x^2}{p}\right) = -1.$$

Here and below $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. Finally, we define $\mathcal{E}_5$ to be the set of positive odd integers less than p which can be represented in the form $(2z)^* \pmod p$ for some $1 \le z \le p_1$ and at the same time in the form

$$(2x-1)^*(2x)^*(2x+1)^* \pmod p$$

for some $1 \le x \le p_1 - 1$ satisfying the conditions

$$\left(\frac{4(2x-1)(2x)(2x+1)+1}{p}\right) = -1, \qquad \left(\frac{1-3x^2}{p}\right) = 1.$$

To each number of the set $\mathcal{E}_i$ we associate the residue class to which this number belongs. With this convention, since $(2x)!(p-2x)! \equiv 2x \pmod p$, we have $\mathcal{E}_1 \subset \mathcal{E}$.

If $u \in \mathcal{E}_4$ or $u \in \mathcal{E}_5$, then $u \equiv (2x-1)^*(2x)^*(2x+1)^* \pmod p$ for some $x \le p_1 - 1$. Together with (2.1) this yields

$$u \equiv (2x-2)! \cdot (p-2x-2)! \pmod p,$$

whence $u \in \mathcal{E}$. Thus, $\mathcal{E}_4 \subset \mathcal{E}$, $\mathcal{E}_5 \subset \mathcal{E}$. The same argument shows that $\mathcal{E}_2 \subset \mathcal{E}$, $\mathcal{E}_3 \subset \mathcal{E}$.

It is also easy to see that $\mathcal{E}_i \cap \mathcal{E}_j = \emptyset$ for $1 \le i \ne j \le 5$. Indeed, if, for example, $u \in \mathcal{E}_3$, then $\left(\frac{4u^*+1}{p}\right) = 1$, while if $u \in \mathcal{E}_4 \cup \mathcal{E}_5$, we have $\left(\frac{4u^*+1}{p}\right) = -1$. Hence $\mathcal{E}_3 \cap \mathcal{E}_4 = \emptyset$, $\mathcal{E}_3 \cap \mathcal{E}_5 = \emptyset$. The other cases are verified similarly. Therefore,

$$|\mathcal{E}| \ge |\mathcal{E}_1| + |\mathcal{E}_2| + |\mathcal{E}_3| + |\mathcal{E}_4| + |\mathcal{E}_5| = \frac{p-1}{2} + |\mathcal{E}_2| + |\mathcal{E}_3| + |\mathcal{E}_4| + |\mathcal{E}_5|.$$

We claim that the following estimates hold:

$$|\mathcal{E}_2| \ge (\tfrac{1}{4} + o(1))p, \qquad |\mathcal{E}_3| \ge (\tfrac{1}{16} + o(1))p, \tag{2.2}$$

$$|\mathcal{E}_4| \ge (\tfrac{1}{32} + o(1))p, \qquad |\mathcal{E}_5| \ge (\tfrac{1}{96} + o(1))p. \tag{2.3}$$

In order to estimate $|\mathcal{E}_4|$, we let I to be the number of solutions of the system of congruences

$$\begin{cases} 2r-1 \equiv (2x-1)^*(2x)^*(2x+1)^* \pmod p \\ 2z \equiv (2x-1)(2x)(2x+1) \pmod p \\ \left(\frac{4(2x-1)(2x)(2x+1)+1}{p}\right) = -1 \\ \left(\frac{1-3x^2}{p}\right) = -1 \end{cases}$$

under the conditions

$$1 \le x \le p_1 - 1, \quad 1 \le z \le p_1, \quad 1 \le r \le p_1.$$

Note that for a given nonzero $\lambda \equiv 2z \pmod p$, if the congruence

$$(2x-1)2x(2x+1) \equiv \lambda \pmod p \tag{2.4}$$

has two distinct nonzero solutions $x \not\equiv y \pmod p$, then we have

$$(2y+x)^2 \equiv 1 - 3x^2 \pmod p.$$

This means that given r, the above system of congruence has at most one solution. This implies that $|\mathcal{E}_4| \ge I$.

Let us analyze the cardinality $|\mathcal{E}_5|$. Denote by J the number of solutions of the system of congruences

$$\begin{cases} 2r-1 \equiv (2x-1)^*(2x)^*(2x+1)^* \pmod p \\ 2z \equiv (2x-1)(2x)(2x+1) \pmod p \\ \left(\frac{4(2x-1)(2x)(2x+1)+1}{p}\right) = -1 \\ \left(\frac{1-3x^2}{p}\right) = 1 \end{cases}$$

with the conditions

$$1 \le x \le p_1 - 1, \quad 1 \le z \le p_1, \quad 1 \le r \le p_1.$$

Given r, we have at most three solutions to this system. Hence, $|\mathcal{E}_5| \ge J/3$, and we have

$$|\mathcal{E}_4| \ge I, \quad |\mathcal{E}_5| \ge \frac{J}{3}. \tag{2.5}$$

For I and J we will obtain the asymptotic formulas

$$I = \frac{p}{32} + O(p^{1/2}\log^3 p), \quad J = \frac{p}{32} + O(p^{1/2}\log^3 p).$$

Denote $g(x) = (2x-1)2x(2x+1)$. Using basic trigonometric identities, we obtain

$$\begin{aligned} I &= \frac{1}{p^2}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{x=1}^{p_1-1}\delta(x)\gamma(x)\sum_{r=1}^{p_1}\sum_{z=1}^{p_1} e^{2\pi i\frac{a}{p}(2r-1-(g(x))^*)}e^{2\pi i\frac{b}{p}(2z-g(x))} \\ &- \frac{1}{p^2}\sum_{a=0}^{p-1}\sum_{b=0}^{p-1}\sum_{x\in\mathcal{A}}\delta(x)\gamma(x)\sum_{r=1}^{p_1}\sum_{z=1}^{p_1} e^{2\pi i\frac{a}{p}(2r-1-(g(x))^*)}e^{2\pi i\frac{b}{p}(2z-g(x))}, \end{aligned}$$

where

$$2\delta(x) = 1 - \left(\frac{4g(x)+1}{p}\right), \qquad 2\gamma(x) = 1 - \left(\frac{1-3x^2}{p}\right)$$

and

$$\mathcal{A} = \{x:\ 1 \le x \le p_1 - 1,\ (4g(x)+1)(1-3x^2) \equiv 0 \pmod{p}\}.$$

Clearly, $|\mathcal{A}| \le 5$. Hence, using the well-known estimate

$$\sum_{a=1}^{p-1} \left| \sum_{n=X+1}^{X+Y} e^{2\pi i a n/p} \right| < p \log p,$$

we derive

$$\left| \frac{1}{p^2} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{x \in \mathcal{A}} \delta(x)\gamma(x) \sum_{r=1}^{p_1} \sum_{z=1}^{p_1} e^{2\pi i \frac{a}{p}(2r-1-(g(x))^*)} e^{2\pi i \frac{b}{p}(2z-g(x))} \right|$$
$$\ll \frac{1}{p^2} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \left| \sum_{r=1}^{p_1} e^{2\pi i \frac{2ar}{p}} \right| \left| \sum_{z=1}^{p_1} e^{2\pi i \frac{2bz}{p}} \right| \ll \log^2 p.$$

Thus,

$$I = \frac{1}{p^2} \sum_{a=0}^{p-1} \sum_{b=0}^{p-1} \sum_{x=1}^{p_1-1} \delta(x)\gamma(x) \sum_{r=1}^{p_1} \sum_{z=1}^{p_1} e^{2\pi i \frac{a}{p}(2r-1-(g(x))^*)} e^{2\pi i \frac{b}{p}(2z-g(x))} + O(\log^2 p).$$

Separating the term corresponding to $a = b = 0$, we obtain

$$I = \frac{p_1^2}{p^2} \sum_{x=1}^{p_1-1} \delta(x)\gamma(x) + R_1 + O(\log^2 p) = \frac{p}{32} + R_1 + R_2 + O(\log^2 p), \tag{2.6}$$

where

$$R_1 \ll \frac{1}{p^2} \sum_{\substack{0 \le a,b \le p-1 \\ (a,b) \ne (0,0)}} \left| \sum_{r=1}^{p_1} e^{2\pi i \frac{a}{p}(2r-1)} \sum_{z=1}^{p_1} e^{2\pi i \frac{b}{p} 2z} \right| S(a,b), \tag{2.7}$$

$$S(a,b) = \left| \sum_{x=1}^{p_1-1} \delta(x)\gamma(x) e^{2\pi i \frac{1}{p}(a(g(x))^* + bg(x))} \right|,$$

$$R_2 \ll \left| \sum_{x=1}^{p_1-1} -\left(\frac{4g(x)+1}{p}\right) - \left(\frac{1-3x^2}{p}\right) + \left(\frac{(4g(x)+1)(1-3x^2)}{p}\right) \right|.$$

Next, we shall prove that, for $0 \le a, b \le p-1$ with $(a,b) \ne (0,0)$,

$$R_1 + R_2 \ll p^{1/2} \log^3 p.$$

Indeed, applying the technique of extending the summation over short intervals to the whole system of residues, we get

$$S(a,b) = \left| \sum_{x=1}^{p_1-1} \sum_{y=0}^{p-1}{}' \delta(y)\gamma(y) e^{2\pi i \frac{1}{p}(a(g(y))^* + bg(y))} \frac{1}{p} \sum_{\nu=0}^{p-1} e^{2\pi i \frac{\nu}{p}(y-x)} \right|$$
$$\le \frac{1}{p} \sum_{\nu=0}^{p-1} \left| \sum_{x=1}^{p_1-1} e^{2\pi i \frac{\nu x}{p}} \right| \left| \sum_{y=0}^{p-1}{}' \delta(y)\gamma(y) e^{2\pi i \frac{1}{p}(a(g(y))^* + bg(y) + \nu y)} \right|,$$

where the dash means that from the indicated range of summation over y the points $0, p_1$ and p_1+1 (which are poles of $g(y)^*$) are excluded. Since

$$4\delta(y)\gamma(y) = 1 - \left(\frac{4g(y)+1}{p}\right) - \left(\frac{1-3y^2}{p}\right) + \left(\frac{(4g(y)+1)(1-3y^2)}{p}\right),$$

in view of the Weil estimate for hybrid character sums with rational arguments (see, for example, [9]), we have

$$\left|\sum_{y=0}^{p-1}{}' \delta(y)\gamma(y) e^{2\pi i \frac{1}{p}(a(g(y))^* + bg(y) + \nu y)}\right| \ll p^{1/2}.$$

Therefore,

$$S(a,b) \ll \frac{p^{1/2}}{p} \sum_{\nu=0}^{p-1} \left|\sum_{x=1}^{p_1-1} e^{2\pi i \frac{\nu x}{p}}\right| \ll p^{1/2}\log p.$$

Inserting this into (2.7), we get

$$R_1 \ll \frac{p^{1/2}\log p}{p^2} \left(\sum_{a=0}^{p-1} \left|\sum_{r=1}^{p_1} e^{2\pi i \frac{a}{p} 2r}\right|\right)^2 \ll p^{1/2}\log^3 p.$$

Similarly, $R_2 \ll p^{1/2}\log p$. Hence, by (2.6), we obtain that

$$I = \frac{p}{32} + O(p^{1/2}\log^3 p).$$

Analogously,

$$J = \frac{p}{32} + O(p^{1/2}\log^3 p).$$

Thus, in view of (2.5), we get

$$|\mathcal{E}_4| \geq \frac{p}{32} + O(p^{1/2}\log^3 p), \quad |\mathcal{E}_5| \geq \frac{p}{96} + O(p^{1/2}\log^3 p),$$

which proves the required estimate (2.2).

The same argument applied to $\mathcal{E}_2, \mathcal{E}_3$ implies (2.3). Thus, we conclude that

$$|\mathcal{E}| \geq \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{1}{96}\right) p + O(p^{1/2}\log^3 p) = \frac{41}{48} p + O(p^{1/2}\log^3 p).$$

Acknowledgement

This work was supported by Project PAPIIT IN 100307 from UNAM.

Received December 19, 2006

Final version received February 06, 2007

Instituto de Matemáticas
Universidad Nacional Autónoma de México
58089, Morelia, Michoacán
México
garci@matmor.unam.mx

References

[1] Y. G. Chen and L. X. Dai, *Congruences with factorials modulo p,* Integers **6** (2006), A21, 3 pp.
[2] C. Cobeli, M. Vâjâitu and A. Zaharescu, *The sequence* $n!$ (mod p), J. Ramanujan Math. Soc. **15** (2000), 135–154.
[3] M. Z. Garaev and V. C. Garcia, *Waring type congruences involving factorials modulo a prime,* Arch. Math. (Basel) **88** (2007), No. 1, 35–41.
[4] M. Z. Garaev and F. Luca, *Character sums and product of factorials modulo p,* J. Théor. Nombres Bordeaux **17** (2005), 161–170.
[5] M. Z. Garaev, F. Luca and I. E. Shparlinski, *Character sums and congruences with* $n!$, Trans. Amer. Math Soc. **356** (2004), 5089–5102.
[6] M. Z. Garaev, F. Luca and I. E. Shparlinski, *Exponential sums and congruences with factorials*, J. Reine Angew. Math. **584** (2005), 29-44.
[7] V. C. Garcia, *Representations of residue classes by factorials, binomial coefficients and harmonic sums modulo a prime,* Preprint (2006).
[8] R. K. Guy, Unsolved Problems in Number Theory, Springer-Verlag, New York, 1994.
[9] E. Kowalski, *Exponential sums over definable subsets of finite fields*, (2005), arXiv:math.NT/0504316.
[10] F. Luca and P. Stănică, *Products of factorials modulo p*, Colloq. Math. **96** (2003), 191–205.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

ON THE DISTRIBUTION OF THE POWER GENERATOR MODULO A PRIME POWER FOR PARTS OF THE PERIOD

EDWIN D. EL-MAHASSNI

ABSTRACT. This paper studies the multidimensional distribution of the power generator of pseudorandom numbers modulo a high power of a fixed prime number for parts of the period. That is, we study a sequence of numbers generated by the power generator when the number of terms in such sequence is smaller than its period. These results compliment some recently obtained distribution bounds of the power generator modulo a high power of fixed prime for the entire period. The case of a prime power modulus, although it does not have any immediate cryptography related applications, may still be of interest for other settings which require quality pseudorandom numbers.

1. Introduction

Let $e \geq 2$, $m \geq 1$ and ϑ be integers such that $\gcd(\vartheta, m) = 1$. Then one can define the sequence (u_n) by the recurrence relation

$$u_n \equiv u_{n-1}^e \pmod{m}, \quad 0 \leq u_n \leq m-1, \qquad n = 1, 2, \dots, \tag{1.1}$$

with the *initial value* $u_0 = \vartheta$. This sequence is known as the *power generator* of pseudorandom numbers. It is obvious that the sequence (1.1) eventually becomes periodic with some period $\tau \leq M$. In this paper we shall assume that $\gcd(e, \varphi(m)) = 1$; and so it follows that the sequence (u_n) is *purely periodic*. Apart from some results such as those in [1, 2, 4, 5, 9, 13, 15, 18, 22, 26, 32, 34] and more specifically in [4, 13, 14] for prime power moduli, very little else is known about the distribution of the sequence of numbers produced by the power generator. And, despite [4, 13, 14] all using different methods, none of them can be adequately applied to the case of multidimensional distributions. Other results concerning the power generator have also been obtained in [3, 11]. Specifically, in [11], a distribution result has recently been established for the sequence generated by (1.1) over the entire period. Often, methods which estimate the bounds for the whole period cannot be extended to subsets, see, for example, [12, 23]. In fact, in some cases, obtaining a bound for subsets of a sequence is a much more difficult problem than for the entire period, e.g. [33]. Furthermore, some publications explicitly set out to obtain results which only deal with such subsets, e.g. [17, 30]. Studying the distribution results of parts of the sequence when equivalent results are already known for the whole period raises a few questions. For instance, do the desirable properties obtained for the entire sequence also apply to subsets of the sequence? And if so, then how small can these subsets be before they 'lose' their distribution bounds?

2000 *Mathematics Subject Classification*: Primary: 11L07, 11K38. Secondary: 11B50, 11K45.
Keywords and phrases: exponentials sums, pseudorandom number generators, discrepancy.

The aim of this paper is to answer such questions. Here we show that the original method of [27, 30], and more recently also used in [7, 8], combined with bounds for exponential sums with sparse polynomials from [11, 31] allows us to study the multidimensional distribution of the power generator of pseudorandom numbers modulo a high power of a small prime number p over parts of the period. Several other results about non-linear pseudorandom number generators have been obtained in [7, 8, 16, 17, 30, 28, 29]. However, these apply to generators of the form $u_n \equiv f(u_{n-1}) \pmod m$ where f is a polynomial or a rational function of small degree, while in this paper we do not impose any restrictions on the size of the exponent e.

2. Preliminaries

For a sequence of N points

$$\Gamma = (\gamma_{1,n}, \ldots, \gamma_{s,n})_{n=1}^{N} \tag{2.1}$$

in the half-open box $[0,1)^s$, denote by Δ_Γ its *discrepancy,* that is,

$$\Delta_\Gamma = \sup_{B\subseteq[0,1)^s} \left| \frac{T_\Gamma(B)}{N} - |B| \right|,$$

where $T_\Gamma(B)$ is the number of points of the sequence Γ which hit the box

$$B = [\alpha_1, \beta_1) \times \cdots \times [\alpha_s, \beta_s) \subseteq [0,1)^s$$

and the supremum is taken over all such boxes. For an integer vector $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{Z}^s$ we put

$$|\mathbf{a}| = \max_{i=1,\ldots,s} |a_i|, \qquad h(\mathbf{a}) = \prod_{i=1}^{s} \max\{|a_i|, 1\}.$$

This discrepancy of a sequence of points in the s-dimensional unit cube can be estimated by the *Erdös–Turán–Koksma inequality* (see Theorem 1.21 of [6]) which we present in the following form.

LEMMA (2.2). *There exists a constant $C_s > 0$ depending only on the dimension s, such that for any integer $L \geq 1$, for the discrepancy of a sequence of points (2.1) the bound*

$$\Delta_\Gamma < C_s \left(\frac{1}{L} + \frac{1}{N} \sum_{0<|\mathbf{a}|\leq L} \frac{1}{h(\mathbf{a})} \left| \sum_{n=1}^{N} \exp\Big(2\pi i \sum_{j=1}^{s} a_j \gamma_{j,n}\Big) \right| \right)$$

holds, where the sum is taken over all integer vectors

$$\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{Z}^s$$

with $0 < |\mathbf{a}| \leq L$.

Let p be a fixed prime number. For an integer vector $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{Z}^s$ we define the exponential sum

$$S(\mathbf{a}, r) = \sum_{n=0}^{N-1} \mathbf{e}_r \left(\sum_{i=1}^{s} a_i u_{n+i} \right),$$

for $1 \le N \le \tau$, with τ being the period of the sequence (u_n) given by (1.1), and where

$$\mathbf{e}_r(z) = \exp(2\pi i z/p^r),$$

for some $r \ge 1$. We obtain a non-trivial upper bound for the sums $S(\mathbf{a}, r)$ and derive (see Theorem (3.1)) the uniformity of distribution modulo $m = p^r$ of the elements u_n, $n = 1, \ldots, N < \tau$. For a t-element set $\mathcal{R} = \{r_1, \ldots, r_t\} \subseteq \mathbb{Z}$ denote by $\Delta(\mathcal{R})$ the following determinant,

$$\Delta(\mathcal{R}) = \det \binom{r_i}{j}_{i,j=1,\ldots,t},$$

where for integers $m \ge 0$ and k we set

$$\binom{k}{m} = \frac{k(k-1)\ldots(k-m+1)}{m!}.$$

Let $\operatorname{ord}_p z$ denote the p-adic order of $z \in \mathbb{Z}$. The arguments for the following bound appear in [11], Lemma 2.1 and [31], Lemma 5.

LEMMA (2.3). *Let p be a prime and let $\alpha \ge 1$ be an integer. Then for any set $\mathcal{R} = \{r_1, \ldots, r_t\} \subseteq \mathbb{Z}$, any $\epsilon > 0$, and any integers $A_1, \ldots, A_t$ with* $\gcd(A_1, \ldots, A_t, p) = 1$*, the bound*

$$\left| \sum_{\substack{x=1 \\ \gcd(x,p)=1}}^{p^\alpha} \mathbf{e}_\alpha\Big(\sum_{1 \le j \le t} A_j x^{r_j} \Big) \right| \le C(p, t, \varepsilon) p^{\alpha(1-1/t+\epsilon)+\gamma}$$

holds, where $\gamma = \operatorname{ord}_p \Delta(\mathcal{R})$ *and the constant $C(p, t, \varepsilon)$ depends only on p, t and ε.*

We say that an integer g is *regular* modulo a prime p with $\gcd(p, g) = 1$ if $g^{p-1} \not\equiv 1 \pmod{p^2}$ for odd p and $g \equiv 5 \pmod 8$ for $p = 2$. The following result has also been proved in [11], [Lemma 2.3].

LEMMA (2.4). *Let $s \ge 1$ and $k \ge s$ be integers and let*

$$\mathcal{R} = \left\{ 1, \ldots, e^{s-1}, e^k, \ldots, e^{k+s-1}, \right\}.$$

If e is regular modulo p, then

$$\operatorname{ord}_p \Delta(\mathcal{R}) \le 8s^2 + s \log k.$$

Now we are prepared to formulate our main estimate.

THEOREM (2.5). *Let $e \ge 2$ and $m = p^r$ where p is a prime such that e is regular modulo p. Assume that the sequence (u_n) given by (1.1) is periodic with period τ, where $1 \le N \le \tau$. Then, for every integer s, any $\epsilon > 0$, and every vector* $\mathbf{a} = (a_0, \ldots, a_{s-1}) \in \mathbb{Z}^s$ *with* $\gcd(a_0, \ldots, a_{s-1}, p^r) = \mu$*, we have*

$$|S(\mathbf{a}, r)| \ll N^{1/2} m^{1/2} \left(m/\mu\right)^{-1/4s(s+1)+\epsilon}$$

where the implied constant depends at most on p, s and ϵ.

Proof. For any integer $k \ge 0$ we have

$$\left| S(\mathbf{a}, r) - \sum_{n=0}^{N-1} \mathbf{e}_r \left(\sum_{j=1}^{s} a_j u_{n+k+j} \right) \right| \le 2k\,.$$

Therefore, for any integer $K \ge 1$,

$$K|S(\mathbf{a}, r)| \le W + K^2,$$

where

$$W = \left| \sum_{n=0}^{N-1} \sum_{k=0}^{K-1} \mathbf{e}_r(\sum_{j=1}^{s} a_j u_{n+k+j}) \right| \le \sum_{n=0}^{N-1} \left| \sum_{k=0}^{K-1} \mathbf{e}_r(\sum_{j=1}^{s} a_j u_{n+k+j}) \right| .$$

Accordingly, applying the Cauchy inequality, we obtain

$$\begin{aligned}
W^2 &\le N \sum_{n=0}^{N-1} \left| \sum_{k=0}^{K-1} \mathbf{e}_r(\sum_{j=1}^{s} a_j u_{n+k+j}) \right|^2 \\
&\le N \sum_{n=1}^{\tau} \left| \sum_{k=0}^{K-1} \mathbf{e}_r(\sum_{j=1}^{s} a_j \vartheta^{e^{n+k+j}}) \right|^2 \\
&\le N \sum_{k=0}^{K-1} \sum_{l=0}^{K-1} \sum_{\substack{x=0 \\ \gcd(x,p)=1}}^{p^r} \mathbf{e}_r(\sum_{j=1}^{s} a_j (x^{e^{k+j}} - x^{e^{l+j}})) \\
&\le N p^{\rho} \sum_{k=0}^{K-1} \sum_{l=0}^{K-1} \sum_{\substack{x=0 \\ \gcd(x,p)=1}}^{p^{r-\rho}} \mathbf{e}_{r-\rho}(\sum_{j=1}^{s} (a_j/p^{\rho})(x^{e^{k-l+j}} - x^{e^{j}})),
\end{aligned}$$

where $p^{\rho} = \mu$ for some integer ρ, with $1 \le \rho \le r$. If $k = l$, then the inner sum is trivially equal to $p^{r-\rho}$. There are K such sums. Otherwise, applying Lemma (2.3) and Lemma (2.4), we obtain

$$\begin{aligned}
W^2 &\ll KNp^{\rho} p^{r-\rho} + Np^{(r-\rho)(1-1/2s+\epsilon)} \sum_{k=0}^{K-1} \sum_{l=0}^{K-1} p^{8s^2 + s\log_p (k-l)} \\
&\ll KNp^{\rho} p^{r-\rho} + Np^{\rho} p^{(r-\rho)(1-1/2s+\epsilon)} K^{s+2} \\
&\ll KNm + Nm^{(1-1/2s+\epsilon)} \mu^{1/2s-\epsilon} K^{s+2}.
\end{aligned}$$

Balancing the two terms above in the above estimate (up to $(m/\mu)^{\epsilon}$) by selecting $K = \lfloor (m/\mu)^{1/2s(s+1)} \rfloor$, we obtain the result claimed. □

If for example $\mu = 1$ then for any $\delta > 0$ the bound of Theorem (2.5) is nontrivial provided that r is sufficiently large in terms of p, s and δ.

3. Main Result

Let D_s denote the discrepancy of the points

$$\left(\left\{ \frac{u_n}{p^r} \right\}, \ldots, \left\{ \frac{u_{n+s-1}}{p^r} \right\} \right), \qquad n = 1, \ldots, N < \tau\,.$$

THEOREM (3.1). *Assume that the sequence* (u_n) *given by (1.1) with* $m = p^r$ *where* p *is a prime such that* e *is regular modulo* p, *is periodic with period* τ *and with* $1 \le N \le \tau$. *Then for every positive integer* s, *and any* $\epsilon > 0$, *the bound*

$$D_s \ll N^{-1/2} m^{1/2-1/4s(s+1)+\epsilon}$$

holds, where the implied constant depends at most on p, s *and* ϵ.

Proof. From Theorem (2.5) and Lemma (2.2), applied with $L = m$ we see

$$\begin{aligned}
D_s &\ll \frac{1}{m} + \frac{1}{N} \sum_{\rho=0}^{r} \sum_{\substack{0<|\mathbf{a}|\le p^r \\ \gcd(a_0,\dots,a_{s-1},p^r)=p^\rho}} \frac{1}{h(\mathbf{a})} N^{1/2} m^{1/2} (m/p^\rho)^{-1/4s(s+1)+\varepsilon/2} \\
&\ll \frac{1}{m} + N^{-1/2} m^{1/2-1/4s(s+1)+\varepsilon/2} \sum_{\rho=0}^{r} p^{\rho(1/4s(s+1)-\varepsilon/2)} \sum_{\substack{0<|\mathbf{a}|\le p^r \\ \gcd(a_0,\dots,a_{s-1},p^r)=p^\rho}} \frac{1}{h(\mathbf{a})} \\
&\ll \frac{1}{m} + N^{-1/2} m^{1/2-1/4s(s+1)+\varepsilon/2} \sum_{\rho=0}^{r} p^{-\rho(1+\varepsilon/2-1/4s(s+1))} \sum_{0<|\mathbf{a}|\le p^{r-\rho}} \frac{1}{h(\mathbf{a})} \\
&\ll \frac{1}{m} + N^{-1/2} m^{1/2-1/4s(s+1)+\varepsilon/2} \sum_{\rho=0}^{r} p^{-\rho(1+\varepsilon/2-1/4s(s+1))} (\log p^{r-\rho})^s
\end{aligned}$$

and after simple calculations we derive the desired statement. □

We remark that for any $\gamma < \delta/2$, the bound of the theorem is $O(m^{-\gamma})$ provided that $N > m^{1-1/2s(s+1)+\delta}$ and m and r are sufficiently large in terms of s and δ.

4. Remarks

Other characteristics of the power generator (1.1) with prime power moduli $m = p^r$ are of interest as well. Also, one could try to find a non-trivial result when $N < m^{1-1/2s(s+1)}$. There is no particular reason for choosing the base p to be a prime number. Although this seems the most natural choice, the methods here also work for moduli m which are products of high powers of several fixed primes. In particular, we can apply [35], Problem 12.d, Chapter 3 to Lemma (2.3) so that we can reduce exponential sums with polynomials and arbitrary denominators to exponential sums with prime power denominators. Hence, the upper bound for Lemma (2.3) becomes $C(p_1, \dots, p_n, t, \varepsilon) m^{(1-1/t+\varepsilon)+\gamma}$, where the modulus m has n prime factors $p_1, \dots, p_n$. This should lead to variants of Theorems (2.5) and (3.1) for such moduli (there is not probably enough interest to such a result to justify unavoidable technical and notational compications). However, neither the method of this work nor that in [10, 14] can be extended to arbitrary composite moduli m. Also, quite clearly, since we should have at least $N > m^{1-1/2s(s+1)+\delta}$, the results hold as long as $\tau > m^{1-1/2s(s+1)+\delta}$. On the other hand, a variant of the method of [13, 25] has led to nontrivial upper bounds of the exponential sums involved in studying the uniformity of distribution of the power generator modulo a composite and to a number of other results. It would also be interesting to extend the results of this paper to the case of the *exponential generator*

$$v_n \equiv g^{v_{n-1}} \pmod m, \quad 0 \le v_n \le m-1, \qquad n = 1, 2, \dots,$$

which also has numerous cryptographic applications [5, 22]. Lastly, one could try to study the distribution not of consecutive s-tuples of the sequence but rather s-tuples $u_{n+g(i)}$, $1 \le i \le s$, for a fixed function g taking integer values distinct modulo τ. This will certainly work for certain simple functions g.

Acknowledgements

The author wishes to thank Igor Shparlinski for suggesting this problem and for helpful comments, advice and the reading of this paper.

Received June 19, 2006

Final version received January 09, 2007

DEPARTMENT OF COMPUTING
MACQUARIE UNIVERSITY
NORTH RYDE, NSW, AUSTRALIA, 2109
edwinelm@ics.mq.edu.au

REFERENCES

[1] L. BLUM, M. BLUM AND M. SHUB, *A simple unpredictable pseudo-random number generator*, SIAM J. Comput. **15** (1986), 364–383.

[2] J. J. BRENNAN AND B. GEIST, *Analysis of iterated modular exponentiation: The orbit of x^α mod N*, Designs, Codes and Cryptography **13** (1998), 229–245.

[3] W. S. CHOU AND I. E. SHPARLINSKI, *On the cycle structure of repeated exponentiation modulo a prime*, J. Number Theory **107** (2004),345–356.

[4] T. W. CUSICK, *Properties of the x^2 mod N pseudorandom number generator*, IEEE Trans. Inform. Theory **41** (1995), 1155–1159.

[5] T. W. CUSICK, C. DING AND A. RENVALL, Stream Ciphers and Number Theory, Elsevier, Amsterdam, 1998.

[6] M. DRMOTA AND R.F. TICHY, Sequences, discrepancies and applications, Springer-Verlag, Berlin, 1997.

[7] E. D. EL-MAHASSNI, I. E. SHPARLINSKI, AND A. WINTERHOF, *Distribution of nonlinear congruential pseudorandom numbers modulo almost squarefree integers*, Monatshefte fuer Mathematik **148** (2006), 297–307.

[8] E. D. EL-MAHASSNI AND A. WINTERHOF, *On the distribution of nonlinear congruential pseudorandom numbers in residue rings*, IJNT **2** 1 (2006), 163–168 .

[9] R. FISCHLIN AND C. P. SCHNORR, *Stronger security proofs for RSA and Rabin bits*, J. Cryptology **13** (2000), 221–244.

[10] J. B. FRIEDLANDER, J. HANSEN AND I. E. SHPARLINSKI, *On character sums with exponential functions*, Mathematika **47** (2000), 75–85.

[11] J. B. FRIEDLANDER, J. HANSEN AND I. E. SHPARLINSKI, *On the distribution of the power generator modulo a prime power*, Proc. DIMACS workshop on unusual applications of number theory 2000, Amer. Math. Soc., (2004), 71–79.

[12] J. B. FRIEDLANDER, S. V. KONYAGIN AND I. E. SHPARLINSKI, *Some doubly exponential sums over $\mathbb{Z}_m$*, Acta Arith. **105** (2002), 349–370.

[13] J. B. FRIEDLANDER, D. LIEMAN AND I. E. SHPARLINSKI, *On the distribution of the RSA generator*, Proc. Intern. Conf. on Sequences and their Applications (SETA'98), Singapore, Springer-Verlag, London, 1999, 205–212.

[14] J. B. FRIEDLANDER AND I. E. SHPARLINSKI, *On the distribution of the power generator*, Math. Comp. **70** (2001), 1575–1589.

[15] F. GRIFFIN AND I. E. SHPARLINSKI, *On the linear complexity profile of the power generator*, IEEE Trans. Inform. Theory **46** (2000), 2159–2162.

[16] F. GRIFFIN, H. NIEDERREITER AND I. E. SHPARLINSKI, *On the distribution of nonlinear recursive congruential pseudorandom numbers of higher orders*, Proc. the 13th Symp. on Appl. Algebra, Algebraic Algorithms, and Error-Correcting Codes, Hawaii, 1999, Lect. Notes in Comp. Sci., Springer-Verlag, Berlin, **1719** (1999), 87–93.

[17] J. GUTIERREZ, H. NIEDERREITER AND I. E. SHPARLINSKI, *On the multidimensional distribution of nonlinear congruential pseudorandom numbers in parts of the period*, Monatsh. Math. **129** (2000), 31–36.

[18] J. HÅSTAD AND M. NÄSLUND, *The security of individual RSA bits*, Proc 39th IEEE Symp. on Foundations of Comp. Sci., 1998, 510–519.

[19] D. KNUTH, The art of computer programming, vol. 2, Addison-Wesley, Reading, MA, 1998.

[20] N. M. KOROBOV, *Exponential sums with exponential functions and the distribution of digits in periodic fractions*, Matem. Zametki **8** (1970), 641–652. Math. Notes **8** (1970), 831–837.

[21] N. M. KOROBOV, *On the distribution of digits in periodic fractions*, Math. Sb. **89** (1972), 654–670. Mat. USSR-Sb. **18** (1972), 659–676.

[22] J. C. LAGARIAS, *Pseudorandom number generators in cryptography and number theory*, Proc. Symp. in Appl. Math., Amer. Math. Soc., Providence, RI, **42** (1990), 115–143.

[23] T. LANGE AND I. E. SHPARLINSKI, *Certain Exponential Sums and Random Walks on Elliptic Curves*, Canad. J. Math. **57** (2005), 338–350.

[24] R. LIDL AND H. NIEDERREITER, Finite fields, Cambridge University Press, Cambridge, 1997.

[25] D. LIEMAN AND I. E. SHPARLINSKI, *On a new exponential sum*, Canad. Math. Bull. **44** (2001), 87–92.

[26] A. J. MENEZES, P. C. VAN OORSCHOT AND S. A. VANSTONE, Handbook of Applied Cryptography, CRC Press, Boca Raton, FL, 1996.

[27] H. NIEDERREITER AND I. E. SHPARLINSKI, *On the distribution and lattice structure of nonlinear congruential pseudorandom numbers*, Finite Fields and Their Appl. **5** (1999), 246–253.

[28] H. NIEDERREITER AND I. E. SHPARLINSKI, *On the distribution of inversive congruential pseudorandom numbers modulo a prime power*, Acta Arith. **92** (2000), 89–98.

[29] H. NIEDERREITER AND I. E. SHPARLINSKI, *On the distribution of pseudorandom numbers and vectors generated by inversive methods*, Appl. Algebra Engrg., Comm. Comput. **10** (2000), 189–202.

[30] H. NIEDERREITER AND I. E. SHPARLINSKI, *On the distribution of inversive congruential pseudorandom numbers in parts of the period*, Math. Comp. **70** (2001), 1569–1574.

[31] I. E. SHPARLINSKI, *On exponential sums with sparse polynomials*, J. Number Theory **60** (1996), 233–244.

[32] I. E. SHPARLINSKI, *On the linear complexity of the power generator*, Des. Codes Cryptogr., **23** (2001), 5–10.

[33] I. E. SHPARLINSKI, *On the uniformity of distribution of the Naor-Reingold pseudo-random function*, Finite Fields Appl. **7** (2001), 318–326.

[34] D. R. STINSON, Cryptography: Theory and Practice, CRC Press, Boca Raton, FL, 1995.

[35] I. M. VINOGRADOV, Elements of Number Theory, Dover Publ., NY, 1954.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

CONTINUOUS CONVERGENCE AND DUALITY OF LIMITS OF TOPOLOGICAL ABELIAN GROUPS

S. ARDANZA-TREVIJANO, M. J. CHASCO

ABSTRACT. We find conditions under which direct and inverse limits of arbitrary indexed systems of topological Abelian groups are related via the duality defined by the continuous convergence structure. This generalizes known results by Kaplan about duality of direct and inverse sequences of locally compact Abelian groups.

1. Introduction

Given a topological Abelian group G, its group of continuous characters ΓG endowed with the compact open topology τ_{co} is another topological group, usually denoted by $G^\wedge$ and called the dual of G. The duality theorem of Pontryagin-van Kampen states that a locally compact Abelian (LCA) group G is topologically isomorphic to its bidual group $(G^\wedge)^\wedge$ by means of the natural evaluation mapping. This theorem lies at the core of abstract harmonic analysis on locally compact Abelian groups and its extension to more general groups gives rise to the notion of reflexive group.

The original results of Pontryagin-van Kampen can be generalized to more general topological Abelian groups by means of two different duality theories. That is, given a topological Abelian group G we may consider ΓG endowed with either the compact open topology τ_{co}, obtaining $G^\wedge$ the Pontryagin dual (P-dual), or the continuous convergence structure Λ_c, obtaining a convergence group denoted by $\Gamma_c G$ that we call the c-dual of G. The convergence structure Λ_c has the advantage of making the evaluation mapping $\omega\colon \Gamma G \times G \to \mathbb{T}$ continuous although it is not usually topological. For a locally compact Abelian group G there is no difference between τ_{co} and Λ_c in ΓG. Hence the theorem of Pontryagin-van Kampen can be understood in the framework of the two dualities. There are many extensions of this theorem obtained for P-duality. We give as examples the ones by Kaplan [9], [10], Smith [15], Banaszczyk [2] or Pestov [14] among others. The approach of c-duality has also been fruitfully used in the works of Binz, Butzmann and others. The recent book of Beattie and Butzmann [3] provides an excellent overview of convergent structures and contains many relevant results in this direction.

A frequently used method to extend a property of a class of groups to a larger class is to take direct or inverse limits. There are situations where this method

2000 *Mathematics Subject Classification*: 18A30, 22A05, 22D35, 43A40.

Keywords and phrases: duality, convergence group, nuclear topological group, direct limit, inverse limit.

The authors acknowledge the financial aid received from MCYT, BFM2003-05878 and FEDER funds.

can be used to extend the known members of the class of reflexive groups. Kaplan proved that sequential direct and inverse limits of locally compact Abelian groups are P-reflexive and also that the P-dual of a sequential direct (inverse) limit is the inverse (direct) limit of the corresponding sequence of P-duals [10]. However, there is an old example due to Leptin [11] of an inverse limit of P-reflexive groups that is not P-reflexive.

The aim of the present article is to show that under some conditions, direct and inverse limits are related via c-duality. Working in the c-duality setting allows us to get rid of the requirement of countability of the index set that is present in Kaplan's results mentioned above. Countability is also needed in [1] where the authors prove that certain direct and inverse limits of sequences of P-reflexive Abelian groups that are metrizable or k_ω-spaces are P-reflexive and dual of each other. These results have been recently extended by Glöckner and Gramlich in [7].

We first study when the c-dual of a direct limit is the inverse limit of the c-dual system. Here, a crucial fact is that in the category of continuous convergence Abelian groups, the natural map η from a group to its c-bidual is continuous.

We then proceed to study under which conditions the c-dual of the limit of an inverse system is the direct limit of the c-dual system. This is a delicate problem that cannot be solved by categorical arguments only. The usual construction of the direct limit as a quotient group of the coproduct of the groups in the system gives a hint of where the difficulties come from. In P-duality the P-dual of the product is not always the coproduct.[1] This difficulty disappears in the framework of c-reflexivity [3]. However further work is needed to prove c-duality between general inverse and direct limits.

2. Convergence groups and c-duality

We introduce in this section the category of convergence Abelian groups denoted by CAG and the notion of c-duality. For an up to date introduction to convergence Abelian groups we refer the reader to the monograph [3].

First recall some basic notions about convergence spaces.

A *convergence structure* on a set X consists of a map $\lambda\colon X \to 2^{\mathbb{F}(X)}$ where $\mathbb{F}$ is the set of all filters on X, such that for all $x \in X$ we have

i) The filter generated by x belongs to $\lambda(x)$.
ii) For all filters $\mathcal{F}, G \in \lambda(x)$, the intersection $\mathcal{F} \cap \mathcal{G}$ belongs to $\lambda(x)$.
iii) If $\mathcal{F} \in \lambda(x)$, then $\mathcal{G} \in \lambda(x)$ for all filters $\mathcal{G}$ on X finer than $\mathcal{F}$.

A *convergence space* (X, Λ) is a set with a convergence structure. See ([3], pp. 2ff), for a more detailed exposition.

The notion of convergence space generalizes that of topological space. A topological space has a natural convergence structure, given by the convergent filters in the topology, which makes it a convergence space. Note that there are well known convergence structures, like the almost sure convergence in measure theory, that do not come from a topology on the supporting set.

[1]Nickolas proved that the P-dual of a product of LCA groups coincides with the coproduct of the P-duals if and only if the index set is countable [13].

Many topological notions that can be stated in terms of convergence of filters (such as continuity, open and closed sets, cluster point, compactness, etc) have their corresponding definitions for convergence spaces.

A *convergence group* is a group endowed with a convergence structure compatible with the group structure. Clearly every topological group is a convergence group and it can be treated in this way.

Let CAG be the category of convergence Abelian groups whose objects are convergence Abelian groups and whose morphisms are continuous homomorphisms. For two objects G and H in CAG, the group of morphisms from G to H will be denoted by CAG(G, H). The category TAG of topological Abelian groups and continuous homomorphisms is a full subcategory of CAG.

Consider the multiplicative group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ with the Euclidean topology and denote by ΓG the group of morphisms CAG$(G, \mathbb{T})$.

We now define a convergence structure that makes ΓG a convergence group with nice properties. The *continuous convergence structure* Λ_c in ΓG is the coarsest convergence structure for which the evaluation mapping $\omega\colon \Gamma G \times G \to \mathbb{T}$ is continuous[2] ($\Gamma G \times G$ has the natural product convergence).

That is: A filter $\varPhi$ of ΓG converges continuously to ϕ if and only if $\omega(\varPhi \times \mathcal{F}) = \varPhi(\mathcal{F})$ converges to $\phi(x)$ in $\mathbb{T}$, for every $\mathcal{F} \to x$ in G. Here $\varPhi \times \mathcal{F}$ denotes the filter generated by the products $\Phi \times F$ and $\omega(\varPhi \times \mathcal{F}) = \varPhi(\mathcal{F})$ denotes the filter generated by the sets $\Phi(F)$, with $\Phi \in \varPhi$, $F \in \mathcal{F}$.

For any object G in CAG, we have that ΓG with the continuous convergence structure Λ_c is a Hausdorff convergence group ([3], 8.1) named the *convergence dual group* of G (c-dual for short) and denoted by $\Gamma_c G$. By Hausdorff we mean that any filter in $\Gamma_c G$ has at most one limit. From now on we will consider all of our groups in the subcategory of Hausdorff convergence Abelian groups HCAG.

For each $f \in$ HCAG(G, H), we can define the adjoint homomorphism $\Gamma_c f \in$ HCAG$(\Gamma_c H, \Gamma_c G)$ by $\Gamma_c f(\chi) = \chi \circ f$ for $\chi \in \Gamma_c H$. Thus $\Gamma_c(-)$ is a contravariant functor from HCAG to HCAG (or a covariant functor from HCAG^{op} to HCAG). There is a natural transformation κ from the identity functor in HCAG to the covariant functor $\Gamma_c\Gamma_c(-) := \Gamma_c(\Gamma_c(-))$. This can be described by $\kappa_G\colon G \to \Gamma_c\Gamma_c G$ where $\left[\kappa_G(x)\right](\chi) = \chi(x)$ for any $x \in G$ and $\chi \in \Gamma_c G$. Note that if the starting group G is a topological group, then the continuous convergence in its c-bidual $\Gamma_c\Gamma_c G$ is also topological (see [6]). A convergence Abelian group G is said to be *c-reflexive* if κ_G is an isomorphism in HCAG. The continuity of $\omega\colon \Gamma_c G \times G \to \mathbb{T}$ implies that κ_G is also continuous and hence a morphism in HCAG$(G, \Gamma_c\Gamma_c G)$.

We now relate c-reflexivity to the classical Pontryagin reflexivity. Recall that for a group G in HTAG, ΓG with the compact open topology τ_{co} is a topological group usually denoted by $G^{\wedge}$. The group G is called *Pontryagin-reflexive* or *P-reflexive*, if the evaluation $\sigma_G \to G^{\wedge\wedge}$ is a topological isomorphism. Note that this evaluation may not even be a morphism in HTAG, since it may not be continuous. The duality theorem of Pontryagin-van Kampen was originally stated for groups in LCA. For a group G in this category, τ_{co} and Λ_c coincide in

[2]Note that in the Pontryagin setting the continuity of $\omega\colon G^{\wedge} \times G \to \mathbb{T}$ is a strong requirement since it forces any reflexive group G to be locally compact [12].

ΓG, hence in LCA there are no differences between P-duality and c-duality.[3] Therefore the original results of Pontryagin-van Kampen can be generalized in two directions. Given a group G, consider in ΓG either the compact open topology to study P-reflexivity (as in Pontryagin duality theory), or the continuous convergence structure to study c-reflexivity. We will adopt the latter point of view in the remaining sections.

3. Direct and inverse limits of convergence groups

A directed set $\mathcal{A}$ can be considered as a category where the objects are the elements $\alpha \in \mathcal{A}$ and the set of morphisms $\mathcal{A}(\alpha, \beta)$ consists of only one element if $\alpha \leq \beta$ and is empty otherwise. A *direct system* in HCAG is a covariant functor D from a directed set $\mathcal{A}$ to HCAG. We use the notation $\{G_\alpha, f_\alpha^\beta, \mathcal{A}\}$ for a direct system, where $G_\alpha = D(\alpha)$ are the groups and $f_\alpha^\beta = D\left(\mathcal{A}(\alpha, \beta)\right)$ the linking maps.

A *direct limit* or *inductive limit* for a direct system $\{G_\alpha, f_\alpha^\beta, \mathcal{A}\}$ in HCAG is a pair $(\varinjlim G_\alpha, \{p_\alpha\}_{\alpha\in\mathcal{A}})$, where $\varinjlim G_\alpha$ is an object in HCAG and the p_α's are morphisms in HCAG$(G_\alpha, \varinjlim G_\alpha)$ such that $p_\alpha = p_\beta \circ f_\alpha^\beta$ for $\alpha \leq \beta$, satisfying the following universal property: Given an object G' in HCAG and morphisms p'_α in HCAG(G_α, G') for all $\alpha \in \mathcal{A}$ such that $p'_\alpha = p'_\beta \circ f_\alpha^\beta$ whenever $\alpha \leq \beta$, there is a unique morphism p in HCAG$(\varinjlim G_\alpha, G')$ such that $p'_\alpha = p \circ p_\alpha$.

Dually, an inverse system in HCAG is a contravariant functor I from $\mathcal{A}$ to HCAG (or equivalently a covariant functor from $\mathcal{A}$ to HCAG^{op}, the opposite category). We will denote a generic inverse system by $\{G_\alpha, g_\beta^\alpha, \mathcal{A}\}$ and an *inverse limit* or *projective limit* by a pair $(\varprojlim G_\alpha, \{\pi_\alpha\}_{\alpha\in\mathcal{A}})$, where $\pi_\alpha \colon \varprojlim G_\alpha \to G_\alpha$.

In order to describe the standard constructions of inverse and direct limits in HCAG we first recall the notions of products and coproducts in this category.

Let $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ be a family in HCAG and let $\prod G_\alpha$ be the (algebraic) product. The *product convergence structure* on the group $\prod G_\alpha$ is the initial convergence structure with respect to the projections $\pi_\alpha \colon \prod G_\alpha \to G_\alpha$. This convergence structure makes $\prod G_\alpha$ an object in HCAG.

A filter $\mathcal{F}$ converges to an element $x \in \prod G_\alpha$ if and only if, for each $\alpha \in \mathcal{A}$, $\pi_\alpha(\mathcal{F})$ converges to $\pi_\alpha(x)$ in G_α. Observe that if all the convergence groups of the family $\{G_\alpha\}_{\alpha\in\mathcal{A}}$ are topological, then its convergence product is also topological.

The inverse limit of an inverse system $\{G_\alpha, g_\beta^\alpha, \mathcal{A}\}$ in HCAG, can be constructed as the following subgroup of the product $\prod G_\alpha$,

$$\left\{(x_\alpha)_{\alpha\in\mathcal{A}} \in \prod G_\alpha : g_\beta^\alpha(x_\beta) = x_\alpha\right\} .$$

The algebraic coproduct of Abelian groups $\bigoplus_{\alpha\in\mathcal{A}} G_\alpha$ is the group of all $x \in \prod G_\alpha$ such that $\{\alpha \in \mathcal{A}\colon\ \pi_\alpha(x) \neq e_{G_\alpha}\}$ is finite. The *coproduct convergence structure* is defined as the finest group convergence structure making the inclusions $i_\alpha \colon G_\alpha \to \bigoplus G_\alpha$ continuous.

The group $\bigoplus G_\alpha$ with the coproduct convergence structure is an object of HCAG called the *coproduct convergence group* of the family $\{G_\alpha\}_{\alpha\in\mathcal{A}}$.

[3] A metrizable topological Abelian group is P-reflexive if and only if it is c-reflexive [5]. However this equivalence is not true in general [6].

Considering the coproduct convergence on $\bigoplus G_\alpha$, the standard construction of the inductive limit in HCAG for a direct system $\{G_\alpha, f_\alpha^\beta, \mathcal{A}\}$ is the following

$$\varinjlim G_\alpha \cong (\bigoplus G_\alpha)/\bar{H},$$

where H is the subgroup generated by $\{i_\beta \circ f_\alpha^\beta(g_\alpha) - i_\alpha(g_\alpha)\colon\ \alpha \leq \beta;\ g_\alpha \in G_\alpha\}$, and $\bar{H}$ is the intersection of all the closed subgroups of G containing H.

4. Duality properties of limits

There are many interesting results published in the literature about c-duality of convergence groups. We will use two of them due to Beattie and Butzmann as the starting point of our study. The first result establishes the isomorphisms $\Gamma_c\left(\prod G_\alpha\right) \cong \bigoplus \Gamma_c G_\alpha$ and $\Gamma_c\left(\bigoplus G_\alpha\right) \cong \prod \Gamma_c G_\alpha$ where $(G_\alpha)_{\alpha\in\mathcal{A}}$, is any family of convergence Abelian groups. Consequently if the convergence groups (G_α) are all c-reflexive, both $\bigoplus G_\alpha$ and $\prod G_\alpha$ are also c-reflexive (pp. 214-215 of [3]).

Remark. Observe that if we work with arbitrary index sets we cannot translate this statement completely to the Pontryagin setting. The product of an arbitrary family of P-reflexive groups is P-reflexive, however the P-dual of the product cannot always be described as the coproduct of the P-dual system, as we noticed in the introduction.

The second result by Beattie and Butzmann (p. 229 of [3]) shows that the limit of an inverse system of locally compact topological groups is c-reflexive. We have further explored the duality relation between direct and inverse limits. Our first result describes the c-dual of the direct limit and it follows directly from categorical arguments.

THEOREM (4.1). *Let $\{G_\alpha, f_\alpha^\beta, \mathcal{A}\}$ be a a direct system of convergence groups. Then*

$$\Gamma_c(\varinjlim G_\alpha) \cong \varprojlim \Gamma_c G_\alpha$$

Proof. For each pair G and H of objects in HCAG and morphism $f\colon G \to \Gamma_c H$, there is a unique morphism $f'\colon H \to \Gamma_c G$ such that $\Gamma_c(f')\circ\kappa_G = f$. In fact, for $h \in H$ and $g \in G$, $f'(h)(g) = f(g)(h)$ and the map $A\colon$ HCAG$(G, \Gamma_c H) \to$ HCAG$(H, \Gamma_c G)$ which maps f to f' is continuous. Hence, the functor $\Gamma_c(-)\colon$ HCAG$^{op} \to$ HCAG is right adjoint to $\Gamma_c(-)\colon$ HCAG $\to$ HCAG^{op} and consequently, the contravariant functor $\Gamma_c(-)\colon$ HCAG $\to$ HCAG transforms direct into inverse limits whenever they exist ([8], p. 307). Hence

$$\Gamma_c(\varinjlim G_\alpha) \cong \varprojlim \Gamma_c G_\alpha \qquad \square$$

The c-dual of the inverse limit cannot be obtained in such a natural way and requires restrictions on the groups and morphisms, which we proceed to describe.

Denote $\mathbb{T}_+ = \{z \in \mathbb{T}\,|\operatorname{Re} z \geq 0\}$. For a convergence group G, the *polar* of a subset $A \subset G$ is the set $A^\triangleright = \{\chi \in \Gamma G\colon \chi(A) \subset \mathbb{T}_+\}$ and the inverse polar of a subset $B \subset \Gamma G$ is $B^\triangleleft = \{x \in G\colon \chi(x) \subset \mathbb{T}_+ \text{ for all } \chi \in B\}$.

Let G be an object of HCAG. A subgroup H of G is called *dually closed* in G if for every $x \in G \setminus H$ there exists a character $\chi \in \Gamma G$ with $\chi(H) = e_\mathbb{T}$ and

$\chi(x) \neq e_{\mathbb{T}}$. A subgroup H of G is called *dually embedded* if every character of H extends to a character of G. Note that a subgroup H of G is dually closed in G if and only if $H = H^{\triangleright\triangleleft}$.

PROPOSITION (4.2). (1) *Let* $\{G_\alpha, f_\alpha^\beta, \mathcal{A}\}$ *be a direct system of convergence groups and* $H = gp\{i_\alpha(x_\alpha) - i_\beta \circ f_\alpha^\beta(x_\alpha)\colon \alpha \leq \beta;\ x_\alpha \in G_\alpha\}$. *Then*

$$H^{\triangleright} = \varprojlim \Gamma_c G_\alpha.$$

(2) *Let* $\{G_\alpha, g_\beta^\alpha, \mathcal{A}\}$ *be an inverse system of convergence groups where the limit maps* π_α *have dense images. Let* $L = gp\{i_\alpha(\varphi_\alpha) - i_\beta \circ \Gamma_c(g_\beta^\alpha)(\varphi_\alpha) : \alpha \leq \beta, \varphi_\alpha \in \Gamma_c G_\alpha\}$. *Then*

$$(\varprojlim G_\alpha)^{\triangleright} = L.$$

Proof. First part:

Given $(\varphi_\alpha)_{\alpha\in\mathcal{A}} \in \prod \Gamma_c G_\alpha$ and $x_\alpha \in G_\alpha$, the following equalities hold:

$$(\varphi_\alpha)(i_\alpha(x_\alpha) - i_\beta \circ f_\alpha^\beta(x_\alpha)) = \varphi_\alpha(x_\alpha) - \varphi_\beta(f_\alpha^\beta(x_\alpha)) = \varphi_\alpha(x_\alpha) - \Gamma_c f_\alpha^\beta(\varphi_\beta)(x_\alpha)\,.$$

From here it follows, on the one hand, that if $(\varphi_\alpha)_{\alpha\in\mathcal{A}} \in \varprojlim \Gamma_c G_\alpha$, then $(\varphi_\alpha)(i_\alpha(x_\alpha) - i_\beta \circ f_\alpha^\beta(x_\alpha)) = e_{\mathbb{T}}$ and on the other hand if $(\varphi_\alpha)_{\alpha\in\mathcal{A}} \in H^{\triangleright}$, then $\Gamma_c f_\alpha^\beta(\varphi_\beta) = \varphi_\alpha$ since $(\Gamma_c f_\alpha^\beta(\varphi_\beta) - \varphi_\alpha)(x_\alpha) = e_{\mathbb{T}}$ for all $x_\alpha \in G_\alpha$.

Second part:

If $(x_\alpha)_{\alpha\in\mathcal{A}} \in \varprojlim G_\alpha$, we have that $g_\beta^\alpha(x_\beta) = x_\alpha$, hence

$$\begin{aligned}(i_\alpha(\varphi_\alpha) - i_\beta \circ \Gamma_c(g_\beta^\alpha)(\varphi_\alpha))(x_\alpha)_{\alpha\in\mathcal{A}} &= \varphi_\alpha(x_\alpha) - \left(\Gamma_c(g_\beta^\alpha)(\varphi_\alpha)\right)(x_\beta) \\ &= \varphi_\alpha(x_\alpha) - \varphi_\alpha\left(g_\beta^\alpha(x_\beta)\right) \\ &= \varphi_\alpha(x_\alpha) - \varphi_\alpha(x_\alpha) = e_{\mathbb{T}},\end{aligned}$$

and we have proven that $L \subset (\varprojlim G_\alpha)^{\triangleright}$.

We are left to prove the opposite inclusion. Any element $(\varphi_\alpha)_{\alpha\in\mathcal{A}} \in (\varprojlim G_\alpha)^{\triangleright}$ can be represented as a finite sum

$$(\varphi_\alpha)_{\alpha\in\mathcal{A}} = i_{\alpha_1}(\varphi_{\alpha_1}) + \cdots + i_{\alpha_k}(\varphi_{\alpha_k})\,.$$

where $\alpha_k \geq \alpha_1, \ldots, \alpha_{k-1}$

Consider now an arbitrary element $x_{\alpha_k} \in \pi_{\alpha_k}(\varprojlim G_\alpha)$ and let $(x_\alpha)_{\alpha\in\mathcal{A}}$ be an element of the inverse limit with α_k coordinate x_{α_k}. We know that $g_\beta^\alpha(x_\beta) = x_\alpha$, $\alpha \leq \beta$ and since $(\varphi_\alpha)_{\alpha\in\mathcal{A}}$ is in the polar of $\varprojlim G_\alpha$, we have

$$\begin{aligned}&((\Gamma_c(g_{\alpha_k}^{\alpha_1}))(\varphi_{\alpha_1}) + \cdots + (\Gamma_c(g_{\alpha_k}^{\alpha_{k-1}}))(\varphi_{\alpha_{k-1}}) + \varphi_{\alpha_k})(x_{\alpha_k}) \\ &\qquad= (\varphi_{\alpha_1} g_{\alpha_k}^{\alpha_1} + \cdots + \varphi_{\alpha_{k-1}} g_{\alpha_k}^{\alpha_{k-1}} + \varphi_{\alpha_k})(x_{\alpha_k}) \\ &\qquad= \varphi_{\alpha_1}(x_{\alpha_1}) + \cdots + \varphi_{\alpha_k}(x_{\alpha_k}) \\ &\qquad= (\varphi_\alpha)_{\alpha\in\mathcal{A}}((x_\alpha)_{\alpha\in\mathcal{A}}) = e_{\mathbb{T}}\end{aligned}$$

and hence, since $\pi_{\alpha_k}(\varprojlim G_\alpha)$ is dense in G_{α_k},

$$\left((\Gamma_c(g_{\alpha_k}^{\alpha_1}))(\varphi_{\alpha_1}) + \cdots + (\Gamma_c(g_{\alpha_k}^{\alpha_{k-1}}))(\varphi_{\alpha_{k-1}}) + \varphi_{\alpha_k}\right) = e_{\Gamma_c G_{\alpha_k}}.$$

We can now subtract this term from the expression for $(\varphi_\alpha)_{\alpha\in\mathcal{A}}$ which is enough to obtain our result. More concretely,

$$\begin{aligned}(\varphi_\alpha)_{\alpha\in\mathcal{A}} &= i_{\alpha_1}(\varphi_{\alpha_1}) + \cdots + i_{\alpha_k}(\varphi_{\alpha_k}) \\ &= i_{\alpha_1}(\varphi_{\alpha_1}) + \cdots + i_{\alpha_k}(\varphi_{\alpha_k}) \\ &\quad - i_{\alpha_k}\Big((\Gamma_c(g^{\alpha_1}_{\alpha_k}))(\varphi_{\alpha_1}) + \cdots + (\Gamma_c(g^{\alpha_{k-1}}_{\alpha_k}))(\varphi_{\alpha_{k-1}}) + \varphi_{\alpha_k}\Big) \\ &= i_{\alpha_1}(\varphi_{\alpha_1}) - i_{\alpha_k}(\Gamma_c(g^{\alpha_1}_{\alpha_k}))(\varphi_{\alpha_1}) + \dots \\ &\quad + i_{\alpha_{k-1}}(\varphi_{\alpha_{k-1}}) - i_{\alpha_k}(\Gamma_c(g^{\alpha_{k-1}}_{\alpha_k}))(\varphi_{\alpha_{k-1}}) + i_{\alpha_k}(\varphi_{\alpha_k}) - i_{\alpha_k}(\varphi_{\alpha_k}),\end{aligned}$$

from which we conclude $(\varprojlim G_\alpha)^\triangleright \subset L$. □

We describe the c-dual of the inverse limits in the class of *Nuclear groups*. Roughly speaking a Hausdorff Abelian group G is Nuclear if each neighborhood of zero contains another neighborhood which is "sufficiently small"[4]. This class of groups, introduced by Banaszczyk in [2], has good permanence properties — subgroups, quotients and products of nuclear groups are nuclear groups. Locally compact groups are nuclear and the groups underlying nuclear locally convex topological vector spaces are also in the class of nuclear groups. Banaszczyk succeeded in generalizing many properties of LCA groups to nuclear groups.

LEMMA (4.3). *Every subgroup H of a nuclear group G is dually embedded and $\Gamma_c i\colon \Gamma_c G \to \Gamma_c H$ is a quotient mapping with kernel $H^\triangleright$.*

Proof. See Corollary 8.3 in [2] and Corollary 8.4.10 in [3]. □

Our first description of the c-dual of an inverse limit also requires some restriction on the limit maps.

THEOREM (4.4). *Let $\{G_\alpha, g^\alpha_\beta; \mathcal{A}\}$ be an inverse system of nuclear groups where the limit maps π_α have dense images. Then*

$$\Gamma_c(\varprojlim G_\alpha) \cong \varinjlim \Gamma_c G_\alpha$$

Proof. We have by (4.2) (2) that

$$(\varprojlim G_\alpha)^\triangleright = gp\{i_\alpha(\varphi_\alpha) - i_\beta \circ \Gamma_c(g^\alpha_\beta)(\varphi_\alpha), : \alpha \le \beta, \varphi_\alpha \in \Gamma_c G_\alpha\}.$$

It follows that $\varinjlim \Gamma_c G_\alpha$ is the quotient convergence group $(\bigoplus \Gamma_c G_\alpha)/(\varprojlim G_\alpha)^\triangleright$. But this is an object in HCAG isomorphic to $\Gamma_c(\prod G_\alpha)/(\varprojlim G_\alpha)^\triangleright$. We still need to prove that $\Gamma_c(\varprojlim G_\alpha)$ is isomorphic to this object. In order to do that we use Lemma (4.3) about subgroups of nuclear groups:

Since all groups G_α are nuclear groups the product $\prod G_\alpha$ is nuclear, therefore by Lemma (4.3), $\Gamma i\colon \Gamma_c(\prod G_\alpha) \to \Gamma_c(\varprojlim G_\alpha)$ is a quotient mapping with kernel $(\varprojlim G_\alpha)^\triangleright$ which induces an isomorphism $\psi\colon \Gamma_c(\prod G_\alpha)/(\varprojlim G_\alpha)^\triangleright \to \Gamma_c(\varprojlim G_\alpha)$ in the category HCAG. Hence the assertion follows. □

[4] A Hausdorff Abelian group is called Nuclear if it satisfies the following condition: Given an arbitrary neighborhood U of e_G, $c > 0$ and $m = 1, 2, \dots$, there exists a vector space E and two pre-Hilbert seminorms p, q on E with $d_k(B_p, B_q) \le ck^{-m}$, where d_k is the kth Kolmogorov diameter and $k = 1, 2, \dots$, ([2] p. 72)

We now give an alternative description of the c-dual of an inverse limit without any condition on the limit maps. Let G be a convergence group. We will say that G *has enough characters* if $\kappa_G\colon G \to \Gamma_c\Gamma_c G$ is injective, i.e., if for all $x \in G$, $x \neq e_G$ there exists $\chi \in \Gamma_c G$ such that $\chi(x) \neq e_{\mathbb{T}}$. Given an arbitrary convergence group G, it is easy to check that $G/\ker(\kappa_G)$ is a convergence group with enough characters.

Denote by $\mathrm{HCAG}_{\kappa_{1:1}}$ the category of convergence groups with enough characters, we can define a full functor $F\colon\ \mathrm{HCAG} \to \mathrm{HCAG}_{\kappa_{1:1}}$ by $F(G) = G/\ker(\kappa_G)$. The functor F is left adjoint to the inclusion functor $\mathrm{HCAG}_{\kappa_{1:1}} \to \mathrm{HCAG}$ and hence it preserve direct limits, i.e., $F(\varinjlim G_\alpha) = \varinjlim(FG_\alpha)$.

LEMMA (4.5). *Let G be a Hausdorff convergence group and H a closed subgroup of G, then $F(G/H) \cong G/H^{\triangleright\triangleleft}$.*

Proof. Since $F(G/H) \cong \frac{G/H}{\ker(\kappa_{G/H})}$, it is enough to see that $\ker(\kappa_{G/H})$ is precisely $H^{\triangleright\triangleleft}/H$. Now for $x \in G$, $\kappa_{G/H}[x] = e_{\Gamma_c\Gamma_c(G/H)}$ iff $\chi[x] = e_{\mathbb{T}}$ for all $\chi \in \Gamma_c(G/H)$ which is the same as the statement: $\widetilde{\chi}(x) = e_{\mathbb{T}}$ for all $\widetilde{\chi} \in \Gamma_c G$ such that $\widetilde{\chi}(H) = e_{\mathbb{T}}$ and this occurs if and only if $x \in H^{\triangleright\triangleleft}$. $\square$

THEOREM (4.6). *Let $\{G_\alpha, g^\alpha_\beta, \mathcal{A}\}$ be an inverse system of complete nuclear topological groups. Then*

$$\Gamma_c(\varprojlim G_\alpha) \cong F(\varinjlim \Gamma_c G_\alpha)\,.$$

Proof. Note that a nuclear group is complete if and only if it is c-reflexive (see [4]). We know that $\varprojlim G_\alpha$ is a subgroup of $\prod G_\alpha$, which in turn is a nuclear group. Hence by 8.4.5 in [3] $\Gamma_c(i)\colon \Gamma_c(\prod G_\alpha) \to \Gamma_c(\varprojlim G_\alpha)$ is a quotient map with kernel $(\varprojlim G_\alpha)^{\triangleright}$. This map induces an isomorphism $\Gamma_c(\prod G_\alpha)/(\varprojlim G_\alpha)^{\triangleright} \to \Gamma_c(\varprojlim G_\alpha)$ in HCAG.

Denote by $L = gp\{i_\alpha(\varphi_\alpha) - i_\beta \circ \Gamma_c(g^\alpha_\beta)(\varphi_\alpha) : \alpha \leq \beta, \varphi_\alpha \in \Gamma_c G_\alpha\}$

Now by (4.2).1 we have that $L^{\triangleright} = \varprojlim(\Gamma_c\Gamma_c G_\alpha) \cong \varprojlim G_\alpha$. Hence $L^{\triangleright\triangleright} \cong (\varprojlim G_\alpha)^{\triangleright}$. The c-reflexivity of $\bigoplus \Gamma_c G_\alpha$ yields $(\varprojlim G_\alpha)^{\triangleright} = L^{\triangleright\triangleleft}$. Finally

$$\begin{aligned}\Gamma_c(\varprojlim G_\alpha) &\cong \frac{\Gamma_c(\prod G_\alpha)}{(\varprojlim G_\alpha)^{\triangleright}} \cong \frac{\bigoplus \Gamma_c G_\alpha}{(\varprojlim G_\alpha)^{\triangleright}}\\ &= \frac{\bigoplus \Gamma_c G_\alpha}{L^{\triangleright\triangleleft}} = F\left(\frac{\bigoplus \Gamma_c G_\alpha}{\overline{L}}\right) = F(\varinjlim \Gamma_c G_\alpha).\end{aligned}$$

$\square$

Received June 27, 2006

Final version received December 15, 2006

UNIVERSITY OF NAVARRA
DEPARTMENT OF PHYSICS AND APPLIED MATHEMATICS
E-31080 PAMPLONA
SPAIN
sardanza@unav.es
mjchasco@unav.es

REFERENCES

[1] S. ARDANZA-TREVIJANO AND M. J. CHASCO, *The Pontryagin duality of sequential limits of topological Abelian groups*, J. Pure Appl. Algebra **202** (2005), 11–21.

[2] W. BANASZCZYK, Additive Subgroups of Topological Vector Spaces, Lecture Notes in Math. 1466, Springer-Verlag, Berlin. 1991.

[3] R. BEATTIE, H.-P. BUTZMANN, Convergence structures and applications to Functional Analysis, Kluwer Academic Publishers Dordrecht. 2001.

[4] M. M. BRUGUERA, M. J. CHASCO, E. MARTIN PEINADOR, V. TARIELADZE, *Completeness properties of locally quasi-convex groups*, Topology Appl. **111** (2001), 81–93.

[5] M. J. CHASCO, *Pontryagin duality for metrizable groups*, Arch. Math. (Basel) **70** (1) (1998), 22–28.

[6] M. J. CHASCO, E. MARTÍN-PEINADOR, *Binz-Butzmann duality versus Pontryagin Duality*. Arch. Math. (Basel) **63** (3) (1994), 264–270.

[7] H. GLÖCKNER, R. GRAMLICH AND T. HARTNICK, *Final group topologies, Phan systems and Pontryagin duality*, preprint, arXiv:math.GR/0603537.

[8] K. H. HOFMANN AND S. A. MORRIS, The structure of compact groups, De Gruyter Studies in Mathematics 25, Berlin. 1998.

[9] S. KAPLAN, *Extensions of the Pontryagin duality I: Infinite products*, Duke Math. J. **15** (1948), 649–658.

[10] S. KAPLAN, *Extensions of the Pontryagin duality II: Direct and inverse sequences*, Duke Math. J. **17** (1950), 419–435.

[11] H. LEPTIN, *Zur Dualitatstheorie projektiver Limites abelscher Gruppen*, Abh. Math. Sem. Univ. Hamburg **19** (1955), 264–268.

[12] E. MARTÍN PEINADOR, *A reflexive admissible topological group must be locally compact*, Pro. Amer. Math. Soc. **123** (1955), 3563–3566.

[13] P. NICKOLAS, *Coproducts of Abelian topological groups*, Topology Appl. **120(3)** (2002), 403–426.

[14] V. PESTOV, *Free Abelian topological groups and the Pontryagin-van Kampen duality*, Bull. Austral Math. Soc. **52(2)** (1995), 297–311.

[15] M. F. SMITH, *The Pontrjagin duality theorem in linear spaces*, Ann. of Math. (2) **56** (1952), 248–253.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

ON DERIVED TAME ALGEBRAS

This paper is dedicated to the memory of Professor Andrey Vladimirovich Roiter

RAYMUNDO BAUTISTA

ABSTRACT. Let Λ be a finite-dimensional algebra over an algebraically closed field k. We prove that the bounded derived category $\mathcal{D}^b(\Lambda)$ has tame representation type (Λ is called derived tame), if and only if the full subcategory of $\mathcal{D}^b(\Lambda)$ whose objects are perfect complexes is of tame representation type. We see that if Λ is derived tame, then almost all isomorphism classes of indecomposable complexes $X^\bullet \in \mathcal{D}^b(\Lambda)$ with fixed homology dimension are perfect and have Auslander-Reiten triangles of the form $X^\bullet \to H^\bullet \to X^\bullet \to X^\bullet[1]$.

1. Introduction

Let Λ be a finite-dimensional algebra over an algebraically closed field k and let $\mathcal{D}^b(\Lambda)$ be its bounded derived category. We consider the category of left Λ-modules $\operatorname{Mod}\Lambda$. We denote by $\operatorname{mod}\Lambda$, $\operatorname{Proj}\Lambda$, $\operatorname{proj}\Lambda$, $\operatorname{Inj}\Lambda$ and $\operatorname{inj}\Lambda$ the full subcategories of $\operatorname{Mod}\Lambda$ consisting of the finitely generated, the projective, the finitely generated projective, the injective and the finitely generated injective Λ-modules, respectively. By $\mathcal{D}^b(\operatorname{Mod}\Lambda)$ we denote the bounded derived category of $\operatorname{Mod}\Lambda$; we recall that $\mathcal{D}^b(\Lambda)$ is the bounded derived category of the category $\operatorname{mod}\Lambda$. If $X = (X^i, d_X^i)_{i\in\mathbb{Z}}$ is an object in $\mathcal{D}^b(\Lambda)$ an invariant of it is given by its homology dimension $\mathbf{h}\dim = (h_i)_{i\in\mathbb{Z}}$ with $h_i = \dim_k H^i(X)$.

A sequence of non negative integers $\mathbf{h} = (h_i)_{i\in\mathbb{Z}}$ is called a homology dimension if for all but finitely many i, $h_i = 0$. We recall that according with [20], $\mathcal{D}^b(\Lambda)$ is called discrete and Λ derived discrete if there are only finitely many isoclasses of indecomposables $X \in \mathcal{D}^b(\Lambda)$ with fixed homology dimension. As for algebras, definitions of tame representation type and of wild representation type have been given in [13] for the category $\mathcal{D}^b(\Lambda)$. The algebra Λ is called derived tame or derived wild if the category $\mathcal{D}^b(\Lambda)$ is of tame representation type or of wild representation type, respectively.

The Happel functor, introduced in [15], from the bounded derived category of a finite-dimensional algebra into the stable category of the corresponding repetitive algebra, has been an important tool in the study of the bounded derived category of an algebra. However this functor is not an equivalence of categories for algebras of infinite global dimension. The methods proposed in this paper overcome this difficulty.

In [20] it has been proved that Λ is derived discrete if and only if $\mathcal{D}^b(\Lambda)_{\mathrm{perf}}$, the full subcategory of $\mathcal{D}^b(\Lambda)$ whose objects are the perfect complexes, is discrete. We prove that a similar fact is also true for the tame case: Λ is derived

2000 *Mathematics Subject Classification*: 16G70, 18E30, 18G35.

Keywords and phrases: derived tame, Auslander-Reiten triangle, bocs.

The author thanks the support of project "43374F" of Fondo Sectorial SEP-Conacyt.

tame if and only if $\mathcal{D}^b(\Lambda)_{\mathrm{perf}}$ is of tame representation type. In fact we prove that almost all isomorphism classes of indecomposable objects in $\mathcal{D}^b(\Lambda)$ of given homology dimension are isomorphism classes of perfect complexes.

We also prove that if Λ is derived tame and $\mathbf{h}$ is a fixed homology dimension, then for almost all isomorphism classes $[Y]$ with Y indecomposable perfect complex and $\mathbf{h}\mathrm{dim}Y = \mathbf{h}$, there is an Auslander-Reiten triangle of the form

$$Y \to H \to Y \to Y[-1].$$

In addition, if $\mathbf{h} = (h_i)$ and n_0 is the integer such that $h_{n_0} \neq 0$ and $h_i = 0$ for $i < n_0$, then $Y^j = 0$ for $j < n_0 - 1$ and $d_Y^{n_0-1}\colon Y^{n_0-1} \to Y^{n_0}$ is a monomorphism. This implies that for Λ derived tame for any fixed non-negative integer, almost all isomorphism classes of indecomposable Λ-modules $[M]$ with $\dim_k M \leq d$, the projective dimension of M is equal to one.

In order to prove the above results, we consider in section 2, $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ which is the category of complexes X of finitely generated projective Λ-modules with $X^i = 0$ for i outside the interval $[1, \ldots, m]$. We denote by $\mathbf{C^1_m}(\mathrm{proj}\,\Lambda)$ the full subcategory of $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ whose objects are the complexes X such that $\mathrm{Im} d_X^{i-1} \subset \mathrm{rad} X^i$ for all $i \in \mathbb{Z}$.

In general if $\mathcal{C}$ is a k-category, a morphism $f\colon M \to N$ in $\mathcal{C}$ is called radical if for any split monomorphism $\sigma\colon X \to M$ and any split epimorphism $\pi\colon M \to Y$, $\pi f \sigma\colon X \to Y$ is not an isomorphism. If P and Q are projective Λ-modules, $f\colon P \to Q$ is a radical morphism if and only if $\mathrm{Im} f \subset \mathrm{rad} Q$.

In section 6 we prove the following two results.

THEOREM (1.1). *For fixed m, either* $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ *is of tame representation type or of wild representation type.*

The proof of this last result is in fact given in [6] and [11], using bocses with relations. We present a different proof using just free triangular bocses. We recall from [3] that we have an exact category $(\mathbf{C_m}(\mathrm{proj}\,\Lambda), \mathcal{E})$ in the sense of [19] or [12], where $\mathcal{E}$ is the class of sequences of morphisms (conflations)

$$X \xrightarrow{u} E \xrightarrow{v} Y$$

such that for all $i \in \mathbb{Z}$ the sequence

$$0 \to X^i \xrightarrow{u^i} E^i \xrightarrow{v^i} Y^i \to 0,$$

is an split exact sequence. The exact category $(\mathbf{C_m}(\mathrm{proj}\,\Lambda), \mathcal{E})$ has enough projectives and injectives and it has almost split sequences (see (a) of Theorem 8.2 of [3]).

Definition (1.2). For a complex $X \in \mathbf{C_m}(\mathrm{proj}\,\Lambda)$ its dimension is given by $\dim_k X = \sum_i \dim_k X^i$.

THEOREM (1.3). *Suppose* $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ *is of tame representation type. Then for almost all isomorphism classes* $[X]$ *of indecomposables in the category* $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$*, with fixed dimension, there is an almost split* $\mathcal{E}$*-sequence in* $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ *of the form* $X \to E \to X$.

For this we use tbocses (introduced in [2]) in a similar way as in [6].

In section 7 we consider generic complexes in $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ in the sense of section 5 of [18]; observe that this definition differs from the one given in

[13]. With our definition we obtain similar results to the ones given in [9] for Λ-modules. In particular each generic complex is closely related to a one-parameter family of objects in $\mathcal{D}^b(\Lambda)$. In addition we prove that if X is a generic complex for a derived tame algebra Λ, X is isomorphic in $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ to a bounded complex of projective Λ-modules.

2. Bounded derived categories

Here we see some consequences of Theorems (1.1) and (1.3) for the derived category $\mathcal{D}^b(\Lambda)$.

In the following a rational algebra is a k-algebra of the form $R = k[x, h(x)^{-1}]$, with $h(x) \in k[x]$. The support of the rational algebra R is defined by $\mathcal{S}(R) = \{\lambda \in k | h(\lambda) \neq 0\}$. For $\lambda \in \mathcal{S}(R)$, the simple R-module $k[x]/(x - \lambda)$ will be denoted by S_λ.

Notation (2.1). For $\mathbf{h}$ a homology dimension we denote by $\mathcal{V}(\mathbf{h})$ the full subcategory of $\mathcal{D}^b(\Lambda)$ whose objects are indecomposables $X \in \mathcal{D}^b(\Lambda)$ with $\mathbf{h}\dim X = \mathbf{h}$.

We recall the following definitions:

(1) Λ is called *derived discrete* if for each homology dimension $\mathbf{h}$, the category $\mathcal{V}(\mathbf{h})$ has only finitely many isomorphism classes.

(2) Λ is called *derived tame* if for each homology dimension $\mathbf{h}$ there is a finite set of rational algebras $R_u, u = 1, \ldots, s$ and for each u a bounded complex M_u of Λ-R_u-bimodules free of finite rank over R_u, such that for almost all isomorphism classes $[X]$ with $X \in \mathcal{V}(\mathbf{h})$ there is a $\lambda \in \mathcal{S}(R_u)$ with $X \cong M_u \otimes_{R_u} S_\lambda$ for some $u \in \{1, \ldots, s\}$.

(3) Λ is called *derived wild* if there is a bounded complex W of Λ-$k\langle x, y\rangle$-bimodules free of finite rank over $k\langle x, y\rangle$ such that the functor

$$W \otimes_{k\langle x,y\rangle} - : \mathrm{mod}\, k\langle x, y\rangle \to \mathcal{D}^b(\Lambda)$$

preserves isoclasses and indecomposables.

Concerning the categories $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ we recall the definitions of finite representation type, tame representation type and wild representation type.

(4) $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ is called of *finite representation type* if it has only a finite number of isomorphism classes of indecomposables.

(5) $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ is called of *tame representation type* if for any given positive integer d there are rational algebras $R_u, u = 1, \ldots, s$ and for each u a complex $M_u = (M_u^i, d_{M_u}^i)$ with M_u^i a Λ-R_u-bimodule free of finite rank over R_u, projective as Λ-module and $M_u^i = 0$ for i outside the set $\{1, \ldots, m\}$, such that for almost all isomorphism class $[Y]$ with Y indecomposable and $\dim_k Y \leq d$ there is a $\lambda \in \mathcal{S}(R_u)$ such that $M_u \otimes_{R_u} S_\lambda \cong Y$.

(6) $\mathbf{C_m}(\mathrm{proj}\,\Lambda)$ is called of *wild representation type* if there is a bounded complex of Λ-$k\langle x, y\rangle$-bimodules free of finite rank over $k\langle x, y\rangle$, projectives as Λ-modules, $W = (W^i, d_W^i)$ with $W^i = 0$ for i outside the set $\{1, \ldots, m\}$, such that the functor:

$$W \otimes_{R_u} - : \mathrm{mod}\, k\langle x, y\rangle \to \mathbf{C_m}(\mathrm{proj}\,\Lambda)$$

preserves isoclasses and indecomposables.

We need the following results (see Lemma 2.1 and Lemma 2.2 of [1]).

LEMMA (2.2). *Suppose* $Y = (Y^i, d_Y^i)_{i\in\mathbb{Z}} \in \mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\operatorname{proj}\Lambda)$ *is such that*

$$\dim_k H^j(Y) \leq c$$

for all j *and for some* $u \in [2,\ldots,m]$, $\dim_k Y^u \leq d_u$. *Then*

$$\dim_k Y^{u-1} \leq (d_u + c)L,$$

with $L = \dim_k\Lambda$.

LEMMA (2.3). *Let* $Y = (Y^i, d_Y^i)_{i\in\mathbb{Z}} \in \mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\operatorname{proj}\Lambda)$ *such that for some fixed* c *and all* $j \in [1,m]$, *we have* $\dim_k H^j(Y) \leq c$. *Then*

$$\dim_k Y \leq c(mL + (m-1)L^2 + (m-2)L^3 + \cdots + 2L^{m-1} + L^m).$$

We denote by $\mathbf{C}^{\leq\mathbf{m},\mathbf{b}}(\operatorname{Proj}\Lambda)$ the category of complexes $X = (X^i, d_X^i)$ with $X^i \in \operatorname{Proj}\Lambda$ and $X^i = 0$ for $i > m$, such that $H^i(X) = 0$ for almost all i. By $\mathbf{K}^{\leq\mathbf{m},\mathbf{b}}(\operatorname{Proj}\Lambda)$ we denote the corresponding homotopy category.

Following [3] we denote by $\mathcal{L}_m$ the full subcategory of $\mathbf{K}^{\leq\mathbf{m},\mathbf{b}}(\operatorname{Proj}\Lambda)$ whose object are those X with $H^i(X) = 0$ for $i \leq 1$.

The functor $F\colon \mathbf{K}^{\leq\mathbf{m},\mathbf{b}}(\operatorname{Proj}\Lambda) \to \mathbf{C}_{\mathbf{m}}(\operatorname{Proj}\Lambda)$ which sends a complex

$$X\colon \cdots \to X^{-1} \overset{d^{-1}}{\to} X^0 \overset{d^0}{\to} X^1 \overset{d^1}{\to} \cdots \to X^m \to 0$$

to

$$F(X)\colon \cdots 0 \to 0 \to X^1 \overset{d^1}{\to} \cdots \to X^m \to 0,$$

induces an equivalence

$$\underline{F}\colon \mathcal{L}_m \to \overline{\mathbf{C}_{\mathbf{m}}}(\operatorname{Proj}\Lambda),$$

where $\overline{\mathbf{C}_{\mathbf{m}}}(\operatorname{Proj}\Lambda)$ is the category with the same objects as $\mathbf{C}_{\mathbf{m}}(\operatorname{Proj}\Lambda)$ and as morphisms those in $\mathbf{C}_{\mathbf{m}}(\operatorname{Proj}\Lambda)$ modulo the ones which are factorized through $\mathcal{E}$-injective objects (see Corollary 5.7 of [3]).

Moreover we have an embedding

$$\tau^{\geq 1}\colon \mathcal{L}_m \to \mathcal{D}^b(\operatorname{Mod}\Lambda).$$

Observe that for $P \in \mathcal{L}_m$, $q\colon P \to \tau^{\geq 1}P$ the natural morphism is a quasi-isomorphism.

For a natural number d we denote by $\mathcal{F}_d$ the full subcategory of $\overline{\mathbf{C}_{\mathbf{m}}}(\operatorname{proj}\Lambda)$ whose objects are those indecomposables X with $\dim_k X \leq d$. We denote by $\mathcal{U}_d$ the full subcategory of $\mathcal{L}_m$ whose objects are those $Y \cong F(P)$ with $P \in \mathcal{F}_d$. By $\mathcal{V}_d$ we denote the full subcategory of $\mathcal{D}^b(\Lambda)$ whose objects are those isomorphic to some $\tau^{\geq 1}P$ with $P \in \mathcal{U}_d$.

By Lemma (2.3), $\mathcal{V}(\mathbf{h}) \subset \mathcal{V}_d$, if $d = |\mathbf{h}|(mL + (m-1)L^2 + \cdots + 2L^{m-1} + L^m)$ with $|\mathbf{h}| = \max\{h_i\}_{i\in\mathbb{Z}}$, $L = \dim_k\Lambda$.

THEOREM (2.4). (a) Λ *is derived discrete if and only if for all* m, $\mathbf{C}_{\mathbf{m}}(\operatorname{proj}\Lambda)$ *is of finite representation type;*

(b) *if* Λ *is derived wild it is not derived tame;*

(c) *if for some* m, $\mathbf{C}_{\mathbf{m}}(\operatorname{proj}\Lambda)$ *is of wild representation type then* Λ *is derived wild;*

(d) Λ *is derived tame if and only if for all* m, $\mathbf{C}_{\mathbf{m}}(\operatorname{proj}\Lambda)$, *is of tame representation type;*

(e) Λ *is either derived tame or derived wild (see Bekkert-Drozd* [6]*).*

Proof. Suppose Λ is derived discrete; then by [20] Λ is derived hereditary of Dynkin type or it is a gentle algebra.

For a Krull-Schmidt category $\mathcal{C}$ we denote by $\operatorname{ind}\mathcal{C}$ the full subcategory of $\mathcal{C}$ whose objects are the indecomposables of $\mathcal{C}$.

If Λ is hereditary of Dynkin type, then $\mathbf{C_2}(\operatorname{proj}\Lambda)$ is of finite representation type; for $m > 2$ we have:

$$\operatorname{ind}\mathbf{C_m}(\operatorname{proj}\Lambda) \subset \operatorname{ind}\mathbf{C_2}(\operatorname{proj}\Lambda) \cup \operatorname{ind}\mathbf{C_2}(\operatorname{proj}\Lambda)[1] \cup \cdots \cup \operatorname{ind}\mathbf{C_2}(\operatorname{proj}\Lambda)[m-1]$$

then $\operatorname{ind}\mathbf{C_m}(\operatorname{proj}\Lambda)$ has only finitely many isomorphism classes; thus it is of finite representation type.

If Λ is derived equivalent to a hereditary algebra A of Dynkin type, there is a bounded complex T over Λ-A-bimodules projective finitely generated over both sides such that the functor

$$- \otimes^{\mathbf{L}} T : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(A)$$

is an equivalence. Then for m there is an n and an l such that we have a functor

$$G(-) = - \otimes_\Lambda T[l] : \mathbf{C_m}(\operatorname{proj}\Lambda) \to \mathbf{C_{m+n}}(\operatorname{proj} A)$$

with the following property: if Y and X are indecomposables in $\mathbf{C_m}(\operatorname{proj}\Lambda)$ which are not $\mathcal{E}$-injectives or $\mathcal{E}$-projectives then their images under G are also indecomposables and $G(Y) \cong G(X)$ implies $Y \cong X$. Here $\mathbf{C_{m+n}}(\operatorname{proj} A)$ is of finite representation type; then also $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of finite representation type.

Now suppose that Λ is a gentle algebra $k(Q, I)$. Then from the description of the objects in $\mathbf{K}^{-,\mathbf{b}}(\operatorname{proj}\Lambda)$ in [7] one can see that if there are generalized strings in Q of arbitrary size corresponding to complexes in $\mathbf{C_m}(\operatorname{proj}\Lambda)$ for some fixed m, then there are generalized bands, but this implies that Λ is not derived discrete, therefore for any m, $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of finite representation type.

Conversely assume $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of finite representation type for all m.

Take $\mathbf{h} = (h_i)$ a homology dimension; we may assume $h_i = 0$ for i outside the set $\{2, \ldots, m\}$. Take $d = |\mathbf{h}|(mL + (m-1)L^2 + \cdots + 2L^{m-1} + L^m)$, then by Lemma (2.3), $\mathcal{V}(\mathbf{h}) \subset \mathcal{V}_d$. The categories $\mathcal{V}_d$, $\mathcal{U}_d$ and $\mathcal{F}_d$ are equivalent, by assumption $\mathcal{F}_d$ has only a finite number of isoclasses, and the same is true for $\mathcal{V}(\mathbf{h})$. Therefore Λ is derived discrete.

The part (b) is proved in Theorem 5.2 of [13].

(c) Suppose that $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of wild representation type. Then there is a bounded complex W of Λ-$k\langle x, y\rangle$-bimodules free of finite rank over the right side, projectives as Λ-modules, with $W^i = 0$ for i outside the set $\{1, \ldots, m\}$ and $\operatorname{Im} d_W^{i-1} \subset \operatorname{rad}\Lambda W^i$, such that the functor $W \otimes_{k\langle x,y\rangle} - : \operatorname{mod} k\langle x, y\rangle \to \mathbf{C_m}(\operatorname{proj}\Lambda)$ preserves iso-classes and indecomposables. The composition of this functor with the composition $\mathbf{C_m}(\operatorname{proj}\Lambda) \to \mathbf{K}^{-,\mathbf{b}}(\operatorname{proj}\Lambda) \to \mathcal{D}^b(\Lambda)$ also preserves iso-classes and indecomposables, consequently Λ is derived wild.

(d) Suppose Λ is derived tame. Then if for some m, $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of wild representation type and then by (c), Λ is derived wild, which contradicts (b). Therefore for all m, $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is not of wild representation type, but this implies, by Theorem (1.1) that for all m, $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of tame representation type.

Conversely assume that for all m, $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is of tame representation type. Let $\mathbf{h}$ be a fixed homology dimension; as before we may assume that if $\mathbf{h} = (h_i)$, we have $h_i = 0$ for i outside the set $\{1, \dots, m\}$. Take $d = |\mathbf{h}|(mL + (m-1)L^2 + \cdots + 2L^{m-1} + L^m)$, so $\mathcal{V}(\mathbf{h}) \subset \mathcal{V}_d$. Therefore there are rational algebras R_u, $u = 1, \dots, s$ and for each u a bounded complex M_u over the Λ-R_u-bimodules free of finite rank over the right side with $M_u^i = 0$ for i outside the set $\{1, \dots, m\}$ such that for almost all isomorphism class $[X]$ in $\mathcal{F}_d$ there is a u and $\lambda \in S(R_u)$ with $X \cong M_u \otimes_{R_u} S_\lambda$.

We may assume that for all u and i, $\operatorname{Im} d_{M_u}^{i-1}$ and $\operatorname{Ker} d_{M_u}^i$ are direct summands of M_u^i as right R_u-modules. Then for each u, $W_u = \tau^{\geq 1} M_u$ is a bounded complex over the Λ-R_u-bimodules which are free of finite rank over the right side.

Take $Y \in \mathcal{V}(\mathbf{h})$. Then there is a $P \in \mathcal{U}_d$ with a quasi-isomorphism $q\colon P \to Y$, such that $\tau^{\geq 1} P \cong Y$ in $\mathcal{D}^b(\Lambda)$.

Clearly $\tau^{\geq 1} P = \tau^{\geq 1} F(P)$, $F(P) \in \mathcal{F}_d$. Therefore $F(P) \cong M_u \otimes_{R_u} S_\lambda$ for some u and some $\lambda \in \mathcal{S}(R_u)$. Thus

$$Y \cong \tau^{\geq 1} P = \tau^{\geq 1} F(P) \cong \tau^{\geq 1}(M_u \otimes_{R_u} S_\lambda) \cong \tau^{\geq 1}(M_u) \otimes_{R_u} S_\lambda = W_u \otimes_{R_u} S_\lambda,$$

consequently Λ is derived tame.

(e) Suppose Λ is not derived wild. Then by (c) for all m, $\mathbf{C_m}(\operatorname{proj} \Lambda)$ is not of wild representation type, so by Theorem (1.1), for all m, $\mathbf{C_m}(\operatorname{proj} \Lambda)$ is of tame representation type. Therefore by (d), Λ is derived tame. □

THEOREM (2.5). *Let Λ be a derived tame algebra and $\boldsymbol{h} = (h_i)$ be a fixed homology dimension such that for n_0, $h_{n_0} \neq 0$ and $h_i = 0$ for $i < n_0$. Then for almost all isomorphism class of indecomposable complexes $X \in \mathcal{D}^b(\Lambda)$ with $\boldsymbol{h}\mathrm{dim} X = \boldsymbol{h}$, X is a perfect complex and there is an Auslander-Reiten triangle of the form*

$$X \to H \to X \to X[1].$$

Moreover $X^i = 0$ for $i < n_0 - 1$ and $d_X^{n_0-1}\colon X^{n_0-1} \to X^{n_0}$ is a monomorphism.

Proof. After a shifting we may assume $h_i = 0$ for $i \leq 1$ and $i > n$, $h_2 \neq 0$. By $\mathcal{U}(\mathbf{h})$ we denote the full subcategory of $\mathbf{K}^{\leq \mathbf{n},\mathbf{b}}(\operatorname{proj} \Lambda)$ whose objects are quasi-isomorphic to complexes $X \in \mathcal{V}(\mathbf{h})$. The categories $\mathcal{U}(\mathbf{h})$ and $\mathcal{V}(\mathbf{h})$ are equivalent. We shall see that for almost all isomorphism classes of objects P in $\mathcal{U}(\mathbf{h})$, P is a finite complex. If $P \in \mathcal{U}(\mathbf{h})$ then $\mathbf{h}\mathrm{dim} P = \mathbf{h}$, thus $\dim_k H^1(P) = h_1 = 0$, therefore $\mathcal{U}(\mathbf{h}) \subset \mathcal{L}_n$.

Recall that we have an equivalence $\underline{F}\colon \mathcal{L}_n \to \overline{\mathbf{C_n}}(\operatorname{proj} \Lambda)$.

Denote by $\mathcal{F}(\mathbf{h})$ the full subcategory of $\overline{\mathbf{C_n}}(\operatorname{proj} \Lambda)$ whose objects are isomorphic to some $\underline{F}(P)$ with $P \in \mathcal{U}(\mathbf{h})$. The categories $\mathcal{U}(\mathbf{h})$ and $\mathcal{F}(\mathbf{h})$ are equivalent categories. By Lemma (2.3), $\mathcal{F}(\mathbf{h}) \subset \mathcal{F}_d$ for $d = |\mathbf{h}|(nL + (n-1)L^2 + \cdots + 2L^{n-1} + L^n)$.

For our purposes it is convenient consider $\mathcal{F}(\mathbf{h})[-1]$ as a full subcategory of $\mathbf{C_m}(\operatorname{proj} \Lambda)$ with $m = n + 3$. If $Y \in \mathcal{F}(\mathbf{h})[-1]$, then $Y^1 = 0, Y^{n+2} = 0, Y^{n+3} = 0$ and $\dim_k Y \leq d$.

By (d) of Theorem (2.4), $\mathbf{C_m}(\operatorname{proj} \Lambda)$ is of tame representation type. Then by Theorem (1.3), for almost all isomorphism classes $[Y]$ with $Y \in \mathbf{C_m}(\operatorname{proj} \Lambda)$ there is an almost split sequence

$$Y \to E \to Y$$

in $\mathbf{C_m}(\operatorname{proj} \Lambda)$.

Following the notation of [3] we recall that $A(Y) \cong Y$. In order to calculate $A(Y)$ we take $Z = \nu(Y)[-1]$ and a quasi-isomorphism $q\colon Q \to \tau^{\leq m}Z$, with $Q \in \mathbf{C}^{\leq \mathbf{m},\mathbf{b}}(\operatorname{proj}\Lambda)$. Then $A(Y) \cong F(Q)$. Moreover by [16] there is an Auslander-Reiten triangle in $\mathcal{D}^b(\Lambda)$:

$$Z \to G \to Y \to Z[1].$$

We have $Z^m = Z^{n+3} = \nu(Y^{n+2}) = 0$, therefore $\tau^{\leq m}Z = Z$.

Here Z is indecomposable, then Q is an indecomposable complex in the category $\mathbf{K}^{\leq \mathbf{m},\mathbf{b}}(\operatorname{proj}\Lambda)$, and we may choose Q an indecomposable object in the category $\mathbf{C}^{\leq \mathbf{m},\mathbf{b}}(\operatorname{proj}\Lambda)$ with $Q^m = 0$, here $Z^m = 0$.

We have $F(Q) \cong A(Y) \cong Y$ in $\mathbf{C}_{\mathbf{m}}(\operatorname{proj}\Lambda)$, thus, $Q^1 \cong Y^1 = 0$. Here Q is indecomposable, which implies that $Q^i = 0$ for $i \leq 1$. Moreover $Z^2 = \nu(Y^1) = 0$, so $H^2(Q) \cong H^2(Z) = 0$. Therefore the morphism $d_Q^2\colon Q^2 \to Q^3$ is a monomorphism and $Q \cong Y$, and $Z \cong Q \cong Y$ in $\mathcal{D}^b(\Lambda)$.

Thus we have an Auslander-Reiten triangle in $\mathcal{D}^b(\Lambda)$:

$$(*) \quad Y \to G \to Y \to Y[1].$$

Now $Y[1] \in \mathcal{F}(\mathbf{h})$, so $Y[1] \cong F(P)$ with $P \in \mathcal{U}(\mathbf{h})$. Therefore $P^1 \cong Y^2 \cong Q^2$, $P^2 \cong Y^3 \cong Q^3$. The morphism $d_Q^2\colon Q^2 \to Q^3$ is isomorphic to the morphism $d_P^1\colon P^1 \to P^2$, thus this last morphism is a monomorphism.

Here $h_1 = \dim_k(\operatorname{Ker} d_P^1/\operatorname{Im} d_P^0) = 0$, then $\operatorname{Im} d_P^0 = \operatorname{Ker} d_P^1 = 0$, consequently $d_P^0 = 0$. But P is indecomposable, therefore $P^i = 0$ for $i \leq 0$. Consequently $P = F(P) \cong Y[-1]$. Thus applying the equivalence $[-1]$ to $(*)$ we obtain our result. □

COROLLARY (2.6). *Suppose Λ is selfinjective. Then either it is derived discrete or derived wild.*

Proof. Suppose Λ is neither derived discrete nor derived wild. Then there are infinitely many isomorphism classes in $\mathcal{V}(\mathbf{h})$ for some homology dimension $\mathbf{h}$. Therefore there is an indecomposable X in $\mathcal{D}^b(\Lambda)$ with an Auslander-triangle of the form $X \to H \to X \to X[1]$ with X an indecomposable object in $\mathbf{C}_{\mathbf{m}}^{\mathbf{1}}(\operatorname{proj}\Lambda)$ and $d_X^1\colon X^1 \to X^2$ a radical morphism which is a monomorphism; since X^1 is injective, this is not possible. □

COROLLARY (2.7). *Let Λ be derived tame. Then for a fixed homology dimension $\boldsymbol{h}$, for almost all isomorphism classes $[X]$ with $X \in \mathcal{D}^b(\Lambda)$ a perfect complex and $\boldsymbol{h}\dim_k X = \boldsymbol{h}$, X is isomorphic to a finite complex of finitely generated injective Λ-modules.*

Remark. Observe that gentle algebras are Gorenstein and in this case all finite complexes of finitely generated projective Λ modules are also isomorphic to finite complexes of finitely generated injective Λ-modules (see [14]).

COROLLARY (2.8). *Let Λ be a derived tame algebra. Suppose P is a bounded complex of Λ-R-bimodules projective over Λ and free of finite rank over R, a rational algebra, such that for all $\lambda \in \mathcal{S}(R)$, $P \otimes_R S_\lambda$ is indecomposable in $\mathcal{D}^b(\Lambda)$, and for $\lambda \neq \mu \in \mathcal{S}(R)$, $P \otimes_R S_\lambda \ncong P \otimes_R S_\mu$ in $\mathcal{D}^b(\Lambda)$. Then if $\boldsymbol{h}\dim_{k(x)} P \otimes_R k(x) = \boldsymbol{h} = (h_i)$ is such that $h_{n_0} \neq 0$ and $h_j = 0$ for $j < n_0$,* *we obtain that the morphism* $d_P^{n_0-1} \otimes 1\colon P^{n_0-1} \otimes_R k(x) \to P^{n_0} \otimes_R k(x)$ *is a monomorphism.*

Proof. We may assume that for all $\lambda \in S(R)$, all $\operatorname{Ker} d^i$ are direct summands of P^i as right R-modules. Thus $\mathbf{h}\dim P \otimes_R S_\lambda = \mathbf{h}$ for all $\lambda \in S(R)$. By Theorem (2.5), we may also assume that for all $\lambda \in S(R)$, $P^i \otimes S_\lambda = 0$ for $i < n_0 - 1$ and $\operatorname{Ker}(d^{n_0-1} \otimes 1\colon P^{n_0-1} \otimes S_\lambda \to P^{n_0} \otimes S_\lambda) = 0$. But this implies our assertion. □

COROLLARY (2.9). *Let* Λ *be a derived tame algebra. Then for any fixed non-negative integer d, almost all isomorphism classes of indecomposable modules* $[M]$ *with* $\dim_k M = d$ *have projective dimension one.*

Proof. For M indecomposable with $\dim_k M = d$, take

$$\cdots \to P_M^{-3} \xrightarrow{d_M^{-3}} P_M^{-2} \xrightarrow{d_M^{-2}} P_M^{-1} \xrightarrow{d_M^{-1}} P_M^0 \xrightarrow{\eta} M \to 0$$

a minimal projective resolution of M. Consider $P_M = (P_M^j, d_M^j)$ with $P_M^j = 0$, for $j > 0$ and $d_M^j = 0$ for $j \geq 0$. Then for $\mathbf{h}\dim M = (h_i)$, we have $h_0 = d$, $h_j = 0$ for $j < 0$. Therefore by Theorem (2.5) for almost all isomorphism classes $[M]$, $P_M^j = 0$ for $j < -1$. This proves our claim. □

3. Bocses

A tbocs is a triple $\mathcal{A} = (R, W, \delta)$, where R is a k-algebra (k is a field), W is an R-bimodule such that $W = W_0 \oplus W_1$ as R bimodules. The elements of W_i are called homogeneous of degree i, $i \in \{0, 1\}$. For $w \in W_i$, we put $\deg(w) = i$.

Take now the tensor algebra

$$T_R(W) = R \oplus W \oplus W^{\otimes^2} \oplus \cdots$$

with the graduation induced by that of W. The R-module generated by the set of homogeneous elements in $T_R(W)$ of degree i is denoted by $T_R(W)_i$. Then δ is an R-bimodule endomorphism of $T_R(W)$ such that

i) $\delta(T_R(W)_i) \subset T_R(W)_{i+1}$;

ii) For a, b homogeneous elements of $T_R(W)$

$$\delta(ab) = \delta(a)b + (-1)^{dega} a\delta(b) \qquad \text{(Leibnitz rule);}$$

iii) $\delta^2 = 0$.

The set of all elements of degree zero, $T_R(W)_0$ is a k-algebra, denoted by $A(\mathcal{A})$. This algebra is identified with $T_R(W_0)$. The set of all elements of degree one $T_R(W)_1$ is an $A(\mathcal{A})$-bimodule, which can be identified with $A(\mathcal{A}) \otimes_R W_1 \otimes_R A(\mathcal{A})$, and denoted by $V(\mathcal{A})$. Thus $T_R(W)$ is a differential graded algebra with differential δ. For v_1, v_2 in $T_R(W)$ we denote its product by v_1v_2, in particular if the above elements are in W, $v_1v_2 = v_1 \otimes v_2$.

Let $\mathcal{A} = (R, W, \delta)$ be a tbocs. The category of representations of $\mathcal{A}$, $\operatorname{Rep}\mathcal{A}$ is defined as follows:

The objects of $\operatorname{Rep}(\mathcal{A})$ are the left $A(\mathcal{A})$-modules.

Given two $A(\mathcal{A})$-modules M and N, a morphism $f\colon M \to N$ in $\operatorname{Rep}\mathcal{A}$ is given by a pair $f = (f^0, f^1)$, where

$$f^0 \in \operatorname{Hom}_R(M, N), \quad f^1 \in \operatorname{Hom}_{A(\mathcal{A}),A(\mathcal{A})}(V(\mathcal{A}), \operatorname{Hom}_k(M, N))$$

such that for all $a \in A(\mathcal{A}), m \in M$,

$$af^0(m) = f^0(am) + f^1(\delta(a))(m).$$

Observe that the pair $(f^0, 0)$ is a morphism in Rep$\mathcal{A}$ iff f^0 is a morphism of $A(\mathcal{A})$-modules.

Now if $f = (f^0, f^1)\colon M \to N$ and $g = (g^0, g^1)\colon N \to L$ are morphisms in Rep$\mathcal{A}$, the pair given by $(g^0 f^0, (gf)^1))$ with

$$(gf)^1(v) = g^1(v)f^0 + g^0 f^1(v) + \sum_{i=1}^{l} g^1(v_i^1) f^1(v_i^2)$$

for $\delta(v) = \sum_{i=1}^{l} v_i^1 v_i^2$, $v_i^1, v_i^2 \in V(\mathcal{A})$, is again a morphism. The composition of f and g is defined by $gf = (g^0 f^0, (gf)^1)$.

Using the properties of δ one can see that Rep$\mathcal{A}$ is a category. The identity morphism of $M \in$ Rep$\mathcal{A}$ is given by the pair $\underline{\mathrm{id}_M} = (\mathrm{id}_M, 0)$.

For a tbocs $\mathcal{A} = (R, W, \delta)$ we have a functor

$$I_{\mathcal{A}}\colon \mathrm{Mod}\, A(\mathcal{A}) \to \mathrm{Rep}\, \mathcal{A}$$

which is the identity on objects and for morphisms $u\colon M \to N$ of $A(\mathcal{A})$-modules, we have $I_{\mathcal{A}}(u) = (u, 0)$.

Let R be a k-algebra and $1 = \sum_{i=1}^{n} e_i$ a decomposition into central primitive orthogonal idempotents. We consider the k-subalgebra of R, $R_0 = \sum_{i=1}^{n} e_i k$. The k-algebra R_0 is a basic semisimple finite dimensional k-algebra.

Throughout this paper if $\mathcal{A} = (R, W, \delta)$ is a tbocs we assume that W is a finitely generated R-bimodule.

Definition (3.1). Let W be an R-bimodule. An R_0-subimodule $\tilde{W}$ of W is said to be an R_0-free generator of W if any morphism of R_0-bimodules $u\colon \tilde{W} \to V$, V a S-bimodule, has a unique extension to a morphism of R-bimodules $v\colon W \to V$. In this case we say that W is an R_0-free R-bimodule.

It is easy to see that $\tilde{W}$ is a R_0-free generator of W iff the morphism

$$\rho\colon R \otimes_{R_0} \tilde{W} \otimes_{R_0} R \to W \text{ given by } \rho(s \otimes w \otimes s_1) = sws_1$$

is an isomorphism. On the other hand if $\sigma\colon R \otimes_{R_0} \tilde{W} \otimes_{R_0} R \to W$ is an isomorphism $\sigma(\tilde{W})$ is an R_0-free generator of W.

Definition (3.2). A tbocs $\mathcal{A} = (R, W, \delta)$ is called R_0-free triangular if the following conditions are satisfied:

T.1 There is a filtration of R-bimodules $\{0\} = W_0^0 \subset \cdots \subset W_0^r = W_0$ such that for $i \geq 1$ $\delta(W_0^i) \subset A_i W_1 A_i$, where A_i is the R-subalgebra of A generated by W_0^{i-1}.

T.2 There is a filtration of R_0-bimodules $\tilde{W}_0^1 \subset \cdots \subset \tilde{W}_0^r = \tilde{W}_0$ such that $\tilde{W}_0^j$ is a R_0-free generator of W_0^j.

T.3 There is a sequence of subbimodules $\{0\} = W_1^0 \subset \cdots \subset W_1^s = W_1$ such that for $i \geq 1$ $\delta(W_1^i) \subset A W_1^{i-1} A W_1^{i-1} A$.

T.4 W_1 is R_0-freely generated by $\tilde{W}_1$.

If a tbocs $\mathcal{A}$ satisfies $T.1$, $T.3$ and $T.4$, we say that $\mathcal{A}$ is weakly triangular.

Throughout the paper R_0-free triangular tbocses are called triangular tbocses. We recall that in the category Rep$\mathcal{A}$ idempotents split; moreover for $f = (f^0, f^1)\colon M \to N$, f is an isomorphism if and only if f^0 is an isomorphism.

Definition (3.3). The k-algebra R is called minimal if there is a decomposition $1 = \sum_i e_i$ into central primitive orthogonal idempotents, such that $e_iR = e_ik$ or e_iR is a rational k-algebra.

Definition (3.4). The tbocs $\mathcal{A} = (R, W, \delta)$ is called minimal if R is a minimal k-algebra and $W_0 = 0$.

If $\mathcal{A} = (R, W, \delta)$ is a minimal tbocs then $A(\mathcal{A}) = R$, $V(\mathcal{A}) = W$, the space of morphisms between two objects $M, N \in$ Rep$\mathcal{A}$ is given by all pairs $f = (f^0, f^1)$ with $f^0 \in \mathrm{Hom}_R(M, N)$, $f^1 \in \mathrm{Hom}_{R-R}(W, \mathrm{Hom}_k(M, N))$.

LEMMA (3.5). *Suppose $\mathcal{A} = (R, W, \delta)$ is a triangular minimal tbocs, and $f\colon M \to M$ a morphism in* Rep $\mathcal{A}$ *of the form $f = (0, f^1)$. Then f is nilpotent.*

Proof. Take $0 = W^0 \subset W^1 \subset \cdots \subset W^s = W$, the filtration of $W = W_1$ given by condition $T.3$ of Definition (3.2). Then we have $f^2 = (0, (f^2)^1)$ and $(f^2)^1(W^1) = 0$. In general $f^r = (0, (f^r)^1)$ and $(f^r)^1(W^{r-1}) = 0$, therefore $f^{s+1} = (0, (f^{s+1})^1)$ and $(f^{s+1})^1(W^s) = (f^{s+1})^1(W) = 0$. Consequently $f^{s+1} = 0$. □

PROPOSITION (3.6). *Suppose $\mathcal{A} = (R, W, \delta)$ is a triangular minimal tbocs. Then an object $M \in$* Rep$\mathcal{A}$ *is indecomposable if and only if ${}_RM$ is indecomposable.*

Proof. If M is indecomposable in Rep$\mathcal{A}$, clearly ${}_RM$ is indecomposable. Suppose now that ${}_RM$ is indecomposable. Take $f = (f^0, f^1)$ an idempotent element in $\mathrm{End}_{\mathcal{A}}(M)$. Then $(f^0)^2 = f^0$, thus $f^0 = 0$ or $f^0 = \mathrm{id}_M$. In the first case $f = (0, f^1)$, thus f is nilpotent, then since f is also idempotent we conclude that $f = 0$. In the second case f is an isomorphism; therefore there is a $g \in \mathrm{End}_{\mathcal{A}}(M)$ with $fg = gf = \mathrm{id}_M$. Then $\mathrm{id}_M = fg = f^2g = f(fg) = f$. Therefore M is indecomposable in Rep$\mathcal{A}$. This proves our result. □

For $\mathcal{A} = (R, W, \delta)$ a minimal tbocs, take $1_R = \sum_{i=1}^n e_i$ a decomposition of 1_R as a sum of central primitive orthogonal idempotents.

PROPOSITION (3.7). *Suppose $\mathcal{A} = (R, W, \delta)$ is a minimal triangular tbocs. Then if $M \in$* Rep$\mathcal{A}$ *is indecomposable there is an e_i with $e_iM = M$*

Proof. Here $R \cong Re_1 \times \cdots \times Re_n$, if M is an indecomposable R-module then $e_iM = M$ for some e_i. Our result follows from our previous proposition. □

4. Reduction Functors

In this section we study full and faithful functors $F\colon \mathrm{Rep}\mathcal{B} \to \mathrm{Rep}\mathcal{A}$ which have been considered in [2].

Let R be a k-algebra, we recall from [2] that a left R-module X is called R-R_X admissible if R_X is a k-subalgebra of $\mathrm{End}_R(X)^{op}$ such that $\mathrm{End}_R(X)^{op} = R_X \oplus \mathcal{R}$ as R_X-bimodules with $\mathcal{R}$ an ideal of $\mathrm{End}_R(X)^{op}$, finitely generated projective as a right R_X-module, and X finitely generated projective as a right R_X-module. We have $X^* = \mathrm{Hom}_{R_X}(X_{R_X}, R_X)$ which is a R_X-R-bimodule and

$\mathcal{R}^* = \mathrm{Hom}_{R_X}(\mathcal{R}_{R_X}, R_X)$, a R_X-bimodule. Take dual bases $\{p_j, \gamma_j\}$ for $\mathcal{R}$ and $\{x_i, u_i\}$ for X as right R_X-modules.

We have morphisms

$$e\colon X \to X \otimes_{R_X} \mathcal{R}^*, \quad a\colon X^* \to \mathcal{R}^* \otimes_{R_X} X^*$$

such that for $u \in X^*, x \in X$, we have

$$e(x) = -\sum_j p_j(x) \otimes \gamma_j, \quad a(u) = \sum_{i,j} u(p_j(x_i))\gamma_j \otimes u_i.$$

Let $\mathcal{A} = (R, W, \delta)$ be a tbocs and X an R-R_X admissible left R-module. Consider the R_X-bimodules $(W_X)_0 = X^* \otimes_{R_X} W_0 \otimes_{R_X} X$, $(W_X)_1 = (X^* \otimes_{R_X} W_1 \otimes_{R_X} X) \oplus \mathcal{R}^*$.

For $u \in X^*$ and $v \in X$ we have k-linear maps

$$\phi^0_{u,v}\colon R \to R_X,$$

for $n \geq 1$:

$$\phi^n_{u,v}\colon W^{\otimes^n} \to T_{R_X}(W_X)$$

given by $\phi^0_{u,v}(r) = u(rv)$, $\phi^n_{u,v}(w_1 \otimes w_2 \otimes \cdots \otimes w_n) = \sum_{i_1, i_2, \ldots, i_{n-1}} u \otimes w_1 \otimes x_{i_1} \otimes u_{i_1} \otimes w_2 \otimes x_{i_2} \otimes u_{i_2} \otimes \cdots \otimes x_{i_{n-1}} \otimes u_{i_{n-1}} \otimes w_n \otimes v$.

These morphisms determine a k-linear map

$$\phi_{u,v}\colon T_R(W) \to T_{R_X}(W_X),$$

such that for $\lambda_1, \lambda_2 \in T_R(W)$ we have $\phi_{u,v}(\lambda_1\lambda_2) = \sum_i \phi_{u,x_i}(\lambda_1)\phi_{u_i,v}(\lambda_2)$. For $u \in X^*, v \in X$ we put for $\lambda \in T_R(W)$, $\phi_{a(u),v}(\lambda) = \sum_{i,j} u(p_j(x_i))\gamma_j\phi_{u_i,v}(\lambda)$ and $\phi_{u,e(v)}(\lambda) = -\sum_j \phi_{u,p_j(x)}(\lambda)\gamma_j$.

There is a differential δ_X in $T_{R_X}(W_X)$ with $\delta_X^2 = 0$, and such that for any t a homogeneous element in $T_R(W)^1 = W \oplus W^{\otimes^2} \oplus \cdots$ and $u \in X^*, v \in X$

$$(*) \qquad \delta_X(\phi_{u,v}(t)) = \phi_{a(u),v}(t) + \phi_{u,v}(\delta(t)) + (-1)^{deg t}\phi_{u,e(v)}(t)\,.$$

For $r \in R, u \in X^*, v \in X$, we have

$$\begin{aligned}\phi_{a(u),v}(r) + \phi_{u,e(v)}(r) &= \sum_{i,j} u(p_j(x_i))\gamma_j u_i(rv) - \sum_j u(rp_j(v))\gamma_j \\ &= \sum_{i,j} u(p_j(x_i u_i(rv)\gamma_j - \sum_j u(p_j(rv)\gamma_j = 0.\end{aligned}$$

Thus the equality $(*)$ holds also for $r \in R$ and consequently for any $t \in A(\mathcal{A})$.

We have a tbocs $\mathcal{A}^X = (R_X, W_X, \delta_X)$ and a functor $F^X\colon \mathrm{Rep}\mathcal{A}^X \to \mathrm{Rep}\mathcal{A}$, such that for $M \in \mathrm{Rep}\mathcal{A}^X$, $F^X(M) = X \otimes_{R_X} M$ as R-modules and for $w \in W_0$, $w(x \otimes m) = \sum_i x_i \otimes \phi_{u_i,x}(w)m$. For $f = (f^0, f^1)\colon M \to N$ a morphism in $\mathrm{Rep}\mathcal{A}$, $F^X(f)$ is given for $x \otimes m \in X \otimes_{R_X} M, w \in W_1$ by

$$(\text{M.1}) \qquad F^X(f)^0(x \otimes m) = x \otimes f^0(m) + \sum_j p_j(x) \otimes f^1(\gamma_j)(m)$$

$$(\text{M.2}) \qquad F^X(f)^1(w)(x \otimes m) = \sum_i f^1(u_i \otimes w \otimes x)(m).$$

Remark (4.1). We recall from Proposition 5.3 of [2] that an object $L \in \operatorname{Rep}\mathcal{A}$ is isomorphic to some $F^X(M)$ iff ${}_RL \cong X \otimes_{R_X} L'$ as R-modules for some R_X-module L'.

Observe that, in the above, if γ is an element of degree 0 in $T_R(W)$ then we have $\gamma x \otimes m = \sum_i x_i \otimes \phi_{u_i,x}(\gamma)m$.

On the other hand if $(f, 0)\colon M \to N$ is a morphism in $\operatorname{Rep}\mathcal{A}^X$, from (M.1) and (M.2) we obtain $F^X((f, 0)) = (\mathrm{id}_X \otimes f, 0)$. Consequently F^X induces a functor $F_0^X\colon \operatorname{Mod} A(\mathcal{A}^X) \to \operatorname{Mod} A(\mathcal{A})$ such that $F^X I_{\mathcal{A}^X} \cong I_{\mathcal{A}} F_0^X$. Here ${}_RF_0^X(M) \cong X \otimes_{R_X} M$, so F_0^X is a right exact functor which commutes with arbitrary direct sums. Therefore we have $F_0^X \cong Y \otimes_{A(\mathcal{A}^X)} -$ with Y the $A(\mathcal{A})$-$A(\mathcal{A}^X)$-bimodule $F_0^X(A(\mathcal{A}^X))$. Thus ${}_RY \cong X \otimes_{R_X} A(\mathcal{A}^X)$ which is a finitely generated projective right $A(\mathcal{A}^X)$-module. Thus Y is an $A(\mathcal{A})$-$A(\mathcal{A}^X)$-bimodule, finitely generated projective on the right side.

Suppose now that $\phi\colon R \to R'$ is an epimorphism in the category of rings. We suppose that $1_R = \sum_{i=1}^n e_i$ is a decomposition into central primitive orthogonal idempotents and that if $\{f_1, \ldots, f_t\}$ is the set of those $\phi(e_i) \neq 0$, then $1_{R'} = \sum_{j=1}^t f_j$ is a decomposition into central primitive orthogonal idempotents. Let $Z = \oplus_{i=1}^l Z_i$ be a finite direct sum of pairwise non isomorphic indecomposable R-modules Z_i having finite dimension over k. Moreover we suppose that for any H in $\operatorname{Mod} R'$, $\operatorname{Hom}_R(Z, H) = 0$ and $\operatorname{Hom}_R(H, Z) = 0$.

We have $\operatorname{End}_R(Z)^{op} = S_Z \oplus \mathcal{R}$, where $\mathcal{R} = \operatorname{radEnd}_R(Z)^{op}$ and S_Z is the k-subalgebra of $\operatorname{End}_R(Z)^{op}$ generated by the idempotents $e(Z_i)$ given by the composition of the projection of Z on Z_i with the inclusion of Z_i in Z.

Take the R-module $X = Z \oplus R'$, then

$$\operatorname{End}_R(X)^{op} = R_X \oplus \mathcal{R}$$

with $R_X \cong S_Z \times R'$. Clearly X is an admissible R-R_X-bimodule.

By Lemma 6.2, Lemma 6.3, Lemma 6.4 and Theorem 6.5 of [2] the functor $F^X\colon \operatorname{Rep}\mathcal{A}^X \to \operatorname{Rep}\mathcal{A}$ is a full and faithful functor.

PROPOSITION (4.2). *Suppose $\mathcal{A} = (R, W, \delta)$ is a weak triangular tbocs and X as above, then $\mathcal{A}^X = (R_X, W_X; \delta_X)$ is a weak triangular tbocs.*

Proof. We first consider condition $T.4$ of Definition (3.2). Then if we denote by R_0 the k-subalgebra of R generated by all the idempotents e_i, we have

$$W_1 \cong R \otimes_{R_0} \hat{W}_1 \otimes_{R_0} R$$

for some finitely generated R_0-bimodules $\hat{W}_1$. We have for 1_X the identity of the k-algebra R_X the following decomposition into central primitive orthogonal idempotents:

$$1_X = \sum_i e(Z_i) + \sum_j f_j.$$

We denote by $(R_X)_0$ the k-subalgebra of R_X generated by all idempotents $e(Z_i)$ and f_j. We have $(R_X)_0 = S_Z \times \phi(R_0)$. We have the R_0-$(R_X)_0$-bimodule

$$U_X = Z \oplus \phi(R_0),$$

and the $(R_X)_0$-R_0-bimodule

$$U'_X = Z^* \oplus \phi(R_0).$$

We obtain that

$$X^* \otimes_R W_1 \otimes_R X \cong R_X \otimes_{(R_X)_0} (U'_X \otimes_{R_0} \hat{W}_1 \otimes_{R_0} U_X) \otimes_{(R_X)_0} R_X.$$

Therefore we have

$$W_X = X^* \otimes_R W_1 \otimes_R X \oplus \mathcal{R}^* \cong R_X \otimes_{(R_X)_0} \hat{W}_X \otimes_{(R_X)_0} R_X,$$

with $\hat{W}_X = U'_X \otimes_{R_0} \hat{W}_1 \otimes_{R_0} U_X \oplus \mathcal{R}^*$. Clearly $\hat{W}_X$ is a finitely generated $(R_X)_0$-bimodule, this proves that the tbocs $\mathcal{A}^X$ holds property $T.4$.

For proving conditions $T.1$ and $T.3$ consider L the natural number such that $\mathcal{R}^L = 0$, but $\mathcal{R}^{L-1} \neq 0$.

For $1 \leq j \leq L$ we put $X^j = X\mathcal{R}^{L-j}$, and $(X^*)^j = \{h \in X^* | h(X^{L-j}) = 0\}$. Take $W_0^0 \subset \cdots \subset W_0^{r_0} = W_0$ and $(W_1)_0 \subset \cdots \subset W_1^{r_1} = W_1$ the corresponding filtrations given by the triangularity of $\mathcal{A}$.

We denote by $B_s(i, v, j)$ the R_X-subbimodule of $X^* \otimes_R W_s \otimes_R X$, generated by the elements of the form $f \otimes w \otimes x$ with $f \in (X^*)^i$, $w \in W_s^v$, $x \in X^j$.

We define

$$\begin{aligned}
(W_X)_0^m &= \sum_{i+2lv+j \leq m} B_0(i, v, j), \\
(W_X)_1^{m+l} &= \sum_{i+2lv+j \leq m} B_1(i, v, j) \oplus \mathcal{R}^*, \\
(W_X)_1^i &= \mathcal{R}_i^* \quad \text{for} \quad i \leq l.
\end{aligned}$$

As in Theorem 8.8 of [2] one can see that $\mathcal{A}^X = (R_X, W_X, \delta_X)$ is a weak triangular tbocs with filtrations

$$\begin{aligned}
0 = (W_X)_0^0 \subset \cdots \subset (W_X)_0^{2l(1+r_0)} = (W_X)_0 \\
0 = (W_X)_1^0 \subset \cdots \subset (W_X)_1^{2l(1+r_1)+l} = (W_X)_1.
\end{aligned}$$

□

In the rest of this section we describe a very useful reduction functor introduced originally in [8]. For this, let $\mathcal{A} = (R, W, \delta)$ be a tbocs with R a minimal k-algebra. Suppose $1 = \sum_{i=1}^n e_i$ is a decomposition into central primitive orthogonal idempotents, and $e_i R = k[x, f_i(x)^{-1}]$ for $i = 1, \ldots, t$, $e_j R = e_j k$ for $j = t+1, \ldots, n$.

Now fix a natural number d and elements $g_1, \ldots, g_t \in k[x]$, with $(g_i, f_i(x)) = 1$ for $i = 1, \ldots, t$.

For p a monic irreducible factor of g_i, $1 \leq i \leq t$ we put $Z_i(p) = e_i R/(p) \oplus \cdots \oplus e_i R/(p^d)$. For $1 \leq i \leq t$ we put $Z_i = \oplus_{p \in I(g_i)} Z_i(p)$, where $I(g_i)$ is the set of monic irreducible factors of g_i. For $i = t+1, \ldots, n$ we put $Z_i = e_i R = e_i k$.

We consider $R' = (e_1 R)_{g_1} \times \cdots \times (e_t R)_{g_t}$, with $(e_i R)_{g_i} = k[x, f_i(x)^{-1}, g_i(x)^{-1}]$. Clearly we have an epimorphism in the category of rings $R \to R'$ and $\mathrm{Hom}_R(Z, H) = 0$, $\mathrm{Hom}_R(H, Z) = 0$ for any $H \in \mathrm{Mod}\, R'$. Take $X = Z \oplus R'$, with $Z = \oplus_{i=1}^n Z_i$, the decomposition of Z into the direct sum of indecomposable R-modules of the form $(e_i R)/(p^u)$ with $1 \leq i \leq t$ and $e_i R$ with $i > t$, and the decomposition of R' into the direct sum of R-modules of the form $(e_i R)_{g_i}$, with $1 \leq i \leq t$, gives a decomposition of X into the direct sum of R-modules

X_j. For each X_j we have the idempotent $e(X_j)$ which is the composition of the projection of X on X_j with the corresponding canonical inclusion in X.

For $1 \leq i \leq t$ and $1 \leq u \leq d$ we put $e_i^u(p) = e((e_iR)/(p^u))$, for p monic irreducible factor of g_i, and $e_i^0 = e((e_iR)_{g_i})$. For $t+1 \leq i \leq n$ we put $\underline{e}_i = e(e_iR)$.

We have for $Z = \oplus_{i=1}^n Z_i$, $\mathrm{End}_R(Z)^{op} = S_Z \oplus \mathcal{R}$ where S_Z is the k-subalgebra of $\mathrm{End}_R(Z)^{op}$ generated by the idempotents $e(Z_i)$, $1 \leq i \leq n$ and $\mathcal{R} = \mathrm{rad}\mathrm{End}_R(Z)^{op}$.

Then $\mathrm{End}_R(X)^{op} = R_X \oplus \mathcal{R}$, where $R_X = S_Z \times R'$. Clearly X is a R-R_X admissible R-module. Then we have a full and faithful functor

$$F^X : \mathrm{Rep}\mathcal{A}^X \to \mathrm{Rep}\mathcal{A},$$

with $\mathcal{A}^X = (R_X, W_X, \delta_X)$.

The identity 1_X of R_X has the following decomposition into central primitive orthogonal idempotents:

$$1_X = \sum_{i=1}^{t} e_i^0 + \sum_{i=1}^{t} \sum_{p \in I(g_i)} \sum_{u=1}^{d} e_i^u(p) + \sum_{i=t+1}^{t+n} \underline{e}_i.$$

We have $e_i^0 R_X = (e_iR)_{g_i}$ for $1 \leq i \leq t$; $e_i^u(p)R_X = ke_i^u(p)$ for $1 \leq i \leq t$; $\underline{e}_iR_X = k\underline{e}_i$, for $t+1 \leq i \leq t+n$. Therefore R_X is a minimal k-algebra.

We recall that $(W_X)_0 = X^* \otimes_R W_0 \otimes_R X$. For $1 \leq i, j \leq t$ we have

(1) $e_i^0(W_X)_0 e_j^0 = (e_iR)_{g_i} \otimes_R e_iW_0e_j \otimes_R (e_jR)_{g_j}$;

(2) $e_i^0(W_X)_0 e_j^u(p) = (e_iR)_{g_i} \otimes_R e_iW_oe_j \otimes_R (e_jR)/(p^u)$;

(3) $e_i^u(p)(W_X)_0 e_j^0 = (e_iR)/(p^u))^* \otimes_R e_iW_oe_j \otimes_R (e_jR)g_j$;

(4) $e_i^u(p)(W_X)_0 e_j^v(q) = (e_iR)/(p^u))^* \otimes_R e_iW_oe_j \otimes_R (e_jR)/(q^v)$.

For $1 \leq i \leq t; t+1 \leq j \leq t+n$ we have

(5) $e_i^0(W_X)_0 \underline{e}_j \cong (e_iR)_{g_i} \otimes_R e_iW_0e_j$;

(6) $\underline{e}_j(W_X)_0) e_i^0 \cong e_jW_0e_i \otimes_R (e_iR)_{g_i}$;

(7) $e_i^u(p)(W_X)_0)\underline{e}_j \cong (e_iR/(p^u))^* \otimes_R e_iW_0e_j$;

(8) $\underline{e}_j(W_X)_0)e_i^u(p) \cong e_jW_0e_i \otimes_R (e_iR/(p^u)$.

Finally for $t+1 \leq i, j \leq n$ we obtain

(9) $\underline{e}_i(W_X)_0\underline{e}_j \cong e_iW_0e_j$.

The reduction functor $F^X : \mathrm{Rep}\mathcal{A}^X \to \mathrm{Rep}\mathcal{A}$ will be called a $(d, g_1, \ldots, g_t)$-unravelling.

Definition (4.3). For $\mathcal{A} = (R, W, \delta)$ a tbocs, an object $M \in \mathrm{Rep}\mathcal{A}$ is an R-E-bimodule with $E = \mathrm{End}_{\mathcal{A}}(M)^{op}$ and the right action of E on M given by $m.f = f^0(m)$ for $m \in M, f = (f^0, f^1) \in E$. Then M is called endofinite if the length of M as right E-module is finite; we denote by endol$M$ the length of M as right E-module.

Suppose now that M is an endofinite object in $\mathrm{Rep}\mathcal{A}$. Then if $1 = \sum_i e_i$ is a decomposition into central primitive orthogonal idempotents of R, each e_iM is a R-E-bimodule and $M = \oplus_i e_iM$ as R-E-bimodules, thus endol$M = \sum_i \mathrm{length}_E(e_iM_E)$.

Assume that $e_iR = R_i = k[x, h^{-1}]$, then $E \subset \mathrm{End}_{R_i}(e_iM) = E_i$. Therefore, $\mathrm{length}_{E_i}(e_iM) \leq \mathrm{length}_E(e_iM)$. Thus if M is endofinite, e_iM is an endofinite R_i-module. Consequently $e_iM_{Ri} \cong \sum_{j \in J} L_j$ with L_j an indecomposable R_i-module and in the set $\{L_j\}$ there are only a finite number of isomorphism classes. The only endofinite indecomposables R_i-modules are $k(x)$ and $k[x]/(x-\lambda)^m$ with $\lambda \in S(R_i)$, here $m \leq \mathrm{endol} M$.

LEMMA (4.4). *If* $F^X : \mathrm{Rep}\mathcal{A}^X \to \mathrm{Rep}\mathcal{A}$ *is a* $(d, g_1, \dots, g_t)$ *unravelling, for each endofinite object* $N \in \mathrm{Rep}\mathcal{A}$ *with* $\mathrm{endol} N \leq d$*, there is a* $M \in \mathrm{Rep}\mathcal{A}^X$ *endofinite with* $\mathrm{endol} M \leq \mathrm{endol} N$ *and* $F(M) \cong N$.

Proof. From the above considerations it follows that for $N \in \mathrm{Rep}\mathcal{A}$ with $\mathrm{endol} N \leq d$, ${}_R N \cong X \otimes_{R_X} Z$, for some R_X-module Z, then there is an $M \in \mathrm{Rep}\mathcal{A}^X$ with $F(M) \cong N$. We will assume that $F(M) = N$. Take $E_M = \mathrm{End}_{\mathcal{A}^X}(M)^{op}$ and $E_N = \mathrm{End}_{\mathcal{A}}(N)^{op}$. There is an isomorphism of k-algebras $\phi \colon E_M \to E_N$ induced by the functor F^X. Take $\mathcal{R} = \mathrm{radEnd}_R(X)^{op}$ and an integer l with $\mathcal{R}^l = 0$.

We have a filtration $\mathcal{F}$ of R-modules of $X \otimes_{R_X} M = N$:

$$N_{l-1} = \mathcal{R}^{l-1} X \otimes_{R_X} M \subset \cdots \subset N_1 = \mathcal{R} X \otimes_{R_X} M \subset N_0 = X \otimes_{R_X} M\,.$$

Clearly $\mathcal{F}$ is a filtration of R-modules. The ring E_M also acts on N by $(x \otimes n)f = x \otimes nf = x \otimes f^0(n)$ for $f = (f^0, f^1) \in E_N$. The filtration $\mathcal{F}$ is also a filtration of R-E_N-bimodules. Now observe that for $n \in N_{l-1}, f \in E_N$, we have $nf = n\phi(f)$. The same happen for $\underline{n} \in N_i/N_{i+1}$ for $i = 0, \dots, l-2$. Then the E_N-length of N is equal to the length of N as E_M-module. Now we recall that there is a decomposition $X = \oplus_{i=1}^s X_i$ with the X_i pairwise non isomorphic indecomposables. Take f_i the composition of the projection on the i-th summand followed of the corresponding injection. Then we have $1_X = \sum_{i=1}^s f_i$ a decomposition into central primitive orthogonal idempotents, $Xf_i = X_i$. Here we have that X is a finitely generated projective right R_X-module, so each X_i is a projective R_X-module, then $X_i \cong n_i f_i R_X$ and $n_i \neq 0$. Therefore

$$\begin{aligned}\mathrm{endol} N = \mathrm{length}_{E_M} N = \mathrm{length}_{E_M} X \otimes_{R_X} M = \sum_{i=1}^{s} \mathrm{length}_{E_M} n_i f_i M \\ \geq \sum_{i=1}^{s} \mathrm{length}_{E_M} f_i M = \mathrm{length}_{E_M} M = \mathrm{endol} M.\end{aligned}$$

This proves our claim. □

Definition (4.5). Let R be a minimal k-algebra. Suppose $1 = \sum_{i=1}^n e_i$ is a decomposition into central primitive orthogonal idempotents, and $e_i R = k[x, f_i^{-1}]$ for $i = 1, .., t$, $e_j R = k$ for $j = t+1, \dots, n$, we say that U an R-bimodule is thin if $e_i U e_j = 0$ for $i \leq t$ and $j \leq t$. A tbocs $\mathcal{A} = (R, W, \delta)$ is called thin if W_0 is a thin R-bimodule.

Observe that taking into account the above relations 1-9, if $\mathcal{A}$ is a thin tbocs, and $F^X : \mathrm{Rep}\mathcal{A}^X \to \mathrm{Rep}\mathcal{A}$ is a $(d, g_1, \dots, g_t)$-unravelling, then $\mathcal{A}^X$ is also a thin tbocs.

In the following if R is a minimal k-algebra and $1 = \sum_{i=1}^n e_i$ is a decomposition into central primitive orthogonal idempotents, we denote by S the k-subalgebra of R generated by all the idempotents e_i. Clearly S is a semisimple k-algebra. We recall that U a R-bimodule is called S- free if there is a S-subbimodule $\hat{U}$ of U such that the morphism of R-bimodules $\mu_U \colon R \otimes_S \hat{U} \otimes_S R \to U$ given by $\mu_U(r_1 \otimes u \otimes r_2) = r_1 u r_2$ is an isomorphism.

LEMMA (4.6). *Suppose U is a thin finitely generated R-bimodule, then U is S-free if for all $1 \leq i \leq t$, Ue_i is free as a right e_iR-module and e_iU is free as a left e_iR-module.*

Proof. Here U is thin. Setting $f = \sum_{i=t+1}^{n} e_i$, we have

$$U = (\oplus_{i=1}^{t} fUe_i) \oplus (\oplus_{i=1}^{t} e_iUf) \oplus fUf$$

a direct sum of R-bimodules.

We have $fUf \cong R \otimes_S fUf \otimes_S R$ as R-bimodules, so fUf is an S-free R-bimodule.

Now we have $fUe_i = \oplus_{j=t+1}^{n} e_jUe_i$, a direct sum of R-bimodules. The bimodule U is a quotient of a finite direct sum of copies of $R \otimes_k R$, so for $j \geq t+1$, e_jUe_i is a quotient of a finite direct sum of copies of $e_jR \otimes_k R$. Here $e_jR \cong k$, then e_jUe_i is a finitely generated e_iR-module, consequently it is a free module of finite rank over Re_i. Consequently $e_jUe_i \cong V \otimes_k Re_i$ for some k-vector space V.

For $1 \leq u \leq n$ consider the morphisms $\phi_u \colon S \to ke_u \to k$. Then the morphisms ϕ_j and ϕ_i induce an structure of S-bimodule on V. We have

$$e_jUe_i \cong R \otimes_S V \otimes_S R$$

as R-bimodules, consequently each e_jUe_i is an S-free R-bimodule. In a similar way one can see that e_iUe_j with $1 \leq i \leq t$ is an S-free R-bimodule. This proves our claim. □

Definition (4.7). Let U be an R-bimodule, a filtration $U^1 \subset \cdots \subset U^r = U$ is called an S-free filtration if for $u = 1, \ldots, r$ there are S-free generators V^u of U^u such that $V^1 \subset \cdots \subset V^r$.

LEMMA (4.8). *Let U be a thin finitely generated R-bimodule. Suppose that for $1 \leq i \leq t$ there are filtrations of R-bimodules $U_i^1 \subset \cdots \subset U_i^r = fUe_i$, ${}_iU^1 \subset \cdots \subset_i U^r = e_iUf$, such that for $1 \leq j \leq n-1$, ${}_iU^j$ is free as a left Re_i-module and a direct summand of ${}_iU^{j+1}$ and U_i^j is free as a right Re_i-module and a direct summand of U_i^{j+1}. Moreover, assume we have a filtration of R-bimodules $U_0^1 \subset \cdots \subset U_0^r = fUf$; then if for $1 \leq u \leq r$, $U^u = \sum_{i \leq t}(U_i^u +_i U^u) + U_0^u$,*

$$U^1 \subset \cdots \subset U^r = U$$

is an S-free filtration for U.

Proof. As in the proof of Lemma 4.6 we have for $1 \leq l \leq r$ an isomorphism

$$\psi_{i,l} \colon {}_iU^l \to R \otimes_S V^l \otimes_S R.$$

For $1 \leq l \leq r-1$, ${}_iU^l$ is a direct summand of ${}_iU^{l+1}$; we may take V^l a direct summand of of V^{l+1} as S-bimodules. Then taking ${}_i\hat{U}^l = (\psi_{i,l})^{-1}(1 \otimes V^l \otimes 1)$, and we have a filtration of S-bimodules

$${}_i\hat{U}^l \subset \cdots \subset {}_i\hat{U}^r$$

with ${}_i\hat{U}^l$ an S-free generator of ${}_iU^l$. Similarly we have a filtration

$$\hat{U}_i^l \subset \cdots \subset \hat{U}_i^r$$

with $\hat{U}_i^l$ an S-free generator of U_i^l. Since fR is a semisimple k-algebra and k is an algebraically closed field, then the filtration for fUf is an S-free filtration. Therefore $U^1 \subset \cdots \subset U^r = U$ is an S-free filtration. □

PROPOSITION (4.9). *Let $\mathcal{A} = (R, W, \delta)$ be a thin weak triangular tbocs. Then given a natural number d, there is a $(d, g_1, \ldots, g_t)$- unravelling*

$$F^X : \mathrm{Rep}\mathcal{A}^X \to \mathrm{Rep}\mathcal{A}$$

such that $\mathcal{A}^X$ is a thin triangular tbocs.

Proof. Here $\mathcal{A}^X$ is a thin tbocs. By Proposition (4.2), it is also a weak triangular tbocs. In order to prove that $\mathcal{A}^X$ is a triangular tbocs it is enough to prove that it satisfies condition T.4.

Since $\mathcal{A}$ is weak triangular, we have a filtration

$$0 = W_0^0 \subset W_0^1 \subset \cdots \subset W_0^r = W_0$$

satisfying the condition T.1 of Definition (3.2). There are elements $g_1, \ldots, g_t$ such that for $1 \le i \le t, 1 \le u \le r$, $(e_iR)_{g_i} \otimes_R W_0^u$ and $W_0^u \otimes_R (e_iR)_{g_i}$ are free left $(e_iR)_{g_i}$-modules and free right $(e_iR)_{g_i}$-modules respectively, and for $1 \le u \le r-1$, $(e_iR)_{g_i} \otimes_R W_0^{u-1}$ is a direct summand as a left $(e_iR)_{g_i}$ -module of $(e_iR)_{g_i} \otimes_R W_0^u$ and $W_0^{u-1} \otimes_R (e_iR)_{g_i}$ is a direct summand as a right $(e_iR)_{g_i}$-module of $W_0^u \otimes_R (e_iR)_{g_i}$. We put $R_i = (e_iR)_{g_i}$.

Here $W_0 \otimes_R R_i = fW_0e_i \otimes_R R_i$ is an S-R_i-bimodule with S semisimple. Then as in Lemma (4.6) and Lemma (4.8) there are S-bimodules $\hat{T}_i$ such that

$$W_0 \otimes_R R_i \cong \oplus_{u=1}^r \hat{T}_u \otimes_S R_i$$

with $W_0^u \otimes_R R_i \cong \hat{T}_u \oplus W_0^{u-1} \otimes_R R_i$.

Take the $(d, g_1, \ldots, g_t)$-unravelling $F^X : \mathrm{Rep}\mathcal{A}^X \to \mathrm{Rep}\mathcal{A}$ with

$$\mathcal{A}^X = (R_X, W_X, \delta_X).$$

Here $X^* \otimes_R W_X \otimes_R X$ is a thin R_X-R_X-bimodule. We have for $i \le t$, $e_i^0 R_X = R_i$, $e_i^u(p)R_X = e_i^u k$ and for $t+1 \le j \le n$, $\underline{e}_j R_X = \underline{e}_j k$.

Observe that $(X^* \otimes_R W_0 \otimes_R X)e_i^0 = X^* \otimes_R W_0 \otimes_R R_i$.

Now $X^* \otimes_R W_0 \otimes_R R_i = X^*f \otimes_S fW_0 \otimes_R R_i$. X^*f is an S_X-S-bimodule with both S_X and S semisimple. Then

$$X^*f = \oplus_{u=1}^r \hat{X}^u$$

with $X^*f \cap (X^*)^u = \hat{X}^u \oplus X^*f \cap (X^*)^{u-1}$.

$$(W_X)_0^m \cong \oplus_{u+L(2u+1)\le m} \hat{X}^u \otimes \hat{T}^l \otimes_S R_i.$$

Therefore

$$X^* \otimes_R W_0 \otimes_R R_i \cong \oplus_{u,l} \hat{X}^u \otimes \hat{T}^l \otimes_S R_i.$$

Now it is clear that

$$(W_X)_0^m e_i^0 = \oplus_{u+L(2l+1)\le m} \oplus_{u,l} \hat{X}^u \otimes \hat{T}^l \otimes_S R_i.$$

Thus $(W_X)_0^m e_i^0$ is a right module of finite rank over $R_X e_i^0$ which is a direct summand of $(W_X)_0^{m+1} e_i^0$. In a similar way one can prove that $e_i^0(W_X)_0^m$ is a free left module of finite rank over $R_X e_i^0$ which is a direct summand of $e_i^0(W_X)_0^{m+1}$. Then from Lemma (4.8) we deduce that the filtration

$$0 = (W_X)_0^0 \subset \cdots \subset (W_X)_0^{2l(1+r)} = (W_X)_0$$

is an S_X-free filtration, proving our result. □

PROPOSITION (4.10). *Let $\mathcal{A} = (R, W, \delta)$ be a thin free triangular tbocs which is not of wild representation type. Then given a natural number d, there is a finite set of full and faithful functors $F_i\colon \operatorname{Rep}\mathcal{B}_i \to \operatorname{Rep}\mathcal{A}$, $i = 1, \dots, l$, such that*

(i) *each $\mathcal{B}_i = (R_i, W^i, \delta_i)$ is a minimal triangular tbocs;*

(ii) *for $M \in \operatorname{Rep}\mathcal{A}$ with $\operatorname{endol} M \le d$, there is an $i \in \{1, \dots, l\}$ and $N \in \operatorname{Rep}\mathcal{B}_i$ with $F_i(N) \cong M$;*

(iii) *for each $i \in \{1, \dots, l\}$ there is an $A(\mathcal{A})$-R_i-bimodule Y_i, finitely generated projective over the right side such that*

$$F_i I_{\mathcal{B}_i} \cong I_{\mathcal{A}}(Y_i \otimes_{R_i} -).$$

Proof. By Proposition (4.9) there is a functor $F^X\colon \operatorname{Rep}\mathcal{A}^X \to \operatorname{Rep}\mathcal{A}$, given by a $(d, g_1, \dots, g_t)$-unravelling such that $\mathcal{A}^X$ is a free triangular tbocs. Moreover for M with $\operatorname{endol} M \le d$ there is a $N \in \operatorname{Rep}\mathcal{A}^X$ with $F^X(N) \cong M$ and $\operatorname{endol}(N) \le \operatorname{endol}(M)$. Since $\mathcal{A}$ is not of wild representation type $\mathcal{A}^X$ is not of wild representation type. Therefore by [9] or by Theorem 11.1 of [5] there is a finite set of full and faithful functors $G_i\colon \operatorname{Rep}\mathcal{B}_i \to \operatorname{Rep}\mathcal{A}^X$ $i \in \{1, \dots, l\}$ satisfying conditions (i), (ii) and (iii). Then using Lemma (4.4) and the second part of Remark 4.1 the full and faithful functors $F_i = F^X G_i\colon \operatorname{Rep}\mathcal{B}_i \to \operatorname{Rep}\mathcal{A}$, $i \in \{1, \dots, l\}$ satisfy (i), (ii) and (iii). □

Remark (4.11). With the notation of Proposition (4.10) suppose $1_R = \sum_{i=1}^s e_i$ is a decomposition into central primitive orthogonal idempotents. We consider $D(\mathcal{A}) = \mathbb{Q}^s$, for $M \in \operatorname{rep}\mathcal{A}$ we put $\underline{\dim} M = (\dim_k e_1 M, \dots, \dim_k e_s M)$.

For $i = 1, \dots, l$, R_i is a minimal k-algebra; thus we have a decomposition of $1_{R_i} = \sum_j^{s(j)} f_{i,j}$ with $f_{i,j}$, $j = 1, \dots, s(j)$ a set of central primitive orthogonal idempotents.

The functor $F_i\colon \operatorname{Rep}\mathcal{B}_i \to \operatorname{Rep}\mathcal{A}$ determines a k-linear map $t_{F_i}\colon D(\mathcal{B}_i) \to D(\mathcal{A})$ such that for $M \in \operatorname{rep}\mathcal{B}_i$ we have $\underline{\dim} F_i(M) = t_{F_i}(\underline{\dim} M)$.

5. A category of morphisms

Let $\mathcal{A} = (R, W, \delta)$ be a minimal triangular tbocs. Supose $1_R = \sum_{j=1}^n e_j$ is a decomposition into central primitive orthogonal idempotents in R. Denote by R_0 the k-subalgebra of R generated by all the idempotents e_i. Suppose that for $j = t+1, \dots, n$, $e_j R = k e_j$, with $t < n$. Then if $e = \sum_{j=t+1}^n e_j$, $eR = Re = eRe = eR_0 e$ is a semisimple k-algebra.

From the triangularity condition $T.3$ of Definition (3.2) we have a filtration $0 \subset W^1 \subset \cdots . \subset W^m = W$.

From condition $T.4$ there exists $\hat{W}$ a R_0-subbimodule of W, such that $W \cong R \otimes_{R_0} \hat{W} \otimes_{R_0} R$, for a finitely generated R_0-bimodule $\hat{W}$.

We consider the following category of radical morphisms $\mathcal{M}$ in $\operatorname{Rep}\mathcal{A}$.

The objects of $\mathcal{M}$ are the radical morphisms $\phi\colon X \to Y$ with $fX = 0$, where $f = \sum_{j=1}^t e_j$. The space of morphisms between two objects $\phi\colon X \to Y$ and

$\phi'\colon X' \to Y'$ of $\mathcal{M}$ is given by the pairs of morphisms $u = (u_1, u_2)$, $u_1\colon X \to X'$, $u_2\colon Y \to Y'$, morphisms in Rep$\mathcal{A}$ such that $u_2\phi = \phi' u_1$.

If $v = (v_1, v_2)$ is a morphism from $\phi'\colon X' \to Y'$ to $\phi''\colon X'' \to Y''$, then $vu = (v_1u_1, v_2u_2)$. Observe that if $\phi\colon X \to Y$ is a morphism object of $\mathcal{M}$, then this morphism has the form $\phi = (0, \phi^1)$. In fact, here $fX = 0$, so we may assume $X = \oplus_{j=t+1}^{n} m_u ke_u$; then if $\phi^0 \neq 0$, there is an inclusion $\sigma_u\colon ke_u \to X$ and a projection $\pi_u\colon X \to ke_u$ with $\pi_u\phi^0\sigma_u\colon ke_u \to ke_u$ not zero, so it is an isomorphism. But this implies that $(\pi_u, 0)\phi(\sigma_u, 0)$ is an isomorphism in Rep $\mathcal{A}$, which is not the case because ϕ is a radical morphism.

Clearly $\mathcal{M}$ is a category. We shall see that this category is equivalent to the category of representations of a weak triangular tbocs.

We first describe the morphisms in the category $\mathcal{M}$.

Suppose $u = (u_1, u_2)\colon \phi \to \phi'$ is a morphism in $\mathcal{M}$ with $\phi = (0, \phi^1)\colon X \to Y$, $\phi' = (0, (\phi')^1)\colon X' \to Y'$. Here $u_1 = (u_1^0, u_1^1)$, $u_2 = (u_2^0, u_2^1)$, $u_2\phi = \phi' u_1$.

For $w \in W_1 = W$ with $\delta(w) = \sum_s w_s^1 \otimes w_s^2$ we have:

$$(\phi')^1(w)u_1^0 + \sum_s (\phi')^1(w_s^1)u_1^1(w_s^2) = u_2^0\phi^1(w) + \sum_s u_1^1(w_s^1)\phi^1(w_s^2).$$

For $w \in W$, $x \in X$,

$$\phi^1(wf)(x) = \phi^1(w)(fx) = 0, \quad \text{therefore} \quad \phi^1(w) = \phi^1(we).$$

In a similar way we have $(\phi')^1(w) = (\phi')^1(we)$. Moreover

$$u_1^1(fw)(x) = fu_1^1(w)(x) = 0, u_1^1(wf)(x) = u_1^1(fx) = 0,$$

therefore $u_1^1(w) = u_1^1(ewe)$.

Then for $w \in W$ with $\delta(w) = \sum_s w_s^1 \otimes w_s^2$, we have

$$(1) \quad (\phi')^1(we)u_1^0 - u_2^0\phi^1(we) = \sum_s u_1^1(w_s^1)\phi^1(w_s^2e) - \sum_s (\phi')^1(w_s^1e)u_1^1(ew_s^2e).$$

Now in order to describe the category $\mathcal{M}$ in terms of a tbocs we introduce the following triangular tbocs, $\mathcal{B} = (S, W_{\mathcal{B}}, \delta_{\mathcal{B}})$, with

$$S = \begin{pmatrix} R & 0 \\ 0 & eRe \end{pmatrix}, \quad (W_{\mathcal{B}})_0 = \begin{pmatrix} 0 & We \\ 0 & 0 \end{pmatrix}, \quad (W_{\mathcal{B}})_1 = \begin{pmatrix} W & 0 \\ 0 & eWe \end{pmatrix}.$$

For $w \in W$ with $\delta(w) = \sum_s w_s^1 \otimes w_s^2$ we put

$$\begin{aligned} \delta_{\mathcal{B}}\begin{pmatrix} 0 & we \\ 0 & 0 \end{pmatrix} &= \sum_s \begin{pmatrix} 0 & w_s^1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & w_s^2e \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & w_s^1e \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & ew_s^2e \end{pmatrix} \\ &= \sum_s \begin{pmatrix} 0 & w_s^1 \otimes w_s^2e - w_s^1e \otimes ew_s^2e \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

$$\delta_{\mathcal{B}}\begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} w_s^1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} w_s^2 & 0 \\ 0 & 0 \end{pmatrix} = \sum_s \begin{pmatrix} w_s^1 \otimes w_s^2 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\delta_{\mathcal{B}}\begin{pmatrix} 0 & 0 \\ 0 & ewe \end{pmatrix} = \sum s \begin{pmatrix} 0 & 0 \\ 0 & ew_s^1e \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & ew_s^2e \end{pmatrix} = \sum_s \begin{pmatrix} 0 & 0 \\ 0 & ew_s^1e \otimes ew_s^2e \end{pmatrix},$$

using the Leibnitz rule one can extend $\delta_{\mathcal{B}}$ to a differential $\delta_{\mathcal{B}}: T_R(W) \to T_R(W)$. In order to see that $\delta_{\mathcal{B}}^2 = 0$, it is enough to prove that for $w \in W$ we have

$$\delta_{\mathcal{B}}^2 \begin{pmatrix} 0 & we \\ 0 & 0 \end{pmatrix} = 0, \quad \delta_{\mathcal{B}}^2 \begin{pmatrix} w & 0 \\ 0 & 0 \end{pmatrix} = 0, \quad \delta_{\mathcal{B}}^2 \begin{pmatrix} 0 & 0 \\ 0 & ewe \end{pmatrix} = 0.$$

Take $w \in W$ with $\delta(w) = \sum_s w_s^1 \otimes w_s^2$ and $\delta(w_s^1) = \sum_j w_{s,j}^{1,1} \otimes w_{s,j}^{1,2}$, $\delta(w_s^2) = \sum_j w_{s,j}^{2,1} \otimes w_{s,j}^{2,2}$. From $\delta^2 = 0$ we obtain

$$\sum_{s,j} w_{s,j}^{1,1} \otimes w_{s,j}^{1,2} \otimes w_s^2 - \sum_{s,j} w_s^1 \otimes w_{s,j}^{2,1} \otimes w_{s,j}^{2,2} = 0. \tag{2}$$

Taking $\delta_{\mathcal{B}}^2 \begin{pmatrix} 0 & we \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}$, we have

$$\begin{aligned} u = \sum_{s,j} w_{s,j}^{1,1} \otimes w_{s,j}^{1,2} \otimes w_s^2 e - \sum_{s,j} w_s^1 \otimes w_{s,j}^{2,1} \otimes w_{s,j}^{2,2} e + \sum_{s,j} w_{s,j}^{1,1} \otimes w_{s,j}^{1,2} e \otimes e w_s^2 e \\ - \sum_{s,j} w_s^1 \otimes w_{s,j}^{2,1} e \otimes e w_{s,j}^{2,2} e + \sum_{s,j} w_{s,j}^{1,1} e \otimes e w_{s,j}^{1,2} e \otimes e w_s^2 e \\ - \sum_{s,j} w_s^1 e \otimes e w_{s,j}^{2,1} e \otimes e w_{s,j}^{2,2} e. \end{aligned}$$

Now taking the projections $W \otimes_R W \otimes_R W \otimes_R W \to W \otimes_R W \otimes_R W \otimes_R We$, given by $w_1 \otimes w_2 \otimes w_3 \to w_1 \otimes w_2 \otimes w_3 e$, $W \otimes_R W \otimes_R W \otimes_R W \to W \otimes_R W \otimes_R We \otimes_R eWe$ given by $w_1 \otimes w_2 \otimes w_3 \to w_1 \otimes w_2 e \otimes e w_3 e$ and $W \otimes_R W \otimes_R W \otimes_R W \to We \otimes_R eWe \otimes_R eWe \otimes_R eWe$ given by $w_1 \otimes w_2 \otimes w_3 \to w_1 e \otimes e w_2 e \otimes e w_3 e$ of (2) we obtain that $u = 0$.

In a similar way we obtain the second and third equalities.

PROPOSITION (5.1). *With the above notation, $\mathcal{B} = (S, W_{\mathcal{B}}, \delta_{\mathcal{B}})$ is a thin weak triangular tbocs.*

Proof. We have in S the idempotents $\eta = \begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix}$, $\sigma = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}$, and the following decomposition into central primitive orthogonal idempotents of $1_S = \sum_{i=1}^n e_i \eta + \sum_{i=1}^n e_i \sigma$. The k-subalgebra of S generated by all the idempotents appearing in the above decomposition is

$$S_0 = \begin{pmatrix} R_0 & 0 \\ 0 & eR_0 e \end{pmatrix}.$$

We have filtrations $\{0\} \subset (W_{\mathcal{B}})_i^1 \subset (W_{\mathcal{B}})_i^2 \subset \cdots \subset (W_{\mathcal{B}})_i^m = (W_{\mathcal{B}})_i$, for $i = 0, 1$, with

$$(W_{\mathcal{B}})_0^i = \begin{pmatrix} 0 & W^i e \\ 0 & 0 \end{pmatrix}, (W_{\mathcal{B}})_1^i = \begin{pmatrix} W^i & 0 \\ 0 & eW^i e \end{pmatrix}.$$

Then $\mathcal{B}$ satisfies conditions $T.1$ and $T.3$ of Definition (3.2). Now there is a R_0-subimodule $\hat{W}$ of W such that $W \cong R \otimes_{R_0} \hat{W} \otimes_{R_0} R$. Then we have the isomorphism $eWe \cong eRe \otimes_{eR_0e} e\hat{W}e \otimes_{eR_0e} eRe$, therefore

$$S \otimes_{S_0} \begin{pmatrix} \hat{W} & 0 \\ 0 & e\hat{W}e \end{pmatrix} \otimes_{S_0} S \cong \begin{pmatrix} W & 0 \\ 0 & eWe \end{pmatrix}.$$

Thus we also have condition $T.4$ of Definition (3.2). This proves our result. □

THEOREM (5.2). *There exists a functor $F\colon \mathrm{Rep}\mathcal{B} \to \mathcal{M}$ which is an equivalence of categories.*

Proof. We have $A(\mathcal{B}) = T_S((W_{\mathcal{B}})_0) = \begin{pmatrix} R & We \\ 0 & eRe \end{pmatrix}$.

Take $V \in \mathrm{Rep}\mathcal{B}$; here V is an $A(\mathcal{B})$-module so $V = \eta V \oplus \sigma V$ as k-modules. Here $V_2 = \eta V$ is an R-module and $V_1 = \sigma V$ is an eRe-module. The action of $A(\mathcal{B})$ on V induces a morphism of R-modules $h\colon We \otimes_{eRe} V_1 \to V_2$. Conversely if V_1 is an eRe-module, V_2 is an R-module and $h\colon We \otimes_{eRe} V_1 \to V_2$ a morphism of R-modules, the triple $(V_1, V_2; h)$ determines an $A(\mathcal{B})$-module V.

We recall we have an isomorphism

$$\psi\colon \mathrm{Hom}_R(We \otimes_{eRe} V_1, V_2) \to \mathrm{Hom}_{R\text{-}eRe}(We, \mathrm{Hom}_k(V_1, V_2)).$$

We are now going to define a functor $F\colon \mathrm{Rep}\mathcal{B} \to \mathcal{M}$. Consider an object V in $\mathrm{Rep}\,\mathcal{B}$, given by the triple $(V_1, V_2; h)$. We define $F(V) = \phi = (0, \phi^1)\colon V_1 \to V_2$ with $\phi^1 = \psi(h)\tau \in \mathrm{Hom}_{R\text{-}R}(W, \mathrm{Hom}_k(V_1, V_2))$, where τ is the projection of W on We. Clearly ϕ is a morphism in $\mathcal{A}$ which is an object in $\mathcal{M}$.

Take now a morphism $z\colon V \to V'$ in $\mathrm{Rep}\,\mathcal{B}$, $z = (z^0, z^1)$. Here z^0 is a morphism of S-modules from V to V', then $z^0 = (z^0_2, z^0_1)$ with $z^0_1\colon V_1 \to V'_1$ a morphism of eRe-modules and $z^0_2\colon V_2 \to V'_2$ a morphism of R-modules. On the other hand

$$z^1\colon \begin{pmatrix} W & 0 \\ 0 & eWe \end{pmatrix} \to \mathrm{Hom}_k(V, V')$$

is a morphism of S-bimodules, then $z^1 = (z^1_2, z^1_1)$ with $z^1_1\colon eWe \to \mathrm{Hom}_k(V_1, V'_1)$ a morphism of eRe-bimodules and $z^1_2\colon W \to \mathrm{Hom}_k(V_2, V'_2)$ a morphism of R-bimodules. Since $z\colon V \to V'$ is a morphism in $\mathrm{Rep}\mathcal{B}$ we have for all $we \in We$ with $\delta(w) = \sum_s w^1_s \otimes w^2_s$ and $v_1 \in V_1, v_2 \in V_2$

$$\begin{pmatrix} 0 & we \\ 0 & 0 \end{pmatrix} z^0 \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} = z^0 \begin{pmatrix} 0 & we \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_2 \\ v_1 \end{pmatrix} + z^1 \delta_{\mathcal{B}} \begin{pmatrix} 0 & we \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_2 \\ v_1 \end{pmatrix}.$$

Then we obtain:

$$\begin{aligned} \begin{pmatrix} h'(w \otimes z^0_1(v_1)) \\ 0 \end{pmatrix} = z^0 \begin{pmatrix} h(w \otimes v_1) \\ 0 \end{pmatrix} + \sum_s z^1 \Bigg[\begin{pmatrix} w^1_s & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & w^2_s e \\ 0 & 0 \end{pmatrix} \\ - \begin{pmatrix} 0 & w^1_s e \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & e w^2_s e \end{pmatrix} \Bigg] \begin{pmatrix} v_2 \\ v_1 \end{pmatrix}, \end{aligned}$$

from which we obtain the equality

$$\begin{aligned} (\phi')^1(w)(z^0_1(v_1)) = z^0_2(\phi^1(w)(v_1)) + \sum_s z^1_1(w^1_s)(\phi^1(w^2_s)(v_1)) \\ - \sum_s (\phi')^1(w^1_s e)(z^1_1(e w^2_s e)(v_1)). \end{aligned} \tag{3}$$

Since $\mathcal{A}$ is a minimal tbocs, $u_1 = (z^0_1, z^1_1\rho)$ is a morphism from V_1 to V'_1 in $\mathrm{Rep}\mathcal{A}$, with $\rho\colon W \to eWe$ the projection given by $\rho(w) = ewe$.

Moreover $u_2 = (z^0_2, z^1_2)$ is a morphism from V_2 to V'_2. Then by (3) and (1) we have that $u = (u_1, u_2)$ is a morphism from $\phi = F(V)$ to $\phi' = F(V')$. We put $F(z) = u$. Now it is clear that if $F(z) = 0$, then $z = 0$. Thus F is a faithful functor, in order to prove that F is also full, take any morphism $u = (u_1, u_2)\colon F(V) = \phi \to F(V') = \phi'$, with $V = (V_1, V_2, h)$, $V' = (V_1', V_2', h')$, $\phi = (0, \phi^1)$, $\phi' = (0, (\phi')^1)$, with $\phi^1 = \psi(h)\tau$, $(\phi')^1 = \psi(h')\tau$. We have

$$u_1 = (u_1^0, u_1^1)\colon V_1 \to V_1', u_2 = (u_2^0, u_2^1)\colon V_2 \to V_2'.$$

Here $u_1^0 \in \mathrm{Hom}_R(V_1, V_1')$, $u_2^0 \in \mathrm{Hom}_R(V_2, V_2')$ and $u_1^1\colon W \to \mathrm{Hom}_k(V_1, V_1')$, $u_2^1\colon W \to \mathrm{Hom}_k(V_2, V_2')$ are morphisms of S-bimodules.

We have $u_1^1(ewe) = u_1^1(w)$, then $u_1^1 = \underline{u}_1^1\rho$ with $\underline{u}_1^1\colon eWe \to \mathrm{Hom}_k(V_1, V_1')$ a morphism of eRe-bimodules and $\rho\colon W \to eWe$ given by $\rho(w) = ewe$.

From $\phi' u_1 = u_2\phi$ we deduce the relation (1) for $(\phi')^1$, ϕ^1, u_1^0, u_2^0 and u_1^1. Consider now the pair of morphisms (u^0, u^1), with

$$u^0 = \begin{pmatrix} u_2^0 & 0 \\ 0 & u_1^0 \end{pmatrix} : V = V_1 \oplus V_2 \to V' = V_1' \oplus V_2',$$

$$u^1 = \begin{pmatrix} u_2^1 & 0 \\ 0 & u_1^1 \end{pmatrix} : W \oplus eWe \to \mathrm{Hom}_k(V, V').$$

Clearly both u^0 and u^1 are morphisms of S-bimodules. Here (1) implies (3) and (3) implies that the pair $z = (u^0, u^1)\colon V \to V'$ is a morphism in $\mathrm{Rep}\,\mathcal{B}$. We have that $F(z) = u$, therefore F is a full and faithful functor.

Finally we prove that F is a dense functor. Then take $\phi\colon V_1 \to V_2$ an object in $\mathcal{M}$. We have $\phi = (0, \phi^1)$, $\phi^1\colon W \to \mathrm{Hom}_k(V_1, V_2)$ a morphism of S-bimodules. We have $\phi^1(we) = \phi^1(w)$, thus there exists a morphism $\underline{\phi}^1\colon We \to \mathrm{Hom}_k(V_1, V_2)$ such that $\phi^1 = \underline{\phi}^1\tau$ with $\tau\colon W \to We$ given by $\tau(w) = we$.

Take $\psi^{-1}(\underline{\phi}^1) = h\colon We \otimes_{eRe} V_1 \to V_2$, then $V = (V_1, V_2, h) \in \mathrm{Rep}\,\mathcal{B}$ and $F(V) = \phi$. □

6. Main Results

This section is devoted to the proofs of Theorem (1.1) and Theorem (1.3).

Notation (6.1). In the following, for a projective Λ-module P we denote by $S(P)$ the complex with $S(P)^1 = P$ and $S(P)^i = 0$ for $i \neq 1$. For $h\colon P \to P'$ a morphism of Λ-modules we denote by $S(h)\colon S(P) \to S(P')$ the morphism of complexes given by $S(h)^1 = h$, $S(h)^i = 0$ for $i \neq 1$. For $n \geq 1$, we consider the following category $\mathcal{M}_n$ of morphisms in $\mathbf{C}_\mathbf{n}^\mathbf{1}(\mathrm{Proj}\,\Lambda)$. The objects of $\mathcal{M}_n$ are radical morphisms $f\colon S(P) \to X$ in $\mathbf{C}_\mathbf{n}^\mathbf{1}(\mathrm{Proj}\,\Lambda)$ with P an object in $\mathrm{Proj}\,\Lambda$ and X any object in $\mathbf{C}_\mathbf{n}^\mathbf{1}(\mathrm{Proj}\,\Lambda)$. The morphisms from $f\colon S(P) \to X$ to $f'\colon S(P') \to X'$ are given by pairs of morphisms $u = (u_1, u_2)$, $u_1\colon P \to P'$, $u_2\colon X \to X'$ such that $u_2 f = f' S(u_1)$. If $u = (u_1, u_2)$ is a morphism from $f\colon S(P) \to X$ to $f'\colon S(P') \to X'$ and $v = (v_1, v_2)$ is a morphism from $f'\colon S(P') \to X'$ to $f''\colon S(P'') \to X''$, then $vu = (v_1u_1, v_2u_2)$. The identity morphism of the object $f\colon S(P) \to X$ is given by the pair $(\mathrm{id}_P, \mathrm{id}_X)$.

Proposition (6.2). *There is a functor $G\colon \mathcal{M}_n \to \mathbf{C}_{\mathbf{n+1}}^\mathbf{1}(\mathrm{Proj}\,\Lambda)$ which is an equivalence of categories.*

Proof. Take $f\colon S(P) \to X$ an object in $\mathcal{M}_n$. We have the morphism $f^1\colon P \to X^1$, f is a radical morphism, thus $\mathrm{Im} f^1 \subset \mathrm{rad} X^1$; moreover f is a morphism of complexes, so we have $d_X^1 f^1 = f^2 d_P^1 = 0$. Therefore we have the complex

$G(f)$ in $\mathbf{C^1_{n+1}}(\operatorname{Proj}\Lambda)$ given by $G(F)^i = 0$ for i outside the set $\{1, \dots, n+1\}$, $G(f)^1 = P$, $G(f)^{i+1} = X^i$ for $i = 1, \dots, n$, $d^1_{G(f)} = f^1$, $d^{i+1}_{G(f)} = d^i_X$ for $i = 1, \dots, n$.

Now if $u = (u_1, u_2)$ is a morphism from $f\colon S(P) \to X$ to $f'\colon S(P') \to X'$, we define $G(u)$ in the following way: $G(u)^i = 0$ for i outside the set $\{1, \dots, n+1\}$, $G(u)^1 = u_1\colon G(f)^1 = P \to G(f')^1 = P'$, $G(u)^{i+1} = u_2^i\colon G(f)^{i+1} = X^i \to G(f')^{i+1} = (X')^i$ for $i = 1, \dots, n$.

We have $d^1_{G(f)} G(u)^1 = (f')^1 u_1 = (u_2)^1 f' = G(u)^2 d^1_{G(f)}$. For $i = 1, \dots, n$ we have $d^{i+1}_{G(f')} G(u)^{i+1} = d^i_{X'} u_2^i = u_2^{i+1} d^i_X = G(u)^{i+2} d^{i+1}_{G(f)}$. From here we conclude that $G(u)\colon G(f) \to G(f')$ is a morphism of complexes, so $G(\mathrm{id}_f) = \mathrm{id}_{G(f)}$. Now if v is a morphism from $f'\colon S(P') \to X'$ to $f''\colon S(P'') \to X''$, $G(v)G(u) = G(vu)$. Clearly G is a full, faithful dense functor. $\square$

Definition (6.3). Take $X \in \mathbf{C_n}(\operatorname{Proj}\Lambda)$. Then $E_X = \operatorname{End}_{\mathbf{C_n}(\operatorname{Proj}\Lambda)}(X)$ acts by the left on each X^i. We say that X has finite endolength if each X^i has finite length as E_X-left module. We define $\operatorname{endol}(X) = \sum_i \operatorname{length}_{E_X} X^i$.

Now suppose $P_1, \dots, P_m$ is a representative system of the isomorphism classes of the indecomposable projective Λ-modules. For H a Λ-module we put $\underline{\dim} H = (\dim_k \operatorname{Hom}(P_1, M), \dots, \dim_k \operatorname{Hom}(P_m, M))$.

For the category $\mathbf{C_n}(\operatorname{proj}\Lambda)$ we consider $c(\mathbf{C_n}(\operatorname{proj}\Lambda)) = \mathbb{Q}^{nm}$. For $X \in \mathbf{C_n}(\operatorname{proj}\Lambda)$, we put $c(X) = (\underline{\dim}(X_1/\operatorname{rad}X_1); \dots; \underline{\dim}(X_n/\operatorname{rad}X_n))$.

If $\underline{a} = (a_{i,j})_{1 \le i \le n, 1 \le j \le m} \in c(\mathbf{C_n}(\operatorname{proj}\Lambda))$ we put $|\underline{a}| = \sum_{1 \le i \le n, 1 \le j \le m} |a_{i,j}|$.

Definition (6.4). Let $\mathcal{C}$ be a k-category and E a k-algebra, a $\mathcal{C}$-E-object is an object $M \in \mathcal{C}$ endowed with a homomorphism of k-algebras $\alpha_M\colon E \to \operatorname{End}_{\mathcal{C}}(M)^{op}$. If M and N are $\mathcal{C}$-E-objects, a morphism of $\mathcal{C}$-E-objects from M to N is a morphism $f\colon M \to N$ in $\mathcal{C}$ such that for all $r \in E$, $f\alpha_M(r) = \alpha_N(r) f$. If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor and M is a $\mathcal{C}$-E-object, then $F(M)$ is a $\mathcal{D}$-E-object, taking $\alpha_{F(M)}$ the composition $E \xrightarrow{\alpha_M} \operatorname{End}_{\mathcal{C}}(M)^{op} \xrightarrow{F} \operatorname{End}_{\mathcal{D}}(F(M))^{op}$. Clearly if $f\colon M \to N$ is a morphism of $\mathcal{C}$-E-objects, $F(f)\colon F(M) \to F(N)$ is a morphism of $\mathcal{D}$-E-objects.

Example 1.

A $\mathbf{C_n}(\operatorname{Proj}\Lambda)$-$E$-object is a complex $X \in \mathbf{C_n}(\operatorname{Proj}\Lambda)$ such that each X^i is a Λ-E-bimodule and for all $i \in \mathbb{Z}$, d^i_X is a morphism of Λ-E-bimodules. If X, Y are $\mathbf{C_n}(\operatorname{Proj}\Lambda)$-$E$-objects, a morphism of complexes $f\colon X \to Y$ is a morphism of $\mathbf{C_n}(\operatorname{Proj}\Lambda)$-$E$-objects if each $f^i\colon X^i \to Y^i$ is a morphism of Λ-E-bimodules.

Example 2.

Let $\mathcal{B}$ and $\mathcal{C}$ be full subcategories of a category $\mathcal{D}$. Denote by $\mathcal{M}$ the category of morphisms $f\colon X \to Y$ in $\mathcal{D}$ with $X \in \mathcal{B}, Y \in \mathcal{C}$. Then $f\colon X \to Y$ is a $\mathcal{M}$-E-object if f is a morphism of $\mathcal{D}$-E-objects. Clearly $u = (u_1, u_2)\colon (f\colon X \to Y) \to (f'\colon X' \to Y')$ is a morphism of $\mathcal{M}$-E-objects if and only if u_1 and u_2 are morphisms of $\mathcal{D}$-E-objects.

Example 3.

Let $\mathcal{A} = (R, W, \delta)$ be a tbocs. We say that M is an $\mathcal{A}$-E-bimodule if it is a Rep$\mathcal{A}$-E-object. Then for $x \in E$ we have $\alpha_M(x) = (\alpha_M(x)^0, \alpha_M(x)^1)$. The $\mathcal{A}$-E-bimodule M is said to be proper if for all $x \in E$, $\alpha_M(x)^1 = 0$. In this case M is an R-E-bimodule with $mx = \alpha_M(x)^0(m)$. Moreover for $a \in A(\mathcal{A}), m \in M$, $(am)x = \alpha_M(x)^0(am) = a\alpha_M(x)^0(m) = a(mx)$, consequently M is an $A(\mathcal{A})$-E-bimodule. Clearly if M is an $A(\mathcal{A})$-E-bimodule then M is a proper $\mathcal{A}$-E-bimodule.

If $f = (f^0, f^1)\colon M \to N$ is a morphism in Rep$\mathcal{A}$ with M and N proper $\mathcal{A} - E$-bimodules, then f is a morphism of $\mathcal{A}$-E-bimodules if and only if f^0 is a morphism of R-E-bimodules and for all $v \in V(\mathcal{A})$, $f^1(v)\colon M \to N$ is a morphism of right E-modules.

THEOREM (6.5). *Assume $\mathbf{C}^1_\mathbf{n}(\mathrm{proj}\,\Lambda)$ is not of wild representation type. Then given a natural number d, there is a finite set of full and faithful functors $F_i\colon \mathrm{Rep}\mathcal{B}_i \to \mathbf{C}^1_\mathbf{n}(\mathrm{Proj}\,\Lambda)$, $i = 1, \dots, l$, such that*

(i) *for $i = 1, \dots, l$, $\mathcal{B}_i = (R_i, W^i, \delta_i)$ is a minimal triangular tbocs;*

(ii) *for $i = 1, \dots, l$ there are complexes Y_i with Y_i^j Λ-R_i bimodules projective from both sides and finitely generated over the right side with $F_i(N) \cong Y \otimes_{R_i} N$;*

(iii) *for any $X \in \mathbf{C}^1_\mathbf{n}(\mathrm{Proj}\,\Lambda)$ with $\mathrm{endol}(X) \leq d$, or $|c(X)| \leq d$, there is an $i \in \{1, \dots, l\}$ and a $N \in \mathrm{Rep}\mathcal{B}_i$ with $F_i(N) \cong X$;*

(iv) *for each $i \in \{1, \dots, l\}$ there is a linear transformation $t_{F_i}\colon D(\mathcal{A}_i) \to \mathbb{Q}^{mn}$ such that for all $N \in \mathrm{rep}\mathcal{A}_i$, $c(F_i(N)) = t_{F_i}(\underline{\dim} N)$.*

Proof. We prove our claim by induction on n. First we consider the case $n = 1$. Clearly $\mathbf{C}^1_\mathbf{1}(\mathrm{Proj}\,\Lambda) \cong \mathrm{Proj}\,\Lambda$.

Take the tbocs $\mathcal{U} = (\Lambda, 0, 0)$, then $\mathrm{Rep}\,\mathcal{U} = \mathrm{Mod}\,\Lambda$. Consider $X = P_1 \oplus \cdots \oplus P_n$, where $P_1, \dots, P_n$ is a representative system of the isomorphism classes of the indecomposable projective Λ-module. Here $\mathrm{End}_\Lambda(X)^{op} \cong S \oplus J$, with $J = \mathrm{radEnd}_\Lambda(X)^{op}$. We have the tbocs $\mathcal{U}^X = (S, W, \delta)$, where $W_0 = 0$, $W_1 = J^* = \mathrm{Hom}_S(J_S, S)$ and δ is the extension to $T_S(W)$, using Leibnitz rule, of the comultiplication $J^* \to J^* \otimes_S J^*$. There is a full and faithful functor $F^X\colon \mathrm{Rep}\,\mathcal{U}^X \to \mathrm{Mod}\,\Lambda$. For $M \in \mathrm{Rep}\,\mathcal{U}^X$, $F^X(M) = \Lambda \otimes_S M$. The full and faithful functor F^X induces an equivalence $F^X\colon \mathrm{Rep}\,\mathcal{U}^X \to \mathrm{Proj}\,\Lambda \cong \mathbf{C}^1_\mathbf{1}(\mathrm{Proj}\,\Lambda)$. Since k is an algebraically closed field, then $S \cong k \times \cdots \times k$, therefore $\mathcal{U}^X$ is a minimal tbocs, thus we have (i). Here X is a Λ-S-bimodule projective finitely generated on both sides, thus we have (ii). Moreover $F^X\colon \mathrm{Rep}\,\mathcal{U}^X \to \mathrm{Proj}\,\Lambda$ is an equivalence and then we have (iii).

Take now $t_{F^X}\colon D(\mathcal{U}^X) = \mathbb{Q}^m \to \mathbb{Q}^m$ given by the diagonal matrix with diagonal elements, $\dim_k(P_1/\mathrm{rad}P_1), \dim_k(P_2/\mathrm{rad}P_2), \dots, \dim_k(P_m/\mathrm{rad}P_m)$, then we have (iv).

Assume now our result proved for n; we will prove it for $n+1$. We are assuming that $\mathbf{C}^1_{\mathbf{n+1}}(\mathrm{Proj}\,\Lambda)$ is not of wild representation type, and this implies that $\mathbf{C}^1_\mathbf{n}(\mathrm{Proj}\,\Lambda)$ is not of wild representation type, so by the induction hypothesis for $i = 1, \dots, l$ there are full and faithful functors $F_i\colon \mathrm{Rep}\mathcal{A}_i \to \mathbf{C}^1_\mathbf{n}(\mathrm{Proj}\,\Lambda)$ with $\mathcal{A}_i = (R_i, W^i, \delta_i)$ minimal tbocses and complexes Y_i of $A(\mathcal{A}_i)$-R_i-bimodules finitely generated projectives over the right side such that $Y_i^j = 0$ for j outside the set $\{1, \dots, n\}$ and $F_i(N) \cong Y_i \otimes_{R_i} N$. Moreover if $X \in \mathbf{C}_\mathbf{n}(\mathrm{Proj}\,\Lambda)$ and

$\mathrm{endol}(X) \leq d'$, or $|c(X)| \leq d$, there is an $N \in \mathrm{Rep}\mathcal{A}_i$ for some $i \in \{1, \dots, l\}$ with $F_i(N) \cong X$.

By (iv) the functors $F_i\colon \mathrm{Rep}\mathcal{A}_i \to \mathbf{C}^{\mathbf{1}}_{\mathbf{n}}(\mathrm{Proj}\,\Lambda)$ induce linear transformations $t_{F_i}\colon D(\mathcal{A}_i) \to \mathbb{Q}^{mn}$, such that for $N \in \mathrm{rep}\,\mathcal{A}_i$, $c(F_i(N)) = t_{F_i}(\underline{\dim} N)$.

Take P an indecomposable projective Λ-module and suppose $Z(P, i) \in \mathrm{Rep}\mathcal{A}_i$ is such that $F_i(Z(P, i)) \cong S(P)$. Then $t_{F_i}(\underline{\dim} Z(P, i)) = (\underline{\dim}(P/\mathrm{rad}P); 0; \dots; 0)$. Take $f_{i,j}$ the only primitive central idempotent of R_i such that $f_{i,j}Z(P, i) \neq 0$. Then if $R_i f_{i,j}$ is not k, there are infinitely many non-isomorphic indecomposable objects T_s in $\mathrm{Rep}\mathcal{A}_i$ such that $\underline{\dim} T_s = \underline{\dim} Z(P, i)$. But then applying F_i this implies that there are infinitely many non-isomorphic indecomposable objects $F_i(T_s)$ in $\mathbf{C}_{\mathbf{n}}(\mathrm{Proj}\,\Lambda)$ with $\underline{\dim} F_i(T_s) = (\underline{\dim} P; 0; \dots; 0)$, which is not possible. Therefore $Rif_{i,j} = k$. Take now the sum f_i of all possible $f_{i,j}$ as before. Then $R_i f_i$ is a semisimple k-algebra.

Now for $i \in \{1, \dots, l\}$ take $\mathcal{N}_i$ the category of radical morphisms $u\colon Z_2 \to Z_1$ in $\mathrm{Rep}\mathcal{A}_i$ with $f_i Z_2 = Z_2$. By Theorem (5.2) there is an equivalence of k-categories $G_i\colon \mathrm{Rep}\mathcal{B}_i \to \mathcal{N}_i$, with $\mathcal{B}_i = (S_i, W_{\mathcal{B}_i}, \delta_{\mathcal{B}_i})$ a thin weak triangular tbocs.

Now consider the category $\mathcal{M}_n$ of Definition (6.4). The functor $F_i\colon \mathrm{Rep}\mathcal{A}_i \to \mathbf{C}^{\mathbf{1}}_{\mathbf{n}}(\mathrm{Proj}\,\Lambda)$ induces a full and faithful functor $\hat{F}_i\colon \mathcal{N}_i \to \mathcal{M}_n$, $\hat{F}_i(u\colon Z_2 \to Z_1) = F_i(u)\colon F_i(Z_2) \to F_i(Z_1)$. Thus we have the full and faithful functor $G\hat{F}_i\colon \mathcal{N}_i \to \mathbf{C}^{\mathbf{1}}_{\mathbf{n+1}}(\mathrm{Proj}\,\Lambda)$. Therefore $\mathcal{N}_i$ is not of wild representation type, which implies that $\mathcal{B}_i$ is not of wild representation type for $1 \leq i \leq l$. Then by Proposition (4.10) there are full and faithful functors $F_{i,j}\colon \mathrm{Rep}\mathcal{A}_{i,j} \to \mathrm{Rep}\mathcal{B}_i$ for $j \in \{1, \dots, l(i)\}$ with $\mathcal{A}_{i,j} = (S_{i,j}, W_{i,j}, \delta_{i,j})$ a minimal triangular tbocs such that for all $M \in \mathrm{Rep}\mathcal{B}_i$ with $\mathrm{endol}(M) \leq d$ or $|\underline{\dim} M| \leq d$ there is a $N \in \mathrm{Rep}\mathcal{A}_{i,j}$ for some $j \in \{1, \dots, l(i)\}$ with $F_{i,j}(N) \cong M$.

We have the following full and faithful functors:

$$\mathrm{Rep}\,\mathcal{A}_{i,j} \xrightarrow{F_{i,j}} \mathrm{Rep}\,\mathcal{B}_i \xrightarrow{G_i} \mathcal{N}_i \xrightarrow{\hat{F}_i} \mathcal{M}_n \xrightarrow{G} \mathbf{C}^{\mathbf{1}}_{\mathbf{n+1}}(\mathrm{Proj}\,\Lambda).$$

We have the proper $\mathcal{A}_{i,j}$-$S_{i,j}$-bimodule $F_{i,j}(S_{i,j}) = V_{i,j}$. Then $V_{i,j}$ is an $A(\mathcal{A}_{i,j})$-$S_{i,j}$-bimodule. We recall that

$$A(\mathcal{B}_i) = \begin{pmatrix} R_i & W^i f_i \\ 0 & f_i R_i f_i \end{pmatrix},$$

$V_{i,j} = (V^1_{i,j}, V^2_{i,j}; h_{i,j})$ with $V^1_{i,j}$ and $V^2_{i,j}$ R_i-$S_{i,j}$-bimodules finitely generated projectives over the right side. The morphism $h_{i,j}\colon W^i f_i \otimes_{R_i} V^2_{i,j} \to V^1_{i,j}$ is a morphism of R_i-$S_{i,j}$-bimodules. Then $V^1_{i,j}$ and $V^2_{i,j}$ are proper $\mathcal{A}_i$-$S_{i,j}$-bimodules and $\phi_{i,j} = (0, \phi^1_{i,j})\colon V^1_{i,j} \to V^2_{i,j}$ with $\phi^1_{i,j}(w)(x) = h_{i,j}(wf_i \otimes x)$ for $w \in W^i$, $x \in V^2_{i,j}$. Since $h_{i,j}$ is a morphism of R_i-$S_{i,j}$-bimodules, then $\phi_{i,j}$ is a morphism of $\mathcal{A}_i$-$S_{i,j}$-bimodules.

By definition $G_i(V_{i,j}) = \phi_{i,j}\colon V^1_{i,j} \to V^2_{i,j}$, $\hat{F}_i(G_i(V_{i,j})) = F_i(\phi_{i,j}) \times Y_i \otimes_{R_i} V^1_{i,j} \to Y_i \otimes_{R_i} V^2_{i,j}$.

Now $f_i V^1_{i,j} = V^1_{i,j}$, then $(Y_i \otimes_{R_i} V^1_{i,j})^1 = Y^1_i \otimes_{R_i} V^1_{i,j}$ and $(Y_i \otimes_{R_{i,j}} V^1_{i,j})^s = 0$ for $s \neq 1$, $(Y_i \otimes_{R_i} V^2_{i,j})^s = Y^s_i \otimes_{R_i} V^2_{i,j}$ for $s \in \mathbb{Z}$, $F_i(h_{i,j})^1 = u_{i,j}$, $F_i(h_{i,j})^s = 0$ for $s \neq 1$.

For $Z = G\hat{F}_i G_i F_{i,j}(R_{i,j})$ we have $Z^s = 0$ for s outside the set $\{1,\dots,n+1\}$, $Z^1 = Y_i^1 \otimes_{R_i} V_{i,j}^1$, $Z^2 = Y_i^1 \otimes_{R_i} V_{i,j}^2$, $\dots$, $Z^{n+1} = Y_i^n \otimes_{R_i} V_{i,j}^2$; and $d_Z^1 = u_{i,j}$, $d_Z^s = d_{Y_i}^{s-1} \otimes 1$ for $s \in \{2,\dots,n+1\}$.

For $M \in \operatorname{Rep}\mathcal{A}_{i,j}$ we have $G\hat{F}_i G_i F_{i,j}(M) \cong Z \otimes_{S_{i,j}} M$.

We shall see that the functors $H_{i,j} = G\hat{F}_i G_i F_{i,j} \colon \operatorname{Rep}\mathcal{A}_{i,j} \to \mathbf{C}^{\mathbf{1}}_{\mathbf{n+1}}(\operatorname{Proj}\Lambda)$ satisfy the conditions (i), (ii), (iii) and (iv). Here $\mathcal{A}_{i,j}$ is a minimal triangular tbocs, thus we have (i). Now for Z we have that for $s \in [1, n+1]$, Z^s is a Λ-$S_{i,j}$-bimodule projective on both sides and finitely generated over the right side, and for $M \in \operatorname{Rep}\mathcal{A}_{i,j}$, $H_{i,j}(M) \cong Z \otimes_{S_{i,j}} M$, thus we have (ii).

For proving (iii) take $X \in \mathbf{C}^{\mathbf{1}}_{\mathbf{n+1}}(\operatorname{Proj}\Lambda)$ with $\operatorname{endol}(X) \le d$. Then by Proposition (6.2), $X \cong G(X_2 \xrightarrow{u} X_1)$ with $X_2 = S(P), X_1 \in \mathbf{C}^{\mathbf{1}}_{\mathbf{n}}(\operatorname{Proj}\Lambda)$. Consider $E = \operatorname{End}_{\mathbf{C_n}(\operatorname{Proj}\Lambda)}(X)^{op}$, X_1 and X_2 are $\mathbf{C_n}(\operatorname{Proj}\Lambda)$-$E$-objects and $\operatorname{endol}(X) = \operatorname{length}_E X_1 + \operatorname{length}_E X_2$.

Moreover $\operatorname{endol}(X_1) \le \operatorname{length}_E X_1$ and $\operatorname{endol}(X_2) \le \operatorname{length}_E X_2$. Therefore $\operatorname{endol}(X_1 \oplus X_2) \le \operatorname{endol}(X_1) + \operatorname{endol}(X_2) \le d$. Then there is an i and $N_1, N_2 \in \operatorname{Rep}\mathcal{A}_i$ such that $F_i(N_1) \cong X_1, F_i(N_2) \cong X_2$. Since F_i is a full functor, there is a morphism $v = (0, v^1) \colon N_1 \to N_2$ such that $F_i(v)$ is isomorphic to u. The morphism v is an object of $\mathcal{N}_i$. Clearly v is an $\mathcal{N}_i$-E-bimodule with $\hat{F}_i(v) \cong u$. Since G_i is an equivalence there is a $N \in \mathcal{B}_i$ with $G_i(N) \cong v$. We may assume $N = (N_1, N_2; h)$, then $\operatorname{endol}(N) \le \operatorname{endol}(N_1) + \operatorname{endol}(N_2) = \operatorname{endol}(X_1) + \operatorname{endol}(X_2) \le d$. Then there is a j and an object $M \in \operatorname{Rep}\mathcal{B}_{i,j}$ with $F_{i,j}(M) \cong N$, therefore $H_{i,j}(M) \cong X$. In case $c(X) \le d$ one proceeds in a similar way, proving (iii).

Finally for proving (iv), observe that

$$D(\mathcal{B}_i) = D(\mathcal{A}_i) \oplus D(\mathcal{A}_i)\,;$$

denote by $\pi_s \colon D(\mathcal{B}_i) \to D(\mathcal{A}_i)$ the corresponding projection for $s = 1, 2$. If V is an object in $\operatorname{rep}\mathcal{B}_i$, given by the triple $(V_1, V_2; h)$, then $\underline{\dim} V = (\underline{\dim} V_1, \underline{\dim} V_2)$. Then for $N \in \operatorname{rep}\mathcal{A}_{i,j}$, we have

$$c(H_{i,j}(N)) = (t_{F_i}\pi_1 t_{F_{ij}}(\underline{\dim} N); 0) + (0; t_{F_i}\pi_2 t_{F_{ij}}(\underline{\dim} N))\,.$$

Consequently, there is a linear transformation $t_{H_{i,j}} \colon D(\mathcal{A}_{i,j}) \to \mathbb{Q}^{(n+1)m}$ such that for all $N \in \operatorname{rep}\mathcal{A}_{i,j}$

$$c(H_{i,j}(N)) = t_{H_{i,j}}(\underline{\dim} N).$$

The above proves (iv). $\square$

Proof of Theorem (1.1). Suppose $\mathbf{C_m}(\operatorname{proj}\Lambda)$ is not of wild representation type, so $\mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\operatorname{proj}\Lambda)$ is not of wild representation type. Given a natural number d if for some $X \in \mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\operatorname{proj}\Lambda)$, $\dim_k X \le d$, then $|c(X)| \le d$. By Theorem (6.5), given a non negative integer d, there is a finite set of full and faithful functors $F_i \colon \operatorname{Rep}\mathcal{B}_i \to \mathbf{C}^{\mathbf{1}}_{\mathbf{n}}(\operatorname{Proj}\Lambda)$, $i = 1,\dots,l$ with conditions (i), (ii), (iii) and (iv). Using the notation of Theorem (6.5), for $i \in \{1,\dots,l\}$ we consider T_i the set of central primitive idempotents $f_{i,j}$ in R_i with $f_{i,j}R_i \ne kf_{i,j}$. For each $f_{i,j} \in T_i$ we have $Yf_{i,j} \in \mathbf{C}^{\mathbf{1}}_{\mathbf{n}}(\operatorname{Proj}\Lambda)$. Each $Y^u f_{i,j}$ is a Λ-$R_i f_{i,j}$-bimodule finitely generated projective as a right $R_i f_{i,j}$-module. Since $R_i f_{i,j}$ is a rational k-algebra, $Y^u f_{i,j}$ is free of finite rank as $R_i f_{i,j}$-module. Thus for almost all isomorphism classes $[X]$ of indecomposable objects in $\mathbf{C_m}(\operatorname{proj}\Lambda)$ with $\dim_k X \le d$, we may assume

$X \in \mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$. Therefore for almost all such $[X]$ we have $X \cong Y_i \otimes_{R_i f_{i,j}} S(\lambda)$ for some $\lambda \in k$ and $f_{i,j} \in T_i$. This proves that $\mathbf{C}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$ is of tame representation type. □

Now we recall that if $Y \to E \to X$ is an almost split sequence in $\mathbf{C}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$, then $Y \cong A(X)$. Here $A(X) \cong F(Q)$ with $Q \in \mathbf{C}^{\leq \mathbf{m},\mathbf{b}}(\mathrm{proj}\,\Lambda)$ quasi-isomorphic to $\tau^{\leq m}\nu(X)[-1]$.

We need the following.

LEMMA (6.6). *There is a constant $c(\Lambda)$ depending only on the algebra Λ such that for any $Y \in \mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$, $\dim_k A(Y) \leq c(\Lambda)\dim_k Y$.*

Proof. Take $L = \dim_k \Lambda$, and the Nakayama functor $\nu\colon \mathrm{proj}\,\Lambda \to \mathrm{inj}\,\Lambda$. We recall that if $1 = \sum_{i=1}^n e_i$ is a decomposition of the identity of Λ into primitive orthogonal idempotents, then $\nu(\Lambda e_i) = D(e_i\Lambda)$. Therefore if $P = \oplus_i n_i \Lambda e_i$, then $\nu(P) = \oplus_i n_i D(e_i\Lambda)$. Thus $\dim_k \nu(P) = \sum_i n_i \dim_k D(e_i\Lambda) \leq \sum_i n_i L \leq L(\sum_i n_i \dim_k \Lambda e_i) = L\dim_k P$. If $W = (W^i, d^i_W)$ is a complex of finitely generated projective Λ- modules then $\nu(W) = (\nu(W^i), \nu(d^i_W))$. If in addition W is a finite complex, $\dim_k \nu(W) = \sum_i \dim_k \nu(W^i) \leq L\dim_k W$.

Now choose a quasi-isomorphism $q\colon Z \to \tau^{\leq m}(\nu(X)[-1])$, with $Z = (Z^i, d_Z)$ such that $\mathrm{Im} d^i_Z \subset \mathrm{rad} Z^{i+1}$.

We have $\dim_k H^j(Z) = \dim_k H^j(\tau^{\leq m}X[-1]) \leq L\dim_k X$. Now $A(X) \cong F(Z)$ in $\mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$, thus $\dim_k A(X) \leq c(\Lambda)\dim_k X$ with $c(\Lambda) = L(mL + (m-1)L^2 + \cdots + 2L^{m-1} + L^m)$. This proves our claim. □

The following result implies Theorem (1.3).

THEOREM (6.7). *Assume that $\mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$ is not of wild representation type. Then given a natural number d, for almost all indecomposable object, $X \in \mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$ with $\dim_k X \leq d$ there is an almost split $\mathcal{E}$-sequence*

$$X \to E \to X.$$

Proof. We may assume X is not $\mathcal{E}$-projective so by Theorem 8.5 of [3], there is an almost split $\mathcal{E}$-sequence

$$A(X) \to E \to X$$

in $\mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$.

Given a natural number d, we take $d' = 2(1+c(\Lambda))d$. By Theorem (6.5) there is a finite number of full and faithful functors $F_i\colon \mathrm{Rep}\mathcal{B}_i \to \mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{Proj}\,\Lambda)$ with $\mathcal{B}_i = (R_i, W^i, \delta_i)$ minimal triangular tbocses such that for any $Y \in \mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{Proj}\,\Lambda)$ with $\dim_k Y \leq d'$ there is a $W \in \mathrm{Rep}\,\mathcal{B}_i$ with $F_i(W) \cong Y$. Consider now the family $\mathcal{S}$ of objects in $\mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$ which are isomorphic to some $F_i(f_s R_i)$ with f_s central primitive idempotent of R_i such that $f_s R_i = k$. In the above family there is only a finite number of isomorphism classes.

Take now an indecomposable object $X \in \mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{proj}\,\Lambda)$ which is not in $\mathcal{S}$ with $\dim_k X \leq d$. Suppose moreover that X is not $\mathcal{E}$-projective. Then there is an almost split $\mathcal{E}$-sequence

$$(a)\colon \quad Y \to E \to X,$$

here, $\dim_k(X \oplus E \oplus Y) \leq d'$, so there is a $U \in \mathrm{Rep}\,\mathcal{B}_i$ with $F_i(U) \cong (X \oplus E \oplus Y)$. Therefore there are objects N, M, W in $\mathrm{Rep}\,\mathcal{B}_i$ with $F_i(M) \cong X$, $F_i(N) \cong$

$Y, F_i(W) \cong E$. Since F_i is full and faithful, thus there is an almost split sequence $N \to W \to M$ whose image is isomorphic to (a). Here M is not isomorphic to some $f_s R_i$ with f_s central primitive idempotent of R_i such that $f_s R_i = k$ thus $N \cong M$ which implies that $X \cong Y$. □

7. Generic Complexes

Here we consider generic complexes in the sense of section 5 of [18]. For a derived tame algebra Λ we shall see the relations between one-parameter families of objects in $\mathcal{D}^b(\Lambda)$ and generic complexes in $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$.

Definition (7.1). A complex $X \in \mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ is called *endofinite* if $H^i(X)$ has finite length as $E(X) = \mathrm{End}_{\mathcal{D}^b(\mathrm{Mod}\,\Lambda)}(X)$-module for all $i \in \mathbb{Z}$.

An endofinite complex X is called generic if it is indecomposable and it is not isomorphic in $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ to a bounded complex of finitely generated Λ-modules.

The homology endolength of an endofinite object X of $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ is defined as

$$\mathbf{h}\mathrm{endol}X = (\mathrm{length}_{E(X)} H^i(X))_{i \in \mathbb{Z}}.$$

Definition (7.2). An infinite family $\mathcal{F}$ of pairwise non-isomorphic indecomposable objects in $\mathcal{D}^b(\Lambda)$, (respectively in $\mathbf{C_n}(\mathrm{mod}\,\Lambda)$) is a called one-parameter family if there is a rational k-algebra R and a bounded complex X of Λ-R-bimodules (respectively X a $\mathbf{C_n}(\mathrm{Proj}\,\Lambda)$-$R$-bimodule) with each X^i free of finite rank over R, such for any $M \in \mathcal{F}$, there is a $\lambda \in \mathcal{S}(R)$ with $M \cong X \otimes_R k[x]/(x-\lambda)$, and for any $\lambda \in \mathcal{S}(R)$ there is a $M \in \mathcal{F}$ with $M \cong X \otimes_R k[x]/(x-\lambda)$. We say that $\mathcal{F}$ is parametrized by Y.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two one-parameter families of complexes in $\mathbf{C_n}(\mathrm{mod}\,\Lambda)$ the set $\mathcal{F}_{1,2}$ of those $X \in \mathcal{F}_1$ such that there is a $Y \in \mathcal{F}_2$ with $X \cong Y$ is either finite or cofinite in $\mathcal{F}_1$. The relation between the one-parameter families defined by $\mathcal{F}_1 \approx \mathcal{F}_2$ if the set $\mathcal{F}_{1,2}$ is infinite is an equivalence relation. We say that $\mathcal{F}_1$ is equivalent to $\mathcal{F}_2$ if $\mathcal{F}_1 \approx \mathcal{F}_2$.

Definition (7.3). If X is a bounded complex of Λ-$k(x)$-bimodules, a realization of X is a bounded complex of Λ-R-bimodules Y, with R a rational k-algebra such that $X \cong Y \otimes_R k(x)$ in the category $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$.

THEOREM (7.4). *Let Λ be a derived tame k-algebra, with k an algebraically closed field. Suppose X is a generic complex in $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$. Then*

(i) X is isomorphic to a bounded complex of finitely generated projective Λ-$k(x)$-bimodules P; moreover $\mathbf{h}\mathrm{endol}X = (\dim_{k(x)} H^i(P))$;

(ii) there is a rational k-algebra R and a complex Y of Λ-R-bimodules free of finite rank over R such that $Y \otimes_R k(x) \cong X$ in $\mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ and $Y \otimes_R - : \mathrm{mod}\,R \to \mathcal{D}^b(\mathrm{mod}\,\Lambda)$ preserves indecomposables and isomorphism classes.

Moreover, if $\mathcal{F}$ is a one-parameter family of indecomposable objects in the category $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$, then there is a generic complex $X \in \mathcal{D}^b(\mathrm{Mod}\,\Lambda)$ and a realization Y of X such that $\mathcal{F}$ is equivalent to the one-parameter family $\{Y \otimes_R R/((x-\lambda)^n)_{\lambda \in \mathcal{S}}\}$ for some n.

Proof. We may assume that for $(h_i) = \mathbf{h}\mathrm{endol}X^\bullet$ we have $h_i = 0$ for $i \leq 2$ and $i > m$, $h_2 \neq 0$. Take now $P \in \mathbf{K}^{\leq \mathbf{m},\mathbf{b}}(\mathrm{Proj}\,\Lambda)$ quasi-isomorphic to X. Then $H^i(P) = 0$ for $i \leq 2$. We have $F(P)$ is indecomposable in $\mathbf{C}^{\mathbf{1}}_{\mathbf{m}}(\mathrm{Proj}\,\Lambda)$, with F the functor given after Lemma (2.2). Now $F(P) = Q = (Q^i, d^i_Q)$ is a complex such that each Q^i has finite length as $\mathrm{End}_Q(Q)$-module, so Q has endofinite length d. Since we have an equivalence $F\colon \mathcal{L}_m \to \overline{\mathbf{C_m}}(\mathrm{Mod}\,\Lambda)$, Q is a generic object. By Theorem (6.5) there is a full and faithful functor $G\colon \mathrm{Rep}\,\mathcal{B} \to \mathbf{C}^1_\mathbf{n}(\mathrm{Proj}\,\Lambda)$ with $\mathcal{B} = (S, W, \delta)$ a minimal triangular tbocs and $G(M) \cong Q$ for some $M \in \mathrm{Rep}\,\mathcal{B}$. Thus M is a generic object in $\mathrm{Rep}\,\mathcal{B}$, then there is a central primitive idempotent $f \in S$ such that $M = k(x)f$.

By (ii) of Theorem (6.5) there is a complex Z of Λ-S-bimodules projective from both sides and finitely generated over the right side such that for all $N \in \mathrm{Rep}\,\mathcal{B}$, $F(N) \cong Z \otimes_S N$, thus $Q \cong Z \otimes_S fk(x) \cong Zf \otimes_{fSf} k(x)$. Here $R = fSf$ is a rational k-algebra and $Y = Zf$ is complex of projective right R-modules, so Y is a complex of free finitely generated right R-modules. Our complex Y satisfies the hypothesis of Corollary (2.8), therefore since $Q \cong Y \otimes_R k(x)$, the morphism $d^1_Q\colon Q^1 \to Q^2$ is a monomorphism. But $d^1_P\colon P^1 \to P^2 = d^1_Q\colon Q^1 \to Q^2$, so d^1_P is a monomorphism. But $H^1(P) = 0$, so $d^0_P = 0$, but this implies that $P^j = 0$ for $j \leq 0$, consequently $P = Q$. We have that the radical of $\mathrm{End}_\mathcal{B}(M)$ is nilpotent and $\mathrm{End}_\mathcal{B}(M)/\mathrm{radEnd}_\mathcal{B}(M) \cong k(x)$, thus for $E_P = \mathrm{End}_{\mathrm{C_m}(\mathrm{Proj}\,\Lambda)}(P)$ we have $E_P/\mathrm{rad}E_P \cong k(x)$. From this we obtain (i). Since G is a full and faithful functor, we obtain (ii).

For the last statement of our theorem suppose that $\mathcal{F}$ is a one-parameter family in $\mathcal{D}^b(\Lambda)$. We may assume that there is a fixed $\mathbf{h} = (h_i)$ such that for all $X \in \mathcal{F}$, $\mathbf{h}\mathrm{dim}X = \mathbf{h}$. We may assume that $\mathcal{F} \subset \mathbf{C}^1_\mathbf{m}(\mathrm{proj}\,\Lambda)$ and there is a fixed d such that $\dim_k X \leq d$ for all $X \in \mathcal{F}$. By Theorem (6.5) there are full and faithful functors $F_i\colon \mathrm{Rep}\,\mathcal{B}_i \to \mathbf{C}^1_\mathbf{m}(\mathrm{proj}\,\Lambda)$ with $\mathcal{B}_i = (R_i, W_i, \delta_i)$ minimal tbocses such that for all $Z \in \mathbf{C}^1_\mathbf{m}(\mathrm{proj}\,\Lambda)$ with $\dim_k Z \leq d$ there is a $N \in \mathrm{Rep}\,\mathcal{B}_i$ with $F_i(N) \cong Z$. Therefore almost all isomorphism classes of indecomposable objects $Z \in \mathbf{C}^1_\mathbf{m}(\mathrm{proj}\,\Lambda)$ with $\dim_k Z \leq d$ are in one-parameter families of the form $\{Y_i f_{i,j} R_i \otimes_{R_i} R_i/((x-\lambda)^n)\}_{\lambda \in \mathcal{S}(R_i)}$. Thus $\mathcal{F}$ is equivalent to one of these families, proving our result. $\square$

Received August 4, 2006

Final version received January 22, 2007

Instituto de Matemáticas
Universidad Nacional Autónoma de México–Unidad Morelia
Apdo. Postal 61-3, Xangari
58089, Morelia, Michoacán
México
raymundo@matmor.unam.mx

References

[1] R. Bautista, *Derived endo-discrete Artin algebras*, Colloq. Math. **105** (2006), 297–310.

[2] R. Bautista, J. Boza, E. Pérez, *Reduction functors and exact structures for bocses*, Bol. Soc. Mat. Mexicana (3) **9**, (2003), 21–60.

[3] R. Bautista, M. J. Souto Salorio and R. Zuazua, *Almost split sequences for complexes of fixed size*, J. Algebra **287** (1)(2005), 140–168.

[4] R. Bautista and Y. Zhang, *Representations of a k-algebra over the rational functions over k*, J. Algebra **267** (2003), 342–358

[5] R. BAUTISTA AND R. ZUAZUA, *One-parameter families of modules for tame algebras and bocses*, Algeb. Represent. Theory **8** (5) (2006), 635–677.

[6] V. BEKKERT AND Y. A. DROZD, *Tame-wild dichotomy for derived categories*, arXiv:math.RT/03/0352.

[7] V. BEKKERT AND H. MERKLEN, *Indecomposables in derived categories of gentle algebras*, Algeb. Represent. Theory **6** 2003, 285–302.

[8] W. W. CRAWLEY-BOEVEY, *On tame algebras and bocses*, Proc. London Math. Soc. **56** (3) (1986), 451–483.

[9] W. W. CRAWLEY-BOEVEY, *Tame algebras and generic modules*, Proc. London Math. Soc. **63** (1991) 241–265.

[10] W. W. CRAWLEY-BOEVEY, *Modules of finite length over their endomorphism rings*, London Math. Soc. Lec. Notes Series **168**, Cambridge University Press, 1992, 127–184

[11] YU. A. DROZD, *Derived tame and derived wild algebras*, Algebra Discrete Math. **1** (2004), 57–74.

[12] P. GABRIEL, A. V. ROITER, Representations of finite dimensional algebras, Encyclopaedia of the Mathematical Sciences. **73**, A. I. Kostrikin and I. V. Shafarevich (Eds.), Algebra VIII, Springer, (1992).

[13] CH. GEISS AND H. KRAUSE, *On the notion of derived tameness*, J. Algebra Appl. **1** (2002) 133-158.

[14] CH. GEISS AND I. REITEN, *Gentle algebras are Gorenstien*, Fields Institute Comm. **45** (2005), 129–133.

[15] D. HAPPEL, Triangulated categories in the representation theory of finite dimensional algebras, London. Math. Soc. Lec. Notes 119 (1988).

[16] D. HAPPEL, *Auslander-Reiten triangles in derived categories of finite-dimensional algebras*, Proceedings of the American Mathematical Society, **112**, no. **3** (1991) 641–648.

[17] H. KRAUSE, *Stable equivalence preserves representation type*, Comment. Math. Helv. **72** (1997) 266–284.

[18] H. KRAUSE, *A duality between complexes of right and left modules*, Representations of Algebras Vol I. Proceedings of the Ninth International Conference Beijing, August 21-September 1, 2000. Edited by D. Happel and Y. B. Zhang. Beijing Normal University Press. (2000) 87–96.

[19] D. QUILLEN, Higher Algebraic K-Theory I, SLNM 341, Springer, Berlin (1973) 85–147.

[20] D. VOSSIECK, *The algebras with discrete derived category*, J. Algebra **243** (2001), 168–176.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

NON-FINITESESS OF TWISTED NILS

RAFAEL RAMOS

ABSTRACT. We prove that the twisted *Nils* $NK_1^{\alpha}(R)$ are infinitely generated, when non-trivial, for any ring R and any ring automorphism $\alpha\colon R \longrightarrow R$ that is of finite order.

Introduction

Let R be a ring with 1. Let $\mathcal{G}$ be a discrete group. Then the *Isomorphism Conjecture* [3] states that the K theory of the ring $R\mathcal{G}$ should be computed from the K theory of the family of virtually cyclic subgroups of $\mathcal{G}$. A group Γ is called virtually cyclic if Γ is either finite or Γ contains an infinite cyclic group of finite index. It is known that the infinite virtually cyclic groups are of two types [8]

$$\Gamma \cong G \rtimes T \tag{1}$$

where G is a finite group and $T \cong \mathbb{Z}$ or

$$\Gamma \cong G_0 *_H G_1 \tag{2}$$

where G_0, G_1 and H are finite groups and $|G_0 : H| = 2 = |G_1 \colon H|$.

If we consider the case (1), $\Gamma \cong G \rtimes T$, we have that

$$R\Gamma \cong RG_{\alpha}[T].$$

So we must study $K_1(RG_{\alpha}[T])$.

On other hand, Farrell and Hsiang [2] proved that

$$Wh(G \rtimes_{\alpha} T) \cong X \ \oplus \ NK_1^{\alpha}(\mathbb{Z}G) \ \oplus \ NK_1^{\alpha^{-1}}(\mathbb{Z}G)$$

where Wh denotes the Whitehead group. In general the groups $NK_1(\mathbb{Z}G)$, $NK_1^{\alpha}(\mathbb{Z}G)$ are very difficult to calculate. We specialize in NK_1^{α} and we give a characterization when G is a finite group.

Our main result, which was also independently proven by Grunewald [6], is the following:

THEOREM. *Let R be any ring with 1. Let $\alpha\colon R \longrightarrow R$ be any ring automorphism of finite order. Then the twisted Nils $NK_1^{\alpha}(R)$ are infinitely generated, when non-trivial.*

As a corollary we get

COROLLARY. *Let $R = \mathbb{Z}G$ where G is a finite group. Let $\alpha\colon G \longrightarrow G$ be any group automorphism. If $NK_1^{\alpha}(R) \neq 0$ then the Nil groups $NK_1^{\alpha}(R)$ are infinitely generated.*

2000 *Mathematics Subject Classification*: 19D35, 19B28.

Keywords and phrases: K-theory, twisted *Nil* groups.

Partially supported by CONACYT and PAPIIT-UNAM research grants.

The case $\alpha = \mathrm{id}$ was proved by Farrell [1] for any ring R with 1. Even though we follow the ideas of Farrell in the twisted case ($\alpha \neq id$) there are some complications and we could not obtain a direct reproduction of the proof for the nontwisted case ($\alpha = \mathrm{id}$) given by Farrell.

I thank the referees for helpful comments that improved the exposition of this work.

1. Preliminaries

Throughout this paper we use the following definitions, notation and results. Let R be a ring with 1 and G a group.

- RG denotes the group ring of G with R coefficients.
- $M_n(R)$ denotes the set of $n \times n$ matrices over the ring R.
- $M(m, n, R)$ denotes the set of $m \times n$ matrices over the ring R.

Definition (1.1). Let $\alpha\colon G \longrightarrow G$ be a group automorphism. In this paper α also denotes the automorphism induced in RG defined by

$$\alpha(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g \alpha(g) \quad r_g \in R,\ g \in G.$$

Definition (1.2). Let $\alpha\colon R \longrightarrow R$ be a ring automorphism. We define the ring $R_\alpha[t]$ as follows: additively, $R_\alpha[t] = R[t]$ and multiplicatively by the condition

$$(rt^i)(st^j) = r\alpha^{-i}(s)t^{i+j} \quad r, s \in R\,.$$

Observation (1.3). Note that we have a ring automorphism in $R_\alpha[t]$ induced by a ring automorphism $\alpha\colon R \longrightarrow R$; this automorphism is also denoted by α and is defined by the condition

$$\alpha(rt^i) = \alpha(r)t^i, \text{ where } r \in R.$$

Note that we use α for three different automorphisms.

Definition (1.4). Let $GL_n(R)$ be the group of invertible matrices over R. Consider the directed system of groups given by the monomorphism of groups

$$\mathrm{GL}_n(R) \longrightarrow \mathrm{GL}_{n+1}(R), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

and define

$$\mathrm{GL}(R) = \operatorname*{colim}_{n\to\infty} \mathrm{GL}_n(R)\,.$$

This means that in definition (1.4) we embed $GL_n(R)$ in $\mathrm{GL}_{n+1}(R)$ and then we can think of $GL(R)$ as an infinite union of the sets $\mathrm{GL}_n(R)$ where each matrix in $\mathrm{GL}(R)$ has finite size. *Note that* $\mathrm{GL}(R)$ *is a group.*

Definition (1.5). Let $a \in R$, $i \neq j$. We define the matrix $e_{ij}(a) \in \mathrm{GL}_n(R)$ for $i \neq j$, $1 \leq i, j \leq n$ as the matrix with only ones on the diagonal, the element a in the (i, j)-slot, and zeros elsewhere. We call these matrices elementary matrices.

Definition (1.6). We denote by $E_n(R)$ the subgroup of $\mathrm{GL}_n(R)$ generated by the set of elementary matrices. We denote by $E(R)$ as $\operatorname*{colim}_{n\to\infty} E_n(R)$. We call $E(R)$ the group of elementary matrices.

Definition (1.7). Let R be a ring with 1. Define $K_1(R)$ as

$$\mathrm{GL}(R)_{ab} = \mathrm{GL}(R)/E(R).$$

Definition (1.8). Let $R_\alpha[t] \xrightarrow{\epsilon} R$ the augmentation defined by the condition $\epsilon(t) = 0$ and let $\alpha\colon R \longrightarrow R$ be a ring automorphism. We define

$$NK_1^\alpha(R) = \mathrm{Kernel}\,(\ K_1(R_\alpha[t]) \xrightarrow{\epsilon_*} K_1(R)\ ).$$

Definition (1.9). Let $\mathcal{P}$ be a category with exact sequences and small skeleton $\mathcal{P}_0$. We define $K_0(\mathcal{P})$ to be the free abelian group generated by the set $Ob(\mathcal{P}_0)$ modulo the following relations:

(i) $[P] = [P']$ if there is an isomorphism $P \xrightarrow{\cong} P'$ in $\mathcal{P}$.

(ii) $[P] = [P_1] + [P_2]$ if there is a short exact sequence

$$0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0$$

in $\mathcal{P}$.

2. Non-finiteness of twisted NILS

Definition (2.1). Let $\alpha\colon R \longrightarrow R$ be a ring automorphism and M_1, M_2 be right $R-$modules. An additive function $f\colon M_1 \longrightarrow M_2$ is called α-linear if $f(mr) = f(m)\alpha(r)\,\forall m \in M, \forall r \in R$.

Let $a \in M(m, n, R)$, and let V, V' be right free R-modules with ordered bases $e = (e_1, \ldots, e_n)$ and $e' = (e'_1, \ldots, e'_m)$. Then the α-linear homomorphism $f\colon V \longrightarrow V'$ associated to a with respect to e and e' is defined by the formula

$$f(\sum_{i=1}^{n} e_i r_i) = \sum_{1\le i\le n,\ 1\le j\le m} e_j a_{ji} \alpha(r_i)$$

where $r_i \in R$.

In terms of the canonical basis for $V = V' = R^n$:

$$\varphi_a(r_1, \ldots, r_n) = a \begin{pmatrix} \alpha(r_1) \\ \vdots \\ \alpha(r_n) \end{pmatrix}.$$

Let f' be a α'-linear homomorphism from V' to a third free R-module V'' corresponding to $a' \in M(k, m, R)$ with respect to e' and to an ordered basis $e'' = (e''_1, \ldots, e''_k)$ for V''.

LEMMA (2.2). *$f'f$ is the $\alpha'\alpha$-linear homomorphism corresponding to $a'\alpha'(a)$ with respect to e and e''.*

Proof. [2], lemma 1. □

Note: The following lemma is a direct generalization of [7], lemma 3.2.21.

LEMMA (2.3). *Let $B \in \mathrm{GL}(R_\alpha[t])$. Then B can be reduced modulo $\mathrm{GL}(R)$ and $E(R_\alpha[t])$ to a matrix of the form $I + At$ where A is a matrix with entries in R such that the α^{-1}-linear homomorphism associated to A, φ_A is nilpotent, i.e., $\exists r \in \mathbb{N}$ such that ${\varphi_A}^r \equiv 0$.*

Proof. Let $B \in \mathrm{GL}(R_\alpha[t])$. Then $B = B_0 + B_1 t + \cdots + B_d t^d$ for some d and $B_i \in M(R)$ for all i. If we can reduce B to a matrix of degree zero, the lemma is trivial. Using induction it is always possible to reduce B to a matrix in $\mathrm{GL}(R_\alpha[t])$ of degree $d \leq 1$. That means we can assume that $B = I + At$, so B^{-1} is of the form $B^{-1} = C_0 + C_1 t + \cdots + C_r t^r$, $C_i \in M(R)$.

Now, using the following facts: $I = (I + At)(C_0 + C_1 t + \cdots + C_r t^r)$ and $AtC_i = A\alpha^{-1}(C_i)t$ we conclude that

$$0 = A\alpha^{-1}(A)\alpha^{-2}(A)\cdots\alpha^{-r}(A) = \underbrace{A\alpha^{-1}(\,A\alpha^{-1}(\,A\alpha^{-1}(\,\ldots A\alpha^{-1}(A)\,)\,)\,)}_{r-times}$$

This means that the α^{-1}-linear homomorphism associated to A, $\varphi_A\colon R^n \longrightarrow R^n$ is such that ${\varphi_A}^r = 0$ by lemma 2.2. □

The following result is well known ([5], theorem 2.1.c.)

THEOREM (2.4). *Let R be a ring with 1. Let $\mathrm{Nil}_\alpha(R)$ be the category whose objects are pairs (R^n, φ) with $n \in \mathbb{N}\cup\{0\}$, and let $\varphi\colon R^n \longrightarrow R^n$ be an α^{-1}-linear nilpotent endomorphism of right R-modules whose morphisms are defined as follows:*

*Given two objects (R^n, φ_1), (R^n, φ_2) a morphism between them is an R-*linear *homomorphism $g\colon R^n \longrightarrow R^m$ of right R-modules such that the diagram*

$$\begin{array}{ccc} R^n & \xrightarrow{\varphi_1} & R^n \\ {\scriptstyle g}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ R^m & \xrightarrow{\varphi_2} & R^m \end{array}$$

commutes.

Note that $Nil_\alpha(R)$ is a category with exact sequences and small skeleton. We denote by $\widetilde{K_0}(\mathrm{Nil}_\alpha(R))$ the reduced K-theory of $K_0(Nil_\alpha(R))$.

Then

(a) $K_1(R_\alpha[t]) \cong K_1(R) \oplus NK_1^\alpha(R)$

(b) $NK_1^\alpha(R) \cong \widetilde{K_0}(Nil_{\alpha^{-1}}(R))$. □

Observation (2.5). Let $n \in \mathbb{N}$ and $p(t) \in R_\alpha[t]$. It is always possible to complete $p(t)$ with zeros and assume that it is of the form $p(t) = \sum_{i=0}^{kn} a_i t^i$ for some $k \in \mathbb{N}\cup\{0\}$. Furthermore $p(t)$ can be written as the following sum:

$$p(t) = (\sum_{i=0}^{k-1} a_{in}t^{in} + a_{kn}t^{kn}) + (\sum_{i=0}^{k-1} a_{in+1}t^{in})t + \cdots + (\sum_{i=0}^{k-1} a_{in+(n-1)}t^{in})t^{n-1}. \quad \square$$

Using observation 2.5 we prove the following.

LEMMA (2.6). *Let $n \in \mathbb{N}$. Then $R_\alpha[t]$ is a free left $R_\alpha[t^n]$-module with rank n, i.e., we have an isomorphism of left $R_\alpha[t^n]$-modules*

$$\varphi\colon R_\alpha[t] \xrightarrow{\cong} \underbrace{R_\alpha[t^n] \oplus \cdots \oplus R_\alpha[t^n]}_{n-times}$$

A basis in $R_\alpha[t]$ is given by $1,\ t,\ t^2,\ \ldots,\ t^{n-1}$. □

Let $\iota_n \colon R_\alpha[t^n] \longrightarrow R_\alpha[t]$ be the inclusion. Then we have the induced homomorphism $(\iota_n)_* : K_1(R_\alpha[t^n]) \longrightarrow K_1(R_\alpha[t])$. We now define a transfer homomorphism $\iota_n^* \colon K_1(R_\alpha[t]) \longrightarrow K_1(R_\alpha[t^n])$. First we define a group homomorphism $\iota_n^* \mathrm{GL_r(R_\alpha[t]) \longrightarrow GL_r(R_\alpha[t^n])}$ as follows:

Definition (2.7). Let $B \in \mathrm{GL}_r(R_\alpha[t])$. Then we define $\iota_n^*(B) = \bar{B}$ where $\bar{B}$ is the matrix associated to the following composition with respect to the canonical basis

$$(R_\alpha[t^n] \oplus \cdots \oplus R_\alpha[t^n])^r \xrightarrow{(\varphi^{-1})^r} (R_\alpha[t])^r \xrightarrow{(\ )B} (R_\alpha[t])^r \xrightarrow{\varphi^r} (R_\alpha[t^n] \oplus \cdots \oplus R_\alpha[t^n])^r$$

where $(\varphi^{-1})^r = \varphi^{-1} \times \cdots \times \varphi^{-1}$ r-times, $\varphi^r = \varphi \times \cdots \times \varphi$ r-times and φ is the isomorphism of lemma (2.6).

Using definition (2.7) we get the following two lemmas:

LEMMA (2.8). $\iota_n^* \colon \mathrm{GL}_r(R_\alpha[t]) \longrightarrow \mathrm{GL}_r(R_\alpha[t^n])$ *is a group homomorphism and* $\iota_n^* \colon K_1(R_\alpha[t]) \longrightarrow K_1(R_\alpha[t^n])$ *is well defined.* □

LEMMA (2.9). *Let* $B \in \mathrm{GL}_r(R_\alpha[t^n])$. *Then* $\iota_n^* \circ (\iota_n)_*([B]) = [B \oplus \alpha^{-1}(B) \oplus \cdots \oplus \alpha^{-(n-1)}(B)]$. □

Using lemma (2.3) we prove

LEMMA (2.10). *Let* $x \in NK_1^\alpha(R)$ *be fixed. Hence by lemma (2.3),* $x = [I + Nt]$ *with* $N \in M_r(R)$ *for some* $r \in \mathbb{N}$ *and* $\varphi_N^n = 0$ *for some* $n \in \mathbb{N}$. *Then* (a) $\iota_n^*([I + Nt]) = M$ *where* M *is the following block matrix,*

$$M = \begin{pmatrix} 1 & N & \cdots & 0 & 0 \\ 0 & 1 & \alpha^{-1}(N) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & & 1 & \alpha^{-(n-2)}(N) \\ \alpha^{-(n-1)}(N)t^n & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

(b) *Let* A *be the block matrix strictly lower-triangular (and therefore elementary) given by*

$$A = \begin{pmatrix} 1 & N & \cdots & 0 & 0 \\ 0 & 1 & \alpha^{-1}(N) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & & 1 & \alpha^{-(n-2)}(N) \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Then MA^{-1} *is strictly lower-triangular and therefore elementary.*

Proof. (a) It follows from a direct calculation using definition (2.7).

(b) Let $\bar{N}$, B be the matrices of n blocks

$$\bar{N} = \begin{pmatrix} 0 & N & \cdots & 0 & 0 \\ 0 & 0 & \alpha^{-1}(N) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & & 0 & \alpha^{-(n-2)}(N) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & & 0 & 0 \\ \alpha^{-(n-1)}(N)t^n & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Note that $\bar{N}^n = 0$, $A = I + \bar{N}$ and $M = I + \bar{N} + B$. Since $A = I + \bar{N}$ then $A^{-1} = I - \bar{N} + \bar{N}^2 + \cdots + (-1)^{n-1}\bar{N}^{n-1}$. Hence we get

$$\begin{aligned} MA^{-1} &= (I + \bar{N} + B)(I - \bar{N} + \bar{N}^2 + \cdots + (-1)^{n-1}\bar{N}^{n-1}) \\ &= I + BI - B\bar{N} + B\bar{N}^2 + \cdots + (-1)^{n-1}B\bar{N}^{n-1}. \end{aligned}$$

After some calculations we get

$$MA^{-1} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}$$

where

$$\begin{aligned} a_1 &= \alpha^{-(n-1)}(N)t^n \\ a_2 &= -\alpha^{-(n-1)}(N)t^n N \\ a_3 &= \alpha^{-(n-1)}(N)t^n N\alpha^{-1}(N) \\ a_n &= 1 + (-1)^{n-1}b \end{aligned}$$

and where

$$b = \alpha^{-(n-1)}(N)t^n N\alpha^{-1}(N)\cdots\alpha^{-(n-2)}(N).$$

But

$$\begin{aligned} b &= \alpha^{-(n-1)}(N)t^n \ N\alpha^{-1}(N)\cdots\alpha^{-(n-2)}(N) \\ &= \alpha^{-(n-1)}(N)\alpha^{-n}(N)\alpha^{-(n+1)}(N)\cdots\alpha^{-(n+(n-2))}(N)t^n \\ &= \alpha^{-(n-1)}(\ N\alpha^{-1}(N)\alpha^{-2}(N)\cdots\alpha^{-(n-1)}(N)\ ) = 0. \end{aligned}$$

Therefore MA^{-1} is is strictly lower-triangular. □

From lemma (2.10), it follows that $\iota_n^*([I+Nt]) = [M] = [MA^{-1}][A] = 0$ for N such that $\varphi_N^n = 0$. Using this fact we prove the following.

PROPOSITION (2.11). *If $NK_1^\alpha(R)$ is finitely generated then there exists an integer n_0 such that $\iota_n^* \equiv 0$ in $NK_1^\alpha(R)$ $\forall n \geq n_0$.* □

Definition (2.12). Let R be a ring with 1. Let $\alpha\colon R \longrightarrow R$ be a ring automorphism. Let M be a right R-module. Then we define $\alpha(M)$ as the right R-module such that additively $\alpha(M) = M$. Scalar multiplication is defined by $m * r = m\alpha(r)\ \forall m \in M, \forall r \in R$.

LEMMA (2.13). *We have a commutative diagram*

$$\begin{array}{ccc} \alpha^{-1}(R^n) & \xrightarrow{\varphi_A} & \alpha^{-1}(R^n) \\ {\scriptstyle\alpha}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle\alpha} \\ R^n & \xrightarrow{\varphi_{\alpha(A)}} & R^n \end{array}$$

Proof. Let ψ be the composition defined by the following diagram,

$$\begin{array}{ccc} \alpha^{-1}(R^n) & \xrightarrow{\varphi_A} & \alpha^{-1}(R^n) \\ {\scriptstyle\alpha}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\big\downarrow{\scriptstyle\alpha} \\ R^n & \xrightarrow{\psi} & R^n \end{array}$$

Note that $\alpha^{-1}\colon R^n \xrightarrow{\cong} \alpha^{-1}(R^n)$ with

$$\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \mapsto \begin{pmatrix} \alpha^{-1}(r_1) \\ \vdots \\ \alpha^{-1}(r_n) \end{pmatrix}$$

and $\alpha\colon \alpha^{-1}(R^n) \xrightarrow{\cong} R^n$ with

$$\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \mapsto \begin{pmatrix} \alpha(r_1) \\ \vdots \\ \alpha(r_n) \end{pmatrix}$$

are R-linear isomorphisms and $\varphi_A\colon R^n \longrightarrow R^n$ is an α^{-1}-linear homomorphism. (Note that φ_A thought of as $\varphi_A\colon \alpha^{-1}(R^n) \to \alpha^{-1}(R^n)$ is also an α^{-1}-linear homomorphism). Further, $\varphi_A\colon \alpha^{-1}(R^n) \longrightarrow \alpha^{-1}(R^n)$ is such that

$$\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} \mapsto A\alpha^{-1}\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} = A\begin{pmatrix} \alpha^{-1}(r_1) \\ \vdots \\ \alpha^{-1}(r_n) \end{pmatrix}.$$

Thus $\alpha \circ \varphi_A \circ \alpha^{-1}$

$$\begin{aligned}\begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix} &= \alpha\varphi_A \begin{pmatrix} \alpha^{-1}(r_1) \\ \vdots \\ \alpha^{-1}(r_n) \end{pmatrix} = \alpha \left(A \begin{pmatrix} \alpha^{-2}(r_1) \\ \vdots \\ \alpha^{-2}(r_n) \end{pmatrix} \right) \\ &= \alpha(A)\alpha \begin{pmatrix} \alpha^{-2}(r_1) \\ \vdots \\ \alpha^{-2}(r_n) \end{pmatrix} = \alpha(A)\alpha^{-1} \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}.\end{aligned}$$

Therefore the matrix associated to the composition $\psi = \alpha \circ \varphi_A \circ \alpha^{-1}$ is $\alpha(A)$, which means $\psi = \varphi_{\alpha(A)}$. □

Using lemma (2.13), theorem (2.4) (b), and proposition 10, page 202, [2], we obtain the following result:

LEMMA (2.14). *Let R be a ring with* 1. *Then* $(NK_1^{\alpha}(R))^{\alpha_*} = NK_1^{\alpha}(R)$.

Proof. The following diagram commutes,

$$\begin{array}{ccc} \widetilde{K_0}(Nil_{\alpha^{-1}}(R)) & \xrightarrow{\cong} & NK_1^{\alpha}(R) \\ {\scriptstyle \alpha_*^{-1}}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ \widetilde{K_0}(Nil_{\alpha^{-1}}(R)) & \xrightarrow{\cong} & NK_1^{\alpha}(R) \end{array}$$

$$\begin{array}{ccc} [(R^n, \varphi_A)] & \xrightarrow{\cong} & [I + At] \\ {\scriptstyle \alpha_*^{-1}}\downarrow & & \downarrow{\scriptstyle \alpha_*} \\ [(\alpha^{-1}(R^n), \varphi_A)] = [(R^n, \varphi_{\alpha(A)})] & \longrightarrow & [I + \alpha(A)t] \end{array}$$

where the equality in the last diagram is given by lemma 2.13. Now by [2], proposition 10, page 202 we have that

$$(\widetilde{K_0}(Nil_{\alpha^{-1}}(R)))^{\alpha_*^{-1}} = \widetilde{K_0}(Nil_{\alpha^{-1}}(R)).$$

Therefore $\alpha_*([I + At]) = [I + At]$ in $NK_1^{\alpha}(R)$. □

Note: Farrell [1] proved that theorem (2.15) is true for any ring R with 1 in the case $\alpha =$Id.

THEOREM (2.15). *Let R any ring with* 1. *Let $\alpha\colon R \longrightarrow R$ be any ring automorphism of finite order. Then the twisted Nils $NK_1^{\alpha}(R)$ are infinitely generated, when non-trivial.*

Proof. Assume that $NK_1^{\alpha}(R) \neq 0$ and that $NK_1^{\alpha}(R)$ is finitely generated. By proposition (2.11) there exists an integer n_0 such that $\iota_n^* \equiv 0$ in $NK_1^{\alpha}(R)$ $\forall n \geq n_0$.

Since $\alpha\colon R \longrightarrow R$ is of finite order, $\exists\ m_0 \neq 0$ such that $\alpha^{m_0} =$id $\Rightarrow \alpha^{km_0} = id\, \forall\ k \in \mathbb{N} \cup \{0\}$. Then we have an isomorphism of rings

$$R_{\alpha}[t] = R_{\alpha^{km_0+1}}[t] \xrightarrow{\cong} R_{\alpha}[t^{km_0+1}]\, \forall\ k \in \mathbb{N} \cup \{0\}.$$

Now we use the following theorem of Dirichlet ([9], theorem 4.5):

THEOREM. Let a, $b \in \mathbb{Z}$ such that $(a, b) = 1$. Then $\{a + kb\}_{k=1}^{\infty}$ contains an infinite number of primes. $\square$

Therefore $\{km_0 + 1\}_{k=1}^{\infty}$ contains a infinite number of primes.

Since we assumed that $NK_1^{\alpha}(R) \neq 0$ and as an abelian group it is finitely generated, given any prime p such that p does not appear in the decomposition of $NK_1^{\alpha}(R)$ (this decomposition given by the Fundamental theorem of finitely generated abelian groups [4], theorem 9.3, page 92.) we have that multiplication by p is injective in $NK_1^{\alpha}(R)$. (With exception of a finite number of primes, all other primes have this property). Then there is a prime p with the following properties:

- The multiplication by p, $p(\ )\colon NK_1^{\alpha}(R) \to NK_1^{\alpha}(R)$ is injective.
- $p = km_0 + 1$ for some $k \in \mathbb{N}$.
- $p > n_0$

Let $[I + Nt] \neq 0$ be in $K_1(R_\alpha[t])$. By lemma (2.14) α_* is invariant in $NK_1^{\alpha}(R)$ and by the comments above $p(\ )\colon NK_1^{\alpha}(R) \xrightarrow{N} K_1^{\alpha}(R)$ is injective. Therefore

$$\begin{aligned}
0 \neq p([I + Nt]) &= [(I + Nt) \oplus \alpha^{-1}(I + Nt) \oplus \cdots \oplus \alpha^{-(p-1)}(I + Nt)] \\
&= [(I + Nt) \oplus I + \alpha^{-1}(N)t \oplus \cdots \oplus I + \alpha^{-(p-1)}(N)t] \\
&= \left[I + \begin{pmatrix} N & & & 0 \\ & \alpha^{-1}(N) & & \\ & & \ddots & \\ 0 & & & \alpha^{-(p-1)}(N) \end{pmatrix} t \right] \in NK_1^{\alpha}(R) \\
&\leq K_1(R_\alpha[t]) \\
&\cong \left[I + \begin{pmatrix} N & & & 0 \\ & \alpha^{-1}(N) & & \\ & & \ddots & \\ 0 & & & \alpha^{-(p-1)}(N) \end{pmatrix} t^p \right] \in K_1(R_\alpha[t^p]) \\
&= \begin{pmatrix} I + Nt^p & & & 0 \\ & \alpha^{-1}(I + Nt^p) & & \\ & & \ddots & \\ 0 & & & \alpha^{-(p-1)}(I + Nt^p) \end{pmatrix} \\
&= [(I + Nt^p) \oplus \alpha^{-1}(I + Nt^p) \oplus \cdots \oplus \alpha^{-(p-1)}(I + Nt^p)] \\
&= \iota_p^* \circ (\iota_p)_*(\ [I + Nt^p]\ )
\end{aligned}$$

by lemma (2.9).

Therefore $\iota_p^* \circ (\iota_p)_*(\ [I + Nt^p]\ ) \neq 0$ with $p > n_0$. On the other hand note that $(\iota_p)_*(\ [I + Nt^p]\ ) = [I + Nt^p] \in NK_1^{\alpha}(R)$ (since if $\epsilon_* : K_1(R_\alpha[t]) \longrightarrow K_1(R)$ is the homomorphism induced by the augmentation $\epsilon(t) = 0$ then $\epsilon_*([I + Nt^p]) = [I]$ for $[I + Nt^p] \in K_1(R_\alpha[t])$). By proposition (2.11) this is a contradiction. $\square$

Now the following result is immediate:

COROLLARY (2.16). *Let $R = \mathbb{Z}G$ where G is a finite group. Let $\alpha\colon G \longrightarrow G$ be any group automorphism. If $NK_1^\alpha(R) \neq 0$ then the Nil groups $NK_1^\alpha(R)$ are infinitely generated.* $\square$

It may be worth noting that the proof of Theorem (2.15) holds also under the following weaker assumption: Assume that $NK_1^\alpha(R) \neq 0$. Furthermore, assume that there is an infinite sequence of positive integers $\{n_k\}$ such that the transfer map

$$\iota_{n_k}^*\colon NK_1^\alpha(R) \longrightarrow K_1(R_\alpha[t^{n_k}])$$

is not the zero map. Under this weaker assumption the Theorem (2.15) is still true.

Received March 22, 2006

Final version received October 22, 2006

INSTITUTO DE MATEMÁTICAS, UNIDAD MORELIA
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
APARTADO POSTAL 61-3 (XANGARI)
58089 MORELIA, MICHOACÁN
MÉXICO
rramos@matmor.unam.mx

REFERENCES

[1] F. T. FARRELL, *The nonfiniteness of Nil*, Proc. Amer. Math. Soc. **65** (1977), 215–216.
[2] F. T. FARRELL, W. C. HSIANG, *A formula for $K_1R_\alpha[T]$*, Proc. Symp. Pure Math. **17**, Applications of Categorical Algebra, Amer. Math. Soc., Providence, (1970).
[3] F. T. FARRELL, L. JONES, *Isomorphism conjectures in algebraic K-theory*, J. Amer. Math. Soc. **6** (2), (1993), 249–298.
[4] J. FRALEIGH, Álgebra abstracta, Addison-Wesley Iberoamericana, Wilmington, (1987).
[5] D. R. GRAYSON, *The K-theory of semilinear endomorphisms*, J. Algebra, 113 No. 2: (1998), 358–372.
[6] J. GRUNEWALD, *Non-Finiteness Results for Nil-Groups*, preprint (2006).
[7] J. ROSENBERG, Algebraic K-Theory and Its Applications, Springer-Verlag, New York, (1996).
[8] P. SCOTT AND T. WALL, *Topological methods in group theory. Homological group theory*, LMS. 36, (1979).
[9] L. C. WASHINGTON, Introduction to Cyclotomic Fields, Springer-Verlag, New York, (1997).

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

INNER AMENABILITY OF FOUNDATION SEMIGROUP ALGEBRAS

ALI GHAFFARI

ABSTRACT. In this paper we shall introduce the inner amenability and topological inner amenability for foundation semigroup algebras and show various necessary and sufficient conditions for foundation semigroup algebras to be inner amenable.

1. Introduction

Let S be a locally compact Hausdorff topological semigroup. Let $M(S)$ be the space of all complex Borel measures on S. It is known that $M(S) = C_0(S)^*$, therefore $M(S)$ is a Banach space and with convolution

$$\mu * \nu(\psi) = \int\int \psi(xy)d\mu(x)d\nu(y)$$

$(\mu, \nu \in M(S), \psi \in C_0(S))$, $M(S)$ is a Banach algebra. The subalgebra $M_a(S)$ of $M(S)$ is defined by $M_a(S) = \{\mu \in M(S);\ x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from S into $M(S)$ are weakly continuous$\}$. A semigroup S is called a *foundation semigroup* if $\bigcup\{\operatorname{supp}\mu;\ \mu \in M_a(S)\}$ is dense in S. A trivial example is a topological group and in this case $M_a(S) = L^1(S)$. Note that $M_a(S)$ is a closed two-sided L-ideal of $M(S)$ [5]. We also note that for $\mu \in M_a(S)$ both mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from S into $M(S)$ are norm continuous [5]. When S is a foundation semigroup with identity, it is known that $M_a(S)$ has a bounded approximate identity [5]. For more details on foundation semigroups, the reader is referred to [1] and [8].

Let $M_a(S)^*$ and $M_a(S)^{**}$ be the first and second duals of $M_a(S)$. With the Arens product, $M_a(S)^{**}$ is a Banach algebra [6]. For $\mu \in M_a(S)$, $\nu \in M(S)$ and $f \in M_a(S)^*$, we define $\langle f\nu, \mu\rangle = \langle f, \nu * \mu\rangle$ and $\langle \nu, f\mu\rangle = \langle f, \mu * \nu\rangle$. In [6] the author defined $B = M_a(S)^* M_a(S)$ which is a Banach subspace of $M_a(S)^*$. Clearly $M(S) \subseteq B^*$.

Let X be a linear subspace of $M_a(S)^*$ containing the constant functional 1, where $\langle 1, \mu\rangle = \mu(S)$, $\mu \in M_a(S)$. We say that X is right (respectively, left) translation invariant if $\delta_x X \subseteq X$ (respectively, $X\delta_x \subseteq X$) for all $x \in S$. X is translation invariant if it is both right and left translation invariant. X is said to be topologically invariant if $\mu f \in X$ and $f\mu \in X$ for all $f \in X$ and $\mu \in P(S)$ ($P(S)$ is convex hull of probability measures in $M_a(S)$, that is, all $\mu \in M_a(S)$ for which $\langle 1, \mu\rangle = 1$ and $\mu \geq 0$).

A linear functional $M \in X^*$ is called a *mean* if $\langle M, f\rangle \geq 0$ whenever $f \geq 0$ and $\langle M, 1\rangle = 1$. M is *topologically inner invariant* (respectively, *inner*

2000 *Mathematics Subject Classification*: Primary: 22A20; Secondary: 43A60.

Keywords and phrases: Banach algebras, locally compact semigroup, inner invariant mean, topologically inner invariant mean.

The author is indebted to the University of Semnan for their support.

invariant) if $\langle M, f\mu\rangle = \langle M, \mu f\rangle$ for any $\mu \in P(S)$ and $f \in X^*$ (respectively, $\langle M, f\delta_x\rangle = \langle M, \delta_x f\rangle$ for any $x \in S$ and $f \in X^*$).

The existence of topologically left invariant means and left invariant means for groups is widely investigated (see [13],[14]). The notion of topological left amenability of semigroup algebras was introduced by Wong in [17] and by Riazi and Wong in [15]. For further details and complementary historical comments see [7]. The study of inner amenability is initiated by Effros [4]. See also [11], [18], and [19]. The inner amenability of groups is investigated by many authors e.g., [4], [9], [10], [18] and [19]. The concept of strict inner amenability was introduced and studied in [12] for an arbitrary Lau algebra.

The purpose of this paper is to introduce and to study a concept of inner amenability and topological inner amenability for foundation semigroup algebras. We obtain necessary and sufficient conditions for $M_a(S)^*$ to have an inner invariant mean. Also we study relations between inner invariant means and topologically inner invariant means on a subspace of $M_a(S)^*M_a(S)\bigcap M_a(S)M_a(S)^*$. It is known that the mapping $T\colon LUC(S) \to M_a(S)^*M_a(S)$ given by $\langle T(f), \mu\rangle = \int f(x)d\mu(x)$ is an isometric isomorphism of $LUC(S)$ onto $M_a(S)^*M_a(S)$ [6].

2. Main results

We start this section by a series of lemmas. All over this section S is a foundation, locally compact, Hausdorff, topological semigroup.

LEMMA (2.1). *The following conditions are equivalent.*

(1) *For every $x \in S$, there exists a mean M such that $\langle M, \delta_x f\rangle = \langle M, f\delta_x\rangle$ for any $f \in M_a(S)^*$.*

(2) $\sup\{\langle \delta_x f - f\delta_x, \nu\rangle;\ \nu \in P(S)\} \geq 0$ *for all $x \in S$ and $f \in M_a(S)^*$.*

Proof. Clearly (1) implies (2).

Now, assume that (2) holds. For x in S, consider the subspace

$$X = \delta_x M_a(S)^* - M_a(S)^*\delta_x$$

of $M_a(S)^*$. Let $\rho\colon M_a(S)^* \to \mathbb{R}$ be defined by

$$\rho(f) = \sup\{\langle \delta_x f - f\delta_x, \nu\rangle;\ \nu \in P(S)\}$$

and M_1 be the zero functional on X. By assumption, $M_1 \leq \rho$ on X. By the Hahn-Banach theorem M_1 extends to a linear functional M on $M_a(S)^*$ that also satisfies $M \leq \rho$. This together with linearity of M, implies that M is a mean on $M_a(S)^*$. Moreover $\langle M, \delta_x f\rangle = \langle M, f\delta_x\rangle$ for any $x \in S$. □

LEMMA (2.2). *The following conditions are equivalent:*

(1) *For every $f \in M_a(S)^*$, there exists a mean M such that $\langle M, \mu f\rangle = \langle M, f\mu\rangle$ for any $\mu \in P(S)$.*

(2) *For any $f \in M_a(S)^*$, the weak*-closure of $\{\mu f - f\mu;\ \mu \in P(S)\}$ contains the zero functional.*

Proof. Let $f \in M_a(S)^*$ and let M be a mean on $M_a(S)^*$ such that $\langle M, f\mu\rangle = \langle M, \mu f\rangle$ for any $\mu \in P(S)$. Since $P(S)$ is weak* dense in the set of means on $M_a(S)^*$, there is a net (μ_α) in $P(S)$ such that $\mu_\alpha \to M$ in the weak*-topology.

We will show that $\mu_\alpha f - f\mu_\alpha \to 0$ in the weak*-topology. Let $\mu \in P(S)$ be fixed. We have

$$\begin{aligned}\lim_\alpha \langle \mu, \mu_\alpha f - f\mu_\alpha\rangle &= \lim_\alpha \langle \mu_\alpha f - f\mu_\alpha, \mu\rangle = \lim_\alpha(\langle \mu_\alpha f, \mu\rangle - \langle f\mu_\alpha, \mu\rangle)\\ &= \lim_\alpha(\langle f, \mu * \mu_\alpha\rangle - \langle f\mu_\alpha, \mu\rangle) = \lim_\alpha \langle f\mu - \mu f, \mu_\alpha\rangle\\ &= \lim_\alpha \langle \mu_\alpha, f\mu - \mu f\rangle = \langle M, f\mu - \mu f\rangle = 0.\end{aligned}$$

This shows that $\mu_\alpha f - f\mu_\alpha \to 0$ in the weak*-topology. Thus (1) implies (2).

Conversely, let $f \in M_a(S)^*$ and let (μ_α) be a net in $P(S)$ such that $\mu_\alpha f - f\mu_\alpha \to 0$ in the weak*-topology. Passing to a subnet if necessary, we can assume that (μ_α) converges weak* to some mean M in $M_a(S)^*$. Observe that for any $\mu \in P(S)$,

$$\begin{aligned}\langle M, f\mu - \mu f\rangle &= \lim_\alpha \langle \mu_\alpha, f\mu - \mu f\rangle = \lim_\alpha \langle f\mu - \mu f, \mu_\alpha\rangle\\ &= \lim_\alpha \langle \mu_\alpha f - f\mu_\alpha, \mu\rangle = 0.\end{aligned}$$

Hence $\langle M, \mu f\rangle = \langle M, f\mu\rangle$. □

We establish a criterion that ensures the existence of topologically inner invariant means using Hahn-Banach theorem, a definitely nonconstructive procedure.

THEOREM (2.3). *If S has an identity, then the following conditions are equivalent.*

(1) *$M_a(S)^*$ has a topologically inner invariant mean.*

(2) *If H consists of all functionals $h \in M_a(S)^*$ having the form*

$$\sum_{i=1}^{n} \mu_i f_i - f_i \mu_i$$

for some $f_1, ..., f_n \in M_a(S)^*$ and $\mu_1, ..., \mu_n \in P(S)$, then $\overline{H} \neq M_a(S)^*$.

Proof. If M is a topologically inner invariant mean on $M_a(S)^*$, then $\langle M, h\rangle = 0$ for any $h \in H$. On the other hand $\langle M, 1\rangle = 1$ and so $\overline{H} \neq M_a(S)^*$.

To prove the converse, let (e_α) be an approximate identity in $P(S)$ (see [5]). Let $1 = \sum_{i=1}^{n} \mu_i f_i - f_i \mu_i$ for some $f_1, ..., f_n \in M_a(S)^*$ and $\mu_1, ..., \mu_n \in P(S)$. Thus

$$\begin{aligned}1 = \lim_\alpha \langle 1, e_\alpha\rangle &= \lim_\alpha \langle \sum_{i=1}^{n} \mu_i f_i - f_i \mu_i, e_\alpha\rangle\\ &= \lim_\alpha \langle \sum_{i=1}^{n} f_i, e_\alpha * \mu_i - \mu_i * e_\alpha\rangle = 0,\end{aligned}$$

so it follows that 1 is not in H. By the Hahn-Banach extension theorem, there is M in $M_a(S)^{**}$ such that $\langle M, 1\rangle = \|M\| = 1$ and $\langle M, h\rangle = 0$ for all $h \in H$. Hence M is a topologically inner invariant mean on $M_a(S)^*$. □

Now let S have an identity. Let $E \in M_a(S)^{**}$ be the weak* limit of a net (e_α) which is a bounded approximate identity for $M_a(S)$ with norm one [5]. Then E is a right identity in $M_a(S)^{**}$. If a right identity E has norm one, the converse holds: E is the weak* limit of a norm one approximate identity in $M_a(S)$ (see [3], proposition 7 on p.146 and its proof). Consequently, every right identity E

with norm one is a topologically inner invariant on $M_a(S)^*$. Indeed, if E is the weak* limit of a norm one approximate identity (e_α) in $M_a(S)$, then for every $f \in M_a(S)^*$ and $\mu \in P(S)$,

$$\begin{aligned}\langle E, f\mu\rangle &= \lim_\alpha \langle e_\alpha, f\mu\rangle = \lim_\alpha \langle f, \mu * e_\alpha\rangle \\ &= \langle f, \mu\rangle = \lim_\alpha \langle f, e_\alpha * \mu\rangle \\ &= \lim_\alpha \langle \mu f, e_\alpha\rangle = \langle E, \mu f\rangle.\end{aligned}$$

On the other hand,

$$\|E\| = 1 = \lim_\alpha \langle e_\alpha, 1\rangle = \langle E, 1\rangle.$$

This shows that E is a topologically inner invariant mean on $M_a(S)$.

THEOREM (2.4). *Let S be a foundation locally compact Hausdorff topological semigroup with identity. Let X be a translation invariant Banach subspace of* $M_a(S)^*M_a(S) \bigcap M_a(S)M_a(S)^*$ *with* $1 \in X$. *Let M be a mean on X. Then M is a topologically inner invariant mean on X if and only if M is an inner invariant mean on X.*

Note that, X is topologically invariant. Indeed, if X is a Banach subspace of $M_a(S)^*M_a(S) \bigcap M_a(S)M_a(S)^*$, *then an argument similar to the proof of Lemma 2.3 in* [6] *shows that, X is translation invariant if and only if X is topologically invariant.*

Proof. Necessity. Let M be a topologically inner invariant mean on X. Let $(e_\alpha)_{\alpha\in I}$ be a bounded approximate identity for $M_a(S)$ (see [5]). Let $f \in X$ and $x \in S$. Then $f = g\mu = \nu h$ where $g, h \in M_a(S)^*$ and $\mu, \nu \in M_a(S)$. We have

$$\begin{aligned}\langle M, \delta_x f\rangle &= \langle M, \delta_x(\nu h)\rangle = \langle M, \delta_x * \nu h\rangle = \lim_\alpha \langle M, \delta_x * e_\alpha * \nu h\rangle \\ &= \lim_\alpha \langle M, (\nu h)\delta_x * e_\alpha\rangle = \lim_\alpha \langle M, f\delta_x * e_\alpha\rangle \\ &= \lim_\alpha \langle M, (g\mu)\delta_x * e_\alpha\rangle = \lim_\alpha \langle M, g\mu * \delta_x * e_\alpha\rangle \\ &= \langle M, g\mu * \delta_x\rangle = \langle M, (g\mu)\delta_x\rangle = \langle M, f\delta_x\rangle.\end{aligned}$$

Consequently, M is an inner invariant mean on X.

Sufficiency. Let M be an inner invariant mean on X. Let $f \in X$, $\mu \in P(S)$. We may assume that $K = \operatorname{supp}\mu$ is compact. Then $\psi\colon K \to X$ defined by $\psi(x) = \delta_x f$ is continuous. So, by Theorem 3.20 and Theorem 3.27 in [16] and Theorem A.1 in [2], we can write

$$\int_K \psi(x)d\mu(x) = \int_K \delta_x f d\mu(x) \in X.$$

Now, let $\nu \in M_a(S)$. By Lemma 2.2 in [6] we have

$$\begin{aligned}\langle \nu, \mu f\rangle = \langle \nu * \mu, f\rangle &= \int_K \langle \nu * \delta_x, f\rangle d\mu(x) \\ &= \int_K \langle \nu, \delta_x f\rangle d\mu(x).\end{aligned}$$

It follows that $\int_K \delta_x f d\mu(x) = \mu f$.

It is easy to see that $\int_K f\delta_x d\mu(x) = f\mu$. On the other hand, by Remark 3.26 in [16], we have

$$\begin{aligned}\langle M, \mu f\rangle &= \langle M, \int_K \delta_x f d\mu(x)\rangle = \int_K \langle M, \delta_x f\rangle d\mu(x)\\ &= \int_K \langle M, f\delta_x\rangle d\mu(x) = \langle M, \int_K f\delta_x d\mu(x)\rangle\\ &= \langle M, f\mu\rangle.\end{aligned}$$

This completes the proof. □

Let A be a left Banach S-module (for more on left Banach S-modules, the reader is referred to [13] and [14]). For each $F \in A^{**}$, $f \in A^*$ and $x \in S$, we define

$$\langle f\cdot x, a\rangle = \langle f, x\cdot a\rangle, \text{ and } \langle x\cdot F, f\rangle = \langle F, f\cdot x\rangle$$

whenever $a \in A$. Also if $\mu \in M(S)$ and $f \in A^*$, we define

$$\langle f\cdot\mu, a\rangle = \int \langle f, x\cdot a\rangle d\mu(x) \text{ and } \langle \mu\cdot F, f\rangle = \langle F, f\cdot\mu\rangle$$

for all $a \in A$ and $F \in A^{**}$. For $\mu \in M_a(S)$, let $T_\mu \in \mathcal{B}(A^{**})$ be defined by $T_\mu(F) = \mu\cdot F$, $F \in A^{**}$. For $x \in S$, let $T_x \in \mathcal{B}(A^{**})$ be defined by $T_x(F) = x\cdot F$, $F \in A^{**}$. We also denote the closure of the set $\{T_\mu;\ \mu \in P(S)\}$ in the weak* operator topology by $\mathcal{P}_{A^{**}}$.

THEOREM (2.5). *Among the following seven properties, the implications*

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \text{ and } (v) \Rightarrow (vi) \Rightarrow (vii)$$

hold. If center $Z(P(S))$ of $P(S)$ is nonempty, where

$$Z(P(S)) = \{\mu \in P(S);\ \mu * \nu = \nu * \mu \text{ for all } \nu \in P(S)\},$$

then also (iv) ⇒ (v). If S has an identity, then also (vii) ⇒ (i), so that all seven properties are equivalent.

(i) *$M_a(S)^*$ has an inner invariant mean.*

(ii) *There exists a net (μ_α) in $P(S)$ such that for all $x \in S$,*

$$\delta_x * \mu_\alpha - \mu_\alpha * \delta_x \to 0$$

in the weak* topology.

(iii) *There exists a net (ν_α) in $P(S)$ such that for all $x \in S$,*

$$\delta_x * \nu_\alpha - \nu_\alpha * \delta_x \to 0$$

in the norm topology.

(iv) *For each $n \geq 1$, $x_1, \ldots, x_n \in S$ and $\epsilon > 0$, there exists a $\mu \in P(S)$ such that*

$$\|\delta_{x_i} * \mu - \mu * \delta_{x_i}\| < \epsilon$$

for all $i = 1, 2, \ldots, n$.

(v) *For any compact subset K of S and $\epsilon > 0$, there exists a $\nu \in P(S)$ such that*

$$\|\delta_x * \nu - \nu * \delta_x\| < \epsilon$$

whenever $x \in K$.

(vi) *There exists a net* (ν_α) *in* $P(S)$ *such that*

$$\|\delta_x * \nu_\alpha - \nu_\alpha * \delta_x\| \to 0$$

uniformly on compact subsets of S.

(vii) *For each left Banach* S*-module* A*, there exists* $T \in \mathcal{P}_{A^{**}}$ *such that* $TT_x = T_xT$ *for all* $x \in S$.

Proof. (i) $\Rightarrow$ (ii). Let M be an inner invariant mean on $M_a(S)^*$ and let $(\mu_\alpha)_{\alpha\in I}$ be a net in $P(S)$ such that $\mu_\alpha \to M$ in the weak* topology. It is easy to see that

$$\langle f, \delta_x * \mu_\alpha - \mu_\alpha * \delta_x\rangle \to 0$$

for every $f \in M_a(S)^*$ and $x \in S$.

(ii) $\Rightarrow$ (iii). Since the difference set $P(S) \setminus P(S)$ is a convex subset of $M_a(S)$, and the weak* topology on $M_a(S)$, as a subset of $M_a(S)^{**}$, is the weak topology, the weak* closure of $P(S) \setminus P(S)$ is the same as the norm closure. Thus, by an standard argument, we obtain a net $(\nu_\beta)_{\beta\in J}$ in $P(S)$ such that each ν_β is a convex combination of the elements of $(\mu_\alpha)_{\alpha\in I}$ and

$$\|\delta_x * \nu_\beta - \nu_\beta * \delta_x\| \to 0$$

for all $x \in S$.

(iii) $\Rightarrow$ (iv) Trivial.

(iv) $\Rightarrow$ (v). Let K be a compact subset of S and let $\epsilon > 0$. Consider a fixed element η in $Z(P(S))$. Since both mapping $x \mapsto |\eta| * \delta_x$ and $x \mapsto \delta_x * |\eta|$ from S into $M(S)$ are norm continuous, so for any $x \in K$, there exist a neighborhood U_x of x such that

$$\|\delta_x * \eta - \delta_y * \eta\| < \epsilon \text{ and } \|\eta * \delta_x - \eta * \delta_y\| < \epsilon,$$

whenever $y \in U_x$. We may determine a subset $\{x_1, ..., x_n\}$ in K such that $K \subseteq \bigcup_{i=1}^n U_{x_i}$, and for all $y \in U_{x_i}$,

$$\|\delta_y * \eta - \delta_{x_i} * \eta\| < \epsilon \text{ and } \|\eta * \delta_y - \eta * \delta_{x_i}\| < \epsilon.$$

Consider $\mu \in P(S)$ such that, for any $i = 1, ..., n$, $\|\delta_{x_i} * \mu - \mu * \delta_{x_i}\| < \epsilon$. Put $\nu = \eta * \mu \in P(S)$. For any $x \in K$, there exist $i \in \{1, ..., n\}$ such that $x \in U_{x_i}$. Then we have

$$\begin{aligned}\|\delta_x * \nu - \nu * \delta_x\| &= \|\delta_x * \eta * \mu - \eta * \mu * \delta_x\| \le \|\delta_x * \eta * \mu - \delta_{x_i} * \eta * \mu\| \\ &+ \|\delta_{x_i} * \eta * \mu - \eta * \mu * \delta_{x_i}\| + \|\eta * \mu * \delta_{x_i} - \eta * \mu * \delta_x\| \\ &\le \|\delta_x * \eta - \delta_{x_i} * \eta\| + \|\delta_{x_i} * \mu * \eta - \mu * \delta_{x_i} * \eta\| \\ &+ \|\mu * \eta * \delta_{x_i} - \mu * \eta * \delta_x\| \le \|\delta_x * \eta - \delta_{x_i} * \eta\| \\ &+ \|\delta_{x_i} * \mu - \mu * \delta_{x_i}\| + \|\eta * \delta_{x_i} - \eta * \delta_x\| < 3\epsilon.\end{aligned}$$

(v) $\Rightarrow$ (vi). By assumption, for each pair (K, ϵ), where $K \subseteq S$ is compact and $\epsilon > 0$, there is a $\nu_{(K,\epsilon)} \in P(S)$ such that

$$\|\delta_x * \nu_{(K,\epsilon)} - \nu_{(K,\epsilon)} * \delta_x\| < \epsilon$$

whenever $x \in K$. Then we define the partial ordering on the index set as $\alpha_1 = (K_1, \epsilon_1) \ge \alpha = (K, \epsilon)$, if $K \subseteq K_1$ and $\epsilon \ge \epsilon_1$. It is easy to see that

$$\|\delta_x * \nu_{(K,\epsilon)} - \nu_{(K,\epsilon)} * \delta_x\|$$

converges to 0 uniformly on compact subsets of S.

(vi) $\Rightarrow$ (vii). Let (ν_α) be a net in $P(S)$ such that $\|\delta_x * \nu_\alpha - \nu_\alpha * \delta_x\|$ converges to 0 uniformly on compact subsets of S. Hence we may find $T \in \mathcal{B}(A^{**})$ with $\|T\| \leq 1$ and a subnet (ν_β) of (ν_α) such that $T_{\nu_\beta} \to T$ in the weak* operator topology. For every $x \in S$ and $F \in A^{**}$, we have

$$\begin{aligned}\|T_x T_{\nu_\beta}(F) - T_{\nu_\beta} T_x(F)\| &= \|T_{\delta_x * \nu_\beta}(F) - T_{\nu_\beta * \delta_x}(F)\| \\ &\leq \|\delta_x * \nu_\beta - \nu_\beta * \delta_x\| \|F\| \to 0.\end{aligned}$$

Consequently $T_x T = T T_x$.

(vii) $\Rightarrow$ (i). Let $A = M_a(S)$ and consider $M_a(S)$ as a left S-module where $x.\mu = \delta_x * \mu$, $x \in S$, $\mu \in M_a(S)$. For $F \in M_a(S)^{**}$, let $T_F \in \mathcal{B}(M_a(S)^{**})$ be defined by $T_F(G) = FG$, $G \in M_a(S)^{**}$ (see [6]). As proved in [9],

$$\mathcal{P}_{M_a(S)^{**}} = \{T_F;\ F \in M_a(S)^{**},\ F \geq 0 \text{ and, } \|F\| = 1\}.$$

By assumption, there exists $T_M \in \mathcal{P}_{M_a(S)^{**}}$ such that $T_x T_M = T_M T_x$ for all $x \in S$. If E is the weak*-limit of a net (e_α) which is a bounded approximate identity for $M_a(S)$, then $Ef = f$ for all $f \in M_a(S)^*$. On the other hand, for every $x \in S$ and $f \in M_a(S)^*$ we have $T_x(E)f = \delta_x f$. Indeed,

$$\begin{aligned}\langle T_x(E)f, \mu\rangle &= \langle x \cdot E, f\mu\rangle = \langle E, f\mu * \delta_x\rangle = \langle Ef, \mu * \delta_x\rangle \\ &= \langle f, \mu * \delta_x\rangle = \langle \delta_x f, \mu\rangle.\end{aligned}$$

for any $\mu \in M_a(S)$, that is $T_x(E)f = \delta_x f$. We will show that M is an inner invariant mean on $M_a(S)^*$. If $f \in M_a(S)^*$ and $x \in S$, then

$$\begin{aligned}\langle M, f\delta_x\rangle &= \langle M, E(f\delta_x)\rangle = \langle ME, f\delta_x\rangle = \langle T_M(E), f\delta_x\rangle \\ &= \langle x \cdot T_M(E), f\rangle = \langle T_x T_M(E), f\rangle = \langle T_M T_x(E), f\rangle \\ &= \langle M, T_x(E)f\rangle = \langle M, \delta_x f\rangle.\end{aligned}$$

Consequently M is an inner invariant mean on $M_a(S)^*$. □

Acknowledgment

The author would like to thank the referee for giving useful comments and suggestions for the improvement of this paper.

Received October 24, 2005

Final version received December 8, 2006

DEPARTMENT OF MATHEMATICS
SEMNAN UNIVERSITY
SEMNAN
IRAN
ghaffari1380@yahoo.com

REFERENCES

[1] M. AMINI AND A. R. MEDGHALCHI, *Fourier algebras on topological foundation *-semigroup,* Semigroup Forum **68** (2004), 322–334.

[2] J. F. BERGLUND AND H. D. JUNGHENN AND P. MILNES, Analysis on semigroups. Function spaces, compactifications, represention, John Wiley & Sons, Inc. New York, 1989.

[3] F. F. BONSAL AND J. DUNCAN, Complete normed algebras, Springer-Verlag, New york, 1973.

[4] E. G. EFFORS, *Property and inner amenability,* Proc. Amer. Math. Soc. **47** (1975), 483–486.

[5] H. A. M. DZINOTYIWEYI, The analogue of the group algebra for topological semigroup, Pitman, Boston, London, 1984.
[6] A. GHAFFARI, *Convolution operators on semigroup algebras,* Southeast Asian Bull. Math. 27 (2003), 1–12.
[7] A. GHAFFARI, *Topologically left invariant mean on semigroup algebras,* Proc. Indian Acad. Sci. **115** (2005), 453–459.
[8] M. LASHKARIZADEH BAMI, *The topological centers of* $LUC(S)^*$ *and* $M_a(S)^*$ *of certain foundation semigroups,* Glasg. Math. J. **42** (2000), 335–343.
[9] A. T. LAU AND A. L. T. PATERSON, *Inner amenable locally compact groups,* Trans. Amer. Math. Soc. **325** (1991), 155–169.
[10] V. LOSERT AND H. RINDLER, *Conjugate invariant means,* Colloq. Math. **15** (1987), 221–225.
[11] R. MEMARBASHI AND A. RIAZI, *Topological inner invariant means,* Studia Sci. Math. Hungar. **40** (2003), 293–299.
[12] R. NASR-ISFAHANI, *Inner amenability of Lau algebras,* Arch. Math. (Brno). **37** (2001), 45–55.
[13] A. L. T. PATERSON, *Amenability,* Amer. Math. Soc. Math. Survey and Monographs 29, Providence, Rhode Island, 1988.
[14] J. P. PIER, Amenable locally compact groups, John Wiley And Sons, New York, 1984.
[15] A. RIAZI AND J. C. S. WONG, *Characterizations of amenable locally compact semigroups,* Pacific J. Math. **108** (1983), 479–496.
[16] W. RUDIN, Functional analysis, McGraw Hill, New York, 1991.
[17] J. C. WONG, *Fixed point theorems for measurable semigroups of operations,* Canad. J. Math. **44** (1992), 652–664.
[18] C. K. YUAN, *Conjugate convolutions and inner invariant means,* J. Math. Anal. Appl. **157** (1991), 166–178.
[19] C. K. YUAN, *The existence of inner invariant means on* $L^\infty(G)$, J. Math. Anal. Appl. **130** (1988), 514–524.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

HÖLDER ESTIMATES FOR THE $\overline{\partial}$-EQUATION ON SURFACES WITH SINGULARITIES OF THE TYPE $\mathbf{E_6}$ AND $\mathbf{E_7}$

F. ACOSTA AND E. S. ZERON

ABSTRACT. Let $\Sigma \subset \mathbb{C}^3$ be a 2-dimensional subvariety with an isolated simple (rational double point) singularity at the origin of the cyclic A_n, dihedral D_n, tetrahedral E_6 or octahedral E_7 type. The main objective of this paper is to solve the $\overline{\partial}$-equation in a neighbourhood of the origin in Σ, such that the solution has a Hölder condition.

1. Introduction

Let $\Sigma \subset \mathbb{C}^3$ be a subvariety with an isolated singularity at the origin, and λ be a $\overline{\partial}$-closed $(0,1)$-differential form defined on $\Sigma \setminus \{0\}$. An open problem in complex variables is to produce a general and effective technique for calculating a solution h to the $\overline{\partial}$-differential equation $\overline{\partial}h = \lambda$ in Σ, including the singular point. Gavosto, Fornæss and Ruppenthal have proposed a general technique for solving the equation $\overline{\partial}h = \lambda$ such that h satisfies an extra Hölder condition on an open neighbourhood of the singular point; see [3], [4] and [7]. Their basic idea was to analyse Σ as a branched covering over $\mathbb{C}^2$, to solve the corresponding $\overline{\partial}$-equation on $\mathbb{C}^2$, and to *lift* the solution from $\mathbb{C}^2$ into Σ again.

In a previous paper [1], we proposed an effective technique for solving the equation $\overline{\partial}g = \lambda$ on surfaces Σ with an isolated simple singularity of the regular cyclic A_{n-1} or dihedral D_{n+2} type, for $n \geq 2$, and such that h satisfies an extra Hölder condition on a neighbourhood of the singular point. The main objective of the present work is to extend the analysis done in [1], in order to solve the $\overline{\partial}$-equation on surfaces Σ with an isolated simple singularity of the exceptional tetrahedral E_6 or octahedral E_7 type. The central idea is to consider $\mathbb{C}^2$ as a branched covering over Σ, instead of analysing Σ as a branched covering over $\mathbb{C}^2$. Moreover, we also improve the Hölder estimates that we presented in [1] for the cyclic A_{2n} type.

The authors recommend the works of Dimca [2] and Slodowy [8, 9] for a deep analysis on isolated simple (rationally double point) singularities. In particular, all surfaces Σ with an isolated simple singularity may be locally characterised as the quotient space $\mathbb{C}^2/\mathcal{G}$ where $\mathcal{G}$ is a finite subgroup of the special linear group $SL_2(\mathbb{C})$; and so we have a natural quotient mapping (branched covering) π from $\mathbb{C}^2$ over the singular surface $\mathbb{C}^2/\mathcal{G}$. We present below all the non-trivial finite subgroups $\mathcal{G}$ of $SL_2(\mathbb{C})$, their cardinalities and the polynomial relations which define, up to biholomorphisms, the singular

2000 *Mathematics Subject Classification*: 32F20, 32W05, 35N15.
Keywords and phrases: Hölder estimates, $\overline{\partial}$-equation, branched covering.
Research supported by Cinvestav (Mexico) and Conacyt (Mexico).

quotient surface $\Sigma \cong \mathbb{C}^2/\mathcal{G}$ embedded in $\mathbb{C}^3$. For $n \geq 2$,

$$(1.1) \qquad \begin{array}{lll} \text{Cyclic}, & |\mathbb{Z}_n| = n, & x_1 x_2 = x_3^n; \\ \text{Dihedral}, & |D_{n+2}| = 4n, & x_2^2 = x_1^2 x_3 + x_3^{n+1}; \\ \text{Tetrahedral}, & |E_6| = 24, & x_1^3 + x_2^2 = x_3^4; \\ \text{Octahedral}, & |E_7| = 48, & x_1^3 x_3 + x_2^2 = x_3^3; \\ \text{Icosahedral}, & |E_8| = 120, & x_1^2 = x_2^3 + x_3^5. \end{array}$$

There is an abuse of notation in the previous table, because D_{n+2} denotes both the binary dihedral subgroup of $SL_2(\mathbb{C})$ with $4n$ elements and the dual resolution graph (or Dynkin diagram) of the singular surface $\mathbb{C}^2/\mathcal{G}$, for $\mathcal{G} \equiv D_{n+2}$. The symbol E_6 (respectively: E_7 and E_8) denotes as well the binary tetrahedral (respectively: octahedral and icosahedral) subgroup of $SL_2(\mathbb{C})$ and the corresponding dual resolution graph. We may now state the main result of this work,

THEOREM (1.2). *Let π be the quotient mapping from $\mathbb{C}^2$ over the singular surface $\Sigma \cong \mathbb{C}^2/\mathcal{G}$ embedded in $\mathbb{C}^3$, where $\mathcal{G} < SL_2(\mathbb{C})$ is the subgroup E_6, E_7, D_{n+2} or $\mathbb{Z}_n$, with $n \geq 2$. Fix $0 < \delta < 1/|\mathcal{G}|$, with the cardinality $|\mathcal{G}|$ presented in (1.1). Given an open ball $B_R \subset \mathbb{C}^2$ of radius $R > 0$ and centre in the origin, we may find a finite positive constant $C_1(R, \delta)$ such that: For every continuous $(0,1)$-differential form λ defined on the compact set $\pi(\overline{B_R}) \subset \Sigma$, and $\overline{\partial}$-closed on the regular part of $\pi(B_R)$, there exists a continuous function h on $\pi(B_R)$ which satisfies both the equation $\overline{\partial} h = \lambda$ on the regular part of $\pi(B_R)$ and the Hölder estimate:*

$$(1.3) \qquad \|h\|_{\pi(B_R)} + \sup_{x,w \in \pi(B_R)} \frac{|h(x) - h(w)|}{\|x - w\|^{\delta}} \leq C_1(R, \delta) \|\lambda\|_{\pi(B_R)}.$$

This theorem is proved in the second section of this paper. We have already presented a partial version of Theorem (1.2) for the cyclic and dihedral groups [1]. Notice that the regular part of $\pi(B_R)$ is obtained by removing the isolated singularity of Σ. A differential form is said to be continuous if all its coefficients are continuous functions, so the operator $\overline{\partial}$ is computed in terms of distributions. Moreover, the notation $\|h\|_E$ stands for the supremum of $|h|$ on the set E; and $\|x - w\|$ stands for the Euclidean distance between x and w; this distance is well defined because the singular surface Σ is embedded in $\mathbb{C}^3$. Thus, since $\|x - w\|$ is less than or equal to the *geodesic* distance between x and w measured *along* the surface Σ, we can assert that inequality (1.3) is indeed a Hölder estimate on Σ itself.

On the other hand, given a finite subgroup $\mathcal{G}$ of $SL_2(\mathbb{C})$, Felix Klein has proved that the algebra of holomorphic polynomials on $\mathbb{C}^2$ invariant under the natural action of $\mathcal{G}$ has three generators $x_k(z)$ which satisfy the respective polynomial relation given in (1.1), see Klein [6] and Slodowy [8, 9]. Whence, the quotient mapping π from $\mathbb{C}^2$ onto the singular surface $\mathbb{C}^2/\mathcal{G}$ is equal to the polynomial triplet (x_1, x_2, x_3). In particular, the automorphisms $z \mapsto Hz$ allow us to jump between the different branches of π, for $H \in \mathcal{G}$. That is, given $w = \pi(z)$, the inverse image $\pi^{-1}(w)$ is equal to $\{Hz : H \in \mathcal{G}\}$; and so $\pi^{-1}(w)$ has the same cardinality as $|\mathcal{G}|$ whenever $w \neq 0$. Finally, we need to recall that the norm $\|Hz\| = \|z\|$ is invariant under the action of each matrix $H \in \mathcal{G}$. We are going to prove this fact in the last four sections of this paper.

The proof of Theorem (1.2) requires an estimate of the distance $\|z-\zeta\|$ with respect to the projections $\|\pi(z) - \pi(\zeta)\|$.

THEOREM (1.4). *Let π be the polynomial quotient mapping from $\mathbb{C}^2$ over the singular surface $\Sigma \cong \mathbb{C}^2/\mathcal{G}$ embedded in $\mathbb{C}^3$, where $\mathcal{G} < SL_2(\mathbb{C})$ is the subgroup E_6, E_7, D_{n+2} or $\mathbb{Z}_n$, with $n \geq 2$. Define $\beta = 1/|\mathcal{G}|$, with the cardinality $|\mathcal{G}|$ presented in (1.1). Given an open ball $B_R \subset \mathbb{C}^2$ of radius $R > 0$ and centre at the origin, there exists a finite positive constant $C_2(R)$ such that: For each pair of points z and ζ in B_R with $\|z-\zeta\|$ less than or equal to $\|z - H\zeta\|$ for every matrix H in the group $\mathcal{G}$, the following inequality holds,*

$$\|\pi(z) - \pi(\zeta)\|^{2\beta} \geq 2C_2(R)\|z-\zeta\|\big(\|z\| + \|\zeta\|\big). \tag{1.5}$$

Notice that Σ is embedded in $\mathbb{C}^3$, so the term $\|\pi(z) - \pi(\zeta)\|$ is well defined. The last four sections of this paper are devoted to proving Theorem (1.4), considering consecutively the cyclic $\mathbb{Z}_n$, binary dihedral D_{n+2}, tetrahedral E_6 and octahedral E_7 groups.

As we have already stated in [1], the proof of Theorem (1.2) is based on two main steps: the explicit calculation of the polynomial quotient mapping π from $\mathbb{C}^2$ over the singular surface Σ; and the calculation of the estimate given in (1.5). In the case of the binary icosahedral subgroup $E_8 < SL_2(\mathbb{C})$, the polynomial quoting mapping π from $\mathbb{C}^2$ over $\mathbb{C}^2/E_8$ is given by the following equations:

$$\begin{aligned}
x_1(z) &= z_1^{30} + z_2^{30} + 522(z_1^{25}z_2^5 - z_1^5z_2^{25}) - 10005(z_1^{20}z_2^{10} + z_1^{10}z_2^{20}),\\
x_2(z) &= z_1^{20} - 228z_1^{15}z_2^5 + 494z_1^{10}z_2^{10} + 228z_1^5z_2^{15} + z_2^{20},\\
x_3(z) &= (1728)^{1/5}(z_1^{11}z_2 + 11z_1^6z_2^6 - z_1z_2^{11}).
\end{aligned}$$

It is easy to calculate that the polynomials $x_k(z)$ presented above satisfy the relation $x_1^2 = x_2^3 + x_3^5$, which defines up to biholomorphisms the surface $\mathbb{C}^2/E_8$ with an isolated simple singularity of the type E_8. We expect that the mapping π given by the triplet (x_1, x_2, x_3) satisfies the estimate (1.5) with $\beta = 1/120$.

The next section of this paper is devoted to the proof of Theorem (1.2); and finally, Theorem (1.4) is shown in the last four sections of this work.

2. Proof of Theorem (1.2)

This proof of Theorem (1.2) partially follows the ideas presented in [1]. Let π be the quotient mapping from $\mathbb{C}^2$ over the singular surface $\Sigma \cong \mathbb{C}^2/\mathcal{G}$ embedded in $\mathbb{C}^3$, where $\mathcal{G} < SL_2(\mathbb{C})$ is the subgroup E_6, E_7, D_{n+2} or $\mathbb{Z}_n$, with $n \geq 2$. We have that π is a polynomial mapping, because, as we have said in the introduction, π is equal to the triplet (x_1, x_2, x_3), with $x_k(z)$ the generators of the algebra of polynomials on $\mathbb{C}^2$ invariant under the natural action of $\mathcal{G}$. Recall Klein's work in [8, 9]. It easy to deduce that the origin in $\mathbb{C}^2$ is the inverse image $\pi^{-1}(0)$ of the isolated singularity $0 \in \Sigma$. Moreover, the mapping π is locally a biholomorphism from $\mathbb{C}^2 \setminus \{0\}$ onto the regular part of Σ. We need the following lemma on $\bar{\partial}$-closed differential forms.

LEMMA (2.1). *Let B any open ball in $\mathbb{C}^2$ with centre at the origin, and $\aleph$ be a continuous $(0,1)$-differential form defined on B and $\overline{\partial}$-closed inside $B\setminus\{0\}$. The form $\aleph$ is then $\overline{\partial}$-closed everywhere in B.*

Proof. The differential $\overline{\partial}\aleph$ is calculated in terms of distributions, so the given hypotheses imply that $\int_B \aleph\wedge\overline{\partial}\sigma$ vanishes for every smooth $(2,0)$-differential form σ with compact support in $B\setminus\{0\}$. And we must prove that the same integral vanishes when the differential form σ has compact support on B. Consider a real smooth function $\xi(z)=\widehat{\xi}(\|z\|^2)$ defined on $\mathbb{C}^2$ such that, for $k=1,2$:

$$0\le\xi(z)\le 1,\quad \xi(z)=\begin{cases}0 \text{ if } \|z\|\le 1,\\ 1 \text{ if } \|z\|\ge 2,\end{cases}\quad\text{and}\quad \left|\frac{\partial\xi(z)}{\partial\overline{z_k}}\right|\le 25.$$

Notice that $\int_B \aleph\wedge\overline{\partial}\big[\xi(rz)\rho\big]$ vanishes for all real numbers $r>0$ and smooth $(2,0)$-differential forms ρ with compact support on B, because $\overline{\partial}\aleph$ vanishes in $B\setminus\{0\}$. Differentiating by parts $\overline{\partial}\big[\xi(rz)\rho\big]$ yields that,

$$\left|\int_B \xi(rz)\,\aleph\wedge\overline{\partial}\rho\right| = \left|\int_B \aleph\wedge\rho\wedge\overline{\partial}\xi(rz)\right| \le 50\frac{8\pi^2}{r^3}\|\aleph\wedge\rho\|_B,$$

where $\overline{\partial}\xi(rz)$ vanishes for $\|z\|>2/r$ and the volume of the ball $\|z\|\le 2/r$ in $\mathbb{C}^2$ is equal to $8\pi^2/r^4$. Moreover, the form $\aleph\wedge\rho$ has finite norm because it is continuous and has compact support on B. On the other hand, we also have that,

$$\begin{aligned}\left|\int_B \aleph\wedge\overline{\partial}\rho\right| &\le \left|\int_B \big[1-\xi(rz)\big]\aleph\wedge\overline{\partial}\rho\right| + \left|\int_B \xi(rz)\,\aleph\wedge\overline{\partial}\rho\right| \\ &\le \frac{8\pi^2}{r^4}\|\aleph\wedge\overline{\partial}\rho\|_B + 50\frac{8\pi^2}{r^3}\|\aleph\wedge\rho\|_B<\infty.\end{aligned}$$

Finally, when $r>0$ converges to infinity, we obtain that $\int_B \aleph\wedge\overline{\partial}\rho$ vanishes for every $(2,0)$-differential form ρ with compact support on B, and so the form $\aleph$ is $\overline{\partial}$-closed everywhere in B. □

We need as well the following Henkin estimates, deduced from Theorems 2.1.5 and 2.2.2 of [5].

THEOREM (2.2). *Given an exponent $0<d<1$ and an open ball $B_R\subset\mathbb{C}^2$ of radius $R>0$ and centre in the origin, there exist two finite positive constants $C_3(R)$ and $C_4(R,d)$ such that: For every continuous $(0,1)$-differential form $\aleph$ defined on $\overline{B_R}$, and $\overline{\partial}$-closed on the interior B_R, the equation $\overline{\partial}g=\aleph$ has a continuous solution g on B_R which also satisfies the following Hölder estimates,*

$$\|g\|_{B_R}+\sup_{z,\zeta\in B_R}\frac{|g(z)-g(\zeta)|}{\|z-\zeta\|^{1/2}} \le C_3(R)\|\aleph\|_{B_R}, \tag{2.3}$$

$$\textit{and}\quad \sup_{z,\zeta\in B_{R/2}}\frac{|g(z)-g(\zeta)|}{\|z-\zeta\|^{d}} \le C_4(R,d)\|\aleph\|_{B_R}. \tag{2.4}$$

Proof. Theorem 2.2.2 of [5] automatically implies the existence of a continuous function g on B_R which satisfies both the equation $\overline{\partial}g=\aleph$ and the inequality (2.3). Further, analysing the proofs of Lemma 2.2.1 and Theorem 2.2.2, in

[5], we have that inequality (2.4) holds whenever there exists a finite positive constant $C_0(R)$ such that:

$$\sup_{z,\zeta\in B_{R/2}} \frac{|E(z)-E(\zeta)|}{\|z-\zeta\|} \leq C_0(R)\|\aleph\|_{B_R}, \tag{2.5}$$

for every function $E(z)$ defined according to equation (2.2.7) of [5, p. 70]. Let Υ be the closed interval which joins z and ζ inside the ball $B_{R/2}$. Then,

$$\begin{aligned} |E(z)-E(\zeta)| &\leq \int_0^1 \left|\frac{d}{dt}E(t\zeta+(1-t)z)\right| dt \\ &\leq \|z-\zeta\| \sup_{y\in\Upsilon} \sum_{k=1}^{2} \left|\frac{\partial E}{\partial y_k}\right| + \left|\frac{\partial E}{\partial \overline{y_k}}\right|. \end{aligned} \tag{2.6}$$

By equation (2.2.9) in [5], we know there exists a finite constant $C_0(R)$ such that all partial derivatives $|\frac{\partial E}{\partial y_k}|$ and $|\frac{\partial E}{\partial \overline{y_k}}|$ are less than or equal to $\frac{C_0(R)}{5}\|\aleph\|_{B_R}$, for every $y\in B_{R/2}$ and each index $k=1,2$. Notice that $D=B_R$ in equations (2.2.7) and (2.2.9), but y lies inside the smaller ball $B_{R/2}$. Thus, equation (2.6) automatically implies that inequalities (2.5) and (2.4) hold, as desired. □

We are now in position to prove Theorem (1.2), recall Theorem (1.4) and (2.2).

Proof of Theorem (1.2). Let π be the quotient mapping from $\mathbb{C}^2$ over the singular surface $\Sigma\cong\mathbb{C}^2/\mathcal{G}$ embedded in $\mathbb{C}^3$, where $\mathcal{G}<SL_2(\mathbb{C})$ is the subgroup E_6, E_7, D_{n+2} or $\mathbb{Z}_n$, with $n\geq 2$. Consider an open ball $B_R\subset\mathbb{C}^2$ of radius $R>0$ and centre in the origin. We have already seen in the introduction that π is a polynomial mapping, so the partial derivatives of π are all continuous and bounded mappings on the compact closure $\overline{B_R}$. Thus, there exists a finite positive constant $C_5(R)$ such that the following inequality holds for any continuous $(0,1)$-differential form λ defined on the compact set $\pi(\overline{B_R})$,

$$\|\pi^*\lambda\|_{B_R} \leq C_5(R)\|\lambda\|_{\pi(B_R)}. \tag{2.7}$$

On the other hand, suppose that λ is $\overline{\partial}$-closed on the regular part of $\pi(B_R)$. The pull-back $\pi^*\lambda$ is then a continuous $(0,1)$-differential form well defined on $\overline{B_R}$, and $\overline{\partial}$-closed in the open set $B_R\setminus\{0\}$, because π is locally a biholomorphism from $B_R\setminus\{0\}$ onto the regular part of $\pi(B_R)$. Whence, considering Lemma (2.1) and Theorem (2.2), we automatically have that the equation $\overline{\partial}g=\pi^*\lambda$ has a continuous solution g on B_R which satisfies the pair of Hölder estimates stated in (2.3) and (2.4) for $0<d<1$ fixed. Define $\beta=1/|\mathcal{G}|$, with the cardinality $|\mathcal{G}|$ given in (1.1). The finite sum $\beta\sum_{\mathcal{G}}H^*g$, added over all matrices $H\in\mathcal{G}$, is constant on the fibres of π (it is invariant under every *pull-back* H^*). Hence, there exists a continuous function h defined on $\pi(B_R)$ such that π^*h is equal to $\beta\sum_{\mathcal{G}}H^*g$. We assert that $\overline{\partial}h=\lambda$ on $\pi(B_R)\setminus\{0\}$. This result follows automatically because

$$\pi^*\overline{\partial}h=\beta\textstyle\sum_{\mathcal{G}}\overline{\partial}H^*g=\beta\sum_{\mathcal{G}}H^*\pi^*\lambda=\pi^*\lambda.$$

Recall that the projection $\pi(Hz)=\pi(z)$ and the norm $\|Hz\|=\|z\|$ are both invariant under the action of every matrix H in the finite group $\mathcal{G}$. Moreover,

$$\|h\|_{\pi(B_R)} = \|\beta\textstyle\sum_{\mathcal{G}}H^*g\|_{B_R} \leq \|g\|_{B_R}. \tag{2.8}$$

Now, given $x, w \in \pi(B_R)$, choose the points $z, \zeta \in B_R$ such that $x = \pi(z)$ and $w = \pi(\zeta)$. Since $\pi(\zeta)$ is equal to $\pi(H\zeta)$, we may even choose $\zeta \in B_R$ so that the norm $\|z - \zeta\|$ is less than or equal to $\|z - H\zeta\|$ for each matrix $H \in \mathcal{G}$. A direct application of equation (1.5) in Theorem (1.4) yields

$$\frac{\|x-w\|^{2\beta}}{2C_2(R)} \;\geq\; \|z-\zeta\|\big(\|z\| + \|\zeta\|\big) \;\geq\; \|z-\zeta\|^2. \tag{2.9}$$

Suppose that the points z and ζ are both inside the ball $B_{R/2}$. We may apply equation (2.4) of Theorem (2.2), with $\aleph = \pi^*\lambda$, in order to deduce the following inequality for $0 < d < 1$ fixed,

$$\frac{|h(x)-h(w)|}{\|x-w\|^{d\beta}} \;\leq\; \frac{\beta\sum_{\mathcal{G}}|g(Hz)-g(H\zeta)|}{[2C_2(R)]^{d/2}\|z-\zeta\|^d} \;\leq\; \frac{C_4(R,d)\|\pi^*\lambda\|_{B_R}}{[2C_2(R)]^{d/2}}. \tag{2.10}$$

Notice that the norm $\|z - \zeta\|$ is equal to $\|Hz - H\zeta\|$ for each matrix $H \in \mathcal{G}$. Suppose now, without lost of generality, that $\|z\| \geq R/2$. Inequality (2.9) then implies that $\|x - w\|^{2\beta}$ is greater than or equal to $RC_2(R)\|z - \zeta\|$. Therefore, equation (2.3) of Theorem (2.2) automatically yields the following,

$$\frac{|h(x)-h(w)|}{\|x-w\|^{\beta}} \leq \frac{\beta\sum_{\mathcal{G}}|g(Hz)-g(H\zeta)|}{(RC_2(R)\|z-\zeta\|)^{1/2}} \leq \frac{C_3(R)\|\pi^*\lambda\|_{B_R}}{[RC_2(R)]^{1/2}}. \tag{2.11}$$

Finally, considering Theorem (2.2) and equations (2.7) to (2.11), we can deduce the existence of a bounded positive constant $C_1(R, \delta)$ such that equation (1.3) in Theorem (1.2) holds, with $\delta = d\beta$ and $0 < d < 1$ fixed. □

We close this section with some observations about Theorem (1.2). First, the procedure presented in this section yields a continuous solution h to the equation $\bar\partial h = \lambda$. Moreover, we are directly using the estimates given in [5], but we may use any integration kernel which produces estimates similar to those presented in equations (2.3) and (2.4) of Theorem (2.2).

3. Proof of theorem (1.4) for the cyclic group

The estimate (1.5) of Theorem (1.4) is one of the main pillars in the proof of Theorem (1.2), as we have already seen in previous section. Nevertheless, Theorem (1.4) is quite important on its own. Since the quotient mapping π from $\mathbb{C}^2$ over $\Sigma \cong \mathbb{C}^2/\mathcal{G}$ is smooth (polynomial), there exists a finite positive constant $C_0(R)$ such that $C_0(R)\|z-\zeta\|$ is greater than or equal to $\|\pi(z)-\pi(\zeta)\|$ for all points z and ζ in the open ball $B_R \subset \mathbb{C}^2$ of radius $R > 0$ and centre in the origin. Thus, Theorem (1.4) yields the opposite inequalities with an appropriate exponent.

On the other hand, a weaker version of Theorem (1.4) has already been proved in [1] for the cyclic subgroup $\mathbb{Z}_n$ of $SL_2(\mathbb{C})$ with n elements. The inequality (1.5) has been proved with the exponent $0 < \delta' < 1/Ev(n)$, where $n = |\mathbb{Z}_n|$ and $Ev(n)$ is the smallest even integer greater than or equal to n. The central part of this section is to improve these inequalities for a new exponent $0 < \delta < 1/n$.

Notice that the cyclic subgroup $\mathbb{Z}_n$ of $SL_2(\mathbb{C})$ with $n \geq 2$ elements is generated by the following matrix,

$$H_1 = \begin{pmatrix} \rho_n, & 0 \\ 0\,, & \overline{\rho_n} \end{pmatrix}, \quad \text{with} \quad \rho_n = \mathrm{e}^{2i\pi/n}. \tag{3.1}$$

We can easily verify that the norm $\|Hz\| = \|z\|$ is preserved for every $z \in \mathbb{C}^2$ and each matrix H in $\mathbb{Z}^n$. Moreover, the polynomial quotient mapping π from $\mathbb{C}^2$ over the singular surface $\Sigma_{\mathbb{Z}} \cong \mathbb{C}^2/\mathbb{Z}_n$ embedded in $\mathbb{C}^3$ is given by

$$\pi(s,t) = (s^n, t^n, st), \quad \text{for} \quad \Sigma_{\mathbb{Z}} \cong \{x_1x_2 = x_3^n\}. \tag{3.2}$$

The mapping π is a natural branched n-covering from $\mathbb{C}^2$ over $\Sigma_{\mathbb{Z}}$, and it is trivially invariant under the natural action of the cyclic group $\mathbb{Z}_n$. The following lemma is the central part in the calculations for the new exponent $\beta = 1/n$.

LEMMA (3.3). *Let $n \geq 2$ be fixed. There exists a finite constant $C_6 > 0$ such that: Given two points $z = (a,b)$ and $\zeta = (s,t)$ in $\mathbb{C}^2$ with $\|z - \zeta\|$ less than or equal to $\|z - H_1^k\zeta\|$ for every natural number k, the following inequality holds*

$$\frac{\max\{|a^n - s^n|, |b^n - t^n|, |ab - st|^{n/2}\}}{\|z-\zeta\|^{n/2}(\|z\| + \|\zeta\|)^{n/2}} \geq C_6, \tag{3.4}$$

Proof. Let $z = (a,b)$ and $\zeta = (s,t)$ be a pair of points in $\mathbb{C}^2$. Notice that the left term of equation (3.4) does not change if we multiply both z and ζ by any complex number $\lambda \neq 0$. Therefore, we only need to prove inequality (3.4) on the compact set $\|z\| + \|\zeta\| = 13$; and we may suppose without loss of generality that $|a-s|$ is greater than or equal to $|b-t|$. We consider three principal cases:

Case I. Suppose that $1/n \geq |a-s| \geq |b-t|$ and $|b| \geq 2|s|$. We have that

$$|ab - st| \geq |b| \cdot |a-s| - |s| \cdot |b-t| \geq |a-s| \cdot |b|/2. \tag{3.5}$$

Notice that $|a|$ and $|t|$ are less than or equal to $|s| + 1/n$ and $|b| + 1/n$, respectively. Whence,

$$13 = \|z\| + \|\zeta\| \leq 2|b| + 2|s| + 2/n \leq 3|b| + 2/n,$$

and so $|b| \geq 4$. Finally, the norm $\|z - \zeta\|$ is less than or equal to $|a-s|\sqrt{2}$, because we have set $z = (a,b)$ and $\zeta = (s,t)$, and the absolute value $|b-t|$ is less than or equal to $|a-s|$. Thus, equation (3.5) implies that inequality (3.4) holds, for

$$|ab - st|^{n/2} \geq \left(\|z - \zeta\|\sqrt{2}\right)^{n/2}.$$

Case II. Suppose that $1/n \geq |a-s| \geq |b-t|$ and $|b| \leq 2|s|$. Recall that $|a|$ and $|t|$ are less than or equal to $|s| + 1/n$ and $|b| + 1/n$, respectively. Thus

$$13 = \|z\| + \|\zeta\| \leq 2|b| + 2|s| + 2/n \leq 6|s| + 2/n,$$

and so $|s| \geq 2$. Consider the n-root of unity $\rho_n = e^{2\pi i/n}$ and any natural number $1 \leq k < n$. We can easily deduce that

$$|s - \rho_n^k s| \geq 2|s|\sin(k\pi/n) \geq 8/n \geq 8|a-s|.$$

Hence, the absolute value $|a - \rho_n^k s|$ is greater than or equal to $|a-s|$ for every exponent k. It is easy to verify that the set Λ of all natural numbers $1 \leq j \leq n$ such that $|s - \rho_n^j s|$ is greater than or equal to $|s|$ is composed of at least $n/2$ elements. Thus, recalling that $1/n \geq |a-s|$,

$$|a - \rho_n^j s| \geq |s| - 1/n \geq 3/2 \quad \text{for all} \quad j \in \Lambda.$$

Finally, since $3/2 > \sqrt{2}$, the cardinality satisfies $|\Lambda| \geq n/2$ and the norm $\|z-\zeta\|$ is less than or equal to $|a-s|\sqrt{2}$, we automatically have that the inequality (3.4) holds, because

$$|a^n - s^n|^2 = \prod_{k=1}^{n} \left|a - \rho_n^k s\right|^2 \geq (3/2)^n |a-s|^n > \|z-\zeta\|^n.$$

Case III. Suppose that $|a-s| \geq 1/n$ **or** $|b-t| \geq 1/n$, where $z = (a, b)$ and $\zeta = (s, t)$. We have that $\|z-\zeta\|$ is greater than or equal to $1/n$ as well. Define the compact set $K \subset \mathbb{C}^4$ composed of the pairs (z, ζ) which satisfy the three conditions: $\|z\| + \|\zeta\| = 13$, the norm $\|z - H_1^k \zeta\|$ is greater than or equal to $\|z-\zeta\|$ for every k, and $\|z-\zeta\| \geq 1/n$, where the matrix H_1 is defined in (3.1).

It is easy to verify that the left term of (3.4) vanishes if and only if $s = \rho_n^k a$ and $t = \overline{\rho_n}^k b$ for some natural number k; that is, if and only if $\zeta = H_1^k z$. Thus, the left term of (3.4) is a continuous and non-vanishing function well defined for every pair (z, ζ) in the compact set $K \subset \mathbb{C}^4$ described in the paragraph above. Therefore, this function is bounded from below by a finite positive constant $C_6 > 0$. In other words, inequality (3.4) holds in this case. □

We may now present the proof of Theorem (1.4) for the particular case of the cyclic subgroup $\mathbb{Z}_n$ of $SL_2(\mathbb{C})$ with n elements.

Proof of Theorem (1.4). for the cyclic group $\mathbb{Z}_n$. As we have stated at the beginning of this section, the singular surface $\Sigma_{\mathbb{Z}} \cong \mathbb{C}^2/\mathbb{Z}_n$ embedded in $\mathbb{C}^3$ is defined by the polynomial relation $x_1 x_2 = x_3^n$. Further the polynomial quotient mapping π from $\mathbb{C}^2$ over Σ is defined by $\pi(z)$ equal to $(z_1^n, z_2^n, z_1 z_2)$. Given any pair of points $z = (a, b)$ and $\zeta = (s, t)$ in $\mathbb{C}^2$, we have that $|a^n - s^n|$ and $|b^n - t^n|$ are both less than or equal to $\|\pi(z) - \pi(\zeta)\|$. Moreover, if z and ζ lie inside the ball B_R of radius $R > 0$ and centre in the origin, we also have that

$$|ab - st|^{n/2} \;\leq\; R^{n-2}|ab - st| \;\leq\; R^{n-2}\|\pi(z) - \pi(\zeta)\|.$$

Recall that $2|ab| \leq |a|^2 + |b|^2 < R^2$ and $n \geq 2$. Finally, if $\|z-\zeta\|$ is less than or equal to $\|z - H\zeta\|$ for every matrix $H \in \mathbb{Z}_n$, a direct application of Lemma (3.3) yields the following version of equation (1.5) for the exponent $\beta = 1/n$, with n the cardinality of the group $\mathbb{Z}_n$,

$$\frac{\|\pi(z) - \pi(\zeta)\|}{\|z-\zeta\|^{n/2}(\|z\| + \|\zeta\|)^{n/2}} \geq C_6 \min\{1, R^{2-n}\}. \tag{3.6}$$

□

4. Proof of theorem (1.4) for the dihedral group

Let D_{d+2} be the binary dihedral subgroup of $SL_2(\mathbb{C})$ with $4d$ elements, for $d \geq 2$, which is generated by the cyclic group $\mathbb{Z}_{2d}$ and the following matrix [8, p. 73],

$$H_2 = \begin{pmatrix} 0\,, 1 \\ -1, 0 \end{pmatrix}. \tag{4.1}$$

We have already seen in the previous section that the norm $\|Hz\| = \|z\|$ is preserved for every $z \in \mathbb{C}^2$ and each matrix H in $\mathbb{Z}_{2d}$; and so it is trivial to deduce that the norm is also preserved for every matrix H in the group D_{d+2}.

The quotient mapping $\dot{\pi}$ from $\mathbb{C}^2$ over the singular surface $\Sigma_D \cong \mathbb{C}^2/D_{d+2}$ is given by the composition $\eta_2 \circ \eta_1$, with

$$\begin{aligned} \eta_1(s,t) &= \left(\tfrac{s^{2d}+t^{2d}}{2i}, \tfrac{s^{2d}-t^{2d}}{2i}, st\right) \quad \text{and} \\ \eta_2(x_1,x_2,x_3) &= \left(x_1, x_3x_2, x_3^2\right). \end{aligned} \tag{4.2}$$

It is easy to see that η_1 is a quotient mapping from $\mathbb{C}^2$ onto the singular surface defined by $x_2^2 - x_1^2 = x_3^{2d}$ in $\mathbb{C}^3$. By fixing $\widehat{x}_2 = x_2x_3$ and $\widehat{x}_3 = x_3^2$, we can easily deduce that the mapping

$$\dot{\pi}(s,t) = \eta_2 \circ \eta_1(s,t) = \left(\tfrac{s^{2d}+t^{2d}}{2i}, st\,\tfrac{s^{2d}-t^{2d}}{2i}, s^2t^2\right) \tag{4.3}$$

is a natural branched $4d$-covering from $\mathbb{C}^2$ over

$$\Sigma_D \cong \{\widehat{x}_2^2 - x_1^2\widehat{x}_3 = \widehat{x}_3^{d+1}\} \quad \text{in} \quad \mathbb{C}^3.$$

We have already proved in [1] that Theorem (1.4) holds for the mapping $\dot{\pi}$, the finite group $\mathcal{G} \equiv D_{d+2}$ and the exponent $\beta = \frac{1}{4d}$. Nevertheless, we include the proof for the sake of completeness. We shall use this proof as a model for showing Theorem (1.4) for the binary tetrahedral and octahedral groups.

Proof of Theorem (1.4) for the binary dihedral group D_{d+2}. We begin by analysing the mapping η_1 given in (4.2). It is easy to deduce the existence of an invertible $[3 \times 3]$ matrix Q such that $Q\eta_1$ is equal to the mapping π defined in (3.2), with $n = 2d$. Moreover, the norm $\|Qx\|$ is less than or equal to $2\|x\|$ for every $x \in \mathbb{C}^3$. Given the open ball $B_R \subset \mathbb{C}^2$ of radius $R > 0$ and centre in the origin, let z and ζ be two points in B_R such that $\|z - \zeta\|$ is less than or equal to $\|z - H\zeta\|$ for every matrix H in the group D_{d+2} with $4d$ elements, with $d \geq 2$.

We have that the binary dihedral group D_{d+2} is generated by $\mathbb{Z}_{2d}$ and the matrix H_2 in (4.1), so we fix $\xi = H_2^k\zeta$ with the exponent $k = 0, 1$. Let J_0 be the matrix in $\mathbb{Z}_{2d}$ such that $\|z - J_0\xi\|$ is less than or equal to $\|z - J\xi\|$ for every J in $\mathbb{Z}_{2d}$. Notice that $\|J_0\xi\| = \|\zeta\|$, the mapping η_1 is invariant under the natural action of J_0 and $\|z - \zeta\|$ is less than or equal to $\|z - J_0\xi\|$ because of the given hypotheses. Moreover, recall that $\pi = Q\eta_1$ and $2\|x\| \geq \|Qx\|$. A direct application of equation (3.6) with $n = 2d$ yields the following inequality, where $C_7(R)$ is some finite positive constant independent of the arbitrary exponent $k = 0, 1$,

$$\begin{aligned} \|\eta_1(z) - \eta_1(H_2^k\zeta)\|^2 &\geq C_7(R)\|z - J_0\xi\|^{2d}(\|z\| + \|J_0\xi\|)^{2d} \\ &\geq C_7(R)\|z - \zeta\|^{2d}(\|z\| + \|\zeta\|)^{2d}, \end{aligned} \tag{4.4}$$

We only need to analyse the mapping η_2 given in (4.2). Let w and x be a pair of points in $\eta_1(B_R) \subset \mathbb{C}^3$, so that

$$\|\eta_2(w) - \eta_2(x)\|^2 = |w_1 - x_1|^2 + |w_2w_3 - x_2x_3|^2 + |w_3^2 - x_3^2|^2. \tag{4.5}$$

We have by the definition of η_1 that $|x_3| \leq R^2$, $|x_1| \leq R^{2d}$ and $x_1^2 + x_3^{2d}$ is equal to x_2^2; similar relations are satisfied by w. Hence, the following inequality holds,

$$\begin{aligned} |w_2^2 - x_2^2| &\leq |w_1^2 - x_1^2| + |w_3^{2d} - x_3^{2d}| \\ &\leq 2R^{2d}|w_1 - x_1| + dR^{2d-2}|w_3^2 - x_3^2| \\ &\leq R^{2d}\left(2 + d/R^2\right)\|\eta_2(w) - \eta_2(x)\|. \end{aligned}$$

Let C_6 be the finite positive constant calculated in Lemma (3.3). Since $2R^{2d}$ is greater than or equal to $|w_1 - x_1|$, we can deduce the existence of a finite positive constant $C_8(R)$ such that

$$\frac{\|\eta_2(w) - \eta_2(x)\|}{2C_8(R)} \geq \max \left\{ \begin{matrix} C_6|w_1 - x_1|^2, |w_2^2 - x_2^2|, \\ |w_3^2 - x_3^2|, |w_2w_3 - x_2x_3| \end{matrix} \right\}. \tag{4.6}$$

The right term in the previous inequality can be analysed using equation (3.4) of Lemma (3.3). We just need to set $n = 2$, to recall that $\|z\| + \|\zeta\|$ is greater than or equal to $\|z - \zeta\|$ and to define the points $\widehat{z}{=}(w_2, w_3)$ and $\widehat{\zeta}{=}(x_2, x_3)$. Thus, a direct application of Lemma (3.3) into equation (4.6) yields that the following inequalities hold whenever $\|w - x\|$ is less than or equal to $\|w - \phi(x)\|$, for the mapping $\phi(x)$ defined by $(x_1, -x_2, -x_3)$,

$$\frac{\|\eta_2(w) - \eta_2(x)\|}{C_8(R)C_6} \geq 2 \max \{|w_1 - x_1|^2, \|\widehat{z} - \widehat{\zeta}\|^2\} \geq \|w - x\|^2. \tag{4.7}$$

On the other hand, it is easy to verify that $\eta_1(H_2\zeta)$ is equal to $\phi(\eta_1(\zeta))$ for the matrix H_2 in (4.1). Thus, given z and ζ in B_R such that $\|z - \zeta\|$ is less than or equal to $\|z - H\zeta\|$ for each H in D_{d+2}, we fix the point $w = \eta_1(z)$. Now, if the distance $\|w - \eta_1(\zeta)\|$ is less than or equal to $\|w - \eta_1(H_2\zeta)\|$, we may set $x = \eta_1(\zeta)$ and $k = 0$ into equations (4.4) and (4.7), in order to deduce the following version of equation (1.5) for the exponent $\beta = \frac{1}{4d}$ and the quotient mapping $\dot{\pi} = \eta_2 \circ \eta_1$ defined in (4.3),

$$\frac{\|\dot{\pi}(z) - \dot{\pi}(\zeta)\|}{\|z - \zeta\|^{2d}(\|z\| + \|\zeta\|)^{2d}} \geq C_8(R)C_7(R)C_6. \tag{4.8}$$

Finally, if $\|w - \eta_1(H_2\zeta)\|$ is less than or equal to $\|w - \eta_1(\zeta)\|$, we may set the point $x = \eta_1(H_2\zeta)$ and the exponent $k = 1$ in equations (4.4) and (4.7), in order to deduce that equation (4.8) holds as well. □

5. Proof of theorem (1.4) for the tetrahedral group

Let E_6 be the binary tetrahedral subgroup of $SL_2(\mathbb{C})$ with 24 elements. This group is generated by the following three matrices [8, p. 74]:

$$\begin{pmatrix} i, & 0 \\ 0, & -i \end{pmatrix}, \quad \begin{pmatrix} 0\,, & 1 \\ -1, & 0 \end{pmatrix} \quad \text{and} \quad H_3 = \frac{1}{1+i}\begin{pmatrix} 1\,, & 1 \\ -i, & i \end{pmatrix}. \tag{5.1}$$

The first two matrices generate the binary dihedral group D_4 with 8 elements. Further, the cube H_3^3 is equal to minus the identity matrix. We have already seen, in the previous section, that the norm $\|Hz\| = \|z\|$ is preserved for every $z \in \mathbb{C}^2$ and each matrix H in the group D_4. Thus, we only need to verify that $\|H_3z\| = \|z\|$, in order to deduce that the norm is preserved as well for every matrix in the binary tetrahedral group E_6. We can directly prove that $\|H_3z\|$ is equal to $\|z\|$ by choosing the point $z = (s, t)$ in $\mathbb{C}^2$ and calculating:

$$\|H_3z\|^2 = \frac{(s + t)\overline{(s + t)} + (t - s)\overline{(t - s)}}{2} = |s|^2 + |t|^2.$$

On the other hand, the polynomial quotient mapping $\ddot{\pi}$ from $\mathbb{C}^2$ over the singular surface $\Sigma_6 \cong \mathbb{C}^2/E_6$ is given by the composition $\eta_4 \circ \eta_3$, with $\theta = 2\sqrt{3}$,

$$(5.2)\qquad \begin{aligned} &\eta_3(s,t) \;=\; \Big(s^2t^2 + \tfrac{s^4+t^4}{i\theta},\, s^2t^2 - \tfrac{s^4+t^4}{i\theta},\, st\,\tfrac{s^4-t^4}{2i}\Big) \\ &\text{and} \quad \eta_4(x_1,x_2,x_3) \;=\; \big(x_2x_1, [x_1^3 - x_2^3]/2, x_3\big)\,. \end{aligned}$$

It is easy to see that η_3 is a quotient mapping from $\mathbb{C}^2$ onto the singular surface defined by $\frac{x_1^3+x_2^3}{2} = x_3^2$ inside $\mathbb{C}^3$. By fixing $\widehat{x}_1 = x_1x_2$ and $\widehat{x}_2 = \frac{x_1^3-x_2^3}{2}$, we can easily deduce that $\ddot{\pi} = \eta_4 \circ \eta_3$ is a natural branched 24-covering from $\mathbb{C}^2$ over

$$(5.3)\qquad \Sigma_6 \;\cong\; \big\{\widehat{x}_1^3 + \widehat{x}_2^2 = x_3^4\big\} \quad \text{in} \quad \mathbb{C}^3.$$

Finally, we may verify that $\eta_3(H\zeta)$ is equal to $\eta_3(\zeta)$ for every ζ in $\mathbb{C}^2$ and each matrix H in the group D_4, for we only need to verify that η_3 is invariant under the natural action of the first two matrices presented in (5.1). Moreover, considering the matrix H_3 in (5.1), we may calculate that $\eta_3(H_3\zeta)$ is equal to $\varphi(\eta_3(\zeta))$ as well, where $\varphi(x)$ is defined by $(\tau x_1, \overline{\tau} x_2, x_3)$ in $\mathbb{C}^3$ and $\tau = \frac{i\sqrt{3}-1}{2}$ is the cubic root of the unity. The proof of Theorem (1.4) for the binary tetrahedral group E_7 and the exponent $\beta = \frac{1}{24}$ follows the same structure than the proof for D_{d+2}.

Proof of Theorem (1.4) for the binary tetrahedral group E_6. We begin by analysing the mapping η_3 given in (5.2). It is easy to deduce the existence of an invertible $[3 \times 3]$ matrix $\widehat{Q}$ such that $\widehat{Q}\eta_3$ is equal to the mapping $\dot{\pi}$ defined in (4.3), with $d = 2$. Moreover, the norm $\|\widehat{Q}x\|$ is less than or equal to $\sqrt{3}\|x\|$ for every $x \in \mathbb{C}^3$. Given the open ball $B_R \subset \mathbb{C}^2$ of radius $R > 0$ and centre at the origin, let z and ζ be a pair of points in B_R such that $\|z - \zeta\|$ is less than or equal to $\|z - H\zeta\|$ for every matrix H in the group E_6 with 24 elements.

We have that the binary tetrahedral group E_6 is generated by D_4 and the matrix H_3 in (5.1), so we fix $\xi = H_3^k\zeta$ for a given exponent k. Let J_0 be a matrix in D_4 such that $\|z - J_0\xi\|$ is less than or equal to $\|z - J\xi\|$ for every J in D_4. Notice that $\|J_0\xi\| = \|\zeta\|$, the mapping η_3 is invariant under the natural action of J_0 and $\|z-\zeta\|$ is less than or equal to $\|z - J_0\xi\|$ because of the given hypotheses. Moreover, recall that $\dot{\pi} = \widehat{Q}\eta_3$ and $\sqrt{3}\|x\| \geq \|\widehat{Q}x\|$. A direct application of equation (4.8) with $d = 2$ yields the following inequality, where $C_9(R)$ is some finite positive constant independent of the arbitrary exponent k,

$$(5.4)\qquad \begin{aligned} \|\eta_3(z) - \eta_3(H_3^k\zeta)\|^3 &\geq C_9(R)\|z - J_0\xi\|^{12}(\|z\| + \|J_0\xi\|)^{12} \\ &\geq C_9(R)\|z - \zeta\|^{12}(\|z\| + \|\zeta\|)^{12}. \end{aligned}$$

We only need to analyse the mapping η_4 given in (5.2). Let w and x be a pair of points in $\eta_3(B_R) \subset \mathbb{C}^3$, so that

$$\|\eta_4(w) - \eta_4(x)\|^2 = |w_1w_2 - x_1x_2|^2 + \big|\tfrac{w_1^2 - w_2^3 - x_1^3 + x_2^3}{2}\big|^2 + |w_3 - x_3|^2.$$

We have by the definition of η_3 that $|x_3| \leq R^6$, $|x_1x_2| \leq 2R^8$ and $\frac{x_1^3+x_2^3}{2}$ is equal to x_3^2; similar relations are satisfied by w. Hence, the following inequality holds,

$$\left\|\frac{w_1^3-x_1^3}{2},\frac{w_2^3-x_2^3}{2}\right\| \leq \left\|\frac{w_1^3+w_2^3-x_1^3-x_2^3}{2},\frac{w_1^3-w_2^3-x_1^3+x_2^3}{2}\right\|$$
$$\leq |w_3^2-x_3^2|+\|\eta_4(w)-\eta_4(x)\|$$
$$\leq (2R^6+1)\|\eta_4(w)-\eta_4(x)\|.$$

Recall that $|w_3 \pm x_3|$ and $|w_1w_2-x_1x_2|^{1/2}$ are less than or equal to $2R^6$ and $2R^4$, respectively. Let C_6 be the finite positive constant calculated in Lemma (3.3), we can deduce the existence of a finite positive constant $C_{10}(R)$ such that,

$$\frac{\|\eta_4(w)-\eta_4(x)\|}{2^{3/2}C_{11}(R)} \geq \max\left\{\begin{matrix} C_6|w_3-x_3|^3, \ |w_1^3-x_1^3|, \\ |w_2^3-x_2^3|, |w_1w_2-x_1x_2|^{3/2}\end{matrix}\right\}. \tag{5.5}$$

The right term in previous inequality can be analysed using equation (3.4) of Lemma (3.3). We just need to set $n=3$, to recall that $\|z\|+\|\zeta\|$ is greater than or equal to $\|z-\zeta\|$ and to define the points $\widehat{z}{=}(w_1,w_2)$ and $\widehat{\zeta}=(x_1,x_2)$. Thus, a direct application of Lemma (3.3) into equation (5.5) yields that the following inequalities hold whenever $\|w-x\|$ is less than or equal to $\|w-\varphi_j(x)\|$, for every mapping $\varphi_j(x)$ defined by $(\tau^j x_1, \bar{\tau}^j x_2, x_3)$, where $\tau = \frac{i\sqrt{3}-1}{2}$ is the cubic root of the unity and j is any natural number,

$$\frac{\|\eta_4(w)-\eta_4(x)\|}{C_{11}(R)C_6} \geq 2^{3/2}\max\{\|\widehat{z}-\widehat{\zeta}\|^3, |w_3-x_3|^3\} \geq \|w-x\|^3. \tag{5.6}$$

On the other hand, we have calculated that $\eta_3(H_3^j\zeta)$ is equal to $\varphi_j(\eta_3(\zeta))$ for the matrix H_3 in (5.1) and every exponent j. Thus, given z and ζ in B_R such that $\|z-\zeta\|$ is less than or equal to $\|z-H\zeta\|$ for each H in E_6, we fix the point $w=\eta_3(z)$ and choose an exponent k such that $\|w-\eta_3(H_3^k\zeta)\|$ is less than or equal to $\|w-\eta_3(H_3^j\zeta)\|$ for every j. We may set the point $x=\eta_3(H_3^k\zeta)$ into equations (5.4) and (5.6), in order to deduce the following version of equation (1.5) for the exponent $\beta=\frac{1}{24}$ and the quotient mapping $\ddot{\pi}=\eta_4\circ\eta_3$ defined in (5.2),

$$\frac{\|\ddot{\pi}(z)-\ddot{\pi}(\zeta)\|}{\|z-\zeta\|^{12}(\|z\|+\|\zeta\|)^{12}} \geq C_{11}(R)C_9(R)C_6. \tag{5.7}$$

□

6. Proof of theorem (1.4) for the octahedral group

Let E_7 be the binary octahedral subgroup of $SL_2(\mathbb{C})$ with 48 elements, which is generated by the binary tetrahedral group E_6 and the following matrix [8, p. 74],

$$H_4 = \begin{pmatrix}\rho_8, & 0 \\ 0\,, & \overline{\rho_8}\end{pmatrix}, \quad \text{with} \quad \rho_8=(1+i)/\sqrt{2}. \tag{6.1}$$

Recall that ρ_8 is the eighth-root of the unity. We have already seen, in previous section, that the norm $\|Hz\|=\|z\|$ is preserved for every $z\in\mathbb{C}^2$ and each matrix H in the group E_6; and so it is trivial to deduce that the norm is also preserved for every matrix H in the group E_7. The quotient mapping $\widetilde{\pi}$ from $\mathbb{C}^2$ over the singular surface $\Sigma_7 \cong \mathbb{C}^2/E_7$ is given by

$$\widetilde{\pi}(z) = \eta_2 \circ \ddot{\pi}(z) = \eta_2 \circ \eta_4 \circ \eta_3(z), \tag{6.2}$$

where each η_k has been defined in (4.2) or (5.2). We know that $\ddot{\pi}$ is a quotient mapping from $\mathbb{C}^2$ onto the singular surface Σ_6 define by $x_1^3 + x_2^2 = x_3^4$ inside $\mathbb{C}^3$. By fixing $\widehat{x}_2 = x_2x_3$ and $\widehat{x}_3 = x_3^2$, we can easily deduce that the mapping $\widetilde{\pi}$ in (6.2) is a natural branched 48-cover from $\mathbb{C}^2$ over

$$\Sigma_7 \cong \left\{x_1^3\widehat{x}_3 + \widehat{x}_2^2 = \widehat{x}_3^3\right\} \quad \text{in} \quad \mathbb{C}^3. \tag{6.3}$$

The proof of Theorem (1.4) for the binary octahedral group E_7 and the exponent $\beta = \frac{1}{48}$ is essentially the same as the proofs for D_{d+2} and E_6, so we only present a sketch. Given the open ball $B_R \subset \mathbb{C}^2$ of radius $R > 0$ and centre in the origin, let z and ζ be two points in B_R such that $\|z - \zeta\|$ is less than or equal to $\|z - H\zeta\|$ for every matrix H in the group E_7. Considering equation (5.7), and working as in the proofs of (4.4) and (5.4), we have that the following inequality holds for the matrix H_4 given in (6.1) and the exponent $k = 0, 1$:

$$\frac{\|\ddot{\pi}(z) - \ddot{\pi}(H_4^k\zeta)\|^2}{\|z-\zeta\|^{24}(\|z\| + \|\zeta\|)^{24}} \geq C_{11}^2(R)C_9^2(R)C_6^2. \tag{6.4}$$

Letting w and x be a pair of points in $\ddot{\pi}(B_R) \subset \mathbb{C}^3$, we automatically have that $|x_1| \leq 2R^8$, $|x_3| \leq R^6$ and x_2^2 is equal to $x_3^4 - x_1^3$; similar relations are satisfied by the vector w. Hence, the following inequality holds,

$$\begin{aligned}|w_2^2 - x_2^2| &\leq |w_1^3 - x_1^3| + |w_3^4 - x_3^4| \\ &\leq 12R^{16}|w_1 - x_1| + 2R^{12}|w_3^2 - x_3^2| \\ &\leq 2R^{12}(6R^4+1)\|\eta_2(w) - \eta_2(x)\|.\end{aligned}$$

Recall equation (4.5) with $\eta_2(x) = (x_1, x_3x_2, x_3^2)$. Let C_6 be the finite positive constant calculated in Lemma (3.3). Since $|w_1 - x_1|$ is less than or equal to $4R^8$, and working as in the proof of equations (4.6), we can deduce the existence of a finite positive constant C_{12} with

$$\frac{\|\eta_2(w) - \eta_2(x)\|}{2C_{12}(R)} \geq \max\left\{\begin{matrix} C_6|w_1 - x_1|^2, |w_2^2 - x_2^2|, \\ |w_3^2 - x_3^2|, |w_2w_3 - x_2x_3| \end{matrix}\right\}. \tag{6.5}$$

Suppose that the norm $\|w - x\|$ is less than or equal to $\|w - \phi(x)\|$, where $\phi(x)$ is defined by $(x_1, -x_2, -x_3)$. Working as in the proof of (4.7), we have that:

$$\|\eta_2(w) - \eta_2(x)\| \geq C_{12}(R)C_6\|w - x\|^2. \tag{6.6}$$

Finally, it is easy to verify that $\ddot{\pi}(H_4\zeta)$ is equal to $\phi(\ddot{\pi}(\zeta))$ for the matrix H_4 in (6.1) and the mapping $\ddot{\pi} = \eta_4 \circ \eta_3$ defined in (5.2). Thus, given z and ζ in B_R such that $\|z - \zeta\|$ is less than or equal to $\|z - H\zeta\|$ for H in D_{d+2}, we fix the point $w = \ddot{\pi}(z)$. Working as in the proof of (4.8) and (5.7), we may set the point $x = \ddot{\pi}(H_4^k\zeta)$ into equation (6.4) and (6.6), with an appropriate k, in order to deduce the following version of equation (1.5) for the exponent $\beta = \frac{1}{48}$ and the quotient mapping $\widetilde{\pi} = \eta_2 \circ \ddot{\pi}$ defined in (6.2),

$$\frac{\|\widetilde{\pi}(z) - \widetilde{\pi}(\zeta)\|}{\|z-\zeta\|^{24}(\|z\| + \|\zeta\|)^{24}} \geq C_{12}(R)C_{11}^2(R)C_9^2(R)C_6^3. \tag{6.7}$$

Received October 16, 2006

Final version received January 23, 2007

DEPARTAMENTO DE MATEMÁTICAS
CINVESTAV
APARTADO POSTAL 14-740
07000 MÉXICO D.F.
MÉXICO
facosta@math.cinvestav.mx
eszeron@math.cinvestav.mx

REFERENCES

[1] F. ACOSTA; E. S. ZERON, *Hölder estimates for the $\overline{\partial}$-equation on surfaces with simple singularities*, Bol. Soc. Mat. Mexicana (3). **in press**.

[2] A. DIMCA, Singularities and topology of hypersurfaces, (Universitext), Springer-Verlag, New York, 1992.

[3] E. A. GAVOSTO, *Hölder estimates for the $\overline{\partial}$-equation in some domains of finite type*, J. Geom. Anal. **7** (4), (1997), 593–609.

[4] J. E. FORNÆSS; E. A. GAVOSTO, *The Cauchy Riemann equation on singular spaces*, Duke Math. J. **93** (3), (1998), 453–477.

[5] G. HENKIN; J. LEITERER, Theory of functions on complex manifolds, (Monographs in Mathematics, 79), Birkhäuser-Verlag, Basel, 1984.

[6] F. KLEIN, Lectures on the icosahedron, Dover Publications, Mineola NY, 2003.

[7] J. RUPPENTHAL, *Zur Regularität der Cauchy-Riemannschen Differentialgleichungen auf komplexen Räumen, Dissertation*. Bonner Mathematische Schriften, 380. Universität Bonn, Bonn, 2006.

[8] P. SLODOWY, Simple singularities and simple algebraic groups, Lecture Notes in Mathematics, 815. Springer, Berlin, 1980.

[9] P. SLODOWY, *Platonic solids, Kleinian singularities, and Lie groups, algebraic geometry* (Ann Arbor, Mich., 1981), Lecture Notes in Math., 1008. Springer, Berlin, 1983. pp. 102–138,

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

ON THE COMPLEMENT OF SETS WITH A SYSTEM OF STEIN NEIGHBOURHOODS

E. S. ZERON

ABSTRACT. Let M be a holomorphic (complex) manifold, and K be a compact subset of M which has a system of Stein open neighbourhoods. The main objective of this paper is to show that the complement of K in M is η-connected, where $\eta \geq 0$ is completely defined by the topological properties of M.

1. Introduction

The general class of compact subsets which have a system of Stein open neighbourhoods plays an extremely important role in complex analysis and approximation theory. This class contains the polynomially and rationally convex subsets of $\mathbb{C}^m$, for example, as well as the totally real submanifolds of $\mathbb{C}^m$. We refer the reader to Stolzenberg [15] and Alexander and Wermer [2] for a general review on the subject. Nevertheless, it is usually a very difficult problem to decide whether a given compact subset has a system of Stein open neighbourhoods. Therefore, results which provide topological obstructions are of special interest to complex analysis.

Recently, Forstnerič [7] proved via Morse theory that the complement of a polynomially convex set in $\mathbb{C}^n$ is $(n-1)$-connected for $n \geq 2$. Let M be a holomorphic (complex) manifold, and K be a compact subset of M which has a system of Stein open neighbourhoods. The main objective of this paper is to show that the complement of K in M is η-connected, where $\eta \geq 0$ is completely defined by the topological properties of M. We strongly recommend the works of May [11] and Aguilar, Gitler and Prieto [1] for references on homotopy theory.

THEOREM (1.1). *Let M be a holomorphic q-connected manifold of complex dimension $m \geq 2$, where $q \geq 0$. Define η to be the minimum of q and $m-2$. Then, the complement of any compact set K in M with a system of Stein open neighbourhoods is η-connected.*

We must note that previous theorem is a modification of the fairly classical results of Andreotti and Frankel [3] and Bott [5]. The space $\mathbb{C}^n$ is the perfect example of a manifold which satisfies the hypotheses of Theorem (1.1), for it is contractible. Recall, for the sake of completeness, that the manifold M is q-connected whenever the homotopy groups $\pi_k(M)$ vanish for every $0 \leq k \leq q$. Besides, a compact subset K has a system of Stein open neighbourhoods if for every open neighbourhood V of K there exists a Stein open subset Ω such that $K \subset \Omega \subset V$.

2000 *Mathematics Subject Classification*: 32Q55, 32E20 or 55E05.
Keywords and phrases: Stein sets, Simply connected, Homotopy.
Research supported by Cinvestav and Conacyt (Mexico), and Université de Montréal (Canada).

One of the main applications of Theorem (1.1) is to produce examples of compact Stein sets which have no system of Stein open neighbourhoods; see Corollary (3.2). Recall that a compact set K is said to be Stein whenever the Theorem B of Cartan and Serre is valid on K [8, p. 100]. It was thought for some time that any compact Stein set has a system of Stein open neighbourhoods, but an counterexample due to Björk [4] has shown that this is not the case. More counterexamples (using topological arcs) have been constructed by Henkin [9].

We close this chapter expressing our deepest gratitude to Professor Samuel Gitler and to one of the referees, who suggested to us the final version of Theorem (1.1) and a simplified proof.

The next section of this paper is completely devoted to the proof of Theorem (1.1). Several applications and counterexamples are presented in the final chapter of this work.

2. Proof of Main Theorem (1.1)

We begin by presenting the following lemma which collects several known results about Stein manifolds. Recall that the reduced $\tilde{H}_k(\cdot)$ and singular $H_k(\cdot)$ homology groups coincide for $k \geq 1$.

LEMMA (2.1). *Let W be a holomorphic manifold of complex dimension $m \geq 2$. Suppose there exist a commutative group G and an index $q \geq 0$ such that the reduced homology groups $\tilde{H}_k(W, G)$ vanish for $0 \leq k \leq q$. Given a compact set K in W which has a system of Stein open neighbourhoods, the following two statements hold:*

$$\check{H}^j(K; G) = 0 \quad \textit{for} \quad j \geq m+1; \tag{2.2}$$

$$\tilde{H}_k(W \setminus K; G) = 0 \quad \textit{for} \quad 0 \leq k \leq \min\{q, m-2\}. \tag{2.3}$$

Proof. The Čech cohomology groups $\check{H}^j(\Omega; G)$ vanish for any m-complex Stein manifold Ω and $j > m$, because Ω has the homotopy type of a CW-complex of real dimension less than or equal to m; see [3], [5] or [12]. Thus equation (2.2) holds, for K has a system of Stein open neighbourhoods partially ordered by inclusions and $\check{H}^j(\cdot)$ is invariant under direct limits; see for example [6, p. 348] or [8]. On the other hand, suppose that $k \leq m-2$. A direct application of the Duality Theorem for general manifolds, [6, p. 351] or [14], and the long exact sequence for reduced homology, [6, p. 185] or [14], yields that

$$0 = \tilde{H}_{k+1}(W, W \setminus K; G) \to \tilde{H}_k(W \setminus K; G) \to \tilde{H}_k(W; G) \to .$$

Thus, equation (2.3) automatically holds because of the hypotheses and the above sequence. □

We may now present the proof of Theorem (1.1). Let M be a holomorphic q-connected manifold of complex dimension $m \geq 2$, where $q \geq 0$, and K be a compact subset of M which has a system of open Stein neighbourhood. Lemma (2.1) automatically implies that the complement of K in M is arcwise connected, for a space is 0-connected if and only if its reduced homology group $\tilde{H}_0(\cdot)$ vanishes.

Proof of Theorem (1.1). We only need to show that the compact set K in M has a system of Stein open neighbourhoods whose complements are all η-connected, where η is the minimum of q and $m-2$. Let $U \subset M$ be any open neighbourhood of K. We may suppose that U is a Stein manifold with a finite number of connected components, for the compact set $K \subset M$ has a system of Stein open neighbourhoods.

There is then a biholomorphism h defined from U onto a closed m-complex submanifold $\widetilde{U}$ of $\mathbb{C}^{2m+1}$; see [8, p. 126] or [10, p. 128]. Choose B_ρ an open ball in $\mathbb{C}^{2m+1}$ with centre at some fixed point $x \in \mathbb{C}^{2m+1}$ and radius $\rho > 0$ large enough such that: $h(K)$ is contained in $\widetilde{U} \cap B_\rho$ and the boundary δB_ρ intersects $\widetilde{U}$ transversely. Hence, the set $\widetilde{U} \cap B_\rho$ is a Stein open manifold bounded by the compact smooth manifold $\widetilde{U} \cap \delta B_\rho$. The works of Andreotti and Frankel [3] and Bott [5] automatically imply that

$$(2.4) \qquad \left(\widetilde{U} \cap \overline{B}_\rho,\ \widetilde{U} \cap \delta B_\rho\right) \quad \text{is} \quad (m-1)\text{-connected.}$$

We may also prove (2.4) via Milnor's work [12]. Let x be a fixed point in $\mathbb{C}^{2m+1}$ such that the square-distance function $L_x \colon \widetilde{U} \to \mathbb{R}$ has no degenerate critical points [12, p. 41]. The index of L_x is then less than or equal to m at each one of its critical points, for $\widetilde{U}$ has real dimension $2m$. On the other hand, since $L_x(z)$ is defined by the norm $\|z-x\|^2$, the compact sets $\widetilde{U} \cap \overline{B_\rho}$ and $\widetilde{U} \cap \delta B_\rho$ are respectively equal to $L_x^{-1}([0, \rho^2])$ and $L_x^{-1}(\rho^2)$. A direct application of Morse Theory [12, §3] yields that $\widetilde{U} \cap \overline{B_\rho}$ is obtained from $\widetilde{U} \cap \delta B_\rho$ by attaching k-cells of dimension $k \geq m$, for the index of $-L_x$ is greater than or equal to m at each one of its critical points [12, p. 41]; and so statement (2.4) holds.

Define $\Omega \subset M$ to be the inverse image $h^{-1}(\widetilde{U} \cap B_\rho)$, where h is the biholomorphism from U onto $\widetilde{U}$. We easily have that Ω is a Stein open set, $K \subset \Omega$, and the boundary $\delta\Omega$ is a smooth compact manifold. Moreover, the compact closure $\overline{\Omega} \subset U$ has a system of Stein open neighbourhoods given by the inverse images $h^{-1}(\widetilde{U} \cap B_\tau)$, for the radii $\tau > \rho$. We can deduce that the sets $M \setminus \overline{\Omega}$ and $M \setminus \Omega$ are both arcwise connected because $\delta\Omega$ is smooth and Lemma (2.1). The previous facts and (2.4) also implies that

$$(2.5) \qquad \begin{array}{ll} \pi_k\left(\overline{\Omega}, \delta\Omega\right) & \text{vanishes for every} \quad 0 \leq k \leq m-1; \\ (M \setminus \Omega, E) & \text{is 0-connected for any} \quad E \subset M \setminus \Omega. \end{array}$$

Since the boundary $\delta\Omega$ is smooth and compact, we may choose an open set V in M such that $\overline{\Omega} \subset V$ and the groups $\pi_k(V, V \setminus \Omega)$ vanish as well for every $k \leq m-1$. We may fix V to be an ϵ-neighbourhood of $\overline{\Omega}$ with $\epsilon > 0$ small enough. Notice that M is equal to the union of V and the interior of $M \setminus \Omega$. Hence, we may use the excisive triad $(M; V, M \setminus \Omega)$, the second statement of (2.5), and the Homotopy Excision Theorem [11, p. 81], in order to deduce the following result for $m \geq 2$,

$$(2.6) \qquad \pi_k(M, M \setminus \Omega) \quad \text{vanishes for} \quad 0 \leq k \leq m-1.$$

We may also prove (2.6) by using the Blakers and Massey's Theorem presented in [1, p. 193]. The inclusions of $\delta\Omega$ into $\overline{\Omega}$ and $M \setminus \Omega$ are both cofibrations because $\delta\Omega$ is a compact smooth manifold [1, p. 94]. Therefore, we may use the triad $(M; \overline{\Omega}, M \setminus \Omega)$, both statements of (2.5), and Blakers and Massey's Theorem [1, p. 193], in order to deduce (2.6) for $m \geq 2$.

Finally, we demand in the hypotheses that M is q-connected, with $q \geq 0$. Therefore, the group $\pi_k(M \setminus \Omega)$ vanishes as well for every $k \leq \eta$, with η equal to the minimum of q and $m-2$, because statement (2.6) and the following long exact sequence [1, p. 87] or [11, p. 63],

$$0 = \pi_{k+1}(M,\ M \setminus \Omega) \rightarrow \pi_k(M \setminus \Omega) \rightarrow \pi_k(M) = 0.$$

We may conclude that the complement of K in M is η-connected, because for every open neighbourhood U of K, we may find a Stein open set Ω in M such that $K \subset \Omega \subset U$ and $\pi_k(M \setminus \Omega)$ vanishes for $0 \leq k \leq \eta$. That is, let f be a continuous mapping defined from the k-dimensional sphere $\mathcal{S}^k$ into the complement $M \setminus K$. We may find a Stein open set Ω in M such that $K \subset \Omega$, the image $f(\mathcal{S}^k)$ does not meet Ω, the complement of Ω is 0-connected, and the group $\pi_k(M \setminus \Omega)$ vanishes. The mapping f is then homotopically trivial in $M \setminus \Omega$ and in the larger set $M \setminus K$. We need specify no base point in $M \setminus \Omega$, for it is 0-connected. $\square$

3. Applications and Counterexamples

We want to finish this paper by presenting several applications and counterexamples. As we have said in the introduction, the space $\mathbb{C}^n$ is the perfect example of a manifold which satisfies the hypotheses of Theorem (1.1), for it is contractible. Thus, we have the following result.

COROLLARY (3.1). *Let $K \subset \mathbb{C}^n$ be a compact set with a system of Stein open neighbourhoods, for $n \geq 2$. The set $\mathbb{C}^n \setminus K$ is then $(n-2)$-connected.*

One of the main applications of Theorem (1.1) is the construction of compact Stein sets which have no system of Stein open neighbourhoods. Recall that a compact set K is said to be Stein whenever Theorem B of Cartan and Serre is valid on K, that is, if and only if all the Čech cohomology groups $\check{H}^q(K, \mathcal{L})$ vanish for every $q \geq 1$ and each coherent analytic sheaf $\mathcal{L}$; see [8, p. 100]. In particular, we may fix $\mathcal{L}$ equal to the sheaf of germs of holomorphic p-forms defined on K.

Thus, whenever K is a compact Stein set in $\mathbb{C}^n$, the Dolbeault theorem yields that the Dolbeault cohomology groups $H^{p,q}_{\bar\partial}(K)$ vanish for $p \geq 0$ and $q \geq 1$. That is, given a $\bar\partial$-closed (p, q)-form λ defined on an open neighbourhood V of K, there exists a second $(p, q-1)$-form g defined on a *smaller* neighbourhood W of K such that $\bar\partial g = \lambda$ inside W. Recall that all previous cohomology groups are calculated as direct limits over systems of neighbourhoods of K. The simplest example of a compact Stein set is a zero-dimensional one, such as a copy of the Cantor set.

COROLLARY (3.2). *Let M be a holomorphic q-connected manifold of complex dimension $m \geq 3$, where $q \geq 1$, and K be a (topological) zero-dimensional compact subset of M whose complement is not simply connected. The set K is then Stein, but it has no system of Stein open neighbourhoods in M.*

We must point out that there always exist zero-dimensional compact subsets in M whose complement is not simply connected. Rushing [13] produces several examples of Cantor sets in $\mathbb{C}^n$ such that the first homotopy group of the complement is non-abelian and infinite.

Proof. The set K is Stein because all cohomology groups $\check{H}^q(K, \mathcal{L})$ vanish whenever $q \geq 1$ and K is a zero-dimensional set. Theorem (1.1) easily implies then that K has no system of Stein open neighbourhoods. □

It is interesting to compare Corollary (3.2) with the original Björk [4] and Henkin's [9] examples of compact Stein sets which have no system of Stein open neighbourhoods. On the other hand, we may also use Theorem (1.1) for estimating how *large* is the intersection of all Stein open neighbourhoods of a given compact set. We need the following lemma on Stein manifolds.

LEMMA (3.3). *Let K be a compact subset of a Stein manifold. The intersection of all Stein open neighbourhoods of K is a compact set.*

Proof. Notice that K has at least one Stein open neighbourhood Ω, because the given hypotheses. Define $\widehat{K}^\Omega$ to be the holomorphically convex hull of K in its neighbourhood Ω, that is,

$$\widehat{K}^\Omega = \{z \in \Omega\,;\, |h(z)| \leq \|h\|_K,\ \forall\, h \in \mathcal{O}(\Omega)\}.$$

Further, let $\widetilde{K}$ be the intersection of all Stein open neighbourhoods of K. We easily have that $\widetilde{K}$ contains the intersection $\bigcap_\Omega \widehat{K}^\Omega$, where Ω runs over all Stein open neighbourhoods of K. We assert that the reverse containment holds. Suppose there exists a point $y \in \widetilde{K}$ and a Stein open neighbourhood U of K such that $y \notin \widehat{K}^U$. We have that $y \in U$ and that there is a holomorphic function $h \in \mathcal{O}(U)$ such that $|h(y)|$ is strictly greater than $\|h\|_K$. Thus, the Stein open set $\{z{\in}U; h(z){\neq}h(y)\}$ contains K, but it does not contain y. This is a contradiction of the fact that $y \in \widetilde{K}$.

The set $\widetilde{K}$ and the intersection $\bigcap_\Omega \widehat{K}^\Omega$ are then equal, where Ω runs over all Stein open neighbourhoods of K. Moreover, every hull $\widehat{K}^\Omega$ is compact, for Ω is Stein [8] or [10, p. 109], and so $\widetilde{K}$ is compact. □

We may deduce that the previous lemma holds as well when K is a compact subset of any complex manifold, and K has at least one Stein open neighbourhood. Recall that a continuous mapping f defined from the k-dimensional sphere $\mathcal{S}^k$ into an open manifold W is homotopically trivial if and only if f has a continuous extension to the $(k{+}1)$-dimensional compact ball bounded by $\mathcal{S}^k$.

COROLLARY (3.4). *Let M be a holomorphic q-connected manifold of complex dimension $m \geq 2$, where $q \geq 0$. Besides, let K be a compact set in M which has at least one Stein open neighbourhood, and f be a continuous non-homotopically trivial function defined from the k-dimensional sphere $\mathcal{S}^k$ into $M \setminus K$. If the dimension k is less than or equal to both q and $m{-}2$, then, the image $f(\mathcal{S}^k)$ meets every Stein open neighbourhoods of K.*

Proof. Define the set $\widetilde{K}$ to be the intersection of all Stein open neighbourhoods of K. This intersection is well defined because there is at least one Stein open set Ω containing K. Lemma (3.3) implies that $\widetilde{K}$ is compact. It is easy to deduce that $\widetilde{K}$ has a system of Stein open neighbourhoods in M, for the intersection of Stein open subsets in Ω is again Stein. Thus, given any open set $V \subset M$ which contains $\widetilde{K}$, we just need to pick up enough Stein open neighbourhoods of $\widetilde{K}$ (or K) in Ω so that the intersection of all of them is contained inside V.

We assert that the image $f(\mathcal{S}^k)$ meets $\widetilde{K}$, and so, it meets every Stein open neighbourhood of K. If the set $f(\mathcal{S}^k)$ does not meets $\widetilde{K}$, then, f is homotopically trivial in the complement of $\widetilde{K}$, after Theorem (1.1). Hence, the function f is also homotopically trivial in the larger set $M \setminus K$, which is a contradiction of the given hypotheses. □

Finally, we close this paper by giving a pair of counterexamples which show that the hypotheses of Theorem (1.1) are *sharp*. Let E be the closed set defined by $z_1 = 0$ in $\mathbb{C}^n$. It is easy to see that E has a system of Stein open neighbourhoods, but its complement is not simply connected. Therefore, we cannot relax the hypothesis that K is a compact subset of a complex manifold in Theorem (1.1).

Moreover, let $\mathcal{T}^2$ be the compact torus defined by $|x|{=}|y|{=}1$ in $\mathbb{C}^2$. We have that $\mathcal{T}^2$ is rationally convex, so it has a system of Stein open neighbourhoods, but its complement is not simply connected. The fundamental group of $\mathbb{C}^n \setminus \mathcal{T}^2$ is indeed isomorphic to the integers. Thus, we cannot relax the hypothesis that $\eta \leq m{-}2$ in Theorem (1.1); and it is trivial to see that we cannot relax the hypothesis that $\eta \leq q$ either.

Received September 27, 2006

Final version received January 11, 2007

Departamento de Matemáticas
CINVESTAV
Apdo. Postal 14-740
07000 México D.F.
México
eszeron@math.cinvestav.mx

References

[1] M. Aguilar, S. Gitler, C. Prieto, Algebraic topology from a homotopical viewpoint, Universitext. Springer-Verlag, New York, 2002.

[2] H. Alexander, J. Wermer, Several complex variables and Banach algebras. Third edition, Grad. Texts in Math. 35. Springer-Verlag, New York, 1998.

[3] A. Andreotti, T. Frankel, *The Lefschetz theorem on hyperplane sections*. Ann. of Math. (2) **69** (1959), 713–717.

[4] J. E. Björk, *Holomorphic convexity and analytic structures in Banach algebras*. Ark. Mat. **9** (1971), 39–54.

[5] R. Bott, *On a theorem of Lefschetz*, Michigan Math. J. **6** (1959), 211–216.

[6] G. E. Bredon, Topology and geometry, Grad. Texts in Math. 139. Springer-Verlag, New York, 1993.

[7] F. Forstnerič, *Complements of Runge domains and holomorphic hulls*, Michigan Math. J. **41** (2), (1994), 297–308.

[8] H. Grauert, R. Remmert, Theory of Stein spaces, Grundlehren der Mathematischen Wissenschaften 236. Springer-Verlag, Berlin, 1979.

[9] G. M. Henkin, Personal communication.

[10] L. Hörmander, Complex Analysis in Several Variables, an introduction. Second Edition, North-Holland, Amsterdam, 1979.

[11] J. P. MAY, A concise course in algebraic topology, Chicago Lectures in Mathematics. University of Chicago Press, Chicago, 1999.
[12] J. MILNOR, *Morse theory. Based on lecture notes by M. Spivak and R. Wells*, Annals of Mathematics Studies 51. Princeton University Press, Princeton, 1963.
[13] T. B. RUSHING, *Topological embeddings*, Pure and Applied Mathematics 52, Academic Press, New York-London, 1973.
[14] E. H. SPANIER, Algebraic Topology, McGraw-Hill, New York, 1966.
[15] G. STOLZENBERG, *Polynomially and rationally convex sets*. Acta Math. **109** (1963), 259–289.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

EXISTENCIA DE SOLUCIONES POSITIVAS PARA PROBLEMAS NO LINEALES CON DISCONTINUIDADES INDEFINIDAS

MARCO CALAHORRANO

RESUMEN. En este artículo presentamos algunos resultados sobre la existencia de soluciones positivas para ecuaciones diferenciales de segundo orden (1-dimensional) con término no-lineal de la forma $\lambda m(x)f(u)$, donde m es discontinua y cambia de signo.

ABSTRACT In this paper we present some results about the existence of positive solutions for second order differential equations (1-dimensional) with nonlinear term of the form $\lambda m(x)f(u)$, where m is discontinuous and sign changing.

A Joaquín Bustoz, entrañable amigo. In memoriam.

1. Introducción

Problemas con no linealidades indefinidas han sido estudiados por S. Alama, M. del Pino, G. Tarantello [1], [2], [3], H. Berestycki, I. Capuzzo-Dolcetta, L. Nirenberg [10], D. Papini, F. Zanolin [22], [23], [24], K. Chang y M. Jiang [18]. El caso de valores propios para pesos indefinidos fue estudiado por Anane, Chakrone y Moussa [8], M. Cuesta [19]. Cuando las no linealidades son indefinidas y discontinuas han contribuido también M. C. y S. González [11]. El caso de ecuaciones semilineales elípticas con no linealidades discontinuas ha sido estudiado extensamente por A. Ambrosetti, C. Stuart, M. Badiale, M. Struwe, D. Arcoya, etc, mirar por ejemplo [4], [25], [7], [6], [9] para una bibliografía más extensa. Para los casos donde las no linealidades aparecen con peso observar [12], [20].

Ahora estudiamos la existencia de soluciones positivas para problemas con valores al borde de la forma:

$$\begin{cases} -u'' = \lambda m(x) f(u), & 0 < x < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

donde $\lambda > 0$, $m \in PC([0,1])^1$, m cambia signo y f es una función no lineal con condiciones de crecimiento en cero y en el infinito.

Por facilidad vamos a suponer:

$$m\colon (0,1) \mapsto \mathbb{R} \tag{1.2}$$

2000 *Mathematics Subject Classification*: 34B15, 34B18, 34B09.

Keywords and phrases: indefinite discontinuous nonlinearities, positive solutions, boundary value problems; no linealidades discontinuas, soluciones positivas, problemas de valores en la frontera.

[1] PC([0,1]) es el conjunto de las funciones reales continuas por tramos definidas en el intervalo [0,1].

tal que

$$m(x) = \begin{cases} 1 & \text{si } 0 < x < \alpha, \\ -1 & \text{si } \alpha < x < 1 \end{cases}$$

con $\alpha \in]0, 1[$. Consideraremos las siguientes hipótesis sobre $f \in C^2$

(A) f es tal que:

$$f''(s) > 0 \quad \text{para} \quad s \geq 0, \tag{1.3}$$

$$f(0) = 0, \tag{1.4}$$

$$f(s) - sf'(s) < 0 \quad \text{para} \quad s > 0\,. \tag{1.5}$$

(B) f verifica las siguientes hipótesis:

$$f(0) = 0\,, \tag{1.6}$$

$$\text{existe} \quad s_0 > 0 \quad \text{tal que} \quad f''(s) > 0 \quad \text{para} \quad s \in [0, s_0) \quad \text{y} \tag{1.7}$$

$$f''(s) \geq 0 \quad \text{para} \quad s \in [s_0, +\infty)\,, \tag{1.8}$$

$$f(s) - sf'(s) > 0 \quad \text{para} \quad s > 0\,. \tag{1.9}$$

Observación (1.10). Para la obtención de nuestros resultados hemos seguido las ideas desarrolladas por A. Castro, R. Shivaji y A. Kurepa en [13], [16], [17] y [14] donde estudian la existencia de soluciones no negativas para problemas de tipo semipositone. Es importante considerar los trabajos de D. Papini y F. Zanolin [22], [23], [24] donde se estudian ecuaciones (1-dimensional) no lineales con peso indefinido; dichos autores, sin embargo, consideran pesos al menos continuos. Tomen en cuenta [23] para un estudio histórico, problemas relacionados, bibliografía más extensa y aplicaciones a la ecuación de Hill.

2. El resultado principal

Antes de enunciar el teorema fundamental del trabajo introduzcamos una definición de solución para (1.1).

Definición (2.1). Diremos que $u \in C([0, 1])$ es una solución de (1.1) si $u \in C^2(]0, \alpha[\cup]\alpha, 1[)$ y verifica (1.1) salvo el punto de discontinuidad.

Observación (2.2). En general, las soluciones definidas como en (2.1) pueden ser llamadas soluciones del problema a "k más dos puntos" si k son los puntos de discontinuidad de m; en el caso particular que nos ocupa podríamos llamarla solución del problema de "tres puntos", $u(0) = u(1) = 0$ y $\lim_{t\to\alpha^-} u(t) = \lim_{t\to\alpha^+} u(t)$.

TEOREMA (2.3). *Sea $f'(s) > 0$ para $s \geq 0$,*

(a) *Si las hipótesis **[A]** se verifican y $\lim_{s\to\infty} \frac{f(s)}{s} = +\infty$ entonces existe $\lambda^* > 0$ tal que (1.1), tiene al menos una solución positiva para $\lambda \in (0, \lambda^*)$.*

(b) *Si las hipótesis **[B]** se verifican y $\lim_{s\to\infty} \frac{f(s)}{s} = C$, ($C$ una constante) y $4f'(0) > C$, entonces existen constantes $0 < \underline{\lambda} < \overline{\lambda}$ tales que (1.1) tiene al menos una solución positiva para $\lambda \in (\underline{\lambda}, \overline{\lambda}) = \Lambda$.*

Observación (2.4). Las soluciones obtenidas en el teorema anterior, generalmente, no son soluciones en el sentido de las distribuciones aunque lo son en el sentido casi todo punto.

3. Demostración del Teorema

Para la demostración del teorema (2.3) nosotros transformamos el problema (1.1) en:

$$\begin{cases} -u'' = \lambda f(u), & 0 < x < \alpha, \\ u(0) = 0,\ u(\alpha) = \rho\,; \end{cases} \tag{3.1}$$

$$\begin{cases} u'' = \lambda f(u), & \alpha < x < 1, \\ u(\alpha) = \rho,\ u(1) = 0 \end{cases} \tag{3.2}$$

con $\rho > 0$.

Observación (3.3). Si permitimos que $\rho \geq 0$, entonces las soluciones serán no-negativas y si ρ es simplemente un real la solución podrá cambiar signo.

LEMA (3.4). *Si las ecuaciones con condiciones al borde (3.1) y (3.2) tienen solución entonces (1.1) también lo tendrá (en el sentido de la definición 2.1).*

Estudiaremos ahora las soluciones de los problemas (3.1) y (3.2), para lo cual primero analizaremos la ecuación de (3.1).

Si multiplicamos por u' y luego integramos la ecuación de (3.1) obtendremos:

$$-\frac{u'^2}{2} = \lambda F(u) + k, \tag{3.5}$$

donde

$$F(u) = \int_0^u f(s)ds. \tag{3.6}$$

Como $f(s) > 0$ para $s > 0$, u es cóncava y por lo tanto buscaré soluciones positivas de (3.1) tal que $u(\alpha) = \rho$ y $u'(\alpha^-) = 0$.

PROPOSICIÓN (3.7). *Si las hipótesis de la parte a) del teorema se verifican entonces existe $\lambda^* > 0$ tal que (3.1) tiene al menos una solución positiva, u, para todo $\lambda \in]0, \lambda^*[$. Además la solución u cumple: $u(0) = 0$, $u(\alpha) = \rho$ ($\rho = \sup_{x\in(0,\alpha)} u(x)$) y $u'(\alpha^-) = 0$.*

Demostración. De (3.5) y $u'(\alpha^-) = 0$

$$u'^2 = 2\lambda[F(\rho) - F(u)]. \tag{3.8}$$

De la positividad y concavidad de u en $]0, \alpha[$ tenemos que:

$$u'(x) = \sqrt{2\lambda[F(\rho) - F(u)]}, \tag{3.9}$$

$$\frac{du}{\sqrt{F(\rho) - F(u)}} = \sqrt{2\lambda}dx, \tag{3.10}$$

que integrando nos produce:

$$\sqrt{\lambda} = \frac{1}{\sqrt{2}\alpha}\int_0^\rho \frac{du}{\sqrt{F(\rho) - F(u)}}. \tag{3.11}$$

Definamos la función G como sigue:

$$G(\rho) = \frac{1}{\sqrt{2}\alpha} \int_0^\rho \frac{du}{\sqrt{F(\rho) - F(u)}}. \tag{3.12}$$

Si en la fórmula anterior se hace el cambio de variables $u = \rho v$ la función G viene transformada en:

$$G(\rho) = \frac{\rho}{\sqrt{2}\alpha} \int_0^1 \frac{dv}{\sqrt{F(\rho) - F(\rho v)}}. \tag{3.13}$$

De la hipótesis (1.5), $f(s) - sf'(s) < 0$, se puede probar fácilmente que: $\frac{dG}{d\rho} < 0$ para $\rho > 0$.

Por otro lado podemos demostrar que:

$$G(\rho) \leq \frac{\rho}{\alpha\sqrt{2F(\rho)}} \int_0^1 \frac{dv}{\sqrt{1-v}}, \tag{3.14}$$

y por lo tanto

$$G(\rho) \leq \frac{\rho\sqrt{2}}{\alpha\sqrt{F(\rho)}}, \tag{3.15}$$

Y como hemos supuesto $\lim_{s\to+\infty} \frac{f(s)}{s} = +\infty$ de la fórmula (3.15) nosotros obtendremos que:

$$G(\rho) \to 0, \quad \text{cuando} \quad \rho \to +\infty. \tag{3.16}$$

De esto es claro que la proposición (3.7) viene inmediatamente. □

Observación (3.17). La fórmula definida en (3.12) está relacionada con una dada por los autores Manásevich y Zanolin denominada Time-mapping, mirar [21].

PROPOSICIÓN (3.18). *Bajo las hipótesis de la parte a) del teorema, es decir las mismas de la proposición (3.7), el problema (3.2) tiene al menos una solución positiva para todo* $\lambda \in]0, \lambda^*[$.

Antes de demostrar la proposición notemos lo siguiente:

Observación (3.19). Buscamos soluciones del problema (1.1) al menos continuas en $[0, 1]$; por tanto $\rho = \sup_{x\in]\alpha,1[} u(x)$ y entonces la solución de (1.1) no es diferenciable en α pues debe verificar la ecuación de (3.2). Del razo-namiento anterior deducimos que cualquier solución de (1.1) en el sentido de la definición 2.1 debe verificar (3.2) con $u'(\alpha^+) < 0$.

Demostración. Demostremos la proposición (3.18). Si tomamos en cuenta la observación (3.19) nosotros debemos suponer que existe al menos un $\widehat{\alpha} \in \mathbb{R}^-$ tal que:

$$u(\alpha^+) = \widehat{\alpha}\rho. \tag{3.20}$$

De la ecuación (3.2) y siguiendo el procedimiento de la demostración de la proposición (3.7) se tiene que:

$$\int_0^\rho \frac{du}{\sqrt{(\widehat{\alpha}\rho)^2 - 2\lambda(F(\rho) - F(u))}} = 1 - \alpha. \tag{3.21}$$

Para que la proposición (3.18) quede demostrada nos basta probar que efectivamente existe un tal $\widehat{\alpha} \in \mathbb{R}^-$ tal que para todo $\lambda \in]0, \lambda^*[$ la integral de la fórmula (3.21) alcance el valor $1-\alpha$, con $\alpha \in]0,1[$. Definamos la función $\widehat{G}_\lambda(\rho)$ de la forma siguiente:

$$\widehat{G}_\lambda(\rho) = \int_0^\rho \frac{du}{\sqrt{(\widehat{\alpha}\rho)^2 - 2\lambda(F(\rho) - F(u))}} \tag{3.22}$$

que con el cambio de variables $u = \rho v$ se transforma en:

$$\widehat{G}_\lambda(\rho) = \rho \int_0^1 \frac{dv}{\sqrt{(\widehat{\alpha}\rho)^2 - 2\lambda(F(\rho) - F(\rho v))}}. \tag{3.23}$$

De la prueba de la proposición (3.7) se puede deducir que:

$$\sqrt{\lambda} \leq \frac{\rho\sqrt{2}}{\alpha\sqrt{F(\rho)}}. \tag{3.24}$$

En efecto, mirar la desigualdad (3.15) para una prueba.

Tomando en cuenta (3.24) y luego de algunos cálculos podemos llegar a:

$$\frac{1}{|\widehat{\alpha}|} \leq \widehat{G}_\lambda(\rho) \leq \frac{1}{\sqrt{\widehat{\alpha}^2 - \frac{4}{\alpha^2}}}. \tag{3.25}$$

Por lo tanto para que se verifique la proposición (3.18) se debe cumplir:

$$\frac{1}{|\widehat{\alpha}|} \leq 1 - \alpha \leq \frac{1}{\sqrt{\widehat{\alpha}^2 - \frac{4}{\alpha^2}}} \tag{3.26}$$

para $\alpha \in]0,1[$.

Y así $\widehat{\alpha}$ deberá cumplir con las desigualdades:

$$|\widehat{\alpha}| \leq \sqrt{\frac{1}{(1-\alpha)^2} + \frac{4}{\alpha^2}}, \tag{3.27}$$

$$|\widehat{\alpha}| \geq \frac{1}{(1-\alpha)}, \tag{3.28}$$

y

$$|\widehat{\alpha}| > \frac{2}{\alpha}, \tag{3.29}$$

que se verifican fácilmente. □

PROPOSICIÓN (3.30). *Supuestas las hipótesis de la parte b) del teorema (2.3), entonces existen constantes* $0 < \underline{\lambda} < \overline{\lambda}$ *tales que (3.1) tiene al menos una solución positiva para* $\lambda \in (\underline{\lambda}, \overline{\lambda}) = \Lambda$.

Demostración. Se sigue el razonamiento de la demostración de la proposición (3.7) hasta llegar a:

$$G(\rho) = \frac{\rho}{\sqrt{2}\alpha} \int_0^1 \frac{dv}{\sqrt{F(\rho) - F(\rho v)}}. \tag{3.31}$$

De la hipótesis $f(s) - sf'(s) > 0$ (1.9) se tiene que $\frac{dG}{d\rho} > 0$, es decir, G es creciente y por tanto:

$$\frac{1}{\alpha^2 f'(0)} \leq \lambda \leq \frac{4}{\alpha^2 C}. \tag{3.32}$$

Y puesto que $4f'(0) > C$ entonces existen constantes $0 < \underline{\lambda} < \overline{\lambda}$ tales que para todo $\lambda \in (\underline{\lambda}, \overline{\lambda})$ el problema (3.1) tiene al menos una solución positiva. □

PROPOSICIÓN (3.33). *Bajo las hipótesis de la proposición (3.30) el problema (3.2) tiene al menos una solución positiva para* $\lambda \in (\underline{\lambda}, \overline{\lambda})$.

Demostración. Se aplica el razonamiento de la demostración de la proposición (3.18). □

Ahora estamos en capacidad de probar el teorema ya enunciado:

TEOREMA (3.34). *Sea* $f'(s) > 0$ *para* $s \geq 0$*,]*

(a) *Si las hipótesis* ***[A]*** *se verifican y* $\lim_{s\to\infty} \frac{f(s)}{s} = +\infty$ *entonces existe* $\lambda^* > 0$ *tal que (1.1), tiene al menos una solución positiva para* $\lambda \in (0, \lambda^*)$.

(b) *Si las hipótesis* ***[B]*** *se verifican y* $\lim_{s\to\infty} \frac{f(s)}{s} = C$*, (C una constante) y* $4f'(0) > C$*, entonces existen constantes* $0 < \underline{\lambda} < \overline{\lambda}$ *tales que (1.1) tiene al menos una solución positiva para* $\lambda \in (\underline{\lambda}, \overline{\lambda}) = \Lambda$.

Demostración. La parte a) sigue de las proposiciones (3.7) y (3.18), y la b) viene de las proposiciones (3.30) y (3.33). □

Observación (3.35). Podrán hacerse extensiones del problema al caso del operador p-Laplaciano en una dimensión y seguramente se obtendrán resultados similares.

Reconocimientos

Este artículo está basado en la conferencia presentada por el autor en la 3er Alfa Meeting, Partial Differential Equations in Industry and Engineering, Viena, junio 2005.

Agradezco la hospitalidad del Departamento de Matemática de la Universidad de Viena y del Instituto Pauli de la Universidad de Viena.

Financiado parcialmente por Alfa Project Partial Differential Equations in Industry and Engineering y Escuela Politécnica Nacional.

Recibida 15 de mayo de 2006

Versión final recibida 27 de febrero de 2007

DEPARTAMENTO DE MATEMÁTICA
ESCUELA POLITÉCNICA NACIONAL
LADRÓN DE GUEVARA E11-253
APARTADO 17-01-2759
QUITO
ECUADOR
calahor@math.epn.edu.ec

REFERENCIAS

[1] S. ALAMA, M. DEL PINO, *Solutions of elliptic equations with indefinite nonlinearities via Morse theory and linking*, Ann. Inst. H. Poincaré Anal. Non Linéaire **13**, (1996), 95–115.

[2] S. ALAMA, G. TARANTELLO, *On semilinear elliptic equations with indefinite nonlinearities*, Calc. Var. Partial Differential Equations **1**, (1993), 439–475.

[3] S. ALAMA, G. TARANTELLO, *Elliptic problems with nonlinearities indefinite in sign*, J. Funct. Anal. **141**, (1996), 159–215.

[4] A. AMBROSETTI, *Critical points and nonlinear variational problems*, Cours de la Chaire Lagrange, Mémoire (nouvelle série) No. 49, Supplément au Bull. Soc. Math. France **120**, (1992), 1–139.

[5] A. AMBROSETTI, M. BADIALE, *The dual variational principle and elliptic problems with discontinuous nonlinearities*, J. Math. Anal. Appl. **140** (1989), 363–373.

[6] A. AMBROSETTI, M. CALAHORRANO, F. DOBARRO, *Global branching for discontinuous problems*, Comment Math. Univ. Carolina, **31**, (1990), 213–222.

[7] A. AMBROSETTI, M. STRUWE, *Existence of steady vortex rings in an ideal fluid*, Arch. Ration. Mech. Anal. **108** (1989), 97–109.

[8] A. ANANE, O. CHAKRONE, M. MOUSSA, *Spectrum of one dimensional p-Laplacian operator with indefinite weight*, EJQTDE, **17**, (2002), 1–11.

[9] D. ARCOYA, M. CALAHORRANO, *Some discontinuous problems with a quasilinear operator*, J. Math. Anal. Appl. **187**, (1994), 1059–1072.

[10] H. BERESTYCKI, I. CAPUZZO-DOLCETTA, L. NIRENBERG, *Variational methods for indefinite superlinear homogeneous elliptic problems*, NoDEA **2**, (1995), 553–572.

[11] M. CALAHORRANO, S. GONZÁLEZ, *Existence of weak solutions for a p-Laplacian type problem with an indefinite discontinuous term*, pre-print EPN (2005), 1–15.

[12] M. CALAHORRANO, J. MAYORGA, *Un problema discontinuo con operador cuasilineal*, Rev. Colombiana Mat. **35**, (2001), 1–11.

[13] A. CASTRO, *Non negative solutions for non-positone problems*, Lect. College on Variational Methods, ICTP, Trieste, (1988) 1–12.

[14] A. CASTRO, A. KUREPA, *Energy analysis of a nonlinear singular differential equation and applications*, Rev. Colombiana Mat. **21**, (1987), 155–166.

[15] A. CASTRO, C. MAYA, R. SHIVAJI, *Nonlinear eigenvalue problems with semipositone structure*, EJDE **Conf. 05**, (2000), 33–49.

[16] A. CASTRO, R. SHIVAJI, *Nonnegative solutions for a class of nonpositone problems*, Proc. Roy. Soc. Edinburgh Sect. A **108**, (1988), 291–302.

[17] A. CASTRO, R. SHIVAJI, *Nonnegative solutions for a class of radially symmetric nonpositone problems*, Proc. Amer. Math. Soc. **106**, (1989), 735–740.

[18] K. C. CHANG, M.Y. JIANG, *Dirichlet problem with indefinite nonlinearities*, Calc. Var. Partial Differential Equations **20**, (2004), 257–282.

[19] M. CUESTA, *Eigenvalue problems for the p-Laplacian with indefinite weights*, Electron. J. Differential Equations **33**, (2001), 1–9.

[20] M. GARCÍA-HUIDOBRO, R. MANÁSEVICH, C. YARUR, *Some results about positive solutions of a nonlinear equation with a weighted Laplacian*, Bol. Soc. Parana. Mat. **22**, (2004), 57–65.

[21] R. MANÁSEVICH, F. ZANOLIN, *Time-mappings and multiplicity of solutions for the one-dimensional p-Laplacian*, Nonlinear Anal. **21**, (1993), 269–291.

[22] D. PAPINI, F. ZANOLIN, *A topological approach to superlinear indefinite boundary value problems*, Topol. Methods Nonlinear Anal. **15**, (2000), 203–233.

[23] D. PAPINI, F. ZANOLIN, *Differential equations with indefinite weight: boundary value problems and qualitative properties of the solutions*, Rend. Sem. Mat. Univ. Pol. Torino, **60**, (2002), 265–296.

[24] D. PAPINI, F. ZANOLIN, *Periodic points and chaotic-like dynamics of planar maps associated to nonlinear Hill's equations with indefinite weight*, Georgian Math. J. **9**, (2002), 339–366.

[25] C. STUART, *Differential equations with discontinuous nonlinearities*, Arch. Rational Mech. Anal. **63**, (1976), 59–75.

[26] C. STUART, J. F. TOLAND, *A variational method for boundary value problems with discontinuous non-linearities*, J. London Math. Soc. **21**, (1980), 319–328.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF POSITIVE PERIODIC SOLUTIONS OF A NICHOLSON'S BLOWFLIES MODEL

JING-WEN LI, SUI SUN CHENG, ZHI-YUAN JIANG

ABSTRACT. In this paper, we derive a sufficient condition as well as a necessary condition for existence of positive periodic solutions of the Nicholson's blowflies model with periodic coefficients

$$\dot{x}(t) = -\delta(t)x(t) + P(t)x(t-\tau(t))e^{-a(t)x(t-\tau(t))}, \quad t \geq 0,$$

where $\delta, P, a \in C(\mathbb{R}^+, (0,\infty))$ and $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$ are T-periodic functions. When $P(t) = \gamma\delta(t)$ with $\gamma > 0$, a sufficient and necessary condition for the existence of a positive T-periodic solution follows.

1. Introduction

Nicholson's blowflies models have been studied by many authors. In particular, the delay differential equation

$$\dot{x}(t) = -\delta x(t) + Px(t-\tau)e^{-ax(t-\tau)}, \quad t \in \mathbb{R}^+ = [0,\infty), \tag{1.1}$$

where $\delta, P, a, \tau > 0,$ is used by Gurney et al. in [3] to describe the dynamics of Nicholson's blowflies. For related works, we refer to [1], [3]–[12] and the references cited therein. In particular, it is known that (1.1) has a unique positive equilibrium $\overline{x} = \frac{1}{a}\ln\left(\frac{P}{\delta}\right)$ if, and only if, $P > \delta,$ and in [7]–[9], [12], global attractivity of the positive equilibrium $\bar{x}$ of Eq.(1.1) has been investigated. In [10], the existence of positive T-periodic solutions of (1.1) is considered, and the following result is obtained: If

$$1 - e^{-\delta T} < PT \leq e(1 - e^{-\delta T}), \tag{1.2}$$

then (1.1) has at least one positive T-periodic solution.

In this paper, we will derive a necessary and sufficient condition for the existence of positive T-periodic solutions of (1.1). We will approach our necessary and sufficient condition in a slightly more general setting by studying the equation

$$\dot{x}(t) = -\delta(t)x(t) + P(t)x(t-\tau(t))e^{-a(t)x(t-\tau(t))}, \quad t \geq 0, \tag{1.3}$$

under the initial condition

$$x(s) = \phi(s),\ \phi \in C([-\tau_M, 0], \mathbb{R}^+) \text{ and } \phi \not\equiv 0, \tag{1.4}$$

where $\delta, P, a \in C(\mathbb{R}^+, (0,\infty))$ and $\tau \in C(\mathbb{R}^+, \mathbb{R}^+)$ are T-periodic functions, and

$$\tau_M = \max_{t\in[0,T]} \tau(t) \geq 0.$$

2000 *Mathematics Subject Classification*: 34D45, 34C25, 92D25.

Keywords and phrases: Nicholson's blowflies, positive periodic solution, existence, coincidence degree, positive equilibrium.

It is known that the initial problem (1.3)-(1.4) has a unique nonnegative solution $x(t)$ on $[0,\infty)$ and that $x(t) > 0$ for $t \geq \tau_M$. See e.g. [4]. In the following, by a solution of (1.4) we will mean a solution of (1.3)-(1.4). Note that when $\delta(t)$, $P(t)$, $a(t)$ and $\tau(t)$ are all constants, (1.3) reduces to (1.1).

By means of coincidence degree theory, we first establish a sufficient condition and a necessary condition so that (1.3) has a positive T —periodic solution. Then in the special case when $\delta(t)$, $P(t)$, $a(t)$ and $\tau(t)$ are constant functions, we obtain our desired necessary and sufficient condition for (1.1). As a bonus, we may also obtain a necessary and sufficient condition in the case when $P(t) = \gamma\delta(t)$ and $\gamma > 0$.

Throughout this paper, we always let

$$a_m = \min_{t\in[0,T]} a(t), \quad a_M = \max_{t\in[0,T]} a(t),$$

and

$$\bar{\delta} = \frac{1}{T}\int_0^T \delta(s)ds, \quad \bar{P} = \frac{1}{T}\int_0^T P(s)ds.$$

2. Sufficient Condition

Let

$$X = Z = \{x(t) \in C(\mathbb{R},\mathbb{R}) : x(t+T) = x(t)\}$$

be the Banach space endowed with the usual linear structure as well as the norm $\|x\| = \sup_{t\in[0,T]} |x(t)|$. Let

$$Lx = \dot{x}, \quad Px = Qx = \frac{1}{T}\int_0^T x(t)dt,$$

and

$$Nx = -\delta(t)x(t) + P(t)x(t-\tau(t))e^{-a(t)x(t-\tau(t))}.$$

Obviously,

$$\operatorname{Dom} L = \{x \in X : x \in C^1(\mathbb{R},\mathbb{R})\}, \quad \operatorname{Ker} L = \mathbb{R},$$

$$\operatorname{Im} L = \left\{ z \in Z : \int_0^T z(t)dt = 0 \right\},$$

and

$$\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L$$

since $\operatorname{Im} L$ is closed in Z and $L\colon \operatorname{Dom} L \subset X \to X$ is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \quad \operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I-Q).$$

In the proof of our existence theorem below, we will use the continuation theorem from Gaines and Mawhin [2].

LEMMA (2.1) (Continuation Theorem). *Let L be a Fredholm mapping of index zero, and Ω be bounded open subset in X such that N is L-compact on $\bar{\Omega}$. Assume further that*

(a) *For each $\lambda \in (0,1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;*

(b) *$QNx \neq 0$ for each $x \in \partial\Omega \cap \operatorname{Ker} L$ and*

$$\deg\{QN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.$$

Then the equation $Lx = Nx$ has at least one solution lying in $\operatorname{Dom} L \cap \bar{\Omega}$.

THEOREM (2.2). *Assume that*

$$P(t) > \delta(t), \quad \text{for } t \in [0, T]. \tag{2.3}$$

Then the initial problem (1.3)-(1.4) has at least one positive T-periodic solution.

Proof. Note that $P(t), \delta(t)$ are T-periodic functions and $P(t) > \delta(t) > 0$. Then we can choose $\gamma > 1$ such that

$$P(t) > \gamma\delta(t), \quad \text{for } t \in \mathbb{R}. \tag{2.4}$$

To use Lemma (2.1), we consider the operator equation $Lx = \lambda Nx$ for $\lambda \in (0, 1)$, that is,

$$\dot{x}(t) = -\lambda\delta(t)x(t) + \lambda P(t)x(t-\tau(t))e^{-a(t)x(t-\tau(t))}, \quad t \geq 0. \tag{2.5}$$

Assume that $x(t)$ is a positive T-periodic solution of (2.5), choose $t_* \in [0, T]$ and $t^* \in [t_*, t_* + T]$ such that

$$x(t^*) = \max_{t\in[0,T]} x(t), \quad x(t_*) = \min_{t\in[0,T]} x(t).$$

From (2.5), we have

$$\left(x(t)e^{\lambda\int_0^t \delta(s)ds}\right)' = \lambda P(t)e^{\lambda\int_0^t \delta(s)ds}x(t-\tau(t))e^{-a(t)x(t-\tau(t))}. \tag{2.6}$$

Integrating (2.6) from $t^* - T$ to t^*,

$$\begin{aligned} x(t^*)e^{\lambda\int_0^{t^*} \delta(s)ds} - x(t^* - T)e^{\lambda\int_0^{t^*-T} \delta(s)ds} \\ = \lambda\int_{t^*-T}^{t^*} P(t)e^{\lambda\int_0^t \delta(s)ds}x(t-\tau(t))e^{-a(t)x(t-\tau(t))}dt, \end{aligned} \tag{2.7}$$

and so,

$$\begin{aligned} x(t^*)\left(1 - e^{-\lambda\int_{t^*-T}^{t^*} \delta(s)ds}\right) &= \lambda\int_{t^*-T}^{t^*} P(t)e^{-\lambda\int_t^{t^*} \delta(s)ds}x(t-\tau(t))e^{-a(t)x(t-\tau(t))}dt \\ &\leq \lambda\int_{t^*-T}^{t^*} P(t)e^{-\lambda\int_t^{t^*} \delta(s)ds}x(t-\tau(t))e^{-a_m x(t-\tau(t))}dt \\ &\leq \frac{\lambda}{ea_m}\int_{t^*-T}^{t^*} P(t)e^{-\lambda\int_t^{t^*} \delta(s)ds}dt. \end{aligned}$$

It follows that

$$\begin{aligned} x(t^*) &\leq \frac{\lambda\int_{t^*-T}^{t^*} P(t)e^{-\lambda\int_t^{t^*} \delta(s)ds}dt}{ea_m\left(1 - e^{-\lambda\int_0^T \delta(s)ds}\right)} \leq \frac{\lambda\int_0^T P(t)dt}{ea_m\left(1 - e^{-\lambda\int_0^T \delta(s)ds}\right)} \\ &< \frac{\int_0^T P(t)dt}{ea_m\left(1 - e^{-\int_0^T \delta(s)ds}\right)} = \frac{\overline{P}T}{ea_m\left(1 - e^{-\bar{\delta}T}\right)} := B, \end{aligned} \tag{2.8}$$

where the third inequality follows from the monotonicity of the numerator as a function of λ. Similarly, we have

$$x(t_*)\left(1-e^{-\lambda\int_{t_*-T}^{t_*}\delta(s)ds}\right)=\lambda\int_{t_*-T}^{t_*}P(t)e^{-\lambda\int_t^{t_*}\delta(s)ds}x(t-\tau(t))e^{-a(t)x(t-\tau(t))}dt$$

$$\begin{aligned}
&\geq \lambda\int_{t_*-T}^{t_*}P(t)e^{-\lambda\int_t^{t_*}\delta(s)ds}x(t-\tau(t))e^{-a_Mx(t-\tau(t))}dt\\
&\geq \lambda\min\left\{x(t_*)e^{-a_Mx(t_*)},x(t^*)e^{-a_Mx(t^*)}\right\}\int_{t_*-T}^{t_*}P(t)e^{-\lambda\int_t^{t_*}\delta(s)ds}dt\\
&> \gamma\min\left\{x(t_*)e^{-a_Mx(t_*)},x(t^*)e^{-a_Mx(t^*)}\right\}\int_{t_*-T}^{t_*}\lambda\delta(t)e^{-\lambda\int_t^{t_*}\delta(s)ds}dt\\
&= \gamma\min\left\{x(t_*)e^{-a_Mx(t_*)},x(t^*)e^{-a_Mx(t^*)}\right\}\left(1-e^{-\lambda\int_0^T\delta(s)ds}\right).
\end{aligned}$$

Thus we have

$$x(t_*)>\gamma\min\left\{x(t_*)e^{-a_Mx(t_*)},x(t^*)e^{-a_Mx(t^*)}\right\}. \tag{2.9}$$

Note that the function xe^{-a_Mx} is increasing on $[0,a_M^{-1}]$ and decreasing on $[a_M^{-1},\infty)$. Therefore, if $\min\left\{x(t_*)e^{-a_Mx(t_*)},x(t^*)e^{-a_Mx(t^*)}\right\}=x(t^*)e^{-a_Mx(t^*)}$, then $x(t_*)\leq a_M^{-1}<x(t^*)<B$, or $a_M^{-1}\leq x(t_*)<x(t^*)<B$, and so $x(t^*)e^{-a_Mx(t^*)}>Be^{-a_MB}$. It follows from (2.9) that

$$x(t_*)>\gamma Be^{-a_MB}:=A_1>0. \tag{2.10}$$

If

$$\min\left\{x(t_*)e^{-a_Mx(t_*)},x(t^*)e^{-a_Mx(t^*)}\right\}=x(t_*)e^{-a_Mx(t_*)},$$

then (2.9) yields that

$$x(t_*)>\gamma x(t_*)e^{-a_Mx(t_*)}.$$

It follows that

$$x(t_*)>\frac{\ln\gamma}{a_M}:=A_2>0. \tag{2.11}$$

Set $A=\min\{A_1,A_2\}$. Then

$$A<x(t)<B. \tag{2.12}$$

Let $\Omega=\{x\in X: A<x(t)<B,\ t\in R\}$. Then Ω satisfies the requirement (a) in Lemma (2.1). In the sequel, we will prove that N is L-compact in $\bar{\Omega}$. In fact, the generalized inverse (to L) K_P: $\operatorname{Im}L\to\operatorname{Ker}P\cap\operatorname{Dom}L$ is given by

$$K_Px=\int_0^t x(s)ds-\frac{1}{T}\int_0^T\int_0^t x(s)dsdt.$$

Clearly,

$$QNx=\frac{1}{T}\int_0^T\left[-\delta(s)x(s)+P(s)x(s-\tau(s)e^{-a(s)x(s-\tau(s))}\right]ds.$$

And

$$
\begin{aligned}
K_P(I-Q)Nx = &\int_0^t [-\delta(s)x(s) + P(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}]ds \\
&- \frac{1}{T}\int_0^T \int_0^t \left[-\delta(s)x(s) + P(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}\right] ds\, dt \\
&- \left(\frac{t}{T} - \frac{1}{2}\right) \int_0^T \left[-\delta(s)x(s) + P(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}\right] ds.
\end{aligned}
$$

Obviously, QN, $K_P(I-Q)N$ are both continuous and $QN(\bar{\Omega})$ is bounded. Using the Arzela-Ascoli theorem, it is not difficult to show that $K_P(I-Q)N$ is compact. Hence N is L-compact on $\bar{\Omega}$. Note that $\operatorname{Ker} L \cap \partial\Omega = \{A, B\}$, and that

$$
\begin{aligned}
QN(A) &= \frac{1}{T}\int_0^T \left[-\delta(t)A + P(t)Ae^{-a(t)A}\right] dt \\
&\geq -\bar{\delta}A + \bar{P}Ae^{-a_M A} \geq \bar{\delta}A\left(\frac{\bar{P}}{\bar{\delta}}e^{-a_M A_2} - 1\right) \\
&= \bar{\delta}A\left(\frac{\bar{P}}{\bar{\delta}}e^{-\ln\gamma} - 1\right) = \bar{\delta}A\left(\frac{\bar{P}}{\bar{\delta}\gamma} - 1\right) > 0,
\end{aligned}
$$

and

$$
\begin{aligned}
QN(B) &= \frac{1}{T}\int_0^T \left[-\delta(t)B + P(t)Be^{-a(t)B}\right] dt \\
&\leq -\bar{\delta}B + \bar{P}Be^{-a_m B} \leq -\frac{\bar{\delta}}{ea_m}\cdot\frac{\bar{P}T}{1-e^{-\bar{\delta}T}} + \frac{\bar{P}}{ea_m} \\
&\leq \frac{\bar{P}}{ea_m}\left(1 - \frac{\bar{\delta}T}{1-e^{-\bar{\delta}T}}\right) < 0.
\end{aligned}
$$

Therefore,

$$\deg\{QN, \Omega\cap \operatorname{Ker} L, 0\} \neq 0.$$

Therefore, $\Omega = \{x \in X : A < x(t) < B,\ t \in R\}$ also satisfies the requirement (b) in Lemma (2.1). Now that we have shown conditions (a) and (b) in Lemma (2.1), the equation $Lx = Nx$ has at least one solution on $\bar{\Omega}$. Thus the definitions of L and N at the beginning of this section show that Eq.(1.3) has at least one positive T-periodic solution. The proof is complete. □

3. Necessary Condition

In this section, we first give the condition which guarantees that every positive solution of Eq.(1.3) tends to zero as $t \to \infty$, and then derive a necessary condition for the existence of positive periodic solutions of Eq.(1.3).

THEOREM (3.1). *Assume that*

$$P(t) \leq \delta(t), \quad \text{for } t \in [0, T]. \tag{3.2}$$

Then every positive solution of Eq.(1.3) tends to zero as $t \to \infty$.

Proof. Let $x(t)$ be any positive solution of Eq.(1.3). Then $x(t) > 0,\ t \geq 0$. From (1.3), we have

$$\left[x(t)\exp\left(\int_0^t \delta(s)ds\right)\right]' = \exp\left(\int_0^t \delta(s)ds\right) P(t)x(t-\tau(t))e^{-a(t)x(t-\tau(t))},\ t \geq 0.$$

Integrating the above from $t_0 > 0$ to $t > t_0$, we obtain

(3.3)

$$x(t) = x(t_0)\exp\left(-\int_{t_0}^t \delta(s)\,ds\right) + \int_{t_0}^t \exp\left(-\int_s^t \delta(\xi)\,d\xi\right) P(s)x(s-\tau(s))e^{-a(s)x(s-\tau(s))}ds.$$

It follows from (3.2) and (3.3) that

$$x(t) \leq x(t_0)\exp\left(-\int_{t_0}^t \delta(s)ds\right) + \frac{1}{ea_m}\left[1-\exp\left(-\int_{t_0}^t \delta(s)ds\right)\right].$$

Set $v = \limsup_{t\to\infty} x(t)$. Then $0 \leq v < \infty$. To complete the proof, we only need to show that $v = 0$. In what follows, we shall prove that $v = 0$ in three possible cases.

Case 1. $\dot{x}(t) > 0$ eventually. Choose $t_0 > 0$ sufficiently large that $\dot{x}(t) > 0$ for $t \geq t_0$. Then $0 < x(t-\tau(t)) < x(t)$ for $t \geq t_0 + \tau_M$. Hence, by (1.3)

$$0 < -\delta(t)x(t) + P(t)x(t-\tau(t))e^{-a(t)x(t-\tau(t))} < [P(t)-\delta(t)]x(t) \leq 0, \quad t \geq t_0 + \tau_M.$$

This contradiction shows that Case 1 is impossible.

Case 2. $\dot{x}(t)$ is oscillatory. In this case, there exists $\{t_n\}$ with $t_n \uparrow \infty$ such that

$$\dot{x}(t_n) = 0, \quad n = 1, 2, \ldots, \quad \lim_{n\to\infty} x(t_n) = v.$$

Then by (1.3) and (3.2), we have

$$\begin{aligned} x(t_n) &= \frac{P(t_n)}{\delta(t_n)} x(t_n - \tau(t_n))e^{-a(t_n)x(t_n-\tau(t_n))} \\ &\leq x(t_n-\tau(t_n))e^{-a(t_n)x(t_n-\tau(t_n))} \\ &\leq x(t_n-\tau(t_n))e^{-a_m x(t_n-\tau(t_n))}. \end{aligned}$$

Set $w = \limsup_{n\to\infty} x(t_n - \tau(t_n))$. Then $w \leq v$ and from the above, we obtain $v \leq we^{-a_m w}$, which implies that $v = 0$.

Case 3. $\dot{x}(t) < 0$ eventually. Choose $t_0 > 0$ enough large such that $\dot{x}(t) < 0$ for $t \geq t_0 - \tau_M$. Then $v < x(t-\tau(t)) \leq x(t_0 - \tau(t_0))$ for $t \geq t_0$, hence, from (3.2) and (3.3), we have

$$x(t) \leq x(t_0)\exp\left(-\int_{t_0}^t \delta(s)ds\right) + x(t_0-\tau(t_0))e^{-a_m v}\left[1-\exp\left(-\int_{t_0}^t \delta(s)ds\right)\right].$$

Let $t \to \infty$ in the above, we obtain

$$v \leq x(t_0 - \tau(t_0))e^{-a_m v}.$$

Again, let $t_0 \to \infty$ in the above, we have $v \leq ve^{-a_m v}$, which yields $v = 0$. The proof is complete. □

From Theorem (3.1), we have the following necessary condition immediately.

COROLLAY (3.4). *If (3.2) holds, then Eq.(1.1) has no positive T-periodic solutions.*

4. Necessary and Sufficient Conditions

Combining Theorem (2.2) and Corollary (3.4), we have the following results immediately.

THEOREM (4.1). *Assume that* $P(t) = \gamma\delta(t)$ *with* $\gamma > 0$*, then Eq.(1.3) has at least one positive* T*-periodic solution if and only if* $\gamma > 1$.

THEOREM (4.2). *Eq.(1.1) has at least one positive* T*-periodic solution if and only if* $P > \delta$.

We now return to condition (1.2). First note that $1 - e^{-\delta T} < \delta T$. When $1 - e^{-\delta T} < PT \leq \delta T$, every positive solution of Eq.(1.1) tends to zero as $t \to \infty$ by Theorem (3.1), and so Eq.(1.1) has no positive T-periodic solutions. When $1 - e^{-\delta T} < \delta T < PT$, Eq.(1.1) has a positive T-periodic solution, but the condition $PT \leq e(1 - e^{-\delta T})$ can be removed by Theorem (2.2). The above discussions show that (1.2) cannot be a sufficient condition for the existence of positive T-periodic solution of (1.1).

In view of our result, we may see that the condition (1.2) is false. The error can be traced to the incorrect equality (5.3) in [10]:

$$\max_{t\in[0,T]} \int_t^{t+T} \frac{e^{\delta(s-t)}}{e^{\delta T} - 1} Px(s-\tau)e^{-ax(s-\tau)}ds = \frac{e^{\delta T}}{e^{\delta T} - 1} PTr_0 e^{-ar_0}, \tag{4.3}$$

where $\max_{t\in[0,T]} x(t) = r_0 \in (0, \frac{1}{a}]$. In fact, we can only assert that

$$\begin{aligned}\int_t^{t+T} \frac{e^{\delta(s-t)}}{e^{\delta T} - 1} Px(s-\tau)e^{-ax(s-\tau)}ds &\leq Pr_0 e^{-ar_0} \int_t^{t+T} \frac{e^{\delta(s-t)}}{e^{\delta T} - 1} ds \\ &= \frac{P}{\delta} r_0 e^{-ar_0} < \frac{e^{\delta T}}{e^{\delta T} - 1} PTr_0 e^{-ar_0}.\end{aligned}$$

Finally, we remark that the existence of positive periodic solutions has been discussed in [1], [10], [11]. In [11], the following differential equation

$$\dot{x}(t) = -\delta(t)x(t) + P(t)x(t)e^{-ax(t)}, \quad t \geq 0, \tag{4.4}$$

is considered, where a is a positive constant, and δ and P are positive T-periodic functions. However, the condition of the existence of positive T-periodic solutions of Eq.(4.4) obtained in [11], i.e., $P_m > \delta_M$, is much stronger than our condition (2.3). Our results also improve those in [1].

Received January 09, 2006

Final version received November 11, 2006

JING-WEN LI
DEPARTMENT OF MATHEMATICS
SHAOYANG UNIVERSITY
SHAOYANG, HUNAN, 422000
P. R. CHINA
email: ljw@163.com

SUI SUN CHENG
DEPARTMENT OF MATHEMATICS
TSINGN HUA UNIVERSITY

HSINCHU, TAIWAN 30043
R. O. CHINA
email: sscheng@math.nthu.edu.tw

ZHI-YUAN JIANG
DEPARTMENT OF APPLIED MATHEMATICS
CENTRAL SOUTH UNIVERSITY
CHANGSHA, HUNAN, 410083
P. R. CHINA

REFERENCES

[1] Y. CHEN, *Periodic solutions of delayed periodic Nicholson's blowflies models*, Can. Appl. Math. Q. **11** (1) (2003), 23–28.
[2] R. E. GAINES AND J. L. MAWHIN, Coincidence Degree and Non-linear Differential Equations, Springer, Berlin, 1972.
[3] W. S. C. GURNEY, S. P. BLYTHE AND R. M. NISBET, *Nicholson's blowflies revisited*, Nature **287** (1980), 17–21.
[4] I. GYÖRI AND G. LADAS, Oscillation Theory of Delay Differential Equations with Applications, Clarendon press, Oxford, 1991.
[5] G. KARAKOSTAS, CH. G. PHILOS AND Y. G. SFICAS, *Stable steady state of some population models*, J. Dynamics and Diff. Eqs. **4** (1992), 161–190.
[6] V. L. KOCIC AND G. LADAS, *Oscillation and global attractivity in a discrete model of Nicholson's blowflies*, Applicable Analysis, **38** (1990), 21–31.
[7] M. R. S. KULENOVIC, G. LADAS AND Y. G. SFICAS, *Global attractivity in Nicholson's blowflies*, Applicable Analysis **43** (1992), 109–124.
[8] J. W. LI, *Global attractivity in Nicholson's blowflies*, Appl. Math.-JCU **11B** (1996), 425–436.
[9] J. W. LI, *Global attractivity in a discrete model of Nicholson's blowfiles*, Ann. of Diff. Eqs. **12** (1996),173–182.
[10] X. L. LIU AND W. T. LI, *Existence and uniqueness of positive periodic solutions of functional differential equations*, J. Math. Anal. Appl. **293** (2004), 28–39.
[11] S. H. SAKER AND S. AGARWAL, *Oscillation and attractivity in a periodic Nicholson's blowflies model*, Mathl. Comput. Modelling **35** (2002), 719–731.
[12] W. H. SO AND J. S. YU, *Global attractivity and uniform persistence in Nicholson's blowflies*, Diff. Eqs. and Dynamical Sys. **2** (1994), 11–18.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

APPROXIMATE DOUBLE CENTRALIZERS ARE EXACT DOUBLE CENTRALIZERS

MOHAMMAD SAL MOSLEHIAN, FREYDOON RAHBARNIA AND PRASANNA K. SAHOO

ABSTRACT. We establish the generalized stability of double centralizers associated with the Cauchy, Jensen, and Trif functional equations in the framework of Banach algebras. We also investigate the superstability of double centralizers of Banach algebras strongly without order.

1. Introduction and preliminaries

Let $\mathcal{A}$ be an algebra. Recall that $A_l(\mathcal{A}) := \{a \in \mathcal{A} : a\mathcal{A} = \{0\}\}$ is the left annihilator ideal and $A_r(\mathcal{A}) := \{a \in \mathcal{A} : \mathcal{A}a = \{0\}\}$ is the right annihilator ideal of $\mathcal{A}$. Annihilator ideals are $\{0\}$ if $\mathcal{A}$ is semiprime and a fortiori if $\mathcal{A}$ is semisimple. Obviously, these ideals vanish if $\mathcal{A}$ is unital or approximately unital. We say a Banach algebra $\mathcal{A}$ is strongly without order if $A_l(\mathcal{A}) = A_r(\mathcal{A}) = \{0\}$.

A left centralizer of $\mathcal{A}$ is a linear mapping $L : \mathcal{A} \to \mathcal{A}$ such that $L(ab) = L(a)b$ for all $a, b \in \mathcal{A}$. Similarly, a right centralizer of $\mathcal{A}$ is a linear mapping $R : \mathcal{A} \to \mathcal{A}$ such that $R(ab) = aR(b)$ for all $a, b \in \mathcal{A}$. A double centralizer of $\mathcal{A}$ is a pair (L, R) where L is a left centralizer, R is a right centralizer and $aL(b) = R(a)b$ for all $a, b \in \mathcal{A}$. For example, (L_c, R_c) is a double centralizer where $L_c(a) := ca$ and $R_c(a) := ac$. The set $\mathcal{D}(\mathcal{A})$ of all double centralizers equipped with the multiplication $(L_1, R_1) \cdot (L_2, R_2) = (L_1L_2, R_2R_1)$ is an algebra. The notion of double centralizer was introduced by Hochschild [9] and (also, independently) by Johnson [12]. It is not hard to see that $\mathcal{D}(C_0(\mathcal{X})) = C_b(\mathcal{X})$, $\mathcal{D}(K(\mathcal{H})) = B(\mathcal{H})$, $\mathcal{D}(L^1(G)) = M(G)$, where $\mathcal{X}, \mathcal{H}, G$ are a locally compact Hausdorff space, a Hilbert space, and a locally compact group. The importance of the study of double centralizers is that it is unital and contains a copy of $\mathcal{A}$ as an ideal, if the annihilator ideal $\mathrm{Ann}(\mathcal{A}) = A_l(\mathcal{A}) \cap A_r(\mathcal{A})$ is $\{0\}$. Johnson [12] proved that if $\mathcal{A}$ is an algebra satisfying $A_l(\mathcal{A}) = A_r(\mathcal{A}) = \{0\}$, and L and R are (not necessarily linear) maps on $\mathcal{A}$ fulfilling $aL(b) = R(a)b$, $(a, b \in \mathcal{A})$, then (L, R) is a double centralizer. In addition, if $\mathcal{A}$ is a Banach algebra then L and R are automatically continuous.

It is easy to see that if $\mathcal{A}^2 = \mathcal{A}$ or $\mathrm{Ann}(\mathcal{A}) = \{0\}$, then $L = R$ if and only if L and R are both left and right centralizers, or equivalently (L, R) belongs to the center of $\mathcal{D}(\mathcal{A})$.

An operator $T : \mathcal{A} \to \mathcal{A}$ is said to be a multiplier (see [17]) if $aT(b) = T(a)b$ for all $a, b \in \mathcal{A}$. Clearly, if $A_l(\mathcal{A}) = \{0\}$ ($A_r(\mathcal{A}) = \{0\}$, respectively) then T is a left (right, respectively) centralizer. Multipliers were first studied by Helgeson [8]

2000 *Mathematics Subject Classification*: Primary 39B52; Secondary 39B82, 47B48.
Keywords and phrases: Stability, superstability, double centralizer, ψ-approximate double centralizer, multiplier, strongly without order.

and then investigated on Banach algebras by Wang [28]. One may be referred to [21] for more information on double centralizers and multipliers.

We say a functional equation is *stable* if any function satisfying that functional equation "approximately" is near to a true solution of that functional equation. The functional equation is called *superstable* if every approximate solution is an exact solution of it (see [4] for another notion of superstability which may be called *superstability modulo the bounded functions*; cf. [11]).

In 1940, Ulam [27] posed the first stability problem concerning the stability of group homomorphisms. In the next year, Hyers [10] gave a partial affirmative answer to the question of Ulam in the context of Banach spaces. In 1950, Aoki [2] generalized Hyers' theorem for approximate additive mappins. In 1978, the generalized Hyers' theorem was independently rediscovered by Th.M. Rassias [23] by obtaining a unique linear mapping under certain continuity assumption; see also [20].

THEOREM (1.1) (Rassias' Theorem). *Suppose that E_1 and E_2 are real normed spaces with E_2 complete and $f : E_1 \to E_2$ is a mapping such that for each fixed $x \in E_1$ the mapping $t \mapsto f(tx)$ is continuous on $\mathbb{R}$. Let there exist $\varepsilon \geq 0$ and $p \in [0, 1)$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon \left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in E_1$. Then there exists a unique linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{|2-2^p|} \|x\|^p$$

for all $x \in E_1$.

This result is still valid in the case where $p < 0$ if we assume that $\|0\|^p = \infty$. In 1990, Th. M Rassias during the 27th International Symposium on Functional Equations asked the question whether the above theorem can be proved for $p \geq 1$. In 1991, Gajda [6] provided an affirmative solution to this question for $p > 1$. It is known that there is no analogue of above result for $p = 1$ (see [6, 25]). In 1994, further generalization was obtained by Găvruta [7]. During the last decades several stability problems of functional equations have been investigated by many mathematicians; cf. [5, 11, 14, 24].

In [18], the stability of multipliers was investigated. In this paper, using some ideas from [3, 13, 18], we establish the generalized stability of double centralizers associated with the Cauchy, Jensen, and Trif functional equations. We introduce the notion of ψ-approximate double centralizer and prove the superstability of double centralizers of Banach algebras strongly without order.

Among others, we generalize the results of [18] in several directions. First, we use a general control function. Second, we investigate double centralizers as a generalization of multipliers, and finally we prove the additivity of our mappings without any additional condition such as approximate additivity.

2. Stability of double centralizers associated to the Cauchy equation

Throughout this section, $\mathcal{A}$ denotes a Banach algebra. Our aim is to establish the generalized stability of double centralizers associated to the additive Cauchy functional equation $f(a+b) = f(a) + f(b)$.

THEOREM (2.1). *Suppose $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : \mathcal{A} \to \mathcal{A}$ with $g(0) = 0$ and functions $\varphi_j : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ $(1 \le j \le 2)$ and $\psi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\widetilde{\varphi}_j(a, b, c, d) := \frac{1}{2} \sum_{n=0}^{\infty} 2^{-n} \varphi_j(2^n a, 2^n b, 2^n c, 2^n d) < \infty \quad (1 \le j \le 2), \tag{2.2}$$

$$\lim_{n\to\infty} 2^{-n} \psi(2^n a, 2^n b) = 0,$$

$$\|f(\lambda a + b + cd) - \lambda f(a) - f(b) - f(c)d\| \le \varphi_1(a, b, c, d), \tag{2.3}$$

$$\|g(\lambda a + b + cd) - \lambda g(a) - g(b) - cg(d)\| \le \varphi_2(a, b, c, d),$$

$$\|af(b) - g(a)b\| \le \psi(a, b) \tag{2.4}$$

for all $\lambda \in \mathbb{T} := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ and $a, b, c, d \in \mathcal{A}$. Then there exists a unique double centralizer (L, R) of $\mathcal{A}$ satisfying

$$\|f(a) - L(a)\| \le \widetilde{\varphi_1}(a, a, 0, 0), \tag{2.5}$$

$$\|g(a) - R(a)\| \le \widetilde{\varphi_2}(a, a, 0, 0)$$

for all $a \in \mathcal{A}$.

Proof. Setting $a = b$, $c = d = 0$ and $\lambda = 1$ in (2.3), we have

$$\|f(2a) - 2f(a)\| \le \varphi_1(a, a, 0, 0)$$

for all $a \in \mathcal{A}$. One can use induction to show that

$$\left\| \frac{f(2^n a)}{2^n} - \frac{f(2^m a)}{2^m} \right\| \le \frac{1}{2} \sum_{k=m}^{n-1} 2^{-k} \varphi_1(2^k a, 2^k a, 0, 0) \tag{2.6}$$

for all $n > m \ge 0$ and $a \in \mathcal{A}$. It follows from (2.6) and (2.2) that the sequence $\left\{ \frac{f(2^n a)}{2^n} \right\}$ is Cauchy. Due to the completeness of the Banach algebra $\mathcal{A}$, this sequence is convergent. Define

$$L(a) := \lim_{n\to\infty} \frac{f(2^n a)}{2^n}. \tag{2.7}$$

Putting $c = d = 0$ and replacing a and b by $2^n a$ and $2^n b$, respectively, in (2.3), we get $\|2^{-n} f(2^n(\lambda a + b)) - \lambda 2^{-n} f(2^n a) - 2^{-n} f(2^n b)\| \le 2^{-n} \varphi_1(2^n a, 2^n b, 0, 0)$. Taking the limit as $n \to \infty$, we obtain

$$L(\lambda a + b) = \lambda L(a) + L(b) \tag{2.8}$$

for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{T}$.

In a similar manner, from (2.3), we deduce that $L(ab) = L(a)b$.

Next, let $\gamma = \theta_1 + \mathbf{i}\theta_2 \in \mathbb{C}$ where $\theta_1, \theta_2 \in \mathbb{R}$. Let $\gamma_1 = \theta_1 - [\theta_1]$ and $\gamma_2 = \theta_2 - [\theta_2]$, where $[\theta]$ denotes the integer part of θ. Then $0 \le \gamma_i < 1$ $(1 \le i \le 2)$. One can represent γ_i as $\gamma_i = \frac{\lambda_{i,1} + \lambda_{i,2}}{2}$ such that $\lambda_{i,j} \in \mathbb{T}$ $(1 \le i, j \le 2)$. Since L satisfies (2.8) we infer that

$$
\begin{aligned}
L(\gamma x) &= L(\theta_1 x) + \mathbf{i} L(\theta_2 x) \\
&= \Big([\theta_1] L(x) + L(\gamma_1 x) \Big) + \mathbf{i} \Big([\theta_2] L(x) + L(\gamma_2 x) \Big) \\
&= \Big([\theta_1] L(x) + \frac{1}{2} L(\lambda_{1,1} x + \lambda_{1,2} x) \Big) + \mathbf{i} \Big([\theta_2] L(x) + \frac{1}{2} L(\lambda_{2,1} x + \lambda_{2,2} x) \Big) \\
&= \Big([\theta_1] L(x) + \frac{1}{2} \lambda_{1,1} L(x) + \frac{1}{2} \lambda_{1,2} L(x) \Big) + \mathbf{i} \Big([\theta_2] L(x) + \frac{1}{2} \lambda_{2,1} L(x) + \frac{1}{2} \lambda_{2,2} L(x) \Big) \\
&= \theta_1 L(x) + \mathbf{i} \theta_2 L(x) \\
&= \gamma L(x)
\end{aligned}
$$

for all $x \in \mathcal{A}$. Hence L is $\mathbb{C}$-linear and so it is a left centralizer of $\mathcal{A}$. Moreover, it follows from (2.6) with $m = 0$ and (2.7) that $\|L(a) - f(a)\| \leq \widetilde{\varphi_1}(a, a, 0, 0)$ for all $a \in \mathcal{A}$. It is well known that the additive mapping L satisfying (2.5) is unique (see [3] or [19]).

A similar argument gives us a unique right centralizer R defined by

$$R(a) := \lim_{n \to \infty} \frac{g(2^n a)}{2^n}$$

with the required property.

Replacing a and b by $2^n a$ and $2^n b$, respectively, in (2.4) and dividing the both sides of the obtained inequality by 4^n we get

$$\left\| a\, 2^{-n} f(2^n b) - 2^{-n} g(2^n a)\, b \right\| \leq 4^{-n} \psi(2^n a,\, 2^n b).$$

Passing to the limit as $n \to \infty$, we conclude that $aL(b) = R(a)b$ for all a, $b \in \mathcal{A}$. $\square$

Using the same method as in the proof of Theorem (2.1) one can prove the following theorem.

THEOREM (2.9). *Suppose $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : \mathcal{A} \to \mathcal{A}$ with $g(0) = 0$ and functions $\varphi_j : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \to [0, \infty)$ $(1 \leq j \leq 2)$ and $\psi : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$
\begin{gathered}
\widetilde{\varphi}_j(a, b, c, d) := \frac{1}{2} \sum_{n=1}^{\infty} 2^n \varphi_j(2^{-n} a, 2^{-n} b, 2^{-n} c, 2^{-n} d) < \infty \qquad (1 \leq j \leq 2), \\
\lim_{n \to \infty} 2^n \psi(2^{-n} a, 2^{-n} b) = 0, \\
\|f(\lambda a + b + cd) - \lambda f(a) - f(b) - f(c) d\| \leq \varphi_1(a, b, c, d), \\
\|g(\lambda a + b + cd) - \lambda g(a) - g(b) - c g(d)\| \leq \varphi_2(a, b, c, d), \\
\|a f(b) - g(a) b\| \leq \psi(a, b)
\end{gathered}
$$

for all $\lambda \in \mathbb{T}$ and $a, b, c, d \in \mathcal{A}$. Then there exists a unique double centralizer (L, R) of $\mathcal{A}$ satisfying

$$
\begin{aligned}
\|f(a) - L(a)\| &\leq \widetilde{\varphi_1}(a, a, 0, 0), \\
\|g(a) - R(a)\| &\leq \widetilde{\varphi_2}(a, a, 0, 0)
\end{aligned}
$$

for all $a \in \mathcal{A}$.

COROLLARY (2.10). *Suppose $f : \mathcal{A} \to \mathcal{A}$ is a mapping for which there exist a mapping $g : \mathcal{A} \to \mathcal{A}$ and constants $\epsilon > 0$, and $0 \leq p \neq 1$*

$$\|f(\lambda a + b + cd) - \lambda f(a) - f(b) - f(c)d\| \leq \epsilon(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p),$$
$$\|g(\lambda a + b + cd) - \lambda g(a) - g(b) - cg(d)\| \leq \epsilon(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p),$$
$$\|af(b) - g(a)b\| \leq \epsilon(\|a\|^p + \|b\|^p)$$

for all $\lambda \in \mathbb{T}$ and $a, b, c, d \in \mathcal{A}$. Then there exists a unique double centralizer (L, R) of $\mathcal{A}$ satisfying

$$\|f(a) - L(a)\| \leq \frac{\epsilon\|a\|^p}{|1 - 2^{p-1}|},$$
$$\|g(a) - R(a)\| \leq \frac{\epsilon\|a\|^p}{|1 - 2^{p-1}|}$$

for all $a \in \mathcal{A}$.

Proof. For $j = 1, 2$, put $\varphi_j(a, b, c, d) = \epsilon(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p)$ and $\psi(a, b) = \epsilon(\|a\|^p + \|b\|^p)$ in Theorems (2.1) and (2.9). □

3. Stability of double centralizers associated to the Jensen equation

Stability of the Jensen equation $2f\left(\frac{a+b}{2}\right) = f(a) + f(b)$ has been studied first by Kominek [16] and then by several other mathematicians; see [13, 15] and references therein. In this section, we study the generalized stability of double centralizers associated to the Jensen equation on the punched space $\mathcal{A}$.

THEOREM (3.1). *Suppose $\mathcal{A}$ is a Banach algebra, $f : \mathcal{A} \to \mathcal{A}$ is a mapping with $f(0) = 0$ for which there exist a mapping $g : \mathcal{A} \to \mathcal{A}$ with $g(0) = 0$ and functions $\varphi_j : (\mathcal{A} - \{0\}) \times (\mathcal{A} - \{0\}) \to [0, \infty)$ $(1 \leq j \leq 2)$ and $\psi_j : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ $(1 \leq j \leq 3)$ such that*

$$\widetilde{\varphi}_j(a, b) := \sum_{n=0}^{\infty} 3^{-n} \varphi_j(3^n a, 3^n b) < \infty \qquad (1 \leq j \leq 2), \tag{3.2}$$

$$\lim_{n \to \infty} 3^{-n} \psi_j(3^n a, 3^n b) = 0 \qquad (1 \leq j \leq 3),$$

$$\left\|2f\left(\frac{\lambda a + \lambda b}{2}\right) - \lambda f(a) - \lambda f(b)\right\| \leq \varphi_1(a, b) \qquad (\lambda \in \mathbb{T}, a, b \in \mathcal{A} - \{0\}), \tag{3.3}$$

$$\left\|2g\left(\frac{\lambda a + \lambda b}{2}\right) - \lambda g(a) - \lambda g(b)\right\| \leq \varphi_2(a, b) \qquad (\lambda \in \mathbb{T}, a, b \in \mathcal{A} - \{0\}),$$

$$\|af(b) - g(a)b\| \leq \psi_1(a, b) \qquad (a, b \in \mathcal{A}),$$
$$\|f(ab) - f(a)b\| \leq \psi_2(a, b) \qquad (a, b \in \mathcal{A}),$$
$$\|g(ab) - ag(b)\| \leq \psi_3(a, b) \qquad (a, b \in \mathcal{A}).$$

Then there exists a unique double centralizer (L, R) of $\mathcal{A}$ satisfying

$$\|f(a) - L(a)\| \leq \frac{1}{3}\left(\widetilde{\varphi_1}(a, -a) + \widetilde{\varphi_1}(-a, 3a)\right), \tag{3.4}$$

$$\|g(a) - R(a)\| \leq \frac{1}{3}\left(\widetilde{\varphi_2}(a, -a) + \widetilde{\varphi_1}(-a, 3a)\right)$$

for all $a \in \mathcal{A}$.

Proof. Letting $\lambda = 1$ and $b = -a$ in (3.3), we get

$$\| -f(a) - f(-a)\| \le \varphi_1(a, -a)$$

for all $a \in \mathcal{A}$. Letting $\lambda = 1$ and replacing b by $3a$ and a by $-a$ in (3.3), we get

$$\|2f(a) - f(-a) - f(3a)\| \le \varphi_1(-a,\ 3a)$$

for all $a \in \mathcal{A}$. Thus

$$\begin{aligned}\left\|f(a) - \frac{1}{3}f(3a)\right\| &\le \frac{1}{3}(\ \|f(a) + f(-a)\| + \|2f(a) - f(-a) - f(3a)\|)\\ &\le \frac{1}{3}(\varphi_1(a, -a) + \varphi_1(-a, 3a))\end{aligned}$$

for all $a \in \mathcal{A}$. So

(3.5)

$$\begin{aligned}\left\|\frac{1}{3^n}f(3^n a) - \frac{1}{3^m}f(3^m a)\right\| &\le \sum_{j=m}^{n-1}\left\|\frac{1}{3^j}f(3^j a) - \frac{1}{3^{j+1}}f(3^{j+1}a)\right\|\\ &\le \frac{1}{3}\sum_{j=m}^{n-1}3^{-j}(\varphi_1(3^j a, 3^j(-a)) + \varphi_1(3^j(-a), 3^j(3a))\end{aligned}$$

for all nonnegative integers m, n with $n > m$ and all $a \in \mathcal{A}$. It follows from (3.2) and (3.5) that the sequence $\left\{\frac{1}{3^n}f(3^n a)\right\}$ is a Cauchy sequence for all $a \in \mathcal{A}$. Since $\mathcal{A}$ is complete, the sequence $\left\{\frac{1}{3^n}f(3^n a)\right\}$ is convergent. So one can define the mapping $L : \mathcal{A} \to \mathcal{A}$ by

$$L(a) := \lim_{n\to\infty}\frac{1}{3^n}f(3^n a)$$

for all $a \in \mathcal{A}$. By (3.3), we have

$$\begin{aligned}\left\|2L\left(\frac{a+b}{2}\right) - L(a) - L(b)\right\| &= \lim_{n\to\infty}\frac{1}{3^n}\left\|2f\left(3^n\frac{a+b}{2}\right) - f(3^n a) - f(3^n b)\right\|\\ &\le \lim_{n\to\infty} 3^{-n}\,\varphi_1(3^n a, 3^n b)\\ &= 0\end{aligned}$$

for all $a, b \in \mathcal{A}$. Thus

$$2L\left(\frac{a+b}{2}\right) = L(a) + L(b)$$

for all $a, b \in \mathcal{A}$. Since $f(0) = 0$, we have $L(0) = 0$. Hence $2\,L(\frac{a}{2}) = L(a)$ for each $a \in \mathcal{A}$ and therefore $L(a) + L(b) = 2L\left(\frac{a+b}{2}\right) = L(a+b)$ for all $a, b \in \mathcal{A}$.

Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (3.5), we get (3.4).

Let $\lambda \in \mathbb{T}$, and replacing both a and b in (3.3) by $3^n a$ and dividing the both sides of the obtained inequality by 3^{-n}, we get

$$\|3^{-n}\,f(\lambda 3^n a) - \lambda\,3^{-n}\,f(3^n a)\| \le \frac{3^{-n}}{2}\,\varphi_1(3^n a,\ 3^n a).$$

Passing to the limit as n tends to infinity, we get $L(\lambda a) = \lambda L(a)$. Similarly, one can find a right centralizer R. Now the same argument as in the proof of

Theorem (2.1) yields that (L, R) is a double centralizer of the Banach algebra $\mathcal{A}$. The uniqueness can be proved in a standard fashion. □

Remark (3.6). There is a result similar to Theorem (3.1) in which the control functions φ_j and ψ_j satisfy $\sum_{n=0}^{\infty} 3^n \varphi_j(3^{-n}a, 3^{-n}b) < \infty$ $(1 \le j \le 2)$ and $\lim_{n\to\infty} 3^n \psi_j(3^{-n}a, 3^{-n}b) = 0$ $(1 \le j \le 3)$ (see, e.g. [13]).

COROLLARY (3.7). *Suppose $\mathcal{A}$ is a Banach algebra, $f : \mathcal{A} \to \mathcal{A}$ is a mapping for which there exist a mapping $g : \mathcal{A} \to \mathcal{A}$, nonnegative constants δ, γ, p, q with $p < 1$ and $q < \frac{1}{2}$ such that*

$$\left\| 2f\left(\frac{\lambda a + \lambda b}{2}\right) - \lambda f(a) - \lambda f(b) \right\| \le \delta \, \|a\|^q \, \|b\|^q \quad (\lambda \in \mathbb{T}, a, b \in \mathcal{A} - \{0\}),$$

$$\left\| 2g\left(\frac{\lambda a + \lambda b}{2}\right) - \lambda g(a) - \lambda \, g(b) \right\| \le \delta \|a\|^q \, \|b\|^q \quad (\lambda \in \mathbb{T}, a, b \in \mathcal{A} - \{0\}),$$

$$\begin{aligned} \|af(b) - g(a)b\| &\le \gamma(\|a\|^p + \|b\|^p), \\ \|f(ab) - af(b)\| &\le \gamma(\|a\|^p + \|b\|^p), \\ \|g(ab) - g(a)b\| &\le \gamma(\|a\|^p + \|b\|^p) \end{aligned}$$

for $\lambda = 1, \mathbf{i}$ and for all $a, b \in \mathcal{A}$. Assume that for every fixed $a \in \mathcal{A}$, there is a positive number r_a such that the real functions $t \mapsto \|f(ta)\|$ and $t \mapsto \|g(ta)\|$ are bounded on the interval $[0, r_a]$. Then there exists a unique double centralizer (L, R) of $\mathcal{A}$ satisfying

$$\begin{aligned} \|f(a) - L(a)\| &\le \frac{(1 + 3^q)\delta \|a\|^{2q}}{3 - 3^{2q}}, \\ \|g(a) - R(a)\| &\le \frac{(1 + 3^q)\delta \|a\|^{2q}}{3 - 3^{2q}} \end{aligned}$$

for all $a \in \mathcal{A}$.

Proof. One may use the same argument as in the proof of Theorem (3.1). The only thing one needs to prove is the homogeneous property of the additive mappings L and R, namely $L(\mathbf{i}a) = \mathbf{i}L(a)$ and $R(\mathbf{i}a) = \mathbf{i}R(a)$.

First fix $a \in \mathcal{A}$ and F in the dual $\mathcal{A}^*$ of $\mathcal{A}$ and define the additive function $\Gamma : \mathbb{R} \to \mathbb{R}$ by $\Gamma(t) = F(L(ta))$. Then the function Γ is bounded on $[0, r_a]$ since

$$\begin{aligned} |\Gamma(t)| &\le \|F\| \, \|L(ta)\| \\ &\le \|F\| \left(\|L(ta) - f(ta)\| + \|f(ta)\| \right) \\ &\le \|F\| \left(\frac{(1 + 3^q)\delta \|ta\|^2}{|3 - 3^{2q}|} + \sup\{\|f(ta)\| : t \in [0, r_a]\} \right) \\ &\le \|F\| \left(\frac{(1 + 3^q) r_a^{2q} \delta \|a\|^{2q}}{|3 - 3^{2q}|} + \sup\{\|f(ta)\| : t \in [0, r_a]\} \right). \end{aligned}$$

It follows from Corollary 2.5 of [1] that $\Gamma(t) = \Gamma(1)t$ for all real numbers t. Hence $F(L(ta)) = F(tL(a))$ for all $t \in \mathbb{R}$ and $F \in \mathcal{A}^*$. Therefore $L(ta) = tL(a)$.

Now, for each complex number $\lambda = u + \mathbf{i}v$ and each $a \in \mathcal{A}$, we have

$$L(\lambda a) = L(ua + \mathbf{i}va) = L(ua) + L(\mathbf{i}va) = uL(a) + \mathbf{i}vL(a) = \lambda L(a).$$

Similarly, one can prove that R is homogeneous. $\square$

4. Stability of double centralizers associated to the Trif equation

T. Trif [26] proved the generalized stability for the so-called Trif functional equation

$$sC_{s-2}^{l-2} f\left(\frac{a_1 + \cdots + a_s}{s}\right) + C_{s-2}^{l-1} \sum_{j=1}^{s} f(a_j) = \sum_{1 \le j_1 < \cdots < j_l \le s} l\, f\left(\frac{a_{j_1} + \cdots + a_{j_l}}{l}\right)$$

where C_r^k denotes $\frac{r!}{k!(r-k)!}$. This functional equation was derived by Trif [26] from an inequality of Popoviciu [22] for convex functions. In this section, we study generalized stability of double centralizers associated to the Trif equation. Let $q = \frac{l(s-1)}{s-l}$ and $r = -\frac{l}{s-l}$ for positive integers l, s with $2 \le l \le s-1$.

THEOREM (4.1). *Let $\mathcal{A}$ be a Banach algebra, $f : \mathcal{A} \to \mathcal{A}$ be a mapping with $f(0) = 0$ for which there exist a mapping $g : \mathcal{A} \to \mathcal{A}$ and functions $\varphi_j : \mathcal{A}^{s+2} \to [0, \infty)$ $(1 \le j \le 2)$ and $\mathcal{A} \times \mathcal{A} \to [0, \infty)$ such that*

$$\widetilde{\varphi_j}(a_1, \cdots, a_s, c, d) := \sum_{j=0}^{\infty} q^{-j} \varphi_j(q^j a_1, \cdots, q^j a_s, q^j c, q^j d) < \infty \quad (1 \le j \le 2),$$

$$\lim_{n\to\infty} 2^{-n} \psi(2^n a, 2^n b) = 0,$$

$$\begin{aligned}(4.2)\qquad &\left\| sC_{s-2}^{l-2} f\left(\frac{\lambda a_1 + \cdots + \lambda a_s}{s} + \frac{cd}{s \cdot C_{s-2}^{l-2}}\right) + C_{s-2}^{l-1} \sum_{j=1}^{s} \lambda f(a_j)\right.\\ &\left. - l \sum_{1 \le j_1 < \cdots < j_l \le s} \lambda f\left(\frac{a_{j_1} + \cdots + a_{j_l}}{l}\right) - f(c)d \right\| \le \varphi_j(a_1, \cdots, a_s, c, d),\\ &\left\| sC_{s-2}^{l-2} g\left(\frac{\lambda a_1 + \cdots + \lambda a_s}{d} + \frac{cd}{s \cdot C_{s-2}^{l-2}}\right) + C_{s-2}^{l-1} \sum_{j=1}^{s} \lambda g(a_j)\right.\\ &\left. - l \sum_{1 \le j_1 < \cdots < j_l \le s} \lambda g\left(\frac{a_{j_1} + \cdots + a_{j_l}}{l}\right) - cg(d) \right\| \le \varphi_j(a_1, \cdots, a_s, c, d),\end{aligned}$$

and

$$\|af(b) - g(a)b\| \le \psi(a, b)$$

for all $\lambda \in \mathbb{T}$ and $a_1, \cdots, a_s, a, b, c, d \in \mathcal{A}$. Then there exists a unique double centralizer (L, R) of $\mathcal{A}$ satisfying

$$\begin{aligned}(4.3)\qquad &\|f(a) - L(a)\| \le \frac{1}{l\, C_{s-1}^{l-1}} \widetilde{\varphi}(qa, ra, \cdots, ra, 0, 0),\\ &\|g(a) - R(a)\| \le \frac{1}{l\, C_{s-1}^{l-1}} \widetilde{\varphi}(qa, ra, \cdots, ra, 0, 0)\end{aligned}$$

for all $a \in \mathcal{A}$.

Proof. Set $c = d = 0$ and $\lambda = 1$ in (4.2). It follows from Trif's Theorem [26] there exists a unique additive mapping L defined by $L(a) := \lim_{n\to\infty} \frac{1}{q^n} f(q^n a)$ such that (4.3) holds for all $a \in \mathcal{A}$.

Let $\lambda \in \mathbb{T}$. Put $a_1 = \cdots = a_s = a$ and $c = d = 0$ in (4.2) to obtain

$$\|sC_{s-2}^{l-2}(f(\lambda a) - \lambda f(a))\| \le \varphi(a, \cdots, a, 0, 0)$$

for all $a \in \mathcal{A}$. Therefore

$$q^{-n}\|s\,C_{s-2}^{l-2}(f(\lambda q^n a) - \lambda f(q^n a))\| \le q^{-n}\varphi(q^n a, \cdots, q^n a, 0, 0)$$

for all $a \in \mathcal{A}$. Since the right hand side tends to zero as $n \to \infty$, we have

$$\left\|q^{-n} f(\lambda q^n a) - \lambda q^{-n} f(q^n a)\right\| \to 0$$

as $n \to \infty$ for all $\lambda \in \mathbb{T}$ and $a \in \mathcal{A}$. Hence

$$L(\lambda a) = \lim_{n\to\infty} \frac{f(q^n \lambda a)}{q^n} = \lim_{n\to\infty} \frac{\lambda f(q^n a)}{q^n} = \lambda L(a)$$

for all $\lambda \in \mathbb{T}$ and $a \in \mathcal{A}$. Obviously, $L(0a) = 0 = 0L(a)$.

Using the same argument as in the proof of Theorem (2.1), one can conclude that L is homogeneous.

Putting $\lambda = 1$ and $a_1 = \cdots = a_s = 0$, and replacing c, d by $q^n c, q^n d$, respectively, in (4.2), we get

$$\frac{1}{q^{2n}}\left\|s\,C_{s-2}^{l-2} f\left(\frac{q^{2n}}{s\cdot C_{s-2}^{l-2}} cd\right) - f(q^n c) q^n d\right\| \le \frac{1}{q^{2n}}\varphi(0, \cdots, 0, q^n c, q^n d)$$

for all $c, d \in \mathcal{A}$. Then

$$\begin{aligned} L(cd) &= s\,C_{s-2}^{l-2} L\left(\frac{1}{s\,C_{s-2}^{l-2}}\, c\, d\right) \\ &= \lim_{n\to\infty} \frac{s\,C_{s-2}^{l-2}}{q^{2n}} f\left(\frac{q^{2n}}{s\,C_{s-2}^{l-2}}\, c\, d\right) \\ &= \lim_{n\to\infty} \frac{f(q^n\, c)}{q^n}\, d \\ &= L(c)\, d, \end{aligned}$$

for all $c, d \in \mathcal{A}$. Therefore L is a left centralizer. Similarly, one can find a right centralizer R. By the same reasoning as the above, one can show that (L, R) is the required unique double centralizer. $\square$

Remark (4.4). There is a result similar to Theorem (4.1) in which the role of q^n and q^{-n} are switched (see, e.g., [26]).

5. Superstability of double centralizers

In this section, we prove the superstability of double centralizers of Banach algebras which are strongly without order. More precisely, we introduce the concept of ψ-approximate double centralizer and show that any ψ-approximate double centralizer is an exact double centralizer. Thus we generalize the result of Johnson [12] (see the introduction) and extend the results of [18].

Definition (5.1). Suppose $\mathcal{A}$ is a normed algebra and $L, R : \mathcal{A} \to \mathcal{A}$ are mappings for which there exist a positive number r and a function $\psi : \mathcal{A} \times \mathcal{A}$ satisfying either

$$\lim_{n\to\infty} r^{-n}\, \psi(r^n a,\ b) = \lim_{n\to\infty} r^{-n}\, \psi(a,\ r^n b) = 0 \qquad (a, b \in \mathcal{A}) \tag{5.2}$$

or

$$\lim_{n\to\infty} r^{n}\, \psi(r^{-n} a,\ b) = \lim_{n\to\infty} r^{n}\, \psi(a,\ r^{-n} b) = 0 \qquad (a, b \in \mathcal{A}) \tag{5.3}$$

such that

$$\|a\, L(b) - R(a)\, b\| \leq \psi(a,\ b)$$

for all $a, b \in \mathcal{A}$. Then (L, R) is called a ψ-approximate double centralizer of $\mathcal{A}$.

THEOREM (5.4). *Let $\mathcal{A}$ be a Banach algebra strongly without order. Then any ψ-approximate double centralizer (L, R) of $\mathcal{A}$ is an exact double centralizer.*

Proof. We assume that (5.2) holds. The proof in the case where (5.3) holds is similar. Let $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. We have

$$\begin{aligned}
\|b(L(\lambda a) - \lambda L(a))\| &\leq r^{-n}\, \|r^n bL(\lambda a) - \lambda r^n bL(a))\| \\
&\leq r^{-n}\|r^n bL(\lambda a) - R(r^n b)\lambda a\| + r^{-n}\|\lambda R(r^n b)a - \lambda r^n bL(a)\| \\
&\leq r^{-n}\psi(r^n b, \lambda a) + r^{-n}|\lambda|\psi(r^n b, a).
\end{aligned}$$

By (5.2), the right hand side of the last inequality tends to zero as $n \to \infty$, so $b(L(\lambda a) - \lambda L(a)) = 0$. Since $\mathcal{A}$ is strongly without order we conclude that $L(\lambda a) = \lambda L(a)$. The additivity of L follows from

$$\begin{aligned}
\|c(L(a+b) - L(a) - L(b))\| &\leq r^{-n}\, \|r^n cL(a+b) - R(r^n c)(a+b)\| \\
&\quad + r^{-n}\, \|r^n cL(a) - R(r^n c)a\| \\
&\quad + r^{-n}\, \|r^n cL(b) - R(r^n c)b\| \\
&\leq r^{-n}\psi(r^n c, a+b) + r^{-n}\psi(r^n c, a) + r^{-n}\psi(r^n c, b).
\end{aligned}$$

Finally

$$\begin{aligned}
\|c(L(ab) - L(a)b)\| &\leq r^{-n}\, \|r^n cL(ab) - R(r^n c)ab\| + r^{-n}\|(r^n cL(a) - R(r^n c)a)b\| \\
&\leq r^{-n}\, \psi(r^n c,\ ab) + r^{-n}\, \|b\|\, \psi(r^n c, a)
\end{aligned}$$

yields that $L(ab) = L(a)b$ for all $a, b \in \mathcal{A}$. Thus L is a left centralizer. One can similarly prove that R is a right centralizer. Since L is homogeneous, $r^{-n}L(r^n a) = L(a)$ for all $a \in \mathcal{A}$ and $n \in \mathbb{N}$, therefore

$$\|aL(b) - R(a)b\| = r^{-n}\|aL(r^n b) - R(a)r^n b\| \leq r^{-n}\psi(a, r^n b)$$

and hence, by (5.2), we infer that $aL(b) = R(a)b$ for all $a, b \in \mathcal{A}$. Thus (L, R) is a double centralizer. □

COROLLARY (5.5). *Suppose $\mathcal{A}$ is a Banach algebra strongly without order, $L, R : \mathcal{A} \to \mathcal{A}$ are mappings for which there exist nonnegative numbers ϵ, δ and real numbers p_1, p_2, q_1, q_2 either all of which are greater than 1 or all of which are less than 1, such that*

$$\|aL(b) - R(a)b\| \leq \epsilon\,(\|a\|^{p_1} + \|b\|^{p_2}) + \delta\|a\|^{q_1}\ \|b\|^{q_2}$$

for all $a, b \in \mathcal{A}$. Then (L, R) is a double centralizer of $\mathcal{A}$.

Proof. Use Theorem (5.4) with $\psi(a,b) = \epsilon\,(\|a\|^{p_1} + \|b\|^{p_2}) + \delta\,\|a\|^{q_1}\,\|b\|^{q_2}$. $\square$

Aknowledgement

The authors would like to thank the referees for their valuable suggestions.

Received February 28, 2006

Final version received January 26, 2007

MOHAMMAD SAL MOSLEHIAN
FREYDOON RAHBARNIA
DEPARTMENT OF MATHEMATICS
FERDOWSI UNIVERSITY OF MASHHAD
P. O. BOX 1159
MASHHAD 91775
IRAN
moslehian@ferdowsi.um.ac.ir AND moslehian@ams.org
rahbarnia@ferdowsi.um.ac.ir

PRASANNA K. SAHOO
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOUISVILLE
LOUISVILLE
KENTUCKY 40292
USA
sahoo@louisville.edu

REFERENCES

[1] J. ACZÉL AND J. DHOMBRES, Functional Equations in Several Variables, Encyclopedia of Mathematics and its Applications, 31. Cambridge University Press, Cambridge, 1989.

[2] T. AOKI, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950) 64–66.

[3] C. BAAK AND M. S. MOSLEHIAN, *Stability of J^*-homomorphisms*, Nonlinear Anal.–TMA **63** (2005), 42–48.

[4] J. BAKER, *The stability of the cosine equation*, Proc. Amer. Math. Soc. **74** (1979), 242–246.

[5] S. CZERWIK (ed.), Stability of Functional Equations of Ulam–Hyers–Rassias Type, Hadronic Press Inc., Palm Harbor, Florida, 2003.

[6] Z. GAJDA, *On stability of additive mappings*, Int. J. Math. Math. Sci. **14** (1991), 431–434.

[7] P. GĂVRUTA, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[8] S. HELGASON, *Multipliers of Banach algebras*, Ann. of Math. **64** (1956), 240–254.

[9] G. HOCHSCHILD, *Cohomology and representations of associative algebras*, Duke Math. J. **14** (1947). 921–948.

[10] D. H. HYERS, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[11] D. H. HYERS, G. ISAC AND TH. M. RASSIAS, Stability of Functional Equations in Several Variables, Birkhäuser, Basel, 1998.

[12] B. E. JOHNSON, *An introduction to the theory of centralizers*, Proc. London Math. Soc. **14** (1964), 299–320.

[13] K.-W. JUN AND H.-M. KIM, *Remarks on the stability of additive functional equation*, Bull. Korean Math. Soc. **38** (4), (2001), 679–687.

[14] S.-M. JUNG, Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press lnc. Palm Harbor, Florida, 2001.

[15] S.-M. JUNG, M. S. MOSLEHIAN AND P. K. SAHOO, *Stability of generalized Jensen equation on restricted domains*, preprint.
[16] Z. KOMINEK, *On a local stability of the Jensen functional equation*, Demonstratio Math. **22** (1989), 499–507.
[17] R. LARSEN, Introduction to the Theory of Multipliers, Springer-Verlag, 1971.
[18] T. MIURA, G. HIRASAWA AND S.-E. TAKAHASI, *Stability of multipliers on Banach algebras* Int. J. Math. Math. Sci. (2004), no. 45-48, 2377–2381.
[19] M. S. MOSLEHIAN, *Approximately vanishing of topological cohomology groups*, J. Math. Anal. Appl. **318** (2), (2006), 758–771.
[20] M. S. MOSLEHIAN AND TH. M. RASSIAS, *Stability of functional equations in non-Archimedean spaces*, Appl. Anal. Disc. Math. 1, (2007) No. 2, 325–334
[21] T. W. PALMER, Banach Algebras and the General Theory of $*$-Algebras, Vol. I. Algebras and Banach Algebras, Encyclopedia of Mathematics and its Applications 49, Cambridge University Press, Cambridge, 1994.
[22] T. POPOVICIU, *Sur certaines inégalités qui caractérisent les fonctions convexes*, An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. **11** (1965) 155û-164.
[23] TH. M. RASSIAS, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
[24] TH. M. RASSIAS (ed.), Functional Equations, Inequalities and Applications, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.
[25] TH. M. RASSIAS AND P. ŠEMRL, *On the behavior of mappings which do not satisfy Hyers–Ulam stability*, Proc. Amer. Math. Soc. **114** (1992), 989–993.
[26] T. TRIF, *On the stability of a functional equation deriving from an inequality of Popoviciu for convex functions*, J. Math. Anal. Appl. **272** (2002), 604–616.
[27] S. M. ULAM, Problems in Modern Mathematics, Chapter VI, Science Editions, Wiley, New York, 1964.
[28] J.-K. WANG, *Multipliers of commutative Banach algebras*, Pacific J. Math. **11** (1961), 1131–1149.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

AN ERGODIC PROPERTY OF AMENABLE HYPERGROUPS

LILIANA PAVEL

ABSTRACT. Let K be a hypergroup with Haar measure. It is known that similarly to the group case, the left (topological) amenability is equivalent to the right (topological) stationarity. Based on this fact we give a characterization of the amenability of hypergroups by an ergodic property which is a variant of Reiter-Glicksberg properties from the group case.

1. Introduction

Hypergroups are locally compact spaces whose bounded Radon measures form an algebra which has properties similar to the convolution measure algebra of a locally compact group. A hypergroup can be viewed as a probabilistic group in the sense that for each pair $x, y \in K$ there exists a probability measure $\delta_x * \delta_y$ on K with compact support, such that $(x, y) \mapsto \operatorname{supp} \delta_x * \delta_y$ is a continuous mapping from $K \times K$ into the space of compact subsets of K. Unlike the groups, $\delta_x * \delta_y$ is not in general a point measure. The substantial development of the theory of hypergroups with the works of Dunkl [2], Spector [14] and Jewett [7] put hypergroups in the right setting for harmonic analysis. In our approach the hypergroup possesses a Haar measure. We notice that it is still unknown if an arbitrary hypergroup admits a Haar measure, but all the known examples, such as commutative hypergroups, compact hypergroups, discrete hypergroups, central hypergroups do. As hypergroups generalize locally compact groups, many basic notions from harmonic analysis on groups carry over to hypergroups. In [13], Skantharajah translating literally the notion of amenabilty from groups to hypergroups, has developed a systematic study of amenable hypergroups, following the main directions from the group case.

In this paper we will give a characterization of the amenability of hypergroups by an ergodic property which can be seen as a variant of Reiter-Glicksberg properties from the locally compact groups case. Our approach is based on the equivalence between hypergroup amenability and hypergroup stationarity dicussed in [10] and avoids translating from the group case the usual techniques connected to this sort of characterizations of the amenability (see various approaches of ergodic properties of amenable locally compact groups for example in [12], [5], [11]).

2. Preliminaries

K always stands for a hypergroup with a fixed left Haar measure m with modular function Δ, symbols like $\int \ldots dx$ will always denote the integration

2000 *Mathematics Subject Classification*: Primary 43A62; Secondary 43A07.
Keywords and phrases: hypergroup, amenability, stationarity.

with respect to m. The notation generally agrees with [7]. However the following notations are different from [7]: $x \mapsto x^\vee$denotes the involution on K and δ_x the Dirac measure concentrated at x. We recall that $\mathcal{M}(K)$ is the algebra of all bounded regular (complex valued) Borel measures on K. In addition we use the notation $E(K)$ for the set of Dirac measures on K. For A a subset in a linear space of functions or measures on K, $\operatorname{co} A$ will denote the convex hull of A.

If f is a Borel function on K and $x, y \in K$ the left translate f_x and the right translate f^y are defined by

$$f_x(y) = f^y(x) = \int f d\delta_x * \delta_y = f(x * y)$$

if the integral exists. The function $f^\vee$ is given by $f^\vee(x) = f(x^\vee)$. If $\mu \in M(K)$, and f is a Borel function, then the convolutions $\mu * f$ and $f * \mu$ are defined on K by

$$(\mu * f)(x) = \int f(y^\vee * x) d\mu(y) \qquad \text{and} \qquad (f * \mu)(x) = \int f(x * y^\vee) d\mu(y).$$

It is immediate that $\delta_{x^\vee} * f = f_x$ and $f * \delta_x = f^x$.

Convolution of two functions f and g on K is given by

$$(f * g)(x) = \int f(x * y) g(y^\vee) dy.$$

The spaces $(L_p(K), \|\cdot\|_p)$, $p \in [1, \infty]$ are defined in the usual way with respect to the Haar measure of K (see for example [3], Ch. 6). If $f \in L_p(K)$, $1 \leq p \leq \infty$, $x \in K$, we denote $f \odot \delta_x = f^{x^\vee} \Delta(x^\vee)$. If $\mu \in \operatorname{co} E(K)$, we naturally extend, by linearity, the previous notation,

$$f \odot \mu = \sum_{i=1}^{n} \alpha_i f \odot \delta_{x_i} \;\text{ where }\; \mu = \sum_{i=1}^{n} \alpha_i \delta_{x_i}.$$

It is known that $\delta_{x^\vee} * f$ and $f \odot \delta_x \in L_p(K)$ and that $\|\delta_{x^\vee} * f\|_p \leq \|f\|_p$ and $\|f \odot \delta_x\|_p \leq \|f\|_p$ (or, more $\|f \odot \mu\|_p \leq \|f\|_p$, if $\mu \in \operatorname{co} E(K)$). As it was noticed in [7], 3.3, these are in general not isometries. However, as each $f \in L_p(K)$ takes only complex values (so is finite) by [7], (3.3F), it follows that

$$\int f_x(y) dy = \int f^{x^\vee} \Delta(x^\vee) dy = \int f(y) dy.$$

In our approach we will be interested only in the spaces $(L_1(K), \|\cdot\|_1)$ and $(L_\infty(K), \|\cdot\|_\infty)$. Identifying $L_\infty(K)$ to $L_1^*(K)$ (whenever this is possible, for example, asking to m to be σ-finite [3], Theorem 6.15) we will consider also the weak*-topology, ω^* ($\omega^* = \sigma(L_\infty(K), L_1(K))$) on $L_\infty(K) = L_1^*(K)$. We will denote by $P(K) = \{\varphi \in L_1(K) | \varphi \geq 0, \|\varphi\|_1 = 1\}$. It is known (see for example [10], Proposition 3.3) that for $f \in L_\infty(K)$, the ω^*-closure of the sets $\operatorname{co}\{f^x | x \in K\}$ and $\operatorname{co}\{f_x | x \in K\}$ coincides with the ω^*-closure of the sets $\{f * \varphi^\vee | \varphi \in P(K)\}$ and $\{\varphi * f | \varphi \in P(K)\}$ respectively.

LEMMA (2.1). *For any $f \in L_1(K)$, $\varphi \in P(K)$*

$$\int_K (f * \varphi)(x) dx = \int_K f(x) dx.$$

Proof. It is enough to prove the equality only for $f \in L_1(K)$, $f \geq 0$. Since f, $\varphi \in L_1(K)$, using ([7], (5.5K) and (6.1E)) we have

$$\int_K (f * \varphi)(x)dx = [(f * \varphi)m](K) = (fm * \varphi m)(K)$$
$$= (fm)(K) \cdot (\varphi m)(K) = \left(\int_K f(x)dx\right)\left(\int_K \varphi(x)dx\right) = \int_K f(x)dx.$$

□

With the definitions of the operations "$*$" and "$\odot$" and of the modular function the next result is clear:

LEMMA (2.2). *Let $\theta \in L_\infty(K)$, f, $g \in L_1(K)$, $x \in K$. Then*

$$\int_K \theta(x)(f * g)(x)dx = \int_K (\theta * g^\vee)(x)f(x)dx$$

and

$$\int_K \theta(y)(f \odot \delta_x)(y)dy = \int_K \theta^x(y)f(y)dy.$$

Let us recapitulate the basic notions and facts regarding the amenability and stationarity of hypergroups. A hypergroup K is called (left) amenable if there exists a left invariant mean M on $L_\infty(K)$. It is known that K is (left) amenable if and only if K is topologically (left) amenable (that is $M(\varphi * f) = M(f)$, $\forall \varphi \in P(K)$). The hypergroup K is (right) stationary if for each $f \in L_\infty(K)$ there exists $\alpha \in \mathbb{R}$ such that $\alpha\mathbf{1}$ is in the ω^*-closure (in $L_\infty(K)$) of the set $\mathrm{co}\{f^x | x \in K\}$ (and topologically (right) stationary if $\alpha\mathbf{1}$ is in the ω^*-closure of the set $\{f * \varphi^\vee | \varphi \in P(K)\}$). We denote by $\mathbf{1}$ the real function on K, $\mathbf{1}(x) = 1$, $\forall x \in K$. It is known ([10], Theorem 4.4) that just as in the semigroup and group case (see [9] and [15] respectively) the (left) (topological) amenability is equivalent to the (right) (topological) stationarity. It is also proved (see [13], Theorem 4.1) that the amenabilty for hypergroups is characterized by Reiter's condition (P_1), which can be formulated as follows: there exists a net $(\varphi_\iota)_\iota \subseteq P(K)$ such that $\|\varphi * \varphi_\iota - \varphi_\iota\|_1 \longrightarrow 0$ for each $\varphi \in P(K)$.

3. Results

THEOREM (3.1). *Let K be a right stationary hypergroup. Then, for each $f \in L_1(K)$,*

$$|\int_K f(x)dx| = \inf\{\|f * \varphi\|_1 | \varphi \in P(K)\}.$$

Proof. Take $f \in L_1(K)$. We may suppose that $\|f\|_1 \neq 0$, otherwise, the equalities are obvious. Let φ be arbitrary in $P(K)$. Using Lemma (2.1),

$$\|f * \varphi\|_1 = \int_K |(f * \varphi)(x)|dx \geq |\int_K (f * \varphi)(x)dx| = |\int_K f(x)dx|.$$

Let us denote by $a = \inf\{\|f * \varphi\|_1 | \varphi \in P(K)\}$, so as $\|f\|_1 \neq 0$ it follows that $a \neq 0$. We have just obtained that

$$a \geq |\int_K f(x)dx|.$$

Further, our arguments are based on the Hahn-Banach Separation Theorem: we adapt to our approach techniques which are familiar in the semigroup case while investigating various kinds of ergodic properties ([4] and [16]). Consider the norm closure in $L_1(K)$ of the convex set $A_f = \{f * \varphi | \varphi \in P(K)\}$. By the Hahn-Banach Separation Theorem ([1], V. 2.8) it results that there exists $F \in L_1(K)^*$ such that $\|F\| = 1$ and $|F(g)| \geq a$, $\forall g \in A_f$. As $L_1(K)^* = L_\infty(K)$, we infer that there exists $\theta \in L_\infty(K)$, $\|\theta\|_\infty = 1$ such that

$$|\int_K \theta(x)(f * \varphi)(x)dx| \geq a, \forall \varphi \in P(K).$$

Applying Lemma (2.2),

$$\int_K \theta(x)(f * \varphi)(x)dx = \int_K (\theta * \varphi^\vee)(x)f(x)dx\,, \forall\, \varphi \in P(K),$$

consequently,

$$|\int_K (\theta * \varphi^\vee)(x)f(x)dx| \geq a\,,\ \forall\, \varphi \in P(K).$$

Since K is right stationary, K is also topologically right stationary [10], Proposition 3.3, so there exists $\alpha \in \mathbb{R}$, such that

$$\alpha\mathbf{1} \in \overline{\{\theta * \varphi^\vee | \varphi \in P(K)\}}^{\omega^*}.$$

It follows that

$$|\int_K \alpha\mathbf{1}(x)f(x)dx| \geq a.$$

Since $\|\theta\|_\infty = 1$, clearly, $|\alpha| \leq 1$.

On the other hand, for any $g = f * \varphi \in A_f$,

$$\begin{aligned}\int_K \alpha\mathbf{1}(x)g(x)dx &= \int_K \alpha\mathbf{1}(x)(f * \varphi)(x)dx = \alpha\int_K (f * \varphi)(x)\,dx \\ &= \alpha\int_K f(x)dx = \int_K \alpha\mathbf{1}(x)f(x)dx.\end{aligned}$$

It results that

$$a \leq |\int_K \alpha\mathbf{1}(x)f(x)dx| = |\int_K \alpha\mathbf{1}(x)(f * \varphi)(x)dx| \leq |\alpha|\|f * \varphi\|_1, \quad \forall \varphi \in P(K),$$

thus $a \leq |\alpha| \cdot \inf\{\|f * \varphi\|_1 | \varphi \in P(K)\} = |\alpha| \cdot a$ and, consequently, $|\alpha| \geq 1$. We infer that $|\alpha| = 1$, so

$$a \leq |\int_K f(x)dx|.$$

The theorem is proven. □

Remark. With almost the same proof one can show also that

$$|\int_K f(x)dx| = \inf\{\|f \odot \mu\|_1 | \mu \in \operatorname{co} E(K)\}.$$

Indeed, we first notice that for $\mu \in \operatorname{co} E(K)$, $\mu = \sum_{i=1}^n \alpha_i\delta_{x_i}$ and $f \in L_1(K)$,

$$\|f \odot \mu\|_1 = \int_K |(f \odot \mu)(x)|dx \geq |\int_K (f \odot \mu)(x)dx|$$

$$= |\sum_{i=1}^{n} \alpha_i \int_K (f \odot \delta_{x_i})(x)| = |\left(\sum_{i=1}^{n} \alpha_i\right) \int_K f(x)dx| = |\int_K f(x)dx|,$$

and, consequently, $\inf\{\|f \odot \mu\|_1 | \mu \in \operatorname{co} E(K)\} \geq |\int_K f(x)dx|$. Further, making the same type of judgement as in the proof of the above theorem for the convex set $\{f \odot \mu | \mu \in \operatorname{co} E(K)\}$ instead of the set A_f, we infer that there exists $\theta \in L_\infty(K)$, $\|\theta\|_\infty = 1$ such that

$$|\int_K \theta(x)(f \odot \mu)(x)dx| \geq a\,, \forall\, \mu \in \operatorname{co} E(K).$$

Here a denotes $\inf\{\|f \odot \mu\|_1 | \mu \in \operatorname{co} E(K)\}$. Applying the second formula from Lemma (2.2) it follows that

$$\int_K \theta(x)(f \odot \mu)(x)dx = \int_K \left(\sum_{i=1}^{n} \alpha_i \theta^{x_i}(x)\right) f(x)dx,$$

where μ arbitrary in $\operatorname{co} E(K)$, $\mu = \sum\limits_{i=1}^{n} \alpha_i \delta_{x_i}$. Consequently we have that

$$|\int_K \left(\sum_{i=1}^{n} \alpha_i \theta^{x_i}(x)\right) f(x)dx| \geq a.$$

Since K is right stationary, there exists $\alpha \in \mathbb{R}$, such that $\alpha \mathbf{1} \in \overline{\operatorname{co}\{\theta^x | x \in K\}}^{\omega^*}$. From this point everything follows identically as in the proof of the Theorem (3.1).

THEOREM (3.2). *Let K be a hypergroup such that*

$$|\int_K f(x)dx| = \inf\{\|f * \varphi\|_1 | \varphi \in P(K)\}, \quad \forall\, f \in L_1(K).$$

Then, there exists a net $(\varphi_\iota)_\iota \subseteq P(K)$ such that $\|\varphi * \varphi_\iota - \varphi_\iota\|_1 \longrightarrow 0$, for each $\varphi \in P(K)$.

Proof. The proof follows the same idea as in the locally compact group case [5], Theorem 3.7.3, working with functions in $P(K)$ instead of convex combinations of Dirac measures. The main tool which makes it possible is the fact that if $f \in L_1(K)$ and

$$\int_K f(x)dx = 0,$$

then, as it follows from Lemma (2.1),

$$\int_K f * \varphi(x)dx = 0\,,\ \forall\, \varphi \in P(K)\,.$$

For the sake of completeness, we give here the complete proof. Let $\varphi \in P(K)$ be arbitary fixed. Consider the family $\Lambda = \{\lambda\}$ where $\lambda = (\varphi_1, \varphi_2, \ldots, \varphi_n; \varepsilon)$, where $\varphi_k \in P(K)$, $n \in \mathbb{N}$ and $\varepsilon > 0$ partially ordered by $\lambda \preceq \lambda'$ if and only if $\{\varphi_1, \varphi_2, \ldots, \varphi_n\} \subseteq \{\varphi'_1, \varphi'_2, \ldots, \varphi'_{n'}\}$ and $\varepsilon \leq \varepsilon'$. By Lemma (2.1) for each $\lambda = (\varphi_1, \varphi_2, ..., \varphi_n; \varepsilon)$ we have that

$$\int_K ((\varphi_k * \varphi)(x) - \varphi(x))dx = 0\,,\ \forall\, k = 1, 2, \ldots, n.$$

Using the hypothesis for $f = \varphi_1 * \varphi - \varphi$ we infer that there exists $\gamma_1 \in P(K)$ such that $\|(\varphi_1 * \varphi - \varphi) * \gamma_1\|_1 < \varepsilon$. One can continue in the same way for $f = (\varphi_2 * \varphi - \varphi) * \gamma_1$, so there exists $\gamma_2 \in P(K)$ such that $\|(\varphi_2 * \varphi - \varphi) * \gamma_1 * \gamma_2\|_1 < \varepsilon$. Proceeding inductively, there exists $\gamma_k \in P(K)$ such that

$$\|(\varphi_k * \varphi - \varphi) * \gamma_1 * \gamma_2 * \cdots * \gamma_k\|_1 < \varepsilon, \text{ with } k = 1, 2, \ldots, n.$$

Put $\gamma_\lambda = \gamma_1 * \gamma_2 * \cdots * \gamma_n$ and define $\varphi_\lambda = \varphi * \gamma_\lambda$. As we have that

$$\begin{aligned}\|(\varphi_k * \varphi - \varphi) * \gamma_\lambda\|_1 &= \|(\varphi_k * \varphi - \varphi) * \gamma_1 * \gamma_2 * \cdots * \gamma_n\|_1 \\ &\leq \|(\varphi_k * \varphi - \varphi) * \gamma_1 * \gamma_2 * \cdots * \gamma_k\|_1 \|\gamma_{k+1} * \gamma_2 * \cdots * \gamma_n\|_1 < \varepsilon,\end{aligned}$$

$\forall\, k = 1, 2, \ldots, n$, it follows that for each $\psi \in P(K)$, $\|\psi * \varphi_\lambda - \varphi_\lambda\|_1 \longrightarrow 0$. □

Combining the two above results with the characterization of the amenability by stationarity and by Reiter's condition (P_1) we have the next theorem:

THEOREM (3.3). *K is (left) amenable if and only if for each $f \in L_1(K)$,*

$$\left|\int_K f(x)dx\right| = \inf\{\|f * \varphi\|_1 | \varphi \in P(K)\}.$$

Remark. In [13] various classes of amenable hypergroups were exhibited. For example all commutative hypergroups, compact hypergroups, central hypergroups are proven to be amenable. Consequently, all our results hold for any hypergroup of this kind.

Acknowledgement

It is a pleasure to thank the referee for his useful remarks.

Received July 27, 2006

Final version received February 12, 2007

FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF BUCHAREST
ACADEMIEI 14
010014 BUCHAREST
ROMANIA
lpavel@fmi.unibuc.ro

REFERENCES

[1] N. DUNFORD AND J. SCHWARTZ, Linear operators (part I), Interscience, New York, 1958.

[2] C. F. DUNKL, *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc. **179** (1973), 331–348.

[3] G. B. FOLLAND, Real analysis, Wiley & Sons, Inc., New York, 1984.

[4] I. GLICKSBERG, *On convex hulls of translates*, Pacific J. Math. **13** (1963), 97–113.

[5] F. P. GREENLEAF, Invariant means on topological groups, Van Nostrand, New York, 1969.

[6] E. HEWITT AND K.A. ROSS, Abstract Harmonic analysis (vol. I), Springer, Berlin-Heidelberg-New York, 1963.

[7] R. I. JEWETT, *Spaces with an abstract convolution of measures*, Adv. Math. **18** (1975), 1–101.

[8] R. LASSER, *Orthogonal polynomials and hypergroups*, Rend. Mat. **3** (1983), 185–209.

[9] T. MITCHELL, *Constant functions and left invariant means on semigroups*, Trans. Amer. Math. Soc. **119** (1965), 244–261.

[10] L. PAVEL, *Stationary hypergroups and amenability*, Rend. Circ. Mat. Palermo Serie II, Suppl. **52** (1998), 695–705.

[11] J. P. PIER, Amenable locally compact groups, Wiley & Sons, Inc., New York, 1984.
[12] H. REITER, Classical harmonic analysis and locally compact groups, Oxford University Press, Oxford, 1968.
[13] M. SKANTHARAJAH, *Amenable hypergroups*, Illinois J. Math. **36** (1992), 15–46.
[14] R. SPECTOR, *Aperçu de la théorie des hypergroupes*, Lect. Notes Math. **497**, pp. 643-673, Springer, Berlin-Heidelberg-New York, 1975.
[15] J. C. S. WONG, *Topologically stationary locally compact groups and amenability*, Trans. Amer. Math. Soc. **144** (1969), 351–363.
[16] J. C. S. WONG, *An ergodic property of amenable semigroups*, Pacific J. Math. **48** (1973), 615–619.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

AN ANALYTIC RADON-NIKODYM PROPERTY RELATED TO SYSTEMS OF VECTOR-VALUED CONJUGATE HARMONIC FUNCTIONS AND CLIFFORD ANALYSIS

EMILIO MARMOLEJO-OLEA AND SALVADOR PÉREZ-ESTEVA

ABSTRACT. The purpose of this paper is to study the existence of boundary limits of systems of conjugate harmonic functions defined in the unit ball in $\mathbb{R}^n$ and with values in a real Banach space E. We approach this problem using the language of Clifford Analysis and consider Hardy spaces in the unit ball of $\mathbb{R}^n$ of monogenic functions with values in a Banach Clifford module. In terms of the so called *Monogenic Measures* on the sphere, we define a Monogenic Radon-Nikodym property which is linked with the existence of radial limits of vector-valued monogenic functions as in the holomorphic case. For Banach lattices we adapt the proof by A.V. Bukhvalov and A.A. Danilevich to show that for any real Banach lattice E, the Clifford module $X = A_n \otimes E$ has the Monogenic Radon-Nikodym property (A_n is the Clifford algebra) if and only c_0 is not a subspace of E, which is equivalent to the Analytic Radon-Nikodym property of $E_{\mathbb{C}}$.

1. Introduction

An analytic measure with values in a complex Banach space X is an *X-valued* Borel measure μ of bounded variation in the unit circle with Fourier coefficients $\widehat{\mu}(n) = \int_{S^1} e^{-in\theta} d\mu(\theta) = 0$ for every $n < 0$. The theorem of F. Riesz and M. Riesz asserts that every analytic measure is absolutely continuous with respect to the Lebesgue measure in S^1. The space X has the Analytic Radon-Nikodym Property ($X \in (RN)_a$) if every analytic measure has a density in the Bochner space $L^1_X(S^1)$.

There is a strong relation between the Analytic Radon-Nikodym Property of X and the existence of boundary limits of *X-valued* holomorphic functions belonging to Hardy spaces $H^p_X(D)$ in the disk. These issues have been extensively studied by several authors (see for example [2], [3], [6], [11]). The main result is that $X \in (RN)_a$ if and only if every function in $H^p_X(D)$ has radial (non tangential) limits almost everywhere in S^1 for every $p \in [1, \infty]$ and this is equivalent to the same statement for a single value of p.

Z. Chen and C. Ouyang extended this result in [7], to *X-valued* Hardy spaces on several complex variables in the unit ball of $\mathbb{C}^n$.

A natural substitute for holomorphy in harmonic analysis is to consider Stein-Weiss systems of conjugate harmonic functions. The motivation of this paper is to explore the boundary limits of these systems of harmonic functions

---

2000 *Mathematics Subject Classification*: Primary 46E15, 46E22; Secondary 42B30, 46B42.

Keywords and phrases: Radon-Nikodym property, Hardy spaces, monogenic function, Dirac operator.

Partially supported by PAPIIT-UNAM IN-101303.

defined in the unit ball of $\mathbb{R}^n$ and with values in a real Banach space E. We will approach this problem using the language of Clifford Analysis. We will extend the theory of monogenic Hardy spaces in the unit ball of $\mathbb{R}^n$ (see [12], [15]) to consider monogenic functions with values in a Banach Clifford module. This includes the conjugate systems as a particular case. We will state and prove a version of the theorem of F. Riesz and M. Riesz in this setting. Then we will define a Monogenic Radon-Nikodym property $(RN)_m$ and we will link this property with the existence of radial limits of vector-valued monogenic functions as in the holomorphic case. We present examples of spaces with and without $(RN)_m$. In Section 4 we study the relation between $(RN)_a$ and $(RN)_m$ for Banach lattices. We will adapt the proof by A.V. Bukhvalov and A.A. Danilevich to show that for a Banach lattice E, the module $X = A_n \otimes E \in (RN)_m$ if and only c_0 is not a subspace of E, which is equivalent to $E + iE \in (RN)_a$ as proved in [6]. In particular we have for Banach lattices that $(RN)_m$ is independent of the dimension of $\mathbb{R}^n$.

2. Preliminaries

Throughout this paper B and S^n will denote respectively the unit ball and the sphere of radius one in $\mathbb{R}^{n+1}$. The normalized Lebesgue measure in the sphere S^n will be denoted by σ. For a real or complex Banach space X, $M_X(S^n)$ will be the space of all the Borel measures on S^n of bounded variation with values in X . For $p > 0$, we will denote by $L^p_X(S^n)$ the space of Bochner measurable X-valued functions f in S^n such that $\int_{S^n} \|f(\eta)\|^p \, d\sigma(\eta) < \infty$. If $p \geq 1$ then $\int_{S^n} f(\eta) d\sigma(\eta)$ will denote the Bochner integral of f (see [10] for details of vector-valued measures and integration). By c_0 we will denote the standard space of real vanishing sequences.

Next we mention basic facts of Clifford Analysis used in this paper. For detailed expositions, the reader is refereed to [1], [9], [12], [15].

We consider the *real* 2^n dimensional Clifford algebra A_n which is defined as the minimal enlargement of $\mathbb{R}^n$ to a unitary algebra not generated by any proper subspace of $\mathbb{R}^n$ with the property that $x^2 = -|x|^2$, for any $x \in \mathbb{R}^n$. In particular if $e_1, \ldots, e_n$ is any orthonormal basis for $\mathbb{R}^n$. Then A_n is defined by the anti-commutation relationship $e_i e_j = -e_j e_i$, $i \neq j$ and $e_i^2 = -1$, $i = 1, 2, \ldots n$.

The elements of the algebra A_n have a unique representation of the form

$$a = \sum_\alpha e_\alpha a_\alpha,$$

where $a_\alpha \in \mathbb{R}$ and where we identify e_α with $e_{j_1} \cdots e_{j_r}$ for $\alpha = \{j_1, \ldots, j_r\} \subset \{1, 2, \ldots, n\}$, $(j_i < j_{i+1})$ and $e_\emptyset$ with $e_0 = 1$. The scalar part of a is defined by $Re(a) = a_0$. We give the natural Euclidean metric to A_n as

$$|a| = (\sum_\alpha a_\alpha^2)^{1/2}.$$

The Clifford conjugation on A_n is defined as the unique real lineal involution with $\overline{e_\alpha} e_\alpha = e_\alpha \overline{e_\alpha} = 1$ for all α. Thus for $a \in A_n$ as above

$$\overline{a} = \sum_\alpha \overline{e_\alpha} a_\alpha,$$

with $\overline{e_\alpha} = (-1)^{|\alpha|(|\alpha|+1)/2} e_\alpha$ and where the length of α is given by $|\alpha| = \sum_i j_i$.

We can also embed $\mathbb{R}^{n+1}$ into A_n by identifying $(x_0, x) \in \mathbb{R} \oplus \mathbb{R}^n = \mathbb{R}^{n+1}$, $x = (x_1, \ldots, x_n)$ with $x_0 + x = \sum_{i=0}^n x_i e_i \in A_n$. It follows that every nonzero $x \in \mathbb{R}^{n+1}$ is invertible with inverse $x^{-1} = \frac{\overline{x}}{|x|^2}$. Observe that $A_1 = \mathbb{C}$, and $A_2 = \mathbb{H}$, the quaternionic division algebra.

Note that for $x \in \mathbb{R}^{n+1}$ and $a, b \in A_n$ we have $|x|^2 = x\overline{x}$ and $|xa| = |x||a|$, but in general $|a|^2 \neq a\overline{a}$ and $|ab| \neq |a||b|$; however $|ab| \leq 2^n |a||b|$.

Recall the *Dirac operator* as the differential operator

$$D = \sum_{i=0}^{n} e_i \frac{\partial}{\partial x_i}.$$

acting on A_n-valued functions F with differentiable components defined in a domain in $\mathbb{R}^{n+1}$.

Definition (2.1). We say that $F = \sum_\alpha e_\alpha F_\alpha$ is *left monogenic* or simply monogenic on a region V of $\mathbb{R}^{n+1}$ if

$$DF = \sum_{i=0}^{n} e_i \frac{\partial F}{\partial x_i} = \sum_{i=0}^{n} \sum_{\alpha} e_i e_\alpha \frac{\partial F_\alpha}{\partial x_i} = 0.$$

An important property of the Dirac operator is that the Laplacian in $\mathbb{R}^{n+1}$ can be factored as $D\overline{D} = \triangle$, hence, each component of a left monogenic function is a harmonic function. Let us also recall the Cauchy transform $\mathcal{C}$ defined for the boundary of a fixed smooth domain Ω by

$$\mathcal{C}f(x) = \int_{\partial\Omega} G(y - x) n(y) f(y) d\sigma(y), \;\; x \in \mathbb{R}^{n+1} \setminus \{\partial\Omega\},$$

where

$$G(x) = \frac{\overline{x}}{|x|^{n+1}}$$

is the Cauchy kernel and f is a Clifford valued function. Here $d\sigma(y)$ denotes the Lebesgue measure on $\partial\Omega$, $n(y)$ stand for the outward unit normal and the integrands are interpreted in the sense of Clifford algebra multiplication.

We say that X is a left Banach A_n*-module* if X is a left A_n module and X is also a real Banach space such that for any $a \in A_n$ and $x \in X$

$$\|ax\|_X \leq \kappa |a| \|x\|_X, \tag{2.2}$$

for some $\kappa > 0$. Similarly one can define a right Banach A_n*-module*.

An important example of a left Banach A_n*-module* can be constructed as follows: if $(E, \| \, \|_E)$ is a real Banach space, then

$$X = A_n \otimes E = \{\sum_\alpha e_\alpha x_\alpha \colon x_\alpha \in E\}$$

is a left Banach A_n*-module* with norm

$$\| \sum_\alpha e_\alpha x_\alpha \|_X^2 = \sum_\alpha \|x_\alpha\|_E^2$$

and with the natural left product

$$ax = \sum_{\alpha,\beta} e_\alpha e_\beta a_\alpha x_\beta,$$

for all $a = \sum_\alpha e_\alpha a_\alpha \in A_n$ and $x = \sum_\beta e_\beta x_\beta \in X$. Clearly we could define with the obvious modifications a right Banach A_n*-module* $X = E \otimes A_n$.

X_l^* will denote the space of all bounded left A_n linear functionals. A function $\ell\colon X \to A_n$ belongs to X_l^* *if* it is $\mathbb{R}-$*linear* and $\ell(ax) = a\ell(x)$ for all $a \in A_n$ and $x \in X$. Notice that X_l^* is a right Banach A_n*-module* when provided with norm of $\mathcal{L}(X, A_n)$, namely $\|\ell\| = sup_{\|x\|\leq 1}|\ell(x)|$.

It will be convenient to consider the X_l^*-*dual* norm in X. We have this useful and simple result.

Lemma (2.3). *Let X be a left Banach A_n-module. Consider the dual norm of $x \in X$*

$$\|x\|_d = \sup\{|\ell(x)| : \ell \in X_l^*, \|\ell\| \leq 1\}.$$

Then $\|\cdot\|_d$ is equivalent to $\|\cdot\|$.

Proof. We clearly have that $\|x\|_d \leq \|x\|$. Let x be a nonzero vector in X and let ℓ_0 be in the $\mathbb{R}$-dual space X^* of X with norm one such that $\ell_0(x) = \|x\|$. If we let

$$\ell(y) = \frac{1}{\kappa 2^n} \sum_\alpha \overline{e_\alpha}\ell_0(e_\alpha y),\ y \in X,$$

then $\ell \in X_l^*$ and $\|\ell\| \leq 1$ by (2.2). Moreover $|\ell(x)| \geq |\mathrm{Re}\,\ell(x)| = (\kappa 2^n)^{-1}\|x\|$. It follows that $\|x\|_d \geq (\kappa 2^n)^{-1}\|x\|$. □

If X is a left Banach A_n*-module,* we can extend the Definition (2.1) to include monogenic functions $F\colon V \subset \mathbb{R}^{n+1} \to X$.

In the case $X = A_n \otimes E$, important examples of monogenic functions come from Stein-Weiss systems of conjugate harmonic functions:

Let V be an open region in $\mathbb{R}^{n+1}$. For $i = 0, \ldots, n$, let $u_i\colon V \to E$. We say that $\{u_i\}_{i=0..n}$ is a system of conjugate harmonic functions if

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i},\ i \neq j,$$
$$\sum_{i=0}^{n} \frac{\partial u_i}{\partial x_i} = 0.$$

A family $\{u_i\}_{i=0,\ldots,n}$ is a system of conjugate harmonic functions in V if and only if $F = -u_0 + \sum_{i=1}^n e_i u_i$ is a monogenic function in V.

Lemma (2.4). *A function $F\colon V \subset \mathbb{R}^{n+1} \to X$ is monogenic if and only if $\ell \circ F$ is a monogenic function for every $\ell \in X_l^*$.*

Proof. If F is monogenic in V then $\ell \circ F$ is clearly a monogenic function for every $\ell \in X_l^*$. Conversely, suppose that this last condition holds. Let $\ell_0 \in X^*$ and $\ell \in X_l^*$ defined by $\ell(y) = \frac{1}{\kappa 2^n}\sum_\alpha \overline{e_\alpha}\ell_0(e_\alpha y),\ y \in X$. Then the fact that $\ell \circ F$ is monogenic implies that $\ell_0 \circ F \in C^\infty(V)$ for every $\ell_0 \in X^*$. Then $F \in C^\infty(V)$ (see [18] for example). Proceeding as in the proof of Lemma (2.3) we see that X_l^* separates points from X. Then since $0 = D(\ell \circ F) = \ell(DF)$ for every $\ell \in X_l^*$, it follows that $DF = 0$. □

With obvious changes we may carry over the theory for right modules and right monogenic functions. We remark that classical results such as Cauchy theorems remain valid in the context of X-valued monogenic functions (see [15] for example).

(2.1) Spaces of surface spherical harmonics $\mathcal{H}_k$, $\mathcal{M}_k^+$, $\mathcal{M}_k^-$**.** $\mathcal{H}_k$ will denote the space of surface spherical harmonics in S^n of degree k with values in A_n. We can decompose (see [1]) $\mathcal{H}_k = \mathcal{M}_k^+ \oplus \mathcal{M}_{k-1}^-$, where the spaces $\mathcal{M}_k^+$ and $\mathcal{M}_k^-$, called respectively inner and outer spherical monogenics of order k are defined as follows: $\mathcal{M}_k^+$ consists of the restrictions to S^n of all the monogenic homogeneous polynomials of degree k, and $\mathcal{M}_k^-$ is defined as the space of restrictions to S^n of all the homogeneous monogenic functions of order $-(k+n)$ in $\mathbb{R}^{n+1}\setminus\{0\}$. The spaces are orthogonal in the standard inner product in $L^2(S^n, A_n)$, namely,

$$(f, g) = \int_{S^n} f(\xi)\bar{g}(\xi)d\sigma(\xi).$$

We have orthogonal projections

$$\begin{aligned}\Pi_k &: L^2(S^n, A_n) \to \mathcal{H}_k,\\ P_k &: L^2(S^n, A_n) \to \mathcal{M}_k^+,\\ Q_k &: L^2(S^n, A_n) \to \mathcal{M}_k^-,\end{aligned}$$

and

$$\Pi_k = P_k + Q_{k-1}.$$

Let $Z_k(\xi,\eta)$, $C_k^+(\xi,\eta)$, $C_k^-(\xi,\eta)$ be the kernels of the integral operators Π_k, P_k, Q_k respectively. The Poisson kernel in B can be written as

$$P(x,\xi) = \sum_k Z_k(x,\xi) = \sum_k r^k Z_k(\eta,\xi) = \frac{1-|x|^2}{|x-\xi|^{n+1}}$$

Here and throughout this paper we will write $x = |x|\eta = r\eta$ and $y = |y|\xi = s\xi$ for $x, y \in R^{n+1}$ and $\eta, \xi \in S^n$.

The functions $C_k^{\pm}(\xi,\eta)$ can be written in terms of Geggenbauer polynomials and we have the estimates (see [1], 11.12)

$$\left|C_k^{\pm}(\xi,\eta)\right| \le Ck^n. \tag{2.5}$$

The spaces $\mathcal{M}_k^+$ and $\mathcal{M}_k^-$ have canonical basis $\{V_\alpha\}$ and $\{W_\alpha\}$ (see [9], Ch. 2.1) where the multi indices $\alpha \in \mathbf{N}^n$ have length $|\alpha| = k$ and $\mathbf{N}$ are the nonnegative integers. The following orthogonality relations are valid for $\alpha, \beta \in \mathbf{N}^n$:

$$\begin{aligned}\int_{S^n} W_\beta(\xi)\xi V_\alpha(\xi)d\sigma(\xi) &= \int_{S^n} \overline{V_\alpha}(\xi)\xi W_\beta(\xi)d\sigma(\xi) = \delta_{\alpha\beta},\\ \int_{S^n} \overline{V_\beta}(\xi)\xi V_\alpha(\xi)d\sigma(\xi) &= \int_{S^n} W_\beta(\xi)\xi W_\alpha(\xi)d\sigma(\xi) = 0,\end{aligned}$$

from which we obtain representations

$$C_k^+(\xi,\eta)\overline{\eta} = \sum_{|\alpha|=k} V_\alpha(\xi)W_\alpha(\eta), \tag{2.6}$$

$$C_k^-(\xi,\eta)\overline{\eta} = \sum_{|\alpha|=k} W_\alpha(\xi)\overline{V_\alpha}(\eta). \tag{2.7}$$

Let X be a left Banach A_n-*module*. We can extend the domain of Π_k, P_k and Q_k to $M_X(S^n)$. For instance,

$$P_k\mu(\xi) = \int_{S^n} C_k^+(\xi,\eta)d\mu(\eta).$$

We have that on $M_X(S^n)$, the projections Π_k, P_k and Q_k take values on the $X-valued$ version of $\mathcal{H}_k$, $\mathcal{M}_k^+$ and $\mathcal{M}_k^-$ respectively. Moreover, for any $\varphi \in C^\infty(S^n)$ and any nonnegative integer N

$$\begin{aligned}\int_{S^n}\varphi(\xi)P_k\mu(\xi)d\sigma(\xi) &= \frac{1}{\lambda_k^N}\int_{S^n}\varphi(\xi)\Delta_\xi^N P_k\mu(\xi)d\sigma(\xi)\\ &= \frac{1}{\lambda_k^N}\int_{S^n}\Delta_\xi^N\varphi(\xi)P_k\mu(\xi)d\sigma(\xi),\end{aligned}$$

where $\lambda_k \sim k^2$ is the $k-th$ eigenvalue of the Laplacian on the sphere. Hence by (2.5) we have

$$\left\|\int_{S^n}\varphi(\xi)P_k\mu(\xi)d\sigma(\xi)\right\|_X \le C_N k^{n-(2N+1)}\|\mu\|_{M_X(S^n)}\left\|\Delta^N\varphi\right\|_\infty.$$

We have the same estimate for $\Pi_k\mu$ and $Q_k\mu$. This implies that the series $\sum_{k=0}^\infty P_k\mu$ and $\sum_{k=1}^\infty Q_k\mu$ are convergent in the sense of X-*valued* distributions, and $\mu = \sum \Pi_k\mu$ as X-valued distributions.

3. The monogenic Hardy space $H_X^1(B)$

Definition (3.1). Let X be a left Banach A_n--*module* and $p > 0$. We denote by $H_X^p(B)$ the space of all left monogenic functions F in the ball with values in X such that

$$\sup_{0\le r<1}\int_{S^n}\|F(r\eta)\|_X^p\,d\sigma(\eta) < \infty.$$

Remark (3.2). a) Let $F\colon B \to X$ be a left monogenic function. If $p > \frac{n-1}{n}$, then $F \in H_X^p(B)$ if and only if the radial maximal function $F^*(\xi) = \sup\{\|F(r\xi)\|_X : 0 \le r < 1\}$ belongs to $L^p(S^n)$. In fact, since $\ell\circ F$ is monogenic for every $\ell \in X_l^*$ then $|\ell\circ F(x)|^\varepsilon$ is subharmonic in B provided $\frac{n-1}{n} < \varepsilon < 1$ (see [12] p.106, noticing that the model of Clifford Analysis used in this reference is slightly different to ours, however the proof of this statement applies in this case). It follows from Lemma (2.3) that $\|F(x)\|_d^\varepsilon$ is also subharmonic in B and the remark can be proved following the proof of the scalar case.

b) If $u\colon B \to X$ is a harmonic function such that

$$\sup_{0\le r<1}\int_{S^n}\|u(r\eta)\|_X\,d\sigma(\eta) < \infty,$$

we may represent

$$u(x) = \int_{S^n} P(x,\xi)d\mu(\xi) \tag{3.3}$$

for some measure $\mu \in M_X(S^n)$. This follows by the standard argument using Banach-Alouglou theorem and the duality

$$C_X(S^n)^* = M_{X^*}(S^n)$$

valid for every Banach space X, (see [19]).

c) If we take $f \in L^1_X(S^n)$ and we let $F(x) = \int_{S^n} P(x,\eta)f(\eta)d\sigma(\eta)$, then the harmonic function F has radial (even nontangential) limits a.e. in S^n, since almost every point of S^n is a Lebesgue point of f (see [10], Th. 2.9).

Definition (3.4). Let X be a left Banach A_n*--module* and $\mu \in M_X(S^n)$. We will say that μ is a monogenic measure if

$$\int_{S^n} P(\eta)\eta d\mu = 0$$

for every $P \in \mathcal{M}_k^+$ with $k > 0$.

THEOREM (3.5). *Let X be a left Banach A_n-module. A measure $\mu \in M_X(S^n)$ is monogenic if and only if the Poisson transform F of μ*

$$F(x) = \int_{S^n} P(x,\xi)d\mu(\xi)$$

belongs to $H^1_X(B)$.

Proof. Let $\mu \in M_X(S^n)$ be monogenic. Then by (2.7) we have that $Q_k\mu = 0$ for all k, since the spaces $\mathcal{M}_k^-$ are self conjugate. We may represent

$$\mu = \sum_{k=0}^{\infty} \Pi_k\mu = \sum_{k=0}^{\infty} P_k\mu + \sum_{k=1}^{\infty} Q_k\mu = \sum_{k=0}^{\infty} P_k\mu.$$

Since $Z_k(\xi,\eta) = C_k^+(\xi,\eta) + C_{k-1}^-(\xi,\eta)$, it follows that

$$\int_{S^n} P(x,\xi)d\mu(\xi) = \sum_{k=0}^{\infty} r^k\Pi_k\mu(\eta) = \sum_{k=0}^{\infty} r^k P_k\mu(\eta)$$

with uniform convergence on compact subsets of B. Hence F is monogenic and by Fubini's theorem,

$$\int_{S^n} \|F(r\eta)\|\, d\sigma(\eta) \le \|\mu\|_{M_X(S^n)}, r \in [0,1).$$

To prove the converse suppose that F above is monogenic. Then for each $P \in \mathcal{M}_k^+$ and $r \in [0,1)$

$$\int_{S^n} P(\xi)\xi F(r\xi)d\sigma(\xi) = 0,$$

by the Cauchy Theorem [1].

Since $F(r\xi)d\sigma(\xi)$ converges to μ as vector-valued distributions conclude that

$$\int_{S^n} P(\xi)\xi d\mu(\xi) = \lim_{r\to 1}\int_{S^n} P(\xi)\xi F(r\xi)d\sigma(\xi) = 0.$$

$\square$

COROLLARY (3.6). *Let X be a left Banach A_n-module. A measure $\mu \in M_X(S^n)$ is monogenic if and only if the Cauchy transform $\mathcal{C}$ of μ in the ball*

$$\mathcal{C}\mu(x) = \int_{S^n} G(x-\xi)\xi d\mu(\xi) \tag{3.7}$$

belongs to $H^1_X(B)$ and $\mathcal{C}\mu(x) = 0$ for $|x| > 1$.

Proof. For $x \in B$ and $\xi \in S^n$ we have ([9] p. 182 (1.9))

$$G(\xi - x) = \sum_{k=0}^{\infty} |x|^k C_k^+(\eta, \xi)\bar{\xi}$$

Then if $\mu \in M_X(S^n)$ is monogenic $\mathcal{C}\mu(x) = P\mu(x)$. Moreover if $|x| > 1$ and $\xi \in S^n$. we have ([9] p. 180 (1.7))

$$G(\xi - x) = \sum_{k=0}^{\infty} \frac{C_k^-(\xi, \eta)\bar{\eta}}{|x|^k}$$

it follows that $\mathcal{C}\mu(x) = 0$ for $|x| > 1$.

To prove the converse suppose that $\mathcal{C}\mu(x) = 0$ for $|x| > 1$. then from the above decomposition of G it follows that $Q_k\mu = 0$ for all k, hence μ is monogenic. □

COROLLARY (3.8). *Let X be a left Banach A_n-module. A function $F: B \to X$ belongs to $H^1_X(B)$ if and only if there exists a monogenic measure $\mu \in M_X(S^n)$ such that F has the representation (3.3) or (3.7).*

THEOREM (3.9) (F. Riesz and M. Riesz). *Every monogenic measure $\mu \in M_X(S^n)$ is absolutely continuous with respect to σ.*

Proof. Suppose that $X = A_n$. If μ is monogenic and we let

$$F(x) = \int_{S^n} P(x, \eta) d\mu(\eta),$$

then $F \in H^1(B)$. But we know in this case (see [15] p. 68) that for almost all $\xi \in S^n$, F has nontangential limit $F(\xi)$ and

$$F(x) = \int_{S^n} P(x, \eta) F(\eta) d\sigma(\eta).$$

Thus $F(\xi)$ is a density for μ.

In the general case, let $\mu \in M_X(S^n)$ be a monogenic measure and G a Borel set of S^n with Lebesgue measure zero. Take $\ell \in X_l^*$. Since $\ell \circ \mu$ is monogenic we have that $\ell \circ \mu(G) = 0$ by the first part of the proof and this implies that $\mu(G) = 0$ since X_l^* separates points from X. □

Definition (3.10). We say that a Banach A_n*-module* X has the monogenic Radon-Nikodym property ($X \in (RN)_m$) if every monogenic measure $\mu \in M_X(S^n)$ has a density in $L^1_X(S^n)$.

Remark (3.11). Theorem (3.9) implies that $X \in (RN)_m$ if X has the Radon-Nikodym property ([10]).

THEOREM (3.12). *Let X by a Banach A_n-module. Then $X \in (RN)_m$ if and only if every function $F \in H^1_X(B)$ has radial boundary limits almost everywhere.*

Proof. The proof is a consequence of Remark (3.2)c and Theorem (3.5). □

THEOREM (3.13). *There exists a function $F \in H^{\infty}_{A_n\otimes c_0}(B)$ without radial boundary limits on a set of positive measure. In particular $A_n \otimes c_0 \notin (RN)_m$.*

Proof. We start our construction in the upper half space

$$\mathbb{R}^{n+1}_+ = \{(x_0, x_1, .., x_n) \in \mathbb{R}^{n+1} : x_n > 0\}$$

and then we pull it to the unit ball through a Möbius transform.

There exists a bounded monogenic function $G: \mathbb{R}^{n+1}_+ \to A_n \otimes c_0$ such that

1. $\lim_{x_n\to 0} G(x) \notin A_n \otimes c_0$ for every x on a set of positive Lebesgue measure in $\mathbb{R}^n$,
2. $|G(x)| \le \frac{C}{(1+|x|)^n}$ for all $x \in \mathbb{R}^{n+1}_+$.

To see this, first consider an atom in $\mathbb{R}^n$ as follows: let $a: \mathbb{R}^n \to \mathbb{R}$ be a C^1 function with support in a cube Q in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} a(x)dx = 0$ and $\|a\|_\infty \le \frac{1}{|Q|}$. Then observe that the Hilbert transform of a

$$\begin{aligned} Ha(x) &= p.v. \int_{\mathbb{R}^n} G(x-y)e_n a(y)dy \\ &= p.v. \int_Q \frac{\overline{x-y}}{|x-y|^{n+1}} e_n a(y)dy, \, x \in \mathbb{R}^n \end{aligned}$$

has zero real part and $Ha \in L^\infty(\mathbb{R}^n)$. In fact, with elementary estimates we see that $|Ha(x)| \le \frac{C}{|x|^n}$, for large values of $|x|$, while $|Ha(x)| \le C\|\nabla a\|_\infty$for $|x|$ small. Hence

$$|Ha(x)| \le \frac{C}{(1+|x|)^n}, \tag{3.14}$$

where C depends on $\|\nabla a\|_\infty$ and on the size and position of the cube Q.

Consider the Cauchy transform of a,

$$A(x) = \mathcal{C}a(x) = \int_{\mathbb{R}^n} G(x-y)e_n a(y)dy.$$

Then A is a monogenic function on $\mathbb{R}^{n+1}_+$ and since A is the Poisson integral of $(a + Ha)/2$ we obtain the estimate

$$|A(x)| \le \frac{C}{(1+|x|)^n} \; x \in \mathbb{R}^{n+1}_+, \tag{3.15}$$

with the same dependence of the constant C on a and Q as in (3.14). The function A has boundary values $\lim_{x_n\to 0} A(x) = \frac{1}{2}(a + Ha)(x)$ and in particular the real part of the boundary function is $(1/2)a(x)$.

Now we proceed to construct G. We can easily find an atom a as before with $Q = [-1/2, 1/2]^n$ and such $a = 1$ in an open rectangle $I \subset (-1/2, 0) \times (-1/2, 1/2)^{n-1}$. For any positive integer k, define $a_k(x) = a(kx)$ and let $A_k = \mathcal{C}a_k$. Since the Hilbert transform H is dilation invariant (it is a combination of the Riesz transforms) then the sequence (A_k) is bounded in $L^\infty(\mathbb{R}^{n+1}_+)$. Also *supp* $a_k \subset \frac{1}{k}Q$ and $\|a_k\|_1 \to 0$. Then we see A_k satisfies the estimate (3.15) uniformly in k since $A_k(x) = A(kx)$ and $A_k(x) \to 0$ pointwise in $\mathbb{R}^{n+1}_+$.

Finally, translating the atoms a_k we can construct an increasing function $\varphi: \mathbb{N} \to \mathbb{N}$ and a sequence (g_m) of atoms and with the following properties

a) *supp* $g_k \subset [-2, 2]^n$, for all $k \in \mathbb{N}$,

b) For each k and every $m \in \{\varphi(k)+1, .., \varphi(k+1)\}$ the atom g_m is a translate of a_k,

c) The translates of the rectangle $\frac{1}{k}I$ used to define g_m, for $m \in \{\varphi(k)+1, \ldots, \varphi(k+1)\}$ are a covering for $[-1/2, 1/2]^n$.

Observe that given any $x \in [-1/2, 1/2]^n$ and for any $k \in \mathbb{N}$ there exists $m \in \{\varphi(k)+1, .., \varphi(k+1)\}$ such that $g_m(x) = 1$.

Define on $\mathbb{R}^{n+1}_+$, $G(x) = (G_m(x))_m$, where $G_m = \mathcal{C}(g_m)$. Then G maps $\mathbb{R}^{n+1}_+$ into c_0 and it is monogenic by Lemma (2.4). G satisfies (1) and (2) due to the point (c) in the construction.

To move from $\mathbb{R}^{n+1}_+$ to B let us recall that composition of a monogenic function with a Möbius transform is not monogenic unless it is multiplied by the covariance factor of the Möbius transform (see [16] for details). Consider the Calvin transform $\phi(x) = (1 - e_n x)(x - e_n)^{-1}$, $x \in \mathbb{R}^{n+1}_+$ with covariance $J(\phi, x) = \frac{x-e_n}{|x-e_n|^{n+1}}$. Notice that ϕ is a bijection of ball B onto R^{n+1}_+. Define $F(x) = J(\phi, x)G(\phi(x))$, $x \in B$. Then F is monogenic in B. The estimate (3.15) for G implies that $|G(\phi(x))| \leq C\,|x - e_n|^n$ for x close to e_n. It follows that F is bounded on B and does not have radial boundary limits on a set of positive measure in the sphere. □

COROLLARY (3.16). *If $X = A_n \otimes E \in (RN)_m$ then E does not have a subspace isomorphic to c_0.*

4. The monogenic Radon-Nikodym for Banach lattices

Let (Ω, Σ, μ) be a measure space. We denote be L^0 the space of all measurable functions, finite almost everywhere modulo μ. We will say that a Banach space $(E, \|\cdot\|)$ is a Banach function space on (Ω, Σ, μ), *BFS* for short, if

1. E is a linear subspace of L^0,
2. $x \in L^0$ and $y \in E$, with $|x| \leq |y|$ implies that $x \in E$ (E is an ideal space),
3. $|x| \leq |y|$ implies that $\|x\| \leq \|y\|$, for every $x, y \in E$ ($\|\cdot\|$ is monotone).

Three possible properties for a *BFS* that will be relevant in this section are (see [6, 13])

(A) If $(x_n)_n$ is a sequence in E such that $x_n \downarrow 0$ then $\|x_n\| \to 0$.

(B) If $(x_n)_n$ is an increasing sequence of functions on E such that $\sup \|x_n\| < \infty$ then there exists $x \in E$ such that $x_n \uparrow x$.

(C) If $x_n \uparrow x$, with $x_n, x \in E$ then $\|x_n\| \to \|x\|$.

(Here the convergence means convergence almost everywhere).

Definition (4.1). [13]. Let X and Y be BFS on $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ such that Y satisfies the condition (C) above. Denote by $X[Y]$ the space of all measurable functions $f(s,t)$ on the product space $\Omega_1 \times \Omega_2$ provided with the product measure such that

1) the function $s \to f(t,s)$ belongs to Y for almost all $t \in \Omega_1$ $[\mu_1]$,
2) the function $w_f(t) = \|f(t,\cdot)\|_Y$ belongs to X.

We provide $X[Y]$ with the norm $\|f\|_{X[Y]} = \left\|w_f\right\|_X$. With this norm $X[Y]$ becomes a BFS (see [4], [13] for properties of this space). We will refer to the standard terminology of Banach lattices (see [13], [17], [14]). We will say that a Banach lattice E is a *KB-space* if for every for every sequence $0 \leq x_n \uparrow$ in E such that $\sup \|x_n\| < \infty$, there exists $x \in E$ such that $x_n \to x$ in norm.

THEOREM (4.2). *([13], Th. X.4.9) The following statements are equivalent for a Banach lattice E*

a) *E is a KB-space*

b) *E does not have a copy of* c_0.

Remark (4.3). Every KB-space has order continuous norm (see [14], Sect. 2.2). If we assume that E is a separable KB-space, then there exists a probability measure space (Ω, Σ, μ) such that we may represent E as a BFS such that $L^\infty(\mu) \subset E \subset L^1(\mu)$ (see [14], Th. 2.7.8 and [17], Prop. 2.6.2). Keeping in mind the description of *KB*-spaces in [13], Chapter X, 4.4, we see that E has properties (A), (B), (C) above.

THEOREM (4.4). *Let E be a real Banach lattice,* $E_{\mathbb{C}}$ *its complexification and* $X = A_n \otimes E$. *Then the following statements are equivalent*

1. $E_{\mathbb{C}} \in (RN)_a$.
2. *E does not has a copy of* c_0.
3. *Every* $F \in H^p_X(B)$ *has radial boundary limits for all* $1 \leq p \leq \infty$ *and every* $n \in \mathbb{N}$.
4. *Every* $F \in H^p_X(B)$ *has radial boundary limits for some* $1 \leq p \leq \infty$ *and some* $n \in \mathbb{N}$.
5. *For every* $n \in \mathbb{N}$, $X \in (RN)_m$.

Proof. The equivalence of (1) and (2) was proved in [6]. Suppose that (2) holds. To prove (3) will let $F \in H^1_X(B)$ and show that it can be represented as a Poisson integral of a function in $L^1_X(B)$. Since the image of F is separable we can assume by Remark (4.3) that X is a BFS on a finite measure space (Ω, Σ, μ).

As a first step we prove that we can find a measurable function f on $\Omega \times B$ such that $F(x) = f(\cdot, x)$ and $f(t, \cdot)$ is monogenic for almost all $t \in \Omega$:

Represent F as a Taylor series and as a spherical harmonic expansion

$$F(x) = \sum_{k=1}^{\infty}(\sum_{|\alpha|=k} V_\alpha(x)x_\alpha) = \sum_{k,j} Y^k_j(x)x_{k,j},$$

where $x_\alpha, x_{k,j} \in X$. We have

$$\sum_{k,j} R^k \left\|x_{k,j}\right\|_X < \infty \tag{4.5}$$

for $0 \leq R < 1$. Then for each k we have

$$\sum_{|\alpha|=k} V_\alpha(x)x_\alpha = \sum_{j=1}^{d_k} Y^k_j(x)x_{k,j}.$$

We can choose a set $A_1 \in \Sigma$ with complete measure such that

$$\sum_{|\alpha|=k} V_\alpha(x)x_\alpha(t) = \sum_{j=1}^{d_k} Y^k_j(x)x_{k,j}(t) \tag{4.6}$$

for every $t \in A_1$, $x \in B$ and $k \geq 0$. To see this we find A_1 such that (4.6) holds for x in a countable dense subset of B, then extend this by continuity in B. Fix any $0 < R < 1$. Since X is a KB-space then (4.5) implies that $\sum_{k,j} R^k \left|x_{k,j}\right| \in X$. Hence for almost all t, say $t \in A_2$,

$$\sum_{k,j} R^k \left|x_{k,j}(t)\right| < \infty$$

and $\sum_{k=1}^{\infty} \sum_{j=1}^{d_k} \; Y_j^k x_{k,j}(t)$ defines a harmonic function on $0 \leq r < R$. Then if $t \in A = A_1 \cap A_2$ it follows that $\sum_{k=1}^{\infty} \sum_{|\alpha|=k} V_\alpha(\cdot) x_\alpha(t)$ is monogenic on $0 \leq r < R < 1$. The number $R \in [0, 1)$ is arbitrary, so it is clear that for almost all $t \in \Omega$ the function above is monogenic on B. Also the function

$$f(t, x) = \sum_{k=1}^{\infty} (\sum_{|\alpha|=k} V_\alpha(x) x_\alpha(t)).$$

is measurable on the product $\Omega \times B$ and $F(x) = f(\cdot, x)$. Remark (3.2) implies the existence of a set $G \subset S^n$ of complete Lebesgue measure such that

$$\|f(\cdot, r\xi)\|_X \leq F^*(\xi) < \infty, \tag{4.7}$$

for all $r \in [0, 1)$ and $\xi \in G$. Observe that for fixed r, the function

$$t \rightarrow \int_{S^n} |f(t, r\xi)| \, d\sigma(\xi)$$

belongs to X. In fact,

$$\int_{S^n} |f(\cdot, r\xi)| \, d\sigma(\xi) \leq \sum_{k,j} r^k d_k \left|x_{k,j}\right|(t) \leq C r^k k^{n-2} \left|x_{k,j}\right|(t).$$

Then by (4.5) $\int_{S^n} |f(\cdot, r\xi)| \, d\sigma(\xi) \in X$ and we can estimate its norm using the *Banach function dual* E' of E:

$$\left\| \int_{S^n} |f(\cdot, r\xi)| \, d\sigma(\xi) \right\|_X \leq \sup_{\substack{x' \in E'_+ \\ \|x'\| \leq 1}} \int_\Omega \int_{S^n} |f(t, r\xi)| \, d\sigma(\xi) x'(t) d\mu(t) \tag{4.8}$$

$$\leq \sup_{\substack{x' \in E'_+ \\ \|x'\| \leq 1}} \int_\Omega \int_{S^n} |f(t, r\xi)| \, x'(t) d\mu(t) d\sigma(\xi) \tag{4.9}$$

$$\leq \int_{S^n} F^*(\xi) d\sigma(\xi) < \infty, \tag{4.10}$$

where E'_+ consists of all measurable functions $x' \geq 0$ such that

$$\sup_{\|x\|_E \leq 1.} \int_\Omega x(t) x'(t) d\mu(t) < \infty.$$

Consider the function

$$x_0(t) = \sup_{0 \leq r < 1} \int_{S^n} |f(\cdot, r\xi)| \, d\sigma(\xi).$$

For $t \in A$ as above, $x_0(t) = \lim_{n \to \infty} \int_{S^n} |f(\cdot, r_n\xi)| \, d\sigma(\xi)$, being r_n any sequence $r_n \uparrow 1$. Then the fact the X is a KB-space and (4.10) implies that $x_0 \in X$. It follows that $x_0(t) < \infty$ for almost all t, then for almost all t, the function $f(t, \cdot)$ belongs to the Clifford $H^1(B)$. By the scalar theory, we know that for almost all $t \in \Omega$ there exists the limit

$$\widetilde{f}(t, \xi) = \lim_{r \to 1} f(t, r\xi)$$

for almost all $\xi \in S^n$ and in the L^1 norm.

Note that

$$\int_{S^n} |f(\cdot, r\xi) - f(\cdot, r'\xi)| \, d\sigma(\xi) \leq 2x_0.$$

Then

$$\left\| \int_{S^n} |f(\cdot, r\xi) - f(\cdot, r'\xi)| \, d\sigma(\xi) \right\|_X \to 0$$

as $r, r' \to 1-$, that is, the family $\{f(\cdot, r\cdot)\}_r$ is a Cauchy net on $X[L^1(S^n)]$ as $r \to 1$. In fact, assume that

$$\left\| \int_{S^n} |f(\cdot, r_n\xi) - f(\cdot, s_n\xi)| \, d\sigma(\xi) \right\|_X \geq \varepsilon > 0,$$

with $r_n, s_n \to 1$. Let $g_n = \int_{S^n} |f(\cdot, r_n\xi) - f(\cdot, s_n\xi)| \, d\sigma(\xi)$. We have $g_n \leq 2x_0$ and $g_n(t) \to 0$ for almost all t. Then (see [13], Ch X.1.4) $g_n \xrightarrow{o} 0$, that is, there exist a sequence $\varphi_n \downarrow 0$ in E such that $g_n \leq \varphi_n$. We have that $g_n \to 0$ in norm ([13], Ch X.4.1), since E satisfies (A), and this is impossible.

The space $X[L^1(S^n)]$ is a Banach space, then the limit $f = \lim_{r \to 1-} f(\cdot, r\cdot)$ exists in $X[L^1(S^n)]$. To conclude the proof we will show that for almost all $\xi \in S^n$, $F_0(\xi) = f(\cdot, \xi) = \lim_{r \to 1-} F(r\xi)$ on X.

By a variation of the argument used in the case of the classical Lebesgue spaces we can prove that since $f = \lim_{r \to 1-} f(\cdot, r\cdot)$ in $X[L^1(S^n)]$, there exists a sequence $\{f(t, r_n\xi)\}$ converging a.e. $[\mu \times \sigma]$ to $f(t, \xi)$. Note that for almost all $t \in \Omega$, $f(t, \xi) = \widetilde{f}(t, \xi)$ for almost all ξ. Also for almost all $\xi \in S^n$, $f(t, r_n\xi) \to f(t, \xi)$ a.e. $[\mu]$ and by the estimate (4.7) and the Lemma X.3.5 of [13] we conclude that $f(\cdot, \xi) \in X$ a.e. $[\sigma]$ and $w(\xi) = \|f(\cdot, \xi)\|_X$ belongs to $L^1(S^n)$, that is, $f \in L^1(S^n)[X]$. By Lemma XI.1.2 of [13] we can easily see that $f(\cdot, \xi)$ is Bochner a measurable function of ξ, hence F_0 above belongs to $L^1_X(S^n)$. To prove (3) it is enough to prove that F is the Poisson integral of F_0. Let $A \in \Sigma$ such that $f(t, \cdot) \in H^1(B)$ for every $t \in A$. Then if $t \in A$

$$f(t, x) = \int_{S^n} P(x, \eta)\widetilde{f}(t, \eta) d\sigma(\eta) = \int_{S^n} P(x, \eta) f(t, \eta) d\sigma(\eta).$$

But (see [4], Lemma 2.1) for almost all $t \in A$ we have

$$\int_{S^n} P(x, \eta) f(t, \eta) d\sigma(\eta) = (\int_{S^n} P(x, \eta) f(\cdot, \eta) d\sigma(\eta))(t).$$

This completes the part (2) $\Longrightarrow$ (3).

By Theorem (3.12) and Theorem (3.13) we have (4) $\Longrightarrow$ (2) and this completes the proof. □

COROLLARY (4.11). *$A_n \otimes L^1[0, 1] \in (RN)_m$ for all n.*

Received May 16, 2006

Final version received October 4, 2006

Instituto de Matemáticas, Unidad Cuernavaca
Universidad Nacional Autónoma de México
A.P. 273-3 ADMON 3
62251 Cuernavaca, Mor.
México
emilio@matcuer.unam.mx
salvador@matcuer.unam.mx

References

[1] F. Brackx, R. Delanghe, and F. Sommen, Clifford Analysis, Pitman Research Notes in Mathematics, No. 76, 1982.

[2] O. Blasco, *Boundary values of functions in vector-valued Hardy Spaces and geometry on Banach spaces*, J. Funct. Anal. **78** (1988), 346–364.

[3] A. V. Bukhvalov, *Hardy spaces of vector-valued functions*, Zap. Nauchn. Sem. Lenningr. Otd. Mat. Inst. Akad. SSSR **65** (1976) (Russian); English transl. J. Soviet Math. **16** (1981), 1051–1059.

[4] A. V. Buhvalov, *On spaces with mixed norm*, Vestnik Leningrad (1973), no. 19, 5-12 (Russian); English transl. Vestnik Leningrad Univ. Math **6** (1979), 303–311.

[5] A. Boccuto, A. V. Bukhvalov and A. R. Sambucini, *Some inequalities in classical spaces with mixed norms*, Positivity **6** (4), (2002), 393–411.

[6] A. V. Bukhvalov and A.A. Danilevich, *Boundary properties of Analytic and Harmonic Functions with value in Banach space*, Mat. Zametki **31** (1982) (Russian); English transl. Math. Notes Acad. Sci. USSR **32** (1982), 104–110.

[7] Z. Chen and C. Ouyang, *Vector-valued holomorphic functions on the complex ball and the analytic Radon-Nikodým property*, Acta Math. Sci. Ser. B Engl. Ed. **20** (2), (2000), 278–288.

[8] A. A. Danilevich, *Some boundary properties of abstract analytic functions and their applications*, Mat. Sb. **100** (1976), 507-533 (Russian); English transl. Math. USSR - Sb. **29** (1976), 453–474.

[9] R. Delangue, F. Sommen and V. Souček, Clifford algebra and spinor-valued functions, Mathematics and its applications, Kluwer Academic Publishers, 1992.

[10] J. Diestel and J. J. Uhl Jr., Vector Measures, Mathematical Surveys and Monographs **15**, Amer. Math. Soc., Providence, R.I, 1977.

[11] W. Hensgen, *Some remarks on boundary values of vector valued harmonic and analytic functions*, Arch. Math. (Basel) **57** (1991), 88–96.

[12] J. Gilbert and M. Murray, Clifford Algebras and Dirac Operators in Harmonic Analysis, Cambridge University Press, 1991.

[13] L. V. Kantorovich and G.P. Akilov, Functional analysis, 2nd rev. ed, 'Nauka', Moscow (1977) (Russian); English transl. Pergamon Press, Oxford 1982.

[14] P. Meyer-Nieberg, Banach lattices, Springer-Verlag, Berlin-Heidelberg 1991.

[15] M. Mitrea, Clifford wavelets, singular integrals, and Hardy spaces, Lecture Notes in Math. no. 1575, Springer-Verlag 1994.

[16] J. Ryan, *Conformally covariant operators in Clifford analysis*, J. Math. Anal. Appl. **14** (1995), 677–704.

[17] H. H. Schaefer, Banach lattices an positive operators, Springer-Verlag 1974.

[18] L. Schwartz, *Espaces de fonctions differentiables à valeurs vectorielles*, J. Anal. Math. **4** (1954-1955), 88–148.

[19] I. Singer, *Sur les applications leneaires integrales des espaces des fonctions continus*, Rev. Romaine Math. Pures Appl. **4** (1954) 391–405.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

COINCIDENCE AND FIXED POINTS OF NONLINEAR HYBRID MAPPINGS

YISHENG LAI

ABSTRACT. The concepts of harmonic for single-valued and multi-valued mappings are defined. Some common fixed and coincidence point theorems for single-valued and multi-valued mappings satisfying a class of conditions are obtained by an iteration scheme. The conditions are not assumed to be a contractive type.

1. Introduction

In recent years there have appeared various papers concerning common fixed and coincidence point theory for single-valued and multi-valued mappings, see, for example, [1-8]. Some authors (see, [1-8]) carried their work out in a framework in which the underlying metric space is a complete , and the single-valued and multi-valued mappings satisfy a contractive type condition. In this case, fixed and coincidence points can be found by a technique from Nadler [9] [also cf. [5, 6, 7, 8]]. However, the method can't be employed if the mappings are not assumed to be a contractive type, and such case also has been seldom discussed.

In this paper, the notion of harmonic for single-valued and multi-valued mappings is given and the concept of compatibility is extended [1, 2]. An iteration scheme for finding coincidence and common fixed point of the hybrid mappings satisfying a Φ-type condition is established. Using the technique, we get several coincidence and common fixed point theorems for a class of hybrid mappings without assuming to be a contractive type. In our theorems, replacing the completeness of the space by a set of weaker conditions, we also drop the compatibility requirement and the assumptions of continuity of mappings in Theorem (3.20).

2. Preliminaries

Let (X, d) be a metric space and R^+ the set of nonnegative real numbers. Let $(CB(X), H)$ and $(CL(X), H)$ denote respectively the hyperspaces of nonempty closed bounded subsets of X, and nonempty closed subsets of X, where H is the Hausdorff-Pompei metric induced by d, i.e.,

$$H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{x\in B} d(x, A)\}$$

2000 *Mathematics Subject Classification*: 47H10, 54H25, 54C60.

Keywords and phrases: Hausdorff-Pompei metric; fixed and coincidence point; compatible and harmonic mappings.

for all $A, B \in CB(X)$ (or $CL(X)$), where $d(x, A) = \inf_{y\in A} d(x, y)$. A set-valued mapping $f: X \to CB(X)$ (or $CL(X)$) is called Hausdorff-Pompei continuous at x_0 if $\lim_{x\to x_0} H(fx, fx_0) = 0$.

It is well known that $(CB(X), H)$ and $(CL(X), H)$ are complete metric spaces, whenever (X, d) is complete. Of course, $(CB(X), H)$ and $(CL(X), H)$ are metric spaces.

Definition (2.1). The mappings $f: X \to X$ and $T: X \to CL(X)$ are said to be compatible if $d(fy_n, Tfx_n) \to 0$ whenever $\{x_n\}, \{y_n\}$ are sequences in X such that $Tx_n \to M \in CL(X)$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = t \in M$, where $y_n \in Tx_n$ for $n = 1, 2, \ldots$.

Definition (2.1) slightly extends Kaneko's and Cho's definitions [1, 2].

Definition (2.2). Let $\psi: R^+ \to R^+$ be a function. The mappings $f: X \to X$ and $T: X \to CL(X)$ are said to be ψ-harmonic if the following conditions are satisfied:

(a) $ft \in M$ whenever there exists some sequence $\{x_n\}$ in X such that $Tx_n \to M \in CL(X)$ and $fx_n \to t \in M$;

(b) for t and M above, $H(M, Tt) > \psi(d(ft, t))$ if $ft \neq t$.

Example (2.3). Let $X = \{x: 0 \leq x \leq 1, x \in Q\} \cup \{2\}$ be endowed with the usual metric. Define

$$\psi y = 3y + \sin y, \quad y \in [0, +\infty),$$

$$fx = \begin{cases} \frac{1}{10}, & x = 0 \\ 1 - x, & x \neq 0, 2, x \in X \\ 0, & x = 2. \end{cases}; \qquad Tx = \begin{cases} \{0, \frac{1}{2}\}, & x \neq 0,\ x \in X \\ \{\frac{9}{10}\}, & x = 0 \end{cases}$$

We will show that f and T are ψ-harmonic.

(a) If $\{x_n\}$ is a sequence in X such that $Tx_n \to M \in CL(X)$ and $fx_n \to t \in M$, then $x_n \to 1$ or $\frac{1}{2}$ by definitions of mappings f and T. Obviously, $ft \in M$ when $x_n \to 1$ or $\frac{1}{2}$.

(b) Assume that $x_n \to 1$, then $t = 0, M = \{0, \frac{1}{2}\}$. Since $d(ft, t) = \frac{1}{10}$, $H(M, Tt) = \frac{2}{5}$, we have $H(M, Tt) > \psi(d(ft, t))$. If $x_n \to \frac{1}{2}$, then $t = \frac{1}{2}, M = \{0, \frac{1}{2}\}$ and so $ft = t = \frac{1}{2}$.

$$H(Tx, Ty) = \begin{cases} 0, & x, y \neq 0 \text{ or } x = y = 0,\ x, y \in X \\ \frac{2}{5}, & x = 0, y \neq 0 \text{ or } x \neq 0, y = 0 \end{cases}$$

PROPOSITION (2.4). *Suppose that the function $\Phi(t_1, t_2, t_3, t_4, t_5): (R^+)^5 \to R^+$ satisfies the following conditions ϕ_1 and ϕ_2:*

ϕ_1: $\Phi(t_1, t_2, t_3, t_4, t_5)$ is a nondecreasing continuous function in each coordinate variable;

ϕ_2: Let $\psi(t) = \Phi(t, t, t, at, bt)$, where $a, b \in \{0, 2\}$ with $a + b = 2$. The series $\sum_{n=1}^{+\infty} \psi^n(t)$ converges for each $t \in R^+$, where $\psi^n(t)$ is the nth iterate of our original value t.

Then

(a) *$\psi(t)$ is an increasing function;*

(b) *$\psi(t) < t$ for all $t \in R^+$ and $\psi(0) = 0$.*

Proof. It is easy to see from condition (ϕ_1) that $\psi(t)$ is an increasing function. If $\psi(t) \geq t$ for some $t \in (0, +\infty)$, then $\psi^n(t) \geq \psi^{n-1}(t) \geq \ldots \geq \psi(t) \geq t$. This contradicts condition (ϕ_2), hence $\psi(t) < t$ for all $t \in (0, +\infty)$. Similarly, we have $\psi(0) = 0$. This completes the proof. □

Definition (2.5). ϕ_1 and ϕ_2 in Proposition (2.4) are called a Φ-type condition.

Example (2.6). Let

$$\Phi(t_1, t_2, t_3, t_4, t_5) = h[aL(t_1, t_2, t_3, t_4, t_5) + (1-a)N(t_1, t_2, t_3, t_4, t_5)] \tag{2.7}$$

where $0 \leq h < 1, 0 \leq a \leq 1$,

$$L(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\},$$
$$N(t_1, t_2, t_3, t_4, t_5) = \max\{t_1^2, t_2t_3, t_4t_5, \frac{1}{2}t_2t_5, \frac{1}{2}t_3t_4\}]^{\frac{1}{2}}.$$

We show that the $\Phi(t_1, t_2, t_3, t_4, t_5)$ satisfies a Φ-type condition.

ϕ_1: Obviously.

ϕ_2: $\psi(t) = \Phi(t, t, t, at, bt) = ht$ and so $\psi^n(t) = h^n t$. This implies that the series $\sum_{n=1}^{+\infty} \psi^n(t)$ converges for each $t \in R^+$.

The following implicit relations are due to V. Popa [3].

Let $Ç_6$ be the set of all real continuous functions $F(t_1, t_2, \ldots, t_6)\colon (R^+)^6 \to R$ satisfying the following conditions G_1 and G_2:

G_1: F is non-increasing in the variable $t_2, \ldots, t_6$ and non-decreasing in variable t_1;

G_2: There exists $h \in (0, 1)$ and $k > 1$ with $hk < 1$ such that $u \leq kt$ and $F(t, v, v, u, u + v, 0) \leq 0$ implies $t \leq hv$.

Remark (2.8). The Φ-type condition is different from the implicit relations above. In fact, let $\Gamma(t_1, t_2, \ldots, t_6) := t_1 - \Phi(t_2, t_3, t_4, t_5, t_6)$, where Φ satisfies a Φ-type condition, but $\Gamma \in Ç_6$ is not assured. For instance, let $\Phi(t_2, t_3, t_4, t_5, t_6) = \frac{t_2t_3}{1+t_2t_3} + t_4t_5t_6$. It is easy to see that the Φ satisfies a Φ-type condition, but Γ does not satisfy condition G_2.

PROPOSITION (2.9). *Let $A, B \in CL(X)$ and $\beta > 0$. Then for each $a \in A$, there exists an element $b \in B$ such that $d(a, b) \leq H(A, B) + \beta^+$, where*

$$\beta^+ = \begin{cases} 0, & \text{for } H(A,B) > d(a,B) \text{ or } H(A,B) = 0 \\ \beta, & \text{for } H(A,B) = d(a,B) > 0\,. \end{cases} \tag{2.10}$$

Proof. By the definition of Hausdorff-Pompei metric, it is clear that $d(a, B) \leq \sup_{x\in A} d(x, B) \leq H(A; B)$. If $H(A, B) = 0$, then $A = B$, and so $d(a, b) = H(A; B)$ by taking $b = a$. If $H(A, B) \neq 0$, since there exists $b' \in B$ such that $d(a, b') < d(a, B) + \varepsilon$ for any given $\varepsilon > 0$, there exists $b \in B$ such that $d(a, b) < H(A; B)$ if $H(A, B) > d(a, B)$ and $d(a, b) < H(A; B) + \beta$ if $H(A, B) = d(a, B)$. This completes the proof. □

3. Coincidence Theorems

In this section we give some coincidence point theorems for nonlinear hybrid mappings satisfying a Φ-type condition by an iteration scheme.

THEOREM (3.1). *Let (X,d) be a metric space, $f,g\colon X \to X$ be continuous mappings and $S,T\colon X \to CL(X)$ be H-continuous mappings such that $T(X) \subset f(X)$, $S(X) \subset g(X)$. Suppose that there exists a function Φ satisfying Φ-type condition such that for all $x, y \in X$,*

$$H(Sx, Ty) \le \Phi(d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)). \tag{3.2}$$

If one of $S(X)$, $T(X)$, $f(X)$ and $g(X)$ is a complete subspace of X and the pair (f,S) and (g,T) are compatible. Then there exists a sequence $\{x_n\}$ in X, such that

(a) *for every n, $fx_{2n-1} \in Tx_{2n-2}$, $gx_{2n} \in Sx_{2n-1}$;*
(b) $\lim_{n\to\infty} gx_{2n} = \lim_{n\to\infty} fx_{2n-1} = z$ *for some $z \in X$;*
(c) *$fz \in Sz$, $gz \in Tz$.*

Proof. Let x_0 be an arbitrary point of X. Since $T(X) \subseteq f(X)$, there exists $x_1 \in X$ such that $fx_1 \in Tx_0$, and so there exists a point $u_1 \in Sx_1$ such that

$$d(u_1, fx_1) \le H(Sx_1, Tx_0) + \beta_1^+,$$

where $\beta_1 = 1$ and β_1^+ has the same meaning as (2.10), which is possible by Proposition (2.9).

Moreover, since $S(X) \subseteq g(X)$, there exists a point x_2 in X such that $u_1 = gx_2$ and

$$d(gx_2, fx_1) \le H(Sx_1, Tx_0) + \beta_1^+.$$

Proceeding in this way, we can obtain a sequence $\{x_n\}$ in X such that for each $n \ge 1$,

$$b_n = d(gx_{2n}, fx_{2n-1}) \le H(Sx_{2n-1}, Tx_{2n-2}) + \beta_{2n-1}^+ \tag{3.3}$$

and

$$a_n = d(fx_{2n+1}, gx_{2n}) \le H(Tx_{2n}, Sx_{2n-1}) + \beta_{2n}^+, \tag{3.4}$$

where

$$gx_{2n} \in Sx_{2n-1}, \quad fx_{2n-1} \in Tx_{2n-2}, \tag{3.5}$$

(3.6)
$$\beta_{2n-1}^+ = \begin{cases} 0 & \text{for } H(Sx_{2n-1}, Tx_{2n-2}) > d(Sx_{2n-1}, fx_{2n-1}) \text{ or} \\ & \quad H(Sx_{2n-1}, Tx_{2n-2}) = 0 \\ \beta_{2n-1} & \text{for } H(Sx_{2n-1}, Tx_{2n-2}) = d(Sx_{2n-1}, fx_{2n-1}) > 0,\ n \ge 2, \end{cases}$$

$$\beta_{2n-1} = \begin{cases} \min\{\eta_{2n-1}, \frac{1}{2n-1}, |\psi(a_{n-1}) - \psi(b_{n-1})|\} & \text{for } a_{n-1} \ne b_{n-1} \\ \min\{\eta_{2n-1}, \frac{1}{2n-1}\} & \text{for } a_{n-1} = b_{n-1}, \end{cases} \tag{3.7}$$

(3.8)
$$\eta_{2n-1} = \frac{1}{2} \min\{t - \psi(t) : t \in [H(Sx_{2n-1}, Tx_{2n-2}), H(Sx_{2n-1}, Tx_{2n-2}) + 1]\}$$

and

(3.9)
$$\beta_{2n}^{+} = \begin{cases} 0 & \text{for } H(Tx_{2n}, Sx_{2n-1}) > d(Tx_{2n}, gx_{2n}) \text{ or } H(Tx_{2n}, Sx_{2n-1}) = 0 \\ \beta_{2n} & \text{for } H(Tx_{2n}, Sx_{2n-1}) = d(Tx_{2n}, gx_{2n}) > 0\,, \end{cases}$$

$$\beta_{2n} = \begin{cases} \min\{\eta_{2n}, \frac{1}{2n}, |\psi(b_n) - \psi(a_{n-1})|\} & \text{for } b_n \neq a_{n-1} \\ \min\{\eta_{2n}, \frac{1}{2n}\} & \text{for } b_n = a_{n-1}\,, \end{cases} \tag{3.10}$$

$$\eta_{2n} = \frac{1}{2} \min\{t - \psi(t)\colon t \in [H(Tx_{2n}, Sx_{2n-1}), H(Tx_{2n}, Sx_{2n-1}) + 1]\}\,. \tag{3.11}$$

β_{2n-1} and β_{2n} above are positive by proposition (2.4). It follows from (3.2) and (3.4) that

$$\begin{aligned} a_n = d(fx_{2n+1}, gx_{2n}) &\leq H(Tx_{2n}, Sx_{2n-1}) + \beta_{2n}^{+} \\ &\leq \Phi(d(gx_{2n}, fx_{2n-1}), d(fx_{2n-1}, Sx_{2n-1}), d(gx_{2n}, Tx_{2n}), d(fx_{2n-1}, Tx_{2n}), \\ &\qquad d(gx_{2n}, Sx_{2n-1})) + \beta_{2n}^{+} \\ &\leq \Phi(d(gx_{2n}, fx_{2n-1}), d(fx_{2n-1}, gx_{2n}), d(fx_{2n+1}, gx_{2n}), d(fx_{2n+1}, gx_{2n}) \\ &\qquad + d(fx_{2n-1}, gx_{2n}), 0) + \beta_{2n}^{+}. \end{aligned}$$

That is,

$$a_n \leq H(Tx_{2n}, Sx_{2n-1}) + \beta_{2n}^{+} \leq \Phi(b_n, b_n, a_n, a_n + b_n, 0) + \beta_{2n}^{+}. \tag{3.12}$$

Applying the same argument as above, we have

$$b_n \leq H(Sx_{2n-1}, Tx_{2n-2}) + \beta_{2n-1}^{+} \leq \Phi(a_{n-1}, b_n, a_{n-1}, 0, a_{n-1} + b_n) + \beta_{2n-1}^{+} \tag{3.13}$$

by (3.2) and (3.3).

We shall verify that

$$a_n \leq b_n \leq a_{n-1}, \quad n \geq 2 \tag{3.14}$$

where $a_n = b_n$(resp. $b_n = a_{n-1}$) if and only if $a_n = b_n = 0$(resp. $b_n = a_{n-1} = 0$).

In fact, if there exists some n such that $a_n > b_n$, then it is easily seen from (3.12) and conditions ϕ_1 and ϕ_2 that

$$a_n \leq \Phi(a_n, a_n, a_n, 2a_n, 0) + \beta_{2n}^{+} = \psi(a_n) + \beta_{2n}^{+}, \tag{3.15}$$

which along with Proposition (2.4) implies that $\beta_{2n}^{+} = \beta_{2n} > 0$. Hence, from (3.4)-(3.5) and (3.9)-(3.11), we have

$$\begin{aligned} 0 < H(Tx_{2n}, Sx_{2n-1}) = d(Tx_{2n}, gx_{2n}) \leq a_n &\leq H(Tx_{2n}, Sx_{2n-1}) + \beta_{2n}^{+} \\ &\leq H(Tx_{2n}, Sx_{2n-1}) + 1 \end{aligned}$$

and so $\eta_{2n} \leq \frac{1}{2}(a_n - \psi(a_n))$ by (3.11). This together with again (3.9)-(3.10), (3.15) and Proposition (2.4) yields that

$$\begin{aligned} a_n &\leq \psi(a_n) + \beta_{2n}^{+} = \psi(a_n) + \beta_{2n} \leq \psi(a_n) + \eta_{2n} \\ &\leq \psi(a_n) + \frac{1}{2}(a_n - \psi(a_n)) \leq \frac{1}{2}(a_n + \psi(a_n)) < a_n, \end{aligned} \tag{3.16}$$

which is a contradiction. Therefore, $a_n \leq b_n$. If $a_n = b_n > 0$, then it is not difficult to see from an argument as above that (3.16) still holds, that is, $a_n = b_n$

if and only if $a_n = b_n = 0$. Applying the same argument as above, we have $b_n \le a_{n-1}$, and $b_n = a_{n-1}$ if and only if $b_n = a_{n-1} = 0$. Hence (3.14) is proven.

We now show that the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are convergent. Obviously, the conclusion is true by (3.14) if there exists an integer $n \ge 2$ such that $a_{n-1} = 0$ or $b_n = 0$. Now assume that $a_{n-1}, b_n \ne 0$ for all $n \ge 2$. Then $a_{n-1} > b_n$ by (3.14), and so $\beta_{2n} \le \psi(a_{n-1}) - \psi(b_n)$ by (3.10), which together with (3.9)–(3.10), (3.12) and (3.14) implies that

(3.17)
$$b_{n+1} \le a_n \le \psi(b_n) + \beta_{2n}^{+} \le \psi(b_n) + \beta_{2n} \le \psi(b_n) + \psi(a_{n-1}) - \psi(b_n) = \psi(a_{n-1}).$$

It follows that

$$b_{n+1} \le a_n \le \psi(a_{n-1}) \le \psi(\psi(a_{n-2})) = \psi^2(a_{n-2}) \le \cdots \le \psi^{n-1}(a_1). \tag{3.18}$$

The series $\sum_{n=1}^{+\infty} \psi^n(a_1)$ converges by condition ϕ_2, Therefore, the series $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} b_n$ also converge, that is, the series $\sum_{n=1}^{\infty} d(gx_{2n}, fx_{2n-1})$ and

$$\sum_{n=0}^{\infty} d(gx_{2n}, fx_{2n+1})$$

are convergent.

It is easily obtained from (3.12)-(3.13) that

$$H(Sx_{2n-1}, Tx_{2n-2}) \le \psi(a_{n-1}), H(Tx_{2n}, Sx_{2n-1}) \le \psi(b_n). \tag{3.19}$$

This implies that the series $\sum_{n=1}^{\infty} H(Sx_{2n-1}, Tx_{2n-2})$ and $\sum_{n=1}^{\infty} H(Tx_{2n}, Sx_{2n-1})$ are also convergent. We thus see that $\{fx_{2n-1}\}$ and $\{gx_{2n}\}$ are two Cauchy sequences in $f(X)$ and $g(X)$ respectively, and the sequences $\{Sx_{2n-1}\}$ and $\{Tx_{2n}\}$ also are in $S(X)$ and $T(X)$ respectively.

Suppose that $f(X)$ is a complete subspace of X, then $\{fx_{2n-1}\}$ has a limit in $f(X)$, call it z, and it is easily seen by the convergent series $\sum_{n=1}^{\infty} d(gx_{2n}, fx_{2n-1})$ that $z = \lim_{n\to\infty} fx_{2n-1} = \lim_{n\to\infty} gx_{2n}$. Since $T(X) \subset f(X)$, this must imply that $\{Tx_{2n}\} \to M$ for some $M \in CL(X)$ and so $\{Sx_{2n-1}\} \to M$ by the convergent series $\sum_{n=1}^{\infty} H(Tx_{2n}, Sx_{2n-1})$. Thus

$$\begin{aligned} d(z, M) &\le d(z, fx_{2n-1}) + d(fx_{2n-1}, M) \\ &\le d(z, fx_{2n-1}) + H(Tx_{2n-2}, M) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Since M is closed, $z \in M$ and the compatibility of f and S implies that $d(fgx_{2n}, Sfx_{2n-1}) \to 0$ as $n \to \infty$. This along with the continuity of f and the H-continuity of S yields that

$$\begin{aligned} d(fz, Sz) &\le d(fz, fgx_{2n}) + d(fgx_{2n}, Sz) \\ &\le d(fz, fgx_{2n}) + d(fgx_{2n}, Sfx_{2n-1}) + H(Sfx_{2n-1}, Sz) \to 0 \end{aligned}$$

as $n \to \infty$, that is, $fz \in Sz$ since Sz is closed. Similarly, we can show that $gz \in Tz$.

When one of $T(X)$, $S(X)$ and $g(X)$ is a complete subspace of X, by noting the fact that $T(X) \subset f(X)$ and $S(X) \subset g(X)$, this case essentially pertains to the previous case. This completes the proof. □

THEOREM (3.20). *Let Y be an arbitrary non-empty set, (X, d) a metric space. Let mappings $f: Y \to X$ and $T: Y \to CL(X)$ be such that $T(Y) \subseteq f(Y)$ and there exists a function Φ satisfying Φ-type condition such that for all $x, y \in X$,*

$$H(Tx, Ty) \leq \Phi(d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)). \tag{3.21}$$

If either $T(Y)$ or $f(Y)$ is a complete subspace of X, then there exists a point $t \in Y$ such that $ft \in Tt$.

Proof. Assuming that $f = g$ and $S = T$ on Y as in Theorem (3.1). By a similar argument to that in the proof of Theorem (3.1), we can obtain a sequence $\{x_n\}$ in Y such that $fx_{n+1} \in Tx_n$ for integers $n = 1, 2, \cdots$, and $\{fx_n\}$ is a Cauchy sequence in $f(Y)$.

If $f(Y)$ is a complete subspace of X, then $\{fx_n\}$ has a limit in $f(Y)$. Call it μ. Let $t \in f^{-1}\mu$, then $ft = \mu$. By (3.21), $fx_{n+1} \in Tx_n$ yields that

$$\begin{aligned} d(fx_{n+1}, Tt) &\leq H(Tx_n, Tt) \\ &\leq \Phi(d(fx_n, ft), d(fx_n, Tx_n), d(ft, Tt), d(fx_n, Tt), d(ft, Tx_n)) \\ &\leq \Phi(d(fx_n, ft), d(fx_n, fx_{n+1}), d(ft, Tt), d(fx_n, Tt), d(ft, fx_{n+1})) \end{aligned}$$

Passing to the limits as $n \to +\infty$, it then follows from conditions ϕ_1 and ϕ_2 that

$$d(ft, Tt) \leq \Phi(0, 0, d(ft, Tt), d(ft, Tt), 0) \leq \psi(d(ft, Tt)),$$

which together with Proposition (2.4) implies that $d(ft, Tt) = 0$, that is, $ft \in Tt$.

When $T(Y)$ is a complete subspace of X, by noting the fact that $T(Y) \subset f(Y)$, this case essentially pertains to the previous case. This completes the proof. □

Remark (3.22). Assuming that the function $\Phi(t_1, t_2, t_3, t_4, t_5)$ in Theorem (3.20) is the same as the function Φ in Example (2.6), then we get the main result of Pathak, Kang and Cho in [7] by Theorem (3.20) and Example (2.6).

Remark (3.23). Theorem (3.20) is different from the main results in the literature [3, 4]. First, in [3, 4] (X, d) is assumed to be a complete metric space. Secondly Φ-type condition is also dissimilar from implicit relations in [3, 4] by Remark (2.8).

THEOREM (3.24). *Let (X, d) be a metric space, $f, g: X \to X$ be continuous mappings and $S, T: X \to CL(X)$ be H-continuous mappings such that $T(X) \subset f(X)$, $S(X) \subset g(X)$. Suppose that there exist functions $\alpha_i: X \times X \to [0, 1)$ with $\sum_{i=1}^{3} \alpha_i(x, y) \leq 1$, Φ_i satisfying Φ-type condition for $i = 1, 2, 3$, and $\Gamma: R^+ \times R^+ \to R^+$ with $\Gamma(u, v) = 0$ whenever $uv = 0$ such that for all $x, y \in X$,*

$$\begin{aligned} H^p(Sx, Ty) \leq \sum_{i=1}^{3} \alpha_i(x, y)\Phi_i^p(d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)) \\ + \Gamma(d(fx, Ty), d(gy, Sx)), \end{aligned} \tag{3.25}$$

where $p \geq 1$. If one of $S(X)$, $T(X)$, $f(X)$ and $g(X)$ is a complete subspace of X and the pair (f, S) and (g, T) are compatible, then there exists a point $z \in X$ such that $fz \in Sz$, $gz \in Tz$.

The proof of Theorem (3.24) is similar to that of Theorem (3.1). We omit it here.

Remark (3.26). Theorem (3.24) generalizes many fixed and coincidence point theorems (cf. [1, 2, 8]).

Example (3.27). Let $X = [1, \infty)$ be with the Euclidean metric and define $fx = 2x^4 - 1$, $gx = 2x^6 - 1$ and $Sx = [1, x^2]$, $Tx = [x, x^2]$ for all $x \leq 1$.

Obviously, f and g (resp. S and T) are continuous (resp. H-continuous) mappings and $f(X) = g(X) = S(X) = T(X) = X$. We claim that f and S are compatible. In fact, If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $Sx_n = [1, x_n^2] \to M \in CL(X)$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty}(2x_n^4 - 1) = \lim_{n\to\infty} y_n = t \in M$, where $y_n \in [1, x_n^2]$ for $n = 1, 2, \ldots$, then $x_n \to 1$.

On the other hand, we can show that $H(fSx_n, Sfx_n) = 2(x_n^4 - 1)^2 \to 0$ if and only if $x_n \to 1$ as $n \to \infty$ and so, since $d(fy_n, Sfx_n) \leq H(fSx_n, Sfx_n)$, we have

$$\lim_{n\to\infty} d(fy_n, Sfx_n) = 0.$$

Therefore, f and S are compatible. By a similar argument as above, we have that g and T are also compatible.

By the definitions of mappings f, g, S and T, we have

$$\begin{aligned}
&H(Sx, Ty) = \max\{|y-1|, |x^2 - y^2|\};\\
&d(fx, gy) = 2|y^6 - x^2| \geq 4|y^3 - x^2| \geq 4(y^2 - x^2) \quad as \quad y \geq x;\\
&d(fx, Sx) = (2x^2 + 1)(x^2 - 1) \geq 3(x^2 - 1) \geq 3(x^2 - y^2) \quad as \quad y < x;\\
&d(gy, Ty) = 2y^6 - y^2 - 1 \geq 10(y - 1).
\end{aligned}$$

Set

$$\Phi(t_1, t_2, t_3, t_4, t_5) := \frac{1}{2} \max\{t_1, t_2, t_3, \frac{1}{2}(t_4 + t_5)\}.$$

It is easily to see that $H(Sx, Ty) \leq \Phi(t_1, t_2, t_3, t_4, t_5)$. Then it follows Theorem (3.1) that there exists $z \in X$ such that $fz \in Sz$, $gz \in Tz$.

Example (3.28). Let $Y = X = \{x : 0 \leq x \leq 1, x \in Q\}$ be endowed with the usual metric. Let $fx = 1 - x$, $Tx = \{0, 1\}$, $x \in X$, and the function $\Phi(t_1, t_2, t_3, t_4, t_5)$ the same as the function Φ in Example (2.6). It is easy to see that all the hypotheses of Theorem (3.20) are satisfied and $ft \in Tt$, $t = 0, 1$.

4. Fixed point theorems

In this section, using Theorems (3.1) and Theorem (3.20), we prove several fixed point theorems for nonlinear hybrid mappings satisfying a Φ-type condition.

THEOREM (4.1). *Let (X, d) be a metric space and let $f\colon X \to X$ be continuous mapping and $T\colon X \to CL(X)$ be H-continuous mapping such that $T(X) \subset f(X)$ and there exists a function Φ satisfying a Φ-type condition such that for all $x, y \in X$, (3.21) is satisfied. Assume that the following conditions are satisfied:*

(i) *$T(X)$ or $f(X)$ is a complete subspace of X and the pair (f, T) is compatible;*

(ii) *for each $x \in X$, $fx \in Tx$ implies that $f^n x \to z$ for some $z \in X$.*

Then f and T have a common fixed point in X.

Proof. By Theorem (3.20), $ft \in Tt$ for some $t \in X$ and so $f^n t \to z$ for some $z \in X$ by condition (ii). We now verify that $f^2 t = fft \in Tft$. In fact, set for each integer $n \geq 1$, $x_n = t$ and $y_n = ft$; it then follows that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} y_n = ft \in Tt, \quad Tx_n \to Tt, \quad y_n \in Tx_n$$

which along with the compatibility of f and T implies that $d(fy_n, Tfx_n) = 0$ and so $f^2 t \in Tft$. Repeating this argument, we obtain $f^n t \in Tf^{n-1}t$ for each n and the continuity of T yields that

$$d(z, Tz) \leq d(z, f^n t) + d(f^n t, Tz) \leq d(z, f^n t) + H(Tf^{n-1}t, Tz) \to 0,$$

that is, $z \in Tz$ since Tz is closed. It is clear that $fz = z$ by the continuity of f. Hence z is a common fixed point of f and T. This completes the proof. □

THEOREM (4.2). *Let (X, d) be a metric space, $f : X \to X$ and $T : X \to CL(X)$ be ψ-harmonic mappings such that $T(X) \subset f(X)$ and there exists a function Φ satisfying Φ-type condition such that for all $x, y \in X$, (3.21) is satisfied, where the function $\psi(t)$ has the same meanings as in proposition (2.4). If either $T(X)$ or $f(X)$ is a complete subspaces of X, then f and T have a common fixed point in X.*

Proof. By a similar argument to that in the proof of Theorem (3.1), we can obtain a sequence $\{x_n\}$ in X such that $Tx_n \to M \in CL(X)$, $fx_n \to t \in M$ and $d(fx_n, fx_{n+1}) \leq H(Tx_n, Tx_{n-1}) + \varepsilon_n$ for each $n \geq 1$, where $\varepsilon_n \to 0$ with $\varepsilon_n \geq 0$ and $\varepsilon_n = 0$ if $H(Tx_n, Tx_{n-1}) = 0$. $ft \in M$ because f and T are ψ-harmonic mappings. It then follows the definition of the Hausdorff-Pompei metric that

$$d(ft, Tt) \leq H(M, Tt), \quad d(t, Tt) \leq H(M, Tt). \tag{4.3}$$

Using (3.21), we have that

$$H(Tx_n, Tt) \leq \Phi(d(fx_n, ft), d(fx_n, Tx_n), d(ft, Tt), d(fx_n, Tt), d(ft, Tx_n)).$$

Passing the limits as $n \to +\infty$ we get

$$H(M, Tt) \leq \Phi(d(t, ft), d(t, M), d(ft, Tt), d(t, Tt), d(ft, M)), \tag{4.4}$$

which together with $ft \in M$, $t \in M$ and (4.3) implies that

$$H(M, Tt) \leq \Phi(d(t, ft), 0, H(M, Tt), H(M, Tt), 0). \tag{4.5}$$

We now show that

$$H(M, Tt) \geq d(t, ft) \tag{4.6}$$

In fact, if $H(M, Tt) < d(t, ft)$, then $ft \neq t$ and it follows from (4.5) that

$$H(M, Tt) \leq \Phi(d(t, ft), 0, d(t, ft), d(t, ft), 0) \leq \psi(d(t, ft)). \tag{4.7}$$

On the other hand, since f and T are ψ-harmonic mappings, $ft \neq t$ yields that $H(M, Tt) > \psi(d(t, ft))$. This contradicts (4.7). Hence (4.6) is proven.

It follows from (4.5) and (4.6) that

$$H(M, Tt) \leq \Phi(H(M, Tt), 0, H(M, Tt), H(M, Tt), 0) \leq \psi(H(M, Tt)), \tag{4.8}$$

which together with the Proposition (2.4) implies that $H(M, Tt) = 0$, that is, $M = Tt$. Now by (4.6) and noting that $t \in M$, we get $ft = t \in Tt$. Therefore t is a common fixed point of f and T in X. This completes the proof. □

THEOREM (4.9). *Let (X, d) be a metric space and let f, g, S and $T: X \to X$ be continuous mappings such that $T(X) \subset f(X)$, $S(X) \subset g(X)$ and there exists a function Φ satisfying Φ-type condition such that for all $x, y \in X$,*

$$d(Sx, Ty) \le \Phi(d(fx, gy), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)). \tag{4.10}$$

If one of $S(X)$, $T(X)$, $f(X)$ and $g(X)$ is a complete subspaces of X and the pair (f, S) and (g, T) are compatible. Assume also that for any given $t > 0$, $\Phi(t, 0, 0, t, t) < t$. Then f, g, S and T have a common fixed point z in X. Further, z is the unique common fixed point of f, S and of g, T.

Proof. The existence of a point t with $ft = St$ and $gt = Tt$ follows from Theorem (3.1). By the condition (4.10), we have

$$\begin{aligned} d(ft, gt) = d(St, Tt) &\le \Phi(d(ft, gt), d(ft, St), d(gt, Tt), d(ft, Tt), d(gt, St)) \\ &= \Phi(d(ft, gt), 0, 0, d(ft, gt), d(ft, gt)), \end{aligned}$$

which together with $\Phi(t, 0, 0, t, t) < t$ whenever $t > 0$ yields that $d(ft, gt) = 0$ and so $ft = St = gt = Tt$. By [2], since f and S are compatible mappings and $ft = St$, we deduce that

$$Sft = SSt = fSt = fft, \tag{4.11}$$

which along with condition (4.10) implies that

$$d(SSt, Tt) \le \Phi(d(SSt, Tt), 0, 0, d(SSt, Tt), d(SSt, Tt)). \tag{4.12}$$

It yields $d(SSt, Tt) = 0$, i.e., $SSt = Tt$. We thus have

$$Sft = SSt = Tt == gt = ft \tag{4.13}$$

and so $ft = z$ is a fixed point of S. Further, (4.11) and (4.13) imply that

$$Sz = SSt = fz = z.$$

Similarly, we conclude from the compatibility of g and T that $Tz = gz = z$. Therefore the point z is a common fixed point of f, g, S and T.

We now show the uniqueness of the common fixed point z. Let z' be another common fixed point of f and S. It follows from condition (4.10) that

$$\begin{aligned} d(z', z) = d(Sz', Tz) &\le \Phi(d(fz', gz), d(fz', Sz'), d(gz, Tz), d(fz', Tz), d(gz, Sz') \\ &= \Phi(d(z', z), 0, 0, d(z', z), d(z', z)), \end{aligned}$$

which together with the condition $\Phi(t, 0, 0, t, t) < t$ whenever $t > 0$ implies that $d(z', z) = 0$ and so $z = z'$. This completes the proof. □

COROLLARY (4.14). *Let (X, d) be a metric space and let mappings $S, T: X \to X$ be such that one of $S(X)$, $T(X)$ is a complete subspace of X. Suppose that there exists a function Φ satisfying Φ-type condition such that for all $x, y \in X$,*

$$d(Sx, Ty) \le \Phi(d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)) \tag{4.15}$$

Assume that for any given $t > 0$, $\Phi(t, 0, 0, t, t) < t$. Then S and T have a common fixed point z in X. Further, z is the unique common fixed point of S and of T.

Proof. Let $fx = gx = x$ in Theorem (3.1), then it follows from Theorem (3.1) that there exists a sequence $\{x_n\}$ in X such that $x_{2n-1} = Tx_{2n-2}$, $x_{2n} = Sx_{2n-1}$ for every n and $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n-1} = z$ for some $z \in X$. We show that z is a common fixed point of S and T.

Since

$$d(Sz, Tx_{2n}) \le \Phi(d(z, x_{2n}), d(z, Sz), d(x_{2n}, Tx_{2n}), d(z, Tx_{2n}), d(x_{2n}, Sz)),$$

taking the limit as $n \to \infty$, we obtain $d(Sz, z) \le \Phi(0, d(z, Sz), 0, 0, d(z, Sz)) < d(Sz, z)$, a contradiction, unless $z = Sz$. A similar argument applied to $d(Sx_{2n-1}, Tz)$ yields $z = Tz$.

As in the proof of Theorem (4.9), we have the uniqueness. This completes the proof. □

Remark (4.16). It is easy to see from the proofs of Theorem (3.1) and Corollary (4.14) and the proof of Theorem 1 in [10] that in Corollary (4.14) Φ-type condition is replaced by $\Phi(t, t, t, at, bt) < t$ for any $t > 0$, where $a, b \in \{0, 1, 2\}$ with $a+b = 2$, the Corollary (4.14) is also true. Thus we improve a main result of Husain and Sehgal [10] by replacing the completeness of the space X by one of $S(X)$, $T(X)$ being a complete subspace of X.

Example (4.17). Let $Y = X = \{x : 0 \le x \le 1, x \in Q\} \cup \{2\}$ be endowed with the usual metric, and let the mappings f and T be the same as f and T in Example (2.3), respectively. Define

$$\Phi(t_1, t_2, t_3, t_4, t_5) = \frac{9}{10}\max\{t_1, t_2, t_3\} + 3t_4t_5.$$

Then $\psi(t) = \frac{9}{10}t$. By a similar argument as in Examples (2.6) and (3.27), we have that f and T are ψ-harmonic and $\Phi(t_1, t_2, t_3, t_4, t_5)$ satisfies a Φ-type condition.

On the other hand, since $d(fx, Tx) = \frac{4}{5}$ when $x = 0$ and

$$H(Tx, Ty) = \begin{cases} 0, & x, y \ne 0 \text{ or } x = y = 0,\ x, y \in X \\ \frac{2}{5}, & x = 0, y \ne 0 \text{ or } x \ne 0, y = 0, \end{cases}$$

it is easy to see that the inequality (3.21) is satisfied. Note that $T(X) = \{0, \frac{1}{2}, \frac{9}{10}\}$ is complete. Thus all the hypothesis of Theorems (4.2) and (3.20) are satisfied, and $ft = t \in Tt$, $t = \frac{1}{2}$, $fz \in Tz$, $z = 1$.

Remark (4.18). The continuity of mappings in Theorems (3.20) and (4.2) is not assumed, and one can replace the completeness of the space by a set of weaker conditions. For instance, see Example (4.17) above.

Acknowledgement

The author would like to thank the anonymous referees for their helpful suggestions which led to an improved presentation of the manuscript.

This work was supported by The YoungTalent Foundation of Zhejiang Gongshang University and The Foundation of Department of Education of Zhejiang Province No. 20070628.

Received September 15, 2005

Final version received February 21, 2007

DEPARTMENT OF INFORMATION AND COMPUTER SCIENCE
ZHEJIANG GONGSHANG UNIVERSITY
HANGZHOU 310018
P. R. CHINA
laiyisheng@yahoo.com.cn

REFERENCES

[1] CHO Y. J., FISHER B., GENGA G. S., *Coincidence theorems for nonlinear hybrid contractions*, Internat. J. Math. Math. Sci. **20** (2), (1997), 249–256.

[2] KANEKO H., SESSA S., *Fixed point theorems for compatible multi-valued and single-valued mapping*, Internat. J. Math. Math. Sci. **12** (1989), 257–262.

[3] POPA V., *Coincidence and fixed points theorems for noncontinuous hybrid contractions*, Nonlinear Anal. Forum **7** (2), (2002), 153–158.

[4] POPA V., *Two general fixed points theorems for multifunctions satifying implicit relations*, Bul. Inst. Politeh. Ia\c si. Sec\c t. I. Mat. Mec. Teor. Fiz. **46**(50), (2000), no. 1-2, 41–46 (2001).

[5] TÜRKOĞLU D., ÖZER O., FISHER B., *A Coincidence Point Theorem For Multi-valued Contractions*, Math. Commun., 7(2002), 39–44.

[6] PATHAK H. K., *Fixed point theorems for weak compatible multi-valued and single-valued mapping*, Acta. Math. Hungar **67** (1-2), (1995), 69–78.

[7] PATHAK H. K., KANG S. M., CHO Y. J., *Coincidence and fixed point theorems for nonlinear hybrid generalized contractions*, Czechoslovak Math. J. **48** (123), (1998), 341–357.

[8] SHAHZAD, NASEER AND KAMRAN,TAYYAB, *Coincidence points and R-weakly commuting maps*, Arch. Math. (Brno), Tomus **37** (2001), 179–183.

[9] NADLER S., *Multivalued contraction mappings*, Pacific J. Math., **20** (1969), 475–488.

[10] HUSAIN S. A., SEHGAL V. M., *On common fixed points for a family mappings*, Bull. Austral. Math. Soc. **13** (1975), 261–267.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

REAL HESSIAN CURVES

ADRIANA ORTIZ-RODRÍGUEZ AND FRANK SOTTILE

ABSTRACT. We give some real polynomials in two variables of degrees 4, 5, and 6 whose hessian curves have more connected components than had been known previously. In particular, we give a quartic polynomial whose hessian curve has 4 compact connected components (ovals), a quintic whose hessian curve has 8 ovals, and a sextic whose hessian curve has 11 ovals.

1. Introduction

The *parabolic curve* on a generic smooth surface S embedded in three-dimensional Euclidean space consists of the points where S has zero Gaussian curvature. It separates *elliptic points* (where the curvature is positive) from *hyperbolic points* (where the curvature is negative). These notions are well-defined for surfaces embedded in affine or even projective space, as the sign of Gaussian curvature is invariant under affine transformations.

If the surface S is expressed locally as the graph $z = f(x, y)$ of a smooth function f, then the sign of its hessian determinant

$$\mathrm{Hess}(f) := \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2,$$

equals the sign of its curvature at the corresponding point. Thus the parabolic curve is the image under f of its *hessian curve*, which is defined by $\mathrm{Hess}(f) = 0$. When the surface S is the graph of a polynomial $f \in \mathbb{R}[x, y]$, this local description is global, and so questions about the disposition of the parabolic curve on S are equivalent to the same questions about the hessian curve in $\mathbb{R}^2$.

Suppose that d is even. Harnack proved [3] that a smooth plane curve of degree d has at most $1 + \binom{d-1}{2}$ connected components in $\mathbb{RP}^2$. This is also the bound for the number of components of a compact curve in $\mathbb{R}^2$ of degree d. A non-compact curve in $\mathbb{R}^2$ of degree d can have at most $\binom{d-1}{2}$ bounded components (*ovals*) and d unbounded components. These unbounded components come from the intersection of the corresponding curve in $\mathbb{RP}^2$ with the line at infinity. Harnack constructed a curve in $\mathbb{RP}^2$ of degree d with $1 + \binom{d-1}{2}$ components which has one component meeting the line at infinity in d points. This Harnack curve shows that the bound for non-compact curves in $\mathbb{R}^2$ is attained, and choosing a different line at infinity shows that the bound for compact curves in $\mathbb{R}^2$ is also attained.

2000 *Mathematics Subject Classification*: 51N10, 53A15.

Keywords and phrases: parabolic curve, problems type Harnack, configurations of hessian curves.

The first named author was supported by grants DGAPA-UNAM PAPIIT-IN105806 and SEP-CONACyT 41339-F.

Sottile supported in part by NSF CAREER grant DMS-0538734.

We are interested in the possible number and disposition of the components of the hessian curve in $\mathbb{R}^2$ of a polynomial $f \in \mathbb{R}[x, y]$ of degree n. This is problem 2001-1 in the list of Arnold's problems [1], attributed to A. Ortiz-Rodríguez. See also the discussion of related problems 2000-1, 2000-2, 2001-1, and 2002-1. The hessian of f has degree at most $2n - 4$. By Harnack's Theorem, a compact hessian curve has at most $(2n-5)(n-3)+1$ ovals and a non-compact hessian curve has at most $(2n-5)(n-3)$ ovals and $2n-4$ unbounded components.

While we know of no additional restrictions on hessian curves, we are not assured that all possible configurations are acheived by hessians. When n is at least 4, simple parameter counting shows that not all polynomials of degree $2n - 4$ arise as hessians of polynomials of degree n. The placement of the set of hessian curves among all curves of degree $2n - 4$ may restrict the possible configurations of hessian curves in $\mathbb{R}^2$. For example, a simple calculation shows that

$$\mathrm{Hess}(f) \;=\; \left(\frac{f_{xx}+f_{yy}}{2}\right)^2 - \left(\frac{f_{xx}-f_{yy}}{2}\right)^2 - f_{xy}^2.$$

Thus the hessian of a polynomial is a linear combination of 3 squares, which shows that the hessians lie in the second secant variety to the veronese embedding of polynomials of degree $n - 2$ in polynomials of degree $2n - 4$ (the veronese consists of the perfect squares).

We also know of no general techniques for constructing hessian curves with a prescribed configuration. One of us (Ortiz-Rodríguez) investigated this question [4, 5] and constructed polynomials $f \in \mathbb{R}[x, y]$ of degree n whose hessians had $\binom{n-1}{2}$ ovals in $\mathbb{R}^2$. When n is 4, 5, and 6, these numbers are 3, 6, and 10, respectively. We do not know if it is possible for a hessian curve to achieve the Harnack bound, or more generally, which configurations are possible for hessian curves.

Here, we present a quartic polynomial f whose hessian achieves the Harnack bound of 4 ovals, a quintic whose hessian has 8 ovals, a sextic whose hessian has 11 ovals, as well as examples of non-compact hessian curves of quartics, quintics, and sextics. These examples show that hessian curves can have more ovals than was previously known. They were found in a computer search, using the software Maple.

Our method was to generate a random polynomial, compute its hessian, and then compute an upper bound on its number of ovals in $\mathbb{RP}^2$, sometimes also screening for the number of unbounded components in $\mathbb{R}^2$. This upper bound was one-half the minimum number of real critical points of a projection to one of the axes, as each oval in $\mathbb{RP}^2$ contributes at least two critical points to the projection. We separately investigated compact hessian curves of sextics. Polynomials whose upper bound for ovals was at least 4, 8, and 11 (for quartics, quintics, and sextics, respectively) were saved for further study. The further investigation largely involved viewing pictures in $\mathbb{R}^2$ of these potentially interesting hessians. In all, only a few hundred polynomials warranted such further scrutiny.

We examined the hessians of 150 million quartics, 40 million each of quintics and sextics, and over 200 million sextics with compact hessians (the different protocol of pre-screening for compactness allowed a greater number to be examined). This required 628 days of CPU time on several computers, most of which were running Linux on Intel Pentium processors with speeds between 1.8 and 3 gigaHertz. We did not find a quartic whose hessian had 3 ovals and 4 unbounded components, nor a quintic whose hessian had more than 8 ovals in $\mathbb{RP}^2$, nor a sextic whose hessian had more than 11 ovals in $\mathbb{RP}^2$. (The examples we give at the end with 12 ovals in $\mathbb{RP}^2$ are pertubations of a curve we found with 11 ovals.) This suggests that it may not be possible for hessian curves in $\mathbb{R}^2$ to achieve the Harnack bounds. Further pictures and computer code are at the web page[1].

Tables 1 and 2 summarize this discussion concerning the number of components of hessian curves. The pairs (o, u) in Table 2 refer to ovals and unbounded components, respectively.

Degree of f	n	3	4	5	6	7
Degree of hessian	$2n-4$	2	4	6	8	10
Harnack bound for hessian	$(2n-5)(n-3)+1$	1	4	11	22	37
Ortiz hessians [4, 5]	$(n-1)(n-2)/2$	1	3	6	10	15
New examples			4	8	11	—

Table 1. Ovals of compact hessian curves.

Degree of f	n	3	4	5	6
Degree of hessian	$2n-4$	2	4	6	8
Harnack bound	$((2n-5)(n-3),\ 2n-4)$	(0,2)	(3,4)	(10,6)	(21,8)
New examples			(2,4)	(6,4)	(10,4)
			(3,2)	(7,2)	(11,2)

Table 2. Configurations of non-compact hessians.

2. Hessian curves with many ovals

We begin with the following observation about hessian polynomials.

PROPOSITION (2.1). *A polynomial $h(x, y)$ is a hessian of some polynomial f if and only if there exist polynomials p, q, r such that $p_y = q_x$, $q_y = r_x$, and $h = pr - q^2$.*

Proof. If h is the hessian of f, then $h = f_{xx}f_{yy} - f_{xy}^2$, and f_{xx}, f_{xy}, and f_{yy} satisfy these conditions. Conversely, if p, q, and r satisfy the conditions, then elementary integral calculus gives polynomials s and t such that $s_x = p$, $s_y = q$, $t_x = q$, and $t_y = r$. Since $s_y = t_x$, there is a polynomial f with $f_x = s$ and $f_y = t$, and thus h is the hessian of f. □

THEOREM (2.2). *There exists a real polynomial of degree 4 in two variables whose hessian curve is smooth, compact, and consists of exactly four ovals.*

[1] `www.math.tamu.edu/~sottile/stories/Hessian/index.html`.

Proof. Let f be the polynomial

$$-2y^2+2xy+12x^2+10y^3+3xy^2-10x^2y-13x^3-11y^4+6xy^3+9x^2y^2-2x^3y-x^4.$$

If we divide its hessian by -4, we obtain the polynomial

$$\begin{aligned} h := 25 - 134x - 374y + 91x^2 + 948xy + 1137y^2 + 429x^3 + 612x^2y \\ - 2313xy^2 - 876y^3 + 63x^4 + 54x^3y - 99x^2y^2 - 234xy^3 + 675y^4. \end{aligned}$$

We claim that the hessian curve, $h(x, y) = 0$, is a compact smooth curve in $\mathbb{R}^2$ with exactly 4 connected components. We provide a picture of the hessian curve in Figure 1. This was drawn by Maple using its `implicitplot` function

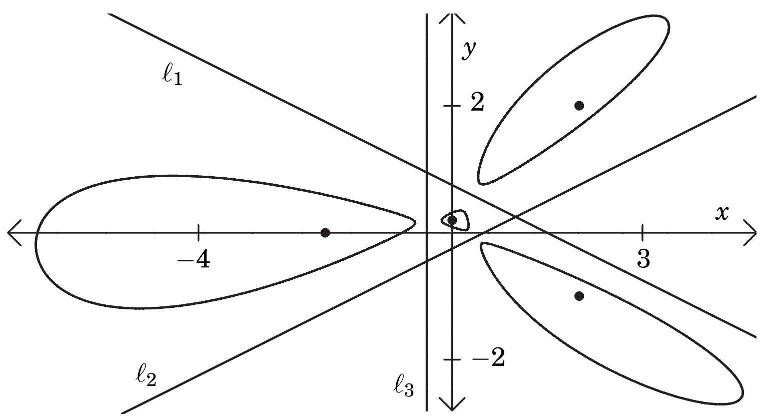


Figure 1. Quartic hessian curve with 4 compact components

with 200×200 grid. We give ad hoc arguments that verify our claim about the hessian curve.

We compute the values of the hessian at the four points inside each oval of Figure 1,

$$h(-2,0) = -7068,\ h(0,\tfrac{1}{5}) = -\tfrac{5124}{125},\ h(2,2) = -8508,\ \text{and}\ h(2,-1) = -6828\,.$$

Next, we shall prove that h is positive on three lines of Figure 1,

$$\ell_1 : y = \frac{3}{4} - \frac{x}{2}, \qquad \ell_2 : y = \frac{x}{2} - \frac{1}{4}, \qquad \text{and} \qquad \ell_3 : x = -\frac{2}{5},$$

and that it is positive on a neighborhood N of infinity.

The complement of the lines ℓ_1, ℓ_2, and ℓ_3 divide $\mathbb{R}^2$ into 7 components. Since h is positive on these lines and on N but is negative at the four points $(-2, 0)$, $(0, \frac{1}{5})$, $(2, 2)$, and $(2, -1)$, which lie in different regions, the hessian curve $h = 0$ is compact and has at least one 1-dimensional component in each region surrounding one of the four points. Since 4 is the maximum number of one-dimensional connected components of a quartic, and such quartics are smooth, we deduce that the hessian curve is smooth, compact, and consists of exactly four ovals.

Note that h contains the monomial term $63x^4$, and so it is positive near infinity along the x-axis. We show that h does not vanish on any of the three lines and that its homogenization does not vanish on the line ℓ_∞ at infinity in $\mathbb{RP}^2$, which implies our claims about the positivity of h. For this, we invoke a classical characterization of when a univariate quartic has no real zeroes. References may be found, for example in [2, §71].

Given a univariate quartic polynomial of the form

$$z^4 + 4\alpha z^3 + \beta z^2 + \gamma z + \delta\,,$$

linear substitution of $(z - \alpha)$ for z gives the reduced quartic

$$z^4 + az^2 + bz + c\,,$$

where $a = \beta - 6\alpha^2$, $b = \gamma - 2\alpha\beta + 8\alpha^3$, and $d = \delta - \alpha\gamma + \alpha^2\beta - 3\alpha^4$. The discriminant of this reduced quartic is

$$\Delta \;:=\; -4a^3b^2 - 27b^4 + 16a^4c - 128a^2c^2 + 144ab^2c + 256c^3\,.$$

This criterion also uses the polynomial

$$L \;:=\; 2a(a^2 - 4c) + 9b^2\,.$$

Then the quartic has no real zeroes if and only if

$$\Delta > 0 \quad \text{and either} \quad a \geq 0 \quad \text{or} \quad L \geq 0\,. \tag{2.3}$$

Homogenizing h, restricting it to the line at infinity, substituting $y = 1$, and dividing by 9 gives the quartic

$$q_\infty \;:=\; 7x^4 + 6x^3 - 11x^2 - 26x + 75\,.$$

(This is just the top-degree homogeneous piece of h.)

Restricting h to the lines ℓ_1, ℓ_2, and ℓ_3 and clearing denominators gives

$$q_1 := 21168x^4 - 157632x^3 + 592264x^2 - 337648x + 58387\,,$$
$$q_2 := 20016x^4 + 4608x^3 + 377320x^2 - 278112x + 52707\,,\ \text{and}$$
$$q_3 := 421875y^4 - 489000y^3 + 1278975y^2 - 411710y + 42073\,.$$

These satisfy the criterion (2.3) to have no real zeroes, as may be seen from Table 3, where we give the values of Δ, L, and a, for each of these polynomials. □

Polynomial	Δ	L	a
q_∞	$\frac{5025022208}{16807}$	$\frac{564896}{2401}$	$\frac{-181}{98}$
q_1	$\frac{10541505901315505865337 6}{198607342807439307}$	$\frac{3692894126604316}{340405734249}$	$\frac{931453}{129654}$
q_2	$\frac{3480737406935818536390 4}{141964610099247963}$	$\frac{4123100447100116}{272136458889}$	$\frac{6549023}{347778}$
q_3	$\frac{1004256582132069221868116 8}{8552615046501159667968 75}$	$\frac{1376823939540422}{40045166015625}$	$\frac{1066423}{421875}$

Table 3. Values of Δ, L, and a.

Each of the remaining curves we discuss is smooth, each oval has exactly two vertical and two horizontal tangents, and each unbounded component has exactly one vertical and one horizontal tangent. These claims are best verified symbolically. For each, we give the polynomial f and display a picture of the hessian curve, drawn with the `implicitplot` function of Maple. These were rendered, at least locally, with a grid size sufficiently small to separate the tangents, and therefore provide a faithful picture of the hessian curves as curves in $\mathbb{R}^2$.

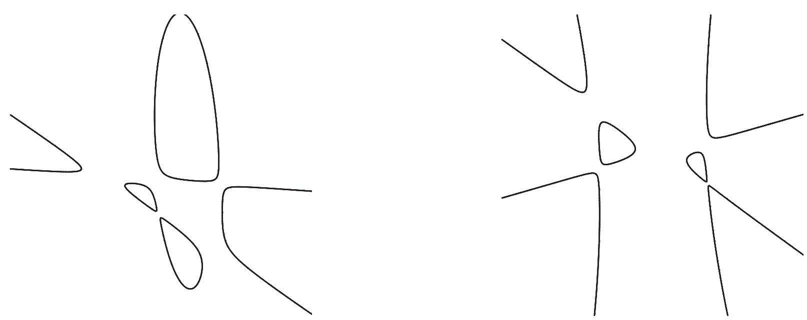
Figure 2(a) displays the hessian curve of the quartic

$$22x^2 + 36xy + 24y^2 - 80x^3 - 10x^2y + 71xy^2 + 39y^3 + 15x^4 + 4x^3y - 3x^2y^2 - 21xy^3 - 17y^4\,,$$

which has 3 ovals and 2 unbounded components. Figure 2(b) displays the hessian curve of the quartic

$$-70x^2 - 35xy - 2y^2 - 93x^3 - 14x^2y + 41xy^2 - 70y^3 + 31x^4 + 7x^3y - 30x^2y^2 + 37xy^3 + 91y^4\,,$$

which has 2 ovals and 4 unbounded components. While we have generated and checked 150 million quartics, we did not find one whose hessian curve achieves the Harnack bound of 3 ovals and 4 unbounded components.



(a) (b)

Figure 2. Hessians of quartics

Figure 3(a) displays the hessian curve of the quintic

$$\begin{aligned}&4y^2 + xy - 6x^2 - 25y^3 + 24xy^2 + 15x^2y - 33x^3 + y^4 - 3xy^3 + 15x^2y^2\\ &- 19x^3y - 26x^4 + 33y^5 - 2xy^4 - 23x^2y^3 - 30x^3y^2 - 26x^4y + 31x^5,\end{aligned}$$

which is compact with 8 ovals.

Figure 3(b) displays the hessian curve of the quintic

$$\begin{aligned}&-54y^2 - 103xy - 26x^2 - 88y^3 + 45xy^2 + 91x^2y - 96x^3 - 12y^4 + 43xy^3\\ &+ 6x^2y^2 + 11x^3y + 49x^4 + 22y^5 - 20xy^4 - 38x^2y^3 - 14x^3y^2 + 45x^4y + 76x^5\,,\end{aligned}$$

which has 7 ovals and 2 unbounded components.

(a) (b)

Figure 3. Hessians of quintics

Figure 4 displays the hessian curve of the quintic

$$60y^2 + 21xy + 76x^2 + 95y^3 - 18xy^2 - 79x^2y + 88x^3 - 25y^4 - 22xy^3$$
$$+ 50x^2y^2 - 9x^3y - 5x^4 - 57y^5 - 50xy^4 + 21x^2y^3 + 87x^3y^2 + 35x^4y - 56x^5,$$

which has 6 ovals and 4 unbounded components. The boxed region on the left has been expanded in the picture on the right.

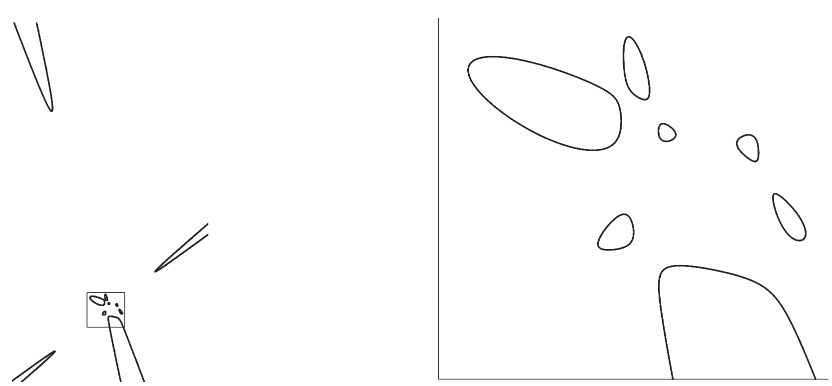


Figure 4. Hessian of a quintic with 6 ovals and 4 unbounded components

These quintics all have 8 ovals in $\mathbb{RP}^2$. While we have generated and checked 40 million quintics, we did not find any with more ovals.

Figure 5 displays the hessian curve of the sextic

$$\begin{aligned} &45y^2 - 47xy - 30x^2 \;+\; 96y^3 - xy^2 + 8x^2y + 54x^3 \\ -\;& 96y^4 - 64xy^3 - 50x^2y^2 - 33x^3y + 91x^4 \\ -\;& 100y^5 + 84xy^4 - 43x^3y^2 + 66x^4y - 58x^5 \\ +\;& 70y^6 + 90xy^5 - 28x^2y^4 - 53x^3y^3 + 43x^4y^2 + 36x^5y - 38x^6\,, \end{aligned}$$

which has 11 ovals. The boxed region on the left has been expanded in the picture on the right.

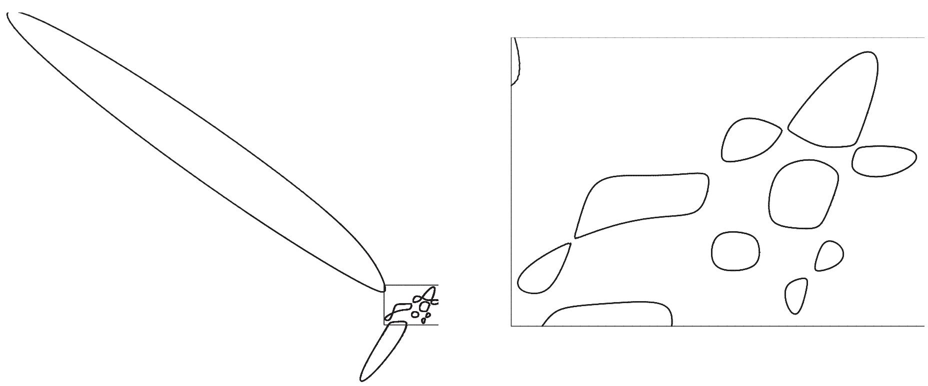


Figure 5. Hessian of a sextic with 11 ovals.

Figure 6(a) displays the hessian curve of the sextic

$$\begin{aligned} -\;& 53y^2 - 31xy + 59x^2 \;-\; 79y^3 + 82xy^2 - 52x^2y + 22x^3 \\ +\;& 75y^4 - 27xy^3 + 63x^2y^2 - 85x^3y - 89x^4 \\ +\;& 80y^5 + 27xy^4 - 69x^2y^3 + 17x^3y^2 - 7x^4y - 43x^5 \\ -\;& 25y^6 + 17xy^5 + 27x^2y^4 - 55x^3y^3 - 37x^4y^2 + 59x^5y + 45x^6\,, \end{aligned}$$

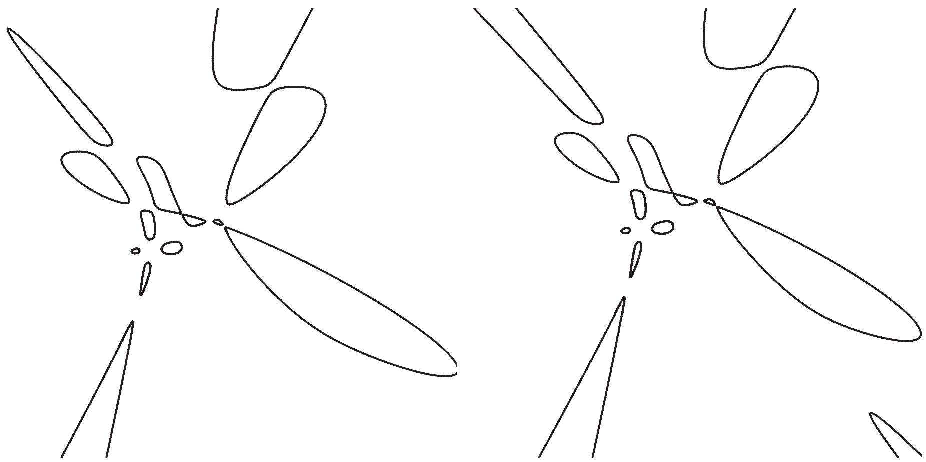
which has 11 ovals and 2 unbounded components.

Figure 6(b) displays the hessian curve of the sextic

$$\begin{aligned} -\;& 80y^2 - 46xy + 89x^2 \;-\; 118y^3 + 123xy^2 - 78x^2y + 33x^3 \\ +\;& 113y^4 - 40xy^3 + 94x^2y^2 - 128x^3y - 133x^4 \\ +\;& 120y^5 + 40xy^4 - 104x^2y^3 + 25x^3y^2 - 10x^4y - 64x^5 \\ -\;& 37y^6 + 25xy^5 + 40x^2y^4 - 82x^3y^3 - 56x^4y^2 + 89x^5y + 67x^6\,, \end{aligned}$$

which has 10 ovals and 4 unbounded components. Both hessian curves have 12 ovals in $\mathbb{RP}^2$.

Despite examining over 240 million sextics, we did not find any sextics whose hessian curves had more than 11 ovals in $\mathbb{RP}^2$. These last two examples, which have 12 ovals in $\mathbb{RP}^2$, are perturbations of a sextic found in the search whose hessian curve had 11 ovals in $\mathbb{RP}^2$.



(a) (b)

Figure 6. Hessians of sextics

Acknowledgments

We want to thank to V.I. Arnold, L. Ortiz-Bobadilla, V. Kharlamov, B. Reznick, and E. Rosales-González for their suggestions and comments. Sottile thanks Universidad Nacional Autónoma de México for its hospitality, and where this investigation was initiated.

Received May 13, 2005

Final version received November 8, 2006

ADRIANA ORTIZ-RODRÍGUEZ
INSTITUTO DE MATEMÁTICAS
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
ÁREA DE LA INV. CIENTÍFICA
CIRCUITO EXTERIOR, CIUDAD UNIVERSITARIA
MÉXICO, D.F. 04510
MÉXICO
aortiz@matem.unam.mx

FRANK SOTTILE
DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
COLLEGE STATION
TX 77843
USA
sottile@math.tamu.edu
http://www.math.tamu.edu/~sottile

References

[1] V. I. Arnold, Arnold's Problems, Springer-Verlag, Berlin, 2004.
[2] Dickson, L. E., First Course in the Theory of Equations, John Wiley, New York, 1922.
[3] Harnack A., *Über vieltheiligkeit der ebenen algebraischen curven*, Math. Ann. **10** (1876), 189–199.
[4] A. Ortiz-Rodríguez, *On the special parabolic points and the topology of the parabolic curve of certain smooth surfaces in* $\mathbb{R}^3$, C. R. Math. Acad. Sci. Paris **334** (6), (2002), 473–478.
[5] A. Ortiz-Rodríguez *Quelques aspects sur la géometrie des surfaces algébriques réelles*, Bulletin des Sciences mathématiques, **127** (2003), 149–177.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

ON L_p-BRUNN-MINKOWSKI TYPE INEQUALITIES OF CONVEX BODIES

FENGHONG LU AND GANGSONG LENG

ABSTRACT. In this paper L_p-Brunn-Minkowski type inequalities for L_p-projection bodies, L_p-centroid bodies, L_p-curvature images and L_p-polar projection bodies are established.

1. Introduction and main results

The classical Brunn-Minkowski inequality (see [4], [17]) states that if K, L are convex bodies in $\mathbb{R}^n$, then

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \tag{1.1}$$

with equality if and only if K and L are homothetic.

In [10], [11] Lutwak showed how Firey L_p-combination (see [3]) leads to the L_p-Brunn-Minkowski theory for $p \geq 1$. Lutwak established the extension of the classical Brunn-Minkowski inequality —the L_p-Brunn-Minkowski inequality— in [10], [11], which states that if K, L are convex bodies containing the origin in their interiors in $\mathbb{R}^n$, and $p > 1$, then

$$V(K +_p L)^{p/n} \geq V(K)^{p/n} + V(L)^{p/n}, \tag{1.2}$$

with equality if and only if K and L are dilates.

The Brunn-Minkowski inequality and its generalizations have in recent decades dramatically extended their influence in many areas of mathematics. Various applications have surfaced, for example, to probability and multivariate statistics, shapes of crystals, geometric tomography, elliptic partial differential equations, and combinatorics, see [1], [2], [4], [5], [17]. An excellent survey on this inequality is provided by Gardner [6].

In recent years, many authors devoted their attention to the L_p-Brunn-Minkowski theory, as a central part of convexity. For a detailed list of references on this subject, see, for instance, [14]. There are natural extensions of centroid bodies, projection bodies, curvatures, and John ellipsoids in the L_p-Brunn-Minkowski theory, see [11]-[15]. The purpose of this paper is to establish some new generalizations of the Brunn-Minkowski inequality to L_p-projection bodies [13], L_p-centroid bodies [12], [13], L_p-curvature images [11], and L_p-polar projection bodies [15], [16]. Our main results are the following theorems.

2000 *Mathematics Subject Classification*: 52A40, 52A39.

Keywords and phrases: L_p-Brunn-Minkowski inequality, L_p-projection body, L_p-centroid body, L_p-curvature image, L_p-polar projection body, polar.

Supported in part by the National Natural Science Foundation of China (Grant No.10671117).

THEOREM (1.3). *If $K,\ L \in \mathcal{K}_s^n$ and $n \neq p \geq 1$, then*

$$V(\Pi_p(K \bar{+}_p L))^{p/n} \geq V(\Pi_p K)^{p/n} + V(\Pi_p L)^{p/n}, \tag{1.4}$$

$$V(\Pi_p^*(K \bar{+}_p L))^{-p/n} \geq V(\Pi_p^* K)^{-p/n} + V(\Pi_p^* L)^{-p/n}, \tag{1.5}$$

with equality in (1.4) and (1.5) if and only if $\Pi_p K$ and $\Pi_p L$ are dilates.

Remark (1.6). If $p = 1$, $K \bar{+}_1 L$ is just the Blaschke linear combination of K and L [8].

THEOREM (1.7). *If $K, L \in \mathcal{S}_o^n$ and $p \geq 1$, then*

$$V(\Gamma_p(K \check{+}_p L))^{p/n} \geq V(\Gamma_p K)^{p/n} + V(\Gamma_p L)^{p/n}, \tag{1.8}$$

$$V(\Gamma_p^*(K \check{+}_p L))^{-p/n} \geq V(\Gamma_p^* K)^{-p/n} + V(\Gamma_p^* L)^{-p/n}, \tag{1.9}$$

with equality in (1.8) and (1.9) if and only if $\Gamma_p K$ and $\Gamma_p L$ are dilates.

Remark (1.10). If $p = 1$, $K \check{+}_1 L$ is just the harmonic Blaschke linear combination of K and L [8].

THEOREM (1.11). *If $K, L \in \mathcal{K}_s^n$ and $n \neq p \geq 1$, then*

(1.12)
$$V(\Gamma_{-p}(K \bar{+}_p L))^{-p/n} \geq \frac{V(K)}{V(K \bar{+}_p L)} V(\Gamma_{-p} K)^{-p/n} + \frac{V(L)}{V(K \bar{+}_p L)} V(\Gamma_{-p} L)^{-p/n},$$

(1.13)
$$V(\Gamma_{-p}^*(K \bar{+}_p L))^{p/n} \geq \frac{V(K)}{V(K \bar{+}_p L)} V(\Gamma_{-p}^* K)^{p/n} + \frac{V(L)}{V(K \bar{+}_p L)} V(\Gamma_{-p}^* L)^{p/n},$$

with equality in (1.12) and (1.13) if and only if $\Gamma_{-p} K$ and $\Gamma_{-p} L$ are dilates.

THEOREM (1.14). *If $K, L \in \mathcal{F}_s^n$ and $n \neq p \geq 1$, then*

$$V(\Lambda_p(K \bar{+}_p L))^{p/n} \geq V(\Lambda_p K)^{p/n} + V(\Lambda_p L)^{p/n}, \tag{1.15}$$

with equality if and only if $\Lambda_p K$ and $\Lambda_p L$ are dilates.

In Section 2, we give the necessary notation, definitions and background material. For reference see Gardner [4] and

Schneider [17]. We shall prove Theorems (1.3), (1.7), (1.11), and (1.14) in Section 3.

2. Notation and preliminaries

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$, for the set of convex bodies containing the origin in their interiors in $\mathbb{R}^n$, write $\mathcal{K}_o^n$. The subset of $\mathcal{K}_o^n$ consisting of the centered convex bodies will be denoted by $\mathcal{K}_s^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$.

If $K \in \mathcal{K}^n$, then its support function, $h_K = h(K, \cdot)\colon \mathbb{R}^n \longrightarrow \mathbb{R}$, is defined by

$$h(K, x) = \max\{x \cdot y\colon y \in K\}, \qquad x \in \mathbb{R}^n, \tag{2.1}$$

where $x \cdot y$ denotes the standard inner product of x and y. The Hausdorff distance, $\delta(K, L)$, between $K, L \in \mathcal{K}^n$, can be defined by $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ is the sup-norm on the space of continuous functions, $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$ which is star-shaped (about the origin), is its radial function, $\rho_K = \rho(K, \cdot)\colon \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$, defined by

$$\rho(K, x) = \max\{\lambda \geq 0\colon \lambda x \in K\}, \qquad x \in \mathbb{R}^n \setminus \{0\}. \tag{2.2}$$

If ρ_K is positive and continuous, K will be called a star body (about the origin). Let $\mathcal{S}_o^n$ denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Two star bodies K and L are said to be dilations (of each other) if $\rho(K, u)/\rho(L, u)$ is independent of all $u \in S^{n-1}$.

If $K \in \mathcal{K}_o^n$, the polar body of K, K^*, is defined by

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1, x \in K\}. \tag{2.3}$$

It is easy to verify that $(K^*)^* = K$. If $K \in \mathcal{K}_o^n$, then the support and radial function of K^* satisfy

$$h_{K^*} = \frac{1}{\rho_K} \qquad \text{and} \qquad \rho_{K^*} = \frac{1}{h_K}. \tag{2.4}$$

L_p-mixed volume. For $p \geq 1$, $K, L \in \mathcal{K}_o^n$ and $\varepsilon > 0$, the Firey L_p-combination $K +_p \varepsilon \cdot L$ is defined as the convex body whose support function is given by

$$h(K +_p \varepsilon \cdot L, \cdot)^p = h(K, \cdot)^p + \varepsilon h(L, \cdot)^p. \tag{2.5}$$

Firey combinations of convex bodies were defined and studied by Firey [3] (who called them p-means of convex bodies).

For $p \geq 1$, the L_p-mixed volume, $V_p(K, L)$, of $K, L \in \mathcal{K}_o^n$ can be defined by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

That this limit exists was demonstrated in [10].

It was shown in [10] that, corresponding to each convex body K in $\mathcal{K}_o^n$, there is a positive Borel measure, $S_p(K, \cdot)$, for $p \geq 1$, on S^{n-1} such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h(Q, u)^p dS_p(K, u), \tag{2.6}$$

for all $Q \in \mathcal{K}_o^n$. The measure $S_1(K, \cdot)$ is just the classical surface area measure of K and usually denoted by $S(K, \cdot)$ or S_K.

For $p \geq 1$, a convex body $K \in \mathcal{K}_o^n$ is said to have a p-curvature function, $f_p(K, \cdot)\colon S^{n-1} \longrightarrow \mathbb{R}$, if $S_p(K, \cdot)$ is absolutely continuous with respect to spherical Lebesgue measure, S, and

$$dS_p(K, \cdot)/dS = f_p(K, \cdot). \tag{2.7}$$

Let $\mathcal{F}_o^n$ denote set of all convex bodies in $\mathcal{K}_o^n$ that have a positive continuous p-curvature function, for $p \geq 1$. The subset of $\mathcal{F}_o^n$ consisting of the centered convex bodies will be denoted by $\mathcal{F}_s^n$.

From the definition of the L_p-mixed volume, it follows immediately that for each $K \in \mathcal{K}_o^n$,

$$V_p(K, K) = V(K). \tag{2.8}$$

We shall require a basic inequality for the L_p-mixed volume. The L_p-Minkowski inequality states that for $K, L \in \mathcal{K}_o^n$ and $p \geq 1$ (see [10, 11])

$$V_p(K, L) \geq V(K)^{(n-p)/n} V(L)^{p/n}, \tag{2.9}$$

with equality if and only if K and L are dilates.

In [10], a solution to the even L_p-Minkowski Problem in $\mathbb{R}^n$ was given for all $p \geq 1$ and $p \neq n$. From this, the L_p-Blaschke addition was defined by Lutwak in [10]. For $n \neq p \geq 1$ and $K, L \in \mathcal{K}_s^n$, the L_p-Blaschke addition $K \bar{+}_p L \in \mathcal{K}_s^n$ was defined in [10] by

$$S_p(K \bar{+}_p L, \cdot) = S_p(K, \cdot) + S_p(L, \cdot), \tag{2.10}$$

From definition (2.7) and (2.10), if $n \neq p \geq 1$, $K, L \in \mathcal{F}_s^n$, we have

$$f_p(K \bar{+}_p L, \cdot) = f_p(K, \cdot) + f_p(L, \cdot), \tag{2.11}$$

L_p**-dual mixed volume.** For star bodies K, L and $p \geq 1$, $\varepsilon > 0$, the L_p-harmonic radial combination $K +_{-p} \varepsilon \diamond L$ is defined as the star body whose radial function is given (see [11]) by

$$\rho(K +_{-p} \varepsilon \diamond L, \cdot)^{-p} = \rho(K, \cdot)^{-p} + \varepsilon \rho(L, \cdot)^{-p}. \tag{2.12}$$

For $p \geq 1$, the L_p-dual mixed volume $V_{-p}(K, L)$ of the star bodies K, L is defined (see [11]) by

$$\frac{n}{-p} V_{-p}(K, L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_{-p} \varepsilon \diamond L) - V(K)}{\varepsilon}. \tag{2.13}$$

The definition above and the polar coordinate formula for volume give the following integral representation of the L_p-dual mixed volume $V_{-p}(K, L)$ of the star bodies K, L (see [11])

$$V_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K^{n+p}(v) \rho_L^{-p}(v) dS(v), \tag{2.14}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From the definition of the L_p-dual mixed volumes, it follows immediately that for each $K \in \mathcal{S}_o^n$,

$$V_{-p}(K, K) = V(K). \tag{2.15}$$

We shall also require a basic inequality for the L_p-dual mixed volume. The L_p-Minkowski inequality for the L_p-dual mixed volumes states that for K, $L \in \mathcal{S}_o^n$ and $p \geq 1$ (see [11])

$$V_{-p}(K, L) \geq V(K)^{(n+p)/n} V(L)^{-p/n}, \tag{2.16}$$

with equality if and only if K and L are dilates.

Suppose $K, L \in \mathcal{S}_o^n$, we introduce the L_p-harmonic Blaschke addition of K and L, $K \check{+}_p L$. First define $\xi > 0$ by

$$\xi^{1/(n+p)} = \frac{1}{n} \int_{S^{n-1}} [V(K)^{-1} \rho(K, u)^{n+p} + V(L)^{-1} \rho(L, u)^{n+p}]^{n/(n+p)} dS(u). \tag{2.17}$$

The body $K \check{+}_p L \in \mathcal{S}_o^n$ is defined as the body whose radial function is given by

$$\xi^{-1} \rho(K \check{+}_p L, \cdot)^{n+p} = V(K)^{-1} \rho(K, \cdot)^{n+p} + V(L)^{-1} \rho(L, \cdot)^{n+p}. \tag{2.18}$$

By equalities (2.17), (2.18) and the polar coordinate formula for volume, we can get $\xi = V(K \check{+}_p L)$. Hence from equality (2.18), we obtain

$$\rho(K \check{+}_p L, \cdot)^{n+p} = \frac{V(K \check{+}_p L)}{V(K)} \rho(K, \cdot)^{n+p} + \frac{V(K \check{+}_p L)}{V(L)} \rho(L, \cdot)^{n+p}. \tag{2.19}$$

L_p-geometric bodies. Let $K \in \mathcal{K}_o^n$, for $p \geq 1$, the L_p-projection body of K, $\Pi_p K$, is the origin-symmetric convex body whose support function, for $u \in S^{n-1}$, is defined (see [13]) by

$$h(\Pi_p K, u)^p = \frac{1}{(n+p)c_{n,p}\omega_n} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v), \tag{2.20}$$

where

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}},$$

and ω_n denotes the n-dimensional volume of the unit ball B in $\mathbb{R}^n$, namely

$$\omega_n = \pi^{\frac{n}{2}} / \Gamma(1 + \frac{n}{2}).$$

If $K \in \mathcal{S}_o^n$, and $p \geq 1$, then the L_p-centroid body $\Gamma_p K$ of K is the origin-symmetric convex body whose support function, for $u \in S^{n-1}$, is given (see [12], [13]) by

$$h(\Gamma_p K, u)^p = \frac{1}{c_{n,p} V(K)} \int_K |\, u \cdot x \,|^p \, dx, \tag{2.21}$$

where the integration is with respect to Lebesgue measure.

If $K \in \mathcal{K}_o^n$ and $p > 0$, then the L_p-polar projection body $\Gamma_{-p} K$ is an origin-symmetric star body whose radial function, for $u \in S^{n-1}$, is given (see [15], [16]) by

$$\rho(\Gamma_{-p} K, u)^{-p} = \frac{1}{V(K)} \int_{S^{n-1}} |u \cdot v|^p dS_p(K, v)\,. \tag{2.22}$$

For $p \geq 1$ the body $\Gamma_{-p} K$ is a convex body [15].

For $p \geq 1$, $K \in \mathcal{F}_o^n$, Lutwak [11] defined the L_p-curvature image, $\Lambda_p K \in \mathcal{S}_o^n$, of K, by

$$\rho(\Lambda_p K, \cdot)^{n+p} = \frac{V(\Lambda_p K)}{\omega_n} f_p(K, \cdot). \tag{2.23}$$

It should be noted that for $p = 1$, this definition of curvature image differs from the definition used by Lutwak in ([7, 8, 9]).

3. Proof of the results

In order to prove these theorems we need the following lemmas.

LEMMA (3.1). *If K, $L \in \mathcal{K}_s^n$ and $p \geq 1$, then*

$$\Pi_p(K \tilde{+}_p L) = \Pi_p K +_p \Pi_p L\,. \tag{3.2}$$

Proof. From definition (2.20), definition (2.10) and definition (2.20) again, definition (2.5), it follows that

$$\begin{aligned} h(\Pi_p(K \tilde{+}_p L), u)^p &= \frac{1}{(n+p)c_{n,p}\omega_n} \int_{S^{n-1}} |u \cdot v|^p dS_p(K \tilde{+}_p L, v) \\ &= \frac{1}{(n+p)c_{n,p}\omega_n} \int_{S^{n-1}} |u \cdot v|^p (dS_p(K, v) + dS_p(L, v)) \\ &= h(\Pi_p K, u)^p + h(\Pi_p L, u)^p = h(\Pi_p K +_p \Pi_p L, u)^p. \end{aligned}$$

□

LEMMA (3.3). *If K, $L \in \mathcal{S}_o^n$ and $p \geq 1$, then*

$$\Gamma_p(K \check{+}_p L) = \Gamma_p K +_p \Gamma_p L\,. \tag{3.4}$$

Proof. From definition (2.21), definition (2.19) and definition (2.21) again, definition (2.5), it follows that

$$\begin{aligned} h(\Gamma_p(K \check{+}_p L), u)^p &= \frac{1}{c_{n,p} V(K \check{+}_p L)} \int_{K \check{+}_p L} |u \cdot x|^p dx \\ &= \frac{1}{(n+p) c_{n,p} V(K \check{+}_p L)} \int_{S^{n-1}} |u \cdot v|^p \rho(K \check{+}_p L, v)^{n+p} dS(v) \\ &= \frac{1}{(n+p) c_{n,p}} \int_{S^{n-1}} |u \cdot v|^p \Big(\frac{\rho(K, v)^{n+p}}{V(K)} + \frac{\rho(L, v)^{n+p}}{V(L)} \Big) dS(v) \\ &= h(\Gamma_p K, u)^p + h(\Gamma_p L, u)^p = h(\Gamma_p K +_p \Gamma_p L, u)^p. \end{aligned}$$

□

LEMMA (3.5). *If K, $L \in \mathcal{K}_s^n$ and $p \geq 1$, then*

$$\Gamma_{-p}(K \bar{+}_p L) = \frac{V(K)}{V(K \bar{+}_p L)} \diamond \Gamma_{-p} K +_{-p} \frac{V(L)}{V(K \bar{+}_p L)} \diamond \Gamma_{-p} L\,. \tag{3.6}$$

Proof. From definition (2.22), definition (2.10) and definition (2.22) again, definition (2.12), it follows that

$$\begin{aligned} \rho(\Gamma_{-p}(K \bar{+}_p L), u)^{-p} &= \frac{1}{V(K \bar{+}_p L)} \int_{S^{n-1}} |u \cdot v|^p dS_p(K \bar{+}_p L, v) \\ &= \frac{1}{V(K \bar{+}_p L)} \int_{S^{n-1}} |u \cdot v|^p (dS_p(K, v) + dS_p(L, v)) \\ &= \frac{V(K)}{V(K \bar{+}_p L)} \rho(\Gamma_{-p} K, u)^{-p} + \frac{V(L)}{V(K \bar{+}_p L)} \rho(\Gamma_{-p} L, u)^{-p} \\ &= \rho\Big(\frac{V(K)}{V(K \bar{+}_p L)} \diamond \Gamma_{-p} K +_{-p} \frac{V(L)}{V(K \bar{+}_p L)} \diamond \Gamma_{-p} L, u \Big)^{-p}. \end{aligned}$$

□

LEMMA (3.7). *If K, $L \in \mathcal{F}_s^n$ and $p \geq 1$, then*

$$\Lambda_p(K \bar{+}_p L) = \left(\frac{V(\Lambda_p(K \bar{+}_p L))}{V(\Lambda_p K \check{+}_p \Lambda_p L)} \right)^{1/(n+p)} (\Lambda_p K \check{+}_p \Lambda_p L)\,. \tag{3.8}$$

Proof. From definition (2.23), equality (2.11), and definition (2.23) again, definition (2.19), it follows that

$$\begin{aligned} \rho(\Lambda_p(K \bar{+}_p L), u)^{n+p} &= \frac{V(\Lambda_p(K \bar{+}_p L))}{\omega_n} f_p(K \bar{+}_p L, u) \\ &= \frac{V(\Lambda_p(K \bar{+}_p L))}{\omega_n} (f_p(K, u) + f_p(L, u)) \\ &= \frac{V(\Lambda_p(K \bar{+}_p L))}{V(\Lambda_p K)} \rho(\Lambda_p K, u)^{n+p} + \frac{V(\Lambda_p(K \bar{+}_p L))}{V(\Lambda_p L)} \rho(\Lambda_p L, u)^{n+p} \\ &= \frac{V(\Lambda_p(K \bar{+}_p L))}{V(\Lambda_p K \check{+}_p \Lambda_p L)} \rho(\Lambda_p K \check{+}_p \Lambda_p L, u)^{n+p}. \end{aligned}$$

□

Proof of Theorem (1.3). Let $K, L \in \mathcal{K}_s^n$ and $n \neq p \geq 1$. From definition (2.6), Lemma (3.1) and the L_p-Minkowski inequality (2.9), for any $M \in \mathcal{K}_o^n$, it follows that

$$\begin{aligned} V_p(M, \Pi_p(K \bar{+}_p L)) &= V_p(M, \Pi_p K +_p \Pi_p L) \\ &= V_p(M, \Pi_p K) + V_p(M, \Pi_p L) \\ &\geq V(M)^{(n-p)/n}(V(\Pi_p K)^{p/n} + V(\Pi_p L)^{p/n}), \end{aligned}$$

with equality if and only if M, $\Pi_p K$ and $\Pi_p L$ are dilates.

Let $M = \Pi_p(K \bar{+}_p L)$, we get

$$V(\Pi_p(K \bar{+}_p L))^{p/n} \geq V(\Pi_p K)^{p/n} + V(\Pi_p L)^{p/n},$$

with equality if and only if $\Pi_p K$ and $\Pi_p L$ are dilates.

Therefore we have proved inequality (1.4).

Let $K, L \in \mathcal{K}_s^n$ and $n \neq p \geq 1$. From the polar coordinate formula for volume, Lemma (3.1) and the Minkowski integral inequality (see [4], [17]), it follows that

$$\begin{aligned} V(\Pi_p^*(K \bar{+}_p L))^{-p/n} &= \left(\frac{1}{n}\int_{S^{n-1}} (h(\Pi_p(K \bar{+}_p L), u)^p)^{-n/p} dS(u)\right)^{-p/n} \\ &= n^{p/n}\|h(\Pi_p K, u)^p + h(\Pi_p L, u)^p\|_{-n/p} \\ &\geq n^{p/n}\|h(\Pi_p K, u)^p\|_{-n/p} + n^{p/n}\|h(\Pi_p L, u)^p\|_{-n/p} \\ &= V(\Pi_p^* K)^{-p/n} + V(\Pi_p^* L)^{-p/n}, \end{aligned}$$

with equality if and only if $\Pi_p K$ and $\Pi_p L$ are dilates.

Therefore we have proved inequality (1.5). $\square$

Proof of Theorem (1.7). Let $K, L \in \mathcal{S}_o^n$ and $p \geq 1$. From definition (2.6), Lemma (3.3) and the L_p-Minkowski inequality (2.9), for any $M \in \mathcal{K}_o^n$, it follows that

$$\begin{aligned} V_p(M, \Gamma_p(K \check{+}_p L)) &= V_p(M, \Gamma_p K +_p \Gamma_p L) \\ &= V_p(M, \Gamma_p K) + V_p(M, \Gamma_p L) \\ &\geq V(M)^{(n-p)/n}(V(\Gamma_p K)^{p/n} + V(\Gamma_p L)^{p/n}), \end{aligned}$$

with equality if and only if M, $\Gamma_p K$ and $\Gamma_p L$ are dilates.

Let $M = \Gamma_p(K \check{+}_p L)$, we get

$$V(\Gamma_p(K \check{+}_p L))^{p/n} \geq V(\Gamma_p K)^{p/n} + V(\Gamma_p L)^{p/n},$$

with equality if and only if $\Gamma_p K$ and $\Gamma_p L$ are dilates.

Therefore we have proved inequality (1.8).

Let $K, L \in \mathcal{S}_o^n$ and $p \geq 1$. From the polar coordinate formula for volume, Lemma (3.3) and the Minkowski integral inequality (see [4], [17]), it follows that

$$\begin{aligned}
V(\Gamma_p^*(K\check{+}_pL))^{-p/n} &= \left(\frac{1}{n}\int_{S^{n-1}}(h(\Gamma_p(K\check{+}_pL),u)^p)^{-n/p}dS(u)\right)^{-p/n}\\
&= n^{p/n}\|h(\Gamma_pK,u)^p + h(\Gamma_pL,u)^p\|_{-n/p}\\
&\geq n^{p/n}\|h(\Gamma_pK,u)^p\|_{-n/p} + n^{p/n}\|h(\Gamma_pL,u)^p\|_{-n/p}\\
&= V(\Gamma_p^*K)^{-p/n} + V(\Gamma_p^*L)^{-p/n},
\end{aligned}$$

with equality if and only if Γ_pK and Γ_pL are dilates.

Therefore we have proved inequality (1.9). $\square$

Proof of Theorem (1.11). Let $K, L \in \mathcal{K}_s^n$ and $n \neq p \geq 1$. From definition (2.14), Lemma (3.5) and the L_p-Minkowski inequality (2.16), for any $M \in \mathcal{S}_o^n$ it follows that

$$\begin{aligned}
V_{-p}(M,\Gamma_{-p}(K\bar{+}_pL)) &= V_{-p}\Big(M, \frac{V(K)}{V(K\bar{+}_pL)}\diamond\Gamma_{-p}K +_{-p} \frac{V(L)}{V(K\bar{+}_pL)}\diamond\Gamma_{-p}L\Big)\\
&= \frac{V(K)}{V(K\bar{+}_pL)}V_{-p}(M,\Gamma_{-p}K) + \frac{V(L)}{V(K\bar{+}_pL)}V_{-p}(M,\Gamma_{-p}L)\\
&\geq V(M)^{(n+p)/n}\left(\frac{V(K)}{V(K\bar{+}_pL)}V(\Gamma_{-p}K)^{-p/n} + \frac{V(L)}{V(K\bar{+}_pL)}V(\Gamma_{-p}L)^{-p/n}\right),
\end{aligned}$$

with equality if and only if M, $\Gamma_{-p}K$ and $\Gamma_{-p}L$ are dilates.

Let $M = \Gamma_{-p}(K\bar{+}_pL)$, we get

$$V(\Gamma_{-p}(K\bar{+}_pL))^{-p/n} \geq \frac{V(K)}{V(K\bar{+}_pL)}V(\Gamma_{-p}K)^{-p/n} + \frac{V(L)}{V(K\bar{+}_pL)}V(\Gamma_{-p}L)^{-p/n},$$

with equality if and only if $\Gamma_{-p}K$ and $\Gamma_{-p}L$ are dilates.

Therefore we have proved inequality (1.12).

Let $K, L \in \mathcal{K}_s^n$ and $n \neq p \geq 1$. From definition (2.6), Lemma (3.5) and the L_p-Minkowski inequality (2.9), for any $M \in \mathcal{K}_o^n$, it follows that

$$\begin{aligned}
V_p(M,\Gamma_{-p}^*(K\bar{+}_pL)) &= \frac{1}{n}\int_{S^{n-1}} h(\Gamma_{-p}^*(K\bar{+}_pL),u)^p dS_p(M,u)\\
&= \frac{1}{n}\int_{S^{n-1}} \rho(\Gamma_{-p}(K\bar{+}_pL),u)^{-p} dS_p(M,u)\\
&= \frac{1}{n}\int_{S^{n-1}}\left(\frac{V(K)\rho(\Gamma_{-p}K,u)^{-p}}{V(K\bar{+}_pL)} + \frac{V(L)\rho(\Gamma_{-p}L,u)^{-p}}{V(K\bar{+}_pL)}\right)dS_p(M,u)\\
&= \frac{V(K)}{V(K\bar{+}_pL)}V_p(M,\Gamma_{-p}^*K) + \frac{V(L)}{V(K\bar{+}_pL)}V_p(M,\Gamma_{-p}^*L)\\
&\geq V(M)^{(n-p)/n}\left(\frac{V(K)}{V(K\bar{+}_pL)}V(\Gamma_{-p}^*K)^{p/n} + \frac{V(L)}{V(K\bar{+}_pL)}V(\Gamma_{-p}^*L)^{p/n}\right),
\end{aligned}$$

with equality if and only if M, Γ_{-p}^*K and Γ_{-p}^*L are dilates.

Let $M = \Gamma_{-p}^*(K\bar{+}_pL)$, we get

$$V(\Gamma_{-p}^*(K\bar{+}_pL))^{p/n} \geq \frac{V(K)}{V(K\bar{+}_pL)}V(\Gamma_{-p}^*K)^{p/n} + \frac{V(L)}{V(K\bar{+}_pL)}V(\Gamma_{-p}^*L)^{p/n},$$

with equality if and only if $\Gamma_{-p}K$ and $\Gamma_{-p}L$ are dilates.

Therefore we have proved inequality (1.13). $\square$

Proof of Theorem (1.14). For $K, L \in \mathcal{F}_s^n$ and $n \neq p \geq 1$. From Lemma (3.7), definition (2.14), definition (2.19) and the L_p-Minkowski inequality (2.16), for any $M \in \mathcal{S}_o^n$, it follows that

$$\begin{aligned} V_{-p}(\Lambda_p(K\bar{+}_pL), M) &= V_{-p}(\left(\frac{V(\Lambda_p(K\bar{+}_pL))}{V(\Lambda_pK\check{+}_p\Lambda_pL)}\right)^{1/(n+p)}(\Lambda_pK\check{+}_p\Lambda_pL), M) \\ &= \frac{V(\Lambda_p(K\bar{+}_pL))}{V(\Lambda_pK)}V_{-p}(\Lambda_pK, M) \\ &\qquad + \frac{V(\Lambda_p(K\bar{+}_pL))}{V(\Lambda_pL)}V_{-p}(\Lambda_pL, M) \\ &\geq \left(\frac{V(\Lambda_p(K\bar{+}_pL))}{V(\Lambda_pK)}V(\Lambda_pK)^{(n+p)/n}\right. \\ &\qquad \left. + \frac{V(\Lambda_p(K\bar{+}_pL))}{V(\Lambda_pL)}V(\Lambda_pL)^{(n+p)/n}\right)V(M)^{-p/n}, \end{aligned}$$

with equality if and only if M, Λ_pK and Λ_pL are dilates.

Let $M = \Lambda_p(K\bar{+}_pL)$, we get

$$V(\Lambda_p(K\bar{+}_pL))^{p/n} \geq V(\Lambda_pK)^{p/n} + V(\Lambda_pL)^{p/n},$$

with equality if and only if Λ_pK and Λ_pL are dilates.

Therefore we have proved inequality (1.15). □

Acknowledgement

The authors wish to thank the referees for their many excellent suggestions for improving the original manuscript.

Received May 19, 2006

Final version received December 11, 2006

FENGHONG LU
DEPARTMENT OF MATHEMATICS
SHANGHAI UNIVERSITY
SHANGHAI, 200444
P. R. CHINA
lufh325@163.com, fenghongaoran@sina.com

DEPARTMENT OF MATHEMATICS
SHANGHAI UNIVERSITY OF ELECTRIC POWER
SHANGHAI, 200090
P. R. CHINA

GANGSONG LENG
DEPARTMENT OF MATHEMATICS
SHANGHAI UNIVERSITY
SHANGHAI, 200444
P. R. CHINA
gleng@staff.shu.edu.cn

REFERENCES

[1] I. J. BAKELMAN, Convex Analysis and Nonlinear Geometric Elliptic Equations, Springer-Verlag, Berlin, 1994.

[2] Y. D. BURAGO AND V. A. ZALGALLER, Geometric Inequalities, Springer-Verlag, New York, 1988.

[3] W. J. FIREY, *p-means of convex bodies*, Math. Scand. **10** (1962), 17–24.

[4] R. J. GARDNER, Geometric Tomography, Encyclopedia of Mathematics and Its Applications, vol. 58, Cambridge University Press, Cambridge, 1995.

[5] R. J. GARDNER AND P. GRONCHI, *A Brunn-Minkowski inequality for the integer lattice*, Trans. Amer. Math. Soc. **353** (2001), 3995–4024.

[6] R. J. GARDNER, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. **39** (2002), 355–405.

[7] E. LUTWAK, *On some affine isoperimetric inequalities*, J. Differential Geom. **23** (1986), 1–13.

[8] E. LUTWAK, *Centroid bodies and dual mixed volumes*, Proc. London Math. Soc. **60** (1990), 365–391.

[9] E. LUTWAK, *Extended affine surface area*, Adv. Math. **85** (1991), 39–68.

[10] E. LUTWAK, *The Brunn-Minkowski-Firey theory I: mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131–150.

[11] E. LUTWAK, *The Brunn-Minkowski-Firey theory II: affine and geominimal surface areas*, Adv. Math. **118** (1996), 244–294.

[12] E. LUTWAK AND G. ZHANG, *Blaschke-Santaló inequalities*, J. Differential Geom. **47** (1997), 1–16.

[13] E. LUTWAK, D. YANG AND G. ZHANG, *L_p affine isoperimetric inequalities*, J. Differential Geom. **56** (2000), 111–132.

[14] E. LUTWAK, D. YANG AND G. ZHANG, *Volume inequalities for subspaces of L_p*, J. Differential Geom. **68** (2004), 159–184.

[15] E. LUTWAK, D. YANG AND G. ZHANG, *L_p John ellipsoids*, Proc. London Math. Soc. **90** (2005), 497–520.

[16] E. LUTWAK, D. YANG AND G. ZHANG, *Optimal Sobolev norms and the L_p Minkowski problem*, Int. Math. Res. Not. (2006), 1–21.

[17] R. SCHNEIDER, Convex Bodies: the Brunn-Minkowski Theory, Encyclopedia of Mathematics and Its Applications, vol. 44, Cambridge University Press, Cambridge, 1993.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

CURVES WITH CONSTANT CURVATURE RATIOS

J. MONTERDE

ABSTRACT. Curves in $\mathbb{R}^n$ for which the ratios between two consecutive curvatures are constant are characterized by the fact that their tangent indicatrix is a geodesic in a flat torus. For $n = 3, 4$, spherical curves of this kind are also studied and compared with intrinsic helices in the sphere.

1. Introduction

The notion of a generalized helix in $\mathbb{R}^3$, a curve making a constant angle with a fixed direction, can be generalized to higher dimensions in many ways. In [7] the same definition is proposed but in $\mathbb{R}^n$. In [4] the definition is more restrictive: the fixed direction makes a constant angle with all the vectors of the Frenet frame. It is easy to check that this definition only works in the odd dimensional case. Moreover, in the same reference, it is proven that the definition is equivalent to the fact that the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \dots$, k_i being the curvatures, are constant. This statement is related with the Lancret Theorem for generalized helices in $\mathbb{R}^3$ (the ratio of torsion to curvature is constant). Finally, in [1] the author proposes a definition of a general helix in a 3-dimensional real-space-form substituting the fixed direction in the usual definition of generalized helix by a Killing vector field along the curve.

In this paper we study the curves in $\mathbb{R}^n$ for which all the ratios $\frac{k_2}{k_1}, \frac{k_3}{k_2}, \frac{k_4}{k_3}, \dots$ are constant. We call them curves with constant curvature ratios or ccr-curves. The main result is that, in the even dimensional case, a curve has constant curvature ratios if and only if its tangent indicatrix is a geodesic in the flat torus. In the odd case, a constant must be added as the new coordinate function.

In the last section we show that a ccr-curve in S^3 is a general helix in the sense of [1] if and only if it has constant curvatures. To achieve this result, we have obtained the characterization of spherical curves in $\mathbb{R}^4$ in terms of the curvatures. Moreover, we have also found explicit examples of spherical ccr-curves with non-constant curvatures.

2. Frenet's elements for a curve in $\mathbb{R}^n$

Let us recall from [5] the definition of the Frenet frame and curvatures.

For C^{n-1} curves, α, which have linearly independent derivatives up to order $n-1$, the moving Frenet frame is constructed as if it were in usual space using the Gram-Schmidt process. Orthonormal vectors $\{\overrightarrow{\mathbf{e_1}}, \overrightarrow{\mathbf{e_2}}, \dots, \overrightarrow{\mathbf{e_{n-1}}}\}$ are obtained and the last vector is added as the unit vector in $\mathbb{R}^n$ such that $\{\overrightarrow{\mathbf{e_1}}, \overrightarrow{\mathbf{e_2}}, \dots, \overrightarrow{\mathbf{e_n}}\}$ is an orthonormal basis with positive orientation.

2000 *Mathematics Subject Classification*: Primary 53A04; Secondary 53C40, 53A05.
Keywords and phrases: spherical curves, generalized helices, theorem of Lancret.

The ith curvature is defined as

$$k_i = \frac{\langle \dot{\overrightarrow{\mathbf{e_i}}}, \overrightarrow{\mathbf{e_{i+1}}} \rangle}{||\alpha'||},$$

for $i = 1, \ldots, n-1$.

Frenet's formulae in n-space can be written as

$$(2.1)\quad \begin{pmatrix} \dot{\overrightarrow{\mathbf{e_1}}}(s) \\ \dot{\overrightarrow{\mathbf{e_2}}}(s) \\ \dot{\overrightarrow{\mathbf{e_3}}}(s) \\ \vdots \\ \dot{\overrightarrow{\mathbf{e_{n-1}}}}(s) \\ \dot{\overrightarrow{\mathbf{e_n}}}(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 & 0 & \ldots & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & \ldots & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & k_{n-1} \\ 0 & 0 & 0 & 0 & \ldots & -k_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{\mathbf{e_1}}(s) \\ \overrightarrow{\mathbf{e_2}}(s) \\ \overrightarrow{\mathbf{e_3}}(s) \\ \vdots \\ \overrightarrow{\mathbf{e_{n-1}}}(s) \\ \overrightarrow{\mathbf{e_n}}(s) \end{pmatrix}.$$

In accordance with [7] we will say that a curve is twisted if its last curvature k_{n-1} is not zero. Sometimes we will also say that the curve is not regular.

3. ccr-curves

Instead of looking for curves making a constant angle with a fixed direction as in [4] or [7], we will study another way of generalizing the notion of helix.

Definition (3.1). A curve $\alpha\colon I \to \mathbb{R}^n$ is said to have constant curvature ratios (that is to say, it is a ccr-curve) if all the quotients $\frac{k_{i+1}}{k_i}$ are constant.

As is well known, generalized helices in $\mathbb{R}^3$ are characterized by the fact that the quotient $\frac{\tau}{\kappa}$ is constant (Lancret's theorem). It is in this sense that ccr-curves are a generalization to $\mathbb{R}^n$ of generalized helices in $\mathbb{R}^3$.

In [4] the author defines a generalized helix in the n-dimensional space (n odd) as a curve satisfying that the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \ldots$ are constant. It is also proven that a curve is a generalized helix if and only if there exists a fixed direction which makes constant angles with all the vectors of the Frenet frame. Obviously, ccr-curves are a subset of generalized helices in the sense of [4].

(3.2) Examples.

3.2.1. *Example with constant curvatures.* The subset of $\mathbb{R}^{2n}$ parametrized by

$$\begin{aligned} &\overrightarrow{\mathbf{x}}(u_1, u_2, \ldots, u_n) \\ &\quad = (r_1 \cos(u_1), r_1 \sin(u_1), r_2 \cos(u_2), r_2 \sin(u_2), \ldots, r_n \cos(u_n), r_n \sin(u_n)) \end{aligned}$$

where $u_i \in \mathbb{R}$ is called a flat torus in $\mathbb{R}^{2n}$.

By analogy, the subset of $\mathbb{R}^{2n+1}$ parametrized by

$$\begin{aligned} &\overrightarrow{\mathbf{x}}(u_1, u_2, \ldots, u_n) \\ &\quad = (r_1 \cos(u_1), r_1 \sin(u_1), r_2 \cos(u_2), r_2 \sin(u_2), \ldots, r_n \cos(u_n), r_n \sin(u_n), a) \end{aligned}$$

where $u_i \in \mathbb{R}$ and a is a real constant, will be called a flat torus in $\mathbb{R}^{2n+1}$.

It is just a matter of computation to show that any curve in a flat torus of the kind

$$\alpha(t) = \overrightarrow{\mathbf{x}}(m_1 t, m_2 t, \ldots, m_n t)$$

has all its curvatures constant (see [6]).

These curves are the geodesics of the flat tori, and it is proven in the cited paper that they are twisted curves if and only if the constants $m_i \neq m_j$ for all $i \neq j$.

3.2.2. *Example with non-constant curvatures.* Now, let $k(s)$ be a positive function. Let us define $g(s) = \int_0^s k(u)du$. If α is a curve parametrized by its arc-length and with constant curvatures, $a_1, a_2, \ldots, a_{n-1}$, then the curve $\beta(s) = \int_0^s \overrightarrow{\mathbf{e_1}}^\alpha(g(u))du$ is a curve whose curvatures are $k_i(s) = a_i k(s)$.

Note that $\dot\beta(s) = \overrightarrow{\mathbf{e_1}}^\alpha(g(s))$. This implies that $\overrightarrow{\mathbf{e_1}}^\beta(s) = \overrightarrow{\mathbf{e_1}}^\alpha(g(s))$. Taking derivatives $k_1^\beta(s)\overrightarrow{\mathbf{e_2}}^\beta(s) = k_1^\alpha(g(s))\overrightarrow{\mathbf{e_2}}^\alpha(g(s))k(s)$. Therefore,

$$\overrightarrow{\mathbf{e_2}}^\beta(s) = \overrightarrow{\mathbf{e_2}}^\alpha(g(s)), \qquad \text{and} \qquad k_1^\beta(s) = a_1 k(s).$$

By similar arguments it is possible to show that $k_i^\beta(s) = a_i k(s)$ for any $i = 1, \ldots, n-1$. Therefore, β is a ccr-curve with non-constant curvatures.

In the next section we will show that every ccr-curve is of this kind.

4. Solving the natural equations for ccr-curves

The Frenet formulae can be explicitly integrated only for some particular cases. Ccr-curves are one of these. In fact, Frenet's formulae are

$$\begin{pmatrix} \dot{\overrightarrow{\mathbf{e_1}}}(s) \\ \dot{\overrightarrow{\mathbf{e_2}}}(s) \\ \dot{\overrightarrow{\mathbf{e_3}}}(s) \\ \vdots \\ \dot{\overrightarrow{\mathbf{e_{n-1}}}}(s) \\ \dot{\overrightarrow{\mathbf{e_n}}}(s) \end{pmatrix} = k_1(s) \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 0 & c_2 & 0 & \ldots & 0 & 0 \\ 0 & -c_2 & 0 & c_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & c_{n-1} \\ 0 & 0 & 0 & 0 & \ldots & -c_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{\mathbf{e_1}}(s) \\ \overrightarrow{\mathbf{e_2}}(s) \\ \overrightarrow{\mathbf{e_3}}(s) \\ \vdots \\ \overrightarrow{\mathbf{e_{n-1}}}(s) \\ \overrightarrow{\mathbf{e_n}}(s) \end{pmatrix},$$

for some constants $c_2, \ldots, c_{n-1}$.

Reparametrization of the curve allows that system to be reduced to an easier one. The reparametrization is given by the inverse function of

$$g(s) = \int_0^s k_1(u)du.$$

Note that $t = g(s)$ is a reparametrization because k_1 is a positive function. The reparametrization we need is the inverse function $s = g^{-1}(t)$. It is a simple matter to verify that, with respect to parameter t, the Frenet formulae are reduced to a linear system of first order differential equations with constant coefficients

$$\begin{pmatrix} \overrightarrow{\mathbf{e_1}}'(t) \\ \overrightarrow{\mathbf{e_2}}'(t) \\ \overrightarrow{\mathbf{e_3}}'(t) \\ \vdots \\ \overrightarrow{\mathbf{e_{n-1}}}'(t) \\ \overrightarrow{\mathbf{e_n}}'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 0 & c_2 & 0 & \ldots & 0 & 0 \\ 0 & -c_2 & 0 & c_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & c_{n-1} \\ 0 & 0 & 0 & 0 & \ldots & -c_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{\mathbf{e_1}}(t) \\ \overrightarrow{\mathbf{e_2}}(t) \\ \overrightarrow{\mathbf{e_3}}(t) \\ \vdots \\ \overrightarrow{\mathbf{e_{n-1}}}(t) \\ \overrightarrow{\mathbf{e_n}}(t) \end{pmatrix}. \tag{4.1}$$

We can apply the well-known methods of integration of systems of linear equations with constant coefficients. Let F_n be the matrix of constant coefficients of this system.

(4.2) Eigenvalues and their multiplicity. The first thing we have to do is to compute the eigenvalues of the coefficient matrix.

Due to the skew symmetry of the matrix, it can have not real eigenvalues other than zero. Due to the fact that the determinant of F_n vanishes only for odd n, we can say that for odd dimensions, 0 is an eigenvalue, whereas for even dimensions, 0 is an eigenvalue only if $k_{n-1} = 0$.

By definition, we have that the constants $c_2, c_3, \dots, c_{n-2}$ are not zero. If the last constant, c_{n-1}, vanishes, then the same happens with the last curvature function k_{n-1}. In this case the curve is included in a hyperspace, so we can consider it to be a curve in an $n-1$ dimensional space.

Therefore, from now on, we shall consider that all the curvatures, and then all the constants c_i, are not zero.

Note that, in this case, for any $x \in \mathbb{C}$, the rank (in $\mathbb{C}$) of the matrix

$$\begin{pmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & x & c_2 & 0 & \dots & 0 & 0 \\ 0 & -c_2 & x & c_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & x & c_{n-1} \\ 0 & 0 & 0 & 0 & \dots & -c_{n-1} & x \end{pmatrix}$$

is at least $n-1$. Therefore, their eigenvalues are all of multiplicity 1.

(4.3) Canonical Jordan form. Let $a_\ell \pm \mathbf{i} b_\ell$, $\ell = 1, \dots, [\frac{n}{2}]$, with $a_\ell, b_\ell \in \mathbb{R}$, be the non-zero eigenvalues of the coefficient matrix. Therefore, for $n = 2k$, the associated canonical Jordan form is of the form

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k \end{pmatrix}$$

where $J_\ell = \begin{pmatrix} a_\ell & -b_\ell \\ b_\ell & a_\ell \end{pmatrix}$.

The matrix can be diagonalized because all the eigenvalues are of multiplicity one. Therefore, there is a orthogonal matrix, S, such that if C is the matrix of constant coefficients, then

$$C = S^{-1} J S.$$

Therefore, the general solution of the system for the first vector is

$$\vec{\mathbf{e}}_1(u) := \sum_{\ell=1}^{k} \vec{A_\ell}\, e^{a_\ell u} \cos(b_\ell\, u) + \vec{B_\ell}\, e^{a_\ell u} \sin(b_\ell\, u),$$

where $\{\vec{A_\ell}, \vec{B_\ell}\}_{\ell=1}^{k}$ is a family of orthogonal vectors.

For $n = 2k + 1$, the associated canonical Jordan form is of the form

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & J_1 & 0 & \dots & 0 \\ 0 & 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & J_k \end{pmatrix}$$

Now, the general solution of the system for the first vector is

$$\vec{\mathbf{e}}_1(u) := \overrightarrow{A_0} + \sum_{\ell=1}^{k} \overrightarrow{A_\ell}\, e^{a_\ell u} \cos(b_\ell\ u) + \overrightarrow{B_\ell}\, e^{a_\ell u} \sin(b_\ell\ u),$$

where $\{\overrightarrow{A_0}\} \cup \{\overrightarrow{A_\ell}, \overrightarrow{B_\ell}\}_{\ell=1}^{k}$ is a family of orthogonal vectors.

(4.4) The eigenvalues are pure imaginaries. The condition $||\vec{\mathbf{e}}_1(u)|| = 1$ for all u implies that all the real parts of the eigenvalues are zero. Indeed, if, for example, $a_1 \neq 0$, then let m be a non-zero coordinate of $\overrightarrow{A_1}$. Bearing in mind that

$$|m|\ e^{a_1 u}\ |\cos(b_1 u)| \leq ||\vec{\mathbf{e}}_1(u)||,$$

and that the left-hand member is an unbounded function, then $||\vec{\mathbf{e}}_1(u)|| \neq 1$.

Therefore, all the real parts of the eigenvalues are zero and the general solution (in the even case) of the system for the first vector is

$$\vec{\mathbf{e}}_1(u) := \sum_{\ell=1}^{k} \overrightarrow{A_\ell} \cos(b_\ell\ u) + \overrightarrow{B_\ell} \sin(b_\ell\ u).$$

Analogously for the odd case.

Moreover, let us recall that the vectors $\{\overrightarrow{A_i}, \overrightarrow{B_i}\}_{i=1}^{k}$ are an orthogonal base of $\mathbb{R}^n$ associated to the canonical Jordan form.

(4.5) The main result. Finally, an isometry of $\mathbb{R}^n$ allows us to state the next result.

THEOREM (4.5.1). *A curve has constant curvature ratios if and only if its tangent indicatrix is a twisted geodesic on a flat torus.*

Note that in the odd dimensional case this result implies that the last coordinate of the tangent indicatrix is a constant. Therefore there is a direction making a constant angle with the curve. Nevertheless, this is not the case in the even dimensional case. There are no fixed directions making a constant angle with the tangent vector.

When all the curvatures are constant, then the curve is also a ccr-curve and its tangent indicatrix is of the kind described in the previous statement. Moreover, the reparametrization $g(s) = \int_0^s k_1(u)du$ is just the product by a constant.

Since the integration of a geodesic on a flat torus in $\mathbb{R}^{2k}$ with respect to its parameter is again a curve of the same kind, we get the following corollary:

COROLLARY (4.5.2). *A curve has constant curvatures if and only if it is*

1. *a twisted geodesic on a flat torus, in the even dimensional case, or*
2. *a twisted geodesic on a flat torus times a linear function of the parameter, in the odd dimensional case.*

(4.6) $n = 3$**.** The eigenvalues of the matrix of coefficients are 0 and $\pm\sqrt{1+c^2}\,\mathbf{i}$ ($c = c_2$, to simplify).

Therefore, the general solution of the system for the first vector is

$$\overrightarrow{\mathbf{e_1}}(u) = \overrightarrow{A_1} + \overrightarrow{A_2}\cos(\sqrt{1+c^2}u) + \overrightarrow{A_3}\sin(\sqrt{1+c^2}u),$$

where $\overrightarrow{A_i}$, $i = 1, 2, 3$ are constant vectors.

Once we have the tangent vector, we only have to undo the reparametrization and to integrate to obtain the curve

$$\alpha(s) = x_0 + \overrightarrow{c_1}s + \overrightarrow{c_2}\int_0^s \cos(\sqrt{1+c^2}g(v))dv + \overrightarrow{c_3}\int_0^s \sin(\sqrt{1+c^2}g(v))dv.$$

(4.7) $n = 4$**.** The eigenvalues are

$$\pm\frac{\mathbf{i}}{\sqrt{2}}\sqrt{1+c_2^2+c_3^2 \pm \sqrt{(1+c_2^2+c_3^2)^2 - 4c_3^2}}.$$

Therefore, the general solution of the system for the first vector is

$$\overrightarrow{\mathbf{e}}_1(u) := \overrightarrow{A_1}\cos(m_+u) + \overrightarrow{B_1}\sin(m_+u) + \overrightarrow{A_2}\cos(m_-u) + \overrightarrow{B_2}\sin(m_-u),$$

where

$$m_\pm = \frac{1}{\sqrt{2}}\sqrt{1+c_2^2+c_3^2 \pm \sqrt{(1+c_2^2+c_3^2)^2 - 4c_3^2}}$$

and where $\overrightarrow{A_i}$, $\overrightarrow{B_i}$, $i = 1, 2$ are constant vectors.

5. Spherical ccr-curves

In order to compare ccr-curves with the definition of generalized helices given in [1], we will try to determine which ccr-curves are included in a sphere.

LEMMA (5.1). *A curve $\alpha\colon I \to \mathbb{R}^4$ is spherical, i.e., it is contained in a sphere of radius R, if and only if*

$$\frac{1}{k_1^2} + \left(\frac{\dot{k}_1}{k_1^2k_2}\right)^2 + \frac{1}{k_3^2}\left(\left(\frac{\dot{k}_1}{k_1^2k_2}\right)^{\cdot} - \frac{k_2}{k_1}\right)^2 = R^2. \tag{5.2}$$

Proof. The proof here is similar to that for spherical curves in $\mathbb{R}^3$. It consists in obtaining information thanks to successive derivatives of the expression $< \alpha(s) - m, \alpha(s) - m >= R^2$, where m is the center of the sphere. In particular, what can be proven is that spherical curves can be decomposed as

$$\alpha(s) = m - \frac{R}{k_1}\overrightarrow{\mathbf{e_2}}(s) + R\frac{\dot{k}_1}{k_1^2k_2}\overrightarrow{\mathbf{e_3}}(s) + R\frac{1}{k_3}\left(\left(\frac{\dot{k}_1}{k_1^2k_2}\right)^{\cdot} + \frac{k_2}{k_1}\right)\overrightarrow{\mathbf{e_4}}(s). \tag{5.3}$$

□

As a corollary we obtain the classical result for spherical three-dimensional curves:

COROLLARY (5.4). *A curve $\alpha\colon I \to \mathbb{R}^3$ is spherical, i.e., it is contained in a sphere of radius R, if and only if*

$$\frac{1}{k_1^2} + \left(\frac{\dot{k}_1}{k_1^2 k_2}\right)^2 = R^2. \tag{5.5}$$

From now on, we shall suppose that $m = 0$ and $R = 1$.

(5.6) Spherical ccr-curves in $\mathbb{R}^3$. In this case, we can rewrite Eq. (5.5) in terms of curvature, $k_1 = \kappa$, and torsion $k_2 = \tau = c\kappa$, c being a constant.

$$\frac{\dot{\kappa}}{\kappa^2\sqrt{\kappa^2 - 1}} = \pm c.$$

Let us consider just the positive sign. This differential equation can be integrated and the solution is

$$\kappa(s) = \frac{1}{\sqrt{1 - (cs + s_0)^2}}.$$

Thanks to a shift of the parameter we get that the curvature and torsion of a spherical generalized helix are given by

$$\kappa(s) = \frac{1}{\sqrt{1 - c^2 s^2}}, \qquad \tau(s) = \frac{c}{\sqrt{1 - c^2 s^2}}.$$

We now need to compute the reparametrization

$$u = g(s) = \int_0^s \kappa(t)dt = \frac{1}{c}\arcsin(cs).$$

With the appropriate initial conditions, the generalized spherical helix is

$$\begin{aligned}\alpha_c(s) = \Bigg(&\sqrt{1 - c^2 s^2}\cos\Big(\frac{\sqrt{1 + c^2}\arcsin(cs)}{c}\Big) + \frac{c^2 s}{\sqrt{1 + c^2}}\sin\Big(\frac{\sqrt{1 + c^2}\arcsin(cs)}{c}\Big),\\ &-\sqrt{1 - c^2 s^2}\sin\Big(\frac{\sqrt{1 + c^2}\arcsin(cs)}{c}\Big) + \frac{c^2 s}{\sqrt{1 + c^2}}\cos\Big(\frac{\sqrt{1 + c^2}\arcsin(cs)}{c}\Big),\\ &\frac{cs}{\sqrt{1 + c^2}}\Bigg)\end{aligned}$$

Note that the curve α_c is defined in the interval $]-\frac{1}{c}, \frac{1}{c}[$. If we change the parameter in accordance with $s = \frac{1}{c}\sin t$, the spherical helix is now parametrized as

$$\begin{aligned}\beta_c(t) = \Bigg(&\cos t\cos\Big(\frac{\sqrt{1 + c^2}}{c}t\Big) + \frac{c}{\sqrt{1 + c^2}}\sin t\sin\Big(\frac{\sqrt{1 + c^2}}{c}t\Big),\\ &-\cos t\sin\Big(\frac{\sqrt{1 + c^2}}{c}t\Big) + \frac{c}{\sqrt{1 + c^2}}\sin t\cos\Big(\frac{\sqrt{1 + c^2}}{c}t\Big), \frac{\sin t}{\sqrt{1 + c^2}}\Bigg)\end{aligned}$$

Now, it is clear that the projection of these curves on the plane xy are arcs of epicycloids. This result was known by W. Blaschke, as is mentioned in [8], where it is also proven by different methods.

(5.7) Spherical ccr-curves in $\mathbb{R}^4$.

5.7.1. *The constant curvatures case.* The curve

$$\alpha(s) = \frac{1}{\sqrt{r_1^2 + r_2^2}}\left(\frac{r_1}{m_1}\sin(m_1 s), -\frac{r_1}{m_1}\cos(m_1 s), \frac{r_2}{m_2}\sin(m_2 s), -\frac{r_2}{m_2}\cos(m_2 s)\right)$$

is a spherical curve (with radius 1), if and only if

$$r_1^2 m_2^2 + r_2^2 m_1^2 = m_1^2 m_2^2 (r_1^2 + r_2^2).$$

5.7.2. *The non-constant case.* In this case, we can rewrite Eq. (5.2) in terms of curvature, k_1, $k_2 = c_2 k_1$ and $k_3 = c_3 k_1$, where c_2, c_3 are constants.

$$\frac{1}{k_1^2} + \left(\frac{\dot{k}_1}{c_2 k_1^3}\right)^2 + \frac{1}{c_3^2 k_1^2}\left(\left(\frac{\dot{k}_1}{c_2 k_1^3}\right)^{\cdot} + c_2\right)^2 = 1. \tag{5.8}$$

By changing $f = \frac{1}{k_1^2}$ the equation is reduced to

$$f + \frac{1}{4c_2^2}\dot{f}^2 + \frac{1}{c_3^2} f\left(-\frac{1}{2c_2}\ddot{f} + c_2\right)^2 = 1. \tag{5.9}$$

Computation of the general solution seems to be a difficult task. Instead, we can try to compute some particular solutions.

For instance, the constant solution $f(s) = \frac{c_3^2}{c_2^2 + c_3^2}$ or the polynomial solutions of degree 2,

$$f(s) = \frac{-2c_2^2 + c_3^2 - c_3\sqrt{-8c_2^2 + c_3^2}}{2(c_2^2 + c_3^2)} + \frac{1}{2}\left(2c_2^2 - c_3^2 - c_3\sqrt{-8c_2^2 + c_3^2}\right) s^2,$$

$$f(s) = 2c_2 s + \frac{1}{2}\left(2c_2^2 - c_3^2 - c_3\sqrt{-8c_2^2 + c_3^2}\right) s^2.$$

For these three particular solutions the reparametrization g, where $g(s) = \int_0^s k_1(t)dt = \int_0^s \frac{1}{\sqrt{f(t)}}dt$, can be computed explicitly. We can thus obtain explicit examples of ccr-curves in S^3 with non-constant curvatures.

A particular case. With $c_2 = \frac{1}{2}$, $c_3 := \frac{\sqrt{3}}{2}$, then $m_1 = \sqrt{\frac{3}{2}}$, $m_2 = \frac{1}{\sqrt{2}}$ and $r_1 = r_2 = \frac{1}{\sqrt{2}}$. The function $f(s) = \frac{1}{2} - 2s^2$ is a solution of Eq. (5.9). Therefore, $k_1(s) = \frac{2}{\sqrt{1-4s^2}}$, and $g(s) = \int_0^s \frac{2}{\sqrt{1-4t^2}}dt = \arcsin(2s)$.

If

$$\overrightarrow{\mathbf{e_1}}(t) = \frac{1}{\sqrt{2}}\left(\cos\left(\sqrt{\frac{3}{2}}t\right), \sin\left(\sqrt{\frac{3}{2}}t\right), \cos\left(\frac{1}{\sqrt{2}}t\right), \sin\left(\frac{1}{\sqrt{2}}t\right)\right),$$

then

$$\alpha(s) = \left(0, -\frac{\sqrt{3}}{2}, 0, \frac{1}{2}\right) + \int_0^s \overrightarrow{\mathbf{e_1}}(\arcsin(2u))du, \quad s \in \left]-\frac{1}{2}, \frac{1}{2}\right[$$

is a spherical ccr-curve with center at the origin of coordinates, with radius 1 and with non-constant curvatures.

6. Intrinsic generalized helices

In [1] the author proposes a definition of general helix on a 3-dimensional real-space-form substituting the fixed direction in the usual definition of generalized helix by a Killing vector field along the curve.

Let $\alpha\colon I \to M$ be an immersed curve in a 3-dimensional real-space-form M. Let us denote the intrinsic Frenet frame by $\{\overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}}\}$. The intrinsic Frenet's formulae are

$$\left\{\begin{aligned} \nabla_{\overrightarrow{\mathbf{t}}}\overrightarrow{\mathbf{t}} &= \kappa\overrightarrow{\mathbf{n}},\\ \nabla_{\overrightarrow{\mathbf{t}}}\overrightarrow{\mathbf{n}} &= -\kappa\overrightarrow{\mathbf{t}} + \tau\overrightarrow{\mathbf{b}},\\ \nabla_{\overrightarrow{\mathbf{t}}}\overrightarrow{\mathbf{b}} &= -\tau\overrightarrow{\mathbf{n}}, \end{aligned}\right. \tag{6.1}$$

where ∇ is the Levi-Civita connection of M and where κ and τ are called the intrinsic curvature and torsion functions of curve α, respectively.

From now on we shall suppose that $M = S^3$. Therefore, any curve on S^3 can also be considered to be a curve in $\mathbb{R}^4$. We shall try to obtain the relationship between the Frenet elements, $\{\overrightarrow{\mathbf{e_1}}, \overrightarrow{\mathbf{e_2}}, \overrightarrow{\mathbf{e_3}}, \overrightarrow{\mathbf{e_4}}, k_1, k_2, k_3\}$, of the curve as a curve in 4-dimensional Euclidian space and the intrinsic Frenet elements $\{\overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}}, \kappa, \tau\}$. Note first that $\overrightarrow{\mathbf{t}} = \overrightarrow{\mathbf{e_1}}$. Then

$$\nabla_{\overrightarrow{\mathbf{t}}}\overrightarrow{\mathbf{t}} = \dot{\overrightarrow{\mathbf{e_1}}} - \langle\dot{\overrightarrow{\mathbf{e_1}}}, \alpha\rangle\alpha = k_1(\overrightarrow{\mathbf{e_2}} - \langle\overrightarrow{\mathbf{e_2}}, \alpha\rangle\alpha),$$

where we have used as the Gauss map of the sphere the identity map.

Therefore

$$\overrightarrow{\mathbf{n}} = \frac{\nabla_{\overrightarrow{\mathbf{t}}}\overrightarrow{\mathbf{t}}}{||\nabla_{\overrightarrow{\mathbf{t}}}\overrightarrow{\mathbf{t}}||} = \frac{1}{\sqrt{1-\langle\overrightarrow{\mathbf{e_2}}, \alpha\rangle^2}}(\overrightarrow{\mathbf{e_2}} - \langle\overrightarrow{\mathbf{e_2}}, \alpha\rangle\alpha), \tag{6.2}$$

and

$$\kappa = \langle\nabla_{\overrightarrow{\mathbf{t}}}\overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}}\rangle = k_1\sqrt{1-\langle\overrightarrow{\mathbf{e_2}}, \alpha\rangle^2} = \sqrt{k_1^2 - 1},$$

which were obtained using Eq. (5.3).

The intrinsic binormal vector is the only vector such that $\{\overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}}, \alpha\}$ is an orthonormal basis of $\mathbb{R}^4$ with positive orientation. Then

$$\overrightarrow{\mathbf{b}} = \alpha \wedge \overrightarrow{\mathbf{t}} \wedge \overrightarrow{\mathbf{n}}.$$

Now, by replacing the intrinsic tangent and normal with $\overrightarrow{\mathbf{t}} = \overrightarrow{\mathbf{e_1}}$ and (6.2), we get

$$\overrightarrow{\mathbf{b}} = \frac{k_1}{\sqrt{k_1^2 - 1}}\,\alpha \wedge \overrightarrow{\mathbf{e_1}} \wedge \overrightarrow{\mathbf{e_2}} = \frac{1}{\sqrt{1-(\frac{1}{k_1})^2}}\,\alpha \wedge \overrightarrow{\mathbf{e_1}} \wedge \overrightarrow{\mathbf{e_2}}.$$

Therefore

$$\dot{\overrightarrow{\mathbf{b}}} = \left(\frac{1}{\sqrt{1-(\frac{1}{k_1})^2}}\right)^{\cdot}\alpha \wedge \overrightarrow{\mathbf{e_1}} \wedge \overrightarrow{\mathbf{e_2}} + \frac{1}{\sqrt{1-(\frac{1}{k_1})^2}}\,\alpha \wedge \overrightarrow{\mathbf{e_1}} \wedge k_2\overrightarrow{\mathbf{e_3}}.$$

A consequence of this computation is that $<\dot{\overrightarrow{\mathbf{b}}}, \alpha >= 0$, and therefore, $\nabla_{\overrightarrow{\mathbf{t}}}\overrightarrow{\mathbf{b}} = \dot{\overrightarrow{\mathbf{b}}}$. Finally,

$$\begin{aligned}\tau &= -\langle \nabla_{\overrightarrow{\mathbf{t}}}\overrightarrow{\mathbf{b}}, \overrightarrow{\mathbf{n}}\rangle = -\left\langle \frac{1}{\sqrt{1-(\frac{1}{k_1})^2}}\,\alpha\wedge\overrightarrow{\mathbf{e_1}}\wedge k_2\overrightarrow{\mathbf{e_3}}, \frac{1}{\sqrt{1-(\frac{1}{k_1})^2}}\overrightarrow{\mathbf{e_2}}\right\rangle \\ &= -\frac{k_2}{1-(\frac{1}{k_1})^2}\langle \alpha\wedge\overrightarrow{\mathbf{e_1}}\wedge\overrightarrow{\mathbf{e_3}}, \overrightarrow{\mathbf{e_2}}\rangle = \frac{k_2}{1-(\frac{1}{k_1})^2} = \frac{k_1^2 k_2}{\kappa^2}.\end{aligned}$$

PROPOSITION (6.3). *The only* 4*-dimensional spherical non-trivial ccr-curves which are also intrinsic generalized helices of* S^3 *are helices, i.e., curves with all curvatures constant.*

Proof. As it is proven in [1], a curve in S^3 is an intrinsic helix if and only if $\tau = 0$ or there exists a constant b such that $\tau = b\kappa \pm 1$.

The case $\tau = 0$ implies that $k_1 k_2 = 0$ and we get a non-regular curve.

In the other case, if the curve is also a ccr-curve (with $k_2 = ck_1$), then

$$\frac{ck_1^3}{\kappa^2} = b\kappa \pm 1.$$

Equivalently

$$\left(\frac{ck_1^3}{k_1^2-1} \mp 1\right)^2 = b(k_1^2 - 1).$$

That is, the function k_1 is the solution of a polynomial equation with constant coefficients; and, therefore, the function k_1 is constant, and so the other two curvatures k_2 and k_3 are also constant. The same happens with κ and τ. We are then in the presence of a helix according to the designation in [1], or a geodesic in a flat torus in $\mathbb{R}^4$ according to [6]. □

Received October 02, 2006

Final version received October 20, 2006

DEPARTMEN DE GEOMETRIA I TOPOLOGIA
UNIVERSITAT DE VALÈNCIA
AVD. VICENT ANDRÉS ESTELLÉS 1
E-46100-BURJASSOT (VALÈNCIA)
SPAIN
monterde@uv.es

REFERENCES

[1] M. BARROS, *General helices and a theorem of Lancret*, Proc. Amer. Math. Soc. **125**, 1503–1509 (1997).
[2] DO CARMO, M. P., *Differential Geometry of curves and surfaces*, Prentice-Hall Inc., (1976).
[3] A. FERNÁNDEZ, A. GIMÉNEZ, P. LUCAS, *Null generalized helices in Lorentz-Minkowski spaces*, J. Physics, A: Math. and General, vol 35, 8243–8251 (2002).
[4] H. A. HAYDEN, *On a generalized helix in a Riemannian n-space*, Proc. London Math. Soc. **32**, 37–45 (1931).
[5] W. KLINGENBERG, A course in Differential Geometry, Springer-Verlag (1978).
[6] S. RODRIGUES COSTA, *On closed twisted curves*, Proc. Amer. Math. Soc. **109** (1990), 205–214.
[7] M.C. ROMERO-FUSTER, E. SANABRIA-CODESAL, *Generalized helices, twistings and flattenings of curves in n-space*, Matemática Contemporanea **17** (1999), 267–280.
[8] STRUICK, D. J., Lectures on Classical Differential Geometry, Dover, New-York, (1988).

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

FOX'S SPREADS ON NEARNESS SPACES

HLENGANI SIWEYA

ABSTRACT. Hunt's uniform spreads, which are generalizations of Fox's spreads, are extended to the category of nearness spaces and uniformly continuous functions. We prove that there is a bijective correspondence between Hunt's spread points in $((X,\xi), f, (Z,\mu))$ and Herrlich's ξ-clusters in (X,ξ), where X and Z are nearness spaces.

1. Historical Background

As a basic concept, Fox's spreads are used in the "uniform sense" of Hunt, thus defined as triples (f, X, Z), where f is a continuous function from a topological T_1-space X into a topological complete space Z with defining completely regular topology and which carries a complete uniformity $\mathfrak{V}$ compatible with the topology. The induced X-component uniformity $\mathfrak{U}$ is then compatible with the given topology on X (explained in Section 2). If we denote by $\{\mathfrak{W}_\lambda\}$ the collection of all open coverings of Z (it is a uniformity base of the existing uniformity) then $\{\mathfrak{U}_\lambda = c[f^{-1}(\mathfrak{W}_\lambda), \mathfrak{R}_X]\}_\lambda$ is a collection of open coverings of X generating the X-component uniformity $\mathfrak{U}$. We recall that $c([f^{-1}(\mathfrak{W}_\lambda), \mathfrak{R}_X]$ is defined as the set of all components of $f^{-1}[W]$ for some $W \in \mathfrak{W}_\lambda$. We note further that f is uniformly continuous.

In this article, Hunt's bijective correspondence between "spread points" in (f, X, Z) and minimal Cauchy filters in $(X, \mathfrak{U})$ (in [10]) is extended to one between spread points in $(X, \mathfrak{U})$ and Herrlich's ξ-clusters in nearness spaces. In fact, a simple consequence of these ideas assures that the definitions of "complete spread" and "completion of a spread" that Hunt formulated in terms of the uniform concepts are topologically invariant.

The main result of this article is then the following.

THEOREM (1.1). *The correspondence between the collection of all spread points in $((X,\xi), f, (Z,\mu))$ with Z carrying a complete nearness μ compatible with its topology and the collection of all ξ_μ-clusters in (X,ξ) defined by*

$$\chi \mapsto \mathcal{A}$$

is a bijection, where $\widehat{\mathcal{A}} = (\mathrm{Im}\ \chi)^+$. □

Spreads (commonly known as Fox's spreads [6], [10]) owe their origin to Fox's paper [5]. For completeness, a continuous function $f : X \to Z$ between T_1-spaces is called a *spread* if the collection of *all* the components of *all* inverse images $f^{-1}(V)$ of open sets V in Z form a base for a topology on X. In this note

2000 *Mathematics Subject Classification*: 54E15, 54E17.
Keywords and phrases: spread point, nearness spread, uniform spread completion.
Financial support from the University of Limpopo is acknowledged.

we denote a spread so defined by (X, f, Z). It is immediate from this definition that X is locally connected. Now given a spread (X, f, Z) with $z \in Z$, denote by $\mathcal{N}_z$ the filter base of open neighbourhoods of z, and let

$$\mathcal{C}_z^V = \{K \subset X \mid K \text{ is a component of } f^{-1}(V)\}$$

for open V in Z. A *spread point* is then a function

$$\chi\colon \mathcal{N}_z \longrightarrow \{K \subset X \mid \exists V' \subset Z \text{ open in } Z, K \text{ is a component of } f^{-1}(V')\}$$

that satisfies

$$SP_1\colon V \in \mathcal{N}_z \text{ implies } \chi(V) \in \mathcal{C}_z^V,$$
$$SP_2\colon U, V \in \mathcal{N}_z \text{ and } U \subset V$$

imply $\chi(U) \subset \chi(V)$.

Remark. (a) Note that SP_2 is equivalent to each of the following:

(i) $\operatorname{Im}\chi$ is a filter base (and the resulting filter will be denoted by $(\operatorname{Im}\chi)^+$).

(ii) $\operatorname{Im}\chi$ has the *finite intersection property*.

(b) The point $z \in Z$ is necessarily unique (see [10] and [16]). The introduction of the term *spread point* by Hunt was necessary in so far as it simplified Fox's canonical spread completion as well as provided for the leap from this completion to Hunt's uniform spread completion — a completion never published but subsequently quoted and extensively used in several results of Hunt [10] after his result on the bijective correspondence between spread points and minimal Cauchy filters. For that reason, another objective is also to present Hunt's uniform spread completion.

We recall a few nearness concepts necessary for this note. (We follow Herlich [7] and Preuβ [15].)

A *nearness space* is a pair (X, μ) where X is a non-empty set and μ is a set of covers of X satisfying

N_1: $\{X\} \in \mu$.

N_2: $\mathcal{U} \leq \mathcal{V}, \mathcal{U} \in \mu$ imply $\mathcal{V} \in \mu$.

N_3: $\mathcal{U}, \mathcal{V} \in \mu$ implies $\mathcal{U} \wedge \mathcal{V} = \{U \cap V \mid U \in \mathcal{U}, V \in \mathcal{V}\} \in \mu$.

N_4: $\mathcal{U} \in \mu$ implies $\operatorname{int}\mathcal{U} = \{\operatorname{int} U \mid U \in \mathcal{U}\} \in \mu$, where

$$\operatorname{int} U = \{x \in X \mid \{X - \{x\}, U\} \in \mu\}.$$

(The relation $\mathcal{U} \leq \mathcal{V}$ is the usual refinement; thus for each $U \in \mathcal{U}$ there exists a $V \in \mathcal{V}$ with $U \subset V$.)

A nearness space (X, μ) is an N_1-*space* if, in addition, μ satisfies

N_5: For any $x \neq y$ in X,

$$\{X - \{x\}, X - \{y\}\} \in \mu.$$

A collection μ_B of covers on X is *a base* for a nearness structure on X if it satisfies the following axioms:

NB_1: $\mathcal{U}, \mathcal{V} \in \mu_B$ imply there exists a $\mathcal{W} \in \mu_B$ such that $\mathcal{W} \leq \mathcal{U} \wedge \mathcal{V}$,

NB_2: $\mathcal{U} \in \mu_B$ implies $\{\operatorname{int}_{\mu_B}\mathcal{U} \mid U \in \mathcal{U}\} \in \mu_B$.

Moreover, each nearness space (X, μ) induces up to bijection a system ξ_μ of so called "near collections" on X as follows:

$$\mathcal{A} \in \xi_\mu \Leftrightarrow \forall \mathcal{U} \in \mu\, \exists\, U \in \mathcal{U},\ U \in \operatorname{Sec} \mathcal{A},$$

where

$$\operatorname{Sec} \mathcal{A} = \{T \subset X \mid \forall\, A \in \mathcal{A},\ T \cap A \neq \emptyset\}.$$

In addition, there also can be established a corresponding system γ_μ of so called "Cauchy systems" as follows:

$$\mathcal{A} \in \gamma_\mu \Leftrightarrow \forall \mathcal{U} \in \mu\, \exists\, U \in \mathcal{U}\, \exists\, A \in \mathcal{A},\ A \subset U.$$

A filter is called a *Cauchy filter* iff it is a Cauchy system. Consequently, a simple bijection between the set of all Cauchy filters and the set of all "near grills" on X will be realized by the "Sec-operator" (as defined above). This induces an isomorphism between the categories of filter merotopic spaces in the sense of Katetov (Katetov [11]) and grill determined prenearness spaces (see also [1]) if we omit Axiom (N_4). (Note that the neighborhood filter $\mathcal{N}_z$ of $z \in Z$ referred to above is a minimal Cauchy filter.)

For a nearness space (X, ξ), the maximal elements of the set $\xi - \{\varnothing\}$, when ordered by inclusion, are called *ξ-clusters*. Thus an *ξ_μ-cluster* in (X, ξ) is a non-empty *maximal ξ-near collection* in X.

Unless specified otherwise, we propose to work in N_1-spaces — those nearness spaces in which the underlying topological space is T_1. Given a nearness space (X, ξ), we set

$$\tilde{\xi} = \{\mathcal{A} \subseteq \mathcal{P}(X) \mid \forall \mathcal{B} \in \bar{\xi}\, \exists\, A \in \mathcal{A}\, \exists\, B \in \mathcal{B}, A \cap B = \varnothing\},$$

where $\mathcal{B} \in \bar{\xi}$ (read $\mathcal{B}$ *is far*) means $\mathcal{B} \in \mathcal{P}(\mathcal{P}(X)) - \xi$.

LEMMA (1.2). *A Cauchy filter ξ in a nearness space (X, μ) is a μ-cluster if and only if, for each $A \in \xi$ there is some $\mathcal{U} \in \mu$ and some $U \in \mathcal{U}$ such that*

$$st\,(U, \mathcal{U}) \subset A\,.$$

Proof. Let $A \in \xi$. By definition there is a $\mathcal{U}$ in $\tilde{\xi}$ and therefore a $U \in \mathcal{U}$ such that $U \cap A \neq \varnothing$ which is all we need. □

Following the calculations in Preuβ [15], namely, that

$$\operatorname{int}(U \cap V) = \operatorname{int} U \cap \operatorname{int} V,$$

for $U \in \mathcal{U},\ V \in \mathcal{V}$ where $\mathcal{U}, \mathcal{V} \subset \mathcal{P}(X)$, we deduce that

COROLLAY (1.3). *For any nearness base $\{\mu_\lambda \mid \lambda \in \Lambda\}$ of (X, μ), if ξ is a μ-cluster in (X, μ), then the collection*

$$\xi \wedge \bigcup_\lambda \mu_\lambda$$

is a nearness base for ξ. □

Definition (1.4). [Herrlich [7], [8]] For a subcollection $\mathcal{A} \subset \mathcal{P}(X)$, we set

$$\widehat{\mathcal{A}} = \{B \subset X \mid \forall\, A \in \mathcal{A}, B \cap A \neq \varnothing\} = \operatorname{Sec} \mathcal{A}$$

In consequence, we have corresponding minimal Cauchy filters by the set γ_μ when ordered by inclusion. In fact, our main result hinges on the following result.

PROPOSITION (1.5) (Herrlich [7], Remark 5.6 (2)). *For a nearness space (X, ξ), if $\mathcal{A}$ is an ξ_μ-cluster, then $\widehat{\mathcal{A}} = Sec\mathcal{A}$ is a minimal Cauchy filter (and so $\widehat{\widehat{\mathcal{A}}} = \mathcal{A}$). Conversely, if (X, μ) is especially "regular", then $\mathcal{C}$ is a minimal Cauchy filter iff there exists a ξ_μ-cluster $\mathcal{A}$ with $\mathcal{C} = \widehat{\mathcal{A}}$. The correspondence*

$$\mathcal{A} \mapsto \widehat{\mathcal{A}}$$

then induces a bijection between the set of all ξ_μ-clusters and the set of all minimal γ_μ-Cauchy filters on X. □

Remark. A nearness space (X, μ) is called *regular* iff it satisfies the following condition (see Herrlich [7]):

(R) For each $\mathcal{U} \in \mu$ there is some (refinement) $\mathcal{V} \in \mu$ such that for each $V \in \mathcal{V}$, there exists some $U \in \mathcal{U}$ with $\{X - V, U\} \in \mu$.

Then, every uniform nearness space is regular and every topological nearness space which is regular as a topological space is regular as a nearness space. Moreover, for a regular nearness space it holds that its induced topology is regular in the original sense. See Preuß [15].

2. Hunt's uniform spread completion

When Hunt introduced uniform spreads, he did not show how a uniform spread completion could be constructed save to relate spread points (used, implicitly, in Fox's canonical spread completion) to minimal Cauchy filters. Moreover, it will follow from our construction presented here that Fox's spread completion is a special case of Hunt's uniform spread completion.

In this section, therefore, we present a uniform spread completion (which we name after Hunt). Such a completion has been shown by Hunt to be unique. In fact, Hunt has taught us many results associated with a uniform spread — most of which have been drawn from algebraic topology. In this connection, motivated by Fox's founding article on spreads, Montesinos-Amilibia [14] gave and studied modified topological definitions of a *branched folded covering* and a *singular covering*. Contrary to Fox spread completion constructed in the presence of local connectedness, Michael [13] showed how to complete a spread without local connectedness. A decade ago, Bunge and Funk [3], also showed that Fox's (complete) spreads have a natural definition in topos theory. A few other topologists (see e.g. [6]) have investigated Fox's spreads in other contexts which we believe are worth noting.

A space X is said to be *locally connected in* a topological space Y if there exists a base $\mathcal{B}$ of Y such that $X \cap B$ is connected for each $B \in \mathcal{B}$ (see [5]). An example of a subspace locally connected but not locally connected in another space is the following: The space $\mathbb{R} - \{0\}$ is a locally connected subspace which fails to be locally connected in $\mathbb{R}$.

We say that a spread (X, f, Z) is *complete* if

$$\bigcap \mathrm{Im}(\chi) \neq \emptyset\,.$$

In [16], it was shown that for a spread $f: X \longrightarrow Z$ to be complete it is necessary and sufficient that whenever $j: X \longrightarrow Y$ is a dense embedding, $j(X)$ is locally connected in Y and $g: Y \longrightarrow Z$ is a spread such that $f = g \circ j$, then j is a homeomorphism.
A *completion* of a spread (X, f, Z) is a complete spread (X_s, g, Z) for which there is a dense embedding $j: X \hookrightarrow X_s$ of X into X_s such that $j(X)$ is locally connected in X_s, where X_s is the locally connected space whose elements are the spread points. See also [9].

Construction: Hunt's uniform spread completion. We recall that a completely regular space is *topologically complete* if some uniformity compatible with its topology is complete — where the uniformity compatible with its topology is one whose neighborhood basis is the set

$$\{\mathrm{St}\,(p, \mathfrak{U}_\alpha \mid \alpha \in A\}.$$

Hunt's uniform spreads (in [10]) arise as follows: Given a spread (X, f, Z) in which τ_X is a topology on X and Z is topologically complete, say Z carries a complete uniformity $\mathcal{W}$ compatible with its topology, we know that $f^{-1}(\mathcal{W})$ is a base for a uniformity on X. Now the uniformity generated by the collection

$$c[f^{-1}(\mathcal{W}), \tau_X]$$

of all components of all sets in $f^{-1}(\mathfrak{W})$ is the τ-component uniformity relative to $\mathcal{W}$ on the space X. We then call the spread f a *uniform spread*. See also [9].

Now suppose that (X, f, Z) is a uniform spread from a uniform space X carrying the uniformity $\mathfrak{U}$ induced by inverse images $f^{-1}(\mathcal{W})$ of $\mathcal{W}$ from a uniformity $\mathfrak{W}$ compatible with the topology on Z. To arrive at a uniform spread completion, consider the uniform completion X_U whose uniformity is that generated by minimal Cauchy filters of X. Then X is densely imbedded in X_U by say, $j_U: X \to X_U$ which maps each $x \in X$ to the minimal Cauchy filter $(\mathrm{Im}\chi)^+$ for which χ is the spread point in (X, f, Z) taking each uniform cover W containing $f(x)$ to the component of $f^{-1}(\mathcal{W})$ that contains x.

Define $f_U: X_U \to Z$ by associating with each $x \in X_U$ the unique point $f(x) \in Z$ for which χ is the spread point.

(i) *$j_U(X)$ is locally connected in X_U*: By definition of the induced uniformity $\mathfrak{U}$ on X, we know that X is uniformly locally connected. But then $j_U(X)$ is uniformly locally connected and, accordingly (from Hunt [9]) it is locally connected in X_U.

(ii) *f_U is a complete uniform spread*: Consider a uniform cover $\mathcal{W}$ in $\mathfrak{W}$ and the collection $\{c[f^{-1}(\mathcal{W}), \mathfrak{U}] \mid \mathcal{W} \in \mathfrak{W}\}$ of all components of $f^{-1}(\mathcal{W})$. Then $f_U^{-1}(\mathcal{W}) = \bigcup\limits_{\mathcal{W}} (\mathrm{Im}\,\chi)^+$.

(iii) *$f_U \circ j = f$*: Take $x \in X$. Then one easily shows that

$$f_U \circ j_U(x) = f_U(\mathrm{Im}\chi)^+ = f(x). \qquad \square$$

Remark. Recall (see e.g. Preuβ [15]) that a topological space X is an $\mathbb{R}_0$-space iff $x \in \overline{\{y\}}$ implies that $y \in \overline{\{x\}}$ for every $x, y \in X$. Since the category $\mathbb{T}$-$\mathbb{N}$**ear** of topological nearness spaces and uniformly continuous functions is isomorphic to the category $\mathbb{R}_0$-$\mathbb{T}$**op** of topological $\mathbb{R}_0$-spaces and continuous functions, it is clear that for an $\mathbb{R}_0$-space X Hunt's uniform spread completion so described reduces to Fox's spread completion.

3. What are nearness spreads?

Hunt's uniform spreads can be extended to the category of nearness spaces and uniformly continuous functions as follows: For a topological space (X, τ) and a collection $\mathcal{A} \subseteq \mathcal{P}(X)$, we denote by

$$c(\mathcal{A}, \tau) = \{K \subseteq_c A \mid A \in \mathcal{A}\}$$

the collection of *all* components of *all* sets in $\mathcal{A}$ ($\subseteq_c$ denotes component). Now consider a nearness structure

$$\mu = \{\mu_\lambda \mid \lambda \in \Lambda\}$$

on X. We then have

PROPOSITION (3.1). *The collection*

$$\{c(\mu_\lambda, \mu) \mid \lambda \in \Lambda\}$$

is a base for a regular nearness on X, and the resulting nearness structure on X is called the τ-component nearness relative to μ. □

Proof. This follows from Hunt [9], Proposition 3.1. □

Proof of Theorem (1.1). In view of Proposition (3.1), we assume that X is a regular nearness space and then generalize Hunt's proof to the nearness case as follows:

The following is a generalization of the original proof of Hunt for uniform spreads.

(i) Suppose that χ is a spread point in (X, f, Z). Then the filter $(\operatorname{Im}\chi)^+$ (generated by the filter base $\operatorname{Im}\chi$) is a Cauchy filter.

We claim that $(\operatorname{Im}\chi)^+$ is a minimal Cauchy filter in (X, ξ)): Given $z \in Z$ and an open neighborhood $W \ni z$ in Z, set $U = \chi(W)$. Since the neighborhood filter of z is a minimal Cauchy filter in (X, ξ), we pick $\mathcal{W}_\alpha$ and some open neighborhood $V \ni z$ with $V \in \mathcal{W}_\alpha$ such that

$$\operatorname{St}(V, \mathcal{W}_\alpha) \subseteq W.$$

We set $S = \chi(V)$. Then (by SP_2)

$$V \subseteq St(V, \mathcal{W}_\alpha) \subseteq W \Rightarrow \chi(V) \subseteq \chi(W)$$

Then $St(S, \mathcal{U}_\lambda) \subseteq U$: For, if $S \cap T \neq \emptyset$ for a component T of $f^{-1}(M)$ with $M \in \mathcal{W}_\alpha$ then $V \cap M \neq \emptyset$, and so $M \subseteq W$. But $T \cap U \neq \emptyset$, so $T \subseteq U$ and then $st(S, \mathcal{U}_\lambda) \subseteq U$, ensuring that $(\operatorname{Im}\chi)^+$ is a minimal Cauchy filter.

We now invoke Proposition (1.5); set $\mathcal{A}$ to be the ξ-cluster for which $\widehat{\mathcal{A}} = (\operatorname{Im}\chi)^+$.

(ii) *The correspondence is surjective*: For, if $\mathcal{A}$ is an ξ-cluster then $\widehat{\mathcal{A}}$ is a minimal Cauchy filter in (X, χ). It follows from the uniform continuity of $f : (X, \chi) \longrightarrow (Z, \mu)$ that the filter $[f(\widehat{\mathcal{A}})]^+$ generated by $f(\widehat{\mathcal{A}})$ is a Cauchy filter in (Z, μ) which is complete by assumption. Accordingly, $[f(\widehat{\mathcal{A}})]^+$ converges to a point $z \in Z$. To arrive at a spread point we proceed as follows: Take an open nhood $W \ni z$, and note that $f(\widehat{\mathcal{A}}) \ni W$. Now Corollary (1.3) ensures that a

$V \in \xi \wedge \bigcup_\lambda \mu_\lambda$ exists such that $f(V) \subseteq W$. Find a component G of $f^{-1}(W)$ such that $V \subseteq G \subseteq f^{-1}(W)$. Such a component U of $f^{-1}(W)$ is unique in $\widehat{\mathcal{A}}$. This then ensures that we define $\chi(W) = U$. Then $\operatorname{Im}\chi$ has the finite intersection property because $\operatorname{Im}\chi \subseteq \widehat{\mathcal{A}}$, and therefore, χ is a spread point in (X, f, Z).

(iii) *The correspondence is injective*: Take two spread points $\chi \neq \chi'$ in (X, f, Z), say,

$$\chi \colon \mathcal{N}_z \longrightarrow \mathcal{C}_z^V; \qquad \chi \colon \mathcal{N}_z \longrightarrow \mathcal{C}_{z'}^V,$$

for $z, z' \in Z$. We find that

(iiia) If $z \neq z'$ then there are disjoint neighborhoods V, W of z and z', respectively, and so $\chi(V) \cap \chi(W) = \emptyset$ making $\chi(W) \notin (\operatorname{Im}\chi')^+$ since $(\operatorname{Im}\chi')^+$ is a filter. This means that $(\operatorname{Im}\chi)^+ \neq (\operatorname{Im}\chi')^+$.

(iiib) On the other hand, if $z \neq z'$ in Z it follows from the choice of $\chi, \chi\prime$ that a neighborhood V of $z = z\prime$ exists for which $\chi(V) \neq \chi\prime(V)$. Since these are components, we must have $\chi(V) \cap \chi\prime(V) = \emptyset$; thus $\chi(V) \notin (\operatorname{Im}\chi')^+$. □

There are other generalizations of results on Hunt's uniform spreads to the realm of nearness spaces, which are a subject of further investigation.

Acknowledgements

The author is indebted to the referee whose suggestions greatly improved the original manuscript. In fact, the referee remarked that the nearness spreads introduced in this paper are extendable to the called "*supernear spaces*" introduced by the referee. However, this has not been explored.

Received January 09, 2006

Final version received September 18, 2006

Department of Mathematics
University of Limpopo
Private Bag X1106
Sovenga, 0727
South Africa
siweyah@ul.ac.za

References

[1] Bentley, L., *The role of Nearness Spaces in Topology*, Categorical Topology, edited by E. Binz and H. Herrlich, Lecture Notes in Math. Vol. **549**, Springer-Verlag, Berlin, 1976, 1–22.

[2] Bentley, H. L. and Herrlich, H., *On the strict completion of a nearness space*, Quaest Math. **24** (2001) 39–50.

[3] Bunge, M and Funk, J., *Spreads and the symmetric topos II*, J. Pure and Appl. Alg. Vol. **130** (1) (1998) 49–84.

[4] Bushaw, D., (Elements of) General Topology, John Wiley, New York, 1963.

[5] Fox, R. H., *Covering spaces with singularities*, pp. 243-257 in "Algebraic Geometry and Topology: A symposium in Honour of S. Lefschetz, edited by R. H. Fox et al", Princeton Univ. Press, Princeton, N. J., 1957.

[6] de Neymet de Christ, S and Gonzales Acuna, F., *A generalization of Fox's spread completion*, in: Sypmosium on Algebraic Topology in honour of José Adem (Oaxtepec, 1981) (Ed. Gitler, S), pp. 271–285, Contemporary Math. **12**, Amer. Math. Soc., Providence, R. I., 1982.

[7] HERRLICH, H., *A concept of Nearness*, Gen. Top. its Appl. **5** (1974) 191–212.
[8] HERRLICH, H., *Topological Structures*, Math. Centre Tract **52** (1974) 59–122.
[9] HUNT, J. H. V., *Branched coverings as uniform completions of unbranched coverings (Résumé)*, Contemporary Mathematics (AMS), Vol. **12** (1982) 141–155.
[10] HUNT, J. H. V., *The uniform properties of Fox's spreads*, Bol. Soc. Mat. Mexicana (2) **34** (1989) no. 1-2, 11–21.
[11] KATETOV, M., *On continuity structures and spaces of mappings*, Comment. Math. Univ. Carolinae **6** (1965) 257–278.
[12] KELLY, J. L., General Topology, Van Nostrand, Princeton, N. J., 1955.
[13] MICHAEL, E., *Completing a spread (in the sense of R. H. Fox) without local connectedness*, Nederl. Akad. Wetensch. Prod. Ser. A66 = Inda. Math. **25** (1963) 629–633.
[14] MONTESINOS-AMILIBIA, JOSÉ MARÍA, *Branched coverings after Fox*, Bol. Soc. Mat. Mexicana 3a. Ser. Vol. **11** No. 1 (2005) 19–64.
[15] PREUβ, G., *Theory of topological spaces (An approach to Categorical Topology)*, Dept. of Math. Free Univ. Berlin, F. R. G. D. Reidel Publ. co, 1988.
[16] SIWEYA, H. J., *A necessary and sufficient condition for a complete spread*, Scient. Math. Jap. **54** (3) (2001) 459–462.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

A COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS IN SYMMETRIC SPACES SATISFYING AN IMPLICIT RELATION

DURAN TURKOGLU AND ISHAK ALTUN

Abstract. In this paper, we prove a common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying an implicit relation and a property (E.A) introduced in [M. Aamri, D. El Moutawakil, Some new common fixed point thorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002) 181-188]. Our theorem generalizes Theorem 1 of [A. Aliouche, A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type], Theorem 2.2 of [M. Aamri, D. El Moutawakil, Common fixed points under contractive conditions in symmetric spaces, Appl. Math. E-Notes 3 (2003) 156-162] and Theorem 2 of [M. Aamri, D. El Moutawakil, Some new common fixed point thorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002) 181-188].

1. Introduction and preliminaries

It is well known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended in many different directions. However, it has been observed in [6] that some of the defining properties of the metric are not needed in the proofs of certain metric theorems. Motivated by this fact, Hicks and Rhoades [6] established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric or semi-metric.

Recall that a symmetric on a set X is a nonnegative real valued function d on $X \times X$ such that
(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$.

Let d be a symmetric on a set X and for $r > 0$ and any $x \in X$, let $B(x, r) = \{y \in X : d(x, y) < r\}$. A topology $t(d)$ on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x, r) \subset U$ for some $r > 0$. A symmetric d is a semi-metric if for each $x \in X$ and each $r > 0$, $B(x, r)$ is a neighborhood of x in the topology $t(d)$. Note that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $t(d)$. The following two axioms were given by Wilson [19]. Let (X, d) be a symmetric space.

(W.3) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

2000 *Mathematics Subject Classification*: Primary 54H25; Secondary 47H10.
Keywords and phrases: common fixed point, weakly compatible mappings, symmetric space, implicit relation.

(W.4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply that $\lim_{n\to\infty} d(y_n, x) = 0$.

It is easy to see that for a semi-metric d, if $t(d)$ is Hausdorff, then $(W.3)$ holds. On the other hand, the notion of the weak commutativity is introduced by Sessa [16] as follows:

Two selfmappings S and T of a metric space (X, d) are said to be weakly commuting if

$$d(STx, TSx) \leq d(Sx, Tx), \text{ for all } x \in X.$$

Jungck [8] extended this concept in the following way: Let S and T be two selfmappings of a metric space (X, d). S and T are said to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Obviously, two weakly commuting mappings are compatible but the converse is not true as is shown in [8]. Recently, Jungck [9] introduced the concept of weakly compatible maps as follows: Two selfmappings S and T of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points; i.e., if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

It is easy to see that two compatible maps are weakly compatible but the converse is not true. All these concepts have been frequently used to prove existence theorems in common fixed point theory.

However, the study of common fixed points of non-compatible maps is also very interesting [10], [11].

On the other hand, Aamri and El Moutawakil [2] have established some new common fixed point theorems under strict contractive conditions on a metric space for mappings satisfying property $(E.A)$ defined as follows: Let S and T be two selfmappings of a metric space (X, d). We say that S and T satisfy property $(E.A)$ if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \in X.$$

The main purpose of this paper is to give a common fixed point theorem for selfmappings of a symmetric space. These self mappings are assumed to satisfy an implicit relation and a new property introduced recently in [2] on a metric space, which generalizes the notion of non-compatible maps in the setting of a symmetric space.

Definition (1.1). [3] Let S and T be two selfmappings of a symmetric space (X, d). S and T are said to be compatible if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0$$

for some $t \in X$.

Definition (1.2). [3] Two selfmappings S and T of a symmetric space (X, d) are said to be weakly compatible if they commute at their coincidence points.

Definition (1.3). [3] Let S and T be two selfmappings of a symmetric space (X, d). We say that S and T satisfy the property $(E.A)$ if there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0$$

for some $t \in X$.

Example (1.4). Let $X = [0, +\infty[$. Let d be a symmetric on X defined by $d(x, y) = e^{|x-y|} - 1$ for all x, y in X. Define $S, T\colon X \to X$ as follows: $Sx = 2x+1$ and $Tx = x+2$, for all $x \in X$. Note that the function d is not a metric. Consider the sequence $x_n = 1 + \frac{1}{n}$, $n = 1, 2, \ldots$.

Clearly

$$\lim_{n\to\infty} d(Sx_n, 3) = \lim_{n\to\infty} d(Tx_n, 3) = 0.$$

Then S and T satisfy property $(E.A)$, but S and T are not weakly compatible.

Example (1.5). Let $X = R$ with the above symmetric function d. It is easy to see that the condition $(W.3)$ holds. Define $S, T\colon X \to X$ by $Sx = x + 1$ and $Tx = x + 2$, for all $x \in X$.

Suppose that property $(E.A)$ holds. Then there exists in X a sequence $\{x_n\}$ satisfying $\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0$ for some $t \in X$. Therefore

$$\lim_{n\to\infty} d(x_n, t-1) = \lim_{n\to\infty} d(x_n, t-2) = 0.$$

In view of $(W.3)$, we conclude that $t-1 = t-2$, which is a contradiction. Hence S and T do not satisfy property $(E.A)$. It is clear from Definition (1.1), that two selfmappings S and T of a symmetric space (X, d) will be non-compatible if there exists at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(Sx_n, t) = \lim_{n\to\infty} d(Tx_n, t) = 0 \text{ for some } t \in X$$

but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or does not exist.

Therefore, two non-compatible selfmappings of a symmetric space(X, d) satisfy property $(E.A)$.

Definition (1.6). [3] Let (X, d) be a symmetric space. We say that (X, d) satisfies property $(H.E)$ if given $\{x_n\}$, $\{y_n\}$ and x in X,

$$\lim_{n\to\infty} d(x_n, x) = 0 \text{ and } \lim_{n\to\infty} d(y_n, x) = 0 \text{ imply } \lim_{n\to\infty} d(x_n, y_n) = 0.$$

Example (1.7). (i) Every metric space (X, d) satisfies property $(H.E)$.

(ii) Let $X = [0, +\infty)$ with the symmetric function d defined in Example (1.4). It is easy to see that the symmetric space (X, d) satisfies property $(H.E)$.

2. Implicit relation

Implicit relations on metric spaces have been used in many articles. (see [4], [7], [12], [13], [14], [17]).

Let R_+ denote the non-negative real numbers and let $\mathcal{F}$ be the set of all continuous functions $F\colon R_+^4 \to R$ satisfying the following conditions:

F_1: there exists an upper semi-continuous and non-decreasing function $f : R_+ \to R_+$, $f(0) = 0$, $f(t) < t$ for $t > 0$, such that for $u \geq 0$,

$$F(u, v, v, 0) \leq 0 \text{ or } F(u, v, 0, v) \leq 0 \text{ or } F(u, 0, v, v) \leq 0$$

implies $u \leq f(v)$.

F_2: $F(u, 0, 0, 0) > 0$ and $F(u, u, u, 0) > 0, \forall u > 0$.

Example (2.1). $F(t_1, t_2, t_3, t_4) = t_1 - \alpha \max\{t_2, t_3, t_4\}$, where $0 < \alpha < 1$.

F_1: Let $u > 0$ and $F(u, v, v, 0) = u - \alpha v \leq 0$, then $u \leq \alpha v$. Similarly, let $u > 0$ and $F(u, v, 0, v) \leq 0$, then $u \leq \alpha v$ and again let $u > 0$ and $F(u, 0, v, v) \leq 0$, then $u \leq \alpha v$. If $u = 0$ then $u \leq \alpha v$. Thus F_1 is satisfied with $f(t) = \alpha t$.

F_2: $F(u, 0, 0, 0) = u > 0, \forall u > 0$ and $F(u, u, u, 0) = u(1 - \alpha) > 0$, $\forall u > 0$.

Thus $F \in \mathcal{F}$.

Example (2.2). $F(t_1, t_2, t_3, t_4) = t_1 - \psi(\max\{t_2, t_3, t_4\})$, where $\psi : R_+ \to R_+$ is upper semi-continuous, non-decreasing and $\psi(0) = 0$, $\psi(t) < t$ for $t > 0$.

F_1: Let $u > 0$ and $F(u, v, v, 0) = u - \psi(v) \leq 0$, then $u \leq \psi(v)$. Similarly, let $u > 0$ and $F(u, v, 0, v) \leq 0$, then $u \leq \psi(v)$ and again let $u > 0$ and $F(u, 0, v, v) \leq 0$, then $u \leq \psi(v)$. If $u = 0$ then $u \leq \psi(v)$. Thus F_1 is satisfied with $f = \psi$.

F_2: $F(u, 0, 0, 0) = u > 0, \forall u > 0$ and $F(u, u, u, 0) = u - \psi(u) > 0$, $\forall u > 0$.

Thus $F \in \mathcal{F}$.

Example (2.3). $F(t_1, t_2, t_3, t_4) = t_1 - (at_2 + bt_3 + ct_4)$, where $a > 0$, $b, c \geq 0$, and $\max\{a + b, a + c, b + c\} < 1$.

F_1: Let $u > 0$ and $F(u, v, v, 0) = u - (a + b)v \leq 0$, then $u \leq (a + b)v$. Similarly, let $u > 0$ and $F(u, v, 0, v) \leq 0$, then $u \leq (a + c)v$ and again let $u > 0$ and $F(u, 0, v, v) \leq 0$, then $u \leq (b + c)v$. If $u = 0$ then $u \leq (b + c)v$. Thus F_1 is satisfied with $f(t) = \max\{a + b, a + c, b + c\}t$.

F_2: $F(u, 0, 0, 0) = u > 0, \forall u > 0$ and $F(u, u, u, 0) = u(1 - a + b) > 0$, $\forall u > 0$.

Thus $F \in \mathcal{F}$.

Example (2.4). $F(t_1, t_2, t_3, t_4) = t_1 - \dfrac{at_3^2 + bt_4^2}{t_2 + t_3 + t_4 + 1}$, where $a, b \geq 0$ and $a + b < 1$.

F_1: Let $u > 0$ and $F(u, v, v, 0) = u - \dfrac{av^2}{2v + 1} \leq 0$, then $u \leq \dfrac{av^2}{2v + 1} \leq av$. Similarly, let $u > 0$ and $F(u, v, 0, v) \leq 0$, then $u \leq bv$ and again let $u > 0$ and $F(u, 0, v, v) \leq 0$, then $u \leq (a + b)v$. If $u = 0$ then $u \leq (a + b)v$. Thus F_1 is satisfied with $f(t) = (a + b)t$.

F_2: $F(u, 0, 0, 0) = u > 0, \forall u > 0$ and $F(u, u, u, 0) = \dfrac{(2 - a)u^2 + u}{2u + 1} > 0$, $\forall u > 0$.

Thus $F \in \mathcal{F}$.

Example (2.5). $F(t_1, t_2, t_3, t_4) = t_1 - \dfrac{\alpha t_2 t_3 + \beta t_2 t_4 + \gamma t_3 t_4}{t_2 + t_3 + t_4 + 1}$, where $\alpha, \beta, \gamma \geq 0$ and $\max\{\alpha, \beta, \gamma\} < 1$.

F_1: Let $u > 0$ and $F(u, v, v, 0) = u - \dfrac{av^2}{2v + 1} \leq 0$, then $u \leq \dfrac{av^2}{2v + 1} \leq av$. Similarly, let $u > 0$ and $F(u, v, 0, v) \leq 0$, then $u \leq \beta v$ and again let $u > 0$ and $F(u, 0, v, v) \leq 0$, then $u \leq \gamma v$. If $u = 0$ then $u \leq \gamma v$. Thus F_1 is satisfied with $f(t) = \max\{\alpha, \beta, \gamma\}t$.

F_2: $F(u,0,0,0) = u > 0, \forall u > 0$ and $F(u,u,u,0) = \dfrac{(2-\alpha)u^2+u}{2u+1} > 0$, $\forall\, u > 0$.

Thus $F \in \mathcal{F}$.

3. Main result

THEOREM (3.1). *Let d be a symmetric for X that satisfies $(W.3)$, $(W.4)$ and $(H.E)$. Let A, B, S and T be self mappings of (X,d) such that*

$$(3.2)\quad F\left(\int_0^{d(Ax,By)}\varphi(t)dt, \int_0^{d(Sx,Ty)}\varphi(t)dt, \int_0^{d(Sx,By)}\varphi(t)dt, \int_0^{d(By,Ty)}\varphi(t)dt\right) \le 0$$

for all $x, y \in X$ where $F \in \mathcal{F}$ and $\varphi\colon R_+ \to R_+$ is a Lebesque-integrable mapping which is summable, non-negative and such that

$$(3.3)\qquad \int_0^{\varepsilon}\varphi(t)dt > 0 \text{ for all } \varepsilon > 0.$$

Suppose that $A(X) \subset T(X)$ and $B(X) \subset S(X)$, $\{A,S\}$ and $\{B,T\}$ are weakly compatible and $\{A,S\}$ or $\{B,T\}$ satisfies property $(E.A)$. If the range of one of the mappings A, B, S and T is a closed subspace of X, then A, B, S and T have common fixed point in X.

Proof. Suppose that B and T satisfy property $(E.A)$. Then, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(Bx_n, z) = \lim_{n\to\infty} d(Tx_n, z) = 0$ for some $z \in X$. Therefore, by $(H.E)$ we have $\lim_{n\to\infty} d(Bx_n, Tx_n) = 0$. Since $B(X) \subset S(X)$, there exists in X a sequence $\{y_n\}$ such that $Bx_n = Sy_n$. Hence, $\lim_{n\to\infty} d(Sy_n, z) = 0$. Let us show that $\lim_{n\to\infty} d(Ay_n, z) = 0$.

Suppose that $\overline{\lim}_{n\to\infty} d(Ay_n, Bx_n) > 0$. Then, using (3.2), we have

$$F\left(\int_0^{d(Ay_n,Bx_n)}\varphi(t)dt, \int_0^{d(Sy_n,Tx_n)}\varphi(t)dt, \int_0^{d(Sy_n,Bx_n)}\varphi(t)dt, \int_0^{d(Bx_n,Tx_n)}\varphi(t)\,dt\right) \le 0$$

and so

$$F\left(\overline{\lim_{n\to\infty}}\int_0^{d(Ay_n,Bx_n)}\varphi(t)\,dt, \overline{\lim_{n\to\infty}}\int_0^{d(Bx_n,Tx_n)}\varphi(t)\,dt, 0, \overline{\lim_{n\to\infty}}\int_0^{d(Bx_n,Tx_n)}\varphi(t)\,dt\right) \le 0.$$

From F_1, there exists an upper semi-continuous and non-decreasing function $f\colon R_+ \to R_+$, $f(0) = 0$, $f(t) < t$ for $t > 0$ such that

$$\overline{\lim_{n\to\infty}}\int_0^{d(Ay_n,Bx_n)}\varphi(t)dt \le f\left(\overline{\lim_{n\to\infty}}\int_0^{d(Bx_n,Tx_n)}\varphi(t)dt\right) < \overline{\lim_{n\to\infty}}\int_0^{d(Bx_n,Tx_n)}\varphi(t)dt.$$

Therefore $\overline{\lim}_{n\to\infty} \int_0^{d(Bx_n,Tx_n)}\varphi(t)dt > 0$ which is a contradiction. Then we have that $\lim_{n\to\infty} \int_0^{d(Ay_n,Bx_n)}\varphi(t)dt = 0$ and (3.3) implies that $\lim_{n\to\infty} d(Ay_n, Bx_n) = 0$. By $(W.4)$, we deduce that $\lim_{n\to\infty} d(Ay_n, z) = 0$. Suppose that $S(X)$ is a closed subspace of X. Then $z = Su$ for some $u \in X$. Consequently, we have

$$\lim_{n\to\infty} d(Ay_n, Bx_n) = \lim_{n\to\infty} d(Bx_n, Su) = \lim_{n\to\infty} d(Tx_n, Su) = \lim_{n\to\infty} d(Sy_n, Su) = 0.$$

We claim that $Au = Su$. Using (3.2),

$$F\left(\int_0^{d(Au,Bx_n)}\varphi(t)dt, \int_0^{d(Su,Tx_n)}\varphi(t)dt, \int_0^{d(Su,Bx_n)}\varphi(t)dt, \int_0^{d(Bx_n,Tx_n)}\varphi(t)dt\right) \leq 0$$

and letting $n \to \infty$, we have

$$F\left(\lim_{n\to\infty}\int_0^{d(Au,Bx_n)}\varphi(t)dt, 0, 0, 0\right) \leq 0$$

which is a contradiction with F_2 if $\lim_{n\to\infty}\int_0^{d(Au,Bx_n)}\varphi(t)dt > 0$. Thus we obtain $\lim_{n\to\infty}\int_0^{d(Au,Bx_n)}\varphi(t)dt = 0$ and (3.3) implies that $\lim_{n\to\infty} d(Au, Bx_n) = 0$. By $(W.3)$ we have $z = Au = Su$. The weak compatibility of A and S implies that $ASu = SAu$; i.e., $Az = Sz$. On the other hand, since $A(X) \subset T(X)$, there exists $v \in X$ such that $Au = Tv$. We claim that $Bv = Tv$. If not, condition (3.2) gives

$$F\left(\int_0^{d(Au,Bv)}\varphi(t)dt, \int_0^{d(Su,Tv)}\varphi(t)dt, \int_0^{d(Su,Bv)}\varphi(t)dt, \int_0^{d(Bv,Tv)}\varphi(t)dt\right) \leq 0$$

and so

$$F\left(\int_0^{d(Au,Bv)}\varphi(t)dt, 0, \int_0^{d(Tv,Bv)}\varphi(t)dt, \int_0^{d(Bv,Tv)}\varphi(t)dt\right) \leq 0.$$

From F_2

$$\int_0^{d(Tv,Bv)}\varphi(t)dt = \int_0^{d(Au,Bv)}\varphi(t)dt \leq f\left(\int_0^{d(Tv,Bv)}\varphi(t)dt\right)$$

which is a contradiction since $\int_0^{d(Tv,Bv)}\varphi(t)dt > 0$ by (3.3). Hence, $z = Au = Su = Bv = Tv$. The weak compatibility of B and T implies that $BTv = TBv$; i.e., $Bz = Tz$. Let us show that z is a common fixed point of A, B, S and T .

If $z \neq Az$, using (3.2), we get

$$F\left(\int_0^{d(Az,Bv)}\varphi(t)dt, \int_0^{d(Sz,Tv)}\varphi(t)dt, \int_0^{d(Sz,Bv)}\varphi(t)dt, \int_0^{d(Bv,Tv)}\varphi(t)dt\right) \leq 0$$

and so

$$F\left(\int_0^{d(Az,z)}\varphi(t)dt, \int_0^{d(Az,z)}\varphi(t)dt, \int_0^{d(Az,z)}\varphi(t)dt, 0\right) \leq 0$$

which is a contradiction with F_2 since $\int_0^{d(Az,z)}\varphi(t)dt > 0$ by (3.3). Thus $z = Az = Sz$.

If $z \neq Bz$, using (3.2), we get

$$F\left(\int_0^{d(Az,Bz)}\varphi(t)dt, \int_0^{d(Sz,Tz)}\varphi(t)dt, \int_0^{d(Sz,Bz)}\varphi(t)dt, \int_0^{d(Bz,Tz)}\varphi(t)dt\right) \leq 0$$

and so

$$F\left(\int_0^{d(z,Bz)}\varphi(t)dt, \int_0^{d(z,Bz)}\varphi(t)dt, \int_0^{d(z,Bz)}\varphi(t)dt, 0\right) \leq 0$$

which is a contradiction with F_2 since $\int_0^{d(z,Bz)} \varphi(t)dt > 0$ by (3.3). Thus $z = Bz = Tz = Az = Sz$.

The proof is similar when $T(X)$ is assumed to be a closed subspace of X. The cases in which $A(X)$ or $B(X)$ is a closed subspace of X are similar to the cases in which $T(X)$ or $S(X)$ respectively is closed since $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

For the uniqueness of z, suppose that $w \neq z$ is another common fixed point of A, B, S and T.

Using (3.2), we obtain

$$F\left(\int_0^{d(Az,Bw)} \varphi(t)dt, \int_0^{d(Sz,Tw)} \varphi(t)dt, \int_0^{d(Sz,Bw)} \varphi(t)dt, \int_0^{d(Bw,Tw)} \varphi(t)dt\right) \leq 0$$

and so

$$F\left(\int_0^{d(z,w)} \varphi(t)dt, \int_0^{d(z,w)} \varphi(t)dt, \int_0^{d(z,w)} \varphi(t)dt, 0\right) \leq 0$$

which is a contradiction with F_2 since $\int_0^{d(z,w)} \varphi(t)dt > 0$ by (3.3). Thus $z = w$. This completes the proof of the theorem. □

If we combine Theorem (3.1) with Example (2.2) we have the following corollary which it is Theorem 1 of [3].

COROLLARY (3.4). *Let d be a symmetric for X that satisfies* (W.3), (W.4) *and* (H.E)*. Let A, B, S and T be self mappings of* (X, d) *such that*

$$\int_0^{d(Ax,By)} \varphi(t)dt \leq \psi\left(\int_0^{\max\{d(Sx,Ty),d(Sx,By),d(By,Ty)\}} \varphi(t)dt\right)$$

for all x, $y \in X$ where $\varphi\colon R_+ \to R_+$ is a Lebesque-integrable mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \varphi(t)dt > 0 \textit{ for all } \varepsilon > 0.$$

Suppose that $A(X) \subset T(X)$ and $B(X) \subset S(X)$, $\{A, S\}$ and $\{B, T\}$ are weakly compatible and $\{A, S\}$ or $\{B, T\}$ satisfies property $(E.A)$. If the range of one of the mappings A, B, S and T is a closed subspace of X, then A, B, S and T have a common fixed point in X.

If $\varphi(t) = 1$, $A = B$ and $S = T$ in Corollary (3.4), we obtain Theorem 2.1 of [1].

If $\varphi(t) = 1$, in Corollary (3.4), we obtain Theorem 2.2 of [1].

Since two non-compatible selfmappings of a symmetric space (X, d) satisfy property $(E.A)$, we get the following result.

COROLLARY (3.5). *Let d be a symmetric for X that satisfies* (W.3) *and* (H.E)*. Let A and S be two non-compatible weakly compatible self mappings of* (X, d) *such that*

$$F\left(\int_0^{d(Ax,Ay)} \varphi(t)dt, \int_0^{d(Sx,Sy)} \varphi(t)dt, \int_0^{d(Sx,Ay)} \varphi(t)dt, \int_0^{d(Ay,Sy)} \varphi(t)dt\right) \leq 0$$

for all $x, y \in X$ where $F \in \mathcal{F}$ and $\varphi\colon R_+ \to R_+$ is a Lebesgue-measurable mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \varphi(t)dt > 0 \textit{ for all } \varepsilon > 0$$

and $A(X) \subset S(X)$. If the range of A or S is a closed subspace of X, then A and S have a common fixed point in X.

If we combine Corollary (3.5) with Example (2.2) we have Corollary 2 of [3].

COROLLARY (3.6). *Let A, B, S and T be self mappings of a metric space (X, d) such that*

$$F\left(\int_0^{d(Ax,By)} \varphi(t)dt, \int_0^{d(Sx,Ty)} \varphi(t)dt, \int_0^{d(Sx,By)} \varphi(t)dt, \int_0^{d(By,Ty)} \varphi(t)dt\right) \leq 0$$

for all $x, y \in X$ where $F \in \mathcal{F}$ and $\varphi\colon R_+ \to R_+$ is a Lebesque-integrable mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \varphi(t)dt > 0 \textit{ for all } \varepsilon > 0.$$

Suppose that $A(X) \subset T(X)$ and $B(X) \subset S(X)$, $\{A, S\}$ and $\{B, T\}$ are weakly compatible and $\{A, S\}$ or $\{B, T\}$ satisfies property $(E.A)$. If the range of one of the mappings A, B, S and T is a closed subspace of X, then A, B, S and T have common fixed point in X.

If we combine Corollary (3.6) with Example (2.2) we have Corollary 3 of [3].
If $\varphi(t) = 1$, in Corollary (3.6) and combine with Example (2.2) we have Theorem 2 of [2].
If we combine Theorem (3.1) with Example (2.4) we have the following corollary.

COROLLARY (3.7). *Let d be a symmetric for X that satisfies $(W.3)$, $(W.4)$ and $(H.E)$. Let A, B, S and T be self mappings of (X, d) such that, for all $x, y \in X$,*

$$\int_0^{d(Ax,By)} \varphi(t)dt \leq \frac{a\left(\int_0^{d(Sx,By)} \varphi(t)dt\right)^2 + b\left(\int_0^{d(By,Ty)} \varphi(t)dt\right)^2}{\int_0^{d(Sx,Ty)} \varphi(t)dt + \int_0^{d(Sx,By)} \varphi(t)dt + \int_0^{d(By,Ty)} \varphi(t)dt + 1}$$

where $a, b \geq 0$, $a + b < 1$ and $\varphi\colon R_+ \to R_+$ is a Lebesque-integrable mapping which is summable, non-negative and such that

$$\int_0^{\varepsilon} \varphi(t)dt > 0 \textit{ for all } \varepsilon > 0.$$

Suppose that $A(X) \subset T(X)$ and $B(X) \subset S(X)$, $\{A, S\}$ and $\{B, T\}$ are weakly compatible and $\{A, S\}$ or $\{B, T\}$ satisfies property $(E.A)$. If the range of one of the mappings A, B, S and T is a closed subspace of X, then A, B, S and T have common fixed point in X.

Remark (3.8). We obtain some new results, if we combine Theorem (3.1) with some examples of F.

Now we give an example.

Example (3.9). Let $X = \{\frac{1}{n} : n \in N\} \cup \{0\}$ with the symmetric defined by $d(x, y) = e^{|x-y|} - 1$ for all $x, y \in X$. It is obvious that the symmetric d satisfies $(W.3)$, $(W.4)$ and $(H.E)$. Define $A, B, S, T : X \to X$ as follows:

$$Ax = Bx = \begin{cases} \dfrac{1}{n+1}, & x = \dfrac{1}{n}, \\ 0, & x = 0 \end{cases} \quad Sx = Tx = x \text{ for all } x \in X.$$

Again it is obvious that $A(X) \subset T(X)$ and $B(X) \subset S(X)$, $\{A, S\}$ and $\{B, T\}$ are weakly compatible and $\{A, S\}$ or $\{B, T\}$ satisfies property $(E.A)$. Also $S(X)$ and $T(X)$ are closed subsets of X.

Now we claim that the mappings A, B, S and T satisfy the condition (3.2) of Theorem (3.1) with $F \in \mathcal{F}$ defined by

$$F(t_1, t_2, t_3, t_4) = t_1 - \tfrac{1}{2} \max\{t_2, t_3, t_4\}$$

and $\varphi : R_+ \to R_+$ defined by

$$\varphi(t) = \begin{cases} \big(\ln(1+t)\big)^{\frac{1}{\ln(1+t)} - 2} \left[\dfrac{1 - \ln(\ln(1+t))}{1+t}\right], & t > 0, \\ 0, \quad t = 0. \end{cases}$$

That is, we claim that the following inequality is satisfied:

$$\text{(3.10)} \quad \int_0^{d(Ax,Ay)} \varphi(t)\, dt \le \frac{1}{2} \max\left\{\int_0^{d(x,y)} \varphi(t)\, dt, \int_0^{d(x,Ay)} \varphi(t) dt, \int_0^{d(y,Ay)} \varphi(t) dt\right\}.$$

We show sufficiently that

$$\text{(3.11)} \qquad \int_0^{d(Ax,Ay)} \varphi(t) dt \le \frac{1}{2} \int_0^{d(x,y)} \varphi(t) dt$$

instead of (3.10). Now, since

$$\int_0^s \varphi(t)\, dt = (\ln(1+s))^{\frac{1}{\ln(1+s)}},$$

the inequality (3.11) is equivalent to

$$\big(\ln(1 + d(Ax, Ay))\big)^{\frac{1}{\ln(1+d(Ax,Ay))}} \le \frac{1}{2} \big(\ln(1 + d(x, y))\big)^{\frac{1}{\ln(1+d(x,y))}}$$

and so, since $d(x, y) = e^{|x-y|} - 1$, the inequality (3.11) is equivalent to

$$\text{(3.12)} \qquad |Ax - Ay|^{\frac{1}{|Ax-Ay|}} \le \tfrac{1}{2} |x - y|^{\frac{1}{|x-y|}}$$

for all $x, y \in X$. Using [5, Example 3.6] we can show that the inequality (3.12) is true for all $x, y \in X$. Thus all conditions of Theorem (3.1) are satisfied and so the mappings A, B, S and T have a common fixed point in X. Note that the results of [1] and [2] are not applicable to this example.

Acknowledgement

The authors are thankful to Prof. Rhoades for his papers [15] and [18] and the referees for their valuable comments in modifying the first version of this paper.

Received May 24, 2006

Final version received February 15, 2007

Duran Turkoglu
Department of mathematics
Faculty of Science and Arts
Gazi University
06500 Teknikokullar
Ankara
Turkey
dturkoglu@gazi.edu.tr

Ishak Altun
Department of mathematics
Faculty of Science and Arts
Kirikkale University
71450 Yahsihan
Kirikkale
Turkey
ialtun@kku.edu.tr, ishakaltun@yahoo.com

References

[1] M. Aamri and D. El Moutawakil, *Common fixed points under contractive conditions in symmetric spaces*, Appl. Math. E-Notes **3** (2003), 156–162.
[2] M. Aamri, D. El Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl. **270** (2002), 181–188.
[3] A. Aliouche, *A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type*, J. Math. Anal. Appl. **322** (2006), 796–802.
[4] I. Altun, H. A. Hancer and D. Turkoglu, *A fixed point theorem for multi-maps satisfying an implicit relation on metrically convex metric spaces*, Math. Commun. **11** (2006), 17–23.
[5] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci. **29** (2002), 531–536.
[6] T. L. Hicks and B. E Rhoades, *Fixed point theory in symmetric spaces with applications to probabilistic spaces*, Nonlinear Anal., **36** (1999), 331–344.
[7] M. Imdad, S. Kumar and M. S. Khan, *Remarks on some fixed point theorems satisfying implicit relations*, Rad. Mat. **11** (1), (2002), 135–143.
[8] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci. **9** (1986), 771–779.
[9] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci. (4) (2), (1996), 199–215.
[10] R. P. Pant, *Common fixed points of contractive maps*, J. Math. Anal. Appl. **226** (1998), 251–258.
[11] R. P. Pant, *Common fixed points of sequences of mappings*, Ganita **4** (1996), 43–49.
[12] V. Popa, *Fixed point theorems for implicit contractive mappings*, Stud. Cercet Stiint Ser. Mat. Univ. Bacau **7** (1997), 127–133.
[13] V. Popa, *A general coincidence theorem for compatible multivalued mappings satisfying an implicit relation*, Demonstratio Math. **33** (1) (2000), 159–164.

[14] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation*, Demonstratio Math. **32** (1) (1999), 157–163.

[15] B. E. Rhoades, *Two fixed point theorems for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci. 2003 (2003), no. 63, 4007–4013.

[16] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. **32** (46), (1982), 149–153.

[17] S. Sharma and B. Desphande, *On compatible mappings satisfying an implicit relation in common fixed point consideration*, Tamkang J. Math. **33** (3), (2002), 245–252.

[18] P. Vijayaraju, B. E. Rhoades and R. Mohanraj, *A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci. **15** (2005), 2359–2364.

[19] W. A. Wilson, *On semi-metric spaces*, Amer. J. Math. **53** (1931), 361–373.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

ON SUBGROUPS OF $\pi_*(L_2T(1) \wedge M(2))$ AT THE PRIME TWO

IPPEI ICHIGI, KATSUMI SHIMOMURA, AND XIANGJUN WANG

ABSTRACT. Let L_2 denote the Bousfield localization functor with respect to the v_2-localized Brown-Peterson spectrum $v_2^{-1}BP$ on the stable homotopy category of spectra at the prime two, and $T(1)$ denote the Ravenel spectrum. Then the Adams-Novikov spectral sequence is a tool to determine the homotopy groups $\pi_*(L_2T(1))$. In [2], for $s > 6$, we determined the E_∞-term E_∞^s of the Adams-Novikov spectral sequence converging to $\pi_*(L_2T(1))$. The s-th line E_2^s of the E_2-term is very complicated if $1 < s \le 6$. Let $M(k)$ denote the mod 2^k Moore spectrum. Then the complicated parts of the E_2-term for $\pi_*(L_2T(1) \wedge M(k))$ also stay in the s-th lines for $1 < s \le 6$. Here we determine the s-th lines of the Adams-Novikov E_∞-term for $s = 0$, 1 and $s > 6$ of the Adams-Novikov spectral sequence converging to $\pi_*(L_2T(1) \wedge M(2))$. The result shows how disordered the structure of the E_2-term for $\pi_*(L_2T(1))$ is.

1. Introduction

Let $\mathcal{S}_{(p)}$ denote the stable homotopy category of spectra localized away from a prime number p, and BP denote the Brown-Peterson spectrum characterized by the coefficient ring $BP_* = \pi_*(BP) = \mathbb{Z}_p[v_1, v_2, \dots]$. Then we have the Bousfield localization functor $L_n\colon \mathcal{S}_{(p)} \to \mathcal{S}_{(p)}$ with respect to $v_n^{-1}BP$, and denote the image of it as $\mathcal{L}_n$. The homotopy groups $\pi_*(L_nS^0)$ of the sphere spectrum S^0 play an important role to understand $\mathcal{L}_n$. The Adams-Novikov spectral sequence is a good tool to determine them. For $n \le 2$, the E_2-term for $\pi_*(L_nS^0)$ is determined in [4], [10], [8] and [9], and the homotopy groups of L_nS^0 are also determined if $n \le 2$, except for the case where $n = 2$ and $p = 2$. Hereafter, we consider the exceptional case, and we set $n = 2$ and $p = 2$. Let $T(1)$ denote the Ravenel spectrum characterized by the BP_*-homology $BP_*(T(1)) = BP_*[t_1] \subset BP_*BP = BP_*[t_1, t_2, \dots]$. Then the homotopy groups $\pi_*(L_2T(1))$ would help us to understand the homotopy groups $\pi_*(L_2S^0)$. Let $\overline{M}(k)$ for $k > 0$ and $\overline{V}(1)$ be cofibers of $2^k\colon T(1) \to T(1)$ and $v_1\colon \Sigma^2\overline{M}(1) \to \overline{M}(1)$, respectively. Note that $\overline{M}(k) = T(1) \wedge M(k)$ for the mod 2^k Moore spectrum $M(k)$, and, in particular, $\overline{M}(1)$ is the Mahowald spectrum $X\langle 1\rangle$ and v_1 is the self map induced from the generator of $\pi_2(X\langle 1\rangle)$ (*cf* [3], [7]). Then the homotopy groups $\pi_*(L_2\overline{V}(1))$ are determined in [3]. We consider the spectrum $\overline{M}(1, \infty)$ defined by the cofiber sequence

$$\overline{M}(1) \xrightarrow{\eta_1} v_1^{-1}\overline{M}(1) \longrightarrow \overline{M}(1, \infty) \tag{1.1}$$

2000 *Mathematics Subject Classification*: Primary 55Q99; Secondary 55Q45, 55Q51.

Keywords and phrases: Ravenel spectrum, Moore spectrum, stable homotopy groups of spheres.

for the localization map η_1 and obtain a cofiber sequence

$$\overline{V}(1) \longrightarrow \overline{M}(1,\infty) \xrightarrow{v_1} \overline{M}(1,\infty) \tag{1.2}$$

We determine $\pi_*(L_2\overline{M}(1,\infty))$ by use of the cofiber sequence (1.2), and then $\pi_*(L_2\overline{M}(1))$ by (1.1) in [7]. Our next target is to determine the homotopy groups $\pi_*(L_2T(1))$ by using the mod 2 Bockstein spectral sequence. It is very hard to compute the Bockstein spectral sequence, but we get a partial results on $\pi_*(L_2T(1))$ in [2]. Here we show how hard the other parts of the E_2-term are. (The paper [5] seems to require some more time to appear, because it is too hard to verify the complicated results.) To do so, we consider the mod 4 Moore spectrum $M(2)$. Similarly, $\overline{M}(2,\infty)$ denotes a cofiber of the localization map $\eta_1\colon \overline{M}(2) \to v_1^{-1}\overline{M}(2)$. The zeroth line of the E_2-term of the Adams-Novikov spectral sequence for $\pi_*(L_2\overline{M}(2,\infty))$ is not so complicated that we determine the structure in Theorem (2.10). Since we know the structure of $E_2^0(L_2\overline{M}(1,\infty))$, it seems easy to determine the structure of $E_2^0(L_2\overline{M}(2,\infty))$ by the exact sequence $0 \to E_2^0(L_2\overline{M}(1,\infty)) \xrightarrow{2} E_2^0(L_2\overline{M}(2,\infty)) \to E_2^0(L_2\overline{M}(1,\infty)) \xrightarrow{\delta} E_2^1(L_2\overline{M}(1,\infty))$, but it is unexpectedly hard to compute the connecting homomorphism. This is why we here employ the exact sequence

$$0 \to E_2^0(L_2\overline{M}(2,1)) \to E_2^0(L_2\overline{M}(2,\infty)) \to E_2^0(L_2\overline{M}(2,\infty)) \xrightarrow{\delta} E_2^1(L_2\overline{M}(2,1)) \tag{1.3}$$

to determine it. Here the homotopy groups $\pi_*(L_2\overline{M}(2,1))$ are determined in Theorem (2.5). It seems easier to use this exact sequence even when we compute the first line $E_2^1(L_2\overline{M}(2,\infty))$ than to use that exact sequence. Since we determine the homotopy groups $\pi_*(v_1^{-1}\overline{M}(2))$ in Proposition (4.1), the result on $E_2^0(L_2\overline{M}(2,\infty))$ gives rise to the zeroth and the first lines of the E_2-term for $\pi_*(L_2\overline{M}(2))$. This displays how complicated the structure of the homotopy groups is. By use of the result [2], we also determine the s-th lines for $s > 5$, and we obtain subgroups of $\pi_*(L_2\overline{M}(2))$. The second line is too complicated to determine here.

This paper is divided into six sections:

1. Introduction
2. Statement of results
3. The change of rings theorem and the relations in $\Sigma(2)$
4. The homotopy groups $\pi_*(v_1^{-1}T(1)\wedge M(2))$ and $\pi_*(L_2\overline{M}(2,1))$
5. The elements x_n, g_n, R_n and X_n, and relations between them
6. The action of the connecting homomorphism

In the next section, we not only state our main results, but also prove some of main results. In section three, we introduce the Hopf algebroids which we consider in this paper and set up formulas on their right units η_R. In section four, we determine homotopy groups $\pi_*(v_1^{-1}\overline{M}(2))$ and prove Theorems (2.5) and (2.6). We introduce some cochains in section five, and compute the behavior of the connecting homomorphism in the last section. We also prove Theorem (2.10) there.

2. Statement of results

We work in the stable homotopy category $\mathcal{S}_{(2)}$ of spectra localized away from the prime two. Let BP denote the Brown-Peterson spectrum with coefficient ring $BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \dots]$. Then $BP_*BP = BP_*[t_1, t_2, \dots]$ is a Hopf algebroid over BP_*(*cf* [6]). We compute homotopy groups of a spectrum X by the Adams-Novikov spectral sequence with E_2-term $E_2^{s,t}(X) = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*(X))$. Let $T(1)$ denote the Ravenel spectrum with $BP_*(T(1)) = BP_*[t_1] \subset BP_*BP$. We have an element $v_1+2t_1 \in \mathrm{Ext}^{0,2}_{BP_*BP}(BP_*, BP_*[t_1]) = E_2^{0,2}(T(1))$, which survives to a homotopy element of $\pi_2(T(1))$. Since $T(1)$ is a ring spectrum, the homotopy element defines a self map $\alpha^j\colon \Sigma^{2j}T(1) \to T(1)$ for each $j > 0$, whose cofiber we denote by $T(1)/v_1^j$. For the mod 2^i Moore spectrum $M(i)$ for $i > 0$, put $\overline{M}(i, j) = M(i) \wedge (T(1)/v_1^j)$. Then, $BP_*(\overline{M}(i, j)) = BP_*[t_1]/(2^i, (v_1 + 2t_1)^j) \subset BP_*BP/(2^i, (v_1 + 2t_1)^j)$, and so we have the Adams-Novikov spectral sequence

$$E_2^{s,t}(L_2\overline{M}(i, j)) = \mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, v_2^{-1}BP_*[t_1]/(2^i, (v_1 + 2t_1)^j)) \Rightarrow \pi_*(L_2\overline{M}(i, j)).$$

Note that $\overline{M}(1, 1)$ is the $\overline{V}(1)$ in the introduction. We first determine the structure of the homotopy groups $\pi_*(L_2\overline{M}(2, 1))$, which is obtained from the cofiber sequence

$$\overline{M}(1, 1) \xrightarrow{2} \overline{M}(2, 1) \xrightarrow{\tilde{i}} \overline{M}(1, 1) \tag{2.1}$$

given by smashing $T(1)/v_1$ with the cofiber sequence

$$\overline{M}(1) \xrightarrow{2} \overline{M}(2) \longrightarrow \overline{M}(1) \tag{2.2}$$

of the Moore spectra. In order to state the result, we set up notation: In [3], the E_2-term of the Adams-Novikov spectral sequence for $\pi_*(L_2\overline{M}(1, 1))$ is determined as

$$E_2^{*,*}(L_2\overline{M}(1, 1)) = K(2)_*[v_3, h_{20}] \otimes \Lambda(h_{21}, h_{30}, h_{31}, \rho_2).$$

Here

$$K(2)_* = \mathbb{Z}/2[v_2, v_2^{-1}], \tag{2.3}$$

and h_{2i}, h_{3i} and ρ_2 are the elements represented by the cocycles, whose leading terms are $t_2^{2^i}$, $t_3^{2^i}$ and $v_2^{-5}t_4 + v_2^{-10}t_4^2$ in the cobar complex

$$\Omega^1_{BP_*BP}v_2^{-1}BP_*[t_1]/(2, v_1),$$

respectively. By use of the generators $\overline{h}_{21}$ and $\overline{h}_{31}$ given in Lemma (4.3), we rewrite the E_2-term $E_2^*(L_2\overline{M}(1, 1))$:

$$E_2^*(L_2\overline{M}(1, 1)) = K_*[v_3^2] \otimes \mathbb{Z}/2[h_{20}] \otimes \Lambda(v_2, v_3, \overline{h}_{21}, h_{30}, \overline{h}_{31}, \rho_2)$$

as a $\mathbb{Z}/2$-module, where

$$K_* = \mathbb{Z}/2[v_2^2, v_2^{-2}]. \tag{2.4}$$

Put

$$\begin{aligned} F^0 &= K_*[v_3^2] \otimes \Lambda(\overline{h}_{31}, \rho_2), \\ F &= K_*[v_3^2] \otimes \Lambda(\overline{h}_{21}, \overline{h}_{31}, \rho_2) \quad \text{and} \\ C &= K_*[v_3^2] \otimes \mathbb{Z}/2[h_{20}] \otimes \Lambda(v_3, \overline{h}_{21}, h_{30}, \overline{h}_{31}, \rho_2). \end{aligned}$$

We notice that $F = F^0 \otimes \Lambda(\overline{h}_{21})$, $C = \mathbb{Z}/2[h_{20}] \otimes F \otimes \Lambda(v_3, h_{30})$ and

$$E_2^*(L_2\overline{M}(1,1)) = C \otimes \Lambda(v_2).$$

Furthermore, the map 2 (resp. $\tilde{i}$) in (2.1) induces a homomorphism

$$2\colon E_2^*(L_2\overline{M}(1,1)) \to E_2^*(L_2\overline{M}(2,1))$$

(resp. $\pi\colon E_2^*(L_2\overline{M}(2,1)) \to E_2^*(L_2\overline{M}(1,1))$), and $2M$ (resp. M) $\subset E_2^*(L_2\overline{M}(2,1))$ for a submodule $M \subset E_2^*(L_2\overline{M}(1,1))$ denotes the image of M under the homomorphism 2 (resp. π^{-1}).

THEOREM (2.5). *The E_2-term $E_2^*(L_2\overline{M}(2,1))$ is isomorphic to*

$$2v_2C \oplus h_{20}C \oplus 2v_3F \oplus h_{30}F \oplus 2v_3h_{30}F^0 \oplus v_3h_{30}\overline{h}_{21}F^0 \oplus \widetilde{F}.$$

Here $\widetilde{F} = \mathbb{Z}/4[v_2^2, v_2^{-2}, v_3^2] \otimes \Lambda(\overline{h}_{21}, \overline{h}_{31}, \rho_2)$.

For describing the homotopy groups $\pi_*(L_2\overline{M}(2,1))$, we introduce the modules

$$\begin{aligned} F'^0 &= K_*[v_3^4] \otimes \Lambda(\overline{h}_{31}, \rho_2), \\ F' &= K_*[v_3^4] \otimes \Lambda(\overline{h}_{21}, \overline{h}_{31}, \rho_2), \\ \widetilde{F'} &= \mathbb{Z}/4[v_2^2, v_2^{-2}, v_3^4] \otimes \Lambda(\overline{h}_{21}, \overline{h}_{31}, \rho_2), \\ C' &= K_*[v_3^4] \otimes \mathbb{Z}/2[h_{20}]/(h_{20}^3) \otimes \Lambda(v_3, \overline{h}_{21}, h_{30}, \overline{h}_{31}, \rho_2) \qquad \text{and} \\ C'' &= K_*[v_3^4] \otimes \mathbb{Z}/2[h_{20}]/(h_{20}^2) \otimes \Lambda(v_3, \overline{h}_{21}, h_{30}, \overline{h}_{31}, \rho_2). \end{aligned}$$

THEOREM (2.6). *The Adams-Novikov E_∞-terms for the homotopy groups $\pi_*(L_2\overline{M}(2,1))$ are isomorphic to the direct sum of the modules*

$$2v_2C',\ h_{20}C'',\ 2v_3F,\ h_{30}F',\ 2v_3h_{30}F^0,\ v_3h_{30}\overline{h}_{21}F^{0\prime},\ \widetilde{F'} \quad \text{and} \quad 2v_3^2F'.$$

These theorems are proved in section four.

Next we consider the spectrum $\overline{M}(2,\infty) = \lim_{\to} \overline{M}(2,n)$, and the exact sequence (1.3). Put

$$E_* = \mathbb{Z}/2[v_1, v_2^2, v_2^{-2}] \quad \text{and} \quad \widetilde{E}_* = \mathbb{Z}/4[v_1, v_2^2, v_2^{-2}]. \tag{2.7}$$

Then K_* in (2.4) is $E_*/(v_1)$ and $\widetilde{E}_*/(2) = E_*$. Let $x_k = v_3^{4^k} + \dots$ denote a cochain in the cobar complex $\Omega^*_{BP_*BP}v_2^{-1}BP_*[t_1]$, which is defined in (5.3), and a_k for each $k \ge 0$ be the integer defined by

$$a_k = 4^k + 2e_k = e_{k+1} + e_k, \quad \text{for} \quad e_k = (4^k - 1)/3. \tag{2.8}$$

We further introduce modules:

$$\begin{aligned}
M &= 2v_2v_3K_*[v_3^2] \oplus 2v_2E_*[v_3^4]\langle v_3^2/v_1^3\rangle \oplus 2v_2E_*[x_n^2]\langle x_n/v_1^{a_n+1}\rangle \\
&\qquad \oplus 2v_2E_*[x_{n+1}]\langle x_n^2/v_1^{2a_n}\rangle,\\
L_{-1}^3 &= 2E_*[v_3^2]\langle v_3/v_1\rangle,\\
L_k^0 &= \widetilde{E}_*[x_{2k}^4]\langle x_{2k}^2/v_1^{c_{2k}}\rangle \oplus 2E_*[x_{2k}^4]\{x_{2k}^2/v_1^j : c_{2k} < j \le 2c_{2k}\},\\
L_k^1 &= \widetilde{E}_*[x_{2k+1}^2]\langle x_{2k+1}/v_1^{2c_{2k}}\rangle \oplus 2E_*[x_{2k+1}^2]\{x_{2k+1}/v_1^j : 2c_{2k} < j \le c_{2k+1}+2\},\\
L_k^2 &= \widetilde{E}_*[x_{2k+1}^4]\langle x_{2k+1}^2/v_1^{c_{2k+1}+2}\rangle\\
&\qquad 2E_*[x_{2k+1}^4]\{x_{2k+1}^2/v_1^j : c_{2k+1}+2 < j \le 2c_{2k+1}+6\},\\
L_k^3 &= \widetilde{E}_*[x_{2k+2}^2]\langle x_{2k+2}/v_1^{2c_{2k+1}+6}\rangle 2E_*[x_{2k+2}^2]\{x_{2k+2}/v_1^j : 2c_{2k+1}+6 < j \le c_{2k+2}\}.
\end{aligned}$$

Here, the integers c_n are defined by

(2.9)
$$c_n = 4^n + 3\times 4^{1+\varepsilon(n)}\bar{e}_{[\frac{n}{2}]} \quad \text{for } n\ge 0,\quad \varepsilon(n) = \frac{1-(-1)^n}{2} \quad\text{and}\quad \bar{e}_k = \frac{16^k-1}{15},$$

and $[x]$ denotes the greatest integer that does not exceed x.

THEOREM (2.10). *The zeroth line of the E_2-term for $\pi_*(L_2\overline{M}(2,\infty))$ is given as follows:*

$$\begin{aligned}
E_2^0(L_2\overline{M}(2,\infty)) = v_1^{-1}\widetilde{E}_*/\widetilde{E}_* &\oplus 2v_2(v_1^{-1}E_*/E_*)\\
&\oplus M \oplus L_{-1}^3 \oplus \bigoplus_{k\ge 0}\left(L_k^0\oplus L_k^1\oplus L_k^2\oplus L_k^3\right).
\end{aligned}$$

We prove this in the last section.

This result shows that the structure of $E_2^0(L_2\overline{M}(2,\infty))$ is far complicated than that of $E_2^0(L_2\overline{M}(1,\infty))$. The difficulty to compute the E_2-term $E_2^1(L_2\overline{M}(2,\infty))$ appears in the cokernel of the connecting homomorphism $\delta\colon E_2^0(L_2\overline{M}(2,\infty)) \to E_2^1(L_2\overline{M}(2,1))$, which involves ρ_2 as in Proposition (6.2).

Let

$$\overline{M}(\infty,\infty) = \varinjlim \overline{M}(k,\infty).$$

Then the Adams-Novikov E_2-term $E_2^s(L_2\overline{M}(\infty,\infty))$ is isomorphic to $(\widetilde{C_0}\otimes\Lambda(\rho_2))^s$ for $s>4$ by [2], where

$$\widetilde{C_0} = v_2v_3K_*[v_3^2, h_{20}]\otimes\Lambda(h_{21},h_{30},h_{31}).$$

This result yields the E_2-terms $E_2^s(L_2\overline{M}(2,\infty))$ for $s>5$ from the long exact sequence

$$\begin{aligned}
E_2^{s-1}(L_2\overline{M}(\infty,\infty)) \xrightarrow{4} E_2^{s-1}(L_2\overline{M}(\infty,\infty)) &\xrightarrow{\delta'} E_2^s(L_2\overline{M}(2,\infty))\\
\xrightarrow{1/4} E_2^s(L_2\overline{M}(\infty,\infty)) &\xrightarrow{4} E_2^s(L_2\overline{M}(\infty,\infty))
\end{aligned}$$

induced from the cofiber sequence $L_2\overline{M}(2,\infty)\xrightarrow{1/4} L_2\overline{M}(\infty,\infty)\xrightarrow{4} L_2\overline{M}(\infty,\infty)$. Indeed, the homomorphism 4 is trivial for $s>5$.

THEOREM (2.11). *For $s > 5$, the s-th line of the E_2-term for $\pi_*(L_2\overline{M}(2,\infty))$ is given as follows:*

$$E_2^s(L_2\overline{M}(2,\infty)) = (\widetilde{C_0}\otimes\Lambda(\rho_2))^s \oplus \delta'((\widetilde{C_0}\otimes\Lambda(\rho_2))^{s-1}).$$

Here the summand $\delta'((\widetilde{C_0}\otimes\Lambda(\rho_2))^{s-1})$ denotes the δ'-image, which is isomorphic to $(\widetilde{C_0}\otimes\Lambda(\rho_2))^{s-1}$.

By definition, we have a cofiber sequence

$$\overline{M}(2) \longrightarrow v_1^{-1}\overline{M}(2) \longrightarrow \overline{M}(2,\infty)\,, \tag{2.12}$$

which induces the long exact sequence

$$\begin{aligned} E_2^{s-1}(v_1^{-1}\overline{M}(2)) \longrightarrow E_2^{s-1}(L_2\overline{M}(2,\infty)) &\xrightarrow{\ \delta\ } E_2^s(L_2\overline{M}(2)) \\ \longrightarrow E_2^s(v_1^{-1}\overline{M}(2)) &\longrightarrow E_2^s(L_2\overline{M}(2,\infty)). \end{aligned}$$

We notice that δ is an isomorphism if $s > 2$, since $E_2^s(v_1^{-1}\overline{M}(2)) = 0$ if $s > 1$ by Proposition (4.1). Now Theorems (2.10), (2.11) and Proposition (4.1) imply the following:

THEOREM (2.13). *The E_2-term for $\pi_*(L_2T(1)\wedge M(2))$ is as follows:*

$$\begin{aligned} E_2^0(L_2\overline{M}(2)) &= \mathbb{Z}/4[v_1,v_2^2]\oplus 2v_2\mathbb{Z}/2[v_1,v_2^2],\\ E_2^1(L_2\overline{M}(2)) &= \mathbb{Z}/4[v_1,v_2^2]/(v_1^\infty,v_2^\infty)\oplus 2v_2\mathbb{Z}/2[v_1,v_2^2]/(v_1^\infty,v_2^\infty)\\ &\quad\oplus\delta\left(M\oplus L_{-1}^3\oplus\bigoplus_{k\ge 0}\left(L_k^0\oplus L_k^1\oplus L_k^2\oplus L_k^3\right)\right),\\ E_2^s(L_2\overline{M}(2)) &= \delta\left(\left(\widetilde{C_0}\otimes\Lambda(\rho_2)\right)^{s-1}\right)\oplus\delta\delta'\left(\left(\widetilde{C_0}\otimes\Lambda(\rho_2)\right)^{s-2}\right)\quad \textit{for } s>6. \end{aligned}$$

Here $\delta(X)$ denotes the δ-image of X that is isomorphic to X.

Recall [2] the module

$$\widehat{C_0} = v_2v_3K_*[v_3^4]\otimes\Lambda(h_{20},h_{21},h_{30},h_{31})\bigoplus v_2v_3h_{20}^2K_*[v_3^4]\otimes\Lambda(h_{30},h_{31}).$$

Then we showed in [2] that this is the submodule of the Adams-Novikov E_∞-term consisting of the survivors of the summand C_0 of the E_2-term, and is also a submodule of $\pi_*(L_2\overline{M}(\infty,\infty))$. We also showed in [2] that the Adams-Novikov differential acts trivially on the other summands of the E_2-term.

COROLLARY (2.14). *The Adams-Novikov E_∞-term for the homotopy groups $\pi_*(L_2\overline{M}(2))$ contains the subgroups isomorphic to*

$$\Sigma\big(M\oplus L_{-1}^3\oplus\bigoplus_{k\ge 0}\left(L_k^0\oplus L_k^1\oplus L_k^2\oplus L_k^3\right)\big)\oplus\Sigma\widehat{C_0}\otimes\Lambda(\rho_2)\oplus\Sigma^2\widehat{C_0}\otimes\Lambda(\rho_2).$$

Here Σ denotes a shift of dimension.

3. The change of rings theorem and the relations in $\Sigma(2)$

Let BP and $BP\langle n\rangle$ denote the Brown-Peterson ring spectrum and the unlocalized Johnson-Wilson spectrum, respectively. Then, BP gives rise to the Hopf algebroid $(A,\Gamma) = (BP_*, BP_*BP) = (\mathbb{Z}_{(2)}[v_1, v_2, \dots], BP_*[t_1, t_2, \dots])$ and $BP\langle n\rangle_* = \mathbb{Z}_{(2)}[v_1, v_2, \dots, v_n] \subset BP_*$. Since $BP\langle 3\rangle$ is a BP-module spectrum, $v_2 \in BP_*$ yields the self map $v_2\colon BP\langle 3\rangle \to BP\langle 3\rangle$. Let $F(2)$ denote the spectrum $v_2^{-1}BP\langle 3\rangle$. Then, $F(2)_* = \pi_*(F(2)) = \mathbb{Z}_{(2)}[v_1, v_2, v_2^{-1}, v_3]$ and $F(2)_*(F(2)) = F(2)_* \otimes_A \Gamma \otimes_A F(2)_*$. The Hopf algebroid structure of (A,Γ) defines the one on $(B,\Sigma) = (F(2)_*, F(2)_*(F(2)))$. Consider the Hopf algebroid $(E(2)_*, E(2)_*E(2))$ associated to the localized Johnson-Wilson spectrum $E(2) = v_2^{-1}BP\langle 2\rangle$. In [1], Hovey and Sadofsky showed the change of rings theorem: $\mathrm{Ext}^s_\Gamma(A,M) = \mathrm{Ext}^s_{E(2)_*E(2)}(E(2)_*, E(2)_* \otimes_A M)$ for a v_2-local Γ-comodule M. In the same manner as the "first proof" of it, the equivalence $F(2) = \bigvee_{k\ge 0}\Sigma^{k|v_3|}E(2)$ yields the isomorphism $\mathrm{Ext}^s_\Sigma(B, B\otimes_A M) = \mathrm{Ext}^s_{E(2)_*E(2)}(E(2)_*, E(2)_*\otimes_A M)$, and then, we have an isomorphism

$$\mathrm{Ext}^s_\Gamma(A,M) = \mathrm{Ext}^s_\Sigma(B, B\otimes_A M)$$

for a v_2-local Γ-comodule M. Indeed, $F(2)$- and $E(2)$-Adams resolutions of a spectrum X induce the same spectral sequence, and so the E_2-terms agree.

Consider $\Gamma(2) = \Gamma/(t_1) = A[t_2, t_3, \dots]$. Then the pair $(A, \Gamma(2))$ is a Hopf algebroid induced from (A,Γ) under the projection $\Gamma \to \Gamma(2)$. Since $BP_*(T(1)) = A[t_1]$, we have $F(2)_*(T(1)) = B[t_1]$, which is expressed as a cotensor product

$$B[t_1] = B\square_{\Sigma(2)}\Sigma \tag{3.1}$$

for $\Sigma(2) = B\otimes_A \Gamma(2)\otimes_A B$. Here $(B,\Sigma(2))$ is a Hopf algebroid induced from $(A,\Gamma(2))$. Write H^*M as $\mathrm{Ext}^*_{\Sigma(2)}(B,M)$ for a $\Sigma(2)$-comodule M, and $H^*(M\otimes_A B)$ for a Γ-comodule M is isomorphic to

$$\begin{aligned}\mathrm{Ext}^s_\Gamma(A, v_2^{-1}M\otimes_A A[t_1]) = \mathrm{Ext}^s_\Sigma(B, M\otimes_A B[t_1]) = \mathrm{Ext}^s_{\Sigma(2)}(B, M\otimes_A B)\\ (= H^*(M\otimes_A B)),\end{aligned}$$

where the second equality follows from (3.1) by the change of rings theorem [6], A1.3.13. In this paper, we employ the same method introduced in [4] to compute $H^*B/(4)$, which is isomorphic to the E_2-term $\mathrm{Ext}^s_\Gamma(A, v_2^{-1}A[t_1]/(4))$. For this sake, we introduce the $\Sigma(2)$-comodules $M^0_2(2)$ and $M^1_1(2)$:

Definition (3.2). $M^0_2(2) = B/(4, v_1)$ and $M^1_1(2)$ is the cokernel of the inclusion $B/(4) \to v_1^{-1}B/(4)$.

Note that the modules in the exact sequence (1.3) are:

$$E_2^*(L_2\overline{M}(2,1)) = H^*M^0_2(2) \quad\text{and}\quad E_2^*(L_2\overline{M}(2,\infty)) = H^*M^1_1(2). \tag{3.3}$$

We study the module $H^*B/(4)$ by the long exact sequence

$$H^{s-1}v_1^{-1}B/(4) \longrightarrow H^{s-1}M^1_1(2) \xrightarrow{\delta} H^sB/(4) \longrightarrow H^sv_1^{-1}B/(4) \longrightarrow H^sM^1_1(2)$$

associated to the short exact sequence $0 \to B/(4) \to v_1^{-1}B/(4) \to M^1_1(2) \to 0$. We will determine $H^*v_1^{-1}B/(4)$ in Proposition (4.1). We compute $H^*M^1_1(2)$ by the v_1-Bockstein spectral sequence associated to the short exact sequence $0 \to M^0_2(2) \xrightarrow{1/v_1} M^1_1(2) \xrightarrow{v_1} M^1_1(2) \to 0$ of $\Sigma(2)$-comodules. For computing the Bockstein spectral sequence, we set up formulas on the structure map $\eta_R\colon B \to \Sigma(2)$ of the Hopf algebroid $\Sigma(2)$ and some relations in $\Sigma(2)$.

We begin with the behavior of the right unit $\eta_R\colon A \to \Gamma(2) = A[t_2, t_3, \dots]$. Here, v_i's are the Hazewinkel generators of A defined by

$$v_i = 2m_i - \sum_{k=1}^{i-1} m_k v_{i-k}^{2^k} \in 2^{-1}A = \mathbb{Q}[m_1, m_2, \dots] \tag{3.4}$$

under the inclusion $A \to 2^{-1}A$. The unit map $i\colon S^0 \to BP$ induces the right unit $\eta_R = (i \wedge 1)_*\colon A = \pi_*(BP) \to \Gamma \to \Gamma(2)$, and then its localization $\eta_R\colon 2^{-1}A \to 2^{-1}\Gamma(2)$, whose action is given by the Quillen formulas

$$\eta_R(m_i) = \sum_{k=0}^{i} m_k t_{i-k}^{2^k}. \tag{3.5}$$

With a routine computation with the formulas (3.4) and (3.5), we see that

LEMMA (3.6). *On the generators v_i for $0 < i \le 6$, $\eta_R\colon A \to \Gamma(2)$ acts as follows:*

$$\begin{aligned}
&\eta_R(v_1) = v_1,\\
&\eta_R(v_2) = v_2 + 2t_2,\\
&\eta_R(v_3) = v_3 + v_1t_2^2 - v_1^4t_2 + 2(t_3 - v_1v_2t_2 - v_1t_2^2),\\
&\eta_R(v_4) \equiv v_4 + v_2t_2^4 + v_2^4t_2 + v_1t_3^2 + v_1^2v_3(t_2^2 + v_1^3t_2) + v_1^6t_2^3 + v_1^8t_3 + v_1^9t_2^2 \bmod (2),\\
&\eta_R(v_5) \equiv v_5 + v_3t_2^8 + v_3^4t_2 + v_2t_3^4 + v_2^8t_3 + v_1t_4^2 + v_1c(4)\\
&\qquad\qquad + v_1v_2^8t_2^2 + v_1^2v_2v_3^2t_2^4 \bmod (2, v_1^3) \quad \textit{and}\\
&\eta_R(v_6) \equiv v_6 + v_4t_2^{16} + t_2\eta_R(v_4)^4 + v_3t_3^8 + v_3^8t_3 + v_2t_4^4\\
&\qquad\qquad + v_2^{16}t_4 + v_2c(4)^2 + v_2^{17}t_2^4 \bmod (2, v_1).
\end{aligned}$$

Here

$$2c(4) \equiv \eta_R(v_4^2) - (v_4^2 + v_2^2t_2^8 + v_2^8t_2^2 + v_1^2t_3^4) \quad \bmod (4, v_1^4).$$

Since $\eta_R(v_i) = 0$ in $\Sigma(2)$ if $i \ge 4$, we obtain relations in $\Sigma(2)$ from Lemma (3.6):
(3.7)

$$\begin{aligned}
&v_2t_2^4 \equiv v_2^4t_2 + v_1t_3^2 + v_1^2v_3t_2^2 + v_1^5v_3t_2 + v_1^6t_2^3 + v_1^8t_3 + v_1^9t_2^2 \bmod (2),\\
&v_2t_3^4 = v_2^8t_3 + v_3t_2^8 + v_3^4t_2 + v_1t_4^2 + v_1c(4) + v_1v_2^8t_2^2 + v_1^2v_2v_3^2t_2^4 \bmod (2, v_1^3) \quad \text{and}\\
&v_2t_4^4 = v_2^{16}t_4 + v_3t_3^8 + v_3^8t_3 + v_2c(4)^2 + v_2^{17}t_2^4 \bmod (2, v_1).
\end{aligned}$$

By the first relation of (3.7), we see that $c(4) \equiv v_2^5t_2^5 + v_1t_3^2(v_2t_2^4 + v_2^4t_2) \equiv v_2^5t_2^5$ mod $(2, v_1^2)$, and so $v_2^{-10}c(4) + v_2^{-2}t_2^2 \equiv v_2^{-5}t_2^5 + v_2^{-2}t_2^2 \equiv v_1v_2^{-6}t_2t_3^2$ mod $(2, v_1^2)$.

Then
(3.8)
$$\begin{aligned}
v_2t_3^4 &\equiv v_2^8t_3 + v_3t_2^8 + v_3^4t_2 + v_1t_4^2 + v_1^2(v_2^4t_2t_3^2 + v_2v_3^2t_2^4) \bmod (2, v_1^3)\\
&\equiv v_2^8t_3 + v_2^6v_3t_2^2 + v_3^4t_2 + v_1t_4^2 + v_1^2(v_2^{-2}v_3t_3^4 + v_2^4t_2t_3^2 + v_2v_3^2t_2^4) \bmod (2, v_1^3)\\
&\equiv v_2^8t_3 + v_2^6v_3t_2^2 + v_3^4t_2 + v_1t_4^2\\
&\qquad + v_1^2\left(v_3(v_2^5t_3 + v_2^3v_3t_2^2 + v_2^{-3}v_3^4t_2) + v_2^4t_2t_3^2 + v_2^4v_3^2t_2\right) \bmod (2, v_1^3) \text{ and}\\
v_2t_4^4 &\equiv v_2^{16}t_4 + v_3t_3^8 + v_3^8t_3 \bmod (2, v_1)\\
&\equiv v_2^{16}t_4 + v_2^{14}v_3t_3^2 + v_3^8t_3 + v_2^{13}v_3^3t_2 + v_2^{-2}v_3^9t_2^2 \bmod (2, v_1).
\end{aligned}$$

4. The homotopy groups $\pi_*(v_1^{-1}T(1)\wedge M(2))$ and $\pi_*(L_2\overline{M}(2,1))$

Note that $T(1)\wedge M(2) = \overline{M}(2)$. The Adams-Novikov E_2-term $E_2^*(v_1^{-1}\overline{M}(2))$ is isomorphic to $\mathrm{Ext}^*_{\Gamma(2)}(BP_*,\ v_1^{-1}BP_*/(4))$ by the change of rings theorem [6, A1.3.13], since $v_1^{-1}BP_*[t_1]/(4) = v_1^{-1}BP_*/(4)\square_{\Gamma(2)}\Gamma$.

PROPOSITION (4.1). *The Adams-Novikov E_2-term for $\pi_*(v_1^{-1}\overline{M}(2))$ is isomorphic to*
$$E_2^*(v_1^{-1}\overline{M}(2)) = \mathbb{Z}/4[v_1^{\pm1}, v_2^2]\otimes\Lambda(v_2h_{20}) \oplus 2v_2\mathbb{Z}/2[v_1^{\pm1}, v_2^2] \oplus h_{20}\mathbb{Z}/2[v_1^{\pm1}, v_2^2].$$
Furthermore, this is isomorphic to the homotopy groups $\pi_*(v_1^{-1}\overline{M}(2))$.

Proof. In [6, 6.5.6], it is shown that $E_2^*(v_1^{-1}\overline{M}(1)) = K(1)_*[v_2]\otimes\Lambda(h_{20})$ for $K(1)_* = \mathbb{Z}/2[v_1, v_1^{-1}]$. Consider the long exact sequence
$$E_2^*(v_1^{-1}\overline{M}(1)) \xrightarrow{2} E_2^*(v_1^{-1}\overline{M}(2)) \longrightarrow E_2^*(v_1^{-1}\overline{M}(1)) \xrightarrow{\delta} E_2^{*+1}(v_1^{-1}\overline{M}(1))$$
of the Adams-Novikov E_2-terms associated to the cofiber sequence (2.2). We make a computation in the cobar complex $\Omega^*_{\Gamma(2)}v_1^{-1}BP_*/(4)$ (*cf* [6, A1.2.11]), and see that the action of the connecting homomorphism δ is obtained from the only relation $\delta(v_2) = h_{20}$, which is verified by the equality $\eta_R(v_2) = v_2 + 2t_2$ in $\Gamma(2)$ in Lemma (3.6). Now the E_2-term is obtained from the above exact sequence.

Since the E_2-term has the horizontal vanishing line $s = 2$, the spectral sequence collapses from the E_2-term, and the extension is trivial. □

We turn to the homotopy groups $\pi_*(L_2\overline{M}(2,1))$. The cofiber sequence (2.1) induces the long exact sequence
$$E_2^*(L_2\overline{M}(1,1)) \xrightarrow{2} E_2^*(L_2\overline{M}(2,1)) \longrightarrow E_2^*(L_2\overline{M}(1,1)) \xrightarrow{\delta} E_2^{*+1}(L_2\overline{M}(1,1)).$$

LEMMA (4.2). *The connecting homomorphism δ acts as follows:*
$$\begin{aligned}
\delta(v_2) &= h_{20},\\
\delta(v_3) &= h_{30},\\
\delta(h_{21}) &= h_{20}^2,\\
\textit{and}\qquad \delta(h_{31}) &= h_{30}^2 = (h_{21} + v_2h_{20})h_{31} + v_2v_3^2h_{20}h_{21}.
\end{aligned}$$
Here, we set $v_2^2 = 1$.

Proof. This follows immediately from Lemma (3.6). The last equality is given in [3]. □

We consider the elements

$$\overline{h}_{21} = h_{21} + v_2 h_{20} \quad \text{and} \quad \overline{h}_{31} = h_{31} + v_3 h_{30}.$$

Then Lemma (4.2) implies the following:

LEMMA (4.3). *The connecting homomorphism δ acts trivially on the elements*

$$h_{20}, \quad \overline{h}_{21}, \quad h_{30} \quad \textit{and} \quad \overline{h}_{31}.$$

Proof of Theorem (2.5). The connecting homomorphism δ acts as $\delta(v_2C) = h_{20}C$, $\delta(F) = 0$, $\delta(v_3F) = h_{30}F$ and $\delta(v_3h_{30}F^0) = v_3h_{30}\overline{h}_{21}F^0$. The last correspondence follows from $\delta(v_3h_{30}) = \overline{h}_{21}(\overline{h}_{31} + v_2v_3^2h_{20} + v_3h_{30}) = v_3h_{30}\overline{h}_{21} + \cdots$. Here, the correspondence is written as the leading term. □

Proof of Theorem (2.6). The Adams-Novikov differential on $E_r^*(L_2\overline{M}(1,1))$ is shown in [3] to act as follows:

$$d_3(v_3^{4s+t}x) = \begin{cases} 0 & t = 0, 1, \\ v_2^2v_3^{4s}xh_{20}^3 & t = 2, \\ v_2^2v_3^{4s+1}xh_{20}^3 & t = 3 \end{cases}$$

for $x \in K(2)_*[h_{20}] \otimes \Lambda(\overline{h}_{21}, h_{30}, h_{31}, \rho_2)$. □

5. The elements x_n, g_n, R_n and X_n, and relations between them

In order to define generators of the cohomology $H^*M_1^1(2)$ of the comodule $M_1^1(2)$ in Definition (3.2), we introduce some elements in this section.

First we redefine the elements x_n, which are used to give generators of $H^*M_1^1 = E_2^*(L_2\overline{M}(1,\infty))$ in [7], and then observe the behavior of them under the differential $d = \eta_R - \eta_L \colon B \to \Sigma(2)$ of the cobar complex $\Omega^*_{\Sigma(2)}B$.

LEMMA (5.1). *Put $x = v_3^4 + v_1^3v_2^6v_3 \in B$. Then $d(x) \equiv v_1^6v_2^{-2}g$ mod (2) in $\Sigma(2)$. Here*

$$g = t_3^4 + v_1T_2 + v_1^8v_3^2t_2^2 + v_1^{10}(t_2^6 + v_2^2t_2^4) + v_1^{14}t_3^2 + v_1^{16}t_2^4 \in \Sigma(2)$$

for $T_2 = v_2^8t_2 + v_1v_3^2t_2^4$.

Proof. This follows from the computation

$$\begin{aligned} d(v_3^4) &\equiv v_1^4t_2^8 + v_1^{16}t_2^4 \bmod (2) \\ &\equiv v_1^4v_2^{-2}(v_2^8t_2^2 + v_1^2t_3^4 + v_1^4v_3^2(t_2^4 + v_1^6t_2^2) + v_1^{12}t_2^6 + v_1^{16}t_3^2 + v_1^{18}t_2^4) \\ &\qquad + v_1^{16}t_2^4 \bmod (2) \quad \text{by (3.7), and} \\ d(v_1^3v_2^6v_3) &\equiv v_1^3v_2^6(v_1t_2^2 + v_1^4t_2) \bmod (2). \end{aligned}$$

□

Hereafter, we put $v_2^2 = 1$ for the sake of simplicity. In fact, we consider $\mathbb{Z}/4$-module structure, and v_2^2 is invariant mod (4) by Lemma (3.6). Therefore, v_2^2 plays only a role adjusting the internal degrees, since every congruence is homogeneous.

LEMMA (5.2). *There is an element y of B such that*

$$d(y) \equiv v_1^{10}g^4 + v_1^{12}xg + v_1^{14}r^4 + v_1^{16}r' \bmod (2, v_1^{17}) \in \Sigma(2)$$

for $r = t_4^2 + t_4$ and $r' = v_3^2t_3^8 + v_3^8t_3^2$.

Proof. Put $y' = x^2 + v_1^9v_2v_3^2 + v_1^{11}v_3^9 + v_1^{13}v_3 + v_1^{13}v_2v_3^{10}$, and we have $d(y') \equiv v_1^{11}T_2 + v_1^{13}v_3t_2^2 + v_1^{14}t_4^4 + v_1^{16}(r' + t_2^2t_3^4 + v_3t_2) \bmod (2, v_1^{17})$. Indeed, it is the sum of the following congruences mod $(2, v_1^{17})$:

$$\begin{aligned}
d(x^2) &\equiv v_1^{12}(t_3^8 + v_1^2t_2^2 + v_1^4v_3^4t_2^8)\\
&\equiv v_1^{12}(t_3^2 + v_3^2t_2^4 + v_3^8t_2^2 + v_1^2t_4^4 + v_1^4v_3^2t_3^8 + v_1^4t_2^2t_3^4 + v_1^2t_2^2),\\
d(v_1^9v_2v_3^2) &\equiv v_1^{11}v_2t_2^4 \equiv v_1^{11}(t_2 + v_1t_3^2 + v_1^2v_3t_2^2 + v_1^5v_3t_2),\\
d(v_1^{11}v_3^9) &\equiv v_1^{12}v_3^8(t_2^2 + v_1^3t_2) \equiv v_1^{12}v_3^8(t_2^2 + v_1^3(v_2t_2^4 + v_1t_3^2)),\\
d(v_1^{13}v_3) &\equiv v_1^{14}t_2^2 \quad \text{and}\\
d(v_1^{13}v_2v_3^{10}) &\equiv v_1^{15}v_2v_3^8t_2^4,
\end{aligned}$$

in which we use relations in (3.7).

We now put $y = y' + v_1^4x + v_1^8v_3^{18} + v_1^9v_3^5 + v_1^{12}v_3^{20}$, and compute mod $(2, v_1^{17})$,

$$\begin{aligned}
d(y') &\equiv v_1^{11}T_2 + v_1^{13}v_3t_2^2 + v_1^{14}t_4^4 + v_1^{16}(r' + t_2^2t_3^4 + v_3t_2),\\
d(v_1^4x) &\equiv v_1^{10}(t_3^4 + v_1T_2) \equiv v_1^{10}(t_3^{16} + v_3^4t_2^8 + v_3^{16}t_2^4 + v_1^4t_4^8 + v_1T_2)\\
&\equiv v_1^{10}(t_3^{16} + v_3^4(t_2^21 + v_1^2t_3^4 + v_1^4v_3^2t_2^4) + v_3^{16}t_2^4 + v_1^4t_4^8 + v_1T_2),\\
d(v_1^8v_3^{18}) &\equiv v_1^{10}v_3^{16}t_2^4 + v_1^{16}v_3^{16}t_2^2,\\
d(v_1^9v_3^5) &\equiv v_1^{10}v_3^4t_2^2 + v_1^{13}v_3^4t_2 + v_1^{13}v_3t_2^8 + v_1^{14}t_2^{10} \quad \text{and}\\
d(v_1^{12}v_3^{20}) &\equiv v_1^{16}v_3^{16}t_2^8 \;\equiv\; v_1^{16}v_3^{16}t_2^2
\end{aligned}$$

to obtain

$$\begin{aligned}
d(y) &\equiv v_1^{10}(t_3^{16} + v_1^4t_2^4) + v_1^{12}v_3^4(t_3^4 + v_1t_2 + v_1^2v_3^2t_2^4) + v_1^{14}(t_4^8 + t_4^4)\\
&\qquad + v_1^{15}v_3(t_3^4 + v_1t_2) + v_1^{16}r'\\
&\equiv v_1^{10}g^4 + v_1^{12}xg + v_1^{14}r^4 + v_1^{16}r'.
\end{aligned}$$

Here we use relations $v_1^{13}v_3t_2^2 + v_1^{13}v_3t_2^8 = v_1^{15}v_3t_3^4$ and $v_1^{16}t_2^2t_3^4 + v_1^{14}t_2^{10} = v_1^{14}t_2^4$, and notice that $x = v_3^4 + v_1^3v_3$ and $g^4 \equiv t_3^{16} + v_1^4t_2^4 \bmod (2, v_1^8)$. □

We now define elements $x_k \in B$ for $k \geq 0$ inductively by

(5.3) $\qquad x_0 = v_3, \quad x_1 = x = v_3^4 + v_1^3v_3 \quad \text{and} \quad x_{k+1} = x_k^4 + v_1^{a_{k+1}-12}x^{4e_{k-1}}y$

for the integers a_n and e_n in (2.8), and consider elements $g_k \in \Sigma(2)$ satisfying

(5.4) $\qquad g_1 = g \quad \text{and} \quad d(x^{4e_{k-1}}y) \equiv v_1^{10}g_k^4 + v_1^{12}g_{k+1} \bmod (2) \quad \text{for } k > 0.$

Note that g_k is a well-defined element if we consider it modulo (2).

LEMMA (5.5). *For $k > 0$,*

$$d(x_k) \equiv v_1^{a_k}g_k \bmod (2).$$

Proof. This is verified inductively by definition. Indeed, $d(x_{k+1}) \equiv d(x_k^4) + v_1^{a_{k+1}-12} d(x^{4e_{k-1}}y) \equiv v_1^{4a_k} g_k^4 + v_1^{a_{k+1}-12}(v_1^{10} g_k^4 + v_1^{12} g_{k+1}) \bmod (2)$, and $a_{k+1} = 4a_k + 2$. □

LEMMA (5.6). *For $k > 1$,*

$$g_k \equiv x^{e_{k-1}} g + v_1^2 x^{4e_{k-2}} r^4 \bmod (2, v_1^3).$$

Here $r = t_4^2 + t_4$ as above.

Proof. For $k = 2$, it follows from Lemma (5.2) and (5.4). Suppose that the lemma holds for $k > 1$. Then, Lemma (5.2) and the definition (5.4) show that

$$\begin{aligned} v_1^{12} g_{k+1} &\equiv d(x^{4e_{k-1}} y) + v_1^{10} g_k^4 \equiv x^{4e_{k-1}}(v_1^{10} g^4 + v_1^{12} xg + v_1^{14} r^4) + v_1^{10} x^{4e_{k-1}} g^4 \\ &\equiv v_1^{12} x^{e_k} g + v_1^{14} x^{4e_{k-1}} r^4 \bmod (2, v_1^{15}). \end{aligned}$$

Since v_1 acts monomorphically on the cobar complex, we obtain the lemma. □

We introduce an element $R_n \in \Sigma(2)$ satisfying

$$v_1^{e_n+1} R_n \equiv g_{n+1} + x_n g_n \quad \bmod (2) \tag{5.7}$$

for $n > 0$. Note also that R_n is well-defined modulo (2).

LEMMA (5.8). $R_1 \equiv r^4 + v_1^2 r' \bmod (2, v_1^3)$ *and* $R_2 \equiv R_1^4 \bmod (2, v_1^6)$*. For $k > 1$, there is a cochain w_k such that*

$$d(w_k) \equiv v_1^{e_{k+1}+13}(R_k^4 + R_{k+1}) \bmod (2, v_1^{a_{k+1}-12}).$$

Proof. The congruences on R_1 and R_2 follow from (5.7), (5.3) and (5.4). Indeed, $v_1^{14} R_1 \equiv v_1^{12} g_2 + v_1^{12} x_1 g_1 \equiv d(y) + v_1^{10} g^4 + v_1^{12} xg \bmod (2)$, which is congruent to $v_1^{14} r^4 + v_1^{16} r' \bmod (2, v_1^{17})$ by Lemma (5.2). Thus, the first congruence follows. For the second congruence, we compute

$$\begin{aligned} d(x^4 y) &\equiv v_1^{10} g_2^4 + v_1^{12} g_3 \bmod (2) \\ &\equiv v_1^{10}(x_1^4 g_1^4 + v_1^8 R_1^4) + v_1^{12}(x_2 g_2 + v_1^6 R_2) \bmod (2) \\ &\equiv x_1^4(v_1^{10} g_1^4 + v_1^{12} g_2) + v_1^{18}(R_1^4 + R_2) \bmod (2, v_1^{26}) \\ &\equiv x_1^4 d(y) + v_1^{18}(R_1^4 + R_2) \bmod (2, v_1^{24}), \end{aligned}$$

and obtain $v_1^{18}(R_1^4 + R_2) \equiv 0 \bmod (2, v_1^{24})$, since $d(x^4 y) \equiv x_1^4 d(y) \bmod (2, v_1^{24})$.

By (5.4) and Lemma (5.5), we see that

$$\begin{aligned} d(x^{4e_k} y) &\equiv v_1^{10} g_{k+1}^4 + v_1^{12} g_{k+2} \bmod (2), \\ d(x_k^4 x^{4e_{k-1}} y) &\equiv v_1^{10} x_k^4 g_k^4 + v_1^{12} x_k^4 g_{k+1} \bmod (2, v_1^{4a_k}). \end{aligned}$$

Put $w_k = x^{4e_k} y + x_k^4 x^{4e_{k-1}} y$. Then,

$$\begin{aligned} d(w_k) &\equiv v_1^{10}(g_{k+1}^4 + x_k^4 g_k^4) + v_1^{12}(g_{k+2} + x_k^4 g_{k+1}) \bmod (2, v_1^{4a_k}) \\ &\equiv v_1^{10+4e_k+4} R_k^4 + v_1^{12+e_{k+1}+1} R_{k+1} \bmod (2, v_1^{4a_k}) \end{aligned}$$

by (5.7). The last congruence now follows from the relations $10 + 4e_k + 4 = 12 + e_{k+1} + 1$ and $4a_k = a_{k+1} - 2$. □

We have homologous relations between R_n's:

LEMMA (5.9). *There are elements u_n and u'_n for $n>0$ such that*

$$d(u_1)\equiv v_1^{10}(R_1^2+R_1+x_1g_1^2) \bmod (2,v_1^{12}),\quad \textit{and}$$

$$d(u_n)\equiv v_1^{10\times 4^{n-1}}(R_n^2+R_n+v_1^{4e_{n-1}}x_ng_n^2+v_1^{a_{n-1}}x_ng_n) \bmod (2,v_1^{3\times 4^n})$$

for $n>1$; and

$$d(u'_n)\equiv v_1^{5\times 4^n}(R_{n+1}+R_n^2+v_1^{2e_n}x_n^2g_{n+1}+v_1^{2a_{n-1}}x_n^2g_n^2) \bmod (2,v_1^{6\times 4^n})$$

for $n>0$.

Proof. Put $u_1=v_1^4v_3^{32}x+v_3^{32}y'+v_1^8v_3^{14}+v_1^{11}v_3^{11}+v_1^9v_3^{37}$. Then we compute

$$\begin{aligned}
d(v_1^4v_3^{32}x)&\equiv v_1^{10}v_3^{32}(t_3^4+v_1T_2)\quad \bmod (2,v_1^{18}),\\
d(v_3^{32}y')&\equiv v_3^{32}(v_1^{11}T_2+v_1^{13}v_3t_2^2+v_1^{14}t_4^4+v_1^{16}(r'+t_2^2t_3^4+v_3t_2)) \bmod (2,v_1^{17}),\\
d(v_1^8v_3^{14})&\equiv v_1^8v_3^8(v_3^4+v_1^4t_2^8)(v_3^2+v_1^2t_2^4)-v_1^8v_3^{14} \bmod (2,v_1^{14})\\
&\equiv v_1^{10}v_3^{12}t_2^4+v_1^{12}v_3^{10}t_2^8\quad \bmod (2,v_1^{14})\\
&\equiv v_1^{10}v_3^{12}t_2^4+v_1^{12}v_3^{10}t_2^2\quad \bmod (2,v_1^{14}) \text{ by (3.7)},\\
d(v_1^{11}v_3^{11})&\equiv v_1^{11}v_3^8(v_3^2+v_1^2t_2^4)(v_3+v_1t_2^2)-v_1^{11}v_3^{11} \bmod (2,v_1^{14})\\
&\equiv v_1^{12}v_3^{10}t_2^2+v_1^{13}v_3^9t_2^4 \bmod (2,v_1^{14})\quad \text{and}\\
d(v_1^9v_3^{37})&\equiv v_1^9v_3^{32}(v_3^4+v_1^4t_2^8)(v_3+v_1t_2^2+v_1^4t_2)-v_1^9v_3^{37}\quad \bmod (2,v_1^{14})\\
&\equiv v_1^{10}v_3^{36}t_2^2+v_1^{13}(v_3^{36}t_2+v_3^{33}t_2^8) \bmod (2,v_1^{14})\\
&\equiv v_1^{10}v_3^{36}(t_2^8+v_1^2t_3^4)+v_1^{13}(v_3^{36}t_2+v_3^{33}t_2^8) \bmod (2,v_1^{14}) \text{ by (3.7)}
\end{aligned}$$

and obtain

$$d(u_1)\equiv v_1^{10}(v_3^{32}t_3^4+v_3^{12}t_2^4+v_3^{36}t_2^8)+v_1^{12}v_3^{36}(t_3^4+v_1t_2)+v_1^{13}v_2v_3^9t_2 \bmod (2,v_1^{14}).$$

Here, we have the relation $0\equiv r^8+r^4+v_3^4t_3^8+v_3^{32}t_3^4+v_3^{12}t_2^4+v_3^{36}t_2^8 \bmod (2,v_1^4)$. Indeed, we compute

$$\begin{aligned}
r^8+r^4&=t_4^{16}+t_4^8+t_4^8+t_4^4\;=\;t_4^{16}+t_4^4\\
&=t_4^4+v_3^4t_3^8+v_3^{32}t_3^4+v_3^{12}t_2^4+v_3^{36}t_2^8+t_4^4 \bmod (2,v_1^4)
\end{aligned}$$

by the relation in (3.8). It follows that $d(u_1)\equiv v_1^{10}(r^8+r^4+v_3^4t_3^8) \bmod (2,v_1^{12})$. Notice that $R_1\equiv r^4 \bmod (2,v_1^2)$ by Lemma (5.8), $v_3^4\equiv x_1 \bmod (2,v_1^3)$ by (5.3) and $g_1\equiv t_3^4 \bmod (2,v_1)$ by (5.4) and Lemma (5.1), and we obtain

$$d(u_1)\equiv v_1^{10}(R_1^2+R_1+x_1g_1^2) \bmod (2,v_1^{12}).$$

We now turn to the case for $n=2$. Square the above congruence, and we have

$$d(u_1^2)\equiv v_1^{20}(R_1^4+R_1^2+x_1^2g_1^4) \bmod (2,v_1^{24}).$$

By Lemmas (5.1) and (5.2),

$$d(v_1^{10}x_1^2y)\equiv v_1^{20}x_1^2(g_1^4+v_1^2g_2)+v_1^{22}yg_1^2 \bmod (2,v_1^{24}),$$

where $x=x_1$ and $g=g_1$ by (5.3) and (5.4). Put $u''_1=u_1^2+v_1^{10}x_1^2y$. Since $y\equiv x_1^2 \bmod (2,v_1^2)$ by the definition in Lemma (5.2), we obtain

$$d(u''_1)\equiv v_1^{20}(R_1^4+R_1^2+v_1^2x_1^2g_2+v_1^2x_1^2g_1^2) \bmod (2,v_1^{24}),$$

whence $u_1' = v_1^2 w_1 + u_1''$ satisfies the lemma for $n = 1$ by Lemma (5.8). Suppose that

$$d(u_n') \equiv v_1^{5\times 4^n}(R_{n+1} + R_n^2 + v_1^{2e_n} x_n^2 g_{n+1} + v_1^{2a_{n-1}} x_n^2 g_n^2) \mod (2, v_1^{6\times 4^n}).$$

Squaring this,

$$d({u_n'}^2) \equiv v_1^{10\times 4^n}(R_{n+1}^2 + R_n^4 + v_1^{4e_n} x_n^4 g_{n+1}^2 + v_1^{4a_{n-1}} x_n^4 g_n^4) \mod (2, v_1^{3\times 4^{n+1}}).$$

The elements x_n^4, g_n^4 and R_n^4 are homologous to x_{n+1}, $v_1^2 g_{n+1}$ and R_{n+1} by (5.3), (5.4) and Lemma (5.8), respectively. We define u_{n+1} to be the sum of ${u_n'}^2$ and the elements that give these homologous relations, and we obtain the congruence on $d(u_{n+1})$. Here we notice that $a_n = 4a_{n-1} + 2$.

Squaring the congruence on $d(u_{n+1})$, we have

$$d(u_{n+1}^2) \equiv v_1^{5\times 4^{n+1}}(R_{n+1}^4 + R_{n+1}^2 + v_1^{8e_n} x_{n+1}^2 g_{n+1}^4 + v_1^{2a_n} x_{n+1}^2 g_{n+1}^2) \mod (2, v_1^{6\times 4^{n+1}}).$$

We also have

$$d(v_1^{5\times 4^{n+1}+8e_n-10} x_{n+1}^2 x^{4e_n} y) \equiv v_1^{5\times 4^{n+1}+8e_n} x_{n+1}^2 (g_{n+1}^4 + v_1^2 g_{n+2}) \mod (2, v_1^{6\times 4^{n+1}}).$$

Since $8e_n + 2 = 2e_{n+1}$, we put

$$u_{n+1}' = u_{n+1}^2 + v_1^{5\times 4^{n+1}-e_{n+2}-13} w_{n+1} + v_1^{5\times 4^{n+1}+8e_n-10} x_{n+1}^2 x^{4e_n} y,$$

and obtain the congruence on $d(u_{n+1}')$ by Lemma (5.8). This completes the induction. □

We also consider the elements $x_{n,1} = 2x_n + v_1^{5\times 4^{n-1}} x_{n-1}^2$ and $x_{n,1}' = 2x_n^2 + v_1^{10\times 4^{n-1}} x_{n-1}^4$.

LEMMA (5.10). *For $n > 1$,*

$$d(x_{n,1}) \equiv 2v_1^{a_n+e_{n-1}+1} R_{n-1}\ (=\ 2v_1^{7\times 4^{n-1}} R_{n-1})$$
$$\mod (4, v_1^{2\times 4^n + e_{n-1}-1}) = (4, v_1^{7\times 4^{n-1}+e_n-1}),$$
$$d(x_{n,1}') \equiv 2v_1^{2a_n+2e_{n-1}+2} R_{n-1}^2\ (=\ 2v_1^{14\times 4^{n-1}} R_{n-1}^2)$$
$$\mod (4, v_1^{4^{n+1}+2e_{n-1}-2}) = (4, v_1^{14\times 4^{n-1}+2e_n-2}).$$

Proof. These follow from the computation:

$$d(2x_n) \equiv 2v_1^{a_n} g_n \equiv 2v_1^{a_n} x_{n-1} g_{n-1} + 2v_1^{a_n+e_{n-1}+1} R_{n-1} \mod (4),$$
$$d(v_1^{5\times 4^{n-1}} x_{n-1}^2) \equiv 2v_1^{5\times 4^{n-1}+a_{n-1}} x_{n-1} g_{n-1}$$
$$\mod (4, v_1^{5\times 4^{n-1}+2a_{n-1}}) = (4, v_1^{2\times 4^n+e_{n-1}-1});$$
$$d(2x_n^2) \equiv 2v_1^{2a_n} g_n^2 \equiv 2v_1^{2a_n} x_{n-1}^2 g_{n-1}^2 + 2v_1^{2a_n+2e_{n-1}+2} R_{n-1}^2 \mod (4),$$
$$d(v_1^{10\times 4^{n-1}} x_{n-1}^4) \equiv 2v_1^{10\times 4^{n-1}+2a_{n-1}} x_{n-1}^2 g_{n-1}^2$$
$$\mod (4, v_1^{10\times 4^{n-1}+4a_{n-1}}) = (4, v_1^{4^{n+1}+2e_{n-1}-2}).$$

□

Note that $d(x_n^2) \equiv 2v_1^{a_n} x_n g_n \mod (4, v_1^{2a_n})$ and $e_n + 1 + 2\times 4^{n-1} = 2a_{n-1} + 2$. Then the above two lemmas imply the following:

LEMMA (5.11). *Put* $\overline{u}_n = 2v_1^{a_{n+1}+e_n+1-10\times 4^{n-1}}u_n + x_{n+1,1} + v_1^{e_{n+2}+a_{n-1}+1}x_n^2$ *and* $\overline{u}'_n = 2v_1^{2a_{n+1}+2e_n+2-5\times 4^n}u'_n + x'_{n+1,1} + v_1^{2e_{n+2}+2a_{n-1}+2}x_n^4$. *Then,*

$$\begin{aligned} d(\overline{u}_n) &\equiv 2v_1^{a_{n+1}+e_n+1}(R_n^2 + v_1^{4e_{n-1}}x_ng_n^2) \bmod (4, v_1^{a_{n+1}+2a_{n-1}+2}) \quad \text{and} \\ d(\overline{u}'_n) &\equiv 2v_1^{2a_{n+1}+2e_n+2}(R_{n+1} + v_1^{2e_n}x_n^2g_{n+1}) \bmod (4, v_1^{2a_{n+1}+a_n+2}) \end{aligned}$$

for $n > 0$.

LEMMA (5.12). *There exist elements* $\widetilde{u}_n$ *and* $\widetilde{u}'_n$ *such that*

$$\begin{aligned} d(\widetilde{u}_n) &\equiv 2v_1^{a_{n+1}+e_n+1}(R_n^2 + v_1^{6\times 4^{n-2}}x_{n-1}^6R_{n-2}^2) \bmod (4, v_1^{a_{n+1}+2a_{n-1}+2}) \quad \text{for } n > 2, \\ d(\widetilde{u}'_n) &\equiv 2v_1^{2a_{n+1}+2e_n+2}(R_{n+1} + v_1^{3\times 4^{n-1}}x_n^3R_{n-1}) \bmod (4, v_1^{2a_{n+1}+a_n+2}) \quad \text{for } n \geq 2. \end{aligned}$$

Proof. For $n > 2$, put $\widetilde{u}_n = \overline{u}_n + v_1^{a_{n+1}+2e_n-2a_{n-2}}x_{n-1}^6x_{n-2}^4$, and the first one follows from

$$\begin{aligned} d(\overline{u}_n) &\equiv 2v_1^{a_{n+1}+e_n+1}(R_n^2 + v_1^{4e_{n-1}}x_ng_n^2) \\ &\equiv 2v_1^{a_{n+1}+e_n+1}(R_n^2 + v_1^{4e_{n-1}}x_{n-1}^6(x_{n-2}^2g_{n-2}^2 + v_1^{2e_{n-2}+2}R_{n-2}^2)) \\ &\qquad \bmod (4, v_1^{a_{n+1}+2e_n+2e_{n-1}+2}), d(v_1^{a_{n+1}+2e_n-2a_{n-2}}x_{n-1}^6x_{n-2}^4) \\ &\equiv 2v_1^{a_{n+1}+2e_n}x_{n-1}^6x_{n-2}^2g_{n-2}^2 \bmod (4, v_1^{a_{n+1}+2e_n+2a_{n-2}}). \end{aligned}$$

Here, note that $2a_{n-1} = 2e_n + 2e_{n-1} = e_n + 2\times 4^{n-1} - 1$.

In a similar way, the second congruence follows from

$$\begin{aligned} d(\overline{u}'_n) &\equiv 2v_1^{2a_{n+1}+2e_n+2}(R_{n+1} + v_1^{2e_n}x_n^2g_{n+1}) \\ &\equiv 2v_1^{2a_{n+1}+2e_n+2}(R_{n+1} + v_1^{2e_n}x_n^3(x_{n-1}g_{n-1} + v_1^{e_{n-1}+1}R_{n-1})) \\ &\qquad \bmod (4, v_1^{2a_{n+1}+5e_n+3}), d(v_1^{2a_{n+1}+4e_n+2-a_{n-1}}x_n^3x_{n-1}^2) \\ &\equiv 2v_1^{2a_{n+1}+4e_n+2}x_n^3x_{n-1}g_{n-1} \quad \bmod (4, v_1^{2a_{n+1}+4e_n+2+a_{n-1}}). \end{aligned}$$

We also notice that $a_n - 1 = 5e_n = 2e_n + 4^n - 1$. □

In the same manner as $x_{n,1}$ and $x'_{n,1}$, we consider

$$y_{n,1} = x_{n-1}^4 + v_1^{2a_{n-1}-2a_{n-2}}x_{n-1}^2x_{n-2}^4 \quad \text{and} \quad y'_{n,1} = x_n^2 + v_1^{a_n-a_{n-1}}x_nx_{n-1}^2.$$

LEMMA (5.13).

$$\begin{aligned} d(y_{n,1}) &\equiv 2v_1^{2a_{n-1}+2e_{n-2}+2}x_{n-1}^2R_{n-2}^2 \;(= 2v_1^{14\times 4^{n-2}}x_{n-1}^2R_{n-2}^2) \\ &\qquad \bmod (4, v_1^{2a_{n-1}+2a_{n-2}}) \quad \text{for } n > 2, \\ d(y'_{n,1}) &\equiv 2v_1^{a_n+e_{n-1}+1}x_nR_{n-1} \;(= 2v_1^{7\times 4^{n-1}}x_nR_{n-1}) \bmod (4, v_1^{a_n+a_{n-1}}) \text{ for } n > 1. \end{aligned}$$

Proof. These follow from

$$\begin{aligned} d(x_{n-1}^4) &\equiv 2v_1^{2a_{n-1}}x_{n-1}^2g_{n-1}^2 \bmod (4, v_1^{4a_{n-1}}) \\ &\equiv 2v_1^{2a_{n-1}}x_{n-1}^2(x_{n-2}^2g_{n-2}^2+v_1^{2e_{n-2}+2}R_{n-2}^2) \bmod (4, v_1^{4a_{n-1}}), \\ d(v_1^{2a_{n-1}-2a_{n-2}}x_{n-1}^2x_{n-2}^4) &\equiv 2v_1^{2a_{n-1}}x_{n-1}^2x_{n-2}^2g_{n-2}^2 \bmod (4, v_1^{2a_{n-1}+2a_{n-2}}); \quad \text{and} \\ d(x_n^2) &\equiv 2v_1^{a_n}x_ng_n \quad \bmod (4, v_1^{2a_n}) \\ &\equiv 2v_1^{a_n}x_n(x_{n-1}g_{n-1}+v_1^{e_{n-1}+1}R_{n-1}) \quad \bmod (4, v_1^{2a_n}), \\ d(v_1^{a_n-a_{n-1}}x_nx_{n-1}^2) &\equiv 2v_1^{a_n}x_nx_{n-1}g_{n-1} \bmod (4, v_1^{a_n+a_{n-1}}). \end{aligned}$$

□

LEMMA (5.14). *$d(x_{n-1}^2) \equiv 2v_1^{a_{n-1}}g_n + 2v_1^{2\times 4^{n-1}}R_{n-1} \bmod (4, v_1^{2a_{n-1}})$ for $n \geq 2$ and $d(x_{n-1}^4) \equiv 2v_1^{2a_{n-1}}g_n^2 + 2v_1^{4^n}R_{n-1}^2 \bmod (4, v_1^{4a_{n-1}})$ for $n \geq 2$.*

Proof. These follow from Lemma (5.5) and the definition (5.7). □

Let $b_{n,k}$ for $n, k > 0$ be integers defined by

$$b_{n,k} = 4^n + 3\times 4^{n-2k+3}\overline{e}_{k-1} - 2\times 4^{n-2k} \tag{5.15}$$

and consider the elements

$$\begin{aligned} X_n &= x_{n,1} + \sum_{k=1}^{[\frac{n}{2}]-1} v_1^{b_{n,k}}x_{n-2k+2}^{3\overline{e}_{k-1}}\widetilde{u}'_{n-2k}, \\ X'_n &= x'_{n,1} + \sum_{k=1}^{[\frac{n}{2}]-1} v_1^{2b_{n,k}}x_{n-2k+2}^{6\overline{e}_{k-1}}\widetilde{u}_{n-2k+1}, \\ Y_n &= y_{n,1} + \sum_{k=1}^{[\frac{n-1}{2}]-1} v_1^{2b_{n-1,k}}x_{n-1}^2x_{n-2k+1}^{6\overline{e}_{k-1}}\widetilde{u}_{n-2k}, \\ Y'_n &= y'_{n,1} + \sum_{k=1}^{[\frac{n}{2}]-1} v_1^{b_{n,k}}x_nx_{n-2k+2}^{3\overline{e}_{k-1}}\widetilde{u}'_{n-2k}, \end{aligned} \tag{5.16}$$

where integers $[x]$ and $\overline{e}_n$ are those given in (2.9).

LEMMA (5.17). *The elements in (5.16) satisfy $X_n \equiv 2x_n$, $X'_n \equiv 2x_n^2$, $Y_n \equiv x_{n-1}^4$ and $Y'_n \equiv x_n^2 \bmod (4, v_1^2)$, and*

$$\begin{aligned} d(X_n) &\equiv 2v_1^{c_n}x_{2+\varepsilon(n)}^{3\overline{e}_{[\frac{n}{2}]-1}}R_{1+\varepsilon(n)} \bmod (4, v_1^{c_n+4^{1+\varepsilon(n)}}), \\ d(X'_n) &\equiv 2v_1^{2c_n}x_{2+\varepsilon(n)}^{6\overline{e}_{[\frac{n}{2}]-1}}R_{1+\varepsilon(n)}^2 \bmod (4, v_1^{2c_n+2\times 4^{1+\varepsilon(n)}}), \\ d(Y_n) &\equiv 2v_1^{2c_{n-1}}x_{n-1}^2x_{2+\varepsilon(n-1)}^{6\overline{e}_{[\frac{n-1}{2}]-1}}R_{1+\varepsilon(n-1)}^2 \bmod (4, v_1^{2c_{n-1}+2\times 4^{1+\varepsilon(n-1)}}) \quad \textit{and} \\ d(Y'_n) &\equiv 2v_1^{c_n}x_nx_{2+\varepsilon(n)}^{3\overline{e}_{[\frac{n}{2}]-1}}R_{1+\varepsilon(n)} \bmod (4, v_1^{c_n+4^{1+\varepsilon(n)}}) \end{aligned}$$

for integers $[x]$, c_n, $\varepsilon(n)$ and $\overline{e}_n$ in (2.9).

Proof. Note that the integers $b_{n,k}$ in (5.15) satisfy

$$b_{n,1} = a_n + e_{n-1} - 2a_{n-1} - 2e_{n-2} - 1 \quad \text{and}$$
$$b_{n,k+1} = b_{n,k} + 2a_{n-2k+1} + 2e_{n-2k} + 3\times 4^{n-2k-1} - 2a_{n-2k-1} - 2e_{n-2k-2};$$
$$2b_{n,1} = 2a_n + 2e_{n-1} - a_n - e_{n-1} + 1 \quad \text{and}$$
$$2b_{n,k+1} = 2b_{n,k} + a_{n-2k+2} + e_{n-2k+1} + 6\times 4^{n-2k-1} - a_{n-2k} - e_{n-2k-1}.$$

Indeed, $a_n + e_{n-1} = 7\times 4^{n-1} - 1$.

The differentials on the elements in (5.16) then follow immediately from Lemmas (5.10), (5.12) and (5.13) as follows:

$$\begin{aligned}
d(X_n) &\equiv 2v_1^{a_n+e_{n-1}+1}R_{n-1}\\
&\quad + \sum_{k=1}^{[\frac{n}{2}]-1} v_1^{b_{n,k}} x_{n-2k+2}^{3\bar{e}_{k-1}}(2v_1^{2a_{n-2k+1}+2e_{n-2k}+2}(R_{n-2k+1} + v_1^{3\times 4^{n-2k-1}}x_{n-2k}^3 R_{n-2k-1}))\\
&\equiv 2v_1^{b_{n,[\frac{n}{2}]-1}+2a_{3+\varepsilon(n)}+2e_{2+\varepsilon(n)}+2+3\times 4^{1+\varepsilon(n)}} x_{2+\varepsilon(n)}^{3\bar{e}_{[\frac{n}{2}]-1}} R_{1+\varepsilon(n)} \bmod (4, v_1^{c_n+4^{1+\varepsilon(n)}}),\\
d(X_n') &\equiv 2v_1^{2a_n+2e_{n-1}+2}R_{n-1}^2\\
&\quad + \sum_{k=1}^{[\frac{n}{2}]-1} v_1^{2b_{n,k}} x_{n-2k+2}^{6\bar{e}_{k-1}}(2v_1^{a_{n-2k+2}+e_{n-2k+1}+1}(R_{n-2k+1}^2 + v_1^{6\times 4^{n-2k-1}}x_{n-2k}^6 R_{n-2k-1}^2))\\
&\equiv 2v_1^{2b_{n,[\frac{n}{2}]-1}+a_{4+\varepsilon(n)}+e_{3+\varepsilon(n)}+1+6\times 4^{1+\varepsilon(n)}} x_{2+\varepsilon(n)}^{6\bar{e}_{[\frac{n}{2}]-1}} R_{1+\varepsilon(n)}^2 \bmod (4, v_1^{2c_n+2\times 4^{1+\varepsilon(n)}}),\\
d(Y_n) &\equiv 2v_1^{2a_{n-1}+2e_{n-2}+2}x_{n-1}^2 R_{n-2}^2\\
&\quad + \sum_{k=1}^{[\frac{n-1}{2}]-1} v_1^{2b_{n-1,k}} x_{n-1}^2 x_{n-2k+1}^{6\bar{e}_{k-1}}(2v_1^{a_{n-2k+1}+e_{n-2k}+1}(R_{n-2k}^2 + v_1^{6\times 4^{n-2k-2}}x_{n-2k-1}^6 R_{n-2k-2}^2))\\
&\equiv 2v_1^{2b_{n-1,[\frac{n-1}{2}]-1}+a_{3+\varepsilon(n-1)}+e_{2+\varepsilon(n-1)}+1+6\times 4^{\varepsilon(n-1)}} x_{n-1}^2 x_{2+\varepsilon(n-1)}^{6\bar{e}_{[\frac{n-1}{2}]-1}} R_{1+\varepsilon(n-1)}^2\\
&\qquad \bmod (4, v_1^{2c_{n-1}+2\times 4^{1+\varepsilon(n-1)}}),\\
d(Y_n') &\equiv 2v_1^{a_n+e_{n-1}+1}x_n R_{n-1}\\
&\quad + \sum_{k=1}^{[\frac{n}{2}]-1} v_1^{b_{n,k}} x_n x_{n-2k+2}^{3\bar{e}_{k-1}}(2v_1^{2a_{n-2k+1}+2e_{n-2k}+2}(R_{n-2k+1} + v_1^{3\times 4^{n-2k-1}}x_{n-2k}^3 R_{n-2k-1}))\\
&\equiv 2v_1^{b_{n,[\frac{n}{2}]-1}+2a_{3+\varepsilon(n)}+2e_{2+\varepsilon(n)}+2+3\times 4^{1+\varepsilon(n)}} x_n x_{2+\varepsilon(n)}^{3\bar{e}_{[\frac{n}{2}]-1}} R_{1+\varepsilon(n)} \bmod (4, v_1^{c_n+4^{1+\varepsilon(n)}}).
\end{aligned}$$

Note that $2[\frac{n}{2}] = n - \varepsilon(n)$ by definition (2.9), and we obtain the lemma. □

LEMMA (5.18). *There exists an element χ such that $d(\chi) \equiv 2v_1^5 g_1^2 + 2v_1^7 R_1 + 2v_1^7 v_2 v_3^3 \bar{t}_{21} + 2v_1^7 v_2 v_3^6 t_2 \bmod (4, v_1^8)$ for $r = t_4 + t_4^2$ in Lemma (5.2) and $\bar{t}_{21} = t_2^2 + v_2 t_2$.*

Proof. Mod $(4, v_1^8)$

$$2v_1^5 g_1^2 \equiv 2v_1^5(t_3^2 + v_3^2 t_2^4 + v_3^8 t_2^2 + v_1^2 t_4^4 + v_1^2 t_2^2),$$
$$d(2v_1^2 v_2 v_3^2) \equiv 2v_1^4 v_2 t_2^4,$$

$$\begin{aligned}
d(v_1^4v_2) &\equiv 2v_1^4t_2,\\
d(v_1^3v_3^4) &\equiv 2v_1^5v_3^2t_2^4 + v_1^7t_2^8,\\
d(2v_1^4v_3^9) &\equiv 2v_1^5v_3^8t_2^2,\\
d(2v_2x_1) &\equiv 2v_1^6v_2(t_3^4 + v_1t_2)\\
&\equiv 2v_1^6(t_3 + v_3t_2^8 + v_3^4t_2 + v_1t_4^2 + v_1v_2t_2) \text{ by (3.7)},\\
d(2v_1^5v_3^2) &\equiv 2v_1^7t_2^4,\\
d(3v_1^6v_3) &\equiv 2v_1^6t_3 + 3v_1^7t_2^2,\\
d(v_1^6v_2v_3^4) &\equiv 2v_1^6v_3^4t_2.
\end{aligned}$$

The sum of these congruences with (3.7) shows the existence of an element χ'' such that $2v_1^5g_1^2 + d(\chi'') \equiv 2v_1^7r^2 \bmod (4, v_1^8)$. We further compute mod $(4, v_1^4)$,

$$\begin{aligned}
2v_1^3t_4^2 &\equiv 2v_1^3(t_4^8 + v_3^2t_3^{16} + v_3^{16}t_3^2)\\
&\equiv 2v_1^3(t_4^8 + v_3^2t_3^4 + v_3^6t_2^{32} + v_3^{18}t_2^4 + v_3^{16}t_3^2) \text{ by (3.7)},\\
d(2v_1^2v_3^7) &\equiv 2v_1^3v_3^6t_2^2,\\
d(v_1v_3^{20}) &\equiv 2v_1^3v_3^{18}t_2^4,\\
d(2v_2v_3^{18}) &\equiv 2v_1^2v_2v_3^{16}t_2^4,\\
d(v_1^2v_2v_3^{16}) &\equiv 2v_1^2v_3^{16}t_2.
\end{aligned}$$

These with (3.7) also imply the existence of an element χ' such that

$$2v_1^5g_1^2 + d(\chi') \equiv 2v_1^7r^4 + 2v_1^7v_3^2t_3^4 \quad \bmod (4, v_1^8).$$

The congruences

$$\begin{aligned}
2v_3^2t_3^4 &\equiv 2v_2v_3^2(t_3 + v_3t_2^8 + v_3^4t_2) \quad \text{and}\\
d(v_2v_3^3) &\equiv 2v_2v_3^2t_3 + 2v_3^3t_2
\end{aligned}$$

show that $2v_3^2t_3^4$ is homologous to $2v_2v_3^3\bar{t}_{21} + 2v_2v_3^6t_2$. Since $r^4 = R_1$ by Lemma (5.8), we obtain the lemma. □

LEMMA (5.19). *There exists an element* $\overline{\chi}$ *such that* $d(\overline{\chi}) \equiv 2v_1^3x_1g_1^2 + 2v_1^3v_3^6\bar{t}_{21}$ mod $(4, v_1^4)$ *for* $\bar{t}_{21}$ *in Lemma (5.18).*

Proof. Note that $t_2^4 \equiv v_2t_2 + v_1v_2t_3^2 \bmod (2, v_1^2)$. Then the lemma follows as the sum of the congruences:

$$\begin{aligned}
2v_1^3x_1g_1^2 &\equiv 2v_1^3x_1(t_3^2 + v_3^2t_2^4 + v_3^8t_2^2),\\
d(2v_2x_1v_3^2) &\equiv 2v_1^2v_2x_1t_2^4,\\
d(v_1^2v_2x_1) &\equiv 2v_1^2x_1t_2,\\
d(2v_1^2x_1v_3^9) &\equiv 2v_1^3x_1v_3^8t_2^2,\\
d(2v_1^2x_1v_3^3) &\equiv 2v_1^3x_1v_3^2t_2^2.
\end{aligned}$$

□

6. The action of the connecting homomorphism

In this section, we determine $E_2^*(L_2\overline{M}(2,\infty)) = H^*M_1^1(2)$ by observing the long exact sequence (1.3) (see also (3.3)) by the method using [4, Remark 3.11]:

LEMMA (6.1). *Suppose that a submodule D^s of $H^sM_1^1(2)$ fits in the exact sequence*

$$H^{s-1}M_1^1(2) \xrightarrow{\delta} H^sM_2^0(2) \xrightarrow{1/v_1} D^s \xrightarrow{v_1} D^s \xrightarrow{\delta} H^{s+1}M_2^0(2).$$

Then, $H^sM_1^1(2) = D^s$.

We read off the zeroth and the first lines of the E_2-term $E_2^*(L_2\overline{M}(2,1)) = H^*M_2^0(2)$ from Theorem (2.5) and (3.3):

$$\begin{aligned}
H^0M_2^0(2) &= 2v_2K_*[v_3^2]\otimes\Lambda(v_3)\oplus 2v_3K_*[v_3^2]\oplus\mathbb{Z}/4[v_2^{\pm 2}, v_3^2] \quad \text{and}\\
H^1M_2^0(2) &= 2v_2K_*[v_3^2]\otimes\Lambda(v_3)\{h_{20}, \overline{h}_{21}, h_{30}, \overline{h}_{31}, \rho_2\}\oplus h_{20}K_*[v_3^2]\otimes\Lambda(v_3)\\
&\quad\oplus 2v_3K_*[v_3^2]\{\overline{h}_{21}, \overline{h}_{31}, \rho_2\}\oplus h_{30}K_*[v_3^2]\\
&\quad\oplus 2v_3h_{30}K_*[v_3^2]\oplus\mathbb{Z}/4[v_2^{\pm 2}, v_3^2]\{\overline{h}_{21}, \overline{h}_{31}, \rho_2\}.
\end{aligned}$$

PROPOSITION (6.2). *For the generators of $H^0M_1^1(2)$, we see the behavior of the connecting homomorphism:*

$$\begin{aligned}
\delta(2v_2v_3^{2t+1}/v_1) &= 2v_2v_3^{2t}\overline{h}_{21},\\
\delta(2v_2v_3^{4t+2}/v_1^3 + v_2v_3^{4t}/v_1) &= 2v_3^{4t+1}h_{30} + 2v_3^{4t}\overline{h}_{31},\\
\delta(2v_2x_n^{2t+1}/v_1^{a_n+1}) &= -x_n^{2t}v_3^{4e_{n-1}}\overline{h}_{21} + 2x_n^{2t}v_3^{4e_{n-1}}\rho_2,\\
\delta(2v_2x_n^{4t+2}/v_1^{2a_n}) &= 2v_2x_n^{4t}v_3^{8e_{n-1}}\overline{h}_{31} + 2v_2x_n^{4t}v_3^{8e_{n-1}+1}h_{30};\\
\delta(2v_3^{2t+1}/v_1) &= 2v_2v_3^{2t}h_{20} + 2v_3^{2t}\overline{h}_{21};\\
\delta(v_3^{4t+2}/v_1) &= 2v_2v_3^{4t+1}h_{20} + 2v_3^{4t+1}\overline{h}_{21},\\
\delta(2v_3^{4t+2}/v_1^2) &= 2v_3^{4t}\overline{h}_{21},\\
\delta(v_3^{8t+4}/v_1^2) &= 2v_3^{8t+2}\overline{h}_{21},\\
\delta(2x_1^{2t+1}/v_1^6) &= 2v_2v_3^{8t+1}\overline{h}_{21},\\
\delta(x_1^{4t+2}/v_1^6) &= 2v_2v_3^{16t+5}\overline{h}_{21},\\
\delta(2x_1^{4t+2}/v_1^{14}) &= 2x_1^{4t}(\rho_2 + v_2v_3^3\overline{h}_{21} + v_2v_3^6h_{20}),\\
\delta(x_2^{2t+1}/v_1^{14}) &= 2x_1^{8t+2}(\rho_2 + v_2v_3^3\overline{h}_{21} + v_2v_3^6h_{20}),\\
\delta(2x_{2k}^{2t+1}/v_1^{c_{2k}}) &= 2v_2x_2^{2\times 4^{2k-2}t+3\overline{e}_{k-1}}v_3^2(v_3\overline{h}_{21} + v_3^4h_{20}) \quad k\ge 1,\\
\delta(x_{2k}^{4t+2}/v_1^{c_{2k}}) &= 2v_2x_2^{4^{2k-2}(4t+1)+3\overline{e}_{k-1}}v_3^2(v_3\overline{h}_{21} + v_3^4h_{20}) \quad k\ge 1,\\
\delta(2x_{2k}^{4t+2}/v_1^{2c_{2k}}) &= 2v_3^{4^{2k+1}t+6\overline{e}_k}\overline{h}_{21} \quad k\ge 1,\\
\delta(x_{2k+1}^{2t+1}/v_1^{2c_{2k}}) &= 2v_3^{4^{2k}(8t+2)+6\overline{e}_k}\overline{h}_{21} \quad k\ge 1,
\end{aligned}$$

$$\begin{aligned}
\delta(2x_{2k+1}^{2t+1}/v_1^{c_{2k+1}+2}) &= 2v_2x_1^{2\times 4^{2k}t+3\bar{e}_k}(v_3\overline{h}_{21}+v_3^4h_{20}) \quad k\geq 1,\\
\delta(x_{2k+1}^{4t+2}/v_1^{c_{2k+1}+2}) &= 2v_2x_1^{4^{2k}(4t+1)+3\bar{e}_k}(v_3\overline{h}_{21}+v_3^4h_{20}) \quad k\geq 1,\\
\delta(2x_{2k+1}^{4t+2}/v_1^{2c_{2k+1}+6}) &= 2x_1^{4^{2k+1}t+6\bar{e}_k}(\rho_2+v_2v_3^3\overline{h}_{21}+v_2v_3^6h_{20}) \quad k\geq 1,\\
\delta(x_{2k}^{2t+1}/v_1^{2c_{2k-1}+6}) &= 2x_1^{4^{2k-2}(8t+2)+6\bar{e}_{k-1}}(\rho_2+v_2v_3^3\overline{h}_{21}+v_2v_3^6h_{20}) \quad k>1.
\end{aligned}$$

Proof. Throughout this proof, we use the relations in (3.7) freely. The first and the second equalities follow from

$$\begin{aligned}
d(2v_2v_3^{2t+1}/v_1^2+v_2v_3^{2t}/v_1) &= 2v_2v_3^{2t}t_2^2/v_1+2v_3^{2t}t_2/v_1 \quad \text{and}\\
d(2v_2v_3^{4t+2}/v_1^4+v_2v_3^{4t}/v_1^2) &= 2v_2v_3^{4t}t_2^4/v_1^2+2v_3^{4t}t_2/v_1^2\\
&= 2v_3^{4t}t_3^2/v_1.
\end{aligned}$$

Turn to the third equality. Suppose first that $n=1$. Since

$$\begin{aligned}
d(2v_2x_1^{2t+1}/v_1^8) &= 2v_2x_1^{2t}g_1/v_1^2 \ = \ 2v_2x_1^{2t}(t_3^4+v_1t_2)/v_1^2\\
&= 2x_1^{2t}(t_3+v_3t_2^8+v_3^4t_2+v_1t_4^2)/v_1^2+2v_2x_1^{2t}t_2/v_1,\\
d(x_1^{2t}v_3/v_1^2) &= 2x_1^{2t}t_3/v_1^2+x_1^{2t}t_2^2/v_1+2x_1^{2t}(v_2t_2+t_2^2)/v_1,\\
d(3x_1^{2t}v_3^2/v_1^3) &= 2x_1^{2t}v_3t_2^2/v_1^2+3x_1^{2t}t_2^4/v_1,\\
d(v_2x_1^{2t}v_3^4/v_1^2) &= 2x_1^{2t}v_3^4t_2/v_1^2,
\end{aligned}$$

we obtain $d(2v_2x_1^{2t+1}/v_1^8+\cdots)=2x_1^{2t}t_4^2/v_1+x_1^{2t}t_2^2/v_1+2x_1^{2t}(v_2t_2+t_2^2)/v_1+x_1^{2t}t_2^4/v_1$. Put $\bar{t}_{21}'=t_2^2+t_2^4+2t_4^2+2v_2t_2+2v_2t_2^5$ and recall $\bar{t}_{21}$ in Lemma (5.18). Then, $3\bar{t}_{21}-\bar{t}_{21}'=2t_4^2+2v_2t_4 \bmod (4,v_1)$, since $\eta_R(v_2v_4)\equiv v_2v_4+v_2^2t_2^4-v_2^5t_2+2v_2t_4+2v_4t_2+2v_2t_2^5+2v_2^4t_2^2 \bmod (4,v_1)$ in $\Gamma(2)$. Thus, we see that $\bar{t}_{21}'$ represents $3\overline{h}_{21}+2\rho_2$. For $n>1$, the equality follows similarly from the computation

$$\begin{aligned}
d(2v_2x_n^{2t+1}/v_1^{a_n+2}) &= 2v_2x_n^{2t}g_n/v_1^2 = 2v_2x_n^{2t}x_1^{e_{n-1}}(t_3^4+v_1t_2)/v_1^2\\
&= 2x_n^{2t}x_1^{e_{n-1}}(t_3+v_3t_2^8+v_3^4t_2+v_1t_4^2)/v_1^2+2v_2x_n^{2t}x_1^{e_{n-1}}t_2/v_1,\\
d(x_n^{2t}x_1^{e_{n-1}}v_3/v_1^2) &= 2x_n^{2t}x_1^{e_{n-1}}t_3/v_1^2+x_n^{2t}x_1^{e_{n-1}}t_2^2/v_1^2+2x_n^{2t}x_1^{e_{n-1}}(v_2t_2+t_2^2)/v_1,\\
d(x_n^{2t}x_2^{e_{n-2}}v_3^6/v_1^3) &= 2x_n^{2t}x_1^{e_{n-1}}v_3t_2^2/v_1^2+x_n^{2t}x_1^{e_{n-1}}t_2^4/v_1+2x_n^{2t}x_1^{e_{n-1}}t_2^4/v_1,\\
d(v_2x_n^{2t}x_1^{e_{n-1}}v_3^4/v_1^2) &= 2x_n^{2t}x_1^{e_{n-1}}v_3^4t_2/v_1^2.
\end{aligned}$$

The fourth equality is verified by

$$\begin{aligned}
d(2v_2x_n^{4t+2}/v_1^{2a_n+1}) &= 2v_2x_n^{4t}g_n^2/v_1 = 2v_2x_n^{4t}x_1^{2e_{n-1}}g_1^2/v_1 = 2v_2x_n^{4t}x_1^{2e_{n-1}}t_3^8/v_1\\
&= 2v_2x_n^{4t}x_1^{2e_{n-1}}(t_3^2+v_3^2t_2^4+v_3^8t_2^2)/v_1,\\
d(v_2x_n^{4t}x_1^{2e_{n-1}}v_3^2/v_1) &= 2x_n^{4t}x_1^{2e_{n-1}}v_3^2t_2/v_1,\\
d(2v_2x_n^{4t}x_1^{2e_{n-1}}v_3^9/v_1^2) &= 2v_2x_n^{4t}x_1^{2e_{n-1}}v_3^8t_2^2/v_1.
\end{aligned}$$

The fifth and the sixth ones follow from $d(2v_3^{2t+1}/v_1^2)=2v_3^{2t}t_2^2/v_1$ and $d(v_3^{4t+2}/v_1^2)=2v_3^{4t+1}t_2^2/v_1$, respectively. The seventh and the eighth are checked as

$$\begin{aligned}
d(2v_3^{4t+2}/v_1^3) &= 2v_3^{4t}t_2^4/v_1 \;=\; 2v_3^{4t}(\bar{t}_{21}+t_2^2)/v_1,\\
d(2v_3^{4t+1}/v_1^2) &= 2v_3^{4t}t_2^2/v_1; \quad \text{and}\\
d(v_3^{8t+4}/v_1^3) &= 2v_3^{8t+2}t_2^4/v_1,\\
d(2v_3^{8t+3}/v_1^2) &= 2v_3^{8t+2}t_2^2/v_1.
\end{aligned}$$

The ninth and the tenth ones follow from the computation:

$$\begin{aligned}
d(2x_1^{2t+1}/v_1^7) &= 2x_1^{2t}g_1/v_1 = 2x_1^{2t}(v_2t_3+v_2v_3t_2^2+v_2v_3^4t_2)/v_1,\\
d(v_2x_1^{2t}v_3/v_1) &= 2x_1^{2t}v_3t_2/v_1+2v_2x_1^{2t}t_3/v_1,\\
d(2x_1^{2t+1}v_3^2/v_1^3) &= 2x_1^{2t+1}t_2^4/v_1; \quad \text{and}\\
d(x_1^{4t+2}/v_1^7) &= 2x_1^{4t+1}g_1/v_1 = 2x_1^{4t+1}(v_2t_3+v_2v_3t_2^2+v_2v_3^4t_2)/v_1,\\
d(v_2x_1^{4t+1}v_3/v_1) &= 2x_1^{4t+1}v_3t_2/v_1+2v_2x_1^{4t+1}t_3/v_1,\\
d(2x_1^{4t+2}v_3^2/v_1^3) &= 2x_1^{4t+2}t_2^4/v_1,
\end{aligned}$$

in which $2v_2x_1^{2t}v_3t_2^2/v_1 + 2x_1^{2t}v_3t_2/v_1 = 2v_2x_1^{2t}v_3\bar{t}_{21}/v_1$ and $2v_2x_1^{4t+1}v_3t_2^2/v_1 + 2x_1^{4t+1}v_3t_2/v_1 = 2v_2x_1^{4t+1}v_3\bar{t}_{21}/v_1$. By Lemma (5.18),

$$\begin{aligned}
d(2x_1^{4t+2}/v_1^{15}) &= 2x_1^{4t}g_1^2/v_1^3,\\
d(x_1^{4t}\chi/v_1^8) &= 2x_1^{4t}g_1^2/v_1^3+2x_1^{4t}(R_1+v_2v_3^3\bar{t}_{21}+v_2v_3^6t_2)/v_1.
\end{aligned}$$

The element ρ_2 is represented by the cocycle r^4, and $r^2 = r^4 + s$ for $s = t_3^4 = v_2(t_3 + v_3t_2^2 + v_3^4t_2)$. Note that $2v_2t_3 = 2v_3t_2$ up to homology. Then $\sigma = [s] = v_2v_3\overline{h}_{21} + v_2v_3^4h_{20}$. It follows that

$$\delta(2x_1^{4t+2}/v_1^{14}) = 2x_1^{4t}\rho_2 + 2v_2x_1^{4t}(v_3^3\overline{h}_{21}+v_3^6h_{20}).$$

In the same manner,

$$\begin{aligned}
d(x_2^{2t}x_1^4/v_1^{15}) &= 2x_2^{2t}x_1^2g_1^2/v_1^3,\\
d(x_2^{2t}x_1^2\chi/v_1^8) &= 2x_2^{2t}x_1^2g_1^2/v_1^3+2x_2^{2t}x_1^2r^2/v_1,
\end{aligned}$$

and we obtain

$$\delta(x_2^{2t+1}/v_1^{14}) = 2x_1^{8t+2}\rho_2 + 2v_2x_1^{8t+2}(v_3^3\overline{h}_{21}+v_3^6h_{20}).$$

For $n = 2k \geq 2$, by Lemma (5.17), we compute

$$\begin{aligned}
d(x_{2k}^{2t}X_{2k}/v_1^{c_{2k}+1}) &= 2x_{2k}^{2t}x_2^{3\bar{e}_{k-1}}R_1/v_1,\\
d(2x_{2k}^{2t}x_2^{3\bar{e}_{k-1}}x_1^2/v_1^{15}+\dots) &= 2x_{2k}^{2t}x_2^{3\bar{e}_{k-1}}(R_1+v_2v_3^3\bar{t}_{21}+v_2v_3^6t_2)/v_1.
\end{aligned}$$

It follows that

$$\delta(2x_{2k}^{2t+1}/v_1^{c_{2k}}) = 2v_2x_2^{2\times 4^{2k-2}t+3\bar{e}_{k-1}}v_3^2(v_3\overline{h}_{21}+v_3^4h_{20}).$$

In the same way, for even $n = 2k \geq 2$,

$$d(x_n^{4t}Y_n'/v_1^{c_n+1}) = 2x_{2k}^{4t+1}x_2^{3\bar{e}_{k-1}}R_1/v_1$$

with $d(2x_{2k}^{4t+1}x_2^{3\bar{e}_{k-1}}x_1^2/v_1^{15}+\dots) = 2x_{2k}^{4t+1}x_2^{3\bar{e}_{k-1}}(R_1+v_2v_3^3\bar{t}_{21}+v_2v_3^6t_2)/v_1$, and we obtain

$$\delta(x_{2k}^{4t+2}/v_1^{c_{2k}}) = 2v_2x_2^{4^{2k-2}(4t+1)+3\bar{e}_{k-1}}v_3^2(v_3\overline{h}_{21}+v_3^4h_{20}).$$

For $n = 2k \geq 2$, by Lemmas (5.17), (5.9) and (5.19),

$$\begin{aligned}
d(x_n^{4t}X'_{2k}/v_1^{2c_{2k}+1}) &= 2x_2^{4^{2k-1}t+6\overline{e}_{k-1}}R_1^2/v_1,\\
d(x_2^{4^{2k-1}t+6\overline{e}_{k-1}}X_2/v_1^{c_2+1}) &= 2x_2^{4^{2k-1}t+6\overline{e}_{k-1}}R_1/v_1,\\
d(2x_2^{4^{2k-1}t+6\overline{e}_{k-1}}u_1/v_1^{11}) &= 2x_2^{4^{2k-1}t+6\overline{e}_{k-1}}(R_1^2+R_1+x_1g_1^2)/v_1,\\
d(x_2^{4^{2k-1}t+6\overline{e}_{k-1}}\overline{\chi}/v_1^4) &= 2x_2^{4^{2k-1}t+6\overline{e}_{k-1}}(x_1g_1^2+v_3^6\overline{t}_{21})/v_1,
\end{aligned}$$

and we have

$$\delta(2x_{2k}^{4t+2}/v_1^{2c_{2k}}) = 2v_3^{4^{2k+1}t+6\overline{e}_k}\overline{h}_{21}.$$

For odd $n=2k+1\geq 3$, by Lemmas (5.17), (5.9) and (5.19),

$$\begin{aligned}
d(x_n^{2t}Y_n/v_1^{2c_{n-1}+1}) &= 2x_{n-1}^{8t+2}x_2^{6\overline{e}_{k-1}}R_1^2/v_1 = 2x_2^{4^{2k-2}(8t+2)+6\overline{e}_{k-1}}R_1^2/v_1,\\
d(x_2^{4^{2k-2}(8t+2)+6\overline{e}_{k-1}}X_2/v_1^{c_2+1}) &= 2x_2^{4^{2k-2}(8t+2)+6\overline{e}_{k-1}}R_1/v_1,\\
d(2x_2^{4^{2k-2}(8t+2)+6\overline{e}_{k-1}}u_1/v_1^{11}) &= 2x_2^{4^{2k-2}(8t+2)+6\overline{e}_{k-1}}(R_1^2+R_1+x_1g_1^2)/v_1,\\
d(x_2^{4^{2k-2}(8t+2)+6\overline{e}_{k-1}}\overline{\chi}/v_1^4) &= 2x_2^{4^{2k-2}(8t+2)+6\overline{e}_{k-1}}(x_1g_1^2+v_3^6\overline{t}_{21})/v_1.
\end{aligned}$$

It follows that

$$\delta(x_{2k+1}^{2t+1}/v_1^{2c_{2k}}) = 2v_3^{4^{2k}(8t+2)+6\overline{e}_k}\overline{h}_{21}.$$

Lemmas (5.17), (5.10) and (5.9) show the equalities

$$\begin{aligned}
d(x_{2k+1}^{2t}X_{2k+1}/v_1^{c_{2k+1}+3}) &= 2x_3^{2\times 4^{2k-2}t+3\overline{e}_{k-1}}R_2/v_1^3,\\
d(x_3^{2\times 4^{2k-2}t+3\overline{e}_{k-1}}x'_{2,1}/v_1^{14\times 4+3}) &= 2x_3^{2\times 4^{2k-2}t+3\overline{e}_{k-1}}R_1^2/v_1^3,\\
d(2x_3^{2\times 4^{2k-2}t+3\overline{e}_{k-1}}u'_1/v_1^{23}) &= 2x_3^{2\times 4^{2k-2}t+3\overline{e}_{k-1}}(R_2+R_1^2+v_1^2x_1^2g_2)/v_1^3\\
&= 2x_3^{2\times 4^{2k-2}t+3\overline{e}_{k-1}}(R_2+R_1^2+v_1^2v_2x_1^3(t_3+v_3t_2^2+v_3^4t_2))/v_1^3,\\
d(v_2v_3x_1^{2\times 4^{2k}t+3\overline{e}_k}/v_1) &= 2x_1^{2\times 4^{2k}t+3\overline{e}_k}(v_3t_2+v_2t_3)/v_1,
\end{aligned}$$

which give rise to

$$\delta(2x_{2k+1}^{2t+1}/v_1^{c_{2k+1}+2}) = 2v_2x_1^{2\times 4^{2k}t+3\overline{e}_k}(v_3\overline{h}_{21}+v_3^4h_{20}).$$

Consider the odd case where $n=2k+1$.

$$\begin{aligned}
d(x_n^{4t}Y'_n/v_1^{c_n+3}) &= 2x_n^{4t+1}x_3^{3\overline{e}_{k-1}}R_2/v_1^3 = 2x_3^{4^{2k-2}(4t+1)+3\overline{e}_{k-1}}R_2/v_1^3,\\
d(x_3^{4^{2k-2}(4t+1)+3\overline{e}_{k-1}}x'_{2,1}/v_1^{14\times 4+3}) &= 2x_3^{4^{2k-2}(4t+1)+3\overline{e}_{k-1}}R_1^2/v_1^3,\\
d(2x_3^{4^{2k-2}(4t+1)+3\overline{e}_{k-1}}u'_1/v_1^{23}) &= 2x_3^{4^{2k-2}(4t+1)+3\overline{e}_{k-1}}(R_2+R_1^2+v_1^2x_1^2g_2)/v_1^3\\
&= 2x_3^{4^{2k-2}(4t+1)+3\overline{e}_{k-1}}(R_2+R_1^2+v_1^2v_2x_1^3(t_3+v_3t_2^2+v_3^4t_2))/v_1^3,\\
d(v_2v_3x_1^{4^{2k}(4t+1)+3\overline{e}_k}/v_1) &= 2x_1^{4^{2k}(4t+1)+3\overline{e}_k}(v_3t_2+v_2t_3)/v_1,
\end{aligned}$$

and so

$$\delta(x_{2k+1}^{4t+2}/v_1^{c_{2k+1}+2}) = 2v_2x_1^{4^{2k}(4t+1)+3\overline{e}_k}(v_3\overline{h}_{21}+v_3^4h_{20}).$$

For $n=2k+1\geq 3$, by Lemmas (5.17), (5.10), (5.9) and (5.18),

$$\begin{aligned}
d(x_n^{4t}X'_{2k+1}/v_1^{2c_{2k+1}+7}) &= 2x_3^{4^{2k-1}t+6\bar{e}_{k-1}}R_2^2/v_1^7,\\
d(x_3^{4^{2k-1}t+6\bar{e}_{k-1}}x_{3,1}/v_1^{7\times 4^2+7}) &= 2x_3^{4^{2k-1}t+6\bar{e}_{k-1}}R_2/v_1^7,\\
d(2x_3^{4^{2k-1}t+6\bar{e}_{k-1}}u_2/v_1^{43}) &= 2x_3^{4^{2k-1}t+6\bar{e}_{k-1}}(R_2^2+R_2+v_1^4x_2g_2^2)/v_1^7\\
&= 2x_3^{4^{2k-1}t+6\bar{e}_{k-1}}(R_2^2+R_2+v_1^4x_1^6g_1^2)/v_1^7,\\
d(x_1^{4^{2k+1}t+6\bar{e}_k}\chi/v_1^8) &= 2x_1^{4^{2k+1}t+6\bar{e}_k}(g_1^2+v_1^2R_1+v_1^2v_2v_3^3\bar{t}_{21}+v_1^2v_2v_3^6t_2)/v_1^3.
\end{aligned}$$

It follows that

$$\delta(2x_{2k+1}^{4t+2}/v_1^{2c_{2k+1}+6}) = 2x_1^{4^{2k+1}t+6\bar{e}_k}(\rho_2+v_2v_3^3\overline{h}_{21}+v_2v_3^6h_{20}).$$

Last, for $n=2k\geq 4$, by Lemmas (5.17), (5.10), (5.9) and (5.18),

$$\begin{aligned}
d(x_n^{2t}Y_n/v_1^{2c_{n-1}+7}) &= 2x_{n-1}^{8t+2}x_3^{6\bar{e}_{k-2}}R_2^2/v_1^7 \;=\; 2x_3^{4^{2k-4}(8t+2)+6\bar{e}_{k-2}}R_2^2/v_1^7,\\
d(x_3^{4^{2k-4}(8t+2)+6\bar{e}_{k-2}}x_{3,1}/v_1^{7\times 4^2+7}) &= 2x_3^{4^{2k-4}(8t+2)+6\bar{e}_{k-2}}R_2/v_1^7,\\
d(2x_3^{4^{2k-4}(8t+2)+6\bar{e}_{k-2}}u_2/v_1^{43}) &= 2x_3^{4^{2k-4}(8t+2)+6\bar{e}_{k-2}}(R_2^2+R_2+v_1^4x_2g_2^2)/v_1^7\\
&= 2x_3^{4^{2k-4}(8t+2)+6\bar{e}_{k-2}}(R_2^2+R_2+v_1^4x_1^6g_1^2)/v_1^7,\\
d(x_1^{4^{2k-2}(8t+2)+6\bar{e}_{k-1}}\chi/v_1^8) &= 2x_1^{4^{2k-2}(8t+2)+6\bar{e}_{k-1}}(g_1^2+v_1^2R_1+v_1^2v_2v_3^3\bar{t}_{21}+v_1^2v_2v_3^6t_2)/v_1^3,
\end{aligned}$$

and we obtain $\delta(x_{2k}^{2t+1}/v_1^{2c_{n-1}+6}) = 2x_1^{4^{2k-2}(8t+2)+6\bar{e}_{k-1}}(\rho_2+v_2v_3^3\overline{h}_{21}+v_2v_3^6h_{20})$. □

Proof of Theorem (2.10). Set D^0 to be the right hand side of the formula for $E_2^0(L_2\overline{M}(2,\infty))$, which is $H^0M_1^1(2)$. Then Proposition (6.2) shows that D^0 satisfies the hypothesis of Lemma (6.1). □

Received June 21, 2004

Final version received September 29, 2006

IPPEI ICHIGI
KATSUMI SHIMOMURA
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
KOCHI UNIVERSITY
KOCHI, 780-8520
JAPAN
95sm004@math.kochi-u.ac.jp
katsumi@math.kochi-u.ac.jp

XIANGJUN WANG
NANKAI UNIVERSITY
DEPARTMENT OF MATHEMATICS
TIANJIN 300071
P. R. CHINA
xiwang@nankai.edu.cn

REFERENCES

[1] M. HOVEY AND H. SADOFSKY, *Invertible spectra in the $E(n)$-local stable homotopy category,* J. London Math. Soc. **60** (1999), 284–302.

[2] I. ICHIGI AND K.SHIMOMURA, *Subgroups of $\pi_*(L_2T(1))$ at the prime two*, Hiroshima Math. J. **33** (2003), 359–369.

[3] M. MAHOWALD AND K. SHIMOMURA, *The Adams-Novikov spectral sequence for the L_2-localization of a v_2-spectrum*, Contemporary Math. **146** (1993), 237–250.

[4] H. R. MILLER, D. C. RAVENEL, AND W. S. WILSON, *Periodic phenomena in the Adams-Novikov spectral sequence*, Ann. of Math. **106** (1977), 469–516.

[5] H. NAKAI AND X. WANG, $\mathrm{Ext}^0_{\Gamma(2)}(BP_*, M_0^2)$ *at the prime 2*, preprint.

[6] D. C. RAVENEL, Complex cobordism and stable homotopy groups of spheres, Academic Press, New York, 1986.

[7] K. SHIMOMURA, *The homotopy groups of the L_2-localized Mahowald spectrum $X\langle 1\rangle$*, Forum Math. **7** (1995), 685–707.

[8] K. SHIMOMURA AND X. WANG, *The homotopy groups $\pi_*(L_2S^0)$ at the prime* 3, Topology **41** (2002), 1183–1198.

[9] K. SHIMOMURA AND X. WANG, *The Adams-Novikov E_2-term for $\pi_*(L_2S^0)$ at the prime* 2, Math. Z. **241** (2002), 271–311.

[10] K. SHIMOMURA AND A. YABE, *The homotopy groups $\pi_*(L_2S^0)$*, Topology **34** (1995), 261–289.

Bol. Soc. Mat. Mexicana (3) Vol. 13, 2007

THE LAW OF A STOCHASTIC INTEGRAL WITH TWO INDEPENDENT FRACTIONAL BROWNIAN MOTIONS

XAVIER BARDINA AND CIPRIAN A. TUDOR

ABSTRACT. Using the tools of the stochastic integration with respect to the fractional Brownian motion, we obtain the expression of the characteristic function of the random variable $\int_0^1 B_s^\alpha dB_s^H$ where B^α and B^H are two independent fractional Brownian motions with Hurst parameters $\alpha \in (0,1)$ and $H > \frac{1}{2}$ respectively. The two-parameter case is also considered.

1. Introduction

The theory of multiple stochastic integrals with respect to Brownian motion is well-known (see for instance [9]), but in general, it is difficult to compute the law of a stochastic integral with respect to the Wiener process when the integrand is not deterministic. There are some known results in particular cases. Let us recall the context. Consider W^1 and W^2 two independent Brownian motions. In [6] and [19] the authors studied the law of the random variable

$$\alpha \int_0^1 W_s^1 dW_s^2 + \beta \int_0^1 W_s^2 dW_s^1.$$

When $\alpha = 1$ and $\beta = 0$ they showed that the characteristic function of the stochastic integral $\int_{[0,1]} W_s^1 dW_s^2$ is given by

$$\Phi(t) = \left(\cosh^2\left(\frac{t}{2}\right) + \sinh^2\left(\frac{t}{2}\right)\right)^{-\frac{1}{2}}. \tag{1.1}$$

In the two-parameter case in [10] (see also [12]) the authors proved that the characteristic function of the integral $\int_{[0,1]^2} W_{\underline{s}}^1 dW_{\underline{s}}^2$ (here W^1 and W^2 denotes two independent Brownian sheets) is given by

$$\Phi(t) = \prod_{k\geq 1} \cosh^{-\frac{1}{2}}\left(\frac{2t}{(2k-1)\pi}\right). \tag{1.2}$$

The aim of the present work is develop a similar study for the fractional Brownian motion. The recent development of the stochastic integration with respect to the fractional Brownian motion (fBm) (see for instance [14]) gives the tools for this analysis. Concretely, we will consider two independent fractional Brownian motion B^H and B^α with Hurst parameter $\alpha \in (0,1)$ and $H > \frac{1}{2}$, and we will find an explicit expression for the characteristic function of the

2000 *Mathematics Subject Classification*: 18A30, 22A05, 22D35, 43A40.

Keywords and phrases: duality, convergence group, nuclear topological group, direct limit, inverse limit.

The authors acknowledge the financial aid received from MCYT, BFM2003-05878 and FEDER funds.

stochastic integral $\int_0^1 B_s^\alpha dB_s^H$. We mention that this kind of integrals appears in the study of stochastic wave equations with fractional noise (see [5]). Related results on the law of this integral have also been proved in [7].

2. Preliminaries: Fractional Brownian motion and Wiener integrals

Let $T = [0, 1]$ be the unit interval and let $\left(B_t^H\right)_{t\in T}$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. Denote by R^H its covariance

$$R^H(t, s) = E\left(B_t^H B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t - s|^{2H}\right).$$

We denote by $\mathcal{H}(H) := \mathcal{H}$ the canonical space of the fractional Brownian motion B^H. That is, $\mathcal{H}$ is the closure of the linear span of the indicator functions $\{1_{[0,t]}, t \in T\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R^H(t, s).$$

The structure of the Hilbert space $\mathcal{H}$ varies upon the values of the Hurst parameter. Let us recall some basic facts about this space.

• if $H > \frac{1}{2}$ the elements of $\mathcal{H}$ may not be functions but distributions of negative order (see [15]). Therefore, it is of interest to know significant subspaces of functions contained in it.

Define the function

$$\theta^H(s, t) = H(2H - 1)|s - t|^{2H-2} \tag{2.1}$$

and let $L^2_H(T)$ be the set of functions $f : T \to \mathbb{R}$ such that

$$\int_T\int_T |f(u)||f(v)|\theta(u, v)dudv < \infty,$$

endowed with the scalar product

$$\langle f, g\rangle_H = \int_T\int_T f(u)g(v)\theta(u, v)dudv\,. \tag{2.2}$$

It has been proved in [15] that $L^2_H(T)$ is a strict subset of $\mathcal{H}$ and the scalar products $\langle\cdot,\cdot\rangle_H$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ coincide on $L^2_H(T)$. Moreover, we have the following inclusion

$$L^{\frac{1}{H}}(T) \subset L^2_H(T) \subset \mathcal{H}\,. \tag{2.3}$$

• If $H < \frac{1}{2}$, then $\mathcal{H}$ is a set of functions; it coincides actually with the set $I_{T-}^{\frac{1}{2}-H}\left(L^2(T)\right)$ where $I_{T-}^{\frac{1}{2}-H}$ is the fractional integral of order $\frac{1}{2} - H$ (see [8], [1], [15]). A significant subspace of $\mathcal{H}$ is the set of Hölder continuous functions of order $\frac{1}{2} - H + \varepsilon$ for all $\varepsilon > 0$,

$$C^{\frac{1}{2}-H+\varepsilon}(T) \subset \mathcal{H} \subset L^2(T) \subset L^{\frac{1}{H}}(T)\,. \tag{2.4}$$

Consider $\mathcal{E}_{\mathcal{H}}$ the class of step functions of the form

$$\varphi(\cdot) = \sum_{i=1}^n a_i 1_{(t_i,t_{i+1}]}(\cdot) \qquad n \geq 1, t_i \in T, a_i \in \mathbb{R}\,. \tag{2.5}$$

It has been proved in [16] that $\mathcal{E}_{\mathcal{H}}$ is dense in $\mathcal{H}$. For $\varphi \in \mathcal{E}_{\mathcal{H}}$ of the form (2.5) we define its Wiener integral with respect to the fBm B^H by

$$\int_0^1 \varphi(s) dB_s^H := \sum_{i=1}^n a_i \left(B_{t_{i+1}}^H - B_{t_i}^H \right) \tag{2.6}$$

The mapping $\varphi \mapsto \int_0^1 \varphi(s) dB_s^H$ provides an isometry between $\mathcal{E}_{\mathcal{H}}$ and the first chaos of the fBm B^H and it can be extended as follows:

• If $H > \frac{1}{2}$, it has been proved in [15] that $\mathcal{E}_{\mathcal{H}}$ is dense in $L_H^2(T)$ with respect to the norm $\|\cdot\|_H$. As a consequence, the Wiener integral $\int_0^1 \varphi(s) dB_s^H$ can be defined in a consistent way as limit in $L^2(\Omega)$ of integrals of elementary functions for any $\varphi \in L_H^2(T)$.

• If $H < \frac{1}{2}$, then $\mathcal{E}_{\mathcal{H}}$ is dense in $\mathcal{H}$ (see [8], [15]) and the integral $\int_0^1 \varphi(s) dB_s^H$ can be defined by isometry for any function $\varphi \in \mathcal{H}$.

We will need in this paper stochastic integrals of the form $\int_T u_s dB_s^H$ where u is a stochastic process independent by B^H. Using the above facts, it follows that this integral can be defined by isometry for any $u \in L^2(\Omega) \times L_H^2(T)$ if $H > \frac{1}{2}$ and for any $u \in L^2(\Omega; \mathcal{H})$ if $H < \frac{1}{2}$.

Remark (2.7). The integral $\int_T u_s dB_s^H$ coincides also with the Skorohod integral introduced in [2], [1] since, by independence, the Malliavin derivative of u with respect to B^H is zero.

More generally, for $H > \frac{1}{2}$, let $L_H^2(T^n)$ be the set of functions $f : T^n \to \mathbb{R}$ such that

$$\int_{T^n} |f(u_1, \ldots, u_n)| |f(v_1, \ldots, v_n)| \left(\prod_{i=1}^n \theta^H(u_i, v_i) \right) du_1 \ldots du_n dv_1 \ldots dv_n < \infty,$$

endowed with the scalar product

$$\langle f, g \rangle_{H^n} = \int_{T^n} f(u_1, \ldots, u_n) g(v_1, \ldots, v_n) \left(\prod_{i=1}^n \theta^H(u_i, v_i) \right) du_1 \ldots du_n dv_1 \ldots dv_n \,. \tag{2.8}$$

Obviously, $L_H^2(T^n)$ is a subset of $\mathcal{H}^{\otimes n}$ and if $f, g \in L_H^2(T^n)$ then we have

$$\langle f, g \rangle_{H^n} = \langle f, g \rangle_{\mathcal{H}^{\otimes n}}.$$

We will denote by $L_{s,H}^2(T^n)$ the set of symmetric functions $f \in L_H^2(T^n)$ and if $f \in L_{s,H}^2(T^2)$ let us introduce the (Hilbert-Schmidt) operator (see [7]) $K_f^H : L_H^2(T) \to L_H^2(T)$ given by

$$\left(K_f^H \varphi \right)(y) = \int_T \int_T f(x, y) \varphi(x') \theta^H(x, x') dx dx' \,. \tag{2.9}$$

Remark (2.10). Note that if f is positive and $H > \frac{1}{2}$, then the operator K_f^H is a positive operator. Indeed, we can write

$$\left(K_f^H \varphi \right)(y) = \int_T A(x', y) \varphi(x') dx'$$

where $A(x', y) = \int_T f(x, y)\theta^H(x, x')dx$ is positive. Thus the eigenvalues of K_f^H are positive.

3. The characteristic function of the double integral

Throughout this section B^H and B^α will denote two independent fractional Brownian motion with parameter H and α respectively. We compute the characteristic function of the random variable

$$S := \int_T B_s^\alpha dB_s^H . \tag{3.1}$$

Note that, when $H > \frac{1}{2}$, the random variables S (3.1) is well-defined since obviously B^α belongs to $L^2(\Omega) \times L_H^2(T)$ for any α. When $H < \frac{1}{2}$, if we assume that $\alpha + H > \frac{1}{2}$, then we have $B^\alpha \in C^{\frac{1}{2}-H+\varepsilon}(T)$. But in the following we will need to restrict ourselves to the situation $H > \frac{1}{2}$.

We start with the following lemma which gives an approximation of the random variable S given by (3.1) when the Hurst parameter of the integrator fbm B^H is bigger than one half.

LEMMA (3.2). *Assume that $H > \frac{1}{2}$ and $\alpha \in (0, 1)$. Denote by*

$$T_n = \sum_{i=0}^{n-1} B_{t_i}^\alpha \left(B_{t_{i+1}}^H - B_{t_i}^H \right) \tag{3.3}$$

where $\pi\colon 0 = t_0 < t_1 < \ldots < t_n = 1$ denotes a partition of $[0, 1]$. Then

$$T_n \to S \text{ in } L^2(\Omega) \text{ as } |\pi| \to 0.$$

Proof. Using the independence of B^α and B^H we can write

$$B_{t_i}^\alpha \left(B_{t_{i+1}}^H - B_{t_i}^H \right) = \int_{t_i}^{t_{i+1}} B_{t_i}^\alpha dB_s^H .$$

To prove the lemma it is enough to prove that

$$\sum_{i=0}^{n-1} B_{t_i}^\alpha 1_{[t_i,t_{i+1}]}(\cdot) \to B_\cdot^\alpha = \sum_{i=0}^{n-1} B_\cdot^\alpha 1_{[t_i,t_{i+1}]}(\cdot) \text{ in } L^2(\Omega) \times L_H^2(T) \text{ as } |\pi| \to 0.$$

Actually in general, to prove the convergence of a sequence of stochastic integrals of divergence type one needs also the convergence of the Malliavin derivatives, but in our caseit is unnecessary due to the independence of the two fBms. We have, using formula (2.2),

$$E\left\|\sum_{i=0}^{n-1}(B^\alpha_{t_i}-B^\alpha_\cdot)1_{[t_i,t_{i+1}]}\right\|^2_H$$
$$=\sum_{i,j=0}^{n-1}H(2H-1)\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}E(B^\alpha_{t_i}-B^\alpha_s)(B^\alpha_{t_j}-B^\alpha_r)|r-s|^{2H-2}drds$$
$$\leq\sum_{i,j=0}^{n-1}H(2H-1)\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|t_i-s|^\alpha|t_j-r|^\alpha|r-s|^{2H-2}drds$$
$$\leq H(2H-1)|\pi|^{2\alpha}\sum_{i,j=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}}|r-s|^{2H-2}drds$$
$$=|\pi|^{2\alpha}\sum_{i,j=0}^{n-1}\langle 1_{[t_i,t_{i+1}]},1_{[t_j,t_{j+1}]}\rangle_H=|\pi|^{2\alpha}$$

and this goes to 0 for every $\alpha\in(0,1)$. □

We will also need to prove the following technical lemma:

LEMMA (3.4). a) *Assume that $\alpha>\frac{1}{2}$ and consider the function*

$$f^H(x,y)=\frac{1}{2}\left((1-x)^{2H}+(1-y)^{2H}-|x-y|^{2H}\right),\qquad x,y\in T=[0,1]. \tag{3.5}$$

Then $f^H\in L^2_{s,\alpha}(T^2)$.

b) *Assume that $H>\frac{1}{2}$ and consider the function*

$$f^\alpha(x,y)=\frac{1}{2}\left(x^{2\alpha}+y^{2\alpha}-|x-y|^{2\alpha}\right),\qquad x,y\in T=[0,1]. \tag{3.6}$$

Then $f^\alpha\in L^2_{s,H}(T^2)$.

Proof. Let us prove first the point 2a); the point 2b) is similar. We have to show that

$$I:=\int_0^1\int_0^1\int_0^1\int_0^1 f^H(x_1,y_1)f^H(x_2,y_2)\theta^\alpha(x_1,x_2)\theta^\alpha(y_1,y_2)dx_1dx_2dy_1dy_2<\infty.$$

Note that

$$\begin{aligned}\left|f^H(x_i,y_i)\right|&=E(B^H_1-B^H_{x_i})(B^H_1-B^H_{y_i})\\&\leq\left(E(B^H_1-B^H_{x_i})^2\right)^{1/2}\left(E(B^H_1-B^H_{y_i})^2\right)^{1/2}\\&=(1-x_i)^H(1-y_i)^H.\end{aligned}$$

The integral I is therefore bounded by

$$\begin{aligned}I&\leq(c(\alpha))^2\int_{[0,1]^4}(1-x_1)^H(1-y_1)^H(1-x_2)^H(1-y_2)^H|x_1-x_2|^{2\alpha-2}\cdot\\&\quad\cdot|y_1-y_2|^{2\alpha-2}dx_1dx_2dy_1dy_2\\&=\left(c(\alpha)\int_0^1\int_0^1(1-x_1)^H(1-x_2)^H|x_1-x_2|^{2\alpha-2}dx_1dx_2\right)^2\end{aligned}$$

with $c(\alpha) = \alpha(2\alpha - 1)$. Now, using the change of variables $z = \frac{x-y}{1-y}$, we get

$$\begin{aligned}
I' &:= \int_0^1 \int_0^1 (1-x)^H (1-y)^H |x-y|^{2\alpha-2} dy dx \\
&= 2 \int_0^1 \int_0^x (1-x)^H (1-y)^H (x-y)^{2\alpha-2} dy dx \\
&= 2 \int_0^1 (1-x)^{2H+2\alpha-1} \left(\int_0^x (1-z)^{-H-2\alpha} z^{2\alpha-2} dz \right) dx \\
&= \frac{1}{H+\alpha} \int_0^1 (1-z)^H z^{2\alpha-2} dz < \infty,
\end{aligned}$$

using that $\alpha > \frac{1}{2}$. □

We state now our main result. The point *b*) allows to consider the situation when the Hurst parameter of the integrand α is less than $\frac{1}{2}$.

THEOREM (3.7). a) *Let $\alpha > \frac{1}{2}$ and $H > \frac{1}{2}$. Then the characteristic function of the random variable S given by (3.1) is*

$$E\left(e^{itS}\right) = \prod_{i \geq 1} \left(\frac{1}{1 + t^2 \mu_i} \right)^{\frac{1}{2}}$$

where $(\mu_i)_{i\geq 1}$ are the eigenvalues of the operator $K^{\alpha}_{f^H}$ given by (2.9) where f^H is defined by (3.5).

b) *Assume that $H > \frac{1}{2}$ and $\alpha \in (0, 1)$. Then the characteristic function of S (3.1) is*

$$E\left(e^{itS}\right) = \prod_{i \geq 1} \left(\frac{1}{1 + t^2 \xi_i} \right)^{\frac{1}{2}} \rightarrow$$

where $(\xi_i)_{i\geq 1}$ are the eigenvalues of the operator $K^{H}_{f^\alpha}$ given by (2.9) and f^α is defined by (3.6).

Remark (3.8). If $\alpha = \frac{1}{2}$, then the operator $K^{\alpha}_{f^H}$ must be replaced by

$$\left(K^{\frac{1}{2}}_{f^H} \varphi\right)(y) = \int_0^1 f^H(x, y) \varphi(x) dx. \tag{3.9}$$

Proof of Theorem (3.7). We prove first a). By Lemma (3.2) we have

$$E\left(e^{itS}\right) = \lim_{n\to\infty} E\left(e^{itT_n}\right)$$

where T_n is given by (3.3) with $t_i = \frac{i}{n}$, for every $i = 0, \ldots, n-1$. Let us compute the characteristic function of the random variable T_n.

We will use the following fact: If X, Y are two independent random variables, then

$$E\left(\Phi(X, Y)/X\right) = \varphi(X)$$

where $\varphi(x) = E(\Phi(x, Y))$. Let us put

$$X = \left(B^{\alpha}_0, B^{\alpha}_{\frac{1}{n}}, \ldots, B^{\alpha}_{\frac{n-1}{n}}\right) \text{ and } Y = \left(B^H_{\frac{1}{n}} - B^H_0, \ldots, B^H_{\frac{n}{n}} - B^H_{\frac{n-1}{n}}\right). \tag{3.10}$$

Therefore, we obtain

$$\varphi(x) = E\left(e^{it\sum_{k=0}^{n-1} x_k Y_k}\right) = e^{-\frac{t^2}{2}x^T A^H x}$$

where the matrix $A^H = \left(A_{k,l}^H\right)_{k,l=0,\dots,n-1}$ is given by

$$\begin{aligned}
A_{k,l}^H &= E\left(B_{\frac{k+1}{n}}^H - B_{\frac{k}{n}}^H\right)\left(B_{\frac{l+1}{n}}^H - B_{\frac{l}{n}}^H\right)\\
&= \frac{1}{2n^{2H}}\left(|k-l+1|^{2H} + |k-l-1|^{2H} - 2|k-l|^{2H}\right).
\end{aligned}$$

We will obtain

$$E\left(e^{itT_n}\right) = E(e^{-\frac{t^2}{2}S_n})$$

where

$$\begin{aligned}
S_n &:= \sum_{k,l=0}^{n-1} A_{k,l}^H B_{\frac{k}{n}}^\alpha B_{\frac{l}{n}}^\alpha\\
&= \sum_{k,l=1}^{n-1} A_{k,l}^H B_{\frac{k}{n}}^\alpha B_{\frac{l}{n}}^\alpha\\
&= \sum_{k,l=1}^{n-1} A_{k,l}^H \left(\sum_{k'=0}^{k-1}\left(B_{\frac{k'+1}{n}}^\alpha - B_{\frac{k'}{n}}^\alpha\right)\right)\left(\sum_{l'=0}^{l-1}\left(B_{\frac{l'+1}{n}}^\alpha - B_{\frac{l'}{n}}^\alpha\right)\right)\\
&= \sum_{k',l'=0}^{n-2}\left(B_{\frac{k'+1}{n}}^\alpha - B_{\frac{k'}{n}}^\alpha\right)\left(B_{\frac{l'+1}{n}}^\alpha - B_{\frac{l'}{n}}^\alpha\right)\sum_{l=l'+1}^{n-1}\sum_{k=k'+1}^{n-1} A_{k,l}^H.
\end{aligned}$$

We calculate first

$$\begin{aligned}
&\sum_{l=l'+1}^{n-1}\sum_{k=k'+1}^{n-1} A_{k,l}^H\\
&= \frac{1}{2n^{2H}}\sum_{l=l'+1}^{n-1}\left[\sum_{k=k'+1}^{n-1}\left(|k-l+1|^{2H} + |k-l-1|^{2H} - 2|k-l|^{2H}\right)\right]\\
&= \frac{1}{2n^{2H}}\sum_{l=l'+1}^{n-1}\left[\sum_{k=k'+1}^{n-1}\left(|k-l+1|^{2H} - |k-l|^{2H}\right) - \sum_{k=k'+1}^{n-1}\left(|k-l|^{2H} - |k-l-1|^{2H}\right)\right]\\
&= \frac{1}{2n^{2H}}\sum_{l=l'+1}^{n-1}\left[|n-l|^{2H} - |k'+1-l|^{2H} - |n-1-l|^{2H} + |k'-l|^{2H}\right]\\
&= \frac{1}{2n^{2H}}\left[\sum_{l=l'+1}^{n-1}\left(|l-k'|^{2H} - |l-1-k'|^{2H}\right) - \sum_{l=l'+1}^{n-1}\left(|l+1-n|^{2H} - |l-n|^{2H}\right)\right]\\
&= \frac{1}{2n^{2H}}\left[(n-k'-1)^{2H} + (n-l'-1)^{2H} - |l'-k'|^{2H}\right]\\
&= f^H\left(\frac{k'+1}{n}, \frac{l'+1}{n}\right)
\end{aligned}$$

where the function f^H is given by (3.5). By combining the above calculations we get

$$S_n = \sum_{k,l=0}^{n-1} f^H\left(\frac{k+1}{n}, \frac{l+1}{n}\right)\left(B^\alpha_{\frac{k+1}{n}} - B^\alpha_{\frac{k}{n}}\right)\left(B^\alpha_{\frac{l+1}{n}} - B^\alpha_{\frac{l}{n}}\right).$$

Let us denote by $(\mu_i)_{i\geq 1}$ the eigenvalues of the operator $K^\alpha_{f^H}$ and by $(g_i)_{i\geq 1}$ the corresponding eigenfunctions. Then, using Lemma (3.4), we can write

$$f^H(x,y) = \sum_{i\geq 1} \mu_i g_i(x) g_i(y)$$

with the vectors $(g_i)_{i\geq 1}$ orthogonal in $L^2_{s,\alpha}(T)$ and the μ_i are square-summable.

The sum S_n becomes

$$\begin{aligned} S_n &= \sum_{k,l=0}^{n-1}\left(\sum_{i\geq 1}\mu_i g_i(\frac{k+1}{n})g_i(\frac{l+1}{n})\right)\left(B^\alpha_{\frac{k+1}{n}} - B^\alpha_{\frac{k}{n}}\right)\left(B^\alpha_{\frac{l+1}{n}} - B^\alpha_{\frac{l}{n}}\right) \\ &= \sum_{i\geq 1}\mu_i\left(\sum_{k=0}^{n-1} g_i(\frac{k+1}{n})\left(B^\alpha_{\frac{k+1}{n}} - B^\alpha_{\frac{k}{n}}\right)\right)^2. \end{aligned}$$

Since $\alpha > \frac{1}{2}$ and $g_i \in L^2_{s,\alpha}(T)$ it follows from [15] that

$$\sum_{k=0}^{n-1} g_i(\frac{k+1}{n})\left(B^\alpha_{\frac{k+1}{n}} - B^\alpha_{\frac{k}{n}}\right) \overset{|\pi|\to 0}{\longrightarrow} \int_0^1 g_i(x)dB^\alpha(x) \text{ in } L^2(\Omega)$$

and therefore we have that

$$S_n \overset{n\to\infty}{\longrightarrow} \sum_{i\geq 1}\mu_i H_i^2 \qquad \text{in } L^1(\Omega)$$

where $\left(H_i = \int_0^1 g_i(x)dB^\alpha(x), i \geq 1\right)$ are independent, standard normal random variables. As a consequence, since the eigenvalues are positive (see Remark (2.10))

$$\begin{aligned} E(e^{itT}) &= E\left(\exp\left(-\frac{t^2}{2}\sum_{i\geq 1}\mu_i H_i^2\right)\right) \\ &= \prod_{i\geq 1} E\left(\exp\left(-\frac{t^2}{2}\mu_i H_i^2\right)\right) \\ &= \prod_{i\geq 1}\left(\frac{1}{1+t^2\mu_i}\right)^{\frac{1}{2}}. \end{aligned}$$

Let us discuss now the point b). We follow the lines of a) by interchanging the roles of X and Y in (3.10). We obtain that $E(e^{itS}) = \lim_{n\to\infty} E\left(e^{-\frac{t^2}{2}S_n}\right)$

where

$$S_n = \sum_{k,l=0}^{n-1} E\left(B^{\alpha}_{\frac{k}{n}} B^{\alpha}_{\frac{l}{n}}\right)\left(B^H_{\frac{k+1}{n}} - B^H_{\frac{k}{n}}\right)\left(B^H_{\frac{l+1}{n}} - B^H_{\frac{l}{n}}\right)$$
$$= \sum_{k,l=0}^{n-1} f^{\alpha}\left(\frac{k}{n}, \frac{l}{n}\right)\left(B^H_{\frac{k+1}{n}} - B^H_{\frac{k}{n}}\right)\left(B^H_{\frac{l+1}{n}} - B^H_{\frac{l}{n}}\right)$$

and where f^{α} is given by (3.6). Now we use Lemma (3.4) b) and we proceed as in the proof of the point a). □

Remark (3.11). As a final comment, let us note that the points *a*). and *b*). of the above theorem agree if α and H are bigger than $\frac{1}{2}$. In fact it can be shown that in this case $K^H_{f^{\alpha}}$ and $K^{\alpha}_{f^H}$ have the same eigenvalues and in this case their characteristic functions coincide term by term. Indeed, let us suppose that $\lambda \neq 0$ is an eigenvalue for $K^H_{f^{\alpha}}$. Then there exists a non identically zero function $\varphi_{\alpha,H} \in L^2_H(T)$ such that

$$(K^H_{f^{\alpha}}\varphi_{\alpha,H})(y) = \lambda\varphi_{\alpha,H}(y)$$

or

$$H(2H-1)\int_0^1\int_0^1 \frac{1}{2}\left(x^{2\alpha} + y^{2\alpha} - |x-y|^{2\alpha}\right)\varphi_{\alpha,H}(x')|x-x'|^{2H-2}dxdx' = \lambda\varphi_{\alpha,H}(y).$$

Let us denote by

$$\chi_{\alpha,H}(y) = \varphi_{H,\alpha}(1-y).$$

It is easy to check that $\chi_{\alpha,H} \in L^2_{\alpha}(T)$ and by using the change of variables $1-x = u$ and $1-x' = v$ we obtain

$$(K^{\alpha}_{f^H}\chi_{\alpha,H})(y) = \lambda\chi_{\alpha,H}(y)$$

which implies that λ is also an eigenvalue for $K^{\alpha}_{f^H}$.

4. The two-parameter case

In this section, we will briefly discuss the case of the fractional Brownian sheet. Let us denote by $\left(B^{\alpha_1,\alpha_2}_{s,t}\right)_{s,t\in T}$ and $\left(B^{H_1,H_2}_{s,t}\right)_{s,t\in T}$ two independent fractional Brownian sheets. We recall that a fractional Brownian sheet $\left(B^{H_1,H_2}_{s,t}\right)_{s,t\in T}$ with Hurst parameters $H_1, H_2 \in (0,1)$ is a centered Gaussian process starting from 0 with covariance given by

$$E\left(B^{H_1,H_2}_{s,t}B^{H_1,H_2}_{u,v}\right) = R^{H_1}(s,u)R^{H_2}(t,v), \qquad s,t,u,v \in T,$$

where R^{H_i} is the covariance of the one-parameter fBm with Hurst index H_i ($i = 1,2$). We refer to [4] or [3] for the basic properties and [17], [18] or [11] for elements of the stochastic calculus with respect to this process. We only point here the following facts:

• the canonical Hilbert space $\mathcal{H}(H_1,H_2)$ of the Gaussian process B^{H_1,H_2} is defined as the closure of the linear vector space generated by the indicator functions $\{1_{[0,s]\times[0,t]}, s,t \in T\}$ with respect to the scalar product

$$\langle 1_{[0,s]\times[0,t]}, 1_{[0,u]\times[0,v]}\rangle_{\mathcal{H}(H_1,H_2)} = R^{H_1}(s,u)R^{H_2}(t,v).$$

• if H_1 or H_2 is bigger than $\frac{1}{2}$, then the elements of $\mathcal{H}(H_1, H_2)$ may not be functions, but distributions. In this case it is convenient to work with the following subspace of $\mathcal{H}(H_1, H_2)$

$$L^2_{H_1,H_2}\left(T^2\right) := L^2_{H_1}(T) \otimes L^2_{H_2}(T)$$

which is a space of functions (and which plays the role played by $L^2_H(T)$ in the one-parameter case). Therefore Wiener integrals with respect to B^{H_1,H_2} can be naturally defined for integrands in $L^2_{H_1,H_2}\left(T^2\right)$.

We prove here the following result.

THEOREM (4.1). a). *Assume that* $H_i > \frac{1}{2}$ *and* $\alpha_i > \frac{1}{2}$, $i = 1, 2$. *Then the characteristic function of the random variable*

$$A := \int_T \int_T B^{\alpha_1,\alpha_2}_{u,v} dB^{H_1,H_2}_{u,v} \tag{4.2}$$

is given by

$$E\left(e^{itA}\right) = \prod_{i,j\geq 1} \left(\frac{1}{1 + t^2 \mu_{i,1}\mu_{j,2}}\right)^{\frac{1}{2}} \tag{4.3}$$

where $(\mu_{i,1})_i$ *are the eigenvalues of the operator* $K^{\alpha_1}_{f^{H_1}}$ *given by (2.9),* $(\mu_{j,2})_j$ *are the eigenvalues of* $K^{\alpha_2}_{f^{H_2}}$ *and* f^{H_1}, f^{H_2} *are defined by* (3.5).

b). *If* $H_i > \frac{1}{2}$ *and* $\alpha_i \in (0, 1)$, *then*

$$E\left(e^{itA}\right) = \prod_{i,j\geq 1} \left(\frac{1}{1 + t^2 \xi_{i,1}\xi_{j,2}}\right)^{\frac{1}{2}}$$

where for $j = 1, 2$, $(\xi_{i,j})_i$ *are the eigenvalues of the operator* $K^{H_j}_{f^{\alpha_j}}$, *where* f^{α_j} *is defined by* (3.6).

Proof. We prove only the first part because the second point is similar. Denote by

$$A_n := \sum_{k,l=0}^{n-1} B^{\alpha_1,\alpha_2}_{\frac{k}{n},\frac{l}{n}} B^{H_1,H_2}(\Delta_{k,l})$$

where

$$B^{H_1,H_2}(\Delta_{k,l}) = B^{H_1,H_2}_{\frac{k+1}{n},\frac{l+1}{n}} - B^{H_1,H_2}_{\frac{k}{n},\frac{l+1}{n}} - B^{H_1,H_2}_{\frac{k+1}{n},\frac{l}{n}} + B^{H_1,H_2}_{\frac{k}{n},\frac{l}{n}}$$

As in Lemma (3.2), we can prove that $A_n \to A$ when $n \to \infty$ in $L^2(\Omega)$ for $\alpha_i > \frac{1}{2}$, $H_i > \frac{1}{2}$, $i = 1, 2$. We obtain, using the methods used in the proof of Lemma (3.2) (see also [10]) that

$$E(e^{itA}) = \lim_{n\to\infty} E\left(e^{itS_n}\right)$$

with

$$S_n = \sum_{k,l=0}^{n-1} \sum_{k',l'=0}^{n-1} f^{H_1}\left(\frac{k+1}{n}, \frac{k'+1}{n}\right) f^{H_2}\left(\frac{l+1}{n}, \frac{l'+1}{n}\right) B^{\alpha_1,\alpha_2}(\Delta_{k,l}) B^{\alpha_1,\alpha_2}(\Delta_{k',l'}).$$

By Lemma (3.4) a) we get that $f^{H_i} \in L^2_{s,\alpha_i}(T)$ $(i=1,2)$ and thus $f_{H_i} = \sum_k \mu_{k,i} g_{k,i}$ where $(g_{k,i})_{k\geq 1}$ are the eigenvectors of $K^{\alpha_i}_{f^{H_i}}$ $(i=1,2)$.

$$S_n = \prod_{i,j\geq 1} \mu_{i,1}\mu_{j,2} \left(\sum_{k,l=0}^{n-1} g_{i,1}(\frac{k+1}{n}) g_{j,2}(\frac{l+1}{n}) B^{\alpha_1,\alpha_2}(\Delta_{k,l}) \right)^2 .$$

Since $g_{i,1} \in L^2_{\alpha_1}(T)$ for every $i \geq 1$ and $g_{j,2} \in L^2_{\alpha_2}(T)$ for every $j \geq 1$, we have that $g_{i,1} \otimes g_{j,2} \in L^2_{\alpha_1,\alpha_2}(T^2)$ and it is not difficult to see that

$$\sum_{k,l=0}^{n-1} g_i(\frac{k+1}{n}) g_j(\frac{l+1}{n}) B^{\alpha_1,\alpha_2}(\Delta_{k,l}) \to_{n\to\infty} \int_T \int_T g_i(x) g_j(y) dB^{\alpha_1,\alpha_2}_{x,y} := H_{i,j}$$

and the random variables $H_{i,j}$ are mutually independent and $N(0,1)$ distributed. The conclusion follows easily. $\square$

Acknowledgements

The first author was partially supported by DGES Grants BFM2003-01345 and BFM2003-00261.

Received May 03, 2006

Final version received December 10, 2006

X. BARDINA
DEPARTAMENT DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
08193-BELLATERRA
BARCELONA
SPAIN
Xavier.Bardina@uab.cat

C. A. TUDOR
SAMOS/MATISSE
CENTRE D'ECONOMIE DE LA SORBONNE
UNIVERSITÉ DE PANTHÉON-SORBONNE PARIS 1
90, RUE DE TOLBIAC
75634 PARIS CEDEX 13
FRANCE
tudor@univ-paris1.fr

REFERENCES

[1] E. ALÒS AND D. NUALART, *Stochastic integration with respect to the fractional Brownian motion*, Stochastics and Stochastics Reports **75** (2002), 129–152

[2] E. ALÒS, O. MAZET AND D. NUALART, *Stochastic calculus with respect to Gaussian processes*, Ann. Probab. **29**, 766–801.

[3] A. AYACHE, S. LEGER AND M. PONTIER, *Drap Brownien fractionnaire*, Potential Analysis, **17** (1), (2002), 31–43.

[4] X. BARDINA, M. JOLIS AND C.A. TUDOR, *Weak convergence to the fractional Brownian sheet and other two-parameter Gaussian processes*, Statsit. Probab. Lett. **34** (3), (2003), 389–400.

[5] B.P. BELINSKY AND P. CAITHAMER, *Energy of a string driven by a Gaussian white noise in time*, J. of Applied Probability **38** (2001), 960–974.

[6] R. BERTHUET, *Loi du logarithme itéré pour certaines intégrales stochastiques*, Ann. Sci. Univ. Clérmont-Ferrand Math. **69** (1981), 9–18.
[7] P. CAITHAMER, *Decoupled double stochastic fractional integrals*, Stochastics and Stochastic Reports **77**)2005), 205–210.
[8] P. CHERIDITO AND D. NUALART, *Stochastic integration of divergence type with respect to the fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$,* Ann. Inst. H. Poincare Probab. Statist. **41** (6) (2003), 1049–1081.
[9] C. HOUDRÉ, V. PÉREZ-ABREU AND A.S. ÜSTÜNEL, Multiple Wiener-Itô integrals: an intriductory survey, Chaos expansions, multiple Wiener-Itô integrals and their applications, CRC Press (1994).
[10] O. JULIÀ AND D. NUALART, *The distribution of a double stochastic integral with respect to two independent Brownian sheets*, Stochastics **25** (1998), 171–182.
[11] I. NOURDIN AND C.A. TUDOR, *Remarks on some linear fractional stochastic equations*, Stochastics **78** (2), (2005), 51–65.
[12] D. NUALART, *On the distribution of a double stochastic integral*, Z. Wahrsch. Verw. Gebiete **65** (1983), 49–60.
[13] D. NUALART, Malliavin calculus and related topics, Springer-Verlag 1995.
[14] V. PÉREZ-ABREU AND C. TUDOR, *Multiple stochastic fractional integrals: a transfer principle for multiple stochastic fractional integrals,* Bol. Soc. Mat. Mexicana **8** (2002), 187–203.
[15] V. PIPIRAS AND M. TAQQU, *Integrations questions related to fractional Brownian motion*, Probab. Theory Related Fields **118** (2), (2000), 251–291.
[16] V. PIPIRAS AND M. TAQQU, *Are classes of deterimistic integrands for the fractional Brownian motion on a finite interval complete?* Bernoulli **7** (6), (2001), 873–897.
[17] C.A. TUDOR AND F. VIENS, *Itô formula and local times for the fractional Brownian sheet*, Electronic Journal of Prob. **8**, paper 14, (2003), 1–31.
[18] C.A. TUDOR AND FREDERI VIENS, *Itô formula for the fractional Brownian sheet using the extended divergence operator*, Preprint SAMOS, Univ. Paris 1, to appear in Stochastics 2004.
[19] M. YOR, *Remarques sur une formule de Paul Lévy*, Lecture Notes in Mathematics **784** (1978), 343–346.