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UNIFORM BEHAVIOR OF FAMILIES OF GALOIS REPRESENTATIONS ON SIEGEL MODULAR FORMS AND THE ENDOSCOPY CONJECTURE

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ABSTRACT. We prove the following uniformity principle: if one of the Galois representations in the family attached to a genus two Siegel cusp form of weight $k > 3$, "semistable" and with multiplicity one, is reducible (for an odd prime p), then all the representations in the family are reducible. This, combined with Serre's conjecture (which is now a theorem) gives a proof of the Endoscopy Conjecture.

1. Introduction

In this article, we will consider a genus two Siegel modular form f of level N and weight $k > 3$ (and multiplicity one) and the family of four dimensional symplectic Galois representations attached to it. We assume also that we are in a case where this family is "semistable". In [D1], we have treated the level 1 case, giving conditions on f to ensure that these Galois representations have generically large image. In particular we have imposed an irreducibility condition on one characteristic polynomial of Frobenius (see [D1], condition (4.8)) to obtain a large image result. Furthermore, with the same irreducibility condition, we showed in [D2], again for the level 1 case, that for every $p > 4k-5$ the p-adic representations are irreducible. The only possible reducible case to be considered is the case of two 2-dimensional irreducible components having the same determinant (all other cases can not occur if f is not of Saito-Kurokawa type, cf. [D1], [D2]), and from the results of [D2] this case can only happen if all characteristic polynomials are reducible, i.e., the 2-dimensional components will have coefficients in the same field as the 4-dimensional representations: the field E generated by the eigenvalues of f.

In section 2 we will generalize the main results of [D1] and [D2] to the semistable case.

One of the consequences of Tate's conjecture on the Siegel threefold is that reducibility for the Galois representations attached to f must be a uniform property: if it is verified at one prime, then all the representations in the family are reducible. In this article, we will prove this uniformity principle:

THEOREM (1.1). *Let f be a genus 2 Siegel cuspidal Hecke eigenform of weight $k > 3$ and level N, having multiplicity one, such that the attached Galois representations $\rho_{f,\lambda}$ are "semistable". Suppose that for some odd prime $\ell_0 \nmid N$,*

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$\lambda_0 \mid \ell_0$, the representation ρ_{f,λ_0} is reducible. Then the representations $\rho_{f,\lambda}$ are reducible for every λ.

Moreover, if this happens, either f is of Saito-Kurokawa type or f is endoscopic.

After excluding the Saito-Kurokawa case, we will prove the result more generally for compatible families of geometric, pure and symplectic four-dimensional Galois representations which are "semistable".

A previous version of this preprint dates from 2003, and since several papers using the results contained there have appeared since then, we prefer to present first (sections 2 and 3) the results contained in that early version, which constitute the "core" of this paper: this corresponds to the proof of "uniformity of reducibility" for "almost every" prime, and with the extra condition $\ell_0 > 4k - 5$. At the end of the paper (section 4) we will indicate how to (easily) remove this assumption on ℓ_0. Finally, we will prove that a standard combination of these results with Serre's conjecture allows us to remove the "almost every" in the result and gives also the Endoscopy Conjecture, i.e., the modularity (up to twist) of the irreducible components.

What follows is a brief description of the tools that will appear in the proofs in the "core" part. We will use (as in [D2], section 4) as starting point Taylor's recent results on the Fontaine-Mazur conjecture and the meromorphic continuation of L-functions for odd two-dimensional Galois representations (see [T2], [T3] and [T4]). Then, we will combine some of the results and techniques in [D1] (in particular the information about the description of the action of inertia obtained via p-adic Hodge theory) with Ribet's results (see [R]) on two-dimensional semistable Galois representations (slightly generalized to higher weights), and finally Cebotarev density theorem, the fundamental theorem of Galois theory, and some group theory will suffice for the proof.

2. Preliminaries

As we already explained, the goal of sections 2 and 3 is to prove a theorem which is weaker than theorem (1.1), namely we will prove the following:

THEOREM (2.1). *Let f be a genus 2 Siegel cuspidal Hecke eigenform of weight $k > 3$ and level N, having multiplicity one, such that the attached Galois representations $\rho_{f,\lambda}$ are "semistable". Suppose that for some prime $\ell_0 > 4k - 5$, $\ell_0 \nmid N$, $\lambda_0 \mid \ell_0$, the representation ρ_{f,λ_0} is reducible. Then the representations $\rho_{f,\lambda}$ are reducible for almost every λ.*

From now on we will make the following assumption: f is a genus 2 level N Siegel cuspidal Hecke eigenform of weight $k > 3$, having multiplicity one, and not of Saito-Kurokawa type (theorem (2.1) is trivial in the Saito-Kurokawa case, where by construction the Galois representations are reducible, with one 2-dimensional and two 1-dimensional components). Let $E = \mathbb{Q}(\{a_n\})$ be the field generated by its Hecke eigenvalues. Then, there is a compatible family of Galois representations constructed by Taylor [T1] and Weissauer [W2] verifying the following:

For any prime number ℓ and any extension λ of ℓ to E we have a continuous Galois representation

$$\rho_{f,\lambda} : G_{\mathbb{Q}} \to \mathrm{GSp}(4, \overline{E}_\lambda)$$

unramified outside ℓN and with characteristic polynomial of $\rho_{f,\lambda}(\mathrm{Frob}\ \ p))$ equal to

$$\mathrm{Pol}_p(x) = x^4 - a_p x^3 + (a_p^2 - a_{p^2} - p^{2k-4})x^2 - a_p p^{2k-3} x + p^{4k-6}$$

for every $p \nmid \ell N$. If $\rho_{f,\lambda}$ is absolutely irreducible, then it is defined over E_λ.

In general, we can not guarantee that the field of definition is E_λ, but the residual representation $\bar{\rho}_{f,\lambda}$ can be formally defined in any case (see [D1]) as a representation defined over the residue field of λ, $\mathbb{F}_\lambda$. Nevertheless, not knowing the field of definition of the representations that we will study is not a serious problem: we can work instead with the "field of coefficients" (*), i.e., the field generated by the coefficients of the characteristic polynomials $\mathrm{Pol}_p(x)$; this field contains all the information we need.

The representations $\rho_{f,\lambda}$ are known to have the following properties (cf [W1], [W2], [D1]): they are pure (Ramanujan conjecture is satisfied) and if $\ell \nmid N$ they are crystalline with Hodge-Tate weights $\{0, k-2, k-1, 2k-3\}$. This last property makes possible, via Fontaine-Laffaille theory, to obtain a precise description of the action of the inertia group at ℓ on the residual representation $\overline{\rho}_{f,\lambda}$: it acts through fundamental characters of level one or two, with exponents equal to the Hodge-Tate weights (see [D1] for more details).

We will need a further restriction: we want the representations to be "semistable" at every prime of N (see the definition below).

Since we will not use the fact that our representations are modular, we can change to the more general setting of a family of four-dimensional symplectic Galois representations $\{\rho_\lambda\}$ with coefficients in a number field E (not necessarily defined over E_λ, see (*)), $\det \rho_\lambda = \chi^{4k-6}$, which are pure, and such that there exists a finite set S with, for every $\ell \notin S$, ρ_λ unramified outside $\{\ell\} \cup S$, crystalline at ℓ with Hodge-Tate weights as above, and "semistable" at primes in S, i.e., verifying the following: ρ_λ restricted to I_q is a unipotent group for every $q \in S$.

For every $p \notin S$ we still denote by $\mathrm{Pol}_p(x)$ the characteristic polynomial of the image of Frob p and a_p the trace of this image. The representations being symplectic, we have the standard factorization

$$\mathrm{Pol}_p(x) = (x^2 - (a_p/2 + \sqrt{d_p})x + p^{2k-3})(x^2 - (a_p/2 - \sqrt{d_p})x + p^{2k-3})\,. \tag{2.2}$$

The results of generically large image and irreducibility proved in previous articles for the level 1 case (see [D1], theorem 4.2, and [D2], theorems 2.1 and 4.1) hold also in this generality:

THEOREM (2.3). *Let $\{\rho_\lambda\}$ be a family of Galois representations verifying the above properties, with $k > 3$. Assume that there is a prime $p \notin S$ such that*

$$\sqrt{d_p} \notin E \tag{2.4}$$

where d_p is defined by formula (2.2). Then for all but finitely many of the primes satisfying

$$d_p \notin (\mathbb{F}_\lambda)^2$$

and, more generally, for all primes λ in E except at most for a set of Dirichlet density 0*, the image of ρ_λ is*

$$A^k_\lambda = \{g \in \mathrm{GSp}(4, \mathcal{O}_{E_\lambda}) : \det(g) \in (\mathbb{Q}^*_\ell)^{4k-6}\},$$

where $\mathcal{O}_{E_\lambda}$ denotes the ring of integers of E_λ.
Keeping condition (2.4) we also have: for every prime $\ell \geq 4k-5$, $\ell \notin S$, $\lambda \mid \ell$, the representation ρ_λ is absolutely irreducible.

Differences from the level 1 case:

The proof of the above results given in [D1] and [D2] extends automatically to the semistable case: recall that the determination of the images is done by considering the image of the residual $\mod \lambda$ representations and eliminating all non-maximal proper subgroups of $\mathrm{GSp}(4, \mathbb{F}_\lambda)$. When considering reducible cases (cf. [D1], sections 4.1 and 4.2) if we allow arbitrary ramification at a finite set S then we have to allow the character appearing as one-dimensional component or determinant of a two-dimensional component of a reducible $\bar{\rho}_\lambda$ to ramify at S, but in the semistable case it is easy to see that this character will not ramify at primes in S. The same applies to the case of image equal to a group G having a reducible index 2 normal subgroup M (cf. [D1], section 4.4), the quadratic Galois character G/M can not ramify at primes of S if we assume semistability. Up to these easy remarks, the whole proof translates word by word to the semistable case.

Remark (2.5). Recall that condition (2.4) was introduced (cf. [D1]) specifically to deal with the case where the image of $\bar{\rho}_\lambda$ is reducible, with two 2-dimensional irreducible components of the same determinant. All other cases of non-maximal image can be discarded, for almost every prime, without using condition (2.4).

Remark (2.6). In [D1], the large images result was proved (for the case of conductor 1) with an additional condition, called "untwisted": this condition was imposed to eliminate the possibility that the projective residual image falls in a smaller symplectic group $\mathrm{PGSp}(4, k')$, k' a proper subfield of k, where k is the field generated by the traces of the residual representation. We have not included a similar condition in the above theorem because in the following lemma, we will explain that this condition is superfluous, i.e., that the case of smaller projective symplectic group can never happen if we assume semistability. In particular, this applies to level 1 Siegel cusp forms, so the condition "untwisted" can be removed from theorem 4.2 of [D1].

LEMMA (2.7). *Let $\{\rho_\lambda\}$ be a compatible family of Galois representations as above (in particular, a semistable family). Then for every prime $q > 2k-2$, $q \notin S$ and Q a prime in E dividing q, if we call G the image of $\bar{\rho}_Q$ and $P(G)$ its projectivization, $P(G)$ lies in* $\mathrm{PGSp}(4, k)$ *if and only if G lies in* $\mathrm{GSp}(4, k)$*, for every subfield k of $\mathbb{F}_Q$.*

Proof. A similar result, for semistable two-dimensional representations, is lemma 2.4 in [R]. The proof given there translates word by word, once we have explained why in our case we also have an element c in the inertia group I_q such that $\chi(c)$ is a generator of $\mathbb{F}^*_q$ and the trace of $\bar{\rho}_Q(c)$ is a non-zero element of $\mathbb{F}_q$ (we know a priori, from the description of the action of I_q, that this trace will be in $\mathbb{F}_q$, what requires a proof is the fact that it is not 0).

We have given in [D1], proposition 3.1, a description of the action of I_q that applies in the current situation, because we are assuming that ρ_Q is symplectic and crystalline with Hodge-Tate weights $\{0, k-2, k-1, 2k-3\}$, and $q > 2k-2$. Let ψ be a level 2 fundamental character, and take $c \in I_q$ such that $\psi(c)$ generates $\mathbb{F}_{q^2}^*$. We have four possibilities for the trace of c, whose values are, after a suitable factorization:

$$(1+\chi(c)^{k-1})(1+\chi(c)^{k-2})$$
$$(\psi(c)^{k-2}+\psi(c)^{(k-2)q})(\psi(c)^{k-1}+\psi(c)^{(k-1)q})$$
$$(1+\psi(c)^{(k-2)+(k-1)q})(1+\psi(c)^{(k-1)+(k-2)q})$$
$$(\psi(c)^{k-2}+\psi(c)^{(k-1)q})(\psi(c)^{k-1}+\psi(c)^{(k-2)q})$$

In all cases, the inequality $q > 2k-2$ implies that these traces are not 0. □

3. Uniformity of reducibility

At this point, we can say that the validity or not of condition (2.4) at some prime $p \notin S$ determines the behavior of the family of representations ρ_λ: If condition (2.4) is satisfied, then we have generically large image and irreducibility for every ℓ sufficiently large compared with the weights.

What happens if condition (2.4) is not satisfied at any prime? This implies that the factorization (2.2) takes place over E, i.e., that for every $p \notin S$, $\mathrm{Pol}_p(x)$ reduces over E. The coefficients of all characteristic polynomials $\mathrm{Pol}_p(x)$ generate an order $\mathcal{O}$ of E, and if we restrict to primes λ not dividing the conductor of this order (we are neglecting only finitely many primes), we see that the field generated by the coefficients of the $\bmod \lambda$ reduction of all the $\mathrm{Pol}_p(x)$ gives the whole $\mathbb{F}_\lambda$. Thus, we see that for almost every prime, the failure of (2.4) implies that $\bar{\rho}_\lambda$ has its image in $\mathrm{GSp}(4, \mathbb{F}_\lambda)$ and not in a smaller symplectic, but all characteristic polynomials reduce over $\mathbb{F}_\lambda$: in this case the image can not be the whole symplectic group, because in the group $\mathrm{GSp}(4, \mathbb{F}_\lambda)$ most of the matrices have IRREDUCIBLE characteristic polynomial, and we know that for almost every prime only one possibility (see remark (2.5) after theorem (2.3), and lemma (2.7)) remains:

LEMMA (3.1). *Let $\{\rho_\lambda\}$ be as in the previous section, and assume that for every $p \notin S$, condition (2.4) is not satisfied. Then, for almost every prime λ, the residual representation $\bar{\rho}_\lambda$ is reducible with two 2-dimensional irreducible components of the same determinant.*

(3.2) A reducible member in the family: Residual consequences. From now on, assume that for a prime $q > 4k-5$, $q \notin S$, $Q \mid q$, the Q-adic representation ρ_Q is reducible. We know (using semistability and purity) that the only possible case is the case of two 2-dimensional irreducible components both with determinant χ^{2k-3}. Thus we have

(3.3) $$\rho_Q \cong \sigma_{1,Q} \oplus \sigma_{2,Q}$$

Since this representation is reducible, the last part of theorem (2.3) implies that condition (2.4) must fail at every prime. Therefore, $\sigma_{1,Q}$ and $\sigma_{2,Q}$ will also have coefficients in E and lemma (3.1) implies that for every prime λ in a cofinite set Λ of primes of E, $\bar{\rho}_\lambda$ will satisfy

$$\bar{\rho}_\lambda \cong \pi_{1,\lambda} \oplus \pi_{2,\lambda}$$

where $\pi_{i,\lambda}$ is an irreducible two dimensional representation defined over $\mathbb{F}_\lambda$ having determinant χ^{2k-3}, for $i = 1, 2$ and for every $\lambda \in \Lambda$.

Moreover, we can determine the image of $\bar{\rho}_\lambda$ for almost every prime in Λ:

LEMMA (3.4). *Keep the above assumptions. For every prime $\lambda \in \Lambda_2$, a cofinite subset of Λ, the image of $\bar{\rho}_\lambda$ is a subgroup of* $\mathrm{GSp}(4, \mathbb{F}_\lambda)$ *conjugated to $M_\lambda = \{A \times B \in \mathrm{GL}(2, \mathbb{F}_{1,\lambda}) \times \mathrm{GL}(2, \mathbb{F}_{2,\lambda}) : \det(A) = \det(B) \in \mathbb{F}_\ell^{2k-3}\}$, where $\mathbb{F}_{1,\lambda}, \mathbb{F}_{2,\lambda} \subseteq \mathbb{F}_\lambda$ are the fields of coefficients of $\pi_{1,\lambda}$ and $\pi_{2,\lambda}$.*

Proof. We have assumed that the representations ρ_λ have a finite ramification set S and they are semistable at every prime $q \in S$. A fortiori, the same applies to their residual components $\pi_{i,\lambda}$. Moreover, these two dimensional representations are irreducible for every $\lambda \in \Lambda$. In a similar situation, Ribet has proved a large image result for $\ell \geq 5$, but he assumes that the action of I_ℓ, given by fundamental characters of level 1 or 2, has weights (i.e., exponents of the fundamental characters) 0 and 1. The main point of his proof is to exclude the dihedral case. In our case, using the information on the Hodge-Tate decomposition, we have this extra condition at I_ℓ also verified by the twisted representation $\pi_{i,\lambda} \otimes \chi^{-k+2}$ for, say, $i = 2$ (cf. [D1],[D2]). On the other hand, for $\pi_{1,\lambda}$ Ribet's result still holds if we restrict to primes $\ell > 4k - 5$, because the weights of the action of I_ℓ being 0 and $2k - 3$, the projectivization of the image if I_ℓ gives a cyclic group of order $(\ell \pm 1)/\gcd(\ell \pm 1, 2k - 3) > 2$, and this is all that you need to follow Ribet's argument. We also have a statement as lemma (2.7) for these two dimensional representations, again adapting lemma 2.4 in [R].

We conclude (cf. [R], theorem 2.5 and the remark after) that for ℓ sufficiently large, the images of both irreducible components are conjugated to the subgroup of matrices in $\mathrm{GL}(2, \mathbb{F}_{i,\lambda})$ with determinant in $\mathbb{F}_\ell^{2k-3}$.

Finally, to prove that the image of $\bar{\rho}_\lambda$ is as we want, it remains to show that the Galois fields corresponding to $P(\pi_{1,\lambda})$ and $P(\pi_{2,\lambda})$ are disjoint (P denotes projectivization). These fields having Galois groups isomorphic to the simple groups $\mathrm{PGL}(2, \mathbb{F}_{i,\lambda})$ or $\mathrm{PSL}(2, \mathbb{F}_{i,\lambda})$, they are either disjoint or equal: the second is not possible because the restriction of these two projective representations to I_ℓ are different, and this proves the result. □

(3.5) A reducible member in the family: λ-adic consequences. In the decomposition (3.3) of ρ_Q it is clear that $\sigma_{1,Q}$ has Hodge-Tate weights $\{0, 2k-3\}$ and $\sigma_{2,Q}$ has Hodge-Tate weights $\{k - 2, k - 1\}$ (or viceversa).

Now, we invoke a result of Taylor (see [T2] and [T3], recall that $q > 4k - 5$) asserting that for a representation such as $\sigma_{1,Q}$ it is possible to find a totally real number field F such that it is modular when restricted to this field, and therefore it agrees on F with the Q-adic motivic irreducible Galois representation (constructed by Blasius and Rogawski) attached to a Hilbert modular form h. This implies that $\sigma_{1,Q}$ appears in the cohomology of the restriction of scalars of the motive M_h associated to h, and it can be checked from the fact

that the Q-adic representation of the absolute Galois group of F attached to h has descended to a 2-dimensional representation of $G_{\mathbb{Q}}$, Cebotarev density theorem, and the fact that all modular Galois representations in the family $\{\sigma_{h,\lambda}\}$ attached to h are known to be irreducible, that the whole family descends to a compatible family $\{\sigma_{1,\lambda}\}$ of Galois representations of $G_{\mathbb{Q}}$ containing $\sigma_{1,Q}$. To do this, one has to write the representation $\sigma_{1,Q}$ as in the proof of theorem 6.6 in [T3], and define the representations $\sigma_{1,\lambda}$ formally in the same way using the strongly compatible families associated to the base change of h to each E_i (recall that, for each i, F/E_i is soluble, cf. [T3]). Then, following an idea suggested to us by R. Taylor, one can check that the virtual representations $\sigma_{1,\lambda}$ constructed this way are true Galois representations by applying the arguments of [T4], section 5.3.3.
It follows from the main result of [T3] that the family $\{\sigma_{1,\lambda}\}$ is a strongly compatible family (cf. [T3] for the definition) of Galois representations. Strong compatibility proves the last steps of the following:

PROPOSITION (3.6). *Let ρ_Q be as above, reducible as in (3.1), and let $\sigma_{1,Q}$ be its irreducible component having Hodge-Tate weights $\{0, 2k-3\}$. Then there exists a compatible family of Galois representations $\{\sigma_{1,\lambda}\}$ containing $\sigma_{1,Q}$, such that for every $\ell \notin S$, $\lambda \mid \ell$, the representation $\sigma_{1,\lambda}$ is unramified outside $\{\ell\} \cup S$, is crystalline at ℓ with Hodge-Tate weights $\{0, 2k-3\}$, and is semistable at every prime of S. Of course, these representations are pure because ρ_Q is.*

Recall that, the representation ρ_λ being symplectic, for every $g \in G_{\mathbb{Q}}$ the roots of $\rho_\lambda(g)$ come in reciprocal pairs: $\{\alpha, \chi^{2k-3}(g)/\alpha, \beta, \chi^{2k-3}(g)/\beta\}$.

The following lemma is a first approach to compare the representations $\sigma_{1,\lambda}$ and ρ_λ:

LEMMA (3.7). *For every $\ell \notin S$, $\lambda \mid \ell$, and every $g \in G_{\mathbb{Q}}$, the roots of $\sigma_{1,\lambda}(g)$ form a pair of reciprocal roots of those of $\rho_\lambda(g)$.*

Proof. From the compatibility of the families $\{\sigma_{1,\lambda}\}$ and $\{\rho_\lambda\}$ and the fact that $\sigma_{1,Q}$ is a component of ρ_Q the lemma is obvious for the dense set of Frobenius elements at unramified places. Then, by continuity and Cebotarev the lemma follows for every element of $G_{\mathbb{Q}}$. □

Recall that Λ_2 denotes the cofinite set of primes of E where lemma (3.4) is satisfied. We will shrink this set again by eliminating a finite set of primes, namely, those primes where the image of $\sigma_{1,\lambda}$ fails to be maximal: in fact, if we call $E' \subseteq E$ the field of coefficients of this family of representations and $\mathcal{O}'$ its ring of integers, using semistability and again the slight modification of the methods of [R] to higher weights (as we did before to obtain lemma (3.4)) we see that for almost every prime $\lambda \in E$ the image of $\sigma_{1,\lambda}$ can be conjugated to the subgroup of $\mathrm{GL}(2, \mathcal{O}'_\lambda)$ of matrices with determinant in $\mathbb{Z}_\ell^{2k-3}$ (after proving the similar result for the residual representations, we apply a lemma of Serre in [S1] that shows that the λ-adic image is also large).

Remark. Here we need to know that the residual representations $\bar{\sigma}_{1,\lambda}$ are almost all irreducible. This follows again from the good properties of the λ-adic family: purity, the fact that they are all crystalline with Hodge-Tate weights

$\{0, 2k-3\}$ (and the uniform description of inertia that one gets from this), and semistability.

Thus, we exclude from Λ_2 the finite set of primes where the image of $\sigma_{1,\lambda}$ fails to be maximal, and we obtain a cofinite set Λ_3 where the residual image of ρ_λ is the full M_λ and the image of $\sigma_{1,\lambda}$ is maximal.

We want to extract more information from the relation derived in lemma (3.7). To start with, we work at the level of residual representations. Observe that the same relation proved in lemma (3.7) holds for the roots of the matrices in the image of the residual representations $\bar{\rho}_\lambda$ and $\bar{\sigma}_{1,\lambda}$:

LEMMA (3.8). *Let λ be a prime in Λ_3. Then in the decomposition $\bar{\rho}_\lambda \cong \pi_{1,\lambda} \oplus \pi_{2,\lambda}$ we have $\pi_{1,\lambda} \cong \bar{\sigma}_{1,\lambda}$.*

Remark. Of course, we should write the above equality with $\pi_{i,\lambda}$ for $i = 1$ or 2. But to fix notation, we will always call $\pi_{1,\lambda}$ the component of $\bar{\rho}_\lambda$ where the inertia group at ℓ acts with weights 0 and $2k-3$ (as we did in section 3.1); this is a good way to distinguish the two components, and of course this is the only component that deserves being compared to $\bar{\sigma}_{1,\lambda}$.

Proof. Take $\lambda \in \Lambda_3$. Let L be the Galois field corresponding to $\bar{\rho}_\lambda$, thus $\mathrm{Gal}(L/\mathbb{Q}) \cong M_\lambda$, and B the one corresponding to $\bar{\sigma}_{1,\lambda}$ Thus if $\mathbb{F}'_\lambda$ is the residue field of $\mathcal{O}'_\lambda$ and $U_\lambda = \{A \in \mathrm{GL}(2, \mathbb{F}'_\lambda) : \det(A) \in \mathbb{F}_\ell^{2k-3}\}$, $\mathrm{Gal}(B/\mathbb{Q}) \cong U_\lambda$. We want to prove that $B \subseteq L$. Let $M = L \cap B$, and consider an element $z \in \mathrm{Gal}(B/M)$. Let $\check{z}$ be a preimage of z in $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$, which we can choose such that $\bar{\rho}_\lambda(\check{z}) = \mathbf{1}_4$ (because it is trivial on $M = L \cap B$). Then, the residual version of lemma (3.7) implies that 1 is a double root of the characteristic polynomial of $\bar{\sigma}_{1,\lambda}(\check{z})$. This implies that the group $\mathrm{Gal}(B/M)$ is unipotent, but this group is a normal subgroup of $\mathrm{Gal}(B/\mathbb{Q}) \cong U_\lambda$, and U_λ has no non-trivial unipotent normal subgroup, thus $B = M$, i.e., $B \subseteq L$.

Then, we have a projection $\phi : \mathrm{Gal}(L/\mathbb{Q}) \to \mathrm{Gal}(B/\mathbb{Q})$, that is to say, ϕ sends M_λ onto U_λ and thus $\bar{\sigma}_{1,\lambda}$ is a quotient of $\bar{\rho}_\lambda$.

Since $\bar{\rho}_\lambda \cong \pi_{1,\lambda} \oplus \pi_{2,\lambda}$ we conclude that $\bar{\sigma}_{1,\lambda} \cong \pi_{i,\lambda}$ with $i = 1$ or 2, and using the information on the Hodge-Tate decompositions we see that $i = 1$. $\square$

(3.9) Proof of Theorem (2.1). We start by observing that part of the proof of lemma (3.8) can be translated to the λ-adic setting. Take $\lambda \in \Lambda_3$, and call L' the (infinite) Galois field corresponding to ρ_λ and B' the one corresponding to $\sigma_{1,\lambda}$. Recall that $\mathrm{Gal}(B'/\mathbb{Q})$ is isomorphic to the subgroup U'_λ of $\mathrm{GL}(2, \mathcal{O}'_\lambda)$ composed of matrices with determinant in $\mathbb{Z}_\ell^{2k-3}$, and therefore again we have a group with no non-trivial unipotent subgroups, thus we conclude from lemma (3.7) as in the proof of lemma (3.8) that $B' \subseteq L'$ and that we have a projection $\phi' : \mathrm{Gal}(L'/\mathbb{Q}) \to \mathrm{Gal}(B'/\mathbb{Q})$. We have $\phi' \circ \rho_\lambda = \sigma_{1,\lambda}$. Let us consider the normal subgroup $\mathrm{Gal}(L'/B')$ of $\mathrm{Gal}(L'/\mathbb{Q})$, i.e., we are considering the restriction $\rho_\lambda|_{\ker \phi'}$. The elements in this subgroup fix B', thus by lemma (3.7) we see that the corresponding matrices in $\mathrm{GSp}(4, \mathcal{O}_\lambda)$ will have 1 as a double root.

On the other hand, we know that the residual representation $\bar{\rho}_\lambda \cong \pi_{1,\lambda} \oplus \pi_{2,\lambda} \cong \bar{\sigma}_{1,\lambda} \oplus \pi_{2,\lambda}$ has maximal image M_λ (see lemmas (3.4) and (3.8)). Moreover, the representation $\sigma_{1,\lambda}$ being a "deformation" of $\pi_{1,\lambda}$ which is disjoint from $\pi_{2,\lambda}$ (in the sense established during the proof of lemma (3.4), i.e., up to the equality of determinants), we see that restricting to $\ker \phi'$ will only shrink the image of $\pi_{2,\lambda}$ by making the determinant trivial, in other words: the residual representation $\overline{\rho_\lambda|_{\ker \phi'}}$ has image

$$\mathrm{SL}(2, \mathbb{F}_{2,\lambda}) \oplus \mathbf{1}_2 \subseteq M_\lambda . \tag{3.10}$$

So, what do we know about $\rho_\lambda|_{\ker \phi'}$? We have determined its residual image and we also know that all matrices in its image have 1 as a double root: this last property extends to the Zariski closure of the image, and using the information we have together with the list of possibilities for this Zariski closure given in [T1], we see that the image of $\rho_\lambda|_{\ker \phi'}$ must be contained in $\mathrm{SL}(2, \mathcal{O}_\lambda) \oplus \mathbf{1}_2$. If we call $\mathcal{O}''_\lambda \subseteq \mathcal{O}_\lambda$ the field generated by the traces of the image of $\rho_\lambda|_{\ker \phi'}$, we can apply a lemma of Serre (cf. [S1]) (and Carayol's lemma for the assertion about the field of definition, cf. [C]) and conclude from (3.10) that the image of $\rho_\lambda|_{\ker \phi'}$ must be conjugated to $\mathrm{SL}(2, \mathcal{O}''{}_\lambda) \oplus \mathbf{1}_2$.

Remark. $\ker \phi'$ fixes B' which is an infinite extension of $\mathbb{Q}$, but Serre's lemma can still be applied because $G_\mathbb{Q}$ is compact and the fixer of B' is a closed subgroup.

Thus, we conclude that $\mathrm{Image}(\rho_\lambda) \subseteq \mathrm{GSp}(4, \bar{E}_\lambda)$ contains a normal subgroup isomorphic to $\mathrm{SL}(2, \mathcal{O}''{}_\lambda) \oplus \mathbf{1}_2$, and the quotient by this subgroup gives U'_λ. But it is easy to see that the normalizer of $\mathrm{SL}(2, \mathcal{O}''{}_\lambda) \oplus \mathbf{1}_2$ in $\mathrm{GSp}(4, \bar{E}_\lambda)$ is contained in the reducible group $\mathrm{GL}(2, \bar{E}_\lambda) \oplus \mathrm{GL}(2, \bar{E}_\lambda)$. Thus, ρ_λ is reducible, for every $\lambda \in \Lambda_3$, and $\sigma_{1,\lambda}$ is one of its two-dimensional irreducible components.

4. From Theorem (2.1) to Theorem (1.1)

Recall that the results and proofs given in previous sections date from 2003. The result of "existence of compatible families" that we proved in section 3.2 were extended in [D3], which is a "sequel" to this paper, where it was applied to prove some cases of the Fontaine-Mazur conjecture.

Moreover, this result is key in the proof of Serre's conjecture given in [D4], [KW1], [K], [D5] and [KW2].

In the case of Hodge-Tate weights $\{0, 1\}$ the results of potential modularity of Taylor do not imply that the representation is motivic, but as observed in [D3] we still can apply the techniques explained in section 3.2 and prove existence of families. In this case, the natural restriction becomes $\ell_0 > 2$.

As in the previous sections, we assume that we are not in the Saito-Kurokawa case (thus the reducible case must be a case of two 2-dimensional irreducible components).

Thus, if ℓ_0 is odd, $\ell_0 \nmid N$, we consider the irreducible component σ_{2,λ_0} of ρ_{λ_0} having Hodge-Tate weights $\{k-2, k-1\}$ and we apply existence of compatible families to $\sigma_{2,\lambda_0} \otimes \chi^{2-k}$ instead of σ_{1,λ_0}.

The rest of the proof given in the previous sections extends word by word, except that σ_{2,λ_0} and σ_{1,λ_0} exchange roles. We conclude that theorem (2.1) is still true if we change the assumption $\ell_0 > 4k - 5$ by $\ell_0 > 2$.

We have shown that for almost every prime λ the representation ρ_λ in our compatible family is reducible, and one of its 2-dimensional irreducible components is $\sigma_{2,\lambda}$, a representation that lies in a strongly compatible family. It is obvious (from the definition of compatibility) that the second components, even if they are a priori defined only for almost every prime, will also form a compatible family, let us call them $\sigma_{1,\lambda}$. Moreover, from the formula:

$$\rho_\lambda \cong \sigma_{1,\lambda} \oplus \sigma_{2,\lambda}$$

we see not only the compatibility of the $\sigma_{1,\lambda}$ but also that the representations $\sigma_{1,\lambda}$ are crystalline if ℓ is not in the ramification set S, of Hodge-Tate weights $\{0, 2k-3\}$, with ramification set contained in S and semistable or unramified locally at primes of S.

At this point, we apply an argument based on Serre's conjecture, which is now a theorem (cf. [D5] and [KW2]). Serre proved in [S2] using his conjecture that compatible families of Galois representations as $\{\sigma_{2,\lambda} \otimes \chi^{2-k}\}$ or $\{\sigma_{1,\lambda}\}$ are modular. The fact that $\{\sigma_{1,\lambda}\}$ is a priori only defined for almost every λ is irrelevant for the argument of Serre: he only needs residual modularity in infinitely many characteristics, not in ALL characteristics (it is a typical patching argument). The essential condition is that the family is compatible, with constant Hodge-Tate weights and uniformly bounded conductor.

We conclude that the families $\{\sigma_{2,\lambda} \otimes \chi^{2-k}\}$ and $\{\sigma_{1,\lambda}\}$ correspond to representations attached to classical modular forms, of weight 2 and $2k-2$, respectively, and obviously this implies in particular that the family $\{\sigma_{1,\lambda}\}$ is also defined for EVERY prime λ. This concludes the proof of theorem (1.1).

Remark (4.1). Observe that in the particular case of a level 1 Siegel cuspform, we conclude irreducibility for $p > 2$ if it is not of Saito-Kurokawa type. The reason is that one of the 2-dimensional components would give rise to a level 1 weight 2 classical modular form.

Remark (4.2). What we have shown is that our result of uniformity of reducibility, combined with Serre's conjecture (now a theorem), implies the truth of the Endoscopy Conjecture in the semistable case.

Remark (4.3). Our result of "uniformity of reducibility" can be extended to the non-semistable case, with the same arguments. In fact, this have been done recently by Skinner and Urban (cf. [SU], section 3). The argument of Serre explained above also applies here, so also the Endoscopy Conjecture follows in this case.

5. Final Remarks

The Galois representations attached to a Siegel cusp form f of level greater than one are known to satisfy the semistability condition when the ramified local components of (the automorphic representation corresponding to) f are of certain particular types (for example, a Steinberg representation), as follows from recent works of Genestier-Tilouine and Genestier (cf. [GT]). Thus, the results in this article apply to these cases. We thank J. Tilouine for pointing out this fact to us.

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OMNIPRESENT EXCEPTIONAL MODULES FOR HYPERELLIPTIC CANONICAL ALGEBRAS

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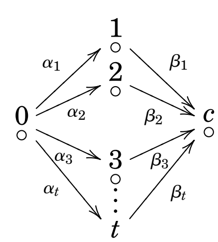
ABSTRACT. A hyperelliptic algebra Λ is a canonical algebra in the sense of Ringel of type $(2, 2, \ldots, 2)$. Using universal extensions we give an explicit description of all but finitely many omnipresent exceptional modules of minimal rank over those algebras. All these modules will be exhibited by matrices involving as coefficients the parameters appearing in the definition of Λ as a bound quiver.

1. Introduction

Let k be a field and $A = kQ/I$ a finite dimensional k-algebra of quiver type. It is well known that a finite dimensional A-module is given by choosing a finite dimensional vector space for each vertex and a linear map for each arrow of Q such that the relations defined by the ideal I are satisfied. We are interested in explicit descriptions of the indecomposable A-modules by matrices.

In [3] Gabriel computed all indecomposable modules for a path algebra $A = kQ$ where Q is a Dynkin quiver. For canonical algebras all indecomposable modules are described in the domestic case in [8] and [6]. Whereas in general one cannot expect explicit descriptions of all indecomposable modules one often can get more information for exceptional modules. Recall that an A-module M is called *exceptional* if $\mathrm{End}(M)$ is a division ring and $\mathrm{Ext}^i_A(M, M) = 0$ for $i \geq 1$. A result of Ringel states that for the path algebra $A = kQ$ of each quiver Q any exceptional module can be exhibited by matrices involving as coefficients 0 and 1 [14]. A similar result was proved by Draexler in [2] for representation-finite algebras.

In this paper we study exceptional modules over a canonical algebra of type $(2, 2, \ldots, 2)$ with t entries. More precisely, let Λ be the quotient kQ/I where Q is the quiver



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and I is the ideal in kQ generated by all paths $\beta_i\alpha_i - \beta_1\alpha_1 - \lambda_i\beta_2\alpha_2,\ i = 3, \dots, t$, where the λ_i are pairwise distinct non-zero elements of k. We always assume that $t \geq 3$. Without loss of generality one can suppose that $\lambda_3 = 1$. If $t \geq 5$ then Λ is of wild representation type and is called *hyperelliptic*. For a general definition of canonical algebras (of quiver type) we refer to [13].

We denote by $\operatorname{mod}(\Lambda)$ the category of finitely generated right Λ-modules. For a Λ-module M the *rank* is given by the formula $\operatorname{rk}(M) = \dim_k M_0 - \dim_k M_c$. It is known from [4] that the indecomposable Λ-modules of rank zero form a separating tubular family in the category of finite dimensional Λ-modules.

Following [11] an exceptional Λ-module M of positive rank is called *omnipresent* if for each non-zero Λ-module S of rank zero there is a non-zero map $M \to S$. In fact one always can assume that a module of positive rank can be considered as omnipresent, possibly considered as a module over a canonical algebra of smaller type (see the discussion in (2.3)). It follows from [11], Proposition 6.3.3, that the rank of an omnipresent exceptional module over a canonical algebra Λ of type $(2, 2, \dots, 2)$, t entries, is greater than or equal to $t - 1$. In this paper we are going to describe by explicit matrices almost all omnipresent exceptional Λ-modules of this *minimal rank* $t - 1$.

In [12] it was shown that for a tubular canonical algebra A each exceptional module over A can be exhibited by matrices involving as coefficients 0, 1 and -1 if A is of type $(3, 3, 3)$, $(2, 4, 4)$ or $(2, 3, 6)$ and by matrices involving as entries $0, 1, -1, \lambda, -\lambda$ and $\lambda - 1$ if A is of type $(2, 2, 2, 2)$ and A is defined by a parameter $\lambda_4 = \lambda$. Also in the hyperelliptic case the described omnipresent exceptional modules of minimal rank will be given by matrices having as coefficients the parameters appearing in the definition of the algebra. Of course one can find other matrices, note however that each exceptional module is uniquely determined by its dimension vector [11], Proposition 4.4.1 and our result can be understood as a solution of a typical problem to find in a certain sense "normal forms".

In order to describe our matrices we will use the following notations. For a natural number n let I_n be the $n \times n$-identity matrix. Define

$$X_n = \left[\begin{array}{ccc} & I_n & \\ \hline 0 & \cdots & 0 \end{array}\right],\ Y_n = \left[\begin{array}{ccc} 0 & \cdots & 0 \\ \hline & I_n & \end{array}\right],\ Z_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \cdots & & & \\ 0 & \cdots & 0 & 0 \end{bmatrix} \in M_{n+1,n}(k)$$

Further, for $i = 3, \dots, t$, we denote $V_n(\lambda_i) = X_n + \lambda_i Y_n \in M_{n+1,n}(k)$ and

$$U_n(\lambda_i) = \begin{bmatrix} 1 & 0 & 0 \cdots & 0 & 0 & 0 \\ \lambda_i & 1 & 0 \cdots & 0 & 0 & 0 \\ 0 & \lambda_i & 1 \cdots & 0 & 0 & 0 \\ & & \ddots & \ddots & & \\ 0 & 0 & 0 \cdots & \lambda_i & 1 & 0 \\ 0 & 0 & 0 \cdots & 0 & \lambda_i & 1 \end{bmatrix} \in M_{n,n}(k).$$

We now define the following right Λ-modules L_n, $n \geq 1$. Put $L_n(0) = k^{(n+1)+(t-2)n}$, $L_n(i) = k^{(t-1)n}$, for $i = 1, \dots, t$, $L_n(c) = k^{n+(t-2)(n-1)}$. The matrices for L_n are given by

$$L_n(\alpha_1) = \begin{bmatrix} X_n & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix}, \quad L_n(\beta_1) = \begin{bmatrix} I_n & & & \\ & X_{n-1} & & \\ & & \ddots & \\ & & & X_{n-1} \end{bmatrix},$$

$$L_n(\alpha_2) = \begin{bmatrix} Y_n & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix}, \quad L_n(\beta_2) = \begin{bmatrix} I_n & & & \\ & Y_{n-1} & & \\ & & \ddots & \\ & & & Y_{n-1} \end{bmatrix},$$

$$L_n(\alpha_i) = \begin{bmatrix} V_n(\lambda_i) & & & Z_n & & \\ & U_n(\lambda_i) & & & & \\ & & \ddots & & & \\ & & & U_n(\lambda_i) & & \\ & & & & \ddots & \\ & & & & & U_n(\lambda_i) \end{bmatrix}, \quad L_n(\beta_i) = \begin{bmatrix} I_n & & & \\ & X_{n-1} & & \\ & & \ddots & \\ & & & X_{n-1} \end{bmatrix}$$

for $i = 3, \ldots, t$. In each matrix there are $(t-2)$ blocks of the same form and in $L_n(\alpha_i)$ the matrix Z_n stands in the first row over the $(i-2)-th$ block $U_n(\lambda_i)$ counting from the left to the right. All the remaining entries are zero.

PROPOSITION (1.1). *For each $n \geq 1$ the module L_n is omnipresent exceptional and of minimal rank $t-1$.*

Our main result will be the following:

THEOREM (1.2). *The matrices given in Theorem* (4.20) *together with those in* 5.2-5.4 *provide explicit descriptions of all but finitely many omnipresent exceptional Λ-modules of minimal rank $t-1$.*

2. Omnipresent exceptional vector bundles for hyperelliptic weighted projective lines

(2.1). In [4] Geigle and Lenzing associated to each canonical algebra of quiver type Λ a *weighted projective line* $\mathbb{X}$ such that the category of *coherent sheaves* $\mathrm{coh}(\mathbb{X})$ is derived equivalent to the category $\mathrm{mod}(\Lambda)$ of finite dimensional Λ-modules. We assume that Λ is of type $(2, 2, \ldots, 2)$, t entries, and defined by parameters $\lambda_3, \ldots, \lambda_t$. Let $\mathbf{L}(\mathbf{p})$ be the rank one abelian group on generators $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_t$ with relations $2\vec{x}_1 = 2\vec{x}_2 = \cdots = 2\vec{x}_t =: \vec{c}$. Then each element of $\mathbf{L}(\mathbf{p})$ can be written in *normal form* $\vec{x} = n\vec{c} + \sum_{i=1}^{t} \varepsilon_i \vec{x}_i$ with $n \in \mathbb{Z}$ and $\varepsilon_i \in \{0, 1\}$. We further denote by $\vec{\omega} = (t-2)\vec{c} - \sum_{i=1}^{t} x_i$ the *dualizing element* of $\mathbf{L}(\mathbf{p})$. We recall that $\mathbf{L}(\mathbf{p})$ is an ordered group with $\sum_{i=1}^{t} \mathbb{N}\vec{x}_i$ as its set of positive elements.

In the hyperelliptic case $\mathbb{X}$ is the projective spectrum of the $\mathbf{L}(\mathbf{p})$ graded algebra

$$S = \frac{k[X_1, \ldots, X_t]}{\langle X_i^2 - X_1^2 - \lambda_i X_2^2,\ i = 3, \ldots, t \rangle}$$

where $deg(X_i) = \vec{x}_i$. The elements $\vec{x}$ of $\mathbf{L}(\mathbf{p})$ are in one to one correspondence with the *line bundles* $\mathcal{O}(\vec{x})$ on $\mathbb{X}$, see [4], Proposition 2.1. Further, for two line bundles $\mathcal{O}_{\mathbb{X}}(\vec{x})$ and $\mathcal{O}_{\mathbb{X}}(\vec{y})$ the space of homomorphisms is given by

$\mathrm{Hom}(\mathcal{O}_{\mathbb{X}}(\vec{x}), \mathcal{O}_{\mathbb{X}}(\vec{y})) = S_{\vec{y}-\vec{x}}$. In particular, if $\vec{x} = n\vec{c} + \sum_{i=1}^{t} \varepsilon_i \vec{x}_i$ is in normal form then we have $\dim_k \mathrm{Hom}(\mathcal{O}, \mathcal{O}_{\mathbb{X}}(\vec{x})) = n+1$ if $n \geq 0$ and $\mathrm{Hom}(\mathcal{O}, \mathcal{O}_{\mathbb{X}}(\vec{x})) = 0$ if $n < 0$.

The derived equivalence $\mathcal{D}^b(\mathrm{coh}(\mathbb{X})) \xrightarrow{\cong} \mathcal{D}^b(\mathrm{mod}(\Lambda))$ is given by a *tilting sheaf* T on $\mathrm{coh}(\mathbb{X})$ of the form

$$T = \mathcal{O} \oplus \bigoplus_{1 \leq i \leq t} \mathcal{O}(\vec{x}_i) \oplus \mathcal{O}(\vec{c})$$

such that $\mathrm{End}(T) \cong \Lambda$.

Denote by $\mathrm{vect}(\mathbb{X})$ (respectively $\mathrm{coh}_0(\mathbb{X})$) the category of vector bundles (respectively finite length sheaves) on $\mathbb{X}$. Moreover, let $\mathrm{coh}_+(\mathbb{X})$ (respectively $\mathrm{coh}_-(\mathbb{X})$) be the full subcategory of $\mathrm{vect}(\mathbb{X})$ formed by all vector bundles whose indecomposable summands F satisfy the condition $\mathrm{Ext}^1_{\mathbb{X}}(T, F) = 0$ (respectively $\mathrm{Hom}_{\mathbb{X}}(T, F) = 0$). Further, we denote by $\mathrm{mod}_+(\Lambda)$ ($\mathrm{mod}_0(\Lambda)$, respectively $\mathrm{mod}_-(\Lambda)$) the full subcategories of $\mathrm{mod}(\Lambda)$ formed by all Λ-modules whose indecomposable summands have positive rank (zero rank, respectively negative rank).

Under the equivalence $\mathcal{D}^b(\mathrm{coh}(\mathbb{X})) \xrightarrow{\cong} \mathcal{D}^b(\mathrm{mod}(\Lambda))$ the subcategory $\mathrm{coh}_+(\mathbb{X})$ (respectively $\mathrm{coh}_0(\mathbb{X})$) corresponds to $\mathrm{mod}_+(\Lambda)$ (respectively $\mathrm{mod}_0(\Lambda)$) by means of $F \mapsto \mathrm{Hom}_{\mathbb{X}}(T, F)$ and $\mathrm{coh}_-(\mathbb{X})[1]$ corresponds to $\mathrm{mod}_-(\Lambda)$ by means of $F[1] \mapsto \mathrm{Ext}^1_{\mathbb{X}}(T, F)$. Note that there are no non-zero morphisms from $\mathrm{mod}_0(\Lambda)$ to $\mathrm{mod}_+(\Lambda)$, from $\mathrm{mod}_-(\Lambda)$ to $\mathrm{mod}_0(\Lambda)$ and from $\mathrm{mod}_-(\Lambda)$ to $\mathrm{mod}_+(\Lambda)$.

The Auslander-Reiten translation $\tau_{\mathbb{X}}$ in $\mathrm{coh}(\mathbb{X})$ is given by shift with the dualizing element $\vec{\omega}$ and gives rise to Serre duality $\mathrm{Ext}^1_{\mathbb{X}}(F, G) \simeq \mathrm{D}\mathrm{Hom}_{\mathbb{X}}(G, \tau_{\mathbb{X}} F)$ where D denotes the standard duality $\mathrm{Hom}_k(-, k)$). For more details concerning sheaves on weighted projective lines we refer to [4].

(2.2). Let $\mathbb{X}$ be a weighted projective line of arbitrary representation type. A sheaf $E \in \mathrm{coh}(\mathbb{X})$ is called *exceptional* if $\mathrm{End}_{\mathbb{X}}(E)$ is a division ring and $\mathrm{Ext}^1_{\mathbb{X}}(E, E) = 0$. It follows from [10] and [7] that $\mathrm{End}_{\mathbb{X}}(E) \simeq k$. Exceptional sheaves on weighted projective lines were studied in [11]. We recall some basic facts.

Definition (2.3). An exceptional vector bundle E on a weighted projective line $\mathbb{X}$ is called *omnipresent* if for each non-zero finite length sheaf $\mathcal{S}$ there is a non-zero map $E \to \mathcal{S}$.

Since each object in $\mathrm{coh}_0(\mathbb{X})$ has finite length and since each vector bundle maps to each simple sheaf of $\tau_{\mathbb{X}}$-order 1 it is sufficient to require in the definition above that there is a non-zero map from E to each simple exceptional sheaf.

Each exceptional vector bundle $E \in \mathrm{coh}(\mathbb{X})$ "is" omnipresent on some weighted projective line $\mathbb{Y}$ of possibly smaller weight type. Indeed, assume that $E \in \mathrm{coh}(\mathbb{X})$ is an exceptional vector bundle which is not omnipresent. Then there is a simple exceptional sheaf $\mathcal{S}$ such that $\mathrm{Hom}_{\mathbb{X}}(E, \mathcal{S}) = 0$. Since there are no non-zero maps from finite length sheaves to vector bundles we conclude by Serre duality $\mathrm{Ext}^1_{\mathbb{X}}(E, \mathcal{S}) \simeq \mathrm{Hom}_{\mathbb{X}}(\mathcal{S}, E(\omega)) = 0$ and therefore E belongs to the left perpendicular category ${}^{\perp}\mathcal{S}$ in the sense of Geigle and Lenzing [5]. Now, the left perpendicular category with respect to all simple exceptional finite length sheaves with the property above is equivalent to a category of coherent sheaves on a weighted projective line $\mathbb{Y}$ of smaller weight type [5], section 9. Since the embedding $\mathrm{coh}(\mathbb{Y}) \hookrightarrow \mathrm{coh}(\mathbb{X})$ is exact and rank preserving E can be considered as an omnipresent exceptional vector bundle on $\mathbb{Y}$. Now, if E is in addition in $\mathrm{coh}_+(\Lambda)$ then the module $\mathrm{Hom}_{\mathbb{X}}(T, E)$ can be considered by "shrinking of arrows" as an omnipresent exceptional module over the canonical algebra corresponding to $\mathbb{Y}$. More precisely, applying the functor $\mathrm{Hom}_{\mathbb{X}}(-, E)$ to the exact sequences

$$0 \to \mathcal{O}(j\vec{x}_i) \to \mathcal{O}((j+1)\vec{x}_i) \to \mathcal{S}_{i,j} \to 0 \tag{2.4}$$

in [5], 2.5.2, one gets isomorphisms for the linear maps $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}((j+1)\vec{x}_i), E) \to \mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(j\vec{x}_i), E)$ for which $\mathrm{Hom}_{\mathbb{X}}(E, S_{i,j}) = 0$. These isomorphisms can be chosen as identities, the corresponding arrows are hence redundant.

PROPOSITION (2.5). *The rank r of an omnipresent exceptional vector bundle on a weighted projective line $\mathbb{X}$ of type $(2, 2, \ldots, 2)$ with t entries satisfies the inequality $r \geq t - 1$.*

This is special case of [11], Theorem 7.3.3.

PROPOSITION (2.6). *Let $\mathbb{X}$ be a weighted projective line of type $(2, 2, \ldots, 2)$ with t entries. Then there is, up to duality and line bundle shift, a unique omnipresent exceptional vector bundle of minimal rank $t - 1$ on $\mathbb{X}$.*

We recall briefly from [11] the construction of those bundles. For this we use the concept of mutations of exceptional sequences (see [15], [10]). The pair $\epsilon = (\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega}))$ is an exceptional pair in $\mathrm{coh}(\mathbb{X})$, i.e. both sheaves are exceptional and $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(-\vec{\omega}), \mathcal{O}(\vec{c})) = 0 = \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(-\vec{\omega}), \mathcal{O}(\vec{c}))$. Now by [11], Theorem 7.3.6, the left mutation of ϵ yields an exceptional bundle E as the middle term of an exact sequence

$$0 \to \mathcal{O}(-\vec{\omega}) \to E \to \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega})) \otimes \mathcal{O}(\vec{c}) \to 0. \tag{2.7}$$

Dually, the right mutation of ϵ yields an exceptional bundle E' as the middle term of an exact sequence

$$0 \to \mathrm{DExt}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega})) \otimes \mathcal{O}(-\vec{\omega}) \to E' \to \mathcal{O}(\vec{c}) \to 0. \tag{2.8}$$

where $\otimes = \otimes_k$. In other terminology these exact sequences are called *universal extensions* (compare section 3).

We have $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega})) \cong \mathrm{DHom}_{\mathbb{X}}(\mathcal{O}, \mathcal{O}(\vec{c} + 2\vec{\omega}))$ and $\vec{c} + 2\vec{\omega} = (t-3)\,\vec{c}$, therefore the vector space $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega}))$ has dimension $t - 2$. Thus $\mathrm{rk}(E) = t - 1 = \mathrm{rk}(E')$. The bundles E and E' are omnipresent and related by vector bundle duality. For the first statement it is sufficient to apply for each simple exceptional sheaf $\mathcal{S}$ the functor $\mathrm{Hom}(-, \mathcal{S})$ to the exact sequences (2.7) and (2.8) and the result follows easily using the exact sequences (2.4) with $\mathcal{S} = \mathcal{S}_{i,j}$. It is proved in [11], Theorem 7.3.6, that each omnipresent exceptional vector bundle on $\mathbb{X}$ is a line bundle shift of E or E'.

3. Universal extensions

(3.1). In [12] we gave a description how the matrices of a module L over a canonical algebra can be computed provided L is a middle term of a universal extension in $\mathrm{mod}(\Lambda)$ and the matrices of the end terms are known. We recall the main idea and specify to the situation of a canonical algebra $\Lambda = kQ/I$ of type $(2, 2, \dots, 2)$, t entries, and defined by parameters $\lambda_3, \dots, \lambda_t$.

Denote by Q_0 the set of vertices and by Q_1 the set of arrows of the quiver Q. Let M and N modules over Λ. Define $C^0(M, N) = \bigoplus_{i\in Q_0} \mathrm{Hom}_k(M(i), N(i))$ and $C^1(M, N) = \bigoplus_{(\gamma:i\to j)} \mathrm{Hom}_k(M(j), N(i))$ where the sum is taken over all arrows $\gamma \in Q_1$. Further, let $\delta_{M,N} : C^0(M, N) \to C^1(M, N)$ be the map given by $\delta_{M,N}((f_i)_{i\in Q_0}) = (f_i M(\gamma) - N(\gamma) f_j)_{\gamma\in Q_1}$. Denote by $U(M, N)$ the subspace of $C^1(M, N)$ consisting of all $(f_\gamma)_{\gamma\in Q_1}$ satisfying the following

$$N(\alpha_1) f_{\beta_1} + f_{\alpha_1} M(\beta_1) + \lambda_i (N(\alpha_2) f_{\beta_2} + f_{\alpha_2} M(\beta_2)) = N(\alpha_i) f_{\beta_i} + f_{\alpha_i} M(\beta_i)$$

for $i = 3, \dots, t$.

Then the image $\Im(\delta_{M,N})$ is contained in $U(M, N)$ and induces a map $\delta_{M,N} : C^0(M, N) \to U(M, N)$ and we have the following result:

PROPOSITION (3.2) ([12], Proposition 3.4). $\mathrm{Ext}^1_\Lambda(M, N) \cong U(M, N)/\Im(\delta_{M,N})$.

(3.3). Let (M, N) be an exceptional pair in $\mathrm{coh}(\mathbb{X})$, i.e. M and N are exceptional objects in $\mathrm{coh}(\mathbb{X})$ and $\mathrm{Hom}_{\mathbb{X}}(M, N) = 0 = \mathrm{Ext}^1_{\mathbb{X}}(M, N) = 0$. Assume further that $\dim_k \mathrm{Ext}^1_{\mathbb{X}}(M, N) = m$. Moreover we suppose that M and N are in $\mathrm{mod}_+(\Lambda)$.

We consider *universal extensions*

$$0 \to N \to L \to \mathrm{Ext}^1_\Lambda(M, N) \otimes M \to 0, \tag{3.4}$$

$$0 \to \mathrm{DExt}^1_\Lambda(M, N) \otimes N \to L' \to M \to 0 \tag{3.5}$$

in $\mathrm{mod}(\Lambda)$ in the sense of [1]. This means in our situation that the connecting homomorphism of the functor $\mathrm{Hom}_\Lambda(M, -)$ applied to (3.4) (respectively of the functor $\mathrm{Hom}_\Lambda(-, N)$ applied to (3.5)) is an isomorphism.

The modules L and L' are exceptional because they are obtained by applying a left respectively right mutation to the exceptional pair (M, N) in the hereditary category $\mathrm{coh}(\mathbb{X})$ [10]. In particular L and L' are uniquely determined by their dimension vectors [11], Theorem 4.4.1, thus by M and N.

If we assume that matrices for M and N are known we have the following receipt to obtain matrices for L and L'. Take a basis $F_1, \dots, F_m$ of a complement of $\Im(\delta_{M,N})$ in $U(M, N)$. Then by Proposition (3.2) the residue classes of the elements $F_1, \dots, F_m$ modulo the image of $\delta_{M,N}$ form a basis of $\mathrm{Ext}^1_\Lambda(M, N)$. Each F_s is of the form $F_s = (f^{(s)}_\gamma)_{\gamma\in Q_1}$ where $f^{(s)}_\gamma : M_j \to N_i$ is a matrix for the arrow $\gamma : i \to j$ (we identify linear maps with matrices).

For each arrow $\gamma : i \to j$ in Q_1 there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N(j) \arrow[r, "\binom{1}{0}"] \arrow[d, "N(\gamma)"'] & L(j) = N(j) \oplus M(j)^{\oplus m} \arrow[r, "(0\ 1)"] \arrow[d, "L(\gamma)"'] & M(j)^{\oplus m} \arrow[r] \arrow[d, "\overline{M}(\gamma)"] & 0 \\
0 \arrow[r] & N(i) \arrow[r, "\binom{1}{0}"'] & L(i) = N(i) \oplus M(i)^{\oplus m} \arrow[r, "(0\ 1)"'] & M(i)^{\oplus m} \arrow[r] & 0,
\end{tikzcd}$$

where $\overline{M}(\gamma) = M(\gamma)^{\oplus m}$, $L(\gamma) = \begin{pmatrix} N(\gamma) & \phi_\gamma \\ 0 & \overline{M}(\gamma) \end{pmatrix}$ and ϕ_γ is a map from $M(j)^{\oplus m}$ to $N(i)$. We put

$$\phi_\gamma = \begin{pmatrix} f_\gamma^{(1)} & f_\gamma^{(2)} & \dots & f_\gamma^{(m)} \end{pmatrix}.$$

Dually, for each arrow $\gamma : i \to j$ in Q_1 there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N(j)^{\oplus m} \arrow[r, "\binom{1}{0}"] \arrow[d, "\overline{N}(\gamma)"'] & L'(j) = N(j)^{\oplus m} \oplus M(j) \arrow[r, "(0\ 1)"] \arrow[d, "L'(\gamma)"'] & M(j) \arrow[r] \arrow[d, "M(\gamma)"] & 0 \\
0 \arrow[r] & N(i)^{\oplus m} \arrow[r, "\binom{1}{0}"'] & L'(i) = N(i)^{\oplus m} \oplus M(i) \arrow[r, "(0\ 1)"'] & M(i) \arrow[r] & 0,
\end{tikzcd}$$

where $\overline{N}(\gamma) = N(\gamma)^{\oplus m}$, $L'(\gamma) = \begin{pmatrix} \overline{N}(\gamma) & \phi'_\gamma \\ 0 & M(\gamma) \end{pmatrix}$ and ϕ'_γ is a map from $M(j)$ to $N(i)^{\oplus m}$. We put

$$\phi'_\gamma = \begin{pmatrix} f_\gamma^{(1)} \\ f_\gamma^{(2)} \\ \dots \\ f_\gamma^{(m)} \end{pmatrix}.$$

THEOREM (3.6). *The modules L and L' described by the matrices above are exceptional.*

For the exact sequence (3.5) the assertion was shown in [12], Proposition 3.7, in the general situation of an arbitrary canonical algebras. The other case follows by dual arguments.

4. Construction of explicit matrices

(4.1). We start with the universal extension in $\mathrm{coh}(\mathbb{X})$

$$0 \to \mathcal{O} \to F \to \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}+\vec{\omega}), \mathcal{O}) \otimes \mathcal{O}(\vec{c}+\vec{\omega}) \to 0 \tag{4.2}$$

which is obtained from (2.7) by shift with $\vec{\omega}$ (note that $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}+\vec{\omega}), \mathcal{O}) \simeq \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega}))$).

For an element $\vec{x} \in \mathbf{L}(\mathbf{p})$ the shift of (4.2) with $\vec{x}$ yields an exact sequence

$$0 \to \mathcal{O}(\vec{x}) \to F(\vec{x}) \to \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}+\vec{\omega}), \mathcal{O}) \otimes \mathcal{O}(\vec{c}+\vec{\omega}+\vec{x}) \to 0 \tag{4.3}$$

which again is a universal extension. Similarly we get a universal extension

$$0 \to \mathrm{DExt}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}+\vec{\omega}), \mathcal{O}) \otimes \mathcal{O}(\vec{x}) \to F'(\vec{x}) \to \mathcal{O}(\vec{c}+\vec{\omega}+\vec{x}) \to 0. \tag{4.4}$$

LEMMA (4.5). *Let $\vec{x} \in \mathbf{L}(\mathbf{p})$ be given in normal form $\vec{x} = n\vec{c} + \sum_{i=1}^{t} \varepsilon_i \vec{x}_i$, $\epsilon_i \in \{0, 1\}$ and assume that exactly r of the entries ε_i are 1, where $0 \leq r \leq t$.*

(a) If $n \geq 0$ and $n + r - 1 \geq 0$ then $\mathcal{O}(\vec{x}), \mathcal{O}(\vec{c} + \vec{\omega} + \vec{x}) \in \mathrm{mod}_+(\Lambda)$ and consequently $F(\vec{x}), F'(\vec{x}) \in \mathrm{mod}_+(\Lambda)$.

(b) If $F(\vec{x}) \in \mathrm{mod}_+(\Lambda)$ (respectively $F'(\vec{x}) \in \mathrm{mod}_+(\Lambda)$) then $n + r - 1 \geq 0$, hence $n \geq 1 - t$.

Proof. Observe first that a bundle G is in $\mathrm{mod}_+(\Lambda)$ if and only if $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), G) = 0$ because for each indecomposable direct summand $\mathcal{O}(\vec{y})$ of T there is a monomorphism $\mathcal{O}(\vec{y}) \hookrightarrow \mathcal{O}(\vec{c})$. We further can assume, up to permutation, that $\vec{x} = n\vec{c} + \sum_{i=t-r+1}^{t} \vec{x}_i$.

(a) By Serre duality we have $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(\vec{x})) \simeq \mathrm{DHom}_{\mathbb{X}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{c} + \vec{\omega})) \simeq \mathrm{DHom}_{\mathbb{X}}(\mathcal{O}, \mathcal{O}(\vec{c} + \vec{\omega} - \vec{x}))$. If $\vec{x}$ is given in normal form as indicated then the normal form of the element $\vec{c} + \vec{\omega} - \vec{x}$ equals $(-1 - n)\vec{c} + \sum_{i=1}^{t-r} \vec{x}_i$. Thus $n \geq 0$ implies $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{x}), \mathcal{O}(\vec{c} + \vec{\omega})) = 0$. Further we have $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(\vec{c} + \vec{\omega} + \vec{x})) \simeq \mathrm{DHom}_{\mathbb{X}}(\mathcal{O}(\vec{c} + \vec{\omega} + \vec{x}), \mathcal{O}(\vec{c} + \vec{\omega})) \simeq \mathrm{DHom}_{\mathbb{X}}(\mathcal{O}, \mathcal{O}(-\vec{x}))$. The normal form of $-\vec{x}$ is $(-n - r)\vec{c} + \sum_{i=t-r+1}^{t} \vec{x}_i$. Since by assumption $-n - r < 0$ we get $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{O}(-\vec{x})) = 0$. Applying the functor $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{c}), -)$ to the exact sequences (4.3) and (4.4) we obtain that $F(\vec{x})$ and $F'(\vec{x})$ belong to $\mathrm{mod}_+(\Lambda)$.

(b) Assume that $F(\vec{x}) \in \mathrm{mod}_+(\Lambda)$ (respectively $F'(\vec{x}) \in \mathrm{mod}_+(\Lambda)$) . Then applying the functor $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}(\vec{c}), -)$ to the exact sequence (4.3) (respectively (4.4)) we see that $\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(\vec{c} + \vec{\omega} + \vec{x})) = 0$. We infer from Serre duality that $\mathrm{Hom}_{\mathbb{X}}(\mathcal{O}, \mathcal{O}(-\vec{x})) = 0$. Since the normal form of $-\vec{x}$ is $(-n - r)\vec{c} + \sum_{i=t-r+1}^{t} \vec{x}_i$ it follows that $-n - r < 0$. $\square$

(4.6). In addition to the matrices described in the introduction we will need the following notations. We define

$$Z_{n+1,n+r} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \cdots & & & \\ 0 & \cdots & 0 & 0 \end{bmatrix} \in M_{n+1,n+r}(k),$$

$$W^{(s)}_{n+1,n+r} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \cdots & & & & & & \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in M_{n+1,n+r}(k)$$

both matrices have only one non-zero entry, for $W^{(s)}_{n+1,n+r}$ this is at the position $(1, s)$.

Take an element $\vec{x} \in \mathbf{L}(\mathbf{p})$, again written in normal form $\vec{x} = n\vec{c} + \sum_{i=1}^{t} \varepsilon_i \vec{x}_i$, and assume that $n \geq 0$ and $n + r - 1 \geq 0$. Moreover, we first assume that $\varepsilon_1 = \varepsilon_2 = 0$. We suppose further that exactly r of the entries ε_i are 1, thus without loss of generality, we can write $\vec{x} = n\vec{c} + \sum_{i=t-r+1}^{t} \vec{x}_i$. The universal extension (4.3) (respectively (4.4)) yield universal extensions in $\mathrm{mod}(\Lambda)$

$$0 \to N \to L \to M^{\oplus\, t-2} \to 0, \tag{4.7}$$

$$0 \to N^{\oplus\, t-2} \to L' \to M^{\oplus\, t-2} \to 0, \tag{4.8}$$

where all terms belong to $\mathrm{mod}_+(\Lambda)$ and $\dim_k \mathrm{Ext}^1_\Lambda(M,N) = m$ (recall that $N = \mathrm{Hom}_{\mathbb{X}}(T, \mathcal{O}(\vec{x}))$ and $M = \mathrm{Hom}_{\mathbb{X}}(T, \mathcal{O}(\vec{x} + \vec{c} + \vec{\omega}))$). Using the formula for the dimension of the homomorphism spaces between line bundles we get the dimension vectors for N and M, namely $N(0) = k^{n+1}$, $N(a) = k^n$ for $a = 1, \dots, t-r$, $N(b) = k^{n+1}$, for $b = t-r+1, \dots, t$, $N(c) = k^n$, and $M(0) = k^{n+r}$, $M(a) = k^{n+r}$ for $a = 1, \dots, t-r$, $M(b) = k^{n+r-1}$ for $b = t-r+1, \dots, t$, $M(c) = k^{n+r-1}$ (recall from (4.1) that $n > 1$ or $n = 0$ and $r > 1$). Explicit matrices for rank 1 modules for arbitrary canonical algebras were described in [12], Proposition 4.3. In our situation one can take the following:

$$\begin{aligned}
&N(\alpha_1) = X_n, \quad N(\alpha_2) = Y_n,\\
&N(\alpha_a) = V_n(\lambda_a) \quad \text{for} \quad a = 3, \dots, t-r,\\
&N(\alpha_b) = U_{n+1}(\lambda_b) \quad \text{for} \quad b = t-r+1, \dots, t,\\
&N(\beta_a) = I_n, \quad \text{for} \quad a = 1, \dots, t-r,\\
&N(\beta_b) = X_n \quad \text{for} \quad b = t-r+1, \dots, t,\\
&M(\alpha_1) = M(\alpha_2) = I_{n+r},\\
&M(\alpha_a) = U_{n+r}(\lambda_a) \quad \text{for} \quad a = 3, \dots, t-r,\\
&M(\alpha_b) = V_{n+r-1}(\lambda_b) \quad \text{for} \quad b = t-r+1, \dots, t\\
&M(\beta_a) = X_{n+r-1} \quad \text{for} \quad a = 1 \quad \text{and} \quad a = 3, \dots, t-r,\\
&M(\beta_2) = Y_{n+r-1}\\
&M(\beta_b) = I_{n+r-1} \quad \text{for} \quad b = t-r+1, \dots, t.
\end{aligned}$$

In order to determine matrices for the modules L and L' as described in section 3 we have to compute a complement of $\Im(\delta_{M,N})$ in $U(M,N)$.

We write an element of $U(M,N)$ as a matrix of matrices $F = \begin{bmatrix} f_{\alpha_1} & f_{\beta_1} \\ f_{\alpha_2} & f_{\beta_2} \\ \dots & \\ f_{\alpha_t} & f_{\beta_t} \end{bmatrix}$ and define the following

$$F_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ Z_{n+1,n+r} & 0 \\ 0 & 0 \\ \dots & \\ \dots & \\ \dots & \\ \dots & \\ 0 & 0 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ Z_{n+1,n+r} & 0 \\ 0 & 0 \\ \dots & \\ \dots & \\ \dots & \\ 0 & 0 \end{bmatrix}, \dots, \quad F_{t-r} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \dots & \\ 0 & 0 \\ Z_{n+1,n+r} & 0 \\ 0 & 0 \\ \dots & \\ 0 & 0 \end{bmatrix}$$

$$F_{t-r+1} = \begin{bmatrix} W^{(1)}_{n+1,n+r} & 0 \\ 0 & 0 \\ W^{(1)}_{n+1,n+r} & 0 \\ \dots & \\ W^{(1)}_{n+1,n+r} & 0 \\ W^{(1)}_{n+1,n+r-1} & 0 \\ \dots & \\ W^{(1)}_{n+1,n+r-1} & 0 \end{bmatrix}, \ F_{t-r+2} = \begin{bmatrix} W^{(2)}_{n+1,n+r} & 0 \\ 0 & 0 \\ W^{(2)}_{n+1,n+r} & 0 \\ \dots & \\ W^{(2)}_{n+1,n+r} & 0 \\ W^{(2)}_{n+1,n+r-1} & 0 \\ \dots & \\ W^{(2)}_{n+1,n+r-1} & 0 \end{bmatrix}, \dots, F_t = \begin{bmatrix} W^{(r)}_{n+1,n+r} & 0 \\ 0 & 0 \\ W^{(r)}_{n+1,n+r} & 0 \\ \dots & \\ W^{(r)}_{n+1,n+r} & 0 \\ W^{(r)}_{n+1,n+r-1} & 0 \\ \dots & \\ W^{(r)}_{n+1,n+r-1} & 0 \end{bmatrix}$$

(For $i = 3, \dots, t-r$ each F_i has only one non-zero matrix, namely $f_{\alpha_i} = Z_{n+1,n+r}$. Moreover, for $s = 1, \dots, r$ in the element F_{t-r+s} we have non-zero matrices for f_{α_a}, $a = 1$ and $a = 3, \dots, t-r$, namely $f_{\alpha_a} = W^{(s)}_{n+1,n+r}$, and for f_{α_b}, $b = t-r+1, \dots, t$, namely $f_{\alpha_b} = W^{(s)}_{n+1,n+r-1}$. It is easily checked that $F_i \in U(M,N)$ for $i = 3, \dots, t$. We denote by H the linear hull of $F_3, \dots, F_t$.

LEMMA (4.9). $H \cap \Im(\delta_{M,N}) = 0$.

Proof. Assume that a linear combination $c_3F_3 + \dots + c_tF_t$ is of the form $\delta_{M,N}(v)$ for some $v \in C^0(M,N)$. Then v is given by linear maps $f_0 : k^{n+r} \to k^{n+1}$, $f_a : k^{n+r} \to k^n$, for $a = 1, 2, \dots, t-r$, $f_b : k^{n+r-1} \to k^{n+1}$, for $b = t-r+1, \dots, t$, and $f_c : k^{n+r-1} \to k^n$. Denote $f_0 = (s_{i,j})_{1\le i\le\, n+1,\, 1\le j\le\, n+r}$, $f_a = (f^{(a)}_{i,j})_{1\le i\le\, n,\, 1\le j\le\, n+r}$, for $a = 1, \dots, t-r$, $f_b = (f^{(b)}_{i,j})_{1\le i\le\, n+1,\, 1\le j\le\, n+r-1}$, for $b = t-r+1, \dots, t$, and $f_c = (t_{i,j})_{1\le i\le\, n,\, 1\le j\le\, n+r-1}$.

Then from $\delta_{M,N}(v) = c_3F_3 + \dots + c_tF_t$ we obtain the following equations

$$f_0M(\alpha_1) - N(\alpha_1)f_1 = c_{t-r+1}W^{(1)}_{n+1,n+r} + c_{t-r+2}W^{(2)}_{n+1,n+r} + \dots + c_tW^{(r)}_{n+1,n+r}, \tag{4.10}$$

$$f_0M(\alpha_2) - N(\alpha_2)f_2 = 0 \tag{4.11}$$

$$\begin{aligned} f_0M(\alpha_a) - N(\alpha_a)f_a = c_aZ_{n+1,n+r} + c_{t-r+1}W^{(1)}_{n+1,n+r} + c_{t-r+2}W^{(2)}_{n+1,n+r} + \dots \\ \dots + c_tW^{(r)}_{n+1,n+r} \quad \text{for} \quad a = 3, \dots, t-r, \end{aligned} \tag{4.12}$$

$$\begin{aligned} f_0M(\alpha_b) - N(\alpha_b)f_b = c_{t-r+1}W^{(1)}_{n+1,n+r-1} + c_{t-r+2}W^{(2)}_{n+1,n+r-1} + \dots \\ \dots + c_tW^{(r)}_{n+1,n+r-1} \quad \text{for} \quad b = t-r+1, \dots, t, \end{aligned} \tag{4.13}$$

$$f_aM(\beta_a) - N(\beta_a)f_c = 0, \quad \text{for} \quad a = 1, \dots, t. \tag{4.14}$$

Taking the matrices for M and N given in section 4.6 we show that all coefficients c_i are zero. First, for $a = 1$ and $a = 3, \dots, t-r$ the equation (4.14) simplifies to $f_aX_{n+r-1} = f_c$ which implies

$$f^{(a)}_{i,j} = t_{i,j} \quad \text{for} \quad 1 \le i \le n,\ 1 \le j \le n+r-1. \tag{4.15}$$

Similarly, again using (4.14) we obtain $f_2Y_{n+r-1} = f_c$ and consequently

$$f^{(2)}_{i,j} = t_{i,j-1} \quad \text{for} \quad 1 \le i \le n,\ 2 \le j \le n+r. \tag{4.16}$$

Now, applying (4.10) we get $f_0 = X_nf_1 + c_{t-r+1}W^{(1)}_{n+1,n+r} + c_{t-r+2}W^{(2)}_{n+1,n+r} + \dots + c_tW^{(r)}_{n+1,n+r}$ and therefore $s_{n+1,j} = 0$ for $j = 1, \dots, n+r$, $s_{i,j} = t_{i,j}$ for $i = 2, \dots, n$ and $j = 1, \dots, n+r-1$, $s_{i,n+r} = f^{(1)}_{i,n+r}$ for $i = 1, \dots, n$, $s_{1,j} = t_{1,j} + c_{t-r+j}$ for $j = 1, \dots, r$ and $s_{1,j} = t_{1,j}$ for $j = r+1, \dots, n+r-1$.

Using this and also (4.11), which simplifies to the equation $f_0 = Y_nf_2$, it follows by induction (from $i = 0$ to $i = n-1$) that

$$f^{(2)}_{n-i,1} = 0 \quad \text{and} \quad t_{n-i,j} = 0 \quad \text{for} \quad j = 1, \dots, n+r-1-i. \tag{4.17}$$

In particular, for $i = n-1$ we obtain $t_{1,j} = 0$ for $j = 1, \dots, r.$ and we infer that

$$c_j = 0 \quad \text{for} \quad j = t-r+1, \dots, t. \tag{4.18}$$

Similarly, it follows by induction (from $i = 1$ to $i = n$) that

$$f^{(1)}_{i,n+r} = 0 \quad \text{and} \quad t_{i,j} = 0 \quad \text{for} \quad j = r+i, \dots, n+r-1. \tag{4.19}$$

Thus we have shown that $f_c = 0$ and $f_0 = 0$. We prove now $c_a = 0$ for $a = 3, \dots t-r$. The equation (4.12) yields $-V(\lambda_a)f_a = c_a Z_{n+1,n+r}$ and from (4.15) and the fact that $f_c = 0$ we see that the in the matrix f_a the first $n+r-1$ columns are zero. Then it follows from $\lambda_a \neq 0$ that $c_a = 0$. This finishes the proof. □

THEOREM (4.20). *The modules L and L' as constructed in* Theorem (3.6) *using the basis elements $F_3, \dots, F_t$ defined in section* 4.6 *are exceptional omnipresent and of rank $t-1$.*

Proof. Using Proposition (3.2) and the fact that $\dim_k \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{O}(\vec{c}), \mathcal{O}(-\vec{\omega}+\vec{c})) = t-2$ Lemma (4.9) implies that $H \oplus \Im(\delta_{M,N}) = U(M, N)$. The elements $F_3, \dots, F_t$ form a basis H and therefore the modules L and L' are exceptional by Theorem (3.6). Obviously we have $\mathrm{rk}(L) = t-1 = \mathrm{rk}(L')$. Moreover L and L' are omnipresent because the bundles E and E' in (2.7) and (2.8) are omnipresent. □

Observe that Proposition (1.1) is a special case of Theorem (4.20), namely for $\vec{x} = n \cdot \vec{c}$, $n \in \mathbb{N}$.

5. Further omnipresent exceptional modules

(5.1). In this chapter we assume that $\vec{x} = n\vec{c} + \sum_{i=1}^{t} \varepsilon_i \vec{x}_i$ is in normal form with $n \geq 0$ but $\varepsilon_1 \neq 0$ or $\varepsilon_2 \neq 0$. We describe rank 1 modules N and M corresponding respectively to line bundles $\mathcal{O}(\vec{x})$ and $\mathcal{O}(\vec{c}+\vec{\omega}+\vec{x})$ in the universal extension (4.3) and (4.4) and give basis elements $F_3, \dots, F_t$ of a complement of $\Im(\delta_{M,N})$ in $U(M, N)$. In this way according to Theorem (3.6) we again define omnipresent exceptional modules of rank $t-1$. The proofs that $H \cap \Im(\delta_{M,N}) = 0$ are similar to that of Lemma (4.9) and are left to the reader.

(5.2). case $\varepsilon_1 = 1,\ \varepsilon_2 = 0$.

We have, up to permutation of the arms of the hypelliptic algebra, $\vec{x} = n\vec{c} + \vec{x}_1 + \sum_{i=t-r+1}^{t} \vec{x}_i$ for some r. In this case the module N differs from that defined in 4.6 only by $N(1) = k^{n+1}$, $N(\alpha_1) = I_{n+1}$ and $N(\beta_1) = X_n$. Moreover we have $M(0) = k^{n+r+1}$, $M(a) = k^{n+r+1}$ for $a = 2, \dots, t-r$, $M(b) = k^{n+r}$ for $b = 1$ and $b = t-r+1, \dots, t$, $M(c) = k^{n+r}$. $M(\alpha_1) = X_{n+r}$, $M(\alpha_2) = I_{n+r}$, $M(\alpha_a) = U_{n+r+1}(\lambda_a)$ for $a = 3, \dots, t-r$, $M(\alpha_b) = V_{n+r}(\lambda_b)$ for $a = t-r+1, \dots, t$, $M(\beta_2) = Y_{n+r}$, $M(\beta_a) = X_{n+r}$ for $a = 3, \dots, t-r$, $M(\beta_b) = X_{n+r}$ for $b = 1$ and $b = t-r+1, \dots, t-r$.

The elements $F_3, \dots, F_{t-r}$ are given as in 4.6 by replacing only the matrix $Z_{n+1,n+r}$ by $Z_{n+1,n+r+1}$ in each F_s and for $s = 1, \dots, r$ we define for F_{t-r+s} by $f_{\alpha_a} = W^{(s)}_{n+1,n+r+1}$ for $a = 1$ and $a = 3, \dots, t-r$ and $f_{\alpha_b} = W^{(s)}_{n+1,n+r}$ for $b = t-r+1, \dots, t$, $f_{\alpha_{a_2}} = 0$ and $f_{\alpha_{\beta_a}} = 0$ for $1 \leq a \leq t$.

(5.3). case $\varepsilon_1 = 0,\ \varepsilon_2 = 1$.

We can assume that $\vec{x} = n\vec{c} + \vec{x}_2 + \sum_{i=t-r+1}^{t} \vec{x}_i$ for some r with $0 \leq r \leq t-2$. Then the modules N and M differ from that in (5.2) only by $N(1) = k^n$, $N(2) = k^{n+1}$, $N(\alpha_1) = X_n$, $N(\alpha_2) = I_{n+1}$, $N(\beta_1) = I_n$, $N(\beta_2) = Y_n$,

$M(1) = k^{n+r+1}$, $M(2) = k^{n+r}$ $M(\alpha_1) = I_{n+r+1}$, $M(\alpha_2) = Y_{n+r}$, $M(\beta_1) = X_{n+r}$, $M(\beta_2) = I_{n+r}$.

The elements $F_3, \dots, F_{t-r}$ are defined as in (5.2). We further define

$$F_{t-r+1} = \begin{bmatrix} W^{(2)}_{n+1,n+r+1} & 0 \\ 0 & 0 \\ W^{(2)}_{n+1,n+r+1} & 0 \\ \cdots & \\ W^{(2)}_{n+1,n+r+1} & 0 \\ W^{(2)}_{n+1,n+r} & 0 \\ \cdots & \\ W^{(2)}_{n+1,n+r} & 0 \end{bmatrix}, \; F_{t-r+2} = \begin{bmatrix} W^{(3)}_{n+1,n+r+1} & 0 \\ 0 & 0 \\ W^{(3)}_{n+1,n+r+1} & 0 \\ \cdots & \\ W^{(3)}_{n+1,n+r+1} & 0 \\ W^{(3)}_{n+1,n+r} & 0 \\ \cdots & \\ W^{(3)}_{n+1,n+r} & 0 \end{bmatrix}, \dots, F_t = \begin{bmatrix} W^{(r+1)}_{n+1,n+r+1} & 0 \\ 0 & 0 \\ W^{(r+1)}_{n+1,n+r+1} & 0 \\ \cdots & \\ W^{(r+1)}_{n+1,n+r+1} & 0 \\ W^{(r+1)}_{n+1,n+r} & 0 \\ \cdots & \\ W^{(r+1)}_{n+1,n+r} & 0 \end{bmatrix}.$$

We remark that

$$\begin{bmatrix} W^{(1)}_{n+1,n+r+1} & 0 \\ 0 & 0 \\ W^{(1)}_{n+1,n+r+1} & 0 \\ \cdots & \\ W^{(1)}_{n+1,n+r+1} & 0 \\ W^{(1)}_{n+1,n+r} & 0 \\ \cdots & \\ W^{(1)}_{n+1,n+r} & 0 \end{bmatrix} \in \Im(\delta_{M,N}).$$

(5.4). case $\varepsilon_1 = 1$, $\varepsilon_2 = 1$.

We can assume that $\vec{x} = n\vec{c} + \vec{x}_1 + \vec{x}_2 + \sum_{i=t-r+1}^{t} \vec{x}_i$ for some r. The module N differs from that in 4.6 only by $N(i) = k^{n+1}$ for $i = 1, 2$, $N(\alpha_1) = N(\alpha_2) = I_{n+1}$, $N(\beta_1) = X_n$, $N(\beta_2) = Y_n$ Moreover, we have $M(0) = k^{n+r+2}$, $M(a) = k^{n+r+2}$ for $a = 3, \dots, t-r$, $M(b) = k^{n+r+1}$ for $b = 1, 2$ and $b = t-r+1, \dots, t$, $M(c) = k^{n+r+1}$. $M(\alpha_1) = X_{n+r+1}$, $M(\alpha_2) = Y_{n+r+1}$, $M(\alpha_a) = U_{n+r+2}(\lambda_a)$ for $a = 3, \dots, t-r$, $M(\alpha_b) = V_{n+r+1}(\lambda_b)$ for $b = t-r+1, \dots, t$, $M(\beta_a) = X_{n+r+1}$ for $a = 3, \dots, t-r$ and $M(\beta_b) = I_{n+r+1}$ for $b = 1, 2$ and $b = t-r+1, \dots, t-r$,

The elements $F_3, \dots, F_{t-r}$ are defined as in 4.6 replacing only in each each F_s the matrix $Z_{n+1,n+r}$ by $Z_{n+1,n+r+2}$. We further define

$$F_{t-r+1} = \begin{bmatrix} W^{(2)}_{n+1,n+r+1} & 0 \\ 0 & 0 \\ W^{(2)}_{n+1,n+r+2} & 0 \\ \cdots & \\ W^{(2)}_{n+1,n+r+2} & 0 \\ W^{(2)}_{n+1,n+r+1} & 0 \\ \cdots & \\ W^{(2)}_{n+1,n+r+1} & 0 \end{bmatrix}, \; F_{t-r+2} = \begin{bmatrix} W^{(3)}_{n+1,n+r+1} & 0 \\ 0 & 0 \\ W^{(3)}_{n+1,n+r+2} & 0 \\ \cdots & \\ W^{(3)}_{n+1,n+r+2} & 0 \\ W^{(3)}_{n+1,n+r+1} & 0 \\ \cdots & \\ W^{(3)}_{n+1,n+r+1} & 0 \end{bmatrix}, \dots, F_t = \begin{bmatrix} W^{(r+1)}_{n+1,n+r+1} & 0 \\ 0 & 0 \\ W^{(r+1)}_{n+1,n+r+2} & 0 \\ \cdots & \\ W^{(r+1)}_{n+1,n+r+2} & 0 \\ W^{(r+1)}_{n+1,n+r+1} & 0 \\ \cdots & \\ W^{(r+1)}_{n+1,n+r+1} & 0 \end{bmatrix}.$$

(5.5). We prove Theorem (1.2). By (2.1) an omnipresent exceptional module of rank $t-1$ over Λ corresponds to an omnipresent exceptional vector bundle of rank $t-1$ over the associated weighted projective line $\mathbb{X}$. Applying Proposition (2.6) such a bundle is obtained by line bundle shift from the bundle E, respectively E', of the exaxt sequence (2.7), respectively (2.8). Now Lemma (4.5) (b) says that if the shifted bundle $F(\vec{x})$ is in $\mathrm{mod}_+(\Lambda)$ (respectively $F'(\vec{x})$ is in $\mathrm{mod}_+(\Lambda)$) and $\vec{x}$ is given in normal form $\vec{x} = n\vec{c} + \sum_{i=1}^{t} \varepsilon_i \vec{x}_i$ then $n \geq 1-t$. Moreover we know from part (a) of Lemma (4.5) that if $n > 0$ or if $n = 0$ and

$r \geq 1$ then the bundles $F(\vec{x})$, $F'(\vec{x})$ are in $\mathrm{mod}_+(\Lambda)$ and are moreover middle terms of universal extensions of objects from $\mathrm{mod}_+(\Lambda)$. In these cases explicit descriptions for their matrices are given in Theorem (4.20) and in (5.2)–(5.4). □

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LARGE ANNIHILATORS IN CAYLEY-DICKSON ALGEBRAS II

This paper is dedicated to the memory of Guillermo Moreno, who made many contributions to the study of Cayley-Dickson algebras.

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ABSTRACT. We establish many previously unknown properties of zero-divisors in Cayley-Dickson algebras. The basic approach is to use a certain splitting that simplifies computations surprisingly.

1. Introduction

Cayley-Dickson algebras are non-associative finite-dimensional $\mathbb{R}$-division algebras that generalize the real numbers, the complex numbers, the quaternions, and the octonions. This paper is a sequel to [DDD], which explores some detailed algebraic properties of these algebras.

Classically, the first four Cayley-Dickson algebras, i.e., $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, are viewed as at least somewhat well-behaved, while the larger Cayley-Dickson algebras are considered to be pathological. There are several different ways of making this distinction. One difference is that the first four algebras do not possess zero-divisors, while the higher algebras do have zero-divisors. Our primary long-term goal is to understand the zero-divisors in as much detail as possible. The specific purpose of this paper is to build directly on the ideas of [DDD] about zero-divisors with large annihilators.

Our motivation for studying zero-divisors is their potential for useful applications in topology; see [Co] for more details. The modern study of Cayley-Dickson algebras has also been taken up in the papers [A], [ES], [M1], [M2], and [M3].

Let A_n be the Cayley-Dickson algebra of dimension 2^n. The central idea of the paper is to use a certain additive splitting of A_n (as expressed indirectly in Definition (3.1)) to simplify multiplication formulas. Multiplication does not quite respect the splitting, but it almost does (see Proposition (4.1)). Theorem (4.5) is the technical heart of the paper; it supplies expressions for multiplication of elements of a codimension 4 subspace of A_n that are simpler than one might expect.

These simple multiplication formulas lead to detailed information about zero-divisors and their annihilators. Section (5) takes a straightforward approach: just write out equations and solve them as explicitly as possible. Our simple multiplication formulas make this feasible. This leads to Theorem (5.10), which almost completely computes the dimension of the annihilator of any element. There are two ways in which the theorem fails to be complete.

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First, it only treats annihilators of elements in a codimension 4 subspace of A_n. Second, rather than determining the dimension of an annihilator precisely, it gives two options, which differ by 4.

We currently have no solution to the first problem. However, in this regard, it was already known that one codimension 2 slice is easy to deal with, so the restriction is really only codimension 2. We intend to address this question in future work.

The second problem has a partial solution in Theorems (6.7) and (6.12), which distinguish between the two possible cases. We find that the answer for A_{n+1} depends inductively not just on an understanding of zero-divisors in A_n but also on a detailed understanding of annihilators in A_n (see Definition (6.1)). Therefore, the description in these theorems is not as explicit as we might like.

Fortunately, we have a complete understanding of zero-divisors and their annihilators in A_4 [KY], Section 3.2, [M1], Corollary 2.14, [DDD], Sections 11 and 12. This allows us to make calculations about zero-divisors in A_5 that are not yet possible for A_n with $n \geq 6$. Section (7) contains the details of these calculations in A_5. Consequently, even though we have not made this result explicit in this article, it is possible to completely understand in geometric terms the zero-divisors in a codimension 4 subspace of A_5. This goes a long way towards completely describing the zero-divisors of A_5.

In addition to the concrete results in Section (7) about A_5, Section (8) gives a number of results about spaces of zero-divisors in A_n for arbitrary n. Consider for a moment only the zero-divisors whose annihilators have dimension differing from the maximum possible dimension by a fixed constant. We show in Theorem (8.12) that, in a certain sense, the space of such zero-divisors does not depend on n. This is a kind of stability result for zero-divisors with large annihilators; it was alluded to in [DDD], Remark 15.8. The basic approach is to use the previous calculations of dimensions of annihilators, together with bounds on the dimensions of annihilators from [DDD] (see Theorem (2.3.2)).

The paper contains a review in Section (2) of the key properties of Cayley-Dickson algebras that we will use. Only some of the material is original; it quotes many results from [DDD] that will be relevant here.

We make one further remark about generalities. Many of our results have hypotheses that eliminate consideration of the classical algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, even though sometimes this is not strictly necessary. From the perspective of this paper, these low-dimensional algebras behave significantly differently than A_n for $n \geq 4$. We eliminate them to avoid awkward but easy special cases.

(1.1) Statement of Results. We now present a summary of our technical results.

Recall that A_n is additively isomorphic to $A_{n-1} \times A_{n-1}$, so elements of A_n are expressions (a, b), where a and b belong to A_{n-1}. See Section (2) for a multiplication formula with respect to this notation. A_n is also isomorphic to $A_{n-1} \times A_{n-1}$ as a real inner product space.

The element $i_n = (0, 1)$ of A_n has many special properties that will be described below. Let $\mathbb{C}_n$ be the linear span of $1 = (1, 0)$ and i_n; it is a subalgebra of A_n isomorphic to the complex numbers. Let $\mathbb{H}_{n+1}$ be the linear span of $(1, 0)$, $(0, 1)$, $(i_n, 0)$, and $(0, i_n)$; it is a subalgebra of A_{n+1} isomorphic to the quaternions.

It turns out that A_n is naturally a Hermitian inner product space. The Hermitian inner product of two elements a and b is the orthogonal projection of ab^* onto $\mathbb{C}_n$. We say that two elements a and b are $\mathbb{C}$-orthogonal if their Hermitian inner product is zero.

Results of [DDD] suggest that we should pay particular attention to elements of A_{n+1} of the form $(a, \pm i_n a)$ with a in the orthogonal complement $\mathbb{C}_n^\perp$ of $\mathbb{C}_n$. Every element of the orthogonal complement $\mathbb{H}_{n+1}^\perp$ of $\mathbb{H}_{n+1}$ can be written uniquely in the form

$$\frac{1}{\sqrt{2}}\Big(a, -i_n a\Big) + \frac{1}{\sqrt{2}}\Big(b, i_n b\Big),$$

where a and b belong to $\mathbb{C}_n^\perp$. We use the notation $\{a, b\}$ for this expression. We insert the ungainly scalars $\frac{1}{\sqrt{2}}$ in order to properly normalize some formulas that appear later. We would like to consider the product of two elements $\{a, b\}$ and $\{x, y\}$ of $\mathbb{H}_{n+1}^\perp$.

PROPOSITION (1.1.1). *Let a, b, x, and y belong to $\mathbb{C}_n^\perp$, and suppose that a and b are $\mathbb{C}$-orthogonal to both x and y. Then*

$$\{a, b\}\{x, y\} = \sqrt{2}\{ax, by\}.$$

This result is proved at the beginning of Section (4). The formula is remarkably simple, but it is not completely general because of the orthogonality assumptions on a, b, x, and y. Most of Section (4) is dedicated to generalizing this formula and understanding the resulting error terms.

Recall that the annihilator $\mathrm{Ann}(x)$ of an element x of A_n is the set of all elements y such that $xy = 0$. Proposition (1.1.1) is the key computational step in the following theorem about annihilators, which is proved in Section (5).

THEOREM (1.1.2). *Let $n \geq 3$, and let a and b be non-zero elements of $\mathbb{C}_n^\perp$. Then the dimension of the annihilator of $\{a, b\}$ is equal to* $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b$ *or* $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b + 4$.

In order to distinguish between the two cases of Theorem (1.1.2), we need the following definition.

Definition (1.1.3). The D-locus is the space of all elements $\{a, b\}$ of A_{n+1} with a and b in $\mathbb{C}_n^\perp$ such that

(1) a and b are $\mathbb{C}$-orthogonal,
(2) a and $\mathrm{Ann}(b)$ are orthogonal, and
(3) b and $\mathrm{Ann}(a)$ are orthogonal.

The following result is proved in Section (6).

THEOREM (1.1.4). *Let a and b be non-zero elements of $\mathbb{C}_n^\perp$. If $\{a, b\}$ does not belong to the D-locus in A_{n+1}, then the dimension of the annihilator of $\{a, b\}$ is* $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b$. *If $\{a, b\}$ belongs to the D-locus in A_{n+1}, then the dimension of the annihilator of $\{a, b\}$ is* $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b + 4$.

For example, if neither a nor b are zero-divisors in A_n and a and b are not $\mathbb{C}$-orthogonal, then $\{a, b\}$ is not a zero-divisor. If neither a nor b are zero-divisors in A_n but are $\mathbb{C}$-orthogonal, then $\{a, b\}$ does belong to the D-locus in A_{n+1} and is thus a zero-divisor. On the other hand, the theorem also shows that if a or b is a zero-divisor, then $\{a, b\}$ is a zero-divisor regardless of whether or not it belongs to the D-locus. In summary, if a and b are $\mathbb{C}$-orthogonal, then $\{a, b\}$ is always a zero-divisor.

In Section (7), we explicitly work out the meaning of Definition (1.1.3) when a and b belong to A_4. The only difficult case occurs when both a and b have non-trivial annihilators, i.e., when both a and b are zero-divisors in A_4. This case is explicitly handled in Theorem (7.5).

Finally, Section (8) provides some general results about zero-divisors with very large annihilators. Recall from [DDD] that the largest annihilators in A_n are $(2^n - 4n + 4)$-dimensional.

Definition (1.1.5). Let $n \geq 4$, and let c be a multiple of 4 such that $0 \leq c \leq 2^n - 4n$. The space $\boldsymbol{T_n^c}$ is the space of elements of unit length in A_n whose annihilators have dimension at least $(2^n - 4n + 4) - c$.

In other words, T_n^c consists of the zero-divisors with annihilators whose dimensions are within c of the maximum.

THEOREM (1.1.6). *Let $n \geq 4$, and let c be a multiple of 4 such that $0 \leq c \leq 2^n - 4n$. If $n \geq \frac{c}{4} + 4$, then T_{n+1}^c is equal to the space of elements of the form $\{a, 0\}$ or $\{0, a\}$ such that a belongs to T_n^c.*

Theorem (1.1.6), which is proved in Section (8), tells us that for sufficiently large n, the space T_{n+1}^c is diffeomorphic to the disjoint union of two copies of T_n^c. The case $c = 0$ was proved in [DDD], Theorem 15.7. An interesting open question is to determine explicitly the geometry of a connected component of T_n^c for n sufficiently large; this connected component depends only on c.

2. Cayley-Dickson algebras

The **Cayley-Dickson algebras** are a sequence of non-associative $\mathbb{R}$-algebras with involution. See [DDD] for a full explanation of their basic properties.

These algebras are defined inductively. We start by defining $\boldsymbol{A_0}$ to be $\mathbb{R}$ with trivial conjugation. Given A_{n-1}, the algebra $\boldsymbol{A_n}$ is defined additively to be $A_{n-1} \times A_{n-1}$. Conjugation in A_n is defined by

$$(a, b)^* = (a^*, -b),$$

and multiplication is defined by

$$(a, b)(c, d) = (ac - d^* b, da + bc^*).$$

One can verify directly from the definitions that A_1 is isomorphic to the complex numbers $\mathbb{C}$; A_2 is isomorphic to the quaternions $\mathbb{H}$; and A_3 is isomorphic to the octonions $\mathbb{O}$.

We implicitly view A_{n-1} as the subalgebra $A_{n-1} \times 0$ of A_n.

(2.1) Complex structure. The element $\boldsymbol{i_n} = (0, 1)$ of A_n enjoys many special properties. One of the primary themes of our long-term project is to fully exploit these special properties.

Let $\mathbb{C}_{\boldsymbol{n}}$ be the $\mathbb{R}$-linear span of $1 = (1, 0)$ and i_n. It is a subalgebra of A_n isomorphic to $\mathbb{C}$.

LEMMA (2.1.1) (DDD, Proposition 5.3). *Under left multiplication, A_n is a $\mathbb{C}_n$-vector space. In particular, if α and β belong to $\mathbb{C}_n$ and x belongs to A_n, then $\alpha(\beta x) = (\alpha\beta)x$.*

As a consequence, the expression $\alpha\beta x$ is unambiguous; we will usually simplify notation in this way.

The **real part $\mathbf{Re}(x)$** of an element x of A_n is defined to be $\frac{1}{2}(x + x^*)$, while the **imaginary part $\mathbf{Im}(x)$** is defined to be $x - \mathrm{Re}(x)$.

The algebra A_n becomes a positive-definite real inner product space when we define $\langle \boldsymbol{a}, \boldsymbol{b} \rangle_{\mathbb{R}} = \mathrm{Re}(ab^*)$ [DDD], Proposition 3.2. If a and b are imaginary and orthogonal, then ab is imaginary. Hence, $ba = b^* a^* = (ab)^* = -ab$. In other words, orthogonal imaginary elements anti-commute. A simple calculation shows that aa^* and $a^* a$ are both equal to $\langle a, a \rangle_{\mathbb{R}}$ for all a in A_n [DDD], Lemma 3.6.

We will need the following slightly technical result.

LEMMA (2.1.2). *Let x and y be elements of A_n such that y is imaginary. Then x and xy are orthogonal.*

Proof. We wish to show that $\mathrm{Re}(x(xy)^*)$ equals zero. This equals $-\mathrm{Re}((xx^*)y)$ by [DDD], Lemmas 2.6 and 2.8, which is zero because y is imaginary and because xx^* is real. □

The real inner product allows us to define a positive-definite Hermitian inner product on A_n by setting $\langle \boldsymbol{a}, \boldsymbol{b} \rangle_{\mathbb{C}}$ to be the orthogonal projection of ab^* onto the subspace $\mathbb{C}_n$ of A_n [DDD], Proposition 6.3. We say that two elements a and b are **$\mathbb{C}$-orthogonal** if $\langle a, b \rangle_{\mathbb{C}} = 0$.

We will frequently consider the subspace $\mathbb{C}_{\boldsymbol{n}}^{\perp}$ of A_n; it is the orthogonal complement of $\mathbb{C}_n$ (with respect either to the real or to the Hermitian inner product). Note that $\mathbb{C}_n^{\perp}$ is a $\mathbb{C}_n$-vector space; in other words, if a belongs to $\mathbb{C}_n^{\perp}$ and α belongs to $\mathbb{C}_n$, then αa also belongs to $\mathbb{C}_n^{\perp}$ [DDD], Lemma 3.8.

LEMMA (2.1.3) ([DDD], Lemmas 6.4 and 6.5). *If a belongs to $\mathbb{C}_n^{\perp}$, then left multiplication by a is $\mathbb{C}_n$-conjugate-linear in the sense that $a \cdot \alpha x = \alpha^* \cdot ax$ for any x in A_n and any α in $\mathbb{C}_n$. Moreover, left multiplication is anti-Hermitian in the sense that $\langle ax, y \rangle_{\mathbb{C}} = -\langle x, ay \rangle_{\mathbb{C}}^*$.*

Similar results hold for right multiplication by a. See also [M2], Lemma 2.1, for a different version of the claim about conjugate-linearity.

The conjugate-linearity of left and right multiplication is fundamental to many later calculations. To emphasize this point, we provide some computational consequences. The next lemma can be interpreted as a restricted kind of bi-conjugate-linearity for multiplication.

LEMMA (2.1.4). *Let a and b be $\mathbb{C}$-orthogonal elements of $\mathbb{C}_n^{\perp}$, and let α and β belong to $\mathbb{C}_n$. Then $\alpha a \cdot \beta b = \alpha^* \beta^* \cdot ab$.*

Proof. By left conjugate-linearity, $\alpha a \cdot \beta b = \beta^*(\alpha a \cdot b)$. Use right conjugate-linearity twice to compute that $\beta^*(\alpha a \cdot b) = \beta^*(ab \cdot \alpha)$. Because a and b are $\mathbb{C}$-orthogonal, ab belongs to $\mathbb{C}_n^\perp$. Therefore, $\beta^*(ab \cdot \alpha) = \beta^*(\alpha^* \cdot ab)$ by left conjugate-linearity again. Finally, this equals $\alpha^*\beta^* \cdot ab$ by Lemma (2.1.1). □

Norms of elements in A_n are defined with respect to either the real or Hermitian inner product: $| a |= \sqrt{\langle a, a\rangle_\mathbb{R}} = \sqrt{\langle a, a\rangle_\mathbb{C}} = \sqrt{aa^*}$; this makes sense because aa^* is always a non-negative real number [DDD], Lemma 3.6. Note also that $| a |=| a^* |$ for all a. We will frequently use that $a^2 = - | a |^2$ if a is an imaginary element of A_n.

LEMMA (2.1.5). *Let a belong to $\mathbb{C}_n^\perp$, and let α and β belong to $\mathbb{C}_n$. Then $\alpha a \cdot \beta a = - | a |^2 \alpha\beta^*$.*

Proof. Follow the same general strategy as in the proof of Lemma (2.1.4). However, instead of using that ab belongs to $\mathbb{C}_n^\perp$, use that $a^2 = - | a |^2$ is real. □

One consequence of Lemma (2.1.5) is that $| \alpha a |=| \alpha || a |$ if α belongs to $\mathbb{C}_n$ and a belongs to $\mathbb{C}_n^\perp$. This follows from the computation $\alpha a \cdot \alpha a = - | a |^2 \alpha\alpha^*$.

(2.2) The subalgebra $\mathbb{H}_n$.

Definition (2.2.1). Let $\mathbb{H}_n$ be the $\mathbb{R}$-linear span of the elements 1, i_n, i_{n-1}, and $i_{n-1}i_n$ of A_n.

The notation reminds us that $\mathbb{H}_n$ is a subalgebra isomorphic to the quaternions. Many of the results that follow refer to $\mathbb{H}_n$ and its orthogonal complement $\mathbb{H}_n^\perp$.

In terms of the product $A_n = A_{n-1} \times A_{n-1}$, $\mathbb{H}_n$ is the $\mathbb{R}$-linear span of $(1, 0)$, $(0, 1)$, $(i_{n-1}, 0)$, and $(0, i_{n-1})$. By inspection, $\mathbb{H}_n$ is a $\mathbb{C}_n$-linear subspace of A_n. It is also equal to $\mathbb{C}_{n-1} \times \mathbb{C}_{n-1}$. Also, $\mathbb{H}_n^\perp$ and $\mathbb{C}_{n-1}^\perp \times \mathbb{C}_{n-1}^\perp$ are equal as subspaces of A_n.

(2.3) Zero-divisors and annihilators. A **zero-divisor** is a non-zero element a of A_n such that there exists another non-zero element b in A_n with $ab = 0$. The **annihilator Ann(a)** of a is the set of all elements b such that $ab = 0$. In other words, Ann(a) is the kernel of left multiplication by a.

LEMMA (2.3.1) ([M1], Corollary 1.9 and [DDD], Lemma 9.5). *If a is a zero-divisor in A_n, then a belongs to $\mathbb{C}_n^\perp$.*

THEOREM (2.3.2) ([DDD], Theorem 9.8 and Proposition 9.10). *The dimension of any annihilator in A_n is a multiple of 4 and is at most $2^n - 4n + 4$.*

See also [M1], Corollary 1.17, for another proof of the first claim.

LEMMA (2.3.3). *Let a belong to $\mathbb{C}_n^\perp$. For any b in A_n, the product ab is orthogonal to* Ann(a).

Proof. Let c belong to Ann(a). Use Lemma (2.1.3) to deduce that $\langle ab, c\rangle_\mathbb{C} = -\langle b, ac\rangle_\mathbb{C}^*$. This equals zero because $ac = 0$. □

Let **Im(a)** be the image of left multiplication by a. Lemma (2.3.3) implies that Im(a) is the orthogonal complement of Ann(a) in A_n.

(2.4) Projections. We still need a few technical definitions and results. We provide complete proofs for the following results because their proofs do not already appear elsewhere.

Definition (2.4.1). For any a in A_n, let $\boldsymbol{\pi_{\mathbb{C}}(a)}$ be the orthogonal projection of a onto $\mathbb{C}_n$, and let $\boldsymbol{\pi_{\mathbb{C}}^{\perp}(a)}$ be the orthogonal projection of a onto $\mathbb{C}_n^{\perp}$.

By definition, $\pi_{\mathbb{C}}(ab^*)$ equals $\langle a, b\rangle_{\mathbb{C}}$ for any a and b.

LEMMA (2.4.2). *Let a and b belong to A_n. Let $b = b' + b''$, where b' is the $\mathbb{C}$-orthogonal projection of b onto the $\mathbb{C}$-linear span of a and where b'' is the $\mathbb{C}$-orthogonal projection of b onto the $\mathbb{C}$-orthogonal complement of a. Then $\pi_{\mathbb{C}}(ab) = ab'$, and $\pi_{\mathbb{C}}^{\perp}(ab) = ab''$. Similarly, $\pi_{\mathbb{C}}(ba) = b'a$, and $\pi_{\mathbb{C}}^{\perp}(ba) = b''a$.*

Proof. Note that $ab = ab' + ab''$. The first term belongs to $\mathbb{C}_n$ by Lemma (2.1.5), while the second term belongs to $\mathbb{C}_n^{\perp}$ because a and b'' are $\mathbb{C}$-orthogonal. Similarly, $ba = b'a + b''a$, where $b'a$ belongs to $\mathbb{C}_n$ and $b''a$ belongs to $\mathbb{C}_n^{\perp}$. □

COROLLARY (2.4.3). *For any a and b in $\mathbb{C}_n^{\perp}$, $\pi_{\mathbb{C}}(ab) = \pi_{\mathbb{C}}(ba)^*$ and $\pi_{\mathbb{C}}^{\perp}(ab) = -\pi_{\mathbb{C}}^{\perp}(ba)$.*

Proof. Write $b = b' + b''$, where b' is the $\mathbb{C}$-orthogonal projection of b onto a and where b'' is the $\mathbb{C}$-orthogonal projection of b onto the $\mathbb{C}$-orthogonal complement of a. By Lemma (2.4.2), $\pi_{\mathbb{C}}(ab) = ab'$ and $\pi_{\mathbb{C}}(ba) = b'a$. It follows from Lemma (2.1.5) that $(ab')^* = b'a$. This finishes the first claim.

For the second claim, Lemma (2.4.2) implies that $\pi_{\mathbb{C}}^{\perp}(ab) = ab''$ and $\pi_{\mathbb{C}}^{\perp}(ba) = b''a$. Because a and b'' are imaginary and orthogonal, $ab'' = -b''a$. □

COROLLARY (2.4.4). *Let a belong to A_n, and let α belong to $\mathbb{C}_n$. Then $\pi_{\mathbb{C}}(\alpha a) = \alpha\pi_{\mathbb{C}}(a) = \pi_{\mathbb{C}}(a\alpha)$.*

Proof. This is an immediate consequence of Lemma (2.4.2) and the fact that $\mathbb{C}_n$ is commutative. □

One way to interpret Corollary (2.4.4) is that $\pi_{\mathbb{C}}$ is a $\mathbb{C}$-linear map.

COROLLARY (2.4.5). *Let a and b belong to A_n. Then ab belongs to $\mathbb{C}_n$ if and only if b belongs to the $\mathbb{C}$-linear span of a and* Ann(a).

Proof. In the notation of Lemma (2.4.2), observe that ab belongs to $\mathbb{C}_n$ if and only if ab'' is zero. □

3. Notation

Definition (3.1). For any a and b in $\mathbb{C}_n^{\perp}$, let $\boldsymbol{\{a, b\}}$ be the element

$$\frac{1}{\sqrt{2}}\big(a + b, i_n(-a + b)\big)$$

of A_{n+1}.

Whenever we write an expression of the form $\{a, b\}$, the reader should automatically assume that a and b belong to $\mathbb{C}_n^{\perp}$; nevertheless, we have tried to be explicit with this assumption. The reason for the factors $\frac{1}{\sqrt{2}}$ will show up in Lemma (3.5) and Lemma (4.1.1), where we study the metric properties of the notation $\{a, b\}$.

LEMMA (3.2). *Let (x, y) belong to $\mathbb{H}_{n+1}^{\perp}$, i.e., let x and y belong to $\mathbb{C}_n^{\perp}$. Then*

$$(x, y) = \frac{1}{\sqrt{2}}\{x + i_n y, x - i_n y\}.$$

The subspace $\mathbb{H}_{n+1}^{\perp}$ of A_{n+1} is equal to the subspace of all elements of the form $\{a, b\}$ with a and b in $\mathbb{C}_n^{\perp}$.

Proof. For the first claim, check the definition. This immediately implies that every element of $\mathbb{H}_{n+1}^{\perp}$ can be written in the form $\{a, b\}$ for some a and b in $\mathbb{C}_n^{\perp}$.

On the other hand, let a and b belong to $\mathbb{C}_n^{\perp}$. Then $a + b$ and $i_n(-a + b)$ also belong to $\mathbb{C}_n^{\perp}$, so $\big(a + b, i_n(-a + b)\big)$ belongs to $\mathbb{H}_{n+1}^{\perp}$. □

Recall that left multiplication makes A_{n+1} into a $\mathbb{C}_{n+1}$-vector space. We now describe multiplication by elements $\mathbb{C}_{n+1}$ with respect to the notation $\{a, b\}$.

Definition (3.3). If α belongs to $\mathbb{C}_n$, then $\tilde{\boldsymbol{\alpha}}$ is the image of α under the $\mathbb{R}$-linear map $\mathbb{C}_n \to \mathbb{C}_{n+1}$ that takes 1 to 1 and i_n to i_{n+1}.

LEMMA (3.4). *Let a and b belong to $\mathbb{C}_n^{\perp}$, and let α belong to $\mathbb{C}_n$. Then*

$$\tilde{\alpha}\{a, b\} = \{\alpha^* a, \alpha b\}.$$

Proof. Compute directly that $i_{n+1}\{a, 0\} = \{-i_n a, 0\}$ and $i_{n+1}\{0, b\} = \{0, i_n b\}$. □

LEMMA (3.5). *For any a and b in $\mathbb{C}_n^{\perp}$,*

$$|\{a, b\}|^2 = |a|^2 + |b|^2 .$$

Proof. According to Definition (3.1), $|\{a, b\}|^2$ equals

$$\frac{1}{2}\big(|a + b|^2 + |i_n(-a + b)|^2 \big).$$

As a consequence of Lemma (2.1.5), this expression equals

$$\frac{1}{2}\big(|a + b|^2 + |-a + b|^2 \big),$$

which simplifies to $|a|^2 + |b|^2$ by the parallelogram law. □

The absence of scalars in the above formula is the primary reason that the scalar $\frac{1}{\sqrt{2}}$ appears in Definition (3.1).

4. Multiplication Formulas

This section is the technical heart of the paper. We establish formulas for multiplication with respect to the notation of Section (3). The rest of the paper consists of many applications of these formulas.

PROPOSITION (4.1). *Let a, b, x, and y belong to $\mathbb{C}_n^\perp$, and suppose that a and b are both $\mathbb{C}$-orthogonal to x and y. Then*

$$\{a, b\}\{x, y\} = \sqrt{2}\{ax, by\}.$$

Proof. We begin by computing that $\{a, 0\}\{x, 0\}$ equals

$$\frac{1}{2}\Big(ax + i_n x \cdot i_n a, -i_n x \cdot a + i_n a \cdot x\Big).$$

Apply Lemma (2.1.4) to simplify this expression to

$$\frac{1}{2}\Big(ax - xa, i_n \cdot xa - i_n \cdot ax\Big).$$

Note that $ax = -xa$ because a and x are imaginary and orthogonal, so this expression further simplifies to

$$(ax, -i_n \cdot ax),$$

which equals $\sqrt{2}\{ax, 0\}$. A similar calculation shows that

$$\{0, b\}\{0, y\} = \sqrt{2}\{0, by\}.$$

Next we compute that $\{a, 0\}\{0, y\}$ equals

$$\frac{1}{2}\Big(ay - i_n y \cdot i_n a, i_n y \cdot a + i_n a \cdot y\Big).$$

Again use Lemma (2.1.4) to simplify to

$$\frac{1}{2}\Big(ay + ya, -i_n \cdot ya - i_n \cdot ay\Big),$$

but this equals zero because $ay = -ya$.

A similar calculation shows that $\{0, b\}\{x, 0\} = 0$. □

Remark (4.2). Proposition (4.1) already gives a sense of how easy it is to express certain zero-divisors using the notation $\{a, b\}$. For example, the product $\{a, 0\}\{0, y\}$ is always zero as long as a and y are $\mathbb{C}$-orthogonal elements of $\mathbb{C}_n^\perp$.

Because of the orthogonality hypotheses on a, b, x, and y, Proposition (4.1) does not quite describe how to multiply arbitrary elements of $\mathbb{H}_{n+1}^\perp$. Therefore, we need more multiplication formulas to handle various special cases.

LEMMA (4.3). *Let a belong to $\mathbb{C}_n^\perp$. Then*

$$\{0, a\}\{a, 0\} = -\{a, 0\}\{0, a\} = |a|^2 (0, i_n).$$

Proof. Compute that $\{0, a\}\{a, 0\}$ equals

$$\tfrac{1}{2}\Big(a^2 - i_n a \cdot i_n a, -2 i_n a \cdot a\Big).$$

Lemma (2.1.5) implies that the first coordinate is zero and that the second coordinate is $|a|^2 i_n$.

Finally, observe that $\{0, a\}$ and $\{a, 0\}$ are orthogonal and imaginary; therefore they anti-commute. □

We write $\tilde{\boldsymbol{\pi}}_{\mathbb{C}}$ for the composition of the projection $A_n \to \mathbb{C}_n$ with the map $\mathbb{C}_n \to \mathbb{C}_{n+1}$ described in Definition (3.3).

COROLLARY (4.4). *Let a and b be $\mathbb{C}_n$-linearly dependent elements of $\mathbb{C}_n^{\perp}$. Then*

(1) $\{a, 0\}\{b, 0\} = \tilde{\pi}_{\mathbb{C}}(ab)^*$.
(2) $\{0, a\}\{0, b\} = \tilde{\pi}_{\mathbb{C}}(ab)$.
(3) $\{a, 0\}\{0, b\} = \tilde{\pi}_{\mathbb{C}}(ab) \cdot (0, i_n)$.
(4) $\{0, a\}\{b, 0\} = -\tilde{\pi}_{\mathbb{C}}(ab)^* \cdot (0, i_n)$.

Proof. Since b belongs to the $\mathbb{C}_n$-linear span of a, we may write $b = \alpha a$ for some α in $\mathbb{C}_n$. Lemma (2.1.5) implies that ab equals $-|a|^2\alpha^*$, so $\tilde{\pi}_{\mathbb{C}}(ab)$ equals $-|a|^2\tilde{\alpha}^*$.

On the other hand, $\{a, 0\}\{\alpha a, 0\}$ equals $\{a, 0\} \cdot \tilde{\alpha}^*\{a, 0\}$ by Lemma (3.4), which also equals $-|\{a, 0\}|^2\tilde{\alpha}$ by Lemma (2.1.5). Finally, this equals $-|a|^2\tilde{\alpha}$ by Lemma (3.5). This establishes formula (1). The calculation for formula (2) is similar.

Next, $\{a, 0\}\{0, \alpha a\}$ equals $\{a, 0\} \cdot \tilde{\alpha}\{0, a\}$ by Lemma (3.4), which also equals $\tilde{\alpha}^* \cdot \{a, 0\}\{0, a\}$ by Lemma (2.1.4). Finally, this equals $-|a|^2\tilde{\alpha}^*(0, i_n)$ by Lemma (4.3), establishing formula (3). The calculation for formula (4) is similar. □

We are now ready to give an explicit formula for multiplication of arbitrary elements of $\mathbb{H}_{n+1}^{\perp}$.

THEOREM (4.5). *Let a, b, x, and y belong to $\mathbb{C}_n^{\perp}$. Then $\{a, b\}\{x, y\}$ equals*

$$\sqrt{2}\{\pi_{\mathbb{C}}^{\perp}(ax), \pi_{\mathbb{C}}^{\perp}(by)\} + \tilde{\pi}_{\mathbb{C}}\big(xa + by\big) + \tilde{\pi}_{\mathbb{C}}\big(ay - xb\big)(0, i_n).$$

Proof. We begin by computing $\{a, 0\}\{x, 0\}$. Write $x = x' + x''$, where x' belongs to the $\mathbb{C}$-linear span of a and x'' is $\mathbb{C}$-orthogonal to a. Then

$$\{a, 0\}\{x, 0\} = \{a, 0\}\{x', 0\} + \{a, 0\}\{x'', 0\}.$$

The first term equals $\tilde{\pi}_{\mathbb{C}}(ax')^*$ by Corollary (4.4), which in turn equals $\tilde{\pi}_{\mathbb{C}}(x'a)$ by Corollary (2.4.3). This is the same as $\tilde{\pi}_{\mathbb{C}}(xa)$ by Lemma (2.4.2). The second term equals $\sqrt{2}\{ax'', 0\}$ by Proposition (4.1), which equals $\sqrt{2}\{\pi_{\mathbb{C}}^{\perp}(ax), 0\}$ by Lemma (2.4.2). The computation for $\{0, b\}\{0, y\}$ is similar.

Now consider the product $\{a, 0\}\{0, y\}$. Write $y = y' + y''$, where y' belongs to the $\mathbb{C}$-linear span of a and y'' is $\mathbb{C}$-orthogonal to a. Then

$$\{a, 0\}\{0, y\} = \{a, 0\}\{0, y'\} + \{a, 0\}\{0, y''\}.$$

The first term equals $\tilde{\pi}_{\mathbb{C}}(ay') \cdot (0, i_n)$ by Corollary (4.4), which is the same as $\tilde{\pi}_{\mathbb{C}}(ay) \cdot (0, i_n)$ by Lemma (2.4.2). The second term equals zero by Proposition (4.1). The computation for $\{0, b\}\{x, 0\}$ is similar. □

Remark (4.6). The three terms in the formula of Theorem (4.5) are orthogonal. The first term belongs to $\mathbb{H}_{n+1}^{\perp}$; the second term belongs to $\mathbb{C}_{n+1}$; and the third term belongs to $\mathbb{H}_{n+1} \cap \mathbb{C}_{n+1}^{\perp}$, which is also the $\mathbb{C}$-linear span of $(0, i_n)$ or the $\mathbb{R}$-linear span of $(i_n, 0)$ and $(0, i_n)$.

Theorem (4.5) shows how to compute the product of two elements of $\mathbb{H}_{n+1}^{\perp}$. On the other hand, it is easy to multiply elements of $\mathbb{H}_{n+1}$; this is just ordinary quaternionic arithmetic. In order to have a complete description of multiplication on A_{n+1}, we need to explain how to multiply elements of $\mathbb{H}_{n+1}$ with elements of $\mathbb{H}_{n+1}^{\perp}$.

Lemma (3.4) shows how to compute the product of an element of $\mathbb{H}_{n+1}^{\perp}$ and an element of $\mathbb{C}_{n+1}$. It remains only to compute the product of an element of $\mathbb{H}_{n+1}^{\perp}$ and an element of $\mathbb{H}_{n+1} \cap \mathbb{C}_{n+1}^{\perp}$, i.e., the $\mathbb{C}$-linear span of $(0, i_n)$. The following lemma makes this computation.

LEMMA (4.7). *Let a and b belong to $\mathbb{C}_n^{\perp}$. Then*

$$(0, i_n)\{a, b\} = -\{a, b\}(0, i_n) = \{b, -a\}.$$

Proof. Compute directly that $(0, i_n)\{a, 0\} = \{0, -a\}$ and that $(0, i_n)\{0, b\} = \{b, 0\}$. Also, use that orthogonal imaginary elements anti-commute. □

(4.1) Inner product computations.

LEMMA (4.1.1). *Let a, b, x, and y belong to $\mathbb{C}_n^{\perp}$. Then*

$$\langle \{a, b\}, \{x, y\} \rangle_{\mathbb{C}} = \langle a, x \rangle_{\mathbb{C}}^{*} + \langle b, y \rangle_{\mathbb{C}}.$$

Proof. We need to compute the projection of the product $-\{a, b\}\{x, y\}$ onto $\mathbb{C}_{n+1}$. Theorem (4.5) immediately shows that this projection equals $-\tilde{\pi}_{\mathbb{C}}(xa + by)$, which is equal to $\langle x, a \rangle_{\mathbb{C}} + \langle b, y \rangle_{\mathbb{C}}$. Finally, recall that $\langle x, a \rangle_{\mathbb{C}} = \langle a, x \rangle_{\mathbb{C}}^{*}$. □

COROLLARY (4.1.2). *Let a, b, x, and y belong to $\mathbb{C}_n^{\perp}$. Then*

$$\langle \{a, b\}, \{x, y\} \rangle_{\mathbb{R}} = \langle a, x \rangle_{\mathbb{R}} + \langle b, y \rangle_{\mathbb{R}}.$$

Proof. Use Lemma (4.1.1), recalling that the real inner product equals the real part of the Hermitian inner product. □

(4.2) Subalgebras. Suppose that a and b are $\mathbb{C}$-orthogonal elements of $\mathbb{C}_n^{\perp}$ that both have norm 1. Suppose also that a and b satisfy the equations $a \cdot ab = -\lambda b$ and $b \cdot ba = -\lambda a$ for some non-zero real number λ. These equations guarantee that a and b generate a 4-dimensional subalgebra of A_n; the subalgebra is isomorphic to $\mathbb{H}$ when $\lambda = 1$. This remark concerns the possible values for λ, and therefore addresses the classification problem for 4-dimensional subalgebras of Cayley-Dickson algebras. See [CD], Section 7 for detailed information on 4-dimensional subalgebras of A_4. In particular, in A_4, the only possible values for λ are 1 and 2 [CD], Theorem 7.1.

Given a and b as in the previous paragraph, compute that

$$\frac{1}{\sqrt{2}}\{a, b\} \cdot \frac{1}{\sqrt{2}}\{b, -a\} = \frac{1}{\sqrt{2}}\{ab, -ba\} + (0, i_n)$$

using Theorem (4.5). Next, compute that

$$\frac{1}{\sqrt{2}}\{a, b\} \left(\frac{1}{\sqrt{2}}\{ab, -ba\} + (0, i_n) \right) = -\frac{\lambda + 1}{\sqrt{2}}\{b, -a\}$$

using Proposition (4.1) and Lemma (4.7). This uses that a and ab are $\mathbb{C}$-orthogonal by Lemma (2.1.2) and also the equations involving a, b, and λ. A similar calculation can be performed with the roles of $\frac{1}{\sqrt{2}}\{a, b\}$ and $\frac{1}{\sqrt{2}}\{b, -a\}$ switched.

We have shown that $\frac{1}{\sqrt{2}}\{a, b\}$ and $\frac{1}{\sqrt{2}}\{b, -a\}$ satisfy the same equations as a and b do, except that λ is replaced by $\lambda + 1$. Using the argument of [CD], Theorem 7.1 (which can be applied even when $n > 4$), it follows that for every positive integer r and every sufficiently large n (depending on r), there is a subalgebra of A_n that is isomorphic to the non-associative algebra with $\mathbb{R}$-basis $\{1, x, y, z\}$ subject to the multiplication rules

$$x^2 = y^2 = z^2 = -1, \; xy = -yx = z\sqrt{r}, \; yz = -zy = x\sqrt{r}, \; zx = -xz = y\sqrt{r}.$$

This algebra is isomorphic to $\mathbb{H}$ when $r = 1$.

Another consequence of our multiplication formulas is the following observation about sets of mutually annihilating elements.

LEMMA (4.2.1). *Let $n \geq 3$. If $\mathbb{C}_n^{\perp}$ contains two sets $\{x_1, \ldots, x_{2^{n-3}}\}$ and $\{y_1, \ldots, y_{2^{n-3}}\}$ of size 2^{n-3} such that $x_i x_j = 0 = y_i y_j$ for all $i \neq j$ and each x_i is $\mathbb{C}$-orthogonal to each y_j, then the product $\{x_i, 0\}\{x_j, 0\}$ is zero when $i \neq j$, and $\{x_i, 0\}\{0, y_j\}$ is zero for all i and j.*

Proof. Compute with Proposition (4.1). □

COROLLARY (4.2.2). *The space $\mathbb{C}_n^{\perp}$ contains 2^{n-3} distinct non-zero elements such that the product of any two distinct elements is zero.*

Proof. We will actually prove the stronger result that $\mathbb{C}_n^{\perp}$ contains two sets $\{x_1, \ldots, x_{2^{n-3}}\}$ and $\{y_1, \ldots, y_{2^{n-3}}\}$ of distinct non-zero elements such that $x_i x_j = 0 = y_i y_j$ for all $i \neq j$ and each x_i is $\mathbb{C}$-orthogonal to each y_j.

The proof is by induction on n, using Lemma (4.2.1). The base case $n = 3$ is trivial; it just calls for the existence of two orthogonal elements of the six-dimensional subspace $\mathbb{C}_3^{\perp}$ of A_3.

Now suppose for induction that the sets $\{x_1, \ldots, x_{2^{n-3}}\}$ and $\{y_1, \ldots, y_{2^{n-3}}\}$ exist in A_n. Consider the subset of A_{n+1} consisting of all elements of the form $\{x_i, 0\}$ or $\{0, y_j\}$. There are 2^{n-2} such elements, and Lemma (4.2.1) implies that the product of any two distinct such elements is zero.

Also consider the subset of A_{n+1} consisting of all elements of the form $\{y_j, 0\}$ or $\{0, x_i\}$. Again, there are 2^{n-2} such elements, and the product of any two distinct such elements is zero.

Finally, by Lemma (4.1.1) and the induction assumption, the elements described in the previous two paragraphs are $\mathbb{C}$-orthogonal. □

Corollary (4.2.2) is also relevant to subalgebras of Cayley-Dickson algebras. The $\mathbb{R}$-linear span of 1 together with a set of mutually annihilating elements is a subalgebra of A_n. These subalgebras are highly degenerate in the sense that $xy = 0$ for any pair of orthogonal imaginary elements. Corollary (4.2.2) implies that A_n contains such a subalgebra of dimension $1 + 2^{n-3}$. In fact, we have shown that A_n contains two such subalgebras whose imaginary parts are $\mathbb{C}$-orthogonal.

Question (4.2.3). Does A_n contain a degenerate subalgebra of dimension larger than $1 + 2^{n-3}$?

5. Annihilation in $\mathbb{H}_{n+1}^{\perp}$

In this section, we apply the multiplication formulas of Section (4) to consider zero-divisors in A_{n+1}.

PROPOSITION (5.1). *Let a, b, x, and y belong to $\mathbb{C}_n^{\perp}$. Then $\{a,b\}\{x,y\}=0$ if and only if*

(i) $\pi_{\mathbb{C}}^{\perp}(ax)=0$,
(ii) $\pi_{\mathbb{C}}^{\perp}(by)=0$,
(iii) $xa+by=0$, *and*
(iv) $\pi_{\mathbb{C}}(ay-xb)=0$.

Proof. Parts (i), (ii), and (iv) are immediate from Theorem (4.5). It follows from (i) and (ii) that $\pi_{\mathbb{C}}(xa+by)=xa+by$. Therefore, part (iii) also follows from Theorem (4.5). □

The conditions of Proposition (5.1) are redundant. For example, condition (i) follows from conditions (ii) and (iii). However, it is more convenient to formulate the proposition symmetrically.

PROPOSITION (5.2). *Let $n\geq 3$. Let a and b be non-zero elements of $\mathbb{C}_n^{\perp}$. Then $\mathbb{H}_{n+1}^{\perp}\cap \operatorname{Ann}\{a,b\}$ is equal to the space of all $\{\alpha a+x,\beta b+y\}$ such that:*

(1) *x belongs to $\operatorname{Ann}(a)$, and y belongs to $\operatorname{Ann}(b)$;*
(2) *α and β belong to $\mathbb{C}_n$;*
(3) $|a|^2\,\alpha+|b|^2\,\beta^*=0$; *and*
(4) $(\beta^*-\alpha)\pi_{\mathbb{C}}(ab)+\pi_{\mathbb{C}}(ay-xb)=0$.

Proof. We want to solve the equation $\{a,b\}\{z,w\}=\{0,0\}$ under the assumption that z and w belong to $\mathbb{C}_n^{\perp}$ (see Lemma (3.2)). Using Proposition (5.1), this is equivalent to solving the four equations

$$\pi_{\mathbb{C}}^{\perp}(az)=0 \tag{5.3}$$

$$\pi_{\mathbb{C}}^{\perp}(bw)=0 \tag{5.4}$$

$$za+bw=0 \tag{5.5}$$

$$\pi_{\mathbb{C}}(aw-zb)=0. \tag{5.6}$$

By Corollary (2.4.5), Equations (5.3) and (5.4) are the same as requiring that z belongs to the $\mathbb{C}$-linear span of a and $\operatorname{Ann}(a)$ and that w belongs to the $\mathbb{C}$-linear span of b and $\operatorname{Ann}(b)$. Therefore, we may write $z=\alpha a+x$ and $w=\beta b+y$ for some α and β in $\mathbb{C}_n$, some x in $\operatorname{Ann} a$, and some y in $\operatorname{Ann} b$. We also know that x and y belong to $\mathbb{C}_n^{\perp}$ by Lemma (2.3.1); this is where we use that a and b are non-zero.

Substitute the expressions for z and w in Equations (5.5) and (5.6) to obtain

$$(\alpha a+x)a+b(\beta b+y)=0 \tag{5.7}$$

$$\pi_{\mathbb{C}}\Big(a(\beta b+y)-(\alpha a+x)b\Big)=0. \tag{5.8}$$

Equation (5.7) simplifies to $-|a|^2\alpha-|b|^2\beta^*=0$ by Lemma (2.1.5) and the fact that $xa=by=0$. This is condition (3) of the proposition.

Equation (5.8) can be rewritten as

$$\pi_{\mathbb{C}}(\beta^* \cdot ab - ab \cdot \alpha) + \pi_{\mathbb{C}}(ay - xb) = 0 \tag{5.9}$$

by Lemma (2.1.3). Apply Corollary (2.4.4) to the second part of the first term of Equation (5.9) to obtain the equation $(\beta^* - \alpha)\pi_{\mathbb{C}}(ab) + \pi_{\mathbb{C}}(ay - xb) = 0$. This is condition (4) of the proposition. □

THEOREM (5.10). *Let $n \geq 3$, and let a and b be non-zero elements of $\mathbb{C}_n^{\perp}$. Then* $\dim \operatorname{Ann}\{a, b\}$ *equals* $\dim \operatorname{Ann} a + \dim \operatorname{Ann} b$ *or* $\dim \operatorname{Ann} a + \dim \operatorname{Ann} b + 4$.

Proof. First we will use Proposition (5.2) to analyze $\mathbb{H}_{n+1}^{\perp} \cap \operatorname{Ann}\{a, b\}$. Let V be the space of elements $\{\alpha a + x, \beta b + y\}$ such that α and β belong to $\mathbb{C}_n$, x belongs to $\operatorname{Ann} a$, and y belongs to $\operatorname{Ann} b$. The dimension of V is equal to $\dim \operatorname{Ann} a + \dim \operatorname{Ann} b + 4$. Recall from Lemma (3.4) that for γ in $\mathbb{C}_n$,

$$\tilde{\gamma}\{\alpha a + x, \beta b + y\} = \{\gamma^* \alpha a + \gamma^* x, \gamma\beta b + \gamma y\}.$$

This shows that V is a $\mathbb{C}_n$-vector space, and Condition (3) of Proposition (5.2) is a non-degenerate conjugate-linear equation in the variables α and β. Hence there is a subspace of V of dimension $\dim \operatorname{Ann} a + \dim \operatorname{Ann} b + 2$ that satisfies condition (3).

Condition (4) of Proposition (5.2) is a conjugate-linear equation in the variables α, β, x, and y, which may or may not be non-degenerate and independent of condition (3). This establishes that

$$\dim \operatorname{Ann} a + \dim \operatorname{Ann} b \leq \dim(\mathbb{H}_{n+1}^{\perp} \cap \operatorname{Ann}\{a, b\}) \leq \dim \operatorname{Ann} a + \dim \operatorname{Ann} b + 2.$$

Lemma (2.3.1) implies that $\operatorname{Ann}\{a, b\}$ is contained in $\mathbb{C}_{n+1}^{\perp}$. Note that $\mathbb{H}_{n+1}^{\perp}$ is a codimension 2 subspace of $\mathbb{C}_{n+1}^{\perp}$. Therefore, the codimension of $\mathbb{H}_{n+1}^{\perp} \cap \operatorname{Ann}\{a, b\}$ in $\operatorname{Ann}\{a, b\}$ is at most 2. This establishes the inequality

$$\dim \operatorname{Ann} a + \dim \operatorname{Ann} b \leq \dim \operatorname{Ann}\{a, b\} \leq \dim \operatorname{Ann} a + \dim \operatorname{Ann} b + 4.$$

The desired result follows from Theorem (2.3.2), which tells us that the dimension of any annihilator is a multiple of 4. □

Theorem (5.10) gives two options for the dimension of $\operatorname{Ann}\{a, b\}$; Section (6) below contains conditions on a and b that distinguish between these two cases.

One might also be concerned that Theorem (5.10) applies only to elements $\{a, b\}$ in which both a and b are non-zero because it relies on Proposition (5.2). For completeness, we also review from [DDD] the simpler situation of elements of the form $\{a, 0\}$ and $\{0, a\}$. The following proposition can be proved with the formulas of Section (4).

PROPOSITION (5.11) ([DDD], Theorem 10.2). *Let $n \geq 4$, and let a belong to $\mathbb{C}_{n-1}^{\perp}$. Then the element $\{a, 0\}$ of A_n is a zero-divisor whose annihilator* $\operatorname{Ann}\{a, 0\}$ *equals the space of all elements $\{x, y\}$ where x belongs to* $\operatorname{Ann}(a)$ *and y is $\mathbb{C}$-orthogonal to* 1 *and a. Similarly, the element $\{0, a\}$ of A_n is a zero-divisor whose annihilator* $\operatorname{Ann}\{0, a\}$ *equals the space of all elements $\{x, y\}$ where y belongs to* $\operatorname{Ann}(a)$ *and x is $\mathbb{C}$-orthogonal to* 1 *and a. In either case, the dimension of the annihilator is* $\dim \operatorname{Ann}(a) + 2^{n-1} - 4$.

In fact, [DDD], Theorem 10.2, was a major inspiration for the notation $\{a, b\}$.

6. The D-locus

In Section (5), we started to consider $\mathrm{Ann}\{a, b\}$ when a and b are arbitrary elements in $\mathbb{C}_n^\perp$, i.e., when $\{a, b\}$ is an arbitrary element of $\mathbb{H}_{n+1}^\perp$. Theorem (5.10) told us that except for some simple well-understood cases covered in Proposition (5.11), the dimension of $\mathrm{Ann}\{a, b\}$ is either $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b$ or $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b + 4$. The goal of this section is to distinguish between these two cases.

Definition (6.1). The ***D*-locus** is the space of all elements $\{a, b\}$ of A_{n+1} with a and b in $\mathbb{C}_n^\perp$ such that

(1) a and b are $\mathbb{C}$-orthogonal,
(2) a and $\mathrm{Ann}(b)$ are orthogonal, and
(3) b and $\mathrm{Ann}(a)$ are orthogonal.

Remark (6.2). Since $\mathrm{Ann}(b)$ is a $\mathbb{C}$-subspace of A_n, a is orthogonal to $\mathrm{Ann}(b)$ if and only if a is $\mathbb{C}$-orthogonal to $\mathrm{Ann}(b)$. Similarly, b is orthogonal to $\mathrm{Ann}(a)$ if and only if b is $\mathbb{C}$-orthogonal to $\mathrm{Ann}(a)$. Thus, conditions (2) and (3) of Definition (6.1) can be rewritten in terms of $\mathbb{C}$-orthogonality.

Also, $\mathrm{Ann}(b)^\perp$ is equal to the image of left multiplication by b (see Lemma (6.9) below), so condition (2) is also equivalent to requiring that $a = bx$ for some x. Similarly, condition (3) is also equivalent to requiring that $b = ay$ for some y.

The point of the following lemma is to determine precisely when condition (4) of Proposition (5.2) holds.

LEMMA (6.3). *Suppose that a and b belong to $\mathbb{C}_n^\perp$. Then $\{a, b\}$ belongs to the D-locus if and only if*

$$(\beta^* - \alpha)\pi_{\mathbb{C}}(ab) + \pi_{\mathbb{C}}(ay - xb) = 0$$

for all α and β in $\mathbb{C}_n$, x in $\mathrm{Ann}(a)$*, and y in* $\mathrm{Ann}(b)$*.*

Proof. Since α, β, x, and y are independent, the displayed expression vanishes if and only if $\pi_{\mathbb{C}}(ab) = 0$, $\pi_{\mathbb{C}}(xb) = 0$ for all x in $\mathrm{Ann}\, a$, and $ay = 0$ for all y in $\mathrm{Ann}\, b$. The first equation just means that a and b are $\mathbb{C}$-orthogonal, the second equation means that b is $\mathbb{C}$-orthogonal to $\mathrm{Ann}(a)$, and the third equation means that a is $\mathbb{C}$-orthogonal to $\mathrm{Ann}(b)$. □

LEMMA (6.4). *If $\{a, b\}$ is non-zero and does not belong to the D-locus, then the dimension of* $\mathrm{Ann}\{a, b\} \cap \mathbb{H}_{n+1}^\perp$ *is equal to* $\dim \mathrm{Ann}(a) + \dim \mathrm{Ann}(b)$*.*

Proof. Let V be the subspace of A_{n+1} consisting of all elements of the form $\{\alpha a + x, \beta b + y\}$, where α and β belong to $\mathbb{C}_n$, x belongs to $\mathrm{Ann}(a)$, and y belongs to $\mathrm{Ann}(b)$. The dimension of V is $\dim \mathrm{Ann}(a) + \dim \mathrm{Ann}(b) + 4$. As in the proof of Theorem (5.10), V is a $\mathbb{C}_n$-vector space.

According to Proposition (5.2), $\mathrm{Ann}\{a, b\} \cap \mathbb{H}_{n+1}^\perp$ is contained in V. In fact, it is the subspace of V defined by the two conjugate-linear equations

$$|a|^2 \alpha + |b|^2 \beta^* = 0 \tag{6.5}$$

$$(\beta^* - \alpha)\pi_{\mathbb{C}}(ab) + \pi_{\mathbb{C}}(ay - xb) = 0. \tag{6.6}$$

Thus, we only need to show that Equations (6.5) and (6.6) are non-degenerate and independent. Equation (6.5) is non-degenerate because $|\, a \,|$ or $|\, b \,|$ is non-zero. Equation (6.6) is non-degenerate by Lemma (6.3).

It remains to show that Equations (6.5) and (6.6) are independent. There are three cases to consider, depending on which part of Definition (6.1) fails to hold for a and b.

If a and b are not $\mathbb{C}$-orthogonal, then $\pi_{\mathbb{C}}(ab)$ is non-zero. Substitute the values $\alpha = -\,|\, b\,|^2$, $\beta = |\, a\,|^2$, $x = 0$, and $y = 0$ into the two equations; note that Equation (6.5) is satisfied, while Equation (6.6) is not satisfied because the left-hand side equals $(|\, a\,|^2 + |\, b\,|^2)\pi_{\mathbb{C}}(ab)$. This shows that the two equations are independent because they have different solution sets.

Next, suppose that a is not orthogonal to $\mathrm{Ann}(b)$. There exists an element y_0 of $\mathrm{Ann}(b)$ such that a and y_0 are not $\mathbb{C}$-orthogonal. This means that $\pi_{\mathbb{C}}(ay_0)$ is non-zero. Substitute the values $\alpha = 0$, $\beta = 0$, $x = 0$, and $y = y_0$ into the two equations; note that Equation (6.5) is satisfied, while Equation (6.6) is not satisfied because the left-hand side equals $\pi_{\mathbb{C}}(ay_0)$. This shows that the two equations are independent because they have different solution sets.

Finally, suppose that b is not orthogonal to $\mathrm{Ann}(a)$. Similarly to the previous case, choose x_0 in $\mathrm{Ann}(a)$ such that $\pi_{\mathbb{C}}(ax_0)$ is non-zero. Substitute the values $\alpha = 0$, $\beta = 0$, $x = x_0$, and $y = 0$ into the two equations; note that Equation (6.5) is satisfied, while Equation (6.6) is not satisfied. □

THEOREM (6.7). *Let a and b be non-zero elements of $\mathbb{C}_n^{\perp}$. If $\{a, b\}$ does not belong to the D-locus, then* $\mathrm{Ann}\{a, b\}$ *is contained in* $\mathbb{H}_{n+1}^{\perp}$. *Moreover, the dimension of* $\mathrm{Ann}\{a, b\}$ *is* $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b$.

Proof. Recall from Lemma (2.3.1) that $\mathrm{Ann}\{a, b\}$ is a subspace of $\mathbb{C}_{n+1}^{\perp}$. Also, $\mathbb{H}_{n+1}^{\perp}$ is a codimension 2 subspace of $\mathbb{C}_{n+1}^{\perp}$. Therefore, the codimension of $\mathrm{Ann}\{a, b\} \cap \mathbb{H}_{n+1}^{\perp}$ in $\mathrm{Ann}\{a, b\}$ is at most 2. Together with Lemma (6.4), this implies that the dimension of $\mathrm{Ann}\{a, b\}$ is at least $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b$ and at most $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b + 2$. However, the dimension of $\mathrm{Ann}\{a, b\}$ is a multiple of 4 by Theorem (2.3.2), so it must equal $\dim \mathrm{Ann}\, a + \dim \mathrm{Ann}\, b$. This shows that $\mathrm{Ann}\{a, b\}$ equals $\mathrm{Ann}\{a, b\} \cap \mathbb{H}_{n+1}^{\perp}$ because their dimensions are equal; in other words, $\mathrm{Ann}\{a, b\}$ is contained in $\mathbb{H}_{n+1}^{\perp}$. □

Theorem (6.7) computes the dimension of $\mathrm{Ann}\{a, b\}$ for any $\{a, b\}$ that does not belong to the D-locus. However, it leaves something to be desired because it does not explicitly describe $\mathrm{Ann}\{a, b\}$ as a subspace of A_{n+1}. The difficulty arises from our use of the fact that the dimension of $\mathrm{Ann}\{a, b\}$ is a multiple of 4.

Question (6.8). Describe $\mathrm{Ann}\{a, b\}$ explicitly when $\{a, b\}$ does not belong to the D-locus.

The rest of this section considers annihilators of elements that belong to the D-locus.

LEMMA (6.9). *Suppose that a and b belong to A_n, and suppose that b is orthogonal to* $\mathrm{Ann}(a)$. *There exists a unique element x such that $ax = b$ and x is orthogonal to* $\mathrm{Ann}(b)$.

Proof. This is a restatement of Lemma (2.3.3). □

Definition (6.10). Let a and b belong to A_n, and suppose that b is orthogonal to $\operatorname{Ann} a$. Then $\frac{b}{a}$ is the unique element such that $a\frac{b}{a} = b$ and such that $\frac{b}{a}$ is orthogonal to $\operatorname{Ann} a$.

Beware that the definition of $\frac{b}{a}$ is not symmetric. In other words, it is not always true that $\frac{b}{a}a = b$.

LEMMA (6.11). *Let a and b be $\mathbb{C}$-orthogonal elements of $\mathbb{C}_n^\perp$, and suppose that b is orthogonal to* $\operatorname{Ann}(a)$*. Then $\frac{b}{a}$ belongs to $\mathbb{C}_n^\perp$ and is $\mathbb{C}$-orthogonal to both a and b.*

Proof. If $a = 0$, then $\operatorname{Ann} a$ is all of A_n so $b = 0$ and $\frac{b}{a}$ also equals 0. In this case, the claim is trivially satisfied. Now assume that a is non-zero.

For the first claim, note that $\langle a, a\frac{b}{a}\rangle_{\mathbb{C}} = \langle a, b\rangle_{\mathbb{C}} = 0$. By Lemma (2.1.3), this equals $-\langle a^2, \frac{b}{a}\rangle_{\mathbb{C}}^*$. But a^2 is a non-zero real number, so $\frac{b}{a}$ is $\mathbb{C}$-orthogonal to 1 as desired.

Next, note that $a\frac{b}{a} = b$ is orthogonal to $\mathbb{C}_n$, so $\left\langle a, \frac{b}{a}\right\rangle_{\mathbb{C}} = \pi_{\mathbb{C}}(b)$ is zero. Also, compute that

$$\left\langle \tfrac{b}{a}, b\right\rangle_{\mathbb{C}} = \left\langle \tfrac{b}{a}, a\tfrac{b}{a}\right\rangle_{\mathbb{C}} = -\left\langle \left(\tfrac{b}{a}\right)^2, a\right\rangle_{\mathbb{C}}^*$$

using Lemma (2.1.3). But $\left(\frac{b}{a}\right)^2$ is a real scalar, which is $\mathbb{C}$-orthogonal to a because we assumed that a belongs to $\mathbb{C}_n^\perp$. □

THEOREM (6.12). *Let a and b be non-zero elements of $\mathbb{C}_n^\perp$, and suppose that $\{a, b\}$ belongs to the D-locus. Then* $\operatorname{Ann}\{a, b\}$ *is the $\mathbb{C}$-orthogonal direct sum of:*

(1) *the space of all elements $\{x, y\}$ such that x belongs to* $\operatorname{Ann}(a)$ *and y belongs to* $\operatorname{Ann}(b)$*;*
(2) *the $\mathbb{C}$-linear span of the element $\left\{|b|^2\, a, -\,|a|^2\, b\right\}$;*
(3) *the $\mathbb{C}$-linear span of $\left\{\frac{b}{a}, -\frac{a}{b}\right\} + \sqrt{2}(0, i_n)$, where $\frac{b}{a}$ and $\frac{a}{b}$ are described in Definition (6.10).*

In particular, the dimension of $\operatorname{Ann}\{a, b\}$ *is equal to* $\dim(\operatorname{Ann} a) + \dim(\operatorname{Ann} b) + 4$.

Proof. It follows from Proposition (5.2) that $\operatorname{Ann}\{a, b\}$ contains the space described in part (1). Recall that Lemma (6.3) implies that condition (4) of Proposition (5.2) vanishes.

Next, note that $\left\{|b|^2\, a, -\,|a|^2\, b\right\}$ satisfies the conditions of Proposition (5.2). It corresponds to $\alpha = |b|^2$, $\beta = -\,|a|^2$, $x = 0$, and $y = 0$.

Finally, we want to show that $\{a, b\}\left\{\frac{b}{a}, -\frac{a}{b}\right\} + \sqrt{2}\{a, b\}(0, i_n)$ is zero. Lemma (6.11) says that Proposition (4.1) applies to the first term, which therefore equals $\sqrt{2}\left\{a\frac{b}{a}, -b\frac{a}{b}\right\}$. This simplifies to $\sqrt{2}\{b, -a\}$. Lemma (4.7) lets us compute that the second term is $\sqrt{2}\{-b, a\}$, as desired.

We have now exhibited a subspace of $\operatorname{Ann}\{a, b\}$ whose dimension is $\dim \operatorname{Ann} a + \dim \operatorname{Ann} b + 4$. Theorem (5.10) implies that we have described the entire annihilator.

Recall that Lemma (4.1.1) describes how to compute Hermitian inner products. Using this lemma, parts (1) and (2) are $\mathbb{C}$-orthogonal because a and b are $\mathbb{C}$-orthogonal to $\mathrm{Ann}(a)$ and $\mathrm{Ann}(b)$ respectively. Parts (1) and (3) are $\mathbb{C}$-orthogonal by Definition (6.10). Parts (2) and (3) are $\mathbb{C}$-orthogonal by Lemma (6.11). □

7. The D-locus in A_5

The goal of this section is to explicitly understand the D-locus in A_5 (see Definition (6.1)). Unlike most of the rest of this paper, this section uses computational techniques that apply in A_4 but have not yet been made to work in general.

Let us consider whether elements of the form $\{a, 0\}$ belong to the D-locus. If a is non-zero, then part (2) of Definition (6.1) fails. Therefore, $\{a, 0\}$ belongs to the D-locus only if $a = 0$. Similarly, $\{0, b\}$ belongs to the D-locus only if $b = 0$.

From now on, we may suppose that a and b are non-zero. If b is not a zero-divisor, then it is easy to determine whether $\{a, b\}$ belongs to the D-locus. Namely, b must be $\mathbb{C}$-orthogonal to a and to $\mathrm{Ann}(a)$ because condition (2) of Definition (6.1) is vacuous. By symmetry, a similar description applies when a is not a zero-divisor. Since annihilators in A_4 are well-understood ([KY], Section 3.2, [M1], Corollary 2.14, [DDD], Sections 11 and 12), it is relatively straightforward to completely describe the elements $\{a, b\}$ belonging to the D-locus in A_5 such that a or b is not a zero-divisor.

There is only one remaining case to consider. It consists of elements of the form $\{a, b\}$, where a and b are both zero-divisors in A_4. We will focus on such elements in the rest of this section. First we need some preliminary calculations in A_4.

Lemma (7.1). *Let a belong to $\mathbb{C}_3^{\perp}$. If b is $\mathbb{C}$-orthogonal to 1 and a, and α belongs to $\mathbb{C}_3$, then the element $\{a, 0\}$ of A_4 is orthogonal to the annihilator of $\{b, \alpha a\}$.*

Proof. Let c be the element of A_3 such that $bc = a$; in other words, $c = -(1/|b|^2)ba$. Note that c is $\mathbb{C}$-orthogonal to both a and b by Lemma (2.1.2).

Using Proposition (4.1), compute that $\{b, \alpha a\}\{\frac{1}{\sqrt{2}}c, 0\} = \{a, 0\}$. Finally, use Lemma (2.3.3) to conclude that $\{a, 0\}$ is orthogonal to $\mathrm{Ann}\{b, \alpha a\}$. □

Lemma (7.2). *Let a be a non-zero element of $\mathbb{C}_3^{\perp}$. If a zero-divisor in A_4 is $\mathbb{C}$-orthogonal to $\{a, 0\}$ and is orthogonal to $\mathrm{Ann}\{a, 0\}$, then it is of the form $\{b, \alpha a\}$, where b is $\mathbb{C}$-orthogonal to a and α belongs to $\mathbb{C}$.*

Proof. Suppose that x is a zero-divisor in A_4 that is $\mathbb{C}$-orthogonal to $\{a, 0\}$ and is orthogonal to $\mathrm{Ann}\{a, 0\}$. Write x in the form $\{b, c\} + (\beta, \gamma)$, where b and c belong to $\mathbb{C}_3^{\perp}$ while β and γ belong to $\mathbb{C}_3$.

Recall from [DDD], Theorem 10.2, that $\mathrm{Ann}\{a, 0\}$ consists of elements of the form $\{0, y\}$, where y is any element of A_3 that is $\mathbb{C}$-orthogonal to 1 and to a. Since x is orthogonal to $\mathrm{Ann}\{a, 0\}$, Lemma (4.1.1) implies that c is $\mathbb{C}$-orthogonal to all such y. In other words, c belongs to the $\mathbb{C}$-linear span of a; i.e., $c = \alpha a$ for some α in $\mathbb{C}_3$.

Since $\{a, 0\}$ and x are $\mathbb{C}$-orthogonal, Lemma (4.1.1) says that b is $\mathbb{C}$-orthogonal to a. Note, in particular, that b and c are $\mathbb{C}$-orthogonal.

Let $x_1 = b + c + \beta$ and $x_2 = -i_3 b + i_3 c + \gamma$ so that $x = (x_1, x_2)$. Recall from [DDD], Proposition 12.1, that since x is a zero-divisor, x_1 and x_2 are imaginary orthogonal elements of A_3 with the same norm.

Multiplication by i_3 preserves norms in A_3. Since x_1 and x_2 have the same norm, it follows that β and γ have the same norm. This uses that b and c are orthogonal, as we have already shown.

Since x_1 and x_2 are orthogonal, it follows that β and γ are orthogonal. This uses that b and c are each orthogonal to both $i_3 b$ and $i_3 c$ since b and c are $\mathbb{C}$-orthogonal.

Next, since x_1 and x_2 are imaginary, it follows that β and γ are $\mathbb{R}$-scalar multiples of i_3. We have shown that β and γ are both orthogonal and parallel and also have the same norm. It follows that β and γ are both zero. □

PROPOSITION (7.3). *Let a, b, and c belong to $\mathbb{C}_3^{\perp}$, and suppose that a is non-zero. Suppose also that $\{b, c\}$ is a zero-divisor in A_4. The element $\{\{a, 0\}, \{b, c\}\}$ belongs to the D-locus in A_5 if and only if b is $\mathbb{C}$-orthogonal to a and c belongs to the $\mathbb{C}$-linear span of a.*

Proof. First suppose that b is $\mathbb{C}$-orthogonal to a and c belongs to the $\mathbb{C}$-linear span of a. Lemma (4.1.1) implies that $\{a, 0\}$ and $\{b, c\}$ are $\mathbb{C}$-orthogonal.

By [DDD], Theorem 10.2, the annihilator of $\{a, 0\}$ consists of elements of the form $\{0, y\}$, where y is $\mathbb{C}$-orthogonal to 1 and a. Therefore, Lemma (4.1.1) implies that $\{b, c\}$ is orthogonal to $\mathrm{Ann}\{a, 0\}$.

Lemma (7.1) implies that $\{a, 0\}$ is orthogonal to $\mathrm{Ann}\{b, c\}$. This finishes one implication.

For the other implication, suppose that $\{\{a, 0\}, \{b, c\}\}$ belongs to the D-locus in A_5. Lemma (7.2) implies that b is $\mathbb{C}$-orthogonal to a and that c belongs to the $\mathbb{C}$-linear span of a. □

Suppose that $a = (a_1, a_2)$ is a zero-divisor in A_4. We recall from [KY], Section 3.2, [M1], Corollary 2.14, [DDD], Sections 11 and 12, some algebraic properties of a. First of all, a_1 and a_2 are imaginary orthogonal elements of A_3 with the same norm. The $\mathbb{R}$-linear span of 1, a_1, a_2, and $a_1 a_2$ is a 4-dimensional subalgebra $\langle\langle a_1, a_2\rangle\rangle$ of A_3 that is isomorphic to the quaternions. The notation indicates that the subalgebra is generated by a_1 and a_2.

The annihilator $\mathrm{Ann}(a)$ is a four-dimensional subspace of A_4 consisting of all elements of the form $(y, -cy)$, where c is the fixed unit vector with the same direction as $a_1 a_2$ and x ranges over the orthogonal complement of $\langle\langle a_1, a_2\rangle\rangle$. The subspace $\langle\langle a_1, a_2\rangle\rangle \times \langle\langle a_1, a_2\rangle\rangle$ is orthogonal to $\mathrm{Ann}(a)$. Let $\mathbf{Eig_2}(\boldsymbol{a})$ be the orthogonal complement of $\mathrm{Ann}(a)$ and $\langle\langle a_1, a_2\rangle\rangle \times \langle\langle a_1, a_2\rangle\rangle$. This space consists of all elements of the form (y, cy), where c and x are as above. Direct calculation shows that $\mathrm{Eig}_2(a)$ is equal to the space of all elements b of A_4 such that $a(ab) = -2b$. From this perspective, it is the 2-eigenspace of the composition of left multiplication by a and left multiplication by $a^* = -a$.

COROLLARY (7.4). *Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be zero-divisors in A_4. Then $\{a, b\}$ belongs to the D-locus in A_5 if and only if b belongs to the $\mathbb{R}$-linear span of $(a_1, -a_2)$, (a_2, a_1), and $\mathrm{Eig}_2(a)$.*

Proof. Since a_1 and a_2 are orthogonal and have the same norm, there exists an imaginary element c of unit length such that $a_2 = ca_1$. There exists an automorphism of A_3 that takes c to $-i_3$. Therefore, we may assume that $c = -i_3$. In other words, we may assume that $a = \{a_1, 0\}$.

Then $\mathrm{Eig}_2(a)$ is equal to the space of all elements of the form $\{y, 0\}$, where y is $\mathbb{C}$-orthogonal to 1 and a. Also, $\{0, a\}$ equals $(a_1, -a_2)$, so the $\mathbb{C}_4$-linear span of $\{0, a\}$ is the same as the $\mathbb{R}$-linear span of $(a_1, -a_2)$ and (a_2, a_1).

Finally, apply Proposition (7.3). $\square$

Recall that $V_2(\mathbb{R}^7)$ is the space of orthonormal 2-frames in $\mathbb{R}^7$. In the following theorem, we identify this space with the space of elements (a_1, a_2) of A_4 such that a_1 and a_2 are orthogonal imaginary unit vectors in A_3.

THEOREM (7.5). *Consider the space X consisting of all elements $\{a, b\}$ belonging to the D-locus in A_5 such that a and b are zero-divisors with unit length. Let ξ be the 4-plane bundle over $V_2(\mathbb{R}^7)$ whose unit sphere bundle has total space diffeomorphic to the 14-dimensional compact simply connected Lie group G_2 (see* [DDD], *Section 7). Then X is diffeomorphic to the unit sphere bundle of $\xi \oplus 2$, where $\xi \oplus 2$ is the fiberwise sum of the vector bundle ξ with the trivial 2-dimensional bundle.*

Proof. First, identify $V_2(\mathbb{R}^7)$ with the space of all zero-divisors in A_4 with unit length. Let η be the bundle over $V_2(\mathbb{R}^7)$ whose fiber over a is the space of all ordered pairs (a, b) such that b is a unit length element of $\mathrm{Eig}_2(a)$. The bundle ξ is also a bundle over $V_2(\mathbb{R}^7)$, but the fiber over a is the space of all ordered pairs (a, b) such that b is a unit length element of $\mathrm{Ann}(a)$.

Using the notation in the paragraphs preceding Corollary (7.4), the isomorphism $\mathrm{Eig}_2(a) \to \mathrm{Ann}(a)\colon (y, cy) \mapsto (y, -cy)$ induces an isomorphism from η to ξ.

Next consider the space of all ordered pairs (a, b) such that a is a unit length zero-divisor and b belongs to the $\mathbb{R}$-span of $(a_1, -a_2)$ and (a_2, a_1), where $a = (a_1, a_2)$. The map that takes (a, b) to a is a trivial 2-plane bundle.

Corollary (7.4) shows that X is the unit sphere bundle of $\eta \oplus 2$. $\square$

Remark (7.6). An obvious consequence of Theorem (7.5) is that X is diffeomorphic to the total space of an S^5-bundle over $V_2(\mathbb{R}^7)$. This bundle is the fiberwise double suspension of the usual S^3-bundle over $V_2(\mathbb{R}^7)$ that is used to construct G_2.

8. Stability

Sections (5) and (6) described many properties of annihilators of elements of the form $\{a, b\}$. This section exploits these properties to study large annihilators, i.e., annihilators in A_n whose dimension is at least 2^{n-1}.

We begin with a result that could have been included in [DDD], but its significance was not apparent at the time.

THEOREM (8.1). *Let $n \geq 3$, and let a belong to A_n. If the dimension of* $\mathrm{Ann}(a)$ *is at least 2^{n-1}, then a belongs to $\mathbb{H}_n^{\perp}$.*

Proof. Let $a = (b, c)$. We claim that b and c are both zero-divisors; otherwise, [DDD], Lemma 9.9, would imply that $\mathrm{Ann}(a)$ has dimension at most $2^{n-1} - 1$. Lemma (2.3.1) implies that b and c belong to $\mathbb{C}_{n-1}^{\perp}$. □

Theorem (8.1) is important in the following way. When searching for zero-divisors with large annihilators, i.e., with annihilators whose dimension is at least half the dimension of A_n, one need only look in $\mathbb{H}_n^{\perp}$. Fortunately, Sections (5) and (6) study zero-divisors in $\mathbb{H}_n^{\perp}$ in great detail.

Next we show by construction that the bound of Theorem (8.1) is sharp in the sense that there exist elements of A_n that do not belong to $\mathbb{H}_n^{\perp}$ but whose annihilators have dimension $2^{n-1} - 4$. Recall that an element a of A_n is alternative if $a \cdot ax = a^2x$ for all x. For every n, there exist elements of A_n that are alternative. For example, a straightforward computation shows that if a is an alternative element of A_{n-1}, then $(a, 0)$ is an alternative element of A_n.

PROPOSITION (8.2). *Let a be any non-zero alternative element of $\mathbb{C}_{n-1}^{\perp}$ such that $|a| = 1$. Then $\mathrm{Ann}(i_{n-1}, a)$ is equal to the set of all elements of the form $(x, ai_{n-1} \cdot x)$ such that x is $\mathbb{C}$-orthogonal to 1 and to a. In particular, the dimension of $\mathrm{Ann}(i_{n-1}, a)$ is equal to $2^{n-1} - 4$.*

Proof. Let x be $\mathbb{C}$-orthogonal to both 1 and a. Using Lemma (2.1.3), compute that the product $(i_{n-1}, a)(x, ai_{n-1} \cdot x)$ is always zero. We have exhibited a subspace of $\mathrm{Ann}(i_{n-1}, a)$ that has dimension $2^{n-1} - 4$. By Theorem (8.1), this subspace must be equal to $\mathrm{Ann}(i_{n-1}, a)$. □

A proof of Proposition (8.2) also appears in [M3], Theorem 4.4.

Question (8.3). Find all of the elements of A_n that have annihilators of dimension $2^{n-1} - 4$.

The paper [DDD] began an exploration of the largest annihilators in A_n. Recall from Theorem (2.3.2) that the annihilators in A_n have dimension at most $2^n - 4n + 4$. Moreover, Theorem 15.7 of [DDD] gives a complete description of the elements whose annihilators have dimension equal to this upper bound. The rest of this section provides more results in a similar vein.

Definition (8.4). Let $n \geq 4$, and let c be a multiple of 4 such that $0 \leq c \leq 2^n - 4n$. The space $\boldsymbol{T_n^c}$ is the space of elements of length one in A_n whose annihilators have dimension at least $(2^n - 4n + 4) - c$.

This is a change in the definition of T_n^c from that used in [DDD]. The elements of T_n^c are unit length zero-divisors whose annihilators are within c dimensions of the largest possible value. The space T_4^0 is diffeomorphic to the Stiefel manifold $V_2(\mathbb{R}^7)$ of orthonormal 2-frames in $\mathbb{R}^7$ [DDD], Section 12.

We have imposed the condition $n \geq 4$ in order to avoid trivial exceptions to our results involving well-known properties of A_n for $n \leq 3$. Also, we have imposed the condition $c \leq 2^n - 4n$ to ensure that every element of T_n^c is always a zero-divisor.

It follows from Lemma (2.3.1) that T_n^c is contained in $\mathbb{C}_n^{\perp}$. Thus, if a and b lie in T_n^c, then it makes sense to talk about $\{a, b\}$. Note that if a is in T_n^c then $\{a, 0\}$ and $\{0, a\}$ lie in T_{n+1}^c. This is because, according to Proposition (5.11), both $\operatorname{Ann}\{a, 0\}$ and $\operatorname{Ann}\{0, a\}$ have dimension equal to $\dim \operatorname{Ann}(a) + 2^n - 4$. Consequently, T^c_{n+1} contains a disjoint union of two copies of T^c_n.

Definition (8.5). The space T^c_n is **stable** if T^c_{n+1} is diffeomorphic to the space of elements of the form $\{a, 0\}$ or $\{0, a\}$ such that a belongs to T^c_n.

For $n \geq 4$, the space T^0_n is stable [DDD], Proposition 15.6; a vastly simpler proof appears below. In fact, our goal is to completely determine which spaces T^c_n are stable.

PROPOSITION (8.6). *Let $n \geq 4$, and let a belong to A_n. If the dimension of* $\operatorname{Ann}(a)$ *is at least $2^n - 8n + 24$, then a is of the form $\{b, 0\}$ or $\{0, b\}$ with b in* $\mathbb{C}^\perp_{n-1}$.

Proof. Suppose that the dimension of $\operatorname{Ann}(a)$ is at least $2^n - 8n + 24$. Note that $2^{n-1} \leq 2^n - 8n + 24$, so the dimension of $\operatorname{Ann}(a)$ is at least 2^{n-1}. By Theorem (8.1), a belongs to $\mathbb{H}^\perp_n$.

Write $a = \{x, y\}$ for some x and y in $\mathbb{C}^\perp_{n-1}$. Assume for contradiction that both x and y are non-zero. By Theorem (5.10), the dimension of $\operatorname{Ann}(a)$ is at most $\dim \operatorname{Ann}(x) + \dim \operatorname{Ann}(y) + 4$. But the dimensions of $\operatorname{Ann}(x)$ and $\operatorname{Ann}(y)$ are at most $2^{n-1} - 4n + 8$ by Theorem (2.3.2), so the dimension of $\operatorname{Ann}(a)$ is at most $2^n - 8n + 20$. This is a contradiction, so either x or y is zero. □

PROPOSITION (8.7). *If $n \geq 4$, $c \geq 0$, and $n \geq \frac{c}{4} + 4$, then T^c_n is stable.*

Proof. It follows from the inequalities that $c \leq 2^n - 4n$.

Let a belong to T^c_{n+1}. Note that $2^{n+1} - 4n - c \geq 2^{n+1} - 8n + 16$, so $\operatorname{Ann}(a)$ has dimension at least $2^{n+1} - 8(n+1) + 24$. Proposition (8.6) implies that a is of the form $\{b, 0\}$ or $\{0, b\}$. The result then follows directly from Proposition (5.11). □

LEMMA (8.8). *For $n \geq 3$, there exists an element $\{a, b\}$ belonging to the D-locus in A_{n+1} such that a and b are elements of $\mathbb{C}^\perp_n$ whose annihilators have dimension $2^n - 4n + 4$.*

Proof. The proof is by induction on n. The base case is $n = 3$. Since every element of A_3 has a trivial annihilator, this case just requires us to choose two $\mathbb{C}$-orthogonal elements from the 6-dimensional space $\mathbb{C}^\perp_3$.

Now suppose that a' and b' are elements of $\mathbb{C}^\perp_n$ whose annihilators have dimension $2^n - 4n + 4$. Suppose also that $\{a', b'\}$ belongs to the D-locus in A_{n+1}.

Consider the elements $a = \{a', 0\}$ and $b = \{b', 0\}$ of A_{n+1}. By Proposition (5.11) and the induction assumption, a and b have annihilators of dimension $2^{n+1} - 4(n+1) + 4$, as desired.

It remains to show that $\{a, b\}$ belongs to the D-locus in A_{n+2}. By Lemma (4.1.1) and the induction assumption, a and b are $\mathbb{C}$-orthogonal. Proposition (5.11) describes $\operatorname{Ann}(a)$ and $\operatorname{Ann}(b)$. By inspection of this description, b is $\mathbb{C}$-orthogonal to $\operatorname{Ann}(a)$ because b' is $\mathbb{C}$-orthogonal to $\operatorname{Ann}(a')$ by the induction assumption. Similarly, a is $\mathbb{C}$-orthogonal to $\operatorname{Ann}(b)$. □

LEMMA (8.9). *For $n \geq 4$, there exist non-zero elements a and b in $\mathbb{C}^\perp_n$ such that* $\operatorname{Ann}\{a, b\}$ *has dimension $2^{n+1} - 8n + 12$.*

Proof. By Lemma (8.8), there exist non-zero elements of $\mathbb{C}_n^{\perp}$ such that $\{a, b\}$ belongs to the D-locus in A_{n+1} and such that $\mathrm{Ann}(a)$ and $\mathrm{Ann}(b)$ both have dimension $2^n - 4n + 4$. Now apply Theorem (6.12) to conclude that $\mathrm{Ann}\{a, b\}$ has dimension $2^{n+1} - 8n + 12$. □

Remark (8.10). Lemma (8.9) shows that the bound of Proposition (8.6) is sharp. Substitute $n - 1$ for n in the lemma to construct an element of A_n whose annihilator has dimension $2^n - 8n + 20$.

PROPOSITION (8.11). *Let $n \geq 4$, let $c \leq 2^n - 4n$, and let $n \leq \frac{c}{4} + 3$. Then T_n^c is not stable.*

Proof. Note that $2^{n+1} - 4(n+1) + 4 - c \leq 2^{n+1} - 8n + 12$. Now apply Lemma (8.9) to construct an element $\{a, b\}$ belonging to T_{n+1}^c such that both a and b are non-zero. □

THEOREM (8.12). *Let $n \geq 4$, and let c be a multiple of 4 such that $0 \leq c \leq 2^n - 4n$. Then T_n^c is stable if and only if $n \geq \frac{c}{4} + 4$.*

Proof. Combine Propositions (8.7) and (8.11). □

We give two illustrations of the theorem.

COROLLARY (8.13). *The space of zero-divisors in A_5 whose annihilators are 16-dimensional is diffeomorphic to two disjoint copies of $V_2(\mathbb{R}^7)$.*

Proof. Apply Theorem (8.12) with $n = 5$ and $c = 0$. □

Corollary (8.13) is the same as [DDD], Corollary 14.7. The proof is vastly more graceful than the one in [DDD]. This demonstrates the power of our computational perspective.

COROLLARY (8.14). *The space of zero-divisors in A_6 whose annihilators are at least 40-dimensional is diffeomorphic to two disjoint copies of the space of zero-divisors in A_5 whose annihilators are at least 12-dimensional.*

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MODEL CATEGORIES AND CUBICAL DESCENT

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ABSTRACT. We prove that the subcategory of fibrant objects of a simplicial model category is a cohomological descent category, in the sense of GuillÚn and Navarro, if and only if an acyclicity criterion holds.

1. Introduction

Let k be a field of characteristic zero, $Sm(k)$ the category of smooth schemes over k and $Sch(k)$ the category of separated and finite type schemes over k. GuillÚn and Navarro have proved [6] an extension result for cohomological functors defined on $Sm(k)$ to cohomological functors defined on $Sch(k)$. Classical cohomogical functors take values in the category of graded abelian groups or in the category of abelian chain complexes. In order to apply their main result to non-abelian situations, such as the rational homotopy type of $\mathbb{C}$-schemes or to motives of singular varieties, they introduced the notion of (cohomological) descent category, a higher category variation of Verdier triangulated categories, as a good class of categories in which cohomology theories take values.

A *descent category* [6] is, essentially, a triple $(\mathcal{D}, E, \mathbf{s})$ given by a cartesian category $\mathcal{D}$ with initial object 0, a saturated class of morphisms E of $\mathcal{D}$, called *weak equivalences*, and for every cubical type $\square$ (see section 4.1) a functor

$$\mathbf{s}_\square : (\square, \mathcal{D}) \to \mathcal{D},$$

called *simple*, from $\square$-diagrams in $\mathcal{D}$ to $\mathcal{D}$, analogous to the limit, that is natural in $\square$ in a precise sense and satisfies the following properties:

1. *Multiplicativity.* The simple of an object X considered as a diagram is isomorphic to X, and for every pair $(\mathbf{X}, \mathbf{Y})$ of $\square$-diagrams there is an isomorphism $\mathbf{s}_\square(\mathbf{X} \times \mathbf{Y}) \to \mathbf{s}_\square\mathbf{X} \times \mathbf{s}_\square\mathbf{Y}$.

2. *Factorisation.* For every $\square \times \square'$-diagram $\mathbf{X} = (\mathbf{X}_{\alpha\beta})$ there is an isomorphism $\mu : \mathbf{s}_{\alpha\beta}\mathbf{X}_{\alpha\beta} \to \mathbf{s}_\alpha\mathbf{s}_\beta\mathbf{X}_{\alpha\beta}$.

3. *Exactness.* Let $f : \mathbf{X} \to \mathbf{Y}$ be a morphism of $\square$-diagrams. If it is a pointwise weak equivalence, then the morphism $\mathbf{s}_\square f : \mathbf{s}_\square\mathbf{X} \to \mathbf{s}_\square\mathbf{Y}$ is a weak equivalence.

4. *Acyclicity criterion.* A morphism $f : X_0 \to X_1$ is a weak equivalence if and only if the simple of the $\square_1$-diagram

$$X_0 \xrightarrow{f} X_1 \leftarrow 0$$

is acyclic, that is, the initial object of $\mathcal{D}$ is weakly equivalent to it.

In fact, an extended acyclicity criterion must hold:

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4'. *Extended acyclicity criterion*. For every augmented diagram $\mathbf{X}^+$ of type $\square_n^+$, the morphism $\mathbf{X}_0 \to \mathbf{s}_{\square_n}\mathbf{X}$ is a weak equivalence if and only if the morphism $0 \to \mathbf{s}_{\square^+}\mathbf{X}^+$ is a weak equivalence.

One problem that appears is to have sufficiently many examples of descent categories. Model categories, with the homotopy limit functor as simple functor, are natural candidates to be descent categories, as the homotopy limit has the factorisation and exactness properties for fibrant objects. We prove that the subcategory of fibrant objects of a simplicial model category is a cohomological descent category if and only if the acyclicity criterion holds. In particular, in a simplicial model category the extended acyclicity criterion is equivalent to the acyclicity criterion.

All stable model categories satisfy the acyclicity criterion. Thus we have as a corollary that the category of fibrant spectra is a cohomological descent category. This result has applications in algebraic K-theory [12].

We remark that the category of topological spaces and homotopy weak equivalences does not verify the acyclicity criterion. We have to take homology isomorphisms. We prove that the category of CW-complexes with h_*-isomorphisms is a homological descent category, where h is a homology theory.

We restrict to simplicial model categories, where a homotopy limit functor is defined. This should not be an important restriction, as it is known that a cofibrantly generated, proper model category with a realization axiom is Quillen equivalent to a simplicial model category [14].

2. Recollections and notations

(2.1) Model categories were introduced by Quillen [13]. We use the definition adopted by Hirschhorn [7], 7.1, which has stronger conditions than Quillen's (all small limits and colimits are required to exist, and also functorial factorisations in axiom five). Modern references of model categories use this definition, but we observe that in our case the stronger conditions are not necessary.

Given $\mathcal{M}$ a simplicial model category, tensoring an object X by a simplicial set K is denoted by $X \otimes K$, and $F(K, X)$ denotes the cotensorisation (usually denoted by X^K).

By adjointness properties [4], II 2.1 and 2.2, $F(K, -)$ preserves limits and $F(-, X)$ converts colimits to limits. We also have $F(K \times L, X) \cong F(K, F(L, X))$ [7], 9.1.11.

(2.2) A model category is *pointed* if the initial and final objects coincide. In this case the initial and final object is denoted by $*$. Let $*$ also denote the one point simplicial set. There are isomorphisms $F(*, X) = X$ and $F(K, *) = *$.

An object X in a pointed model category is *acyclic* if either of the natural morphisms $* \to X$ or $X \to *$ is a weak equivalence, in which case both are.

(2.3) Recall that X is *fibrant* if $X \to *$ is a fibration. Fibrant objects are closed by cotensoring and by taking finite products.

(2.4) The homotopy category $Ho(\mathcal{C})$ of a model category $\mathcal{C}$ is obtained by inverting the weak equivalences. Weak equivalences are *saturated*: a morphism $f : X \to Y$ is a weak equivalence if, and only if, it is an isomorphism in the homotopy category [7], 8.3.10. See [6], 1.5.1 for the notion of saturated in the context of descent categories.

(2.5) If $\mathcal{C}$ is a small category, $B\mathcal{C}$ denotes the classifying space or nerve of $\mathcal{C}$. If α is an object of $\mathcal{C}$, $\mathcal{C}\downarrow\alpha$ denotes the category of objects of $\mathcal{C}$ over α. We use the notations and definitions of [7] concerning classifying spaces, overcategories, homotopy limits and homotopy cofinal functors.

3. Homotopy limits

In this section we recall briefly the definition of the homotopy limit of a diagram in a simplicial model category and its main properties.

(3.1) Given $\mathcal{C}$ a small category and $\mathcal{M}$ a category, a $\mathcal{C}$-codiagram of $\mathcal{M}$, or a *codiagram of type* $\mathcal{C}$ is a functor $\mathcal{C}\to\mathcal{M}$ and $(\mathcal{C},\mathcal{M})$ denotes the category of codiagrams of type $\mathcal{C}$.

If X is an object of $\mathcal{M}$, $\mathcal{C}\times X$ denotes the constant codiagram of type $\mathcal{C}$ (with all morphisms equal to the identity of X).

We use the prefix co- here to be consistent with the terminology used in [6], where a diagram of type $\mathcal{C}$ is a functor $\mathcal{C}^{op}\to\mathcal{M}$.

Definition (3.1.1)**.** Let $\mathbf{X}$ be $\mathcal{C}$-codiagram of a simplicial model category $\mathcal{M}$. The *homotopy limit* of $\mathbf{X}$, $\operatorname{holim}\mathbf{X}$, is the equaliser of the morphisms

$$\prod_{\alpha\in Ob(\mathcal{C})} F(B(\mathcal{C}\downarrow\alpha),\mathbf{X}_\alpha)\ \overset{\phi}{\underset{\psi}{\rightrightarrows}}\ \prod_{(\sigma:\alpha\to\alpha')\in\mathcal{C}} F(B(\mathcal{C}\downarrow\alpha),\mathbf{X}_{\alpha'})$$

where the projection of ϕ in the factor $\sigma\colon\alpha\to\alpha'$ is the composition of the natural projection from the product with the morphism

$$\sigma_*^{1_{B(\mathcal{C}\downarrow\alpha)}}\colon F(B(\mathcal{C}\downarrow\alpha),\mathbf{X}_\alpha)\to F(B(\mathcal{C}\downarrow\alpha),\mathbf{X}_{\alpha'})$$

and the projection of ψ to the factor $\sigma\colon\alpha\to\alpha'$ is the composition of the natural projection from the product with the morphism

$$F(B(\sigma_*),1_{\mathbf{X}_{\alpha'}})\colon F(B(\mathcal{C}\downarrow\alpha'),\mathbf{X}_{\alpha'})\to F(B(\mathcal{C}\downarrow\alpha),\mathbf{X}_{\alpha'}),$$

where $\sigma_*:(\mathcal{C}\downarrow\alpha)\to(\mathcal{C}\downarrow\alpha')$.

Example (3.1.2)**.** Given an object X of a simplicial model category $\mathcal{M}$, $\operatorname{holim}\mathcal{C}\times X=F(B\mathcal{C},X)$, as is easily seen from [2], XI, 2.3.

(3.2) We recall the basic properties of homotopy limits in a simplicial model category $\mathcal{M}$. Observe that homotopy invariance and cofinality require pointwise fibrant diagrams. We could drop the fibrant hypothesis doing a functorial fibrant replacement in the definition of the homotopy limit. In that case functorial factorisations in the definition of model category are necessary.

3.2.1. The homotopy limit is an end. The homotopy limit of a $\mathcal{C}$-codiagram is the end of the functor $\mathcal{C}^{op}\times\mathcal{C}\to\mathcal{M}$, $(\alpha,\alpha')\mapsto F(B(\mathcal{C}\downarrow\alpha),\mathbf{X}_{\alpha'})$. We can write

$$\operatorname{holim}\mathbf{X}=\int_\alpha F(B(\mathcal{C}\downarrow\alpha),\mathbf{X}_\alpha)$$

with the notation of [10] (cf. [7], 18.3.2 and 18.3.6). Therefore end properties as the Fubini theorem hold [10].

3.2.2. The homotopy limit is functorial with respect to both variables:

a) If $\mathbf{X}$ and $\mathbf{Y}$ are $\mathcal{C}$-codiagrams, a morphism $f:\mathbf{X}\to\mathbf{Y}$ induces a morphism
$$\operatorname{holim} f:\ \operatorname{holim}\mathbf{X}\to\operatorname{holim}\mathbf{Y}.$$

b) If $F:\mathcal{C}\to\mathcal{D}$ is a functor between small categories and $\mathbf{X}$ a $\mathcal{D}$-codiagram, a $\mathcal{C}$-codiagram $F^*\mathbf{X}$ is induced. Then there is a natural morphism
$$\operatorname*{holim}_{\mathcal{D}}\mathbf{X}\to\operatorname*{holim}_{\mathcal{C}}F^*\mathbf{X}$$
induced by the morphisms $F_*: B(\mathcal{C}\downarrow\alpha)\to B(\mathcal{D}\downarrow F\alpha)$ [7], 19.1.8.

3.2.3. The homotopy limit is particularly well behaved with respect to pointwise fibrant diagrams:

i) If $\mathbf{X}$ and $\mathbf{Y}$ are pointwise fibrant and f is a pointwise fibration, then $\operatorname{holim} f$ is a fibration [7], 18.5.1. In particular, the homotopy limit of a pointwise fibrant diagram is fibrant.

ii) *Homotopy invariance* [7], 18.5.3. If $\mathbf{X}$ and $\mathbf{Y}$ are pointwise fibrant and f is a pointwise weak equivalence, then $\operatorname{holim} f$ is a weak equivalence of fibrant objects.

iii) *Cofinality theorem* [7], 19.6.7b. If the functor $F:\mathcal{C}\to\mathcal{D}$ is homotopy left cofinal and $\mathbf{X}$ is pointwise fibrant, the natural morphism $\operatorname{holim}_{\mathcal{D}}\mathbf{X}\to\operatorname{holim}_{\mathcal{C}}F^*\mathbf{X}$ is a weak equivalence.

3.2.4. The homotopy limit considered as a functor from the category of $\mathcal{C}$-codiagrams to $\mathcal{M}$ preserves limits. This property is deduced as in [16], lemma 5.11, where is stated for spectra.

(3.3) Homotopy fibre and acyclicity criterion. We define the *homotopy fibre* of a morphism $f: X\to Y$ in a pointed simplicial model category $\mathcal{M}$ as $\operatorname{hofib} f=\operatorname{holim}(X\to Y\leftarrow *)$. The *fibre* of f is $\lim(X\to Y\leftarrow *)$.

Definition (3.3.1) (Acyclicity criterion)**.** A simplicial model category $\mathcal{M}$ satisfies the *acyclicity criterion* if a morphism $f: X\to Y$ with X and Y fibrant is a weak equivalence if and only if the homotopy fibre $\operatorname{hofib} f$ is acyclic.

4. Simple functor of a cubical codiagram

In this section we define the simple functor of a codiagram by the homotopy limit. This functor is defined over the category $Codiag_{\Pi}\mathcal{M}$ of cubical codiagrams with variable type in the category Π. We begin by recalling the definition of the category Π of cubical types [6], 1.1.1. After defining the category of codiagrams we define the simple functor and then we explain how to extend the definition to augmented cubical codiagrams.

(4.1) Cubical types. Associate to a non-empty set S the set of non-empty subsets, ordered by inclusion: that defines the category $\square_S$. The category $\square_{\{0,1,\ldots,n\}}$ is denoted by $\square_n$. An element α of $\square_n$ is identified with a non zero element of $\{0,1\}^{n+1}$

Given S and T two finite sets, any injective map $u: S\to T$ defines a functor $\square_u:\square_S\to\square_T$.

Associate to a family $S=(S_i)_{i\in I}$, with I a finite set, the cartesian product $\Pi_{i\in I}\square_{S_i}$ with the product order. Write $\square_S=\Pi_{i\in I}\square_{S_i}$.

The objects of the category Π of *cubical types* are the families $(S_i)_{i\in I}$ of non-empty finite sets, with I a finite set. Given $S = (S_i)_{i\in I}$ and $T = (T_j)_{j\in J}$, a morphism $u\colon S \to T$ of Π is an injective map $u\colon \Pi_i S_i \to \Pi_j T_j$ such that, for every $\alpha = (\alpha_i) \in \square_S$, exists $\beta = (\beta_j) \in \square_T$ such that $u(\Pi\alpha_i) = \Pi\beta_j$.

Now we turn to augmented cubical types. Associate to a finite set S, possibly empty, the set $\square_S^+$ of subsets of S, ordered by inclusion. Observe that $\square_n^+$ is a cube of dimension $(n+1)$.

For example, the categories $\square_2$ and $\square_2^+$ are represented by

$$\begin{tikzcd}
& 111 & & 110 \arrow[ll] \\
101 \arrow[ur] & & 100 \arrow[ll] \arrow[ur] & \\
& 011 \arrow[uu] & & 010 \arrow[ll] \arrow[uu] \\
001 \arrow[uu] \arrow[ur] & & &
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
& 111 & & 110 \arrow[ll] \\
101 \arrow[ur] & & 100 \arrow[ll] \arrow[ur] & \\
& 011 \arrow[uu] & & 010 \arrow[ll] \arrow[uu] \\
001 \arrow[uu] \arrow[ur] & & 000 \arrow[ll] \arrow[uu] \arrow[ur] &
\end{tikzcd},$$

where identities and compositions of two non-identity morphisms are not represented.

(4.2) The category $Codiag_\Pi\mathcal{M}$. Given $\delta\colon \square \to \square'$ a morphism of Π, there is an induced inverse image functor

$$\delta^*\colon (\square', \mathcal{M}) \to (\square, \mathcal{M})$$

defined by $F \mapsto \delta^*(F) := F\circ\delta$.

The category $Codiag_\Pi\mathcal{M}$ of *cubical codiagrams* is defined as follows. An object is a pair $(X, \square)$, where $\square \in Ob\Pi$ and $X\colon \square \to \mathcal{M}$ is a $\square$-codiagram. A morphism $(\mathbf{Y}, \square') \to (\mathbf{X}, \square)$ is a pair (a, δ) where $\delta\colon \square \to \square'$ is a morphism of Π and $a\colon \delta^*\mathbf{Y} \to \mathbf{X}$ is a natural transformation of functors of $(\square, \mathcal{M})$.

The category $Codiag_\Pi\mathcal{D}$ of codiagrams is analogous to the category $Diag_\Pi\mathcal{D}$ of diagrams [6], 1.2.1.

(4.3) Simple functor of a codiagram. Let $\mathcal{M}_f$ be the category of fibrant objects of a simplicial model category. Let $\square$ be an object of Π. The *simple functor*

$$\mathbf{s}\colon Codiag_\Pi\mathcal{M}_f \to \mathcal{M}_f$$

associates to a $\square$-codiagram $\mathbf{X}$ the object defined by the homotopy limit $\mathbf{s}_\square(\mathbf{X}) = \operatorname{holim}\mathbf{X}$.

If we have $(\mathbf{Y}, \square') \to (\mathbf{X}, \square)$ a morphism in $Codiag_\Pi\mathcal{M}_f$ given by $\delta\colon \square \to \square'$ and $a\colon \delta^*\mathbf{Y} \to \mathbf{X}$, the functorial properties of the homotopy limit (see section 3.2.3) allow us to define the composition

$$\mathbf{s}_{\square'}\mathbf{Y} = \operatorname*{holim}_{\square'}\mathbf{Y} \to \operatorname*{holim}_{\square}\delta^*\mathbf{Y} \to \operatorname*{holim}_{\square}\mathbf{X} = \mathbf{s}_\square\mathbf{X},$$

which gives the covariance of the functor.

A codiagram $\mathbf{X}$ is *acyclic* if the object $\mathbf{sX}$ is acyclic.

5. Simple of an augmented cubical codiagram

In this section we explain first how the simple functor is extended to augmented cubical codiagrams. Then we prove that the acyclicity criterion is equivalent to an extended acyclicity criterion in a simplicial model category. The proof generalises a property of cubes of spectra, following [17], 1.1. For a similar result for the case of topological spaces see [5], 1.1.

First observe that the homotopy limit of an augmented cubical codiagram $\mathbf{X}^+$ is $\mathbf{X}_0$, as the codiagram has an initial object. Thus the simple functor of an augmented cubical codiagram is not the homotopy limit.

The functor $\mathbf{s}$ is extended to augmented cubical codiagrams by using the cone construction [6], 1.4.3. It can be calculated as follows. Given a $\square_n^+$-codiagram $\mathbf{X}^+$, view it as a morphism of two $\square_{n-1}^+$-codiagrams, $f\colon \mathbf{X}_0^+ \to \mathbf{X}_1^+$. The simple object associated to $\mathbf{X}^+$ is obtained as the simple of the $\square_{n-1}^+$-codiagram which in each degree α has the homotopy fibre of f_α. This construction does not depend on the order chosen [6], 1.4.3.

In particular if $f\colon X \to Y$ is a $\square_0^+$-codiagram, its simple is the homotopy fibre of f.

The acyclic augmented cubes are also called *homotopy cartesian* cubes.

Augmented cubical types allow induction, as $\square_n^+ = \square_1^+ \times \square_{n-1}$. Non-augmented cubical types do not have this property. The lemmas below give homotopy left cofinal functors which allow induction in some cases.

LEMMA (5.1). *The functor $f\colon \square_n \to \square_{n-1}^+$ defined by $(i, j) \mapsto (j)$ is homotopy left cofinal.*

Proof. Given $\alpha = (1, j) \in \square_n$, the category $(f \downarrow f(\alpha))$ is isomorphic to the category $(\square_n \downarrow \alpha)$ and thus contractible. $\square$

LEMMA (5.2). *The functor $f\colon \square_1 \times \square_{n-1} \to \square_n$ defined by $f((0, 1), k) = (0, k)$, $f((1, 1), k) = (1, k)$ and $f((1, 0), k) = (1, 0, \dots, 0)$ is homotopy left cofinal.*

Proof. It is easy to see that the required categories are contractible: the category $(f \downarrow (1, 0, \dots, 0))$ is isomorphic to $\square_{n-1}$, the category $(f \downarrow (0, k))$ is isomorphic to $(\square_{n-1} \downarrow (k))$ and the category $(f \downarrow (1, k))$ is isomorphic to $(\square_n \downarrow (1, k))$. $\square$

LEMMA (5.3). *Let $\mathcal{M}$ be a pointed simplicial model category. Given a functor $\mathbf{E}\colon \square_n \to \mathcal{M}$ such that $\mathbf{E}_{1,k}$ is acyclic for every $k \in \square_{n-1}$, then there is a weak equivalence* $\operatorname{holim}_{\square_n} \mathbf{E} \simeq \operatorname{holim}_{\square_{n-1}} \mathbf{E}_{0,-}$.

Proof. Let f be the functor of lemma (5.2). The result is deduced by cofinality, Fubini and homotopy invariance ($\mathbf{E}_{1,k}$ is acyclic):

$$\operatorname*{holim}_{\square_n} \mathbf{E} \xrightarrow{\simeq} \operatorname*{holim}_{\square_1 \times \square_{n-1}} f^*\mathbf{E} \cong \operatorname*{holim}_{j \in \square_1} \operatorname*{holim}_{k \in \square_{n-1}} \mathbf{E}_{f(j,k)} \simeq$$

$$\simeq \operatorname{holim}(* \to * \leftarrow \operatorname*{holim}_{k \in \square_{n-1}} \mathbf{E}_{0,k}) \cong \operatorname*{holim}_{\square_{n-1}} \mathbf{E}_{0,-}\,.$$

$\square$

PROPOSITION (5.4). *Let $\mathcal{M}_f$ be the category of fibrant objects of a pointed simplicial model category where the acyclicity criterion holds. There is a weak equivalence*

$$\mathbf{s}_{\square^+}\mathbf{X}^+ \simeq \mathrm{hofib}(\mathbf{X}_0 \to \operatorname*{holim}_{\square_n} \mathbf{X})\,.$$

Proof. We use induction on n. The case $n = 0$ holds by definition.

Define $\mathbf{E}^+ : \square_n^+ \to \mathcal{M}_f$ by $\mathbf{E}_{0,\mathbf{j}} = \mathrm{hofib}(\mathbf{X}_{0,\mathbf{j}} \to \mathbf{X}_{1,\mathbf{j}})$, and $\mathbf{E}_{1,\mathbf{j}} = \mathrm{hofib}(\mathbf{X}_{1,\mathbf{j}} \to \mathbf{X}_{1,\mathbf{j}})$. Thus, $\mathbf{s}_{\square_n^+}\mathbf{X}^+ = \mathbf{s}_{\square_{n-1}^+}\mathbf{E}_{0,-}^+$.

By induction and lemma (5.3) above $\mathbf{s}_{\square_{n-1}^+}\mathbf{E}_{0,-}^+ \simeq \mathrm{hofib}(\mathbf{E}_0 \to \mathrm{holim}_{\square_{n-1}} \mathbf{E}_{0,-})$ $\simeq \mathrm{hofib}(\mathbf{E}_0 \to \mathrm{holim}_{\square_n} \mathbf{E})$.

To finish it is enough to see that $\mathrm{hofib}(\mathbf{X}_0 \to \mathrm{holim}_{\square_n} \mathbf{X})$ is weakly equivalent to $\mathrm{hofib}(\mathbf{E}_0 \to \mathrm{holim}_{\square_n} \mathbf{E})$.

To this end, define $\mathbf{Y}\colon \square_n^+ \to \mathcal{M}_f$ by $\mathbf{Y}_{0,\mathbf{j}} = \mathbf{Y}_{1,\mathbf{j}} = \mathbf{X}_{1,\mathbf{j}}$, where $\mathbf{j} \in \square_{n-1}^+$. Consider the functor $f\colon \square_n \to 1 \times \square_{n-1}^+$ defined by $(i, \mathbf{j}) \mapsto (1, \mathbf{j})$. By lemma (5.1) above this functor is homotopy left cofinal. As $\square_{n-1}^+$ has an initial object and by cofinality theorem we have:

$$\mathbf{X}_{1,0,\ldots,0} \xrightarrow{\simeq} \operatorname*{holim}_{\square_{n-1}^+} \mathbf{X}_{1,-}^+ \xrightarrow{\simeq} \operatorname*{holim}_{\square_n} f^*(\mathbf{X}_{1,-}^+) = \operatorname*{holim}_{\square_n} \mathbf{Y}\,.$$

Consider the following commutative diagram:
$$\begin{tikzcd}
\mathbf{X_0} \arrow[r] \arrow[d, "\beta"'] & \mathbf{Y_0} = \mathbf{X}_{1,0,\ldots,0} \arrow[d, "\gamma"', "\simeq"] \\
\mathrm{holim}_{\square_n} \mathbf{X} \arrow[r] & \mathrm{holim}_{\square_n} \mathbf{Y}
\end{tikzcd}.$$

If we calculate its simple by rows we obtain $\mathrm{hofib}(\mathbf{E_0} \to \mathrm{holim}_{\square_n} \mathbf{E})$, by definition of $\mathbf{E}$. If we calculate its simple by columns we obtain the homotopy fiber of the first column, $\mathrm{hofib}(\mathbf{X_0} \to \mathrm{holim}_{\square_n} \mathbf{X})$, as the homotopy fiber of the second column is trivial by the acyclicity criterion. □

COROLLARY (5.5). *Let $\mathcal{M}_f$ be the category of fibrant objects of a simplicial model category where the acyclicity criterion holds. An augmented codiagram $\mathbf{X}^+\colon \square_n^+ \to \mathcal{M}_f$ is acyclic if, and only if, the canonical morphism $\mathbf{X}_0 \to \mathrm{holim}_{\square_n} \mathbf{X}$ is a weak equivalence.*

Proof. This follows from the above proposition and the acyclicity criterion. □

6. Main result

A *cohomological descent category* [6], 1.5.3, 1.7.1, is given by $(\mathcal{D}, E, \mathbf{s}, \mu, \lambda)$ satisfying the eight properties $(\mathrm{CD1})^{op}$ to $(\mathrm{CD8})^{op}$ below, which are stated here for $\mathcal{D} = \mathcal{M}_f$ and E the class of weak equivalences.

Let $\mathcal{M}$ be a pointed simplicial model category and $\mathcal{M}_f$ the subcategory of fibrant objects. Let $\mathbf{s}$ be the simple functor defined by the homotopy limit.

By the general properties of ends [10], given $\square, \square' \in \Pi$, we have a natural transformation of functors

$$\mu_{\square,\square'} : \mathbf{s}_\square \circ \mathbf{s}_{\square'} \to \mathbf{s}_{\square\times\square'}$$

such that $\mu_{\square,\square'}(\mathbf{X}) : \mathbf{s}_\square \circ \mathbf{s}_{\square'}(\mathbf{X}) \to \mathbf{s}_{\square\times\square'}(\mathbf{X})$ is an isomorphism for every $\square \times \square'$-codiagram $\mathbf{X}$. This isomorphism is called the Fubini isomorphism.

The morphism $\lambda_{\Box_S}(X) : X = F(*, X) \to F(B(\Box_S), X)$ is the one induced by the simplicial set morphism $B(\Box_S) \to *$.

THEOREM (6.1). *The category $\mathcal{M}_f$ of fibrant objects of a pointed simplicial model category where the acyclicity criterion holds, with the class E of weak equivalences, the simple functor* $\mathbf{s}$ *and* μ *and* λ *defined above, is a cohomological descent category.*

Proof. See $(\mathrm{CD1})^{op}$ to $(\mathrm{CD8})^{op}$ below. □

(CD1)^{op}. $\mathcal{M}_f$ is a cartesian category with initial object.

Proof. Recall that a category is cartesian if it has all finite products, and $\mathcal{M}_f$ has them. □

(CD2)^{op}. The class of weak equivalences is a saturated class of morphisms, stable by products: if $f : X \to X'$ and $g : Y \to Y'$ are weak equivalences, then $f \times g : X \times Y \to X' \times Y'$ is a weak equivalence.

Proof. In every model category, the class of weak equivalences is saturated (see section 2.4).

The stability by products is seen obtaining $X \times Y$ as a homotopy limit of a discrete diagram and using $(\mathrm{CD5})^{op}$. □

If $\delta : \Box \to \Box'$ is a morphism of Π, there is a *direct image* functor

$$\delta_* : (\Box, \mathcal{M}) \to (\Box', \mathcal{M})$$

such that if $\mathbf{X}$ is a $\Box$-codiagram of $\mathcal{M}$ then $\delta_*\mathbf{X}$ is the $\Box'$-codiagram defined by

$$(\delta_*\mathbf{X})_\beta = \begin{cases} \mathbf{X}_\alpha & \text{if } \beta = \delta(\alpha), \alpha \in \Box \\ * & \text{if } \beta \in \Box' \setminus \delta(\Box) \end{cases}$$

with the evident morphisms. This definition is dual to the one for diagrams [6], 1.2.2.

(CD3)^{op}. $\mathbf{s} : \mathit{Codiag}_\Pi\mathcal{M}_f \to \mathcal{M}_f$ is a covariant functor such that if $\delta : \Box \to \Box'$ is a morphism of Π and $\mathbf{X}$ is a $\Box$-codiagram of $\mathcal{M}_f$, the morphism $\mathbf{s}_{\Box'}\delta_*\mathbf{X} \to \mathbf{s}_\Box\mathbf{X}$ is a weak equivalence.

Proof. The functor $\mathbf{s}$ has been defined in section 4.3.

We may assume that $\Box = \Box_S = \Box_{S_1} \times \cdots \times \Box_{S_s}$ and $\Box' = \Box_T = \Box_{T_1} \times \cdots \times \Box_{T_t}$ with $t \geq s$, and that the morphism δ is induced by inclusions $S_i \subset T_i$ and constants $\gamma_{s+1} \in T_{s+1}, \ldots, \gamma_t \in T_t$.

Thus δ embeds $\Box_S$ as a full subcategory of $\Box_T$. Moreover, there are no arrows from a vertex of $\Box_T \setminus \delta(\Box_S)$ to a vertex of $\delta(\Box_S)$. These properties and the definition of δ_* gives us an isomorphism $\operatorname{holim}_{\Box_T}(\delta_*\mathbf{X}) \cong \operatorname{holim}_{\Box_S}\mathbf{X}$. □

Recall that a monoidal functor between monoidal categories is *strong* [10, XI.2] if the Knneth and unit morphisms are isomorphisms. If the second category has a saturated class of morphisms, we say that the functor is *quasistrong* if the Knneth and unit morphisms belong to the saturated class (GuillÚn and Navarro say *quasi-strict* [6, 1.5.2]) .

Recall that an *op-monoidal* functor is defined similarly to a monoidal functor with the directions of the Knneth and unit morphisms reversed.

(CD4)^{op}. For every object $\square$ of Π, the functor $\mathbf{s}_\square : (\square, \mathcal{M}_f) \to \mathcal{M}_f$ is op-monoidal and quasistrong.

Proof. In fact we prove that $\mathbf{s}_\square$ is a strong op-monoidal functor. The Knneth morphism $\sigma = \sigma_\square(\mathbf{X}, \mathbf{Y}) : \mathbf{s}_\square(\mathbf{X} \times \mathbf{Y}) \to \mathbf{s}_\square\mathbf{X} \times \mathbf{s}_\square\mathbf{Y}$ is an isomorphism because holim preserves limits:

$$\mathbf{s}_\square(\mathbf{X} \times \mathbf{Y}) = \operatorname{holim}(\mathbf{X} \times \mathbf{Y}) \cong \operatorname{holim}(\mathbf{X}) \times \operatorname{holim}(\mathbf{Y}) = \mathbf{s}_\square(\mathbf{X}) \times \mathbf{s}_\square(\mathbf{Y}).$$

Observe that σ is natural in $(\mathbf{X}, \mathbf{Y})$.

The unit morphism $\sigma^1_\square : \mathbf{s}_\square(1 \times \square) \to 1$ is clearly an isomorphism: the realization of a constant codiagram in the initial object 1 is $F(K, 1) = 1$ (see example 3.1.2).

Finally, it is clear that σ and σ^1 verify the associativity and unit restrictions and that $\mathbf{s}_\square$ is an op-monoidal functor. $\square$

(CD5)^{op}. *If $f : \mathbf{X} \to \mathbf{Y}$ is a morphism of $\square$-codiagrams of $\mathcal{M}_f$ such that for every $\alpha \in \square$, f_α is a weak equivalence, then $\mathbf{s}_\square f : \mathbf{s}_\square\mathbf{X} \to \mathbf{s}_\square\mathbf{Y}$ is a weak equivalence.*

Proof. This is exactly the homotopy invariance property of the homotopy limit (3.2.3). $\square$

We introduce now the category $Coreal_\Pi\mathcal{M}$ of corealisations, which is analogous to the category $Real_\Pi\mathcal{M}$ of realisations [6, 1.2.1]

Given $\delta : \square \to \square'$ a morphism of Π, there is an induced direct image functor

$$\delta_* : ((\square, \mathcal{M}), \mathcal{M}) \to ((\square', \mathcal{M}), \mathcal{M})$$

defined by $f \mapsto \delta_*(f) := f \circ \delta^*$.

The category $Coreal_\Pi\mathcal{M}$ of corealisations of cubical codiagrams is defined as follows. An object is a functor $s_\square \in ((\square, \mathcal{M}), \mathcal{M})$. A morphism from $s_{\square'} \in ((\square', \mathcal{M}), \mathcal{M})$ to $s_\square \in ((\square, \mathcal{M}), \mathcal{M})$ is a morphism $\delta : \square \to \square'$ of Π and a natural transformation of functors $s_{\square'} \to \delta_* s_\square$ of $((\square', \mathcal{M}), \mathcal{M})$.

Remark (6.2)**.** The category $Coreal_\Pi\mathcal{M}$ has a structure of monoidal category: Given $\square, \square' \in \Pi$, $\mathbf{s}_\square \in ((\square, \mathcal{M}), \mathcal{M})$, $\mathbf{s}_{\square'} \in ((\square', \mathcal{M}, \mathcal{M}))$, the composition

$$\mathbf{s}_\square \circ \mathbf{s}_{\square'} : ((\square \times \square'), \mathcal{M}) \to \mathcal{M}$$

is defined by

$$\mathbf{s}_\square \circ \mathbf{s}_{\square'}(\mathbf{X}) = \mathbf{s}_\square(\alpha \mapsto \mathbf{s}_{\square'}(\beta \mapsto \mathbf{X}_{\alpha\beta})).$$

The unit object is the evaluation functor $Av : (\square_0, \mathcal{M}) \to \mathcal{M}$.

(CD6)^{op}. $(\mathbf{s}, \mu, \lambda_0) : \Pi^{op} \to Coreal_\Pi\mathcal{M}_f$, $\square \mapsto (\mathbf{s}_\square : (\square, \mathcal{M}_f) \to \mathcal{M}_f)$*, is a quasistrong monoidal functor.*

Proof. In fact we prove that $(\mathbf{s}, \mu, \lambda_0)$ is a strong monoidal functor. A (strong) monoidal functor $(\mathbf{s}, \mu, \lambda_0) : (\Pi, \times, \square_0) \to (Codiag_\Pi\mathcal{M}_f, \circ, Av)$ is given by

(i) a functor $\mathbf{s} : \Pi \to Coreal_\Pi\mathcal{M}_f$,

(ii) for every pair $(\square, \square')$ of $\Pi \times \Pi$, a (iso)morphism of $Coreal_\Pi\mathcal{M}_f$ (i.e. a natural transformation of functors)

$$\mu_{\square,\square'} : \mathbf{s}_\square \circ \mathbf{s}_{\square'} \to \mathbf{s}_{\square\times\square'}$$

natural in $(\square, \square')$, and

(iii) a (iso)morphism of $Coreal_{\Pi}\mathcal{M}_f$

$$\lambda_0 : Av \to \mathbf{s}_{\Box_0}$$

compatible with the associativity and unit restrictions.

If we consider $X \in \mathcal{M}_f$ as a $\Box_0$-codiagram, $\mathbf{s}_{\Box_0}X = F(*, X) \cong X$ and $Av(X) = X$. It is clear that we have a natural transformation of functors $\lambda_0 : Av \to \mathbf{s}_{\Box_0}$ such that $\lambda_0(X)$ is an isomorphism.

It is easy to see that μ and λ_0 satisfy the associativity and unit restrictions, so we are done. $\Box$

Given S a non-empty finite set, $\mathbf{s}_{\Box_S}(\Box_S \times X) = F(B(\Box_S), X) = F(\Delta^S, X)$ (see example 3.1.2). For $S = \prod_i S_i$, if we set $\Delta^S = \prod_i \Delta^{S_i}$, the equality also holds.

We denote by $i_\Box$ the functor $X \mapsto \Box \times X$.

(CD7)^{op}. *The morphism λ is a monoidal natural transformation from the functor $G : \Box \mapsto id_{\mathcal{M}_f}$ to the monoidal functor $H : \Box \mapsto \mathbf{s}_\Box \circ i_\Box$, which coincides with λ_0 over $\Box_0$.*

Proof. For every $\Box \in \Pi$ and $X \in \mathcal{M}_f$, the morphism $\lambda_\Box(X)$ is natural in $\Box$ and X, and therefore defines a natural transformation from $G : \Pi^{op} \to (\mathcal{M}_f, \mathcal{M}_f), \Box \mapsto id_{\mathcal{M}_f}$, to $H : \Pi^{op} \to (\mathcal{M}_f, \mathcal{M}_f), \Box \mapsto \mathbf{s}_\Box \circ i_\Box$.

The naturality of λ in $\Box$ is clear: for every morphism $\Box_S \to \Box_T$ of Π, the diagram

$$\begin{tikzcd}
X \arrow[r, "id"] \arrow[d, "\lambda_{\Box_S}(X)"] & X \arrow[d, "\lambda_{\Box_T}(X)"] \\
\mathbf{s}_{\Box_T}(i_{\Box_T}X) = F(\Delta^T, X) \arrow[r] & \mathbf{s}_{\Box_S}(i_{\Box_S}X) = F(\Delta^S, X)
\end{tikzcd}$$

commutes.

It is also clear that λ is natural in X: if $X \to Y$ is a morphism of $\mathcal{M}_f$,

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\lambda_\Box(X)"] & Y \arrow[d, "\lambda_\Box(Y)"] \\
F(\Delta, X) \arrow[r] & F(\Delta, Y)
\end{tikzcd}$$

commutes.

We have to see that the natural transformation λ is monoidal. The functors G and H are between monoidal categories: $(\Pi^{op}, \times, \Box_0)$ and $((\mathcal{M}_f, \mathcal{M}_f), \circ, id_{\mathcal{M}_f})$.

We observe that, given $X \in \mathcal{M}_f$,

$$H(\Box) \circ H(\Box')(X) = (\mathbf{s}_\Box \circ i_\Box) \circ (\mathbf{s}_{\Box'} \circ i_{\Box'})(X) = \mathbf{s}_\Box(i_\Box[\mathbf{s}_{\Box'}(i_{\Box'}X)]) =$$

$$= F(\Delta, F(\Delta', X)) \cong F((\Delta \times \Delta'), X),$$

and that

$$H(\Box \times \Box')(X) = \mathbf{s}_{\Box\times\Box'} \circ i_{\Box\times\Box'}(X) = \mathbf{s}_{\Box\times\Box'}(i_{\Box\times\Box'}X) = F((\Delta \times \Delta'), X)$$

where $\Delta = B\Box$ and $\Delta' = B\Box'$. This defines the morphism in $(\mathcal{M}_f, \mathcal{M}_f)$

$$H_2(\Box, \Box') : \mathbf{s}_{\Box\times\Box'} \circ i_{\Box\times\Box'} \to (\mathbf{s}_\Box \circ i_\Box) \circ (\mathbf{s}_{\Box'} \circ i_{\Box'}).$$

For all $X \in \mathcal{M}_f$ the isomorphism $\mathbf{s}_{\Box_0}(i_{\Box_0}X) = F(*, X) \to X$ defines a morphism in $(\mathcal{M}_f, \mathcal{M}_f)$
$$H_0 : id_{\mathcal{M}_f} \to \mathbf{s}_{\Box_0} \circ i_{\Box_0}.$$
We have that (H, H_2, H_0) is a monoidal functor.

Now we see that the natural transformation λ is monoidal and that it coincides with λ_0 over $\Box_0$. $\Box$

(CD8)^{op}. *Suppose that the acyclicity criterion holds in $\mathcal{M}_f$. For every codiagram $\mathbf{X} : \Box_S \to \mathcal{M}_f$, where S is a finite non-empty set and every augmentation $\varepsilon : \mathbf{X}_0 \to \mathbf{X}$, the morphism $\lambda_\varepsilon := \mathbf{s}_\Box(\varepsilon) \circ \lambda_\Box(\mathbf{X}_0) : \mathbf{X}_0 \to \mathbf{s}_\Box\mathbf{X}$ is a weak equivalence if and only if the canonical morphism $0 \to \mathbf{s}_{\Box^+}\mathbf{X}^+$ is a weak equivalence.*

Proof. This result is exactly corollary 5.5. $\Box$

Everything can be dualised, and we obtain the following dual statement of the main theorem:

THEOREM (6.3) (Dual statement). *The category $\mathcal{M}_c$ of cofibrant objects of a pointed simplicial model category where the dual acyclicity criterion holds, with the class E of weak equivalences, the simple functor defined by the homotopy colimit and μ and λ defined dually as those above, is a homological descent category.*

7. Examples

Our initial target was to prove that the category of fibrant spectra, in the sense of Bousfield and Friedlander [1], is a cohomological descent category with the homotopy limit. The result applies to any of the model categories of spectra available, including symmetric spectra [9] and orthogonal spectra [11]. The acyclicity criterion follows from the fact that model categories of spectra are stable.

(7.1) Stable model categories. Recall that the homotopy category of a pointed model category supports a suspension functor Σ with a right adjoint loop functor Ω. A *stable model category* is a pointed model category where the functors Ω and Σ in the homotopy category are inverse equivalences.

The homotopy category of a stable model category is triangulated [8, 7.1.6]. The cofibre and fibre sequences of [8, 6.2.6] (see also [13, 1.3]) coincide up to sign [8, 7.1.11] and define the distinguished triangles.

PROPOSITION (7.1.1) (Acyclicity criterion for stable simplicial model categories). *Let $\mathcal{M}$ be a stable simplicial model category. A morphism $f : X \to Y$ with X and Y fibrant is a weak equivalence if and only if the homotopy fibre* hofib f *is acyclic.*

Proof. By saturation and passing to the homotopy category, the result follows from the following property of triangulated categories: If
$$X \xrightarrow{f} Y \xrightarrow{g} Z \to \Sigma X$$
is a distinguished triangle in a triangulated category, then g is an isomorphism if and only if $X \cong 0$. $\Box$

The dual acyclicity criterion also holds, replacing the homotopy fibre by the homotopy cofibre. Thus stable simplicial model categories are homological and cohomological descent categories with the homotopy colimit and the homotopy limit respectively.

Example (7.1.2)**.** See [15] for a list of examples of interesting stable simplicial model categories, to which our main result applies, including modules over ring spectra, presheaves of spectra, and spectra categories related to equivariant stable homotopy theory and motivic stable homotopy of schemes.

(7.2) Simplicial sets and topological spaces. We have worked with pointed model categories for simplicity, but everything can be done in unpointed model categories. In that case the extra hypothesis of fibrant initial object is needed.

Topological spaces with homology isomorphisms is the second example of homological descent category of [6]. Usual weak homotopy equivalences can not be used because the acyclicity criterion does not hold if spaces are not arc-wise connected.

Here we recover this example and, furthermore, we obtain the same result for simplicial spaces and CW-complexes with h_*-equivalences respect to a homology theory h.

We only have the homological case because there is not an acyclicity criterion with the homotopy fibre for topological or simplicial spaces.

PROPOSITION (7.2.1). *Let h be a homology theory on the category of simplicial sets. The category of simplicial sets with the homotopy colimit and the h_*-equivalences is a homological descent category.*

Proof. Bousfield localisation respect to a homology theory h_* gives a model category structure on simplicial sets where weak equivalences are those morphisms which induce isomorphism in homology and cofibrations are the usual ones.

Moreover, with its usual simplicial structure it is a simplicial model category [4, X.3]. As the simplicial structure does not change, the homotopy colimit in this simplicial structure is the usual. As cofibrations remain unchanged, all simplicial sets are h_*-cofibrant.

The result follows from the dual statement of the main theorem, observing that The mapping cone sequence gives the acyclicity criterion for any h_*. □

The same result is true for CW-complexes, as Bousfield localisation can be done in topological spaces and the simplicial structure part of the proof of [4, X.3] only relies in the Mayer-Vietoris sequence.

In the case of singular homology we do not need to take CW-complexes because the homotopy colimit has the homotopy invariance property with respect to homology equivalences without the assumption of pointwise cofibrant diagrams [3, 5.16].

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EQUIVALENT NORMS ON DIRICHLET SPACES OF POLYHARMONIC FUNCTIONS ON THE BALL IN $\mathbb{R}^N$

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ABSTRACT. We define $\mathcal{D}^p_{\alpha,k}(B)$, where B is the unit ball in $\mathbb{R}^N$, to be the class of those polyharmonic functions f of order k on B for which

$$|f(0)| + \left(\int_B |\nabla f(x)|^p (1-|x|^2)^\alpha \, dV(x)\right)^{1/p} < \infty,$$

and we present four equivalent norms on $\mathcal{D}^p_{\alpha,k}(B)$. We also consider some equivalent norms on Bloch type spaces.

1. Introduction

Let $\mathbb{R}^N$ $(N \geq 2)$ denote the N-dimensional Euclidean space. A (real-valued) function f defined in a domain $G \subset \mathbb{R}^N$ is said to be polyharmonic of order $k \geq 1$ if $\Delta^k f \equiv 0$ in G, where Δ denotes the ordinary Laplacian. The class of all such functions is denoted by $\mathcal{H}_k(G)$. In this note we consider the case $G = B$, where $B = B_N$ is the unit ball centered at the origin. By the Almansi representation theorem [1], if $f \in \mathcal{H}_k(B)$, then there exist unique harmonic functions $A_m f$ such that

$$f(x) = \sum_{m=0}^{k-1} (1-|x|^2)^m A_m f(x), \qquad x \in B. \tag{1.1}$$

Conversely, if $f_m \in \mathcal{H}(B) := \mathcal{H}_1(B)$, then the function $f(x) = \sum_{m=0}^{k-1}(1 - |x|^2)^m f_m(x)$ is polyharmonic of order k.

For $0 < p < \infty$, and $\alpha > -1$, we define the Dirichlet type space $\mathcal{D}^p_{\alpha,k}(B)$ to be the class of $f \in \mathcal{H}_k(B)$ for which

$$\|f\|_{\mathcal{D}^p_\alpha} := |f(0)| + \left(\int_B |\nabla f(x)|^p (1-|x|^2)^\alpha \, dV(x)\right)^{1/p} < \infty,$$

where dV is the normalized Lebesgue measure on B, and ∇f is the gradient of f,

$$\nabla f = (D_1 f, \ldots, D_N f),$$

$|x|^2 = x_1^2 + \ldots + x_N^2$, and

$$D_i f(x) = \frac{\partial f}{\partial x_i}.$$

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In this note we present some equivalent norms on $\mathcal{D}^p_\alpha$. For a function $f \in C^1(B)$ we denote by Rf the radial derivative of f,

$$Rf(x) = \sum_{i=1}^{N} x_i D_i f(x), \qquad x = (x_1, \dots, x_N),$$

and

$$R_s f = sf + Rf, \qquad s \geq 0.$$

Thus $R_0 = R$.

The main results of our paper are the following theorems.

THEOREM (1.2). *For $0 < p < \infty$, $\alpha > -1$, and $k \geq 2$, the following quantities are equivalent norms on $\mathcal{D}^p_{\alpha,k}(B)$:*

$$Q_1(f) = |f(0)| + \left(\int_B |Rf(x)|^p (1 - |x|^2)^\alpha \, dV(x) \right)^{1/p},$$

$$Q_2(f) = \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{i,j=1}^{N} \left(\int_B |T_{ij} f(x)|^p (1 - |x|^2)^\alpha \, dV(x) \right)^{1/p}.$$

Our second result concerns two equivalent norms which involve the "Almansi coordinates".

THEOREM (1.3). *For $0 < p < \infty$, $\alpha > -1$, and $k \geq 2$, the following quantities are equivalent norms on $\mathcal{D}^p_{\alpha,k}(B)$:*

$$Q_3(f) = \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{m=0}^{k-1} \left(\int_B |\nabla A_m f(x)|^p (1 - |x|^2)^{\alpha + mp} \, dV(x) \right)^{1/p},$$

$$Q_4(f) = \|A_0 f\|_{\mathcal{D}^p_\alpha} + \sum_{m=1}^{k-1} \left(\int_B |A_m f(x)|^p (1 - |x|^2)^{\alpha + (m-1)p} \, dV(x) \right)^{1/p}.$$

Note that Q_3 can be written as

$$Q_3(f) = \sum_{m=0}^{k-1} \|A_m f\|_{\mathcal{D}^p_{\alpha+mp}},$$

while

$$Q_4(f) = \|A_0 f\|_{\mathcal{D}^p_\alpha} + \sum_{m=0}^{k-1} \|A_m f\|_{L^p_{\alpha+(m-1)p}},$$

where

$$\|f\|_{L^p_\alpha} = \left(\int_B |f(x)|^p (1 - |x|^2)^\alpha \, dV(x) \right)^{1/p}. \tag{1.4}$$

(We will use the expression (1.4) even if f is a vector-valued function.) Thus the equivalence $\|f\|_{\mathcal{D}^p_\alpha} \asymp Q_3(f)$ means that the space $\mathcal{D}^p_{\alpha,k}$ is the direct sum of the harmonic spaces $\mathcal{D}^p_{\alpha+mp}$, $m = 0, \dots, k-1$, while the equivalence $\|f\|_{\mathcal{D}^p_\alpha} \asymp Q_4(f)$ means that $\mathcal{D}^p_{\alpha,k}$ is the direct sum of the harmonic space $\mathcal{D}^p_\alpha$ and the harmonic Bergman spaces $\mathcal{H}(B) \cap L^p_{\alpha+(m-1)p}$, $m = 1, \dots k-1$.

The equivalence $\|f\|_{\mathcal{D}^p_\alpha} \asymp Q_1(f)$ can be viewed as a generalization of a theorem of Hardy and Littlewood on harmonic conjugates [3, Theorem 5], which we state as follows:

THEOREM (A). *Let F be a function holomorphic in the unit disk $\mathbb{D} := B_2 \subset \mathbb{C} = \mathbb{R}^2$. If the function $(\partial/\partial r)(\operatorname{Re} F)(re^{i\theta})$ belongs to $L^p_\alpha(\mathbb{D})$, where $p > 0$, $\alpha > -1$, then so does the derivative F'.*

Note that in this situation, $R(\operatorname{Re} f) = r(\partial/\partial r)(\operatorname{Re} F)$ and $|F'| = |\nabla(\operatorname{Re} f)|$ and so Theorem (A) coincides with the equivalence $\|f\|_{\mathcal{D}^p_\alpha} \asymp Q_1(f)$ for $N = 2$, $k = 1$.

2. Some formulas

The tangential derivatives $T_{ij}f$ $(1 \le i, j \le N)$ are defined by

$$T_{ij}f(x) = x_i D_j f(x) - x_j D_i f(x).$$

If $f \in C^2(B)$, then there hold the formulas:

$$D_i Rf = D_i f + R D_i f = R_1 D_i f; \tag{2.1}$$

$$\begin{gathered} D_i T_{ij} f = D_j f + T_{ij} D_i f, \\ D_j T_{ij} f = -D_i f + T_{ij} D_j f, \\ D_k T_{ij} f = T_{ij} D_i f \qquad (k \ne i,\ k \ne j); \end{gathered} \tag{2.2}$$

$$RRf = Rf + \sum_{i,j=1}^{N} x_i x_j D_i D_j f; \tag{2.3}$$

$$\begin{aligned} T_{ij}T_{ij}f = {} & x_i \delta_{ij} D_j f + x_i^2 D_j D_j f - x_i D_i f - x_i x_j D_i D_j f \\ & - x_j D_j f - x_i x_j D_i D_j f + x_j \delta_{ij} D_i f + x_j^2 D_i D_i f; \end{aligned}$$

where δ_{ij} is the Kronecker delta. From this and (2.3) we get

$$RRf + \frac{1}{2} \sum_{i,j=1}^{N} T_{ij} T_{ij} f = (2 - N) Rf + |x|^2 \Delta f, \tag{2.4}$$

where Δ is the ordinary Laplacian,

$$\Delta f = \sum_{i=1}^{N} D_i D_i f.$$

By successive application of (2.1) and (2.2) we get

$$\Delta^k Rf = 2k \Delta^k f + R \Delta^k f,$$

$$\Delta T_{ij} f = T_{ij} \Delta f.$$

As a consequence we get the well known facts:

PROPOSITION (A). *If f is in $\mathcal{H}_k(B)$, then so are Rf and $T_{ij}f$ $(1 \le i, j \le N)$.*

Using the fact that T_{ij} annihilates radial functions, we get:

PROPOSITION (B). *If f is given by* (1.1)*, then*

$$T_{ij}f(x) = \sum_{m=0}^{k-1}(1-|x|^2)^m T_{ij}A_m f(x), \qquad x \in B. \tag{2.5}$$

This provides another proof that T_{ij} preserves $\mathcal{H}_k(B)$.

(We write $A(s) \asymp B(s)$ to denote that $A(s)/B(s)$ lies between two positive constants independent of s.)

3. Subharmonic behavior

The class $QNS(B)$. Let $QNS(B)$ denote the class of non-negative measurable functions u on B for which there exists a constant $Q = Q(u)$ such that

$$u(a) \le Q\varepsilon^{-N}\int_{B(a,\varepsilon)} u\,dV \tag{3.1}$$

whenever

$$B(a,\varepsilon) := \{x\colon |x-a| < \varepsilon\} \subset B.$$

Members of $QNS(B)$ are called quasi-nearly subharmonic functions [9, 10]. The class QNS contains non-negative subharmonic functions. Observe that (3.1) implies

$$\sup_{B(a,\varepsilon)} u \ \le C\varepsilon^{-N}\int_{B(a,2\varepsilon)} u\,dV, \qquad B(a,2\varepsilon) \subset B. \tag{3.2}$$

THEOREM (B). [5, 10] *Let $p > 0$. If $u \in QNS(B)$, then $u^p \in QNS(B)$, and $Q(u^p) \le C_{p,N}Q(u)$.*

The class $HC^1(G)$. This class consists of all locally Lipschitz functions f on B for which there exists a constant $Q' = Q'(f)$ such that

$$|\nabla f(a)| \le \frac{Q'}{\varepsilon}\sup_{B(a,\varepsilon)}|f|, \qquad \text{whenever } B(a,\varepsilon) \subset B. \tag{3.3}$$

Note that a locally Lipschitz function is differentiable almost everywhere and in particular the gradient is defined almost everywhere. If $\nabla f(a)$ does not exist, then we interpret $|\nabla f(a)|$ as

$$|\nabla f(a)| = \limsup_{x\to a}\frac{|f(x)-f(a)|}{|x-a|}.$$

The class $OC^1(B)$. This is the subclass of $HC^1(B)$ consisting of those f for which

$$|\nabla f(a)| \le \frac{Q''}{\varepsilon}\sup\{\,|f(x)-f(a)|\ : x \in B(a,\varepsilon)\},$$

where $Q'' = Q''(f)$ is a constant.

THEOREM (C). [5] (a) *If $f \in HC^1(B)$, then $|f| \in QNS(B)$, and $Q(|f|) \le C_N Q'(f)$.*

(b) *If $f \in OC^1(B)$, then both $|f|$ and $|\nabla f|$ belong to $QNS(B)$, and $Q(|\nabla f|) \le C_N Q''(u)$.*

THEOREM (D). [6, 7] *If f is a function polyharmonic in B, then $f \in OC^1(B)$. Moreover if $f \in \mathcal{H}_k(B)$, then $Q''(u) \le C_{k,N}$ (where $C_{k,N}$ depends only on k and N).*

As a consequence we have the following generalization of a theorem of Hardy and Littlewood [3] ($N = 2$) and Fefferman and Stein [2] ($N \geq 3$).

THEOREM (E). *If $u = |f|^p$, $u = |Rf|^p$, $u = |\nabla f|^p$, or $u(x) = |T_{i,j}f(x)|^p$, where $f \in \mathcal{H}_k(B)$, and $p > 0$, then u satisfies (3.2) with C depending only on p, k and N.*

4. L^p-inequalities for QNS-functions

The following theorem was proved in [7] in the case of polyharmonic functions. However the proof was based only on the condition $f \in HC^1(B)$. Therefore we omit the proof.

THEOREM (4.1). *If $f \in HC^1(B)$, $p > 0$ and $\alpha \in (-\infty, \infty)$, then*

$$\| \, |\nabla f| \, \|_{L^p_{\alpha+p}} \leq C\|f\|_{L^p_\alpha}. \tag{4.2}$$

In order to state a maximal theorem we let

$$u^+(\rho y) = \sup_{0\leq r\leq\rho} |u(ry)| = \sup_{0\leq t\leq\rho} |u(t\rho y)|, \qquad 0 \leq \rho < 1, \; y \in \partial B.$$

THEOREM (4.3). *If $u \in QNS(B)$, $\alpha > -1$ and $p > 0$, then*

$$\|u^+\|_{L^p_\alpha} \leq C\|u\|_{L^p_\alpha}. \tag{4.4}$$

where C depends only on p, α and $Q(u)$.

Let $P(x, y)$ denote the Poisson kernel,

$$P(x, y) = \frac{1 - |x|^2}{|x - y|^N}.$$

Since $\int_{\partial B} P(x, y)\, d\sigma(y) = 1$, where $d\sigma$ is the normalized surface measure on ∂B, we see that Theorem (4.3) is obtained from the following lemma, by using integration in polar coordinates, i.e., the formula

$$\int_B \phi(x)\, dV(x) = N \int_0^1 r^{N-1} dr \int_{\partial B} \phi(ry)\, d\sigma(y).$$

LEMMA (4.5). *If $u \in QNS(B)$, $\alpha > -1$ and $p > 0$, then*

$$\int_0^1 r^{N-1} u^+(ry)^p (1 - r^2)^\alpha \, dr \leq C \int_B u(x)^p (1 - |x|^2)^\alpha P(x, y)\, dV(x), \qquad y \in \partial B.$$

For the proof we need the following elementary lemma.

LEMMA (A). [4] *Let $\beta > 0$, and $\{A_j\}_0^\infty$ a sequence of real numbers. Then*

$$\sum_{j=0}^\infty 2^{-j\beta} |A_{j+1}|^p \leq C|A_0|^p + C \sum_{j=0}^\infty 2^{-j\beta} |A_{j+1} - A_j|^p,$$

where C depends only on β.

Proof of Lemma (4.5). We can assume that $p = 1$ because $u \in QNS(B)$ implies $u^p \in QNS(B)$, by Theorem (B). Let $r_j = 1 - 2^{-j}$ for $j \geq 0$. Then, by Lemma (A),

$$\begin{aligned}\int_0^1 r^{N-1}(1-r^2)^\alpha u^+(ry)\,dr &\leq C\sum_{j=0}^{\infty} 2^{-j(\alpha+1)}u^+(r_{j+1}y)\\ &\leq Cu(0) + C\sum_{j=0}^{\infty} 2^{-j(\alpha+1)}\left(\, u^+(r_{j+1}y) - u^+(r_jy)\,\right)\\ &\leq C\sum_{j=0}^{\infty} 2^{-j(\alpha+1)} \sup_{r_j\leq r\leq r_{j+1}} u(ry).\end{aligned}$$

By (3.2) with $a = a_j := (r_j + r_{j+1})y/2$ and $\varepsilon = (r_{j+1} - r_j)/2 = 2^{-j-2}$,

$$2^{-j(\alpha+1)} \sup_{r_j\leq r\leq r_{j+1}} u(ry) \leq C2^{-j(\alpha+1)}2^{jN}\int_{B(a_j,2^{-j-1})} u(x)\,dV(x). \tag{4.6}$$

On the other hand, simple calculation shows that $|x - a_jy| \leq 2^{-j-1}$ implies

$$2^{-j-2} \leq 1 - |x|, \qquad |x - y| \leq 2^{-j+1}$$

Hence

$$2^{-j}2^{jN} \leq 2^{N+2}\,P(x, y), \quad \text{ for } x \in B(a_j, 2^{-j-1}).$$

From this and (4.6) we get

$$\begin{aligned}2^{-j(\alpha+1)} \sup_{r_j\leq r\leq r_{j+1}} u(ry) &\leq C2^{-j\alpha}\int_{r_{j-1}\leq|x|\leq r_{j+2}} P(x,y)\,u(x)\,dV(x)\\ &\leq C\int_{r_{j-1}\leq|x|\leq r_{j+2}} (1-|x|)^\alpha P(x,y)\,u(x)\,dV(x)\end{aligned}$$

($r_{-1} = 0$) where we have used the inclusion

$$\{x\colon |x - a_j| \leq 2^{-j-1}\} \subset \{x\colon r_{j-1} \leq |x| \leq r_{j+2}\}.$$

Now the desired conclusion is easily obtained by summation from $j = 0$ to ∞. □

THEOREM (4.7). *For a Borel measurable function u on B and $s > 0$, let*

$$I_su(x) = \int_0^1 t^{s-1}u(tx)\,dt, \qquad x \in B.$$

If $u \in QNS(B)$, $p > 0$ and $\alpha > -1$, then

$$\|I_su\|_{L^p_\alpha} \leq C\|u\|_{L^p_{\alpha+p}}.$$

Proof. Write I_su as

$$I_su(\rho y) = \frac{1}{\rho^s}\int_0^\rho t^{s-1}u(ty)\,dt, \qquad y \in \partial B,\ 0 < \rho < 1.$$

Hence

$$|I_su(\rho y)| \leq \frac{1}{s}\sup_{B(0,1/2)} u + 2^{s+1}\int_0^\rho u(ty)\,dt, \qquad y \in \partial B,\ 0 < \rho < 1.$$

Since, by (3.2),

$$\sup_{B(0,1/2)} u \le C\|u\|_{L^p_{\alpha+p}},$$

it suffices to prove that

$$\|Ju\|_{L^p_\alpha} \le C\|u\|_{L^p_{\alpha+p}},$$

where

$$Ju(\rho y) = \int_0^\rho u(ry)\,dr.$$

To show this we proceed in a similar way as in the proof of Lemma (4.5). Namely,

$$\|f\|^p_{L^p_\alpha} = \sum_{j=0}^\infty \int_{r_j}^{r_{j+1}} Nr^{N-1}dr \int_{\partial B} (Ju)(ry)^p(1-r^2)^{\alpha-1}\,d\sigma(y).$$

On the other hand, it is easy to show that

$$\sum_{j=0}^\infty \int_{r_j}^{r_{j+1}} Nr^{N-1}(Ju)(ry)^p(1-r^2)^{\alpha-1}dr \le C\sum_{j=0}^\infty \left(\int_0^{r_{j+1}} u(ry)\,dr\right)^p 2^{-j\alpha}.$$

Now we use Lemma (A) to show that the last quantity is equivalent to

$$A := \sum_{j=0}^\infty \left(\int_{r_j}^{r_{j+1}} u(ry)\,dr\right)^p 2^{-j\alpha}.$$

Since

$$A \le \sum_{j=0}^\infty 2^{-jp}2^{-j\alpha} \sup_{r_j<r<r_{j+1}} u(ry)^p,$$

we can proceed as in the proof of Lemma 1 to get

$$\int_0^1 r^{N-1}(Ju(ry))^p(1-r^2)^\alpha\,dr \le C\int_B u(x)^p(1-|x|^2)^{\alpha+p}P(x,y)\,dV(x),\ \ y\in\partial B.$$

Now integration in polar coordinates gives

$$\|Ju\|_{L^p_\alpha} \le C\|u\|_{L^p_{\alpha+p}},$$

which completes the proof of the theorem. □

5. Inequalities for polyharmonic functions

In [7], the following decomposition theorem for polyharmonic Bergman spaces is proved.

THEOREM (F). *If f is given by* (1.1)*, $0<p<\infty$, and $\alpha>-1$, then*

$$\|f\|_{L^p_\alpha} \asymp \sum_{m=0}^{k-1} \|A_m f\|_{L^p_{\alpha+mp}}, \qquad f\in\mathcal{H}_k(B).$$

Here we prove:

THEOREM (5.1). *Let $\alpha>-1$ and $p>0$. Then*

$$\|f-f(0)\|_{L^p_\alpha} \asymp \|Rf\|_{L^p_{\alpha+p}} \asymp \|\nabla f\|_{L^p_{\alpha+p}}, \qquad f\in\mathcal{H}_k(B).$$

Proof. We have

$$f(x) - f(0) = \int_0^1 \frac{Rf(tx)}{t}\,dt.$$

Hence

$$|f(x) - f(0)| \le \sup_{|x|<1/4} |\nabla f(x)| + 4\int_0^1 |Rf(tx)|\,dt.$$

On the other hand,

$$\nabla f(x) = \int_0^1 (\nabla Rf)(tx)\,dt,$$

whence

$$\sup_{|x|<1/4} |\nabla f(x)| \le \sup_{|x|<1/4} |\nabla Rf(x)|.$$

But

$$\sup_{|x|<1/4} |\nabla Rf(tx)| \le C \sup_{|x|<1/2} |Rf(x)|$$

because Rf is polyharmonic and therefore belongs to HC^1. Also since $Rf \in QNS(B)$, we have

$$\sup_{|x|<1/2} |Rf(x)| \le C\|Rf\|_{L^p_{\alpha+p}}.$$

Combining the above inequalities we get

$$|f(x) - f(0)| \le C\|Rf\|_{L^p_{\alpha+p}} + 4\int_0^1 |Rf(tx)|\,dt.$$

Now the inequality

$$\|f - f(0)\|_{L^p_\alpha} \le C\|Rf\|_{L^p_{\alpha+p}}$$

follows from Theorem (4.7) ($s = 1$). Since $|Rf(x)| \le |x||\nabla f(x)|$ and $\|\nabla f\|_{L^p_{\alpha+p}} \le C\|f\|_{L^p_\alpha}$, by Theorem (4.1), we see that the proof is finished. □

6. Proof of the main results

THEOREM (6.1). *For $0 < p < \infty$, $\alpha > -1$, and $k \ge 2$, we have $Q_1(f) \asymp \|f\|_{\mathcal{D}^p_\alpha}$.*

Proof. The inequality $Q_1(f) \le C\|f\|_{\mathcal{D}^p_\alpha}$ is obvious. On the other hand, we have

$$\|D_i Rf\|_{L^p_{\alpha+p}} \le C\|Rf\|_{L^p_\alpha}, \qquad 1 \le i \le N,$$

by Theorem (4.1). But since $D_i Rf = R_1 D_i f$, we can use the formula

$$I_s R_s u = R_s I_s u = u \tag{6.2}$$

(with $u = D_i f$) together with Theorem (4.7) to get

$$\|D_i f\|_{L^p_\alpha} \le C\|Rf\|_{L^p_\alpha},$$

which concludes the proof. □

LEMMA (6.3). *If f is a harmonic function on B, $p > 0$ and $\alpha > 0$, then*

$$Q_2(f) \asymp \|f\|_{\mathcal{D}^p_\alpha}.$$

Proof. The inequality $Q_2(f) \leq C\|f\|_{\mathcal{D}^p_\alpha}$ is obvious. To prove the reverse inequality, observe that (2.4) implies

$$\frac{1}{2}\sum_{i,j=1}^{N} T_{ij}T_{ij}f = -R_{N-2}Rf. \tag{6.4}$$

Since

$$Q_2(f) = \sum_{m=0}^{N} |A_m f(0)| + \sum_{i,j=1}^{N} \|T_{ij}f\|_{L^p_\alpha}$$

we see that (6.4) and Theorem (4.1) imply

$$\|R_{N-2}Rf\|_{L^p_{\alpha+p}} \leq CQ_2(f).$$

Now we use the formula (6.2) with $u = Rf$, $s = N-2$ together with Theorem (4.7) to conclude that

$$\|Rf\|_{L^p_\alpha} \leq C\|R_{N-2}Rf\|_{L^p_{\alpha+p}} \leq CQ_2(f),$$

for $N \geq 3$. In the case $N = 2$ we can use Theorem (5.1) to get

$$\|Rf\|_{L^p_\alpha} \leq C\|RRf\|_{L^p_{\alpha+p}} \leq CQ_2(f),$$

Now the result follows from Theorem (6.1). □

THEOREM (6.5). *Under the hypotheses of Theorem (1.2) we have* $Q_2(f) \asymp \|f\|_{\mathcal{D}^p_\alpha}$.

Proof. First we prove that $\|f\|_{\mathcal{D}^p_\alpha} \leq CQ_2(f)$. By Proposition (B) and Theorem (F) we have

$$Q_2(f) \asymp \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{m=0}^{k-1}\sum_{i,j=1}^{N} \|T_{ij}A_m f\|_{L^p_{\alpha+mp}}.$$

Hence, by Lemma (6.3),

$$Q_2(f) \asymp \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{m=0}^{k-1} \|\,|\nabla A_m f|\,\|_{L^p_{\alpha+mp}} \qquad (= Q_3(f)).$$

Since $\|\,|\nabla A_m f|\,\|_{L^p_{\alpha+mp}} \asymp \|A_m f - A_m f(0)\|_{L^p_{\alpha+(m-1)p}}$ for $m \geq 1$, we have

$$\begin{aligned} Q_2(f) &\asymp \sum_{m=0}^{k-1} |A_m f(0)| + \|\,|\nabla A_0 f|\,\| + \sum_{m=1}^{k-1} \|A_m f - A_m f(0)\|_{L^p_{\alpha+(m-1)p}} \\ &\asymp \sum_{m=0}^{k-1} |A_m f(0)| + \|\,|\nabla A_0 f|\,\|_{L^p_\alpha} + \sum_{m=1}^{k-1} \|A_m f\|_{L^p_{\alpha+(m-1)p}} \qquad (6.6) \\ &\asymp |A_0 f(0)| + \|\,|\nabla A_0 f|\,\|_{L^p_\alpha} + \sum_{m=1}^{k-1} \|A_m f\|_{L^p_{\alpha+(m-1)p}} \qquad (= Q_4(f)). \end{aligned}$$

On the other hand, since

$$|\nabla f(x)| \leq |\nabla A_0 f(x)| + \sum_{m=1}^{m-1}(1-|x|^2)^m|\nabla A_m f(x)| + 2m(1-|x|^2)^{m-1}|A_m f(x)|,$$

we have, by Theorem (4.1),

$$\| \, |\nabla f| \, \|_{L^p_\alpha} \le C\|A_0 f\|_{\mathcal{D}^p_\alpha} + C\sum_{m=1}^{k-1} \|A_m f\|_{L^p_{\alpha+(m-1)p}}.$$

This inequality and (6.6) give the required inequality.

In order to prove the reverse inequality it suffices to prove that

$$\sum_{m=0}^{k-1} |A_m f(0)| \le C\|f\|_{\mathcal{D}^p_\alpha}. \tag{6.7}$$

Indeed, since $|\nabla f| \in QNS$, we have

$$|f(0)| + \int_B |\nabla f(x)|^p (1-|x|^2)^\alpha \, dV(x) \ge c(|f(0)| + \sup_{|x|<1/2} |\nabla f(x)|),$$

and hence

$$\|f\|_{\mathcal{D}^p_\alpha} \ge c(|f(0)| + \sup_{|x|<1/2} |f(x) - f(0)|) \ge c \sup_{|x|<1/2} |f(x)|,$$

where c is a positive constant. On the other hand,

$$\int_{\partial B} f(ry)\, d\sigma(y) = \sum_{m=0}^{k-1} A_m f(0)(1-r^2)^m, \qquad 0 < r < 1.$$

But the quantity $K(a_0, \ldots, a_{k-1}) = \sup_{0<r<1/2} | \sum_{m=0}^{k-1} a_m (1-r^2)^m |$ is a norm on $\mathbb{R}^k$, and this implies

$$\sum_{m=0}^{k-1} |A_m f(0)| \le C\, K(f) \le C \sup_{0<r<1/2} \int_{\partial B} |f(ry)|\, d\sigma(y) \le C \sup_{|x|<1/2} |f(x)|,$$

which completes the proof. □

7. Bloch type spaces

We define $\mathcal{B}_{\alpha,k}$ ($\alpha > 0$, $k = 1, 2, \ldots$) to be the class of those $f \in \mathcal{H}_k(B)$ for which

$$\|f\|_{\mathcal{B}_\alpha} := |f(0)| + \sup_{x\in B} (1-|x|^2)^\alpha |\nabla f(x)| \; < \infty. \tag{7.1}$$

The space $\mathcal{B}_{1,1}$ is known as the harmonic Bloch space. The following theorem is proved in a similar way as Theorem (1.2); the proof is even simpler and therefore is omitted.

THEOREM (7.2). *For $\alpha > 0$, and $k \ge 2$, the following quantities are equivalent norms on $\mathcal{B}_{\alpha,k}(B)$:*

$$P_1(f) = |f(0)| + \sup_{x\in B} |Rf(x)|^p (1-|x|^2)^\alpha,$$

$$P_2(f) = \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{i,j=1}^{N} \sup_{x\in B} |T_{ij} f(x)| (1-|x|^2)^\alpha,$$

$$P_3(f) = \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{m=0}^{k-1} \sup_{x\in B} |\nabla A_m f(x)| (1-|x|^2)^{\alpha+m},$$

$$P_4(f) = \|A_0 f\|_{\mathcal{B}_\alpha} + \sum_{m=1}^{k-1} \sup_{x\in B} |A_m f(x)|(1-|x|^2)^{\alpha+(m-1)}.$$

In order to state another result, we define the L^q-oscillation $(0 < q \le \infty)$ of f over the ball $B(x, r) \subset B$ as

$$\operatorname{osc}_q(f, x, r) = \left(\frac{1}{r^N}\int_{B(x,r)} |f(z) - f(x)|^q dV(z)\right)^{1/q}. \tag{7.3}$$

Note that $V(B(x, r)) = r^N$. In the case $q = \infty$, equation (7.3) should be interpreted as

$$\operatorname{osc}(f, x, r) = \sup_{z\in B(x,r)} |f(z) - f(x)|.$$

THEOREM (7.4). *Let $0 < q \le \infty$, $\alpha > 0$ and $0 < c < 1$. Then the quantity*

$$P_5(f) = |f(0)| + \sup_{x\in B}(1-|x|^2)^{\alpha-1}\operatorname{osc}_q(f, x, c(1-|x|))$$

is an equivalent norm on $\mathcal{B}_{\alpha,k}$.

Proof. We consider only the case $q < \infty$. It is easy to check that a QNS-function u on B satisfies the condition

$$\sup_{|z-x|<\varepsilon/2} u(z)^q \le \frac{C}{\varepsilon^N}\int_{B(x,\varepsilon)} u(z)^q\, dV(z), \quad B(x,\varepsilon) \subset B.$$

Since a polyharmonic function f belongs to $OC^1(B)$ (Theorem (D)), this implies

$$\begin{aligned} |\nabla f(x)|^q &\le \frac{C}{\varepsilon^{N+q}}\int_{B(x,\varepsilon)} |f(z) - f(x)|^q dV(x) \\ &= \frac{C}{\varepsilon^q}\left(\operatorname{osc}_q(f, x, \varepsilon)\right)^q. \end{aligned} \tag{7.5}$$

where C is independent of f. Now we take $\varepsilon = c(1-|x|)$ and use the hypothesis

$$(1-|x|^2)^{\alpha-1}\operatorname{osc}_q(f, x, c(1-|x|)) \le P_5(f)$$

to get

$$|\nabla f(x)| \le \frac{CP_5(f)}{c(1-|x|)}(1-|x|)^{(1-\alpha)} = \frac{CP_5(f)}{c}(1-|x|)^{-\alpha}. \tag{7.6}$$

This proves part of the theorem.

In the other direction, assume that

$$M(f) := \sup_{x\in B}(1-|x|^2)^{\alpha}|\nabla f(x)| < \infty.$$

It is enough to prove that

$$\sup_{x\in B}(1-|x|^2)^{\alpha-1}\operatorname{osc}(f, x, c(1-|x|)) \le CM(f),$$

because $\operatorname{osc}_q(f, x, c(1-|x|)) \le \operatorname{osc}(f, x, c(1-|x|))$. For we have, by Lagrange's theorem,

$$\operatorname{osc}(f, x, c(1-|x|)) \le c(1-|x|) \sup_{z\in B(x,c(1-|x|))} \|\nabla f(z)\|,$$

whence, by hypothesis,

$$\operatorname{osc}(f, x, c(1-|x|)) \le c(1-|x|)M(f) \sup_{z\in B(x,c(1-|x|))} (1-|z|^2)^{-\alpha}.$$

Now the desired result follows from the inequality

$$1-c < \frac{1-|z|}{1-|x|} < 1+c, \qquad z \in B(x, c(1-|x|)),$$

which is easily deduced from the inequalities $|x|-|z| < c(1-|x|)$ and $|z|-|x| < c(1-|x|)$ valid for $z \in B(x, c(1-|x|))$. This concludes the proof of the theorem. □

Finally we define the mean values of $|f|$ on the ball $B(x, r)$ by

$$m_q(f, x, r) = \left(\frac{1}{r^N}\int_{B(x,r)} |f|^q \, dV\right)^{1/q}, \qquad 0 < q \le \infty.$$

Using Theorems (E) and (7.2) one can prove the following:

THEOREM (7.7). *Let $\alpha > 0$, $0 < q \le \infty$, and $k \ge 2$, and $0 < c < 1$, then the following quantities are equivalent norm on $\mathcal{B}_{\alpha,k}$:*

$$P_6(f) = |f(0)| + \sup_{x\in B} m_q(Rf, x, c(1-|x|))(1-|x|^2)^{\alpha},$$

$$P_7(f) = \sum_{m=0}^{k-1} |A_m f(0)| + \sum_{i,j=1}^{N} \sup_{x\in B} m_q(T_{ij}f, x, c(1-|x|))(1-|x|^2)^{\alpha}.$$

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JOINT SPECTRA IN WAELBROECK ALGEBRAS

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ABSTRACT. In the paper we give a survey of the theory of joint spectra for a locally convex Waelbroeck algebra W. It includes the Gelfand theory, the description of joint spectra on a commutative algebra W and the Harte theorem in the general case. The subspectra generated by W in their subalgebras are also studied. The rôle of regularities in the spectral theory is emphasized.

Introduction

In developing the theory of spectra and joint spectra of Banach algebras, the following two properties of these algebras are specially frequently used:

(1) The set $G(B)$ of invertible elements of the Banach algebra B is open.

(2) The inversion mapping $G(B) \ni x \to x^{-1} \in G(B)$ is continuous.

A topological algebra W is a Waelbroeck algebra if it has both properties. In the original papers of Waelbroeck [22], [23] the algebras, called *algebras of continuous inverse*, are supposed moreover to be locally convex.

What part of the spectral theory of Banach algebras is valid for Waelbroeck algebras? The subject of the present paper is to gather the known results concerning this question, and to add several new facts.

Besides the direct consequences of basic properties presented in Section 1, we study in Section 2 a generalization of the Gelfand-Mazur theorem and of the of Gleason-Kahane-Żelazko theorem. Section 3 presents the generalization of the Gelfand theory of maximal ideals. The results presented in sections 1-3 are well known, in several cases even in a more general form. For this material the reader is refered to the monographs [13], [8] and the articles [1], [2], [3], [24].

In Section 4 we define the joint spectra and we prove the analogues of a theorem of Sołtysiak about the existence of semispectra and of the Żelazko theorem about the general form of a subspectrum in a commutative Banach algebra.

Section 5 is devoted to the one-way spectral mapping theorem valid for the joint spectra defined by families of ideals.

In Section 6 the complete spectral mapping formula, that is, the Harte theorem for Waelbroeck algebras is discussed. The method used is based on the Schur lemma in its algebraic part and on the Gelfand-Mazur theorem as the analytic argument. This approach is much simpler than the proof from [29]; moreover it is applicable to other topological algebras.

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In Sections 7 and 8 we present other methods of describing subspectra. One of them consists in generating subspectrum in a subalgebra of a Waelbroeck algebra. In the commutative case we study the regularities in a Waelbroeck algebra and the associated subspectra.

Section 9 describes the rôle of the spectrum of generators. One proves that in the case of a Q-algebra (in particular for Waelbroeck algebras) the spectrum of generators is rationally convex.

Most of the results mentioned depend upon the existence of continuous linear functionals on the algebra. Locally convex Waelbroeck algebras is the class of topological algebras most exhaustively studied in the present paper.

1. Spectrum of a single element

All algebras considered here are associative, complex and unital, with the unit denoted by e. Topological algebra A is an algebra provided with the Hausdorff topology which makes A a topological linear space and such that the mapping $A \times A \ni (x, y) \to xy \in A$ is continuous. A topological algebra is called an F-algebra if its topology is metrizable and complete. It is an LC algebra if the topology is locally convex. If the topology of A is given by a norm which is submultiplicative: $\|ab\| \leq \|a\|\|b\|$, then A is a normed algebra. Banach algebra is a complete normed algebra. A locally multiplicatively convex algebra (LMC-algebra) is a topological algebra whose topology can be specified by a family of submultiplicative seminorms.

An element $a \in A$ is invertible if there exists $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = e$. By $G(A)$ is denoted the subset of elements invertible in A. A topological algebra is a Q-algebra if $G(A)$ is an open set.

Not all normed algebras are Q-algebras. The necessary and sufficient condition for a normed algebra $(B, \|\cdot\|)$ to be Q-algebra is that $\sum_{j=1}^{\infty} a^j$ converges in B for all $a \in B$ such that $\|a\| < 1$. The inverse $x \to x^{-1}$ is continuous in a normed algebra $(B, \|\cdot\|)$ thanks to the elementary inequality

$$\|b^{-1} - x^{-1}\| \leq 2\|x^{-1}\|^2\|b - x\|.$$

As proved by Banach (see [32]) in an algebra which is an F- and Q-algebra the inversion is continuous, so it is a Waelbroeck algebra. However the field $C(t)$ of rational functions with the topology defined by Williamson in [31] is obviously a Q-algebra but the inverse is not continuous in it.

On the other hand all LMC-algebras have the inversion continuous, but not all of them are Q-algebras. The algebra $C(\mathbb{R})$ of continuous functions on the real line provided with the compact-open topology is one of the simplest examples of this situation.

Let E be a topological unital algebra. For a given $x \in E$ the set

$$\sigma(x) = \{\lambda \in \mathbb{C} |\ x - \lambda e \notin G(E)\}$$

is the spectrum of x in E.

In what follows we simplify the notation writing $x - \lambda$ in place of $x - \lambda e$.

In general the spectrum of an element can be empty (as in the case of a nonconstant element of $C(t)$). The spectrum of the function $f(x) = \exp x$ in the algebra $C(\mathbb{C})$ is open and unbounded.

THEOREM (1.1). *If E is a Q algebra then $\sigma(x)$ is compact for an arbitrary $x \in E$.*

Proof. Let $x \in E$. If $r \in \mathbb{R}$ is sufficiently large then $x\mu^{-1} - e \in G(E)$ for every $\mu \in \mathbb{C}$ satisfying $|\mu| \geq r$, because $G(E)$ is open. Then $x - \mu = \mu(x\mu^{-1} - e) \in G(E)$. So the spectrum of x is bounded. If $\lambda \notin \sigma(x)$ then by the same argument there exists $\varepsilon > 0$ such that for $|\mu| < \varepsilon$ we have $x - (\lambda + \mu) \in G(E)$. The complement of $\sigma(x)$ is open. The spectrum is closed. $\square$

Let A be an algebra and let $x \in A$. The set $\Re(x) = \mathbb{C} \setminus \sigma(x)$ is called the resolvent set of x and the function $R_x(\lambda) = (x - \lambda)^{-1}$ defined on $\Re(x)$ is the resolvent of x.

PROPOSITION (1.2). (Resolvent identity) *For arbitrary $\lambda, \mu \in \Re(x)$*

$$R_x(\lambda) - R_x(\mu) = (\lambda - \mu)R_x(\lambda)R_x(\mu).$$

THEOREM (1.3). *Let A be a topological algebra. Suppose that the inverse is continuous on $G(A)$ and that there exists a nonzero linear continuous functional on A. Then for every $x \in A$,*

$$\sigma(x) \neq \varnothing.$$

Proof. Let $g \in A'$ be such that $g(y) \neq 0$ for some $y \in A$. The form $\varphi : A \ni a \to g(ya)$ is also linear, continuous and satisfies $\varphi(e) \neq 0$. For $x \in A$, let us define $f(\lambda) = \varphi(R_x(\lambda))$. By the resolvent identity we obtain

$$\begin{aligned}\lim_{\lambda\to\mu} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} &= \lim_{\lambda\to\mu} \frac{\varphi(R_x(\lambda)) - \varphi(R_x(\mu))}{\lambda - \mu} \\ &= \lim_{\lambda\to\mu} \varphi\left(\frac{R_x(\lambda) - R_x(\mu)}{\lambda - \mu}\right) = \lim_{\lambda\to\mu} \varphi(R_x(\lambda)R_x(\mu)) = \varphi(R_x(\mu)^2).\end{aligned}$$

The function f is holomorphic on the whole domain $\Re(x)$.

Suppose that $\sigma(x) = \varnothing$, so $\Re(x) = \mathbb{C}$. If we represent

$$f(\lambda) = \varphi((x - \lambda)^{-1}) = \frac{1}{\lambda}\varphi((x\lambda^{-1} - e)^{-1})$$

we observe that $\lim_{\lambda\to\infty} f(\lambda) = 0$. By Liouville's theorem $f = 0$. However, the same formula also implies

$$\lim_{\lambda\to\infty} \varphi((x\lambda^{-1} - e)^{-1}) = \varphi(e) \neq 0,$$

so $f(\lambda) \neq 0$ for λ sufficiently large. This is a contradiction. $\square$

In the case of a Waelbroeck algebra we obtain:

THEOREM (1.4). *Let W be a Waelbroeck algebra such that $W' \neq \{0\}$. Then $\sigma(x)$ is a compact nonvoid set for every $x \in W$.*

2. Gelfand-Mazur Theorem. Multiplicative functionals

The results obtained in the previous section lead to one of generalized versions of the celebrated Gelfand-Mazur Theorem.

THEOREM (2.1). *Let A be a complex unital topological algebra with continuous inverse and such that $A' \neq 0$. If A is a division algebra, then $A \cong \mathbb{C}$.*

Proof. Zero is the only non-invertible element in A. On the other hand every element has a nonzero spectrum. Then for an arbitrary $a \in A$ there exists $\lambda \in \mathbb{C}$ such that $0 = a - \lambda$. The proof follows. □

One of the most important consequences of the Gelfand-Mazur theorem is the relation between maximal ideals and multiplicative functionals in the case of commutative algebras. In order to obtain it we need a result about a quotient algebra of a Wealbroeck algebra.

Let I be a closed two-sided ideal in a topological algebra E. The space E/I provided with the natural algebraic and topological structures is a topological algebra. If E is a Q-algebra then E/I is of the same class. The property of having continuous inversion is not necesserely conserved when we pass to the quotient algebra. In the case of the Waelbroeck algebra it does.

THEOREM (2.2). [24] *Let W be a Waelbroeck algebra and let I be a closed two-sided ideal in W. Then the quotient algebra W/I is a Waelbroeck algebra.*

Proof. The natural projection $\pi\colon W \to W/I$ is open; hence the group $\pi(G(E))$ is open and $G(W/I)$ is also open. It remains to prove that the inverse in E/I is continuous.

We prove first that the inverse $[x] \to [x]^{-1}$ is continuous at $[e]$. Let $\tilde{U}$ be a neighbourhood of $[e]$ in W/I and let $U = \pi^{-1}(\tilde{U})$. By the continuity of the inverse in W there exists a neighbourhood O of e such that $x^{-1} \in U$ for $x \in O$. Then $[x]^{-1} = [x^{-1}] \in \tilde{U}$ for $[x]$ in the neighbourhood $\pi(O)$ of $[e]$. The continuity of the inverse at $[e]$ is proved.

Suppose that $[x]$ has an inverse $[x]^{-1} = [y]$ in E/I. Take a neighbourhood of $[x]^{-1}$ of the form $[y]\,\tilde{U}$, where $\tilde{U}$ is a neighbourhood of $[e]$. If O is the neighbourhood of e chosen above, then for every $g \in O$ the class $[gx]$ is invertible with $[gx]^{-1} = [x]^{-1}[g]^{-1} \in [y]\,\tilde{U}$. Whence $(\pi(O)[x])^{-1} \subset [y]\,\tilde{U}$. The proof follows. □

Recall basic facts about multiplicative functionals on Q-algebras.

PROPOSITION (2.3). *Let A be a Q-algebra and let $\varphi : A \to \mathbb{C}$ be a nonzero linear multiplicative functional. Then φ is continuous and $\varphi(e) = 1$.*

Proof. If $x \in A$ is such that $\varphi(x) \neq 0$ then $\varphi(e)\varphi(x) = \varphi(ex) = \varphi(x)$ what implies $\varphi(e) = 1$. If $a \in A$ is invertible then $\varphi(a)\varphi(a^{-1}) = \varphi(e) = 1$, so $\varphi(a) \neq 0$. The kernel of φ consists of non-invertible elements. In particular it is not dense in A what implies that φ is continuous. □

The kernel of a multiplicative non-zero linear functional is obviously a maximal ideal. In the case of a commutative Waelbroeck algebra every maximal ideal is of this form.

THEOREM (2.4). *Let W be a unital commutative LC Waelbroeck algebra. Then every maximal ideal of W is of codimension one and it is equal to the kernel of a multiplicative linear functional on W.*

Proof. If I is a maximal ideal in W then it is equal to its closure, because in general a closure of a proper ideal in a Q-algebra is proper. The quotient algebra W/I is a division algebra and a locally convex algebra. Then W/I is isomorphic to $\mathbb{C}$ by Theorem (2.1) and I is of the codimension one as a kernel of a continuous multiplicative functional $W \ni x \to [x] \in W/I \cong \mathbb{C}$. □

Theorem (2.4) permits us to identify the space of maximal ideals of a complex locally convex unital commutative Waelbroeck algebra W with the space of linear multiplicative functionals on W. This space will be denoted by $\mathfrak{M}(W)$.

THEOREM (2.5). *Let W be a unital commutative LC Waelbroeck algebra. Then the space $\mathfrak{M}(W)$ is nonvoid.*

Proof. If W is a division algebra then $W \cong \mathbb{C}$ and the corresponding isomorphism is an element of $\mathfrak{M}(W)$. If W contains a non-invertible element x then Wx is an ideal. By the Kuratowski-Zorn Lemma it follows that each ideal in W is contained in a maximal ideal. In particular Wx does. Applying Theorem (2.4) we obtain an element of $\mathfrak{M}(W)$. □

The greater part of the Gelfand theory of maximal ideals generalizes to commutative LC Waelbroeck algebras. The next section will be devoted to this subject.

The theorem of Gleason-Kahane-Żelazko characterizing multiplicative functionals on a unital Banach algebras is also valid for Waelbroeck algebras.

THEOREM (2.6). *Let W be a unital complex Waelbroeck algebra. If φ is a lineal functional on W such that* $\ker \varphi$ *consists of noninvertible elements and $\varphi(e) = 1$, then φ is continuous and multiplicative.*

Proof. In the paper [16] the Gleason-Kahane-Żelazko characterization of multiplicative functionals was extended to all complex unital algebras A such that $\sigma(x)$ is bounded for all $x \in A$. By Theorem 1.1 the proof follows. □

3. Gelfand transform

In this section A denotes a unital complex commutative LC Waelbroeck algebra.

By Theorem (2.5) it follows that the space $\mathfrak{M}(A)$ is nonempty. For every $a \in A$ and $\varphi \in \mathfrak{M}(A)$ the value $\varphi(a)$ belongs to the spectrum of a. Every φ can be treated as an element of the space $P = \prod_{a \in A} \sigma(a)$. According to Theorem 1.4 $\sigma(a)$ is compact for every $a \in A$. By Tychonoff theorem the product topology of P is compact. It is easily seen that the space $\mathfrak{M}(A)$ is a closed subspace of P. Provided with the product topology the space $\mathfrak{M}(A)$ is compact.

The Gelfand transform on A is the mapping which assigns to $a \in A$ the function $\widehat{a}$ on $\mathfrak{M}(A)$ defined by the formula

$$\widehat{a}(\varphi) = \varphi(a).$$

Obviously $\widehat{ab}(\varphi) = \widehat{a}(\varphi)\widehat{b}(\varphi)$. The product topology in $\mathfrak{M}(A)$ is just the weakest one for which all functions $\widehat{a}$, $a \in A$ are continuous. The Gelfand transform is a homomorphism of A into $C(\mathfrak{M}(A))$, the algebra of continuous functions on $\mathfrak{M}(A)$.

THEOREM (3.1). *Let A be a commutative LC Waelbroeck algebra. Then the Gelfand transform on A is continuous.*

For the proof of this theorem we refer the reader to [13], VI, Corollary 1.3.

The following simple observations relate the invertibility of an element of the algebra with the properties of its Gelfand transform.

PROPOSITION (3.2). *Let A be a commutative LC Waelbroeck algebra. For every $a \in A$,*

(1) $\widehat{a}(\mathfrak{M}(A)) = \sigma(a)$.

(2) $a \in G(A)$ *if and only if* $\widehat{a}(\varphi) \neq 0$ *for all* $\varphi \in \mathfrak{M}(A)$. *This being the case,*

$$(a^{-1})\widehat{\ }(\varphi) = (\widehat{a}(\varphi))^{-1}.$$

4. Subspectra. Żelazko theorem.

For a given complex unital algebra B let us denote by B^k_{com} the space of k-tuples $(a_1, \ldots, a_k) \in B^k$ such that the elements a_i, $1 \leq i \leq k$ mutually commute. If $(a_1, \ldots, a_k) \in B^k_{\mathrm{com}}$ and $p = (p_1, \ldots, p_m) : \mathbb{C}^k \to \mathbb{C}^m$ is a polynomial mapping, then $p(a_1, \ldots, a_k)$ is the element of B^m obtained by putting a_i in place of the variable x_i in each polynomial p_j.

Definition. A subspectrum on B is a mapping which assigns to an arbitrary $(a_1, \ldots, a_k) \in B^k_{\mathrm{com}}$ a compact non-empty set $\tau(a_1, \ldots, a_k) \subset \mathbb{C}^k$ such that the following properties are valid:

(i) $\tau(a_1, \ldots, a_k) \subset \prod_{1 \leq j \leq k} \sigma(a_j)$.

(ii) For every $(a_1, \ldots, a_k) \in B^k_{\mathrm{com}}$ and for every polynomial mapping

$$p : \mathbb{C}^k \to \mathbb{C}^m$$

the *the spectral mapping formula* holds:

$$p(\tau(a_1, \ldots, a_k)) = \tau(p(a_1, \ldots, a_k)). \tag{4.1}$$

Let us consider particular cases of the spectral mapping formula.

The *translation formula*

$$\tau(a + \lambda) = \tau(a) + \lambda,$$

is obtained by taking $p(x) = x + \lambda$.

By considering the monomial $p(x, y) = xy$ we deduce the formula

$$\tau(ab) = \{\lambda\mu | (\lambda, \mu) \in \tau(a, b)\}$$

for a, b commuting.

In the case of the projection mapping

$$\pi : \mathbb{C}^{k+1} \ni (x_1, \ldots, x_k, x_{k+1}) \to (x_1, \ldots, x_k) \in \mathbb{C}^k$$

we obtain *the projection property* of the subspectrum:

$$\tau(a_1, \ldots, a_k) = \pi(\tau(a_1, \ldots, a_k, a_{k+1}))$$

for an arbitrary $(k+1)$-tuple of commuting elements.

The majority of subspectra are defined uniquely for k-tuples of mutually commuting elements; however there exist interesting cases of joint spectra which at least can be defined for general k-tuples of elements of the algebra, although the extended function rarely conserves all properties of subspectra. These objects however are worthy to be studied because their structure explains the rôle played by ideals in the construction of joint spectra. This is the case of the approximative point spectrum in Banach algebras.

Let us call *semispectrum* on a complex unital algebra A a function which assigns to an arbitrary $(a_1, \dots, a_k) \in A^k$ a compact non-empty set $\omega(a_1, \dots, a_k) \subset \mathbb{C}^k$ such that

(1) $\omega(a_1, \dots, a_k) \subset \prod_{1 \le j \le k} \sigma(a_j)$.

(2) For every $(a_1, \dots, a_k) \in A^k$ and for every polynomial mapping $p : \mathbb{C}^k \to \mathbb{C}^m$

$$p(\omega(a_1, \dots, a_k)) \subset \omega(p(a_1, \dots, a_k)). \tag{4.2}$$

In this noncommutative case it is neccessary to explain the meaning of the expression $p(a_1, \dots, a_k)$ for p being a polynomial of k variables.

Consider the algebra $\mathfrak{A}$ of all linear mappings $f : A^k \to A$ with the operations:

$$(f + \lambda g)(a_1, \dots, a_k) = f(a_1, \dots, a_k) + \lambda g(a_1, \dots, a_k),$$
$$(fg)(a_1, \dots, a_k) = f(a_1, \dots, a_k) g(a_1, \dots, a_k).$$

Distinguish in this space the constant mapping: $(a_1, \dots, a_k) \to e$ and the coordinate mappings

$$x_i : (a_1, \dots, a_k) \to a_i, \quad 1 \le i \le k.$$

By a polynomial mapping from A^k to A^m we mean the map of the form

$$(p_1, \dots, p_m)(a_1, \dots, a_k) = (p_1(a_1, \dots, a_k), \dots, p_m(a_1, \dots, a_k)),$$

where p_i, $1 \le i \le k$ belong to the subalgebra of $\mathfrak{A}$ generated by the constant and the coordinate mappings.

The relation (2) is called the *one-way spectral mapping formula*.

Formula (2) implies the translation formula as well as the one-way projection formula

$$\tau(a_1, \dots, a_k) \subset \pi(\tau(a_1, \dots, a_k, a_{k+1})).$$

The next theorem, which generalizes some results of A. Sołtysiak [18], [19] determines a necessary and sufficient condition for the existence of a semispectrum on a locally convex Waelbroeck algebra. A theorem of Sołtysiak says that every semispectrum ω on a Banach algebra W produces a multiplicative functional on W of special type.

THEOREM (4.3). *Let ω be a semispectrum on a unital complex Waelbroeck algebra W. Then there exists a linear multiplicative functional φ on W such that $(\varphi(a_1), \dots, \varphi(a_k)) \in \omega(a_1, \dots, a_k)$ for every $a_1, \dots, a_k \in W$.*

Proof. Let us consider again the space $P = \prod_{a \in W} \sigma(a)$, which is compact by Tichonoff's theorem. The elements of P can be treated as functions on W. For an arbitrary finite system of elements $a_1, \dots, a_k \in W$ let us consider the space V linearly generated by $\{a_i\}_{1 \le i \le k}$. Let $\{b_j\}_{1 \le j \le m}$ be a basis of V.

The set $\omega(b_1, \ldots, b_m)$ is nonempty. Let us choose $(\lambda_1, \ldots, \lambda_m) \in \omega(b_1, \ldots, b_m)$. By the one-way spectral mapping formula applied to the polynomial mapping $(x_1, \ldots, x_m) \to (\sum_{j=1}^m \mu_{1j}x_j, \ldots, \sum_{j=1}^m \mu_{lj}x_j)$ we obtain

$$(\sum_{j=1}^{m} \mu_{1j}\lambda_j, \ldots, \sum_{j=1}^{m} \mu_{lj}\lambda_j) \in \omega(\sum_{j=1}^{m} \mu_{1j}b_j, \ldots, \sum_{j=1}^{m} \mu_{lj}b_j). \tag{4.4}$$

Let us define a linear form on V by the formula

$$f(\sum_{j=1}^{m} \mu_j b_j) = \sum_{j=1}^{m} \mu_j \lambda_j.$$

Formula (3) states that f has the property

$$(f(c_1), \ldots, f(c_l)) \in \omega(c_1, \ldots, c_l) \tag{4.5}$$

for an arbitrary system of elements of V. Obviously we can extend f to function on W conserving at least the property $f(c) \in \omega(c)$.

For an arbitrary system $a_1, \ldots, a_k \in W$ let us denote by $\gamma(a_1, \ldots, a_k)$ the set of all elements $f \in P$ which are linear when restricted to the linear span of $\{a_j\}_{1\le j\le k}$ and which satisfy $(f(a_1), \ldots, f(a_k)) \in \omega(a_1, \ldots, a_k)$. The set $\gamma(a_1, \ldots, a_k)$ is compact and non-empty. Moreover it follows by the definition that

$$\gamma(a_1, \ldots, a_k, b_1, \ldots, b_m) \subset \gamma(a_1, \ldots, a_k) \cap \gamma(b_1, \ldots, b_m).$$

The family of compact sets $\gamma(a_1, \ldots, a_k)$ has the finite intersection property hence there exists $f \in P$ such that $f \in \gamma(a_1, \ldots, a_k)$ for an arbitrary system $(a_1, \ldots, a_k) \in W$. The function f is linear on W and for $a_1, \ldots, a_k \in A$ we have $(f(a_1), \ldots, f(a_k)) \in \omega(a_1, \ldots, a_k) \subset \sigma(a_1, \ldots, a_k)$. In particular for a single element $a \in \ker f$ we have $0 \in \omega(a) \subset \sigma(a)$. The kernel of f consists of non-invertible elements. By Theorem (2.6) the functional f is multiplicative. □

Note that, thanks to the thorem of Roitman and Sternfeld mentioned in the proof of Theorem (2.6), the last theorem is also valid in a more general case of a unital complex algebra W such that $\sigma(a)$ is compact for all $a \in W$.

The complete description of all subspectra on a commutative Banach algebra was given by Żelazko in [33]. This result generalizes in a natural way to commutative LC Waelbroeck algebras. A generalization to other class of algebras was studied by Kokk [12].

THEOREM (4.6). *Let W be a unital commutative LC Waelbroeck algebra. Let τ be a subspectrum on W. Then there exists a unique nonempty compact set $K \subset \mathfrak{M}(W)$ such that*

$$\tau(a_1, \ldots, a_k) = \{(\varphi(a_1), \ldots, \varphi(a_k)) \in \mathbb{C}^k \mid \varphi \in K\}.$$

Proof. Let us denote by $\mathcal{K}$ the set of all ideals I in W such that $0 \in \tau(a_1, \ldots, a_k)$ for an arbitrary k-tuple $(a_1, \ldots, a_k) \in I^k$.

There is a simple construction of elements of $\mathcal{K}$. Let $a_1, \ldots, a_k \in W$ and let $(\mu_1, \ldots, \mu_k) \in \tau(a_1, \ldots, a_k)$. Then $0 \in \tau(a_1 - \mu_1, \ldots, a_k - \mu_k)$ by the translation formula. Let I be the ideal generated by elements $a_1 - \mu_1, \ldots, a_k - \mu_k$.

Then for an arbitrary m-tuple $c_j = \sum_{i=1}^k b_{j,i}(a - \mu_i) \in I$, $1 \le j \le m$ we obtain by the spectral mapping formula that $\tau(c_1, \ldots, c_j)$ consists of all elements of the form $(\sum_{i=1}^k \lambda_{1,i}\nu_i, \dots, \sum_{i=1}^k \lambda_{m,i}\nu_i)$ where $(\lambda_{1,1}, \dots, \lambda_{1,k}, \dots, \lambda_{m,1}, \dots, \lambda_{m,k}, \nu_1, \dots, \nu_k)$ belongs to the set

$$\tau(b_{1,1}, \dots, b_{1,k}, \dots, b_{m,1}, \dots, b_{m,k}, a_1 - \mu_1, \dots, a_k - \mu_k).$$

By the projection property the latter set contains an element with the last k coordinates equal to zero, so $0 \in \tau(c_1, \dots, c_k)$. It follows that $I \in \mathcal{K}$.

For an arbitrary $I \in \mathcal{K}$, let $\mathcal{K}_I$ be the set of elements of $\mathcal{K}$ which contain I. If (J_β) is a increasing linearly ordered family of elements of $\mathcal{K}_I$ then $J = \bigcup_\beta J_\beta$ belongs to $\mathcal{K}_I$. By the Kuratowski-Zorn Lemma $\mathcal{K}_I$ contains maximal elements. We shall prove that maximal elements of $\mathcal{K}_I$ are maximal ideals of A. Let J be a maximal element of $\mathcal{K}_I$. Suppose that J is not of codimension one in W. There exists $c \in W$ such that $\lambda c \notin J$.

By the projection formula we know that for an arbitrary k-tuple $a_1, \dots, a_k \in J$ there exists $\mu \in \mathbb{C}$ such that $(0, \dots, 0, \mu) \in \tau(a_1, \dots, a_k, c)$, so by the translation formula $0 \in \tau(a_1, \dots, a_k, c - \mu)$. Let us introduce

$$\delta(a_1, \dots, a_k) = \{\mu \in \mathbb{C} |\ 0 \in \tau(a_1, \dots, a_k, c - \mu)\}.$$

This set is nonempty for an arbitrary k-tuple $(a_1, \dots, a_k) \in J^k$. It is compact because it can be represented as the intersection of a compact set with a closed set:

$$\{\mu \in \mathbb{C} | (0, \dots, 0, \mu) \in \tau(a_1, \dots, a_k, c)\} = \tau(a_1, \dots, a_k, c) \cap \{0\} \times \mathbb{C},$$

where $\{0\}$ denotes the origin in $\mathbb{C}^k$. Since

$$\varnothing \neq \delta(a_1, \dots, a_k, b_1, \dots, b_m) \subset \delta(a_1, \dots, a_k) \cap \delta(b_1, \dots, b_m),$$

the family of the compact sets $\delta(a_1, \dots, a_k)$ has the finite intersection property. Let $\lambda \in \mathbb{C}$ be an element of the intersection of all sets $\delta(a_1, \dots, a_k)$, $a_1, \dots, a_k \in J$.

For arbitrary $a_1, \dots, a_k \in J$ we have $0 \in \tau(a_1, \dots, a_k, c - \lambda)$. The ideal generated by $a_1, \dots, a_k, c - \lambda$ contains J properly and belongs to $\mathcal{K}_I$, what is a contradiction. It follows that $J \in \mathfrak{M}(W)$.

Every element of $\mathcal{K}$ is contained in an element of $\mathcal{K} \cap \mathfrak{M}(W)$. Denote $K = \mathcal{K} \cap \mathfrak{M}(W)$.

Now, $(\mu_1, \dots, \mu_k) \in \tau(a_1, \dots, a_k)$ if and only if there exists $J \in K$ such that $a_i - \mu_i \in J$, $1 \le i \le k$, which implies $\mu_i = \varphi(a_i)$, where φ is the multiplicative functional corresponding to the maximal ideal J.

It remains to prove that K is compact.

Suppose that $J \notin K$. There exists a k-tuple $a_1, \dots, a_k \in J$ such that $0 \notin \tau(a_1, \dots, a_k)$. By the projection property there exists i such that $\varphi(a_i) \neq 0$, where φ is the multiplicative functional corresponding to J. The neighbourhood $V = \{\psi \in \mathfrak{M}(W) | \psi(a_i) \neq 0\}$ of J belongs to $\mathfrak{M}(W) \setminus K$. The latter set is open, hence K is compact. $\square$

One of the most important conclusions of the theorem of Żelazko is that every subspectrum in a commutative Waelbroeck algebra W is uniquely determined by a family of ideals in W. If $K \subset \mathfrak{M}(W)$ is the compact set which corresponds to a subspectrum τ by means of Theorem (4.2) then

$$\tau(a_1, \dots, a_k) = \{(\mu_1, \dots, \mu_k) \in \mathbb{C}^k |\ I(a_1, \dots, a_k) \subset J \text{ for some } J \in K\}.$$

By $I(a_1, \ldots, a_k)$ is denoted the ideal generated in W by elements $a_1, \ldots, a_k$. As we will see later, in the noncommutative case there exist important subspectra constructed in an analogous way.

The monograph [15] provides a comprehensive description of the history and actual state of the spectral theory of Banach algebras including the theory of subspectra. It presents also two principal methods of describing the spectra: in the case of spectra defined only for single elements the concept of a *regularity* is used, while the joint spectra are described by means of so called *spectral systems*.

In section 8 we describe the subspectra in a commutative Waelbroeck algebra using the concept of regularity.

In the case of a noncommutative Waelbroeck algebra W which is not a Banach algebra the existence of a subspectrum on W is not obvious. As proved recently in [28] at least the construction of the Harte spectrum conserves all properties of the Banach algebra case.

For a given k-tuple $(a_1, \ldots, a_k) \in W^k$ in an algebra W we denote by $I_l(a_1, \ldots, a_k)$ $(I_r(a_1, \ldots, a_k))$ the left (resp. right) ideal of W generated by elements $a_1, \ldots, a_k$.

Definition. The *left joint spectrum* on a unital complex algebra W is defined as follows:

$$\sigma_l(a_1, \ldots, a_k) = \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k |\ I_l(a_1 - \lambda_1, \ldots, a_k - \lambda_k) \neq W\}$$

for $(a_1, \ldots, a_k) \in W^k_{\text{com}}$, $k = 1, 2, \ldots$

The *right joint spectrum* $\sigma_r(a_1, \ldots, a_k)$ is defined in an analogous way with the use of the right ideals.

The spectrum $\sigma_H(a_1, \ldots, a_k) = \sigma_l(a_1, \ldots, a_k) \cup \sigma_r(a_1, \ldots, a_k)$ is called the *Harte spectrum* on W.

The Harte spectrum was introduced in [10] in the case of a Banach algebra. It was proved that σ_l, σ_r and σ_H are subspectra. The following generalization of this result was proved in [28].

THEOREM (4.7). *Let W be a unital complex locally convex Waelbroeck algebra. Then $\sigma_l(a_1, \ldots, a_k)$, $\sigma_r(a_1, \ldots, a_k)$, $\sigma_H(a_1, \ldots, a_k)$ are subspectra.*

In section 6 we will prove this theorem in a different way. First we need additional information about subspectra defined by a family of ideals.

5. Subspectra defined by families of ideals

The left spectrum, right spectrum and the Harte spectrum are examples of spectra defined by means of a family of ideals. This method of constructing joint spectra can be generalized in the following form:

Definition. Let B be a unital complex algebra and let $\mathcal{U}$ be a family of ideals in B. Let

$$\sigma^{\mathcal{U}}(a_1, \ldots, a_k) = \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k | \exists I \in \mathcal{U},\ (a_1 - \lambda_1, \ldots, a_k - \lambda_k) \in I^k\},$$

for $(a_1, \ldots, a_k) \in B^k_{\text{com}}$.

The one-way spectral mapping formula is valid for all joint spectra of this form.

PROPOSITION (5.1). *Let B be a unital complex algebra. For an arbitrary family of ideals $\mathcal{U}$ in B the joint spectrum $\sigma^{\mathcal{U}}$ satisfies the formula*

$$p(\sigma^{\mathcal{U}}(a_1, \ldots, a_k)) \subset \sigma^{\mathcal{U}}(p(a_1, \ldots, a_k)),$$

for $(a_1, \ldots, a_k) \in B^k_{\text{com}}$ and p being a polynomial mapping.

Proof. Suppose that the polynomial mapping $p : \mathbb{C}^k \to \mathbb{C}^m$ has the form $p = (p_1, \ldots, p_m)$. For an arbitrary $(\mu_1, \ldots, \mu_k)$ there exist polynomials f_{ij} such that

$$p_j(x_1, \ldots, x_k) - p_j(\mu_1, \ldots, \mu_k) = \sum_{i=1}^{k} f_{ji}(x_1, \ldots, x_k)(x_i - \mu_i)$$

for $1 \leq j \leq m$.

If $(\mu_1, \ldots, \mu_k) \in \sigma^{\mathcal{U}}(a_1, \ldots, a_k)$ then $a_i - \mu_i \in I$ for some $I \in \mathcal{U}$ and $1 \leq i \leq k$. By the formula above $p_j(a_1, \ldots, a_k) - p_j(\mu_1, \ldots, \mu_k) \in I$ for $1 \leq j \leq m$. Hence $p(\mu_1, \ldots, \mu_k) \in \sigma^{\mathcal{U}}(p(a_1, \ldots, a_k))$. The proof follows. □

It is natural to look for a condition for $\sigma^{\mathcal{U}}$ to satisfy the spectral mapping formula.

THEOREM (5.2). *Let B be a unital complex algebra. The joint spectrum $\sigma^{\mathcal{U}}$ satisfies the spectral mapping formula if and only if the family of ideals $\mathcal{U}$ satisfies the following condition (P):*

for every k-tuple $(a_1, \ldots, a_k) \in B^k_{\text{com}}$ such that $I_l(a_1, \ldots, a_k) \subset I \in \mathcal{U}$ and for an arbitrary $c \in B$ commuting with a_i, $1 \leq i \leq k$ there exists $\mu \in \mathbb{C}$ and $J \in \mathcal{U}$ such that $a_1, \ldots, a_k, c - \mu \in J$.

Proof. Assume that the condition (P) is valid. By elementary induction we can deduce from condition (P) its generalization which will be called condition (P_1):

for every $I \in \mathcal{U}$ and for every k-tuple $(a_1, \ldots, a_k) \in I^k_{\text{com}}$ and an arbitrary m tuple $c_1, \ldots, c_m \in B$ such that $(a_1, \ldots, a_k, c_1, \ldots, c_m) \in B^{k+m}_{\text{com}}$ there exist $(\mu_1, \ldots, \mu_m) \in \mathbb{C}^m$ and $J \in \mathcal{U}$ such that

$$a_1, \ldots, a_k, c_1 - \mu_1, \ldots, c_m - \mu_m \in J.$$

In terms of the joint spectrum $\sigma^{\mathcal{U}}$ this is exactly the projection property of the spectrum.

If $(\lambda_1, \ldots, \lambda_k) \in \sigma^{\mathcal{U}}(a_1, \ldots, a_k)$ then by the translation formula $0 \in \sigma^U(a_1 - \lambda_1, \ldots, a_k - \lambda_k)$. If all elements $a_1, \ldots, a_k, c_1, \ldots, c_m$ commute, then by (P_1) there exists $(\mu_1, \ldots, \mu_m) \in \mathbb{C}^m$ such that

$$0 \in \sigma^{\mathcal{U}}(a_1 - \lambda_1, \ldots, a_k - \lambda_k, c_1 - \mu_1, \ldots, c_m - \mu_m)$$

and

$$(\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_m) \in \sigma^{\mathcal{U}}(a_1, \ldots, a_k, c_1, \ldots, c_m).$$

Let $(a_1, \ldots, a_k)$ be a commuting k-tuple of elements of B. By Proposition (5.1)

$$p(\sigma^{\mathcal{U}}(a_1, \ldots, a_k)) \subset \sigma^{\mathcal{U}}(p(a_1, \ldots, a_k)),$$

for an arbitrary polynomial mapping $p = (p_1, \ldots, p_m)$.

For simplicity we shall write $p_j(a)$ instead of $p_j(a_1, \ldots, a_k)$.

Consider the $(k+m)$-tuple of elements $(p_1(a), \dots, p_m(a), a_1, \dots, a_k)$ which mutually commute. Let

$$(\lambda_1, \dots, \lambda_m) \in \sigma^{\mathcal{U}}(p_1(a), \dots, p_m(a)).$$

By the projection property proved above there exists $(\mu_1, \dots, \mu_k) \in \mathbb{C}^k$ such that

$$(\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_k) \in \sigma^{\mathcal{U}}(p_1(a), \dots, p_m(a), a_1, \dots, a_k).$$

There exists $J \in \mathcal{U}$ such that $p_j(a) - \lambda_j \in J$ for all $1 \le j \le m$ and $a_i - \mu_i \in J$ for all $1 \le i \le k$. The one-side projection property implies $\mu = (\mu_1, \dots, \mu_k) \in \sigma^{\mathcal{U}}(a_1, \dots, a_k)$.

For every j we can represent

$$p_j(a) = p_j(\mu) + \sum_{i=1}^{k} f_{ji}(a)(a_i - \mu_i).$$

We obtain $p_j(a) - \lambda_j = p_j(\mu) + \sum_{i=1}^{k} f_{ji}(a)(a_i - \mu_i) - \lambda_j \in J$. However, $\sum_{i=1}^{k} f_{ji}(a)(a_i - \mu_i) \in J$ because J is an ideal. It follows that $p_j(\mu) - \lambda_j \in J$ what means that $p_j(\mu) = \lambda_j$ for all $1 \le j \le m$.

This proves

$$p(\sigma^{\mathcal{U}}(a_1, \dots, a_k)) \supset \sigma^{\mathcal{U}}(p(a_1, \dots, a_k)). \qquad \square$$

The results of this section are purely algebraic. They concern the spectral mapping formula and have nothing to do with the compactness of subspectra.

Let us observe that the spectra which satisfy the spectral mapping formula admit "pull backs" in the following sense.

PROPOSITION (5.3). *Let A, B be unital complex algebras and let $\phi : A \to B$ be a homomorphism satisfying $\phi(e) = e$. If τ is a joint spectrum defined in B_{com} which has the spectral mapping property*

$$p(\tau(b_1, \dots, b_k)) = \tau(p(b_1, \dots, b_k))$$

then the mapping defined in A_{com} by the formula

$$\phi^{-1}\tau(a_1, \dots, a_k) = \tau(\phi(a_1), \dots, \phi(a_k))$$

has the spectral mapping property.

If $\mathcal{U}$ is a family of ideals in B and $\sigma^{\mathcal{U}}$ the corresponding spectrum in B then the spectrum $\phi^{-1}\sigma^{\mathcal{U}}$ coincides with $\sigma^{\mathcal{V}}$ where $\mathcal{V}$ is the family of ideals in A which are inverse images of elements of $\mathcal{U}$:

$$\mathcal{V} = \{\phi^{-1}(I) |\ I \in \mathcal{U}\}.$$

Proof. The proof of the first part of the proposition is a direct calculus. Let $p = (p_1, \dots, p_m)$ be a m-tuple of polynomials of k variables. Then for an arbitrary k-tuple $(a_1, \dots, a_k) \in A_{\mathrm{com}}$ we have

$$\begin{aligned}
p(\phi^{-1}\tau(a_1, \dots, a_k)) &= p(\tau(\phi(a_1), \dots, \phi(a_k))) \\
&= \tau(p_1(\phi(a_1), \dots, \phi(a_k)), \dots, p_m(\phi(a_1), \dots, \phi(a_k))) \\
&= \tau(\phi(p_1(a_1, \dots, a_k)), \dots, \phi(p_m(a_1, \dots, a_k))) \\
&= \phi^{-1}\tau(p(a_1, \dots, a_k)).
\end{aligned}$$

By the definition $0 \in \phi^{-1}\sigma^{\mathcal{U}}(a_1, \dots, a_k)$ if and only if there exists $J \in \mathcal{U}$ such that $\phi(a_1), \dots, \phi(a_k) \in J$. The last property is equivalent to the condition $a_1, \dots, a_k \in \phi^{-1}(J)$ which means $0 \in \sigma^{\mathcal{V}}(a_1, \dots, a_k)$. By the translation property the spectra $\phi^{-1}\sigma^{\mathcal{U}}(a_1, \dots, a_k)$ and $\sigma^{\mathcal{V}}(a_1, \dots, a_k)$ coincide. □

6. Harte theorem for Waelbroeck algebras

This section is dedicated to the proof of Theorem (4.3). We begin with the purely algebraic aspect of the problem, so suppose first that B is a complex algebra with unit e. Let I be a left ideal in B and let E be a commutative subalgebra of B consisting of such that $IE \subset I$.

Consider the set $\mathfrak{I}$ ordered by inclusion of all ideals J in B which satisfy $I \subset J$ and $JE \subset J$. By the Kuratowski-Zorn Lemma there exists a maximal element M in $\mathfrak{I}$.

Consider the quotient space $X = B/M$ with the natural action of B given by the formula $l_b[x] = [bx]$ and with the action of the algebra E defined by $r_b[x] = [xb]$. Finally, we define a representation of $B \times E$ in the carrier space X given by

$$T_{(b,c)}[x] = l_b r_c[x] = [bxc].$$

PROPOSITION (6.1). *The representation T of $B \times E$ on X is irreducible.*

Proof. Denote by $\pi \colon B \to X$ the natural projection. Let $V \subsetneq X$ be a $B \times E$-invariant vector space. In particular V is B-invariant, so $J = \pi^{-1}(V)$ is a left ideal of B containing M. The invariance of V under the action of E implies $JE \subset J$. By the maximality property of M it follows that $J = M$. □

Denote by D the commutant of the representation T, that is the set of all linear operators on X which commute with every operator $T_{(b,c)}$, $(b, x) \in B \times E$. The Schur Lemma (see e.g. [5]) leads immediately to the following theorem.

THEOREM (6.2). *The space D is a division algebra.*

The algebra D can be described in terms of the algebra B. Denote

$$A = \{b \in B |\, Mb \subset M, \text{ and } bx - xb \in M, \text{ for all } x \in E\}.$$

It is clear that A is a unital subalgebra of B and M is a two-sided ideal in A. The right action of A on X can be defined by the formula $r_b[x] = [xb]$. It is convenient to consider also a modified algebraic product in A. For $a, b \in A$ set $a \star b = ba$. The algebra $(A, \star)$ is denoted by $\mathcal{A}$, while $\mathcal{M}$ denotes $(M, \star)$.

THEOREM (6.3). *Every element T of D is of the form $T[x] = [xb]$ for some $b \in \mathcal{A}$. Moreover $D \cong \{u \in B/M |\, l_c u = r_c u, \forall\, c \in E\} \cong \mathcal{A}/\mathcal{M}$.*

Proof. Let $T \in D$. Observe $T[x] = Tl_x[e] = l_x T[e]$ for an arbitrary $x \in B$. If we denote $T[e] = [b]$ then for every $m \in M$ we obtain

$$0 = T[m] = l_m[b] = [mb],$$

giving $Mb \subset M$.

The operator T commutes with all r_c, $c \in E$, whence

$$[bc] = r_c T[e] = Tr_c[e] = T[c] = l_c[b] = [cb],$$

so $b \in \mathcal{A}$ and $u = [b]$ satisfies the condition $l_c u = r_c u$ for all $c \in E$.

All elements of D are of the form $R_{[b]}[x] := [xb]$ for some $b \in \mathcal{A}$. The map $\mathcal{A} \ni b \to R_{[b]}$ is an algebra homomorphism because

$$R_{[b \star c\,]}[x] = [xcb] = R_{[b]}[xc] = R_{[b]}R_{[c]}[x].$$

The kernel of this homomorphism consists of the elements of $\mathcal{M}$, so $D \cong \mathcal{A}/\mathcal{M}$. □

Suppose now that B is a topological algebra. The results we are interested in rest on a theorem of Gelfand-Mazur type, hence the topology of the quotient $D \cong \mathcal{A}/\mathcal{M}$ is essential.

THEOREM (6.4). *Let W be a locally convex Waelbroeck algebra. Then for every left ideal I of W and for every set $S \subset B$ of mutually commuting elements such that $Ic \subset I$ for all $c \in E$ there exists a function $S \ni c \to \mu(c) \in \mathbb{C}$ such that $I + I_l(\{c - \mu(c)\}_{c \in S})$ is a proper ideal of B.*

Proof. Denote by E the subalgebra generated in B by elements of S. Let us construct the ideal M which is the subject of Proposition (6.1) and the division algebra $D = \mathcal{A}/\mathcal{M}$. Let $a \in \mathcal{A}$ be invertible in B. Define $M' = M + Ma^{-1}$. Obviously M' is a left ideal of B. If $e \in M'$, that is, if $e = m + m'a^{-1}$ with $m, m' \in M$, then $a = ma + m' \in M$, which contradicts the invertibility of a. The ideal M' is a proper left ideal. We claim that $M' \in \mathfrak{I}$. For an arbitrary $b \in E$ there exists $m \in M$ such that $ba = ab + m$, hence

$$\begin{aligned} M'b = (M + Ma^{-1})b &\subset M + Ma^{-1}baa^{-1} = M + Ma^{-1}(ab + m)a^{-1} \\ &= M + Mb + Ma^{-1}ma^{-1} \subset M + Ma^{-1} = M'. \end{aligned}$$

By the maximality of M in $\mathfrak{I}$ it follows that $M' = M$, which gives $Ma^{-1} \subset M$. Note that $ba^{-1} - a^{-1}b = a^{-1}(ab - ba)a^{-1} = a^{-1}ma^{-1} \in M$ giving finally $a^{-1} \in \mathcal{A}$. The algebra $\mathcal{A}$ is a subalgebra of B closed under inverse. This means that $\mathcal{A}$ is also a locally convex Waelbroeck algebra as well as its quotient algebra $D = \mathcal{A}/\mathcal{M}$. In particular the inverse is continuous in D, so by the Gelfand-Mazur theorem $D \cong \mathbb{C}$.

For every $s \in S$ the operator r_s belongs to D, hence there exists $\mu(s) \in \mathbb{C}$ such that $[xs] = \mu(s)[x]$ for every $x \in B$. This means that $B(c - \mu(s)) \subset M$ for every $s \in S$, which ends the proof. □

Clearly the right ideal versions of the results are also valid.

Theorem (6.4) is even stronger than the usual projection property (P) of the family of all ideals of a locally convex Waelbroeck algebra B. Thanks to Theorem (5.2) it follows that the left (and right) joint spectrum have the spectral mapping property. The proof of Theorem (4.3) follows.

7. Subspectra generated by algebra extensions

In this section we describe a very special construction of a subspectrum by means of a family of ideals. It concerns certain class of subalgebras of LC Waelbroeck algebras including commutative ones.

Assume that A is a subalgebra of a unital algebra B. We say that A is a unital subalgebra of B if $e \in A$. Let $\mathfrak{I}^l_B(A)$ be the set of left ideals in A which do not intersect with $G(B)$. An ideal $I \in \mathfrak{I}_B^l(A)$ consists of elements which are not invertible even in B.

Let B be a unital algebra and $A \subset B$ be a unital subalgebra of B such that $aAa^{-1} \subset A$ for every $a \in A$ which is invertible in B.

Under the last condition the smallest subalgebra of B which contains A and is inverse-closed can be described as an algebra of "fractions".

Denote

$$R(A) = \{b^{-1}a|\ b \in A \cap G(B),\ a \in A\}.$$

If $x = b^{-1}a,\ y = d^{-1}c \in R(A)$ then

$$x + y = b^{-1}a + d^{-1}c = (db)^{-1}(da + (db)b(db)^{-1}c) \in R(A)$$

because $da + (db)b(db)^{-1}c \in A$ according to our supposition. The space $R(A)$ is a linear space.

On the other hand

$$xy = b^{-1}ad^{-1}c = b^{-1}d^{-1}dad^{-1}c = (db)^{-1}dad^{-1}c \in R(A)$$

because $dad^{-1}c \in A$.

The space $R(A)$ is a unital subalgebra of B which obviously contains A. An element $x = b^{-1}a \in R(A)$ is invertible in B if and only if $a \in G(B)$. This being the case $x^{-1} = a^{-1}b \in R(A)$, so the algebra $R(A)$ is an inverse-closed subalgebra of B.

If I is a left ideal in A then the left ideal generated in $R(A)$ by I is equal to $J = \{b^{-1}a|\ b \in G(B) \cap A,\ a \in I\}$. In particular J is proper if and only if I does not intersect $G(B)$.

We have proved

PROPOSITION (7.1). *Let B be a unital algebra. Let $A \subset B$ be a unital subalgebra of B such that $aAa^{-1} \subset A$ for every $a \in A \cap G(B)$. Then for $(a_1, \ldots, a_k) \in A^k_{\text{com}}$ and $\mathcal{U} = \mathfrak{I}_B^l(A)$*

$$\sigma^{\mathcal{U}}(a_1, \ldots, a_k) = \sigma^l_{R(A)}(a_1, \ldots, a_k).$$

COROLLARY (7.2). *Let W be a unital LC Waelbroeck algebra. Let $A \subset W$ be a unital subalgebra of W such that $aAa^{-1} \subset A$ for every $a \in A \cap G(W)$. If we put $\mathcal{U} = \mathfrak{I}_W^l(A)$ then $\sigma^{\mathcal{U}}$ is a subspectrum in A.*

Proof. The relation $\sigma^{\mathcal{U}}(a_1, \ldots, a_k) \subset \prod_{i=1}^k \sigma(a_i)$ is obvious. The algebra $R(A)$ is an inverse-closed subalgebra of W hence $G(R(A)) = R(A) \cap W$ and $R(A)$ is a LC Waelbroeck algebra. By Theorem (4.3) $\sigma^l_{R(A)}$ has the spectral mapping property, so by Theorem (5.2) the proof follows. □

Example. Let B be a unital algebra. The *Waelbroeck spectrum*, called also the *rational spectrum* of $(a_1, \ldots, a_k) \in B^k_{\text{com}}$, is the set

$$\begin{aligned}\sigma_R(a_1, \ldots, a_k) = \{&(\mu_1, \ldots, \mu_k) \in \mathbb{C}| \\ &p(\mu_1, \ldots, \mu_k) \in \sigma(p(a_1, \ldots, a_k)) \text{ for every polynomial } p\}.\end{aligned}$$

As before $\sigma(a)$ denotes the usual spectrum of a single element $a \in B$.

Let us denote by $A(a_1, \ldots, a_k)$ the unital (commutative) subalgebra generated in B by $a_1, \ldots, a_k$.

According to the definition, $(\mu_1, \ldots, \mu_k) \in \sigma_R(a_1, \ldots, a_k)$ if and only if $p(a_1, \ldots, a_k) - p(\mu_1, \ldots, \mu_k)$ is not invertible in B. We can represent

$$p(a_1, \ldots, a_k) - p(\mu_1, \ldots, \mu_k) = \sum_{i=1}^{k} f_i(a_1, \ldots, a_k)(a_i - \mu_i)$$

where f_i, $1 \leq i \leq k$ are polynomials.

The condition $(\mu_1, \ldots, \mu_k) \in \sigma_R(a_1, \ldots, a_k)$ means that the ideal generated by $a_1 - \mu_1, \ldots, a_k - \mu_k$ in $A(a_1, \ldots, a_k)$ does not intersect $G(B)$. By Proposition (7.1) $\sigma_R(a_1, \ldots, a_k)$ coincides with the usual joint spectrum of $(a_1, \ldots, a_k)$ in the smallest division-closed subalgebra of B containing $A(a_1, \ldots, a_k)$.

In the case of a Banach algebra B this was proved by Arveson [4].

By Corollary (7.2) the Waelbroeck spectrum in a LC Waelbroeck algebra is nonempty.

8. Regularities and joint spectra

The definition of the left joint spectrum suggests a natural generali- zation. The condition $I_l(a_1 - \lambda_1, \ldots, a_k - \lambda_k) \neq W$ means that the ideal $I_l(a_1 - \lambda_1, \ldots, a_k - \lambda_k)$ does not intersect $G(W)$. Let $\mathcal{R} \subset W$ and let

$$\sigma_l^{\mathcal{R}}(a_1, \ldots, a_k) = \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k |\ I_l(a_1 - \lambda_1, \ldots, a_k - \lambda_k) \cap \mathcal{R} = \varnothing\}.$$

for $(a_1, \ldots, a_k) \in W^k_{\text{com}}$. Taking $\mathcal{R} = G(W)$ we obtain as $\sigma_l^{\mathcal{R}}$ the left joint spectrum.

Note that the set $G(W)$ can be described in terms of the left spectrum as $\{x \in W |\ 0 \notin \sigma_l(x)\}$. A number of questions arise.

What conditions should a subset $\mathcal{R} \subset W$ satisfy for the corresponding $\sigma_l^{\mathcal{R}}$ to be a subspectrum?

If τ is a subspectrum and $\mathcal{R}_\tau = \{x \in W |\ 0 \notin \tau(x)\}$, are $\sigma_l^{\mathcal{R}_\tau}$, $\sigma_r^{\mathcal{R}_\tau}$ subspectra? In the affirmative case, which is the relation between them and the original subspectrum τ?

Satisfactory answers are known only in the case of a commutative Banach algebra ([7]). The results can be extended on the commutative LC Waelbroeck algebras.

Let us suppose again that W is a unital commutative LC Waelbroeck algebra. For an arbitrary nonvoid $A \subset W$ let us define

$$A^{\#} = \{x \in W |\ \forall \varphi \in W',\ \varphi(x) = 0 \Rightarrow\ 0 \in \varphi(A)\}.$$

Definition. A nonempty set $\mathcal{R} \subsetneq W$ is called, *a regularity* if

(1). If $a,\ b \in W$ then $ab \in \mathcal{R}$ if and only if $a \in \mathcal{R}$ and $b \in \mathcal{R}$.

(2). $\mathcal{R}^{\#} = \mathcal{R}$.

Examples. (1) The set of invertible elements of W is a regularity. Obviously $ab \in G(W)$ if and only if $a \in G(W)$, $b \in G(W)$. For every $a \notin G(W)$ there exists a multiplicative functional φ such that $\varphi(a) = 0$. Since $\ker \varphi \cap G(W) = \varnothing$ we obtain $a \notin G(W)^{\#}$. Hence $G(W)^{\#} = G(W)$.

(2) Let A be a unital closed subalgebra of W and let $\mathcal{R} = G(W) \cap A$. Using the same arguments as in Example 1 we can see that R is a regularity in A.

THEOREM (8.1). *Let $\mathcal{R}$ be a regularity in W. Then*
(1). *$\mathcal{R}$ is open and contains $G(W)$,*
(2). *The set $K = \{\varphi \in \mathfrak{M}(W) |\ \ker \varphi \cap \mathcal{R} = \varnothing\}$ is compact and*

$$\mathcal{R}^c = \bigcup_{\varphi \in K} \ker \varphi. \tag{8.2}$$

Proof. Let $a \in \mathcal{R}$. By the first property defining $\mathcal{R}$ we have $e \cdot x = x \in \mathcal{R}$, so $e \in \mathcal{R}$. For an arbitrary $x \in G(W)$ it follows $x \cdot x^{-1} = e \in \mathcal{R}$ hence in particular $x \in \mathcal{R}$.

Let V be a neighbourhood of e contained in $G(W)$. If $x \in \mathcal{R}$ and $y \in V$ then $xy \in \mathcal{R}$. The set xV is a neighbourhood of x contained in $\mathcal{R}$, so the regularity is open.

In order to prove the second part of the theorem, take $x \in \mathcal{R}^c$. Since $\mathcal{R}^{\#} = \mathcal{R}$ it follows that there exists a linear functional φ on W such that $\varphi(x) = 0$ but $\varphi(y) \neq 0$ for every $y \in \mathcal{R}$. The kernel of φ consists of non-invertible elements, so by Theorem (2.6) φ is continuous and multiplicative. The ideal $\ker \varphi$ belongs to $\mathfrak{M}(W)$ and does not intersects $\mathcal{R}$. This proves the formula (8.2).

It remains to prove that $K = \{\varphi \in \mathfrak{M}(W) |\ \ker \varphi \cap \mathcal{R} = \varnothing\}$ is compact.

If $\varphi \notin K$, then there exists $x \in \mathcal{R}$ such that $\varphi(x) = 0$. The set $\mathcal{R}$ is open hence there exists $\epsilon > 0$ such that for $|\mu| < \epsilon$ we have $x + \mu \in \mathcal{R}$. Let $U = \{\psi \in \mathfrak{M}(W) |\ |\psi(x)| < \epsilon\}$. It is an open neighbourhood of φ. If $\psi \in U$ then $\psi(x - \psi(x)) = 0$. However $x - \psi(x) \in \mathcal{R}$, so $\psi \notin K$. $\mathfrak{M}(W) \setminus K$ is open and K is compact. $\square$

We have associated to an arbitrary regularity $\mathcal{R}$ a compact subset $K \subset \mathfrak{M}(W)$ which defines a subspectrum on W by the formula

$$\tau(a_1, \ldots, a_k) = \{(\varphi(a_1), \ldots, \varphi(a_k)) \in \mathbb{C}^k |\ \varphi \in K\}. \tag{8.3}$$

We are going to prove that the subspectrum τ defined by the formula (6) coincides with $\sigma^{\mathcal{R}}$.

THEOREM (8.4). *Let $\mathcal{R}$ be a regularity in a commutative LC Waelbroeck algebra W. Then $\sigma^{\mathcal{R}}$ is a subspectrum and*

$$\sigma^{\mathcal{R}}(a_1, \ldots, a_k) = \{(\varphi(a_1), \ldots, \varphi(a_k)) \in \mathbb{C}^k |\ \varphi \in K\},$$

where K is the set of maximal ideals of W which do not intersect $\mathcal{R}$.

Proof. Let us denote by A the algebra of all functions on K of the form $\widehat{x}|K$, $x \in W$. This is a subalgebra of the Banach algebra $C(K)$. Let $R = \{\widehat{x}|K \mid x \in \mathcal{R}\}$. By the definition of K it follows that the elements of R are functions which do not achieve zero on K so they are invertible in $C(K)$.

The condition $\mathcal{R}^{\#} = \mathcal{R}$ implies that every function of the form $\widehat{x}|K$ which nowhere vanishes on K is an element of R. Hence $R = A \cap G(C(K))$. By Theorem (7.1) the mapping defined by the formula

$$\sigma^{\mathcal{U}}(f_1, \ldots, f_k) = \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{C}^k | I(f_1 - \lambda_1, \ldots, f_k - \lambda_k) \cap R = \varnothing\}$$

is a subspectrum on A.

Proposition (5.3) implies that the "pull back" of $\sigma^{\mathcal{U}}$ under the mapping $\phi : W \ni x \to \widehat{x}|K$ is also a subspectrum. By the very definition it follows that $\phi^{-1}\sigma^{\mathcal{U}} = \sigma^{\mathcal{R}}$. This proves that $\sigma^{\mathcal{R}}$ is a subspectrum.

According to Theorem (4.2) there exists a unique compact set $K_1 \subset \mathfrak{M}(W)$ such that

$$\sigma^{\mathcal{R}}(a_1, \dots, a_k) = \{(\varphi(a_1), \dots, \varphi(a_k)) \in \mathbb{C}^k | \ \varphi \in K_1\}.$$

It remains to prove that $K = K_1$. Let $\varphi \in K_1$ and $a \in \ker \varphi$. Hence $\sigma^{\mathcal{R}}(a)$ contains the value $\varphi(a) = 0$. By the definition of $\sigma^{\mathcal{R}}$ the ideal $I(a)$ does not intersect $\mathcal{R}$. In particular $a \notin \mathcal{R}$ so $\varphi \in K$. We have proved that $K_1 \subset K$.

Now suppose that $\varphi \notin K_1$. Then there exists $(a_1, \dots, a_k) \in W^k$ such that

$$(\varphi(a_1), \dots, \varphi(a_k)) \notin \sigma^{\mathcal{R}}(a_1, \dots, a_k).$$

By the translation formula $0 \notin \sigma^{\mathcal{R}}(a_1 - \varphi(a_1), \dots, a_k - \varphi(a_k))$. The ideal $I(a_1 - \varphi(a_1), \dots, a_k - \varphi(a_k))$ which belongs to the kernel of φ contains an element of $\mathcal{R}$. Hence $\varphi \notin K$. It proves that $K \subset K_1$. □

The essential part of this theorem asserts that every finitely generated ideal of W which does not intersect $\mathcal{R}$ is contained in a maximal ideal not intersecting $\mathcal{R}$.

We have proved that every regularity defines a subspectrum. If τ is an arbitrary subspectrum, then $\mathcal{R}_\tau = \{x \in W | \ 0 \notin \tau(x)\}$ is a regularity. In fact, by Theorem (4.2) there exists a compact set $K_\tau \subset \mathfrak{M}(W)$ such that for $x \in W$ we have $0 \in \tau(x)$ if and only if $\varphi(x) = 0$ for some $\varphi \in K_\tau$. It assures both properties defining a regularity:

(1). $\mathcal{R}_\tau^{\#} = \mathcal{R}_\tau$,

(2). $ab \in \mathcal{R}_\tau$ if and only if $a \in \mathcal{R}_\tau$, $b \in \mathcal{R}_\tau$.

In order to describe the relation between the spectra τ and $\sigma^{\mathcal{R}_\tau}$ we relate the compact sets corresponding to both objects according to Theorem (4.2).

Theorem (8.2) describes the compact set corresponding to $\sigma^{\mathcal{R}_\tau}$ as the set K of the maximal ideals of W which do not intersect $\mathcal{R}_\tau$. In terms of multiplicative functionals it means that $\varphi \in K$ if and only if for an arbitrary $a \in W$ $\varphi(a) = 0$ implies $\phi(a) = 0$ for some $\phi \in K_\tau$. This leads to the following result.

Proposition (8.5). *If τ is a subspectrum on a commutative LC Waelbroeck algebra W given by*

$$\tau(a_1, \dots, a_k) = \{(\varphi(a_1), \dots, \varphi(a_k)) \in \mathbb{C}^k | \ \varphi \in K_\tau\}$$

and

$$\sigma^{\mathcal{R}_\tau}(a_1, \dots, a_k) = \{(\varphi(a_1), \dots, \varphi(a_k)) \in \mathbb{C}^k | \ \varphi \in K\},$$

then

$$K = \{\varphi \in \mathfrak{M}(W) | \, \forall \, a \in W \ \hat{a}(\varphi) = 0 \ \Rightarrow \ 0 \in \hat{a}(K_\tau)\}.$$

The last result suggests the following definition.

Definition. Let W be a commutative LC Waelbroeck algebra. Let $K \subset \mathfrak{M}(W)$ and let

$$\widetilde{K} = \{\varphi \in \mathfrak{M}(W) | \, \forall \, a \in W \ \varphi(a) = 0 \Rightarrow \ 0 \in \hat{a}(K)\}.$$

The set $\widetilde{K}$ will be called the $\widehat{W}$-rationally convex hull of K.

Obviously $\widetilde{\widetilde{K}} = \widetilde{K}$. If $\widetilde{K} = K$ the set K is called $\widehat{W}$-rationally convex.

Proposition (8.3) asserts that the set K defining the subspectrum $\sigma^{\mathcal{R}_\tau}$ is the $\widehat{W}$-rationally convex hull of K. In particular K is $\widehat{W}$-rationally convex.

Example. Let $A = A(a_1, \dots, a_k)$ be a commutative Banach algebra generated by elements $(a_1, \dots, a_k)$. Let $M = \sigma(a_1, \dots, a_k) \subset \mathbb{C}^k$. It is well known (see e.g. [34] p. 78) that the set M can be identified with $\mathfrak{M}(A)$ by means the of the mapping which associates to $z = (z_1, \dots, z_k) \in M$ the multiplicative functional defined by the formula

$$\varphi_z(p(a_1, \dots, a_k)) = p(z_1, \dots, z_k)$$

on the dense subset of A consisting of polynomials of the elements $a_1, \dots, a_k$.

Let $K \subset M$ be a compact set which is $\widehat{A}$-rationally convex. If $\varphi_z \notin K$ then $\varphi_z \notin \widetilde{K}$ neither. There exists $a \in A$ such that $\varphi_z(a) = 0$ and $0 \notin \hat{a}(K)$. We can assume that a is of the form $a = p(a_1, \dots, a_k)$. So $p(z_1, \dots, z_k) = 0$ but $p(w_1, \dots, w_k) \neq 0$ for $w \in K$. The $\widehat{A}$-rationally convex subsets of M are rationally convex in the usual sense.

9. Spectra of generators

A topological unital algebra B is generated by $G \subset B$ if the smallest unital and closed subalgebra containing G is B itself. It is a classical result (see e.g. [34]) that for a commutative Banach algebra B generated by a finite set $(a_1, \dots, a_k)$ the space $\mathfrak{M}(B)$ can be identified with the joint spectrum $\sigma(a_1, \dots, a_k)$. Moreover, the latter set is polynomially convex.

V. Müller and A. Sołtysiak [14] have proved that in the case of a noncommutative finitely generated Banach algebra B the identification of $\mathfrak{M}(B)$ with $\sigma(a_1, \dots, a_k)$ is still valid. The spectrum of generators is nonvoid if and only if the two sided ideal generated by the elements $a_i a_j - a_j a_i$, $1 \le i, j \le k$ is not dense in B.

It was observed by W. Żelazko [36] that in the general case of a finitely topological algebra B there is an isomorphism

$$\mathfrak{M}(B) \cong \sigma^t(a_1, \dots, a_k)$$

where $\mathfrak{M}(B)$ is the set of continuous multiplicative functionals on B and

$$\sigma^t(a_1, \dots, a_k) = \{(\mu_1, \dots, \mu_k) | e \notin \sum_{i=1}^{k} B(a_i - \mu_i)^-\}.$$

In this section we generalize this relation in two directions. The finitness of the set of generators can be is removed if we extend the definition of the spectrum on infinite sets in a natural way. Moreover, we obtain an interpretation of the set $\sigma_{\mathcal{U}}(G)$ for an arbitrary family of closed ideals by means of a an apriopriate subset of $\mathfrak{M}(B)$.

For an arbitrary Waelbroeck algebra B we prove that $\sigma_{\mathcal{U}}(G)$ is a rationally convex set. As observed by R. M. Brooks [6], if B is not a Banach algebra, the spectrum of generators can fail to be polynomially convex.

A topological algebra generated by a subset S is denoted by $A(S)$.

For a given family $\mathcal{U}$ of ideals in a unital topological algebra B we define the joint spectrum of a set S as follows:

$$\sigma_{\mathcal{U}}(S) = \{(\mu(s))_{s \in S} | \exists I \in \mathcal{U},\ \forall s \in S\ s - \mu(s) \in I\}.$$

Let

$$\mathfrak{M}_{\mathcal{U}} = \{\varphi \in \mathfrak{M}(B) | \ker \varphi \in \mathcal{U}\}.$$

THEOREM (9.1). *Let $\mathcal{A} = \mathcal{A}(S)$ be a unital topological algebra. Let $\mathcal{U}$ be an arbitrary family of closed left ideals in $\mathcal{A}$. Then*

$$\sigma_{\mathcal{U}}(S) = \{(\varphi(s))_{s\in S} |\, \varphi \in \mathfrak{M}_{\mathcal{U}}\}.$$

Proof. Let $\varphi \in \mathfrak{M}_{\mathcal{U}}$. Thus $s - \varphi(s) \in \ker\varphi$. By assumption the two-sided ideal $\ker\varphi$ belongs to $\mathcal{U}$, so the function $(\varphi(s))_{s\in S}$ is an element of $\sigma_{\mathcal{U}}(S)$.

Now, suppose that $(\lambda_s)_{s\in S} \in \sigma_{\mathcal{U}}(S)$. We define a functional on the dense subalgebra of $\mathcal{A}$ generated by S and e. Every element of this algebra is of the form $p(s_1, \dots, s_k)$, where $p \in \mathcal{P}$ and $s_1, \dots, s_k \in S$.

Let

$$\varphi(p(s_1, \dots, s_k)) = p(\lambda_{s_1}, \dots, \lambda_{s_k}).$$

First of all it is neccessary to prove that this definition is correct. According to the remainder theorem (see [10]) for every polynomial p there exist polynomials q_i such that

$$p(s_1, \dots, s_k) - p(\lambda_{s_1}, \dots, \lambda_{s_k}) = \sum_{i=1}^{k} q_i(s_1, \dots, s_k)(s_i - \lambda_{s_i}),$$

If we suppose $p(s_1, \dots, s_k) = 0$ we obtain

$$-p(\lambda_{s_1}, \dots, \lambda_{s_k}) = \sum_{i=1}^{k} q_i(s_1, \dots, s_k)(s_i - \lambda_{s_i}).$$

The right hand side is an element of the ideal $\sum_{s\in S} \mathcal{A}(s - \lambda_s)^-$ which by assumption belongs to some element I of $\mathcal{U}$, so in particular it is a proper ideal.

The left hand side is proportional to the unit e, hence it must be zero. The definition of f is correct.

By the same remainder formula, the kernel of f consists of elements of the form $\sum_{i=1}^{k} q_i(s_1, \dots, s_k)(s_i - \lambda_{s_i})$, that belong to the same closed ideal $I \in \mathcal{U}$. It follows that f is continuous, so it extends to a multiplicative continuous functional whose kernel is an element of $\mathcal{U}$. □

A set $B \subset \mathbb{C}^k$ is rationally convex if it coincides with its rationally convex hull

$$r(B) = \{z \in \mathbb{C}^k |\, \text{for every polynomial } p, \quad p(z) = c \ \Rightarrow\ c \in p(B)\}.$$

Let us extend this concept on subset of $\mathbb{C}^S$. Let $F \subset \mathbb{C}^S$ and let $\mathbf{z} = (z_s)_{s\in S} \in \mathbb{C}^S$.

We say that $\mathbf{z} \in r(F)$ if for every polynomial p in k variables and for every set $\{s_1, \dots, s_k\} \subset S$

$$p(z_{s_1}, \dots, z_{s_k}) = c \ \Rightarrow\ \exists\, \mathbf{b} \in F, \quad c = p(b_{s_1}, \dots, b_{s_k}).$$

The set $F \subset \mathbb{C}^S$ is rationally convex if $r(F) = F$.

Let us equip $\mathbb{C}^S$ with the topology of pointwise convergence.

Notice that for a Q-algebra $\mathcal{A}$ the spectra $\sigma_l^{(t)}$ and σ_l coincide and for $a_1, \dots, a_k \in \mathcal{A}$ the spectrum $\sigma_l(a_1, \dots, a_k)$ is compact.

THEOREM (9.2). *Let $\mathcal{A} = \mathcal{A}(S)$ be a complex Q-algebra. Let $\mathcal{U}$ be a family of left closed ideals. Then $r(\sigma_{\mathcal{U}}(S)) = \overline{\sigma_{\mathcal{U}}(S)}$. In particular, if $\mathfrak{M}_{\mathcal{U}}$ is closed, $\sigma_{\mathcal{U}}(S)$ is rationally convex.*

Proof. If $\sigma_{\mathcal{U}}(S) = \varnothing$, we have nothing to prove. So we suppose $\sigma_{\mathcal{U}}(S) \neq \varnothing$. Let $\mathbf{z} \in r(\sigma_{\mathcal{U}}(S))$. Define a functional on the subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ of elements of the form $p(s_1, \dots, s_k)$, $s_i \in S$ setting

$$f(p(s_1, \dots, s_k)) = p(z_{s_1}, \dots, z_{s_k}).$$

The correctness of this definition is proved by arguments used in the proof of Theorem (9.1). Suppose $p(s_1, \dots, s_n) = 0$.

There exists $\lambda \in \sigma_{\mathcal{U}}(S)$ such that $p(\lambda_{s_1}, \dots, \lambda_{s_k}) = p(z_{s_1}, \dots, z_{s_k})$. Then by the remainder formula

$$-p(z_{s_1}, \dots, z_{s_k}) = p(s_1, \dots, s_k) - p(\lambda_{s_1}, \dots, \lambda_{s_k}) = \sum_{i=1}^{k} q_i(s_1, \dots, s_k)(s_i - \lambda_{s_i}).$$

The right hand side belongs to an ideal $I \in \mathcal{U}$, while on the right we have element proportional to the unit. Hence

$$f(p(s_1, \dots, s_k)) = p(z_{s_1}, \dots, z_{s_k}) = 0.$$

The multiplicative functional f is well defined on $\mathcal{A}_0$. As we have seen, its kernel consists of elements not invertible in $\mathcal{A}$. The algebra $\mathcal{A}$ is a $\boldsymbol{Q}$-algebra, so this kernel is not dense in $\mathcal{A}_0$. It follows that f is continuous and it extends to a multiplicative, continuous functional on $\mathcal{A}$. The proof can be ended here if the spectrum in question is σ_l. In this case by Theorem 9.1 $\mathbf{z} = f|S \in \sigma_l(S)$, so $r(\sigma_l(S)) = \sigma_l(S)$.

In the case of a generic $\mathcal{U}$ we claim that the functional f belongs to $\overline{\mathfrak{M}_{\mathcal{U}}}$.

The Gelfand transform associates to $a \in \mathcal{A}$ the function $\widehat{a}(\varphi) = \varphi(a)$. The function $\widehat{a}$ is continuous on $\mathfrak{M}(\mathcal{A})$ and in the case of a $\boldsymbol{Q}$-algebra the mapping $\mathcal{A} \ni a \to C(\mathfrak{M}(\mathcal{A}))$ is continuous (see [13]).

Consider tha algebra A consisting of the functions $\widehat{a}|\overline{\mathfrak{M}_{\mathcal{U}}}$. This is a subalgebra of $C(\overline{\mathfrak{M}_{\mathcal{U}}})$. Denote by ψ the superposition of the Gelfand transform with the operator of restriction to $\overline{\mathfrak{M}_{\mathcal{U}}}$. Then $A = \psi(\mathcal{A}_0)$. The image $J = \psi(\ker f)$ is an ideal in the commutative algebra A. As the above calculation shows, every element of J vanishes at some point of $\mathfrak{M}_{\mathcal{U}}$. By Theorem 3.3 [26] it follows that there exists $g \in \overline{\mathfrak{M}_{\mathcal{U}}}$ such that $I \in \ker\, g$. The kernels of the multiplicative functionals f and g in $\mathcal{A}_0$ coincide. Hence $f = g$. We have obtained $f|S = \mathbf{z} \in \overline{\sigma_{\mathcal{U}}(S)}$. □

In the particular case of a Waelbroeck algebra we obtain

COROLLARY (9.3). *Let $W = A(a_1, \dots, a_k)$ be a finitely generated Waelbroeck algebra. Then $\sigma_l(a_1, \dots, a_k) \cong \mathfrak{M}(W)$ is a rationally convex set in $\mathbb{C}^k$.*

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REPRODUCING KERNELS OF WEIGHTED POLY-BERGMAN SPACES ON THE UPPER HALF-PLANE

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ABSTRACT. Let Π be the upper half-plane. The weighted poly-Bergman spaces on Π consist of all functions on $L_2(\Pi, (\lambda+1)(2y)^{\lambda}dx\,dy)$ satisfying the equation $(\partial/\partial\bar{z})^n f = 0$. Reproducing kernels of the weighted poly-Bergman spaces are obtained via the Fourier transform.

1. Introduction

This paper concerns representations of poly-Bergman and anti-poly-Bergman projections on L_2 space over the upper half-plane Π with the usual area measure $dxdy$ and the weighted area measure $(\lambda+1)(2y)^{\lambda}dxdy$. Recall that the Bergman space of a domain $G \subset \mathbb{C}$ is defined as the space of all analytic functions on G belonging to $L_2(G)$. The Bergman space is denoted by $A^2(G)$. By definition, the Bergman projection B is the orthogonal projection from $L_2(G)$ onto $A^2(G)$. The following integral representation of B is well known ([1])

$$(Bf)(z) = \int_G k(z,\zeta)f(\zeta)d\mu(\zeta),$$

where $k(z,\zeta)$ is the Bergman kernel of G, and $d\mu = dxdy$ is the Lebesgue measure.

For G bounded with smooth boundary, the Bergman projection of G has another representation ([1, 3]), which is expressed in terms of the so-called two-dimensional Hilbert transform

$$(S_{\mathbb{R}^2}f)(z) = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{f(\zeta)}{(z-\zeta)^2}\, d\mu(\zeta).$$

This representation is given by

$$B = I - \chi_G S_{\mathbb{R}^2}\chi_G S^*_{\mathbb{R}^2}\chi_G I + L, \tag{1.1}$$

where χ_G is the characteristic function of G and L is a certain compact operator on $L_2(G)$.

A natural generalization ([3]) of the Bergman space is the space $\mathcal{A}^2_n(G)$ of all n-analytic functions. This space is defined as the subspace of $L_2(G)$ consisting of all functions $f(z) = f(x,y)$ which satisfy the equation

$$\left(\frac{\partial}{\partial\bar{z}}\right)^n f = 0,$$

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where, as usual, $2\partial/\partial\bar{z} = \partial/\partial x + i\partial/\partial y$. The set $\mathcal{A}_n^2(G)$ thus defined is a closed subspace of $L_2(G)$ ([8]), and is called the n-poly-Bergman space of G. Note that $\mathcal{A}_1^2(G)$ is the Bergman space of G. From now on the orthogonal projection

$$B_n : L_2(G) \to \mathcal{A}_n^2(G)$$

will be called the n-poly-Bergman projection of G. The true-n-analytic function space was introduced in [8] as $\mathcal{A}_{(n)}^2(G) = \mathcal{A}_n^2(G) \ominus \mathcal{A}_{n-1}^2(G)$, where by convention $\mathcal{A}_0^2(G) = \{0\}$. The true-$n$-poly-Bergman projection $B_{(n)}$ is defined as the orthogonal projection from $L_2(G)$ onto $\mathcal{A}_{(n)}^2(G)$. Obviously

$$B_n = \sum_{k=1}^{n} B_{(k)}.$$

The n-anti-analytic and true-n-anti-analytic function spaces $\tilde{\mathcal{A}}_n^2(G)$ and $\tilde{\mathcal{A}}_{(n)}^2(G)$ are defined along the same lines, with $\partial/\partial\bar{z}$ changed to $\partial/\partial z$. The corresponding orthogonal projections are denoted by $\tilde{B}_n$ and $\tilde{B}_{(n)}$.

A similar representation to (1.1) has been established for B_n when G is bounded with smooth boundary ([3]). Fortunately, B_n can be expressed in terms of the two-dimensional Hilbert transform in a very simple way when G is the upper half-plane Π, as is established in the following theorem.

THEOREM (1.2). *([4]) The n-poly-Bergman projection and the n-anti-poly-Bergman projection of the upper half-plane admit the following representations:*

$$B_n = I - P_\Pi S_{\mathbb{R}^2}^n P_\Pi (S_{\mathbb{R}^2}^*)^n P_\Pi,$$

$$\tilde{B}_n = I - P_\Pi (S_{\mathbb{R}^2}^*)^n P_\Pi S_{\mathbb{R}^2}^n P_\Pi,$$

where $P_\Pi = \chi_\Pi I$.

Our aim is to study *weighted* poly-Bergman spaces of the upper half-plane. These kind of spaces are defined in Section 2, and they are completely described via isomorphisms therein. Their corresponding reproducing kernels are given in Section 3. The Christoffel-Darboux identity plays an important role when obtaining reproducing kernels of weighted poly-Bergman spaces. Let us introduce a preliminary result from which this work has been inspired.

Let F be the Fourier transform on $L_2(\mathbb{R})$:

$$(Ff)(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ity} dt.$$

Consider the following unitary operators on $L_2(\mathbb{R}^2)$:

$$U_1 = F \otimes I,$$

and U_2 defined by

$$(U_2 f)(x, y) = \frac{1}{\sqrt{2|x|}} f\left(x, \frac{y}{2|x|}\right).$$

Let $\chi_\pm$ be the characteristic function of $\mathbb{R}_\pm = \{x : \pm x > 0\}$, and let P_n be the orthogonal projection of $L_2(\mathbb{R}_+)$ onto L_n, where L_n denotes the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $\ell_n(y) = L_n(y)e^{-y/2}\chi_+(y)$. As usual $L_n(y)$ stands for the Laguerre polynomial of degree n, which is defined by

$$L_n(y) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-y)^k}{k!}.$$

This work has been developed following ideas used in [8, 9]. Theorem (1.3) below was proved in [8], and it is important to our purpose.

THEOREM (1.3) ([8]). *The operator* $U = U_2U_1$ *isomorphically maps* $\mathcal{A}^2_{(n)}(\Pi)$ *onto* $L_2(\mathbb{R}_+) \otimes L_{n-1}$, *and maps* $\tilde{\mathcal{A}}^2_{(n)}(\Pi)$ *onto* $L_2(\mathbb{R}_-) \otimes L_{n-1}$. *Consequently,*

$$\begin{aligned} UB_{(n)}U^* &= \chi_+ I \otimes P_{n-1}, \\ UB_nU^* &= \chi_+ I \otimes \sum_{k=1}^{n} P_{k-1}, \\ U\tilde{B}_{(n)}U^* &= \chi_- I \otimes P_{n-1}, \\ U\tilde{B}_nU^* &= \chi_- I \otimes \sum_{k=1}^{n} P_{k-1}. \end{aligned}$$

As usual $\Gamma(z)$ denotes the gamma function. We will be using the following integral formulae ([5]), which are valid for $\operatorname{Re}\zeta > 0$:

$$\int_0^\infty e^{-t\zeta} t^\kappa dt = \frac{\Gamma(\kappa+1)}{\zeta^{\kappa+1}}, \tag{1.4}$$

$$F^*\left(\frac{k!}{(\zeta \pm iy)^{k+1}}\right) = \sqrt{2\pi}|y|^k e^{-\zeta|y|}\chi_\pm(y). \tag{1.5}$$

2. Poly-Bergman Spaces and Isomorphisms

For $\lambda \in (-1, \infty)$, we will define the weighted poly-Bergman spaces. Consider the measure $d\nu_\lambda(y) = (\lambda+1)(2y)^\lambda$ on $\mathbb{R}_+$. Let $d\mu_\lambda$ stand for the measure $dxd\nu_\lambda(y) = (\lambda+1)(2y)^\lambda dxdy$ on the upper half-plane Π. An n-analytic function on Π is a function satisfying the equation

$$\left(\frac{\partial}{\partial \overline{z}}\right)^n f = 0.$$

The weighted poly-Bergman space $\mathcal{A}^2_{n\lambda}(\Pi)$ consists of all n-analytic functions on the space $L_2(\Pi, d\mu_\lambda)$. Analogously, $\tilde{\mathcal{A}}^2_{n\lambda}(\Pi)$ consists of all n-anti-analytic functions on $L_2(\Pi, d\mu_\lambda)$, that is, those satisfying the differential equation $(\partial/\partial z)^n f = 0$.

The true-n-analytic and true-n-anti-analytic function spaces are defined by

$$\mathcal{A}^2_{(n)\lambda}(\Pi) = \mathcal{A}^2_{n\lambda}(\Pi) \ominus \mathcal{A}^2_{n-1,\lambda}(\Pi),$$

$$\tilde{\mathcal{A}}^2_{(n)\lambda}(\Pi) = \tilde{\mathcal{A}}^2_{n\lambda}(\Pi) \ominus \tilde{\mathcal{A}}^2_{n-1,\lambda}(\Pi),$$

where by convention $\mathcal{A}^2_{0\lambda}(\Pi) = \tilde{\mathcal{A}}^2_{0\lambda}(\Pi) = \{0\}$. The purpose of this section is to establish an integral representation of the orthogonal projection of each space just defined above. The following operators stand for the surjective orthogonal projection in the appropiate space:

$$
\begin{aligned}
B_{n\lambda}: &\quad L_2(\Pi, d\mu_\lambda) \longrightarrow \quad \mathcal{A}^2_{n\lambda}(\Pi),\\
B_{(n)\lambda}: &\quad L_2(\Pi, d\mu_\lambda) \longrightarrow \quad \mathcal{A}^2_{(n)\lambda}(\Pi),\\
\tilde{B}_{n\lambda}: &\quad L_2(\Pi, d\mu_\lambda) \longrightarrow \quad \tilde{\mathcal{A}}^2_{n\lambda}(\Pi),\\
\tilde{B}_{(n)\lambda}: &\quad L_2(\Pi, d\mu_\lambda) \longrightarrow \quad \tilde{\mathcal{A}}^2_{(n)\lambda}(\Pi).
\end{aligned}
$$

Obviously

$$
B_{n\lambda} = \sum_{k=1}^{n} B_{(k)\lambda},
$$

$$
\tilde{B}_{n\lambda} = \sum_{k=1}^{n} \tilde{B}_{(k)\lambda}.
$$

The Laguerre polynomials of order λ play an important role in the study of the weighted poly-Bergman spaces. Recall the way they are defined:

$$
\begin{aligned}
L^\lambda_n(y) &= e^y \frac{y^{-\lambda}}{n!}\frac{d^n}{dy^n}(e^{-y}y^{n+\lambda})\\
&= \sum_{j=0}^{n} \frac{\Gamma(\lambda+n+1)}{\Gamma(\lambda+j+1)}\frac{(-y)^j}{j!(n-j)!}.
\end{aligned}
$$

The system of functions

$$
\ell_{n\lambda}(y) = \sqrt{\frac{n!}{\Gamma(\lambda+n+1)}} L^\lambda_n(y) y^{\lambda/2} e^{-y/2}\chi_+(y), \qquad n = 0, 1, 2...
$$

form an orthonormal basis for $L_2(\mathbb{R}_+)$. Let $L_{n\lambda}$ be the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by $\ell_{n\lambda}(y)$. Let $P_{n\lambda}$ denote the orthogonal projection from $L_2(\mathbb{R}_+)$ onto $L_{n\lambda}$, which is given by

$$
(P_{n\lambda}g)(y) = \ell_{n\lambda}(y)\int_0^\infty g(t)\ell_{n\lambda}(t)dt.
$$

Define now the spaces

$$
\begin{aligned}
A^2_{(n)\lambda}(\Pi) &= L_2(\mathbb{R}_+) \otimes L_{n-1,\lambda},\\
A^2_{n\lambda}(\Pi) &= L_2(\mathbb{R}_+) \otimes \bigoplus_{k=0}^{n-1} L_{k\lambda},\\
\tilde{A}^2_{(n)\lambda}(\Pi) &= L_2(\mathbb{R}_-) \otimes L_{n-1,\lambda},\\
\tilde{A}^2_{n\lambda}(\Pi) &= L_2(\mathbb{R}_-) \otimes \bigoplus_{k=0}^{n-1} L_{k\lambda}.
\end{aligned}
$$

Let us introduce the operator $T\colon L_2(\mathbb{R}_+, d\nu_\lambda) \to L_2(\mathbb{R}_+, dy)$ given by

$$
(Tf)(y) = \sqrt{\lambda+1}(2y)^{\lambda/2} f(y),
$$

which is an isometric isomorphism. Thus $I \otimes T$ is a unitary operator from $L_2(\Pi, d\mu_\lambda)$ onto $L_2(\Pi, dxdy)$.

THEOREM (2.1). *The following restrictions of the unitary operator* $V = U_2(I \otimes T)U_1 \colon L_2(\Pi, d\mu_\lambda) \to L_2(\Pi, dxdy)$ *are isometric isomorphisms:*

$$\begin{array}{llll} V: & \mathcal{A}^2_{n\lambda}(\Pi) & \to & A^2_{n\lambda}(\Pi), \\ V: & \mathcal{A}^2_{(n)\lambda}(\Pi) & \to & A^2_{(n)\lambda}(\Pi), \\ V: & \tilde{\mathcal{A}}^2_{n\lambda}(\Pi) & \to & \tilde{A}^2_{n\lambda}(\Pi), \\ V: & \tilde{\mathcal{A}}^2_{(n)\lambda}(\Pi) & \to & \tilde{A}^2_{(n)\lambda}(\Pi). \end{array}$$

Proof. Since $L_2(\Pi, d\mu_\lambda) = L_2(\mathbb{R}, dx) \otimes L_2(\mathbb{R}_+, d\nu_\lambda)$, the operator $U_1 = F \otimes I$ is an isometric isomorphism on $L_2(\Pi, d\mu_\lambda)$. The space $U_1(\mathcal{A}^2_{n\lambda}(\Pi))$ consists of all functions $\varphi(z) = \varphi(x, y) \in L_2(\Pi, d\mu_\lambda)$ satisfying the equation

$$U_1 \left(\frac{\partial}{\partial \overline{z}}\right)^n U_1^{-1} \varphi = \frac{i^n}{2^n} \left(x + \frac{\partial}{\partial y}\right)^n \varphi = 0,$$

whose general solution is a linear combination of functions of the form $\varphi_k(x, y) = \tilde{\varphi}_k(x) y^k e^{-xy}$, $k = 0, ..., n-1$. Making use of formula (1.4) we infer that $\varphi_k(x, y)$ belongs to $L_2(\Pi, d\mu_\lambda)$ if and only if it has the form

$$\varphi_k(x, y) = \chi_+(x) \theta_{\lambda,k}(x) \phi_k(x) y^k e^{-xy},$$

where $\phi_k(x) \in L_2(\mathbb{R}, dx)$ and

$$\theta_{\lambda,k}(x) = \sqrt{\frac{(2x)^{2k} 2x^{\lambda+1}}{(\lambda+1)\Gamma(\lambda+2k+1)}}.$$

Furthermore $\|\varphi_k\|_{L_2(\Pi, d\mu_\lambda)} = \|\phi_k\|_{L_2(\mathbb{R}_+)}$.

A direct computation shows that the function $(U_2(I \otimes T)\varphi_k)(x, y)$ is given by

$$\frac{1}{\sqrt{\Gamma(\lambda+2k+1)}} \chi_+(x) \phi_k(x) y^k y^{\lambda/2} e^{-y/2}. \tag{2.2}$$

Thus any function in the space $U_2(I \otimes T)U_1(\mathcal{A}^2_{n\lambda}(\Pi))$ is a linear combination of functions of the form (2.2):

$$f(x, y) = \sum_{k=0}^{n-1} \frac{1}{\sqrt{\Gamma(\lambda+2k+1)}} \chi_+(x) \phi_k(x) y^k y^{\lambda/2} e^{-y/2}.$$

Rearranging polynomials in y, $f(x, y)$ can be written in terms of the Laguerre polynomials of order λ:

$$\begin{aligned} f(x, y) &= \sum_{k=0}^{n-1} \chi_+(x) \psi_k(x) \sqrt{\frac{n!}{\Gamma(\lambda+n+1)}} L_k^\lambda(y) y^{\lambda/2} e^{-y/2} \\ &= \sum_{k=0}^{n-1} \chi_+(x) \psi_k(x) \ell_{k\lambda}(y), \end{aligned}$$

where $\psi_k \in L_2(\mathbb{R}_+)$. We have just proved that

$$U_2(I \otimes T)U_1(\mathcal{A}^2_{n\lambda}(\Pi)) = L_2(\mathbb{R}_+) \otimes \bigoplus_{k=0}^{n-1} L_{k\lambda} = A^2_{n\lambda}(\Pi).$$

Analogously, it can be proved that the space $U_2(I \otimes T)U_1(\tilde{\mathcal{A}}^2_{n\lambda}(\Pi))$ consists of all functions of the form

$$g(x, y) = \sum_{k=0}^{n-1} \chi_-(x)\psi_k(x)\ell_{k\lambda}(y),$$

where $\psi_k \in L_2(\mathbb{R}_-)$. Therefore $U_2(I \otimes T)U_1(\tilde{\mathcal{A}}^2_{n\lambda}(\Pi)) = \tilde{A}^2_{n\lambda}(\Pi)$. □

COROLLARY (2.3). *The surjective orthogonal projections*

$$B^1_{(n)\lambda} \colon L_2(\Pi, dxdy) \longrightarrow A^2_{(n)\lambda}(\Pi),$$

$$\tilde{B}^1_{(n)\lambda} \colon L_2(\Pi, dxdy) \longrightarrow \tilde{A}^2_{(n)\lambda}(\Pi),$$

are given by

$$B^1_{(n)\lambda} = VB_{(n)\lambda}V^* = \chi_+ I \otimes P_{n-1,\lambda},$$

$$\tilde{B}^1_{(n)\lambda} = V\tilde{B}_{(n)\lambda}V^* = \chi_- I \otimes P_{n-1,\lambda}.$$

We have explicitly

$$(B^1_{(n)\lambda}f)(x, y) = \chi_+(x)\ell_{n-1,\lambda}(y)\int_0^\infty f(x, t)\ell_{n-1,\lambda}(t)dt,$$

$$(\tilde{B}^1_{(n)\lambda}f)(x, y) = \chi_-(x)\ell_{n-1,\lambda}(y)\int_0^\infty f(x, t)\ell_{n-1,\lambda}(t)dt.$$

The Laguerre polynomials of order λ form an orthonormal basis for $L_2(\mathbb{R}_+)$, which is equivalent to $L_2(\mathbb{R}_+) = \bigoplus_{k=0}^{\infty} L_{k\lambda}$. Thus

$$L_2(\mathbb{R}_+ \times \mathbb{R}_+) = \bigoplus_{k=1}^{\infty} A^2_{(k)\lambda}(\Pi),$$

$$L_2(\mathbb{R}_- \times \mathbb{R}_+) = \bigoplus_{k=1}^{\infty} \tilde{A}^2_{(k)\lambda}(\Pi).$$

and therefore

$$L_2(\Pi, d\mu_\lambda) = \bigoplus_{k=1}^{\infty} \mathcal{A}^2_{(k)\lambda}(\Pi) \oplus \bigoplus_{k=1}^{\infty} \tilde{\mathcal{A}}^2_{(k)\lambda}(\Pi).$$

3. Reproducing Kernels of Poly-Bergman Spaces

Let R stand for the operator $U_2(I \otimes T)$. This operator is given by

$$(Rf)(x, y) = \sqrt{\frac{\lambda + 1}{2|x|}}\left(\frac{y}{|x|}\right)^{\lambda/2} f(x, \frac{y}{2|x|}),$$

$$(R^*f)(x, y) = \sqrt{\frac{2|x|}{\lambda + 1}}\frac{1}{(2y)^{\lambda/2}} f(x, 2|x|y).$$

THEOREM (3.1). *The poly-Bergman projection* $B_{(n)\lambda}: L_2(\Pi, d\mu_\lambda) \to ! \mathcal{A}^2_{(n)\lambda}(\Pi)$ *admits the integral representation*

$$(B_{(n)\lambda}f)(z) = \int_\Pi f(\zeta)K_{(n)\lambda}(z,\zeta)d\mu_\lambda(\zeta),$$

where

$$K_{(n)\lambda}(z,\zeta) = \frac{1}{(i\overline{\zeta}-iz)^{\lambda+2}}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\lambda_{jkn}\left(\frac{z-\overline{z}}{z-\overline{\zeta}}\right)^j\left(\frac{\zeta-\overline{\zeta}}{z-\overline{\zeta}}\right)^k$$

and

$$\lambda_{jkn} = \frac{(-1)^{j+k}\Gamma(n)\Gamma(\lambda+n)\Gamma(\lambda+j+k+2)}{\pi(\lambda+1)\Gamma(\lambda+j+1)j!(n-1-j)!\Gamma(\lambda+k+1)k!(n-1-k)!}.$$

Proof. Since $RU_1B_{(n+1)\lambda}U_1^*R^* = \chi_+ I \otimes P_{n\lambda}$, we have

$$U_1B_{(n+1)\lambda}U_1^* = R^*(\chi_+ I \otimes P_{n\lambda})R.$$

To simplify our notation, let us define $C_{(n)\lambda} = U_1B_{(n)\lambda}U_1^*$. Thus

$$\begin{aligned}
(C_{(n+1)\lambda}f)(x,y) &= R^*[(\chi_+ I \otimes P_{n\lambda})Rf](x,y)\\
&= \sqrt{\frac{2|x|}{\lambda+1}}\frac{1}{(2y)^{\lambda/2}}((\chi_+ I \otimes P_{n\lambda})Rf)(x, 2|x|y)\\
&= \sqrt{\frac{2|x|}{\lambda+1}}\frac{1}{(2y)^{\lambda/2}}\chi_+(x)\ell_{n\lambda}(2|x|y)\int_0^\infty (Rf)(x,t)\ell_{n\lambda}(t)dt\\
&= \chi_+(x)\ell_{n\lambda}(2|x|y)\int_0^\infty\left(\frac{t}{2|x|y}\right)^{\lambda/2} f(x,\frac{t}{2|x|})\ell_{n\lambda}(t)dt\\
&= \chi_+(x)\ell_{n\lambda}(2|x|y)\int_0^\infty f(x,\tau)\left(\frac{\tau}{y}\right)^{\lambda/2}\ell_{n\lambda}(2|x|\tau)2|x|d\tau.
\end{aligned}$$

From the definition of U_1 and the integral representation of $C_{(n+1)\lambda}$ we obtain an integral representation for $U_1^*C_{(n+1)\lambda}$ as follows. For $z = x+iy$, the function $h(z) = (U_1^*C_{(n+1)\lambda}f)(z)$ is given by

$$\begin{aligned}
h(z) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\left(\chi_+(t)\ell_{n\lambda}(2|t|y)\int_0^\infty f(t,\tau)\left(\frac{\tau}{y}\right)^{\lambda/2}\ell_{n\lambda}(2|t|\tau)2|t|d\tau\right)e^{ixt}dt\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\int_0^\infty f(t,\tau)\chi_+(t)\left(\frac{\tau}{y}\right)^{\lambda/2}\ell_{n\lambda}(2|t|y)\ell_{n\lambda}(2|t|\tau)2|t|e^{ixt}d\tau dt\\
&= \frac{n!}{\sqrt{2\pi}\Gamma(n+\lambda+1)}\int_\Pi f(t,\tau)\chi_+(t)2t(2t\tau)^\lambda L_n^\lambda(2ty)L_n^\lambda(2t\tau)e^{-t(\tau-iz)}dtd\tau.
\end{aligned}$$

Let $K_{(n)\lambda}(z,\zeta)$ be the reproducing kernel of $\mathcal{A}^2_{(n)\lambda}(\Pi)$, that is, $(B_{(n)\lambda}f)(z) = \langle f, \overline{K_{(n)\lambda}(z,\cdot)}\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the inner product on $L_2(\Pi, d\mu_\lambda)$. Then $h(z) =$

$(B_{(n+1)\lambda}U_1^*f)(z)$ has another representation,

$$\begin{aligned} h(z) &= \langle (U_1^*f)(\zeta), \overline{K_{(n+1)\lambda}(z,\zeta)}\rangle \\ &= \langle f(\zeta), U_1\overline{K_{(n+1)\lambda}(z,\zeta)}\rangle, \qquad \zeta = t+i\tau \\ &= \int_\Pi f(t,\tau)\overline{[(F\otimes I)_\zeta \overline{K_{(n+1)\lambda}}](z,\zeta)}(\lambda+1)(2\tau)^\lambda dt d\tau, \end{aligned}$$

where $(F\otimes I)_\zeta$ means that $F\otimes I$ is acting with respect to the complex variable $\zeta = t+\tau i$. From a comparison of both integral representations of $h = U_1^*C_{(n+1)\lambda}f$ we obtain the relationship between the kernels involved:

$$\overline{[(F\otimes I)_\zeta \overline{K_{(n+1)\lambda}}](z,\zeta)}(\lambda+1)(2\tau)^\lambda = \frac{n!2t(2t\tau)^\lambda}{\sqrt{2\pi}\Gamma(\lambda+n+1)}L_n^\lambda(2ty)L_n^\lambda(2t\tau)e^{-t(\tau-iz)}\chi_+(t)$$

or

$$\begin{aligned} [(F\otimes I)_\zeta \overline{K_{(n+1)\lambda}}](z,\zeta) &= \frac{n!(\lambda+1)^{-1}}{\sqrt{2\pi}\Gamma(\lambda+n+1)}2t^{\lambda+1}L_n^\lambda(2ty)L_n^\lambda(2t\tau)e^{-t(\tau+i\overline{z})}\chi_+(t) \\ &= \sum_{j,k=0}^n \tilde{\lambda}_{jk}(2y)^j(2\tau)^k\left(\sqrt{2\pi}t^{\lambda+j+k+1}e^{-t(\tau+i\overline{z})}\chi_+(t)\right), \end{aligned}$$

where

$$\tilde{\lambda}_{jk} = \frac{(-1)^{j+k}n!\Gamma(\lambda+n+1)}{\pi(\lambda+1)\Gamma(\lambda+j+1)j!(n-j)!\Gamma(\lambda+k+1)k!(n-k)!}.$$

By formula (1.4) we have

$$\overline{K_{(n+1)\lambda}(z,\zeta)} = \sum_{j,k=0}^n \tilde{\lambda}_{jk}(2y)^j(2\tau)^k\frac{\Gamma(\lambda+j+k+2)}{(\tau+i\overline{z}-it)^{\lambda+j+k+2}},$$

which means

$$\begin{aligned} K_{(n+1)\lambda}(z,\zeta) &= \sum_{j,k=0}^n \tilde{\lambda}_{jk}(2y)^j(2\tau)^k\frac{\Gamma(\lambda+j+k+2)}{(i\overline{\zeta}-iz)^{\lambda+j+k+2}} \\ &= \frac{1}{(i\overline{\zeta}-iz)^{\lambda+2}}\sum_{j,k=0}^n \lambda_{jk,n+1}\left(\frac{2yi}{z-\overline{\zeta}}\right)^j\left(\frac{2\tau i}{z-\overline{\zeta}}\right)^k, \end{aligned}$$

where $\lambda_{jk,n+1} = \tilde{\lambda}_{jk}\Gamma(\lambda+j+k+2)$. □

COROLLARY (3.2). *Let $K_{(n)\lambda}(z,\zeta)$ be the reproducing kernel of $\mathcal{A}^2_{(n)\lambda}(\Pi)$. Then the Fourier transform of $\overline{K_{(n+1)\lambda}(z,\zeta)}$ with respect to the real part of $\zeta = t+i\tau$ is given by*

$$[(F\otimes I)_\zeta \overline{K_{(n+1)\lambda}}](z,\zeta) = \frac{n!(\lambda+1)^{-1}}{\sqrt{2\pi}\Gamma(\lambda+n+1)}\chi_+(t)2t^{\lambda+1}L_n^\lambda(2ty)L_n^\lambda(2t\tau)e^{-t(\tau+i\overline{z})},$$

where $z = x+iy$.

THEOREM (3.3). *Let $\tilde{K}_{(n)\lambda}(z,\zeta)$ be the reproducing kernel of $\tilde{\mathcal{A}}^2_{(n)\lambda}(\Pi)$. The poly-Bergman projection $\tilde{B}_{(n)\lambda} : L_2(\Pi, d\mu_\lambda) \longrightarrow \tilde{\mathcal{A}}^2_{(n)\lambda}(\Pi)$ admits the integral representation*

$$(\tilde{B}_{(n)\lambda}f)(z) = \int_\Pi f(\zeta)\tilde{K}_{(n)\lambda}(z,\zeta)d\mu_\lambda(\zeta),$$

where

$$\tilde{K}_{(n)\lambda}(z,\zeta) = \frac{1}{(i\bar{z}-i\zeta)^{\lambda+2}}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\lambda_{jkn}\left(\frac{z-\bar{z}}{\zeta-\bar{z}}\right)^j\left(\frac{\zeta-\bar{\zeta}}{\zeta-\bar{z}}\right)^k$$

and λ_{jkn} is given in Theorem (3.1).

COROLLARY (3.4). *The Fourier transform of $\overline{\tilde{K}_{(n+1)\lambda}(z,\zeta)}$ with respect to the real part of $\zeta = t+i\tau$ is given by*

$$[(F\otimes I)_\zeta\overline{\tilde{K}_{(n+1)\lambda}}](z,\zeta) = \frac{n!(\lambda+1)^{-1}}{\sqrt{2\pi}\Gamma(\lambda+n+1)}\chi_-(t)2|t|^{\lambda+1}L_n^\lambda(2|t|y)L_n^\lambda(2|t|\tau)e^{-|t|(\tau-iz)},$$

where $z = x+iy$.

We will now establish an integral representation of the poly-Bergman projection $B_{n\lambda}$. Let $K_{n\lambda}$ stand for the reproducing kernel of $\mathcal{A}^2_{n\lambda}(\Pi)$. Obviously

$$K_{n\lambda}(z,\zeta) = \sum_{k=1}^{n} K_{(k)\lambda}(z,\zeta).$$

By Corollary (3.2), the Fourier transform of $K_{n+1,\lambda}(z,\zeta)$ with respect to the real part of ζ is given by

(3.5)

$$[(F\otimes I)_\zeta\overline{K_{n+1,\lambda}}](z,\zeta) = \frac{2\chi_+(t)t^{\lambda+1}}{\sqrt{2\pi}(\lambda+1)}e^{-t(\tau+i\bar{z})}\sum_{k=0}^{n}\frac{k!}{\Gamma(\lambda+k+1)}L_k^\lambda(2ty)L_k^\lambda(2t\tau).$$

Let us introduce the following linear functional on the space of all polynomials:

$$\mathcal{F}(p) = \int_0^\infty p(t)t^\lambda e^{-t}dt.$$

The monic polynomials $P_n(x) = (-1)^n n! L_n^\lambda(x)$ (n=0,1,...) satisfy the recurrence relation $P_n(x) = (x-\lambda-2n+1)P_{n-1}(x)-(n-1)(n-1+\lambda)P_{n-2}(x)$, from which interesting properties about $P_n(x)$ can be derived ([2, 5]). The norm of $P_n(x)$ with respect to the functional $\mathcal{F}$ is given by $\|P_n\| = \sqrt{\mathcal{F}(P_n^2)} = \sqrt{n!\Gamma(\lambda+n+1)}$. Thus, the system of polynomials

$$p_n(x) = (-1)^n\sqrt{\frac{n!}{\Gamma(\lambda+n+1)}}L_n^\lambda(x), \quad n = 0,1,2...$$

is normalized with respect to $\mathcal{F}$, that is, $\mathcal{F}(p_np_m) = \delta_{mn}$. Letting $k_n = \|P_n\|^{-1}$, the Christoffel-Darboux identity ([2]) applied to these polynomials takes the form

$$\sum_{k=0}^{n} p_k(x)p_k(u) = \frac{k_n}{k_{n+1}}\frac{p_{n+1}(x)p_n(u)-p_n(x)p_{n+1}(u)}{x-u}$$

or

$$\sum_{k=0}^{n} \frac{k!}{\Gamma(\lambda+k+1)} L_k^\lambda(x) L_k^\lambda(u) = -\frac{(n+1)!}{\Gamma(\lambda+n+1)} \frac{N(x,u)}{x-u},$$

where $N(x,u) = L_{n+1}^\lambda(x)L_n^\lambda(u) - L_n^\lambda(x)L_{n+1}^\lambda(u)$.

LEMMA (3.6). *We have*

$$(n+1)N(x,u) = uL_n^{\lambda+1}(u)L_n^\lambda(x) - xL_n^{\lambda+1}(x)L_n^\lambda(u). \tag{3.7}$$

Proof. It is well known ([2, 5]) that

$$x\frac{d}{dx}L_{n+1}^\lambda(x) = (n+1)L_{n+1}^\lambda(x) - (\lambda+n+1)L_n^\lambda(x), \tag{3.8}$$

$$\frac{d}{dx}L_{n+1}^\lambda(x) = -L_n^{\lambda+1}(x). \tag{3.9}$$

Multiplying equation (3.8) by $L_n^\lambda(u)$ and taking into account equation (3.9) we obtain

$$-xL_n^{\lambda+1}(x)L_n^\lambda(u) = (n+1)L_{n+1}^\lambda(x)L_n^\lambda(u) - (\lambda+n+1)L_n^\lambda(x)L_n^\lambda(u).$$

Interchanging x and u,

$$-uL_n^{\lambda+1}(u)L_n^\lambda(x) = (n+1)L_{n+1}^\lambda(u)L_n^\lambda(x) - (\lambda+n+1)L_n^\lambda(u)L_n^\lambda(x).$$

The last two equations imply equation (3.7). □

THEOREM (3.10). *Let $K_{n\lambda}(z,\zeta)$ be the reproducing kernel of $\mathcal{A}^2_{n\lambda}(\Pi)$. The poly-Bergman projection $B_{n\lambda} : L_2(\Pi, d\mu_\lambda) \longrightarrow \mathcal{A}^2_{n\lambda}(\Pi)$ admits the integral representation*

$$(B_{n\lambda}f)(z) = \int_\Pi f(\zeta)K_{n\lambda}(z,\zeta)d\mu_\lambda(\zeta),$$

where

$$K_{n\lambda}(z,\zeta) = \frac{i(z-\overline{z}-\zeta+\overline{\zeta})^{-1}}{(i\overline{\zeta}-iz)^{\lambda+1}} \sum_{k=0}^{n}\sum_{j=0}^{n-1} \gamma_{jkn} \frac{(z-\overline{z})^j(\zeta-\overline{\zeta})^k - (z-\overline{z})^k(\zeta-\overline{\zeta})^j}{(z-\overline{\zeta})^{j+k}}$$

and

$$\gamma_{jkn} = \frac{(-1)^{j+k}\Gamma(n+1)\Gamma(\lambda+n+1)\Gamma(\lambda+j+k+1)}{\pi(\lambda+1)\Gamma(\lambda+j+1)j!(n-1-j)!\Gamma(\lambda+k+1)k!(n-k)!}.$$

Proof. Applying Christoffel-Darboux identity in equality (3.5) we infer that the function $h(z,\zeta) = [(F\otimes I)_\zeta\overline{K_{n+1,\lambda}}](z,\zeta)$ has the form

$$h(z,\zeta) = -\frac{\Gamma(n+2)}{\sqrt{2\pi}(\lambda+1)\Gamma(\lambda+n+1)}\chi_+(t)t^\lambda e^{-t(\tau+i\overline{z})}\frac{N(2ty,2t\tau)}{y-\tau}.$$

From the definition of L_n^λ and L_{n+1}^λ we get

$$N(2ty,2t\tau) \quad = \quad \sum_{k=0}^{n+1}\sum_{j=0}^{n} c_{jk}(2t)^{j+k}(\tau^j y^k - y^j\tau^k),$$

where

$$c_{jk} = \frac{(-1)^{j+k}\Gamma(\lambda+n+1)\Gamma(\lambda+n+2)}{\Gamma(\lambda+j+1)j!(n-j)!\Gamma(\lambda+k+1)k!(n+1-k)!}.$$

Letting $\tilde{\gamma}_{jk} = c_{jk}\Gamma(n+2)[\pi(\lambda+1)\Gamma(\lambda+n+1)]^{-1}$ we have

$$\begin{aligned} h(z,\zeta) &= -\sum_{k=0}^{n+1}\sum_{j=0}^{n}\tilde{\gamma}_{jk}2^{j+k}\frac{\tau^j y^k - y^j\tau^k}{2y-2\tau}\sqrt{2\pi}\chi_+(t)t^{\lambda+j+k}e^{-t(\tau+i\overline{z})} \\ &= -\sum_{k=0}^{n+1}\sum_{j=0}^{n}\tilde{\gamma}_{jk}2^{j+k}\frac{\tau^j y^k - y^j\tau^k}{2y-2\tau}(F\otimes I)_\zeta\left(\frac{\Gamma(\lambda+j+k+1)}{(\tau+i\overline{z}-it)^{\lambda+j+k+1}}\right). \end{aligned}$$

Therefore

$$\overline{K_{n+1,\lambda}(z,\zeta)} = -\sum_{k=0}^{n+1}\sum_{j=0}^{n}\tilde{\gamma}_{jk}2^{j+k}\frac{\tau^j y^k - y^j\tau^k}{2y-2\tau}\frac{\Gamma(\lambda+j+k+1)}{(i\overline{z}-i\zeta)^{\lambda+j+k+1}},$$

that is,

$$K_{n+1,\lambda}(z,\zeta) = -\sum_{k=0}^{n+1}\sum_{j=0}^{n}\gamma_{jk}\frac{(2\tau)^j(2y)^k - (2y)^j(2\tau)^k}{2y-2\tau}\frac{1}{(i\overline{\zeta}-iz)^{\lambda+j+k+1}},$$

where $\gamma_{jk,n+1} = \tilde{\gamma}_{jk}\Gamma(\lambda+j+k+1)$. The desired formula for the Bergman kernel follows now immediately. □

Making use of Lemma (3.6) we obtain a symmetric form of the Bergman kernel $K_{n\lambda}$, as shown in the following theorem.

THEOREM (3.11). *The reproducing kernel $K_{n\lambda}(z,\zeta)$ is also given by*

$$K_{n\lambda}(z,\zeta) = \frac{(z-\overline{z}-\zeta+\overline{\zeta})^{-1}}{(i\overline{\zeta}-iz)^{\lambda+2}}\sum_{j=0}^{n-1}\sum_{k=0}^{n-1}\alpha_{jkn}\left(\frac{z-\overline{z}}{\lambda+j+1} - \frac{\zeta-\overline{\zeta}}{\lambda+k+1}\right)\frac{(z-\overline{z})^j(\zeta-\overline{\zeta})^k}{(z-\overline{\zeta})^{j+k}},$$

where $\alpha_{jkn} = (\lambda+n)\lambda_{jkn}$.

Proof. By using Lemma (3.6) and the property $z\Gamma(z) = \Gamma(z+1)$ we obtain

$$N(2ty,2t\tau) = 2t\sum_{j=0}^{n}\sum_{k=0}^{n}a_{jk}(2t)^{j+k}y^j\tau^k\left(\frac{\tau}{\lambda+k+1} - \frac{y}{\lambda+j+1}\right),$$

where

$$a_{jk} = \frac{(-1)^{j+k}\Gamma(\lambda+n+1)\Gamma(\lambda+n+2)}{(n+1)\Gamma(\lambda+j+1)j!(n-j)!\Gamma(\lambda+k+1)k!(n-k)!}.$$

Thus $[(F\otimes I)_\zeta\overline{K_{n+1,\lambda}}](z,\zeta)$ equals

$$\frac{-1}{y-\tau}\sum_{j=0}^{n}\sum_{k=0}^{n}\tilde{\alpha}_{jk}(2y)^j(2\tau)^k\left(\frac{\tau}{\lambda+k+1} - \frac{y}{\lambda+j+1}\right)(\sqrt{2\pi}\chi_+(t)t^{\lambda+j+k+1}e^{-t(\tau+i\overline{z})})$$

where $\tilde{\alpha}_{jk} = a_{jk}\Gamma(n+2)[\pi(\lambda+1)\Gamma(\lambda+n+1)]^{-1}$. Therefore $\overline{K_{n+1,\lambda}(z,\zeta)}$ equals

$$\frac{-1}{y-\tau}\sum_{j=0}^{n}\sum_{k=0}^{n}\tilde{\alpha}_{jk}(2y)^j(2\tau)^k\left(\frac{\tau}{\lambda+k+1} - \frac{y}{\lambda+j+1}\right)\frac{\Gamma(\lambda+j+k+2)}{(\tau+i\overline{z}-it)^{\lambda+j+k+2}}$$

or

$$K_{n+1,\lambda}(z,\zeta) = \frac{(2yi-2\tau i)^{-1}}{(i\overline{\zeta}-iz)^{\lambda+2}}\sum_{j=0}^{n}\sum_{k=0}^{n}\alpha_{jk,n+1}\frac{(2y)^j(2\tau)^k}{(i\overline{\zeta}-iz)^{j+k}}\left(\frac{2yi}{\lambda+j+1} - \frac{2\tau i}{\lambda+k+1}\right),$$

where $\alpha_{jk,n+1} = \tilde{\alpha}_{jk}\Gamma(\lambda + j + k + 2)$. It is easy to see that $\alpha_{jkn} = (\lambda + n)\lambda_{jkn}$. $\square$

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QUOTIENT SUBSPACES OF ASYMMETRIC NORMED LINEAR SPACES

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ABSTRACT. If X is an asymmetric normed linear space and H is a linear subspace of X, we give conditions under which the quotient space X/H is also an asymmetric normed space. We study the canonical factorization of a linear continuous mapping in this setting and, finally, we characterize the dual of the asymmetric normed space X/H. The results obtained extend well-known properties of normed spaces to the realm of asymmetric normed spaces.

Quotient subspaces of asymmetric normed linear spaces appear in a natural way in the theory of these spaces that has been developed in recent years, mainly motivated by the applications of these topological structures in Theoretical Computer Science and Functional Analysis (see the bibliography at the end of the paper). However, no systematic studies of the quotient spaces have been done up to this moment, although quotient structures have been shown to be useful in at least two particular matters regarding the general theory of asymmetric normed linear spaces. The first one is the canonical decomposition of an asymmetric normed space in a Hausdorff space and a purely non-Hausdorff space (see [10]). The second one is related to the duality theory of asymmetric normed spaces, which is studied in section 4 of the present paper (see [11]). Thus, this work is devoted to develop a fundamental theory supporting the quotient constructions.

1. Introduction and preliminaries

Let X be a real linear space . A function $p\colon X \to \mathbb{R}^+$, is an *asymmetric norm* on X ([9], [11]) if for all $x, y \in X$ and $r \in \mathbb{R}^+$,

(i) $p(x) = p(-x) = 0$ if and only if $x = 0$.

(ii) $p(rx) = rp(x)$.

(iii) $p(x + y) \leq p(x) + p(y)$.

The pair (X, p) is called *asymmetric normed linear space*. Asymmetric norms are also called quasi-norms in [5],[1], [15] etc. and nonsymmetric norms in [2] .

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If p is an asymmetric norm on X, then the function p^{-1} defined on X by $p^{-1}(x) = p(-x)$ is also an asymmetric norm on X, called the *conjugate* of p. The function p^s defined on X by $p^s(x) = \max\{p(x), p^{-1}(x)\}$ is a norm on X.

A subset M of a linear space is called *cone* or *semilinear space* if for every $x, y \in M$ and $a \in R^+$, $x+y \in M$ and $ax \in M$. If p is a function on M satisfying the conditions of the definition of an asymmetric norm, p will be also called an asymmetric norm on M.

A quasi-metric on a nonempty set A is a function $d\colon A \times A \to \mathbb{R}^+$ that satisfies

(1) $d(a, b) = d(b, a) = 0$ if and only if $a = b$.

(2) For every $a, b, c \in A$, $d(a, b) \leq d(a, c) + d(c, b)$.

Each quasi-metric d on A generates a T_0 topology $T(d)$ on A, which has as basic open sets the d-balls,

$$B_d(a, r) = \{b \in A : d(a, b) < r\}, \qquad a \in A, \quad r > 0.$$

The reader might consult [6] and [12] for more information about quasi-metric spaces.

An asymmetric norm p on a linear space X induces the quasi-metric d_p by means of the formula

$$d_p(x, y) := p(y - x), \qquad x, y \in X.$$

The d_p-ball $B_{d_p}(x, r)$, will be simply denoted by $B^p_r(x)$ and the topology $T(d_q)$ will be denoted by τ_p.

Thus, the sets

$$B^p_\varepsilon(0) = \{x \in X : p(x) < \varepsilon\}, \qquad \varepsilon > 0,$$

define a fundamental system of neighborhoods of zero for the topology τ_p, and for all $y \in X$, the sets $B^p_\varepsilon(y) = y + B^p_\varepsilon(0)$ define a fundamental system of neighborhoods of y. The terms p-neighborhood, p-open, p-closed, etc., will refer to the corresponding topological concepts with respect to that topology.

In [5] there is exhibited a natural class of examples of asymmetric normed spaces, the normed linear lattices. In fact, it is proved that whenever $(X, \|\cdot\|)$ is a normed lattice, then $p(x) = \|x^+\|$ with $x^+ = \sup\{x, 0\}$ is an asymmetric norm on X. Moreover p determines the topology and order of X in the sense of [6]. This class of asymmetric normed spaces is the most interesting one from the point of view of applications.

In the last years several authors have applied both asymmetric normed linear spaces and other related structures from topological algebra and non-symmetric functional analysis to construct suitable mathematical models in theoretical computer science ([9, 14, 16], etc) as well as some questions in approximation theory ([2, 15, 19], etc).

The properties of the spaces with asymmetric norms have been investigated in a series of papers emphasizing the similarities with normed spaces, as well as the differences, see [10, 11, 8, 3, 4, 5, 1] and the references quoted therein.

In the present paper we shall continue the investigation of the properties of the spaces with asymmetric norms. We study the topological structure of the linear subspaces in this setting. If (X, p) is an asymmetric normed space we introduce the concept of (p, p^{-1})-closed subset of X and we show that the

(p, p^{-1})-closed linear subspaces of X play in our context a similar role that the closed subspaces have played in normed spaces. We define in a natural way an asymmetric norm on the quotient space X/H whenever H is a (p, p^{-1})-closed linear subspace of X. We also study the properties of the quotient topology in this context. Thus, we analyze the canonical factorization of a linear continuous function between two asymmetric normed spaces and we give a complete description of the dual space of X/H.

All linear spaces under consideration are assumed to be defined over the field of real numbers.

2. Linear subspaces and continuity

If (X, p) is an asymmetric normed space and A is a subset of X we denote by $\overline{A}^p$ the closure of A in the topological space (X, τ_p).

In a normed space the closure of a linear subspace is a linear subspace. If (X, p) is an asymmetric normed space and H is a linear subspace of X, then the closure $\overline{H}^p$ may fail to be a linear subspace of X, as the next example shows.

Example (2.1). In $\mathbb{R}^2$ we define the asymmetric norm $p((x, y)) := \max\{x^+, y^+\}$, where $x^+ := x \vee 0$; remark that the topology τ_{p^s} is the usual topology of $\mathbb{R}^2$. If we consider the subspace $H = \text{span}\,\{(-1, 1)\}$ it is easy to prove that $\overline{H}^p = \{(x, y) : x \geq -y\}$, which is not a linear subspace of $\mathbb{R}^2$.

The following proposition provides information about the closure of a linear subspace in the context of asymmetric normed spaces.

PROPOSITION (2.2). *Let (X, p) be an asymmetric normed linear space and H a linear subspace of X, then:*
(1) *$\overline{H}^p$ is a cone.*
(2) $\overline{H}^p = -\overline{H}^{p^{-1}}$.
(3) *$\overline{H}^p \cap \overline{H}^{p^{-1}}$ is a linear subspace of X.*
(4) *H is p-closed if and only if H is p^{-1}-closed.*

Proof. (1) Let $x, y \in \overline{H}^p$. Then given $\varepsilon > 0$, there exist $h, h' \in H$ such that $p(h - x) < \frac{\varepsilon}{2}$ and $p(h' - y) < \frac{\varepsilon}{2}$. Therefore

$$p(h + h' - (x + y)) \leq p(h - x) + p(h' - y) < \varepsilon.$$

Then $h + h'$ belongs to $H \cap B^p_\varepsilon(x + y)$, and so $x + y \in \overline{H}^p$.

If $x \in \overline{H}^p$ and $\alpha > 0$, there is $h \in H \cap (x + B^p_{\frac{\varepsilon}{\alpha}}(0))$. Then

$$p(\alpha h - \alpha x) = \alpha p(h - x) < \varepsilon,$$

and we conclude that $\alpha x \in \overline{H}^p$.

(2) If $x \in \overline{H}^p$, then for all $\varepsilon > 0$, $H \cap (x + B^p_\varepsilon(0)) \neq \emptyset$. Since $H = -H$, then $H \cap (-x + B^{p^{-1}}_\varepsilon(0)) \neq \emptyset$ and so $-x \in \overline{H}^{p^{-1}}$.

(3) and (4) are direct consequences of (2). □

Definition (2.3). If (X, p) is an asymmetric normed space and A is a subset of X we say that A is (p, p^{-1})-closed if $A = \overline{A}^p \cap \overline{A}^{p^{-1}}$.

The linear subspace H considered in Example (2.1) is (p, p^{-1})-closed, since $\overline{H}^{p} = \{(x, y) : x \geq -y\}$, $\overline{H}^{p^{-1}} = \{(x, y) : x \leq -y\}$, and so

$$\overline{H}^{p} \cap \overline{H}^{p^{-1}} = \{(x, y) : x = -y\} = H.$$

PROPOSITION (2.4). *Let (X, p) be an asymmetric normed space and H a linear subspace of X. If H is (p, p^{-1})-closed then H is p^s-closed.*

Proof. Since the topologies τ_p and $\tau_{p^{-1}}$ are coarser than the topology τ_{p^s} it satisfies that $\overline{H}^{p^s} \subset \overline{H}^{p}$ and $\overline{H}^{p^s} \subset \overline{H}^{p^{-1}}$. Then $\overline{H}^{p^s} \subset \overline{H}^{p} \cap \overline{H}^{p^{-1}} = H$, and so $\overline{H}^{p^s} = H$.

The converse of this result is not true. In fact, if we consider the asymmetric normed linear space $(\mathbb{R}^2, p)$ of Example (2.1), then the linear subspace $H = \operatorname{span}\{(1, 1)\}$ is p^s-closed and $H \neq \overline{H}^{p} \cap \overline{H}^{p^{-1}}$, since $\overline{H}^{p} = \overline{H}^{p^{-1}} = \mathbb{R}^2$. □

If (X, p) is an asymmetric normed space, by X^* is denoted the set

$$X^* = \{f : (X, p) \to (\mathbb{R}, u) : f \text{ is linear and continuous}\}$$

where $(\mathbb{R}, u)$ is the asymmetric normed space induced by the asymmetric norm u defined on $\mathbb{R}$ by $u(x) = x^+$.

Note that $f \in X^*$ if and only if it is a linear and upper semicontinuous form on (X, τ).

The set X^* is not necessarily a linear space, but it is a cone in the algebraic dual of X. The function

$$p^*(f) = \sup\{f(x) : p(x) \leq 1\}$$

defines an asymmetric norm on the cone X^* and (X^*, p^*) is called the *dual space* of (X, p).

An interesting study of X^* can be found in [11] and [4].

Now we study the relationship between the elements of X^* and certain linear subspaces of X. This will be done with the help of the following result ([5]).

LEMMA (2.5). *Let (X, p) be an asymmetric normed space. Then $f \in X^*$ if and only if there is $M > 0$ such that $f(x) \leq Mp(x)$, for all $x \in X$.*

If f is a linear form on X, we denote by $\ker f$ the f-null space of X, that is,

$$\ker f = \{x \in X : f(x) = 0\}.$$

PROPOSITION (2.6). *Let (X, p) be an asymmetric normed space and let f be a linear functional on X. If $f \in X^*$, then $\ker f$ is (p, p^{-1})-closed.*

Proof. Suppose that $f \in X^*$. By Lemma (2.5), there is $M > 0$ such that $f(x) \leq Mp(x)$ for all $x \in X$. Let $x \in \overline{\ker f}^{p} \cap \overline{\ker f}^{p^{-1}}$. Then if $n \in \mathbb{N}$ there exist $x_n, y_n \in \ker f$ such that $p(x_n - x) < 1/n$ and $p(x - y_n) < 1/n$. Hence

$$f(x) = f(x - y_n + y_n) = f(x - y_n) + f(y_n) \leq Mp(x - y_n) < M/n,$$

$$-f(x) = f(x_n - x - x_n) = f(x_n - x) + f(-x_n) \leq Mp(x_n - x) < M/n,$$

for all $n \in \mathbb{N}$ and so $f(x) = 0$. □

The following example shows that the converse of this result is not true in our setting.

Example (2.7). If $(X, \|\cdot\|)$ is a normed lattice, and $p(x) = \|x^+\|$ is the asymmetric norm associated to the norm, then $f \in X^*$ if and only if f is positive and f is in the classical topological dual of $(X, \|\cdot\|)$ (see Corollary 1 of [1].) If we consider the asymmetric normed space given in Example (2.1) and the linear functional on $\mathbb{R}^2$ defined by $f(x, y) = -x - y$, then ker f is (p, p^{-1})-closed and $f \notin X^*$, since f is not a positive linear form on (X, τ_p).

Now we are going to prove that the (p, p^{-1})-closedness of ker f implies the semicontinuity of f. We begin by establishing an algebraic lemma.

LEMMA (2.8). *Let X be a real linear space, f a linear functional on X, and $a \in X$ such that $f(a) = -1$ and $V = f^{-1}(]-\infty, 1[)$. If U is a subset of X such that $\alpha U \subset U$ for $0 < \alpha < 1$, then $(a + U) \cap \ker f = \emptyset$ if and only if $U \subset V$.*

Proof. Suppose that $U \subset V$. Then if $x \in U$, $f(a+x) = -1+f(x) \neq 0$, because $x \in V$, and so $(a+U) \cap \ker f = \emptyset$. Conversely, suppose that $x \in U$ and $f(x) \geq 1$. Then $x/f(x) \in U$ and $f(a + x/f(x)) = 0$, hence $(a + U) \cap \ker f \neq \emptyset$. □

PROPOSITION (2.9). *If (X, p) is an asymmetric normed space and* ker f *is (p, p^{-1})-closed, then $f \in X^*$ or $-f \in X^*$.*

Proof. Let $V = f^{-1}(]-\infty, 1[)$. If f is not the zero linear map (in which case it is upper semicontinuous) there is $a \in X$ such that $f(a) = -1$. Obviously $a \notin \ker f = \overline{\ker f}^{p} \cap \overline{\ker f}^{p^{-1}}$.

If $a \notin \overline{\ker f}^{p}$, there exists a p-neighborhood U of zero such that $(a + U) \cap \ker f = \emptyset$ and by Lemma (2.8), $U \subset V$ and so V is a p-neighborhood of zero. Hence f is upper semicontinuous, that is, $f \in X^*$.

If $a \notin \overline{\ker f}^{p^{-1}}$, there exists a p-neighborhood of 0, W, such that $(a - W) \cap \ker f = \emptyset$ and by Lemma (2.8), $-W \subset V$ and so V is p^{-1}-neighborhood of zero. Hence f is lower semicontinuous, that is, $-f \in X^*$. □

COROLLARY (2.10). *If X is a normed space and f is a linear form on X, then f is continuous if and only if* ker f *is closed.*

3. Quotient spaces

If X is a linear space and H is a linear subspace of X, the relation defined by xRy if $x - y \in H$ is an equivalence relation on X. The quotient set is a linear space called the quotient of X by H and it is denoted by X/H. If $x \in X$ we denote the equivalence class of x by $\hat{x}$, that is

$$\hat{x} = \{x + h : h \in H\} = x + H.$$

We will denote by π the canonical mapping of X onto X/H, that is, $\pi(x) = \hat{x}$ for all $x \in X$.

The following result is quite simple to prove.

PROPOSITION (3.1). *Let (X, p) be an asymmetric normed linear space and H a linear subspace of X. The function $\hat{p} : X/H \longrightarrow \mathbb{R}$ defined by*

$$\hat{p}(\hat{x}) := \inf\{p(y) : y \in \hat{x}\} = \inf\{p(x+h) : h \in H\}$$

is positive, subadditive and positively homogeneous.

If p is a norm on X, then $\hat{p}$ is a norm on X/H if and only if H is closed. In the setting of asymmetric normed spaces, there are many examples of spaces where the only p-closed linear subspace is the whole space. This is the case, for instance, of the asymmetric normed spaces defined by normed lattices. This fact has motivated the notion of (p, p^{-1})-closed subspace which has been introduced in the above section.

PROPOSITION (3.2). *Let (X, p) be an asymmetric normed vector space and H a linear subspace of X. If H is (p, p^{-1})-closed, then $\hat{p}$ is an asymmetric norm on X/H.*

Proof. By Proposition (3.1), we only need to prove that the equality $\hat{p}(\hat{x}) = \hat{p}(-\hat{x}) = 0$ implies $\hat{x} = \hat{0}$.

If $\hat{p}(\hat{x}) = 0$, then for $\varepsilon > 0$ there is $h \in H$ such that $p(x+h) < \varepsilon$. Thus $x + h \in B^p_\varepsilon(0)$ and so $h \in H \cap (-x + B^p_\varepsilon(0))$. Therefore $-x \in \overline{H}^p$ and, by (2) of Proposition (2.2), $x \in \overline{H}^{p^{-1}}$.

If $\hat{p}(-\hat{x}) = 0$, it can be proved with a similar argument that $x \in \overline{H}^p$ and so $x \in \overline{H}^p \cap \overline{H}^{p^{-1}} = H$. Hence $\hat{x} = \hat{0}$. □

It is well known that each asymmetric normed space is T_0. However, these spaces are not T_1 in general. The following proposition shows that the existence of p-closed linear subspaces of (X, p) characterizes the T_1 axiom of $(X/H, \hat{p})$.

PROPOSITION (3.3). *The asymmetric normed space $(X/H, \hat{p})$ satisfies the T_1 separation axiom if and only if H is p-closed.*

Proof. In [7] it is proved that an asymmetric normed space (X, p) has the T_1 property if and only if $p(x) = 0$ implies $x = 0$. We will use this argument for our proof.

Assuming that $(X/H, \hat{p})$ is a T_1 space, we prove by contradiction that H is p-closed. Suppose that $x \in \overline{H}^p$ and $x \notin H$.

If $x \in \overline{H}^p$ then for all $n \in \mathbb{N}$ there is $h_n \in H$ such that $h_n \in x + B^p_{\frac{1}{n}}(0)$. Therefore $h_n = x + z_n$ with $z_n \in B^p_{\frac{1}{n}}(0)$.

If $x \notin H$, then $z_n = h_n - x \notin H$, therefore $\widehat{z_n} \neq \hat{0}$, for all $n \in \mathbb{N}$. Furthermore, $\widehat{z_n} = \widehat{z_m}$, for all $n, m \in \mathbb{N}$, because $z_n - z_m = h_n - h_m$. Then,

$$0 \leq \hat{p}(\widehat{z}_{n_0}) = \inf\{p(z) : z \in \widehat{z}_{n_0}\} \leq p(z_n) \leq \frac{1}{n},$$

for all $n \in \mathbb{N}$ and this implies that $\hat{p}(\widehat{z}_{n_0}) = 0$ with $\widehat{z}_{n_0} \neq \hat{0}$, which contradicts the hypothesis.

Assuming that H is a p-closed set, we will prove by contradiction that $(X/H, \hat{p})$ has the T_1 property. Suppose that $\hat{p}(\hat{x}) = 0$ and $\hat{x} \neq \hat{0}$.

If $\hat{x} \neq \hat{0}$ then $-x \notin H$. Since H is p-closed, there is $\varepsilon > 0$ such that

$$(-x + B^p_\varepsilon(0)) \cap H = \emptyset. \tag{3.4}$$

Since $\hat{p}(\hat{x}) = 0$, given $\varepsilon > 0$ there is $h \in H$ such that $p(x+h) < \varepsilon$. Thus, $(x+h) \in B^p_\varepsilon(0)$ and we conclude that

$$h \in H \cap (-x + B^p_\varepsilon(0)),$$

which contradicts (3.4). □

The following lemma, given in [5], characterizes the continuous linear mappings between asymmetric normed linear spaces and it will be used later on.

LEMMA (3.5). *Let (X, p) and (Y, q) be two asymmetric normed linear spaces and let f be a linear mapping from X into Y. Then f is continuous if and only if there is $M > 0$ such that $q(f(x)) \leq Mp(x)$ for all $x \in X$.*

The following proposition is quite simple to prove and it shows that the topology generated by $\hat{p}$ coincides with the classical quotient topology on X/H.

PROPOSITION (3.6). *Let H be a (p, p^{-1})-closed linear subspace of an asymmetric normed space (X, p). Then*
(1) $B^{\hat{p}}_\varepsilon(\hat{0}) = \pi(B^p_\varepsilon(0))$, *and*
(2) $\pi\colon (X, p) \longrightarrow (X/H, \hat{p})$ *is a continuous and open linear mapping.*

If X and Y are two linear spaces and f is a linear mapping from X into Y, then $f = i \circ \hat{f} \circ \pi$ where π is the canonical mapping of X onto $X/\ker f$, i is the embedding of $f(X)$ into Y and $\hat{f}$ a linear and bijective mapping from $X/\ker f$ on $f(X)$, defined by $\hat{f}(\hat{x}) = f(x)$.

PROPOSITION (3.7). *Let (X, p) and (Y, q) be two asymmetric normed spaces and f a continuous linear mapping between X and Y. Then:*
(1) $\ker f$ *is (p, p^{-1})-closed.*
(2) $(X/\ker f, \hat{p})$ *is an asymmetric normed linear space.*
(3) *The mapping* $\hat{f}\colon (X/\ker f, \hat{p}) \to (f(X), q)$ *is continuous.*
(4) f *is an open map if and only if* $\hat{f}$ *is an open map.*

Proof. (1) By Lemma (3.5) , if f is continuous then there is $M > 0$ such that $q(f(x)) \leq Mp(x)$. Let $x \in \overline{\ker f}^{p} \cap \overline{\ker f}^{p^{-1}}$, then for $\varepsilon > 0$, there are h, $h' \in \ker f$ such that $p(x+h) \leq \frac{\varepsilon}{M}$ and $p(-x+h') \leq \frac{\varepsilon}{M}$. Thus,

$$q(f(x+h)) \leq Mp(x+h) < \varepsilon,$$

$$q(f(-x+h')) \leq Mp(-x+h') < \varepsilon,$$

and by the linearity of f,

$$q(f(x) + f(h)) < \varepsilon,$$

$$q(f(-x) + f(h')) < \varepsilon.$$

Hence $q(f(x)) = 0 = q(-f(x))$ and so $f(x) = 0$, because q is an asymmetric norm.

(2), (3) and (4) are direct consequences of Proposition (3.2), Lemma (3.5) and Proposition (3.6), respectively. □

If (X, p) is an asymmetric normed linear space and H is a (p, p^{-1})-closed linear subspace of X, the function $p^s(x) = \max\{p(x), p^{-1}(x)\}$ is a norm on X, and since H is p^s-closed we have that $\|\hat{x}\| = \inf\{p^s(x) : x \in \hat{x}\}$ is a norm on X/H.

On the other hand we have that $\hat{p}^s(x) = \max\{\hat{p}(x), \hat{p}^{-1}(x)\}$ is also a norm on X/H. Now, the natural question is to know the relationship between $\|\cdot\|$ and $\hat{p}^s$.

If H has finite codimension, then X/H is finite dimensional and therefore the norms on X/H are equivalent. In a general case, since $B_\varepsilon^{p^s}(0) = B_\varepsilon^{p}(0) \cap B_\varepsilon^{p^{-1}}(0)$, we have that

$$B_\varepsilon^{\|\cdot\|}(\hat{0}) = \pi(B_\varepsilon^{p^s}(0)) = \pi(B_\varepsilon^{p}(0) \cap B_\varepsilon^{p^{-1}}(0))$$

$$\subset \pi(B_\varepsilon^{p}(0)) \cap \pi(B_\varepsilon^{p^{-1}}(0)) = B_\varepsilon^{\hat{p}}(\hat{0}) \cap B_\varepsilon^{\hat{p}^{-1}}(\hat{0}) = B_\varepsilon^{\hat{p}^s}(\hat{0}).$$

Hence the topology generated by the norm $\|\cdot\|$ is finer than the topology generated by the norm $\hat{p}^s$ and moreover, $\hat{p}^s \leq \|\cdot\|$.

4. The dual space of $(X/H, \hat{p})$

Let (X, p) be an asymmetric normed linear space. In Section 2 we have denoted by (X^*, p^*) the dual space of (X, p), where

$$p^*(f) = \sup\{f(x) : p(x) \leq 1\}.$$

It is easy to prove that we can compute $p^*(f)$ by taking the supremum of $f(x)$ in the open ball $B_1^p(0)$, that is

$$p^*(f) = \sup\{f(x) : p(x) < 1\}.$$

If H is a (p, p^{-1})-closed linear subspace of X then $((X/H)^*, \hat{p}^*)$ is the dual space of the asymmetric normed space $(X/H, \hat{p})$, where

$$(X/H)^* = \{\hat{f} : (X/H, \hat{p}) \to (\mathbb{R}, u) : \ f \text{ is linear and continuous}\},$$

and

$$\hat{p}^*(\hat{f}) = \sup\{\hat{f}(\hat{x}) : \hat{p}(\hat{x}) \leq 1\} = \sup\{\hat{f}(\hat{x}) : \hat{p}(\hat{x}) < 1\}.$$

We recall the definition of isometric cones [13].

Definition (4.1). Two asymmetric cones (X, p) and (Y, q) are said to be isometric if there is a bijection ϕ between (X, p) and (Y, q) such that for all $x, y \in X$ and $a \in \mathbb{R}^+$:

(i) $\phi(x + y) = \phi(x) + \phi(y)$,
(ii) $\phi(ax) = a\phi(x)$,
(iii) $q(\phi(x)) = p(x)$.

THEOREM (4.2). *$((X/H)^*, \hat{p}^*)$ is isometric to $(H^\perp, p^*)$, where*

$$H^\perp := \{f \in X^* : f(x) = 0, \forall x \in H\}.$$

We omit the proof of this result since it is a straightforward generalization of the well known proof of the analogous result in normed context.

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ON TWO NON-TRIVIAL PRODUCTS IN THE STABLE HOMOTOPY GROUPS OF SPHERES

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ABSTRACT. In this paper we consider the convergence of some products in the Adams spectral sequence and show that two products $h_0k_0h_m \in E_2^{4,2(p_1-1)(p_1^m+2p_1+2)}$, $k_0h_0h_n\tilde{\gamma}_s \in E_2^{s+4,2(p_2-1)(p_2^n+sp_2^2+(s+1)p_2+s)+s-3}$ both are permanent cycles in the Adams spectral sequence and converge to two new nontrivial homotopy elements in the stable homotopy groups of spheres π_*S respectively, where $p_1 \geqslant 5$ is a prime, $p_2 \geqslant 7$ is a prime, $m \geqslant 3$, $n \geqslant 4$ and $3 \leqslant s < p_2 - 2$.

1. Introduction and the main results

Let A denote the mod p Steenrod algebra and S denote the sphere spectrum localized at an odd prime p. To determine the homotopy groups π_*S of spheres S at p is one of the central problems in the stable homotopy theory. The Adams spectral sequence (ASS, for short).

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \pi_{t-s}S$$

has been an invaluable tool in studying the stable homotopy groups of spheres, where the $E_2^{s,t}$-term is the cohomology of A. If a family of generators x_i in $E_2^{s,*}$ converges nontrivially in the ASS, then we get a family of nontrivial homotopy elements f_i in π_*S. We say that f_i is represented by $x_i \in E_2^{s,*}$ and has filtration s in the ASS. So far, not so many families of homotopy elements in π_*S have been detected. Recently, Lin got a series of results and detected some new families in π_*S (cf. [1-4]).

Throughout this paper we always let $q = 2(p-1)$.

From [7], $\mathrm{Ext}_A^{1,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(Z_p, Z_p)$, $h_i \in \mathrm{Ext}_A^{1,p^iq}(\mathbb{Z}_p, \mathbb{Z}_p)$ for all $i \geqslant 0$ and $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$-basis consisting of α_2, a_0^2, $a_0h_i(i > 0)$, $g_i(i \geqslant 0)$, $k_i(i \geqslant 0)$, $b_i(i \geqslant 0)$, and $h_ih_j(j \geqslant i+2, i \geqslant 0)$ whose internal degrees are $2q+1$, 2, p^iq+1, $p^{i+1}q+2p^iq$, $2p^{i+1}q+p^iq$, $p^{i+1}q$ and p^iq+p^jq respectively.

Let M be the Moore spectrum modulo a prime $p \geqslant 5$ given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i} M \xrightarrow{j} \Sigma S\,.$$

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Let $\alpha\colon \Sigma^q M \to M$ be the Adams map and K be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i'} K \xrightarrow{j'} \Sigma^{q+1} M.$$

This spectrum which we briefly write as K is known to be the Toda-Smith spectrum $V(1)$. Let $V(2)$ be the cofibre of $\beta\colon \Sigma^{(p+1)q} K \to K$ given by the cofibration

$$\Sigma^{(p+1)q} K \xrightarrow{\beta} K \xrightarrow{\bar{i}} V(2) \xrightarrow{\bar{j}} \Sigma^{(p+1)q+1} K\,.$$

Let $\gamma\colon \Sigma^{q(p^2+p+1)} V(2) \to V(2)$ be the v_3-map. Recall we have the γ-family $\{\gamma_t \in \pi_{2(tp^3-t-p^2-p)+1}(S), t \geqslant 1\}$ in $\pi_*^s(S)$ localized at the prime p, where $\gamma_t = jj'\bar{j}\gamma^t\bar{i}i'i$ (see [8], Theorem 2.12).

In [3], Lin proved the existence of third periodicity element in the stable homotopy groups of spheres $\pi_{p^{n+2}q+(p^n-s)(p+1)q-q-3}S$ which is of order p and is represented by $\gamma_{p^n/s}$ in the $E_2^{3,*}$-term of the Adams-Novikov spectral sequence (ANSS, for short). On the way of proving the main result, he detected a new family in the stable homotopy groups of $V(1)$, which is a spectrum closely related to S. He gave the following theorem:

THEOREM (1.1). *Let $p \geqslant 5$, $n \geqslant 1$ and $h_n \in \mathrm{Ext}_A^{1,p^nq}(\mathbb{Z}_p, \mathbb{Z}_p)$ be the known generator in* [7]*. Then*

$$(i'i)_*(h_n) \in \mathrm{Ext}_A^{1,p^nq}(H^*V(1), \mathbb{Z}_p)\,,$$

the reduction of $h_n \in \mathrm{Ext}_A^{1,p^nq}(\mathbb{Z}_p, \mathbb{Z}_p)$, is a permanent cycle in the ASS and converges to a nontrivial element $\varpi_n \in \pi_{p^nq-1}V(1)$.

In this paper, we will base on the family of homotopy elements in $\pi_*V(1)$ in [3] to detect two new families of filtration 4 and $s+4$ in the stable homotopy groups of spheres π_*S. Our main results can be stated as follows.

THEOREM (1.2). *Let $p \geqslant 5$, $n \geqslant 3$, then the product*

$$h_0k_0h_n \neq 0 \in \mathrm{Ext}_A^{4,q(p^n+2p+2)}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the ASS and converges to a nontrivial element $\alpha_1 jj'\beta^2\varpi_n \in \pi_{q(p^n+2p+2)-4}S$ of order p, where $\alpha_1 = j\alpha i$.

THEOREM (1.3). *Let $p \geqslant 7$, $n \geqslant 4$ and $3 \leqslant s < p-2$, then the product*

$$\tilde{\gamma}_s h_0k_0h_n \neq 0 \in \mathrm{Ext}_A^{s+4,q(p^n+sp^2+(s+1)p+s)+s-3}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the ASS and converges to a nontrivial element $\gamma_s\alpha_1 jj'\beta^2\varpi_n \in \pi_{q(p^n+sp^2+(s+1)p+s)-7}S$ of order p, where $\tilde{\gamma}_s$ was given in [5] *and $\alpha_1 = i\alpha j$.*

The May spectral sequence (MSS, for short) and the ASS play very important roles in the proofs of the main theorems.

The paper is arranged as follows: after giving some useful propositions on the MSS in Section 2, we will make use of the MSS and the ASS to obtain some low-dimensional Ext groups which will be used in the proofs of the main theorems in Section 3. Section 4 is devoted to showing the main theorems.

2. The ASS and the MSS

For the sake of completeness, in this section we first review some knowledge on the ASS and the MSS. Then we will show some important theorems on the MSS which will be often used in the proofs of the main theorems.

One of the main tools to determine the stable homotopy groups of spheres $\pi_* S$ is the ASS.

Let p be a prime, X a spectrum of finite type and Y a finite dimensional spectrum. Then there is a natural spectral sequence $\{E_r^{s,t}, d_r\}$, which is called Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}((H^*X;\mathbb{Z}_p), H^*(Y;\mathbb{Z}_p)) \Rightarrow ([Y,X]_{t-s})_p\,,$$

where

$$d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}\,.$$

If X and Y are sphere spectra S, then the Adams spectral sequence(ASS)

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow (\pi_{t-s}S)_p\,.$$

If S is localized at p, then the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \pi_{t-s}S\,.$$

There are three problems in using the ASS: calculation of the E_2-term, computation of the differentials and determination of the nontrivial extensions from E_∞ to $\pi_* S$. So, for computing the stable homotopy groups of spheres with the ASS, we must compute the E_2-term of the ASS, $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$. The most successful method for computing $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ is the MSS.

From [9], there is a May spectral sequence(MSS) $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p)$ with E_1-term

$$(2.1)\quad E_1^{*,*,*} = E(h_{m,i}|m>0, i\geqslant 0)\bigotimes P(b_{m,i}|m>0, i\geqslant 0)\bigotimes P(a_n|n\geqslant 0)\,,$$

where E is the exterior algebra, P is the polynomial algebra, and

$$h_{m,i}\in E_1^{1,2(p^m-1)p^i,2m-1},\, b_{m,i}\in E_1^{2,2(p^m-1)p^{i+1},p(2m-1)},\, a_n\in E_1^{1,2p^n-1,2n+1}\,.$$

One has $d_r\colon E_r^{s,t,u}\to E_r^{s+1,t,u-r}$ and if $x\in E_r^{s,t,*}$ and $y\in E_r^{s',t',*}$, then

$$d_r(x\cdot y) = d_r(x)\cdot y + (-1)^s x\cdot d_r(y)\,.$$

There exists a graded commutativity of the MSS:

$$x\cdot y = (-1)^{ss'+tt'} y\cdot x$$

for $x, y = h_{m,i}, b_{m,i}$ or a_n. The first May differential d_1 is given by

$$\begin{aligned} d_1(h_{i,j}) &= \sum_{0<k<i} h_{i-k,k+j}h_{k,j}\,,\\ d_1(a_i) &= \sum_{0\leqslant k<i} h_{i-k,k}a_k\,,\\ d_1(b_{i,j}) &= 0\,. \end{aligned}$$

For each element $x \in E_1^{s,t,*}$, we define $\dim x = s$, $\deg x = t$. Then we have:

$$
(2.2) \qquad \begin{cases} \dim h_{i,j} = \dim a_i = 1, \dim b_{i,j} = 2\,, \\ \deg h_{i,j} = 2(p^i - 1)p^j = 2(p-1)(p^{i+j-1} + \cdots + p^j)\,, \\ \deg b_{i,j} = 2(p^i - 1)p^{j+1} = 2(p-1)(p^{i+j} + \cdots + p^{j+1})\,, \\ \deg a_i = 2p^i - 1 = 2(p-1)(p^{i-1} + \cdots + 1) + 1, \\ \deg a_0 = 1\,, \end{cases}
$$

where $i \geqslant 1$, $j \geqslant 0$.

By the knowledge on p-adic expression in number theory, we can have that for each integer $t \geqslant 0$, it can be always expressed uniquely as

$$t = q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0) + e\,,$$

where $0 \leqslant c_i < p$ $(0 \leqslant i < n)$, $p > c_n > 0$, $0 \leqslant e < q$.

In the proofs of the main theorems in Section 3, we need the following two theorems on the MSS.

Let s and t be two arbitrary positive integers. Suppose $t = q(c_n p^n + c_{n-1}p^{n-1} + \cdots + c_1 p + c_0) + e$, where $0 \leqslant c_i < p\,(0 \leqslant i < n)$, $p > c_n > 0, 0 \leqslant e < q$. Suppose $h = x_1 x_2 \ldots x_s \in E_1^{s,t,*}$ and $h \in E(h_{m,i}|m > 0, i \geqslant 0) \bigotimes P(a_n|n \geqslant 0)$, where x_i is one of a_k or $h_{l,j}$, $0 \leqslant k \leq n+1$, $0 \leqslant l + j \leqslant n+1$, $l > 0, j \geqslant 0$. By (2.2) we can assume $\deg x_i = q(c_{i,n}p^n + \cdots + c_{i,1}p + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. Then we have

$$\deg h = \sum_{i=1}^{s} \deg x_i = q((\sum_{i=1}^{s} c_{i,n})p^n + \cdots + (\sum_{i=1}^{s} c_{i,1})p + (\sum_{i=1}^{s} c_{i,0})) + (\sum_{i=1}^{s} e_i).$$

Denote $\sum_{i=1}^{s} c_{i,j}$ and $\sum_{i=1}^{s} e_i$ by $\bar{c}_j$ and $\bar{e}$, $0 \leq j \leq n$, respectively. Then we have the following theorem which was given in [6].

THEOREM (2.3) ([6], Proposition 2.1). *With notation as above. If* $\bar{c}_0 - \bar{e} > n+1$, *then h cannot exist, i.e.,* $E_1^{s,t,*} = 0$. □

THEOREM (2.4). *With notation as above. If for some j with* $0 < j \leqslant n$, $\bar{c}_j = s$.

(1) *If there also exist two integers i_1 and i_2 such that* $0 \leqslant i_1 < i_2 < j$ *and* $s \geqslant \bar{c}_{i_1} > \bar{c}_{i_2}$, *then h cannot exist.*
(2) *If there also exists an integer i such that* $0 \leqslant i < j$ *and* $s \geqslant \bar{e} > \bar{c}_i$, *then h cannot exist.*
(3) *If there also exist two integers i'_1 and i'_2 such that* $j < i'_1 < i'_2 \leqslant n$ *and* $s \geqslant \bar{c}_{i'_2} > \bar{c}_{i'_1}$, *then h cannot exist.*

Proof. (1) By (2.2), from $\bar{c}_j = s$ we can have that in $h = x_1 x_2 \cdots x_s$, $\deg x_i =$ higher terms $+p^j q+$lower terms for each $1 \leqslant i \leqslant s$.

By $\sum_{i=1}^{s} c_{i,i_1} = \bar{c}_{i_1}$ and (2.2), we can easily see that there exist $\bar{c}_{i_1}$ factors in h such that deg = higher terms $+p^{i_1}q+$lower terms.

From the above discussion and (2.2), we see that there exist at least $\bar{c}_{i_1}$ factors in h such that deg =higher terms$+p^j q + p^{j-1}q + \cdots + p^{i_2}q + \cdots + p^{i_1}q+$lower terms. It follows that $\sum_{i=1}^{s} c_{i,i_2} \geqslant \bar{c}_{i_1}$. That is to say, $\bar{c}_{i_2} \geqslant \bar{c}_{i_1}$. But we also have $\bar{c}_{i_1} > \bar{c}_{i_2}$. This is a contradiction. It follows that h cannot exist. This completes the proof of (1).

(2) the proof of (2) is similar to that of (1)
(3) the proof of (3) is similar to that of (1) □

3. Some useful Ext groups

In this section, we make use of the MSS to determine some Ext groups which will be used in the proofs of the main theorems.

To show that the product $k_0h_0h_n\tilde{\gamma}_s \in \mathrm{Ext}_A^{s+4,t}(\mathbb{Z}_p, \mathbb{Z}_p)$ is nontrivial, it suffices to prove that the representative of $k_0h_0h_n\tilde{\gamma}_s$ in the MSS is a permanent cycle and cannot be hit by any differential in the MSS. Thus we first give the following lemma.

LEMMA (3.1). *Let $p \geqslant 7$, $n \geqslant 4$, $0 \leqslant s < p-5$, then in the MSS,*

$$E_1^{s+6,p^nq+(s+3)p^2q+(s+4)pq+(s+3)q+s,*} = 0\,.$$

Proof. Let $t = p^nq + (s+3)p^2q + (s+4)pq + (s+3)q + s$. Consider $h = x_1x_2\cdots x_m \in E_1^{s+6,t,*}$ in the MSS, where x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leqslant k \leqslant n+1$, $0 \leqslant l+j \leqslant n+1$, $0 \leqslant u+z \leqslant n$, $l > 0$, $j \geqslant 0$, $u > 0$, $z \geqslant 0$. Assume that $\deg x_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1} + \cdots + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. It follows that $\dim h = \sum_{i=1}^m \dim x_i = s+6$ and

$$\begin{aligned}
\deg h &= \sum_{i=1}^m \deg x_i \\
&= q((\sum_{i=1}^m c_{i,n})p^n + \cdots + (\sum_{i=1}^m c_{i,1})p + (\sum_{i=1}^m c_{i,0})) + (\sum_{i=1}^m e_i) \qquad (3.2)\\
&= q(p^n + (s+3)p^2 + (s+4)p + s + 3) + s\,.
\end{aligned}$$

By virtue of $0 \leqslant s, s+3, s+4 < p$ and the knowledge on the p-adic expression in number theory, we have from (3.2)

$$(3.3)\qquad \begin{cases}
\sum_{i=1}^m e_i = s + \lambda_{-1}q, & \lambda_{-1} \geqslant 0;\\
\sum_{i=1}^m c_{i,0} + \lambda_{-1} = s + 3 + \lambda_0 p, & \lambda_0 \geqslant 0;\\
\sum_{i=1}^m c_{i,1} + \lambda_0 = s + 4 + \lambda_1 p, & \lambda_1 \geqslant 0;\\
\sum_{i=1}^m c_{i,2} + \lambda_1 = s + 3 + \lambda_2 p, & \lambda_2 \geqslant 0;\\
\sum_{i=1}^m c_{i,3} + \lambda_2 = 0 + \lambda_3 p, & \lambda_3 \geqslant 0;\\
\cdots & \cdots\\
\sum_{i=1}^m c_{i,n-1} + \lambda_{n-2} = 0 + \lambda_{n-1} p, & \lambda_{n-1} \geqslant 0;\\
\sum_{i=1}^m c_{i,n} + \lambda_{n-1} = 1. &
\end{cases}$$

Case 1. $0 \leqslant s < p-6$.

By the facts that $\dim h_{i,j} = \dim a_i = 1$ and $\dim b_{i,j} = 2$, we can have

$$6 \leqslant m \leqslant s + 6 < p - 6 + 6 = p$$

from $\dim h = \sum_{i=1}^{m} \dim x_i = s + 6$. Notice that $e_i = 0$ or 1, $c_{i,j} = 0$ or 1, and $m < p$. From (3.3), we have

$$0 \leqslant \sum_{i=1}^{m} e_i, \sum_{i=1}^{m} c_{i,j} \leqslant m < p\,.$$

It follows that the number sequence $(\lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_{n-2}, \lambda_{n-1})$ must equal the sequence $(0, 0, 0, 0, \cdots, 0, 0)$. Then (3.3) can turn into

$$\begin{cases} \sum_{i=1}^{m} e_i = s, & \sum_{i=1}^{m} c_{i,0} = s + 3, \\ \sum_{i=1}^{m} c_{i,1} = s + 4, & \sum_{i=1}^{m} c_{i,2} = s + 3, \\ \sum_{i=1}^{m} c_{i,3} = \cdots = \sum_{i=1}^{m} c_{i,n-1} = 0, & \sum_{i=1}^{m} c_{i,n} = 1\,. \end{cases}$$

By (2.2), it is easy to know that there exists a factor $h_{1,n}$ or $b_{1,n-1}$ among h. By virtue of the graded commutativity of $E_1^{*,*,*}$, we can denote by x_m the factor $h_{1,n}$ or $b_{1,n-1}$. Then $h' = x_1 x_2 \ldots x_{m-1} \in E_1^{l,t-p^n q,*}$, where $l = s+5$ (if $x_m = h_{1,n}$) or $s + 4$ (if $x_m = b_{1,n-1}$). And we have

$$\begin{cases} \sum_{i=1}^{m-1} e_i = s, & \sum_{i=1}^{m-1} c_{i,0} = s + 3, \\ \sum_{i=1}^{m-1} c_{i,1} = s + 4, & \sum_{i=1}^{m-1} c_{i,2} = s + 3. \end{cases} \tag{3.4}$$

By $c_{i,1} = 0$ or 1, we can get

$$m \geqslant s + 5$$

from $\sum_{i=1}^{m-1} c_{i,1} = s + 4$. On the other hand, we also have

$$m \leqslant s + 6\,.$$

Thus m can equal $s + 5$ or $s + 6$.

Since $\sum_{i=1}^{m-1} e_i = s$, $\deg h_{i,j} \equiv 0 (\mathrm{mod}\ q)$ $(i > 0, j \geqslant 0)$, $\deg a_i \equiv 1 (\mathrm{mod}\ q)$ $(i \geqslant 0)$ and $\deg b_{i,j} \equiv 0 (\mathrm{mod}\ q)$ $(i > 0, j \geqslant 0)$, then by the graded commutativity of $E_1^{*,*,*}$, up to sign h' must have a factor $a_{j_1} a_{j_2} \cdots a_{j_s}$ $(0 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_s)$. Notice that the degrees of a_i's. We can assume that $h' = a_0^x a_1^y a_2^z a_3^k x_{s+1} \cdots x_{m-1}$, where $0 \leqslant x, y, z, k \leqslant s$, $x + y + z = s$. Then from (3.4) we have

$$\begin{cases} x + y + z + k + \sum_{i=s+1}^{m-1} e_i = s, & y + z + k + \sum_{i=s+1}^{m-1} c_{i,0} = s + 3, \\ z + k + \sum_{i=s+1}^{m-1} c_{i,1} = s + 4, & k + \sum_{i=s+1}^{m-1} c_{i,2} = s + 3. \end{cases}$$

Then

$$h'' = x_{s+1} \cdots x_{m-1} \in E_1^{l-s,t',*}$$

where $t' = (s+3-k)p^2q + (s+4-z-k)pq + (s+3-y-z-k)q$, and

$$
\left\{
\begin{array}{ll}
\sum\limits_{i=s+1}^{m-1} e_i = 0, & \sum\limits_{i=s+1}^{m-1} c_{i,0} = s+3-y-z-k, \\
\sum\limits_{i=s+1}^{m-1} c_{i,1} = s+4-z-k, & \sum\limits_{i=s+1}^{m-1} c_{i,2} = s+3-k.
\end{array}
\right. \tag{3.5}
$$

Subcase 1.1. If $h = x_1x_2\cdots x_{m-1}h_{1,n}$, then $h' = a_0^x a_1^y a_2^z a_3^k x_{s+1}\cdots x_{m-1} \in E_1^{s+5,t-p^nq,*}$, $h'' = x_{s+1}\cdots x_{m-1} \in E_1^{5,t',*}$.

When $m = s+5$, (3.5) can turn into

$$
\left\{
\begin{array}{ll}
\sum\limits_{i=s+1}^{s+4} e_i = 0, & \sum\limits_{i=s+1}^{s+4} c_{i,0} = s+3-y-z-k, \\
\sum\limits_{i=s+1}^{s+4} c_{i,1} = s+4-z-k, & \sum\limits_{i=s+1}^{s+4} c_{i,2} = s+3-k.
\end{array}
\right. \tag{3.6}
$$

One the one hand, by virtue of $c_{i,2} = 0$ or 1, from $\sum\limits_{i=s+1}^{s+4} c_{i,2} = s+3-k$ we have

$$k = s+3-\sum_{i=s+1}^{s+4} c_{i,2} \geqslant s+3-4 = s-1,$$

i.e., $k \geqslant s-1$. On the other hand, from $\sum\limits_{i=s+1}^{s+4} c_{i,1} = s+4-z-k$, we have that

$$z+k = s+4-\sum_{i=s+1}^{s+4} c_{i,1} \geqslant s+4-4 = s,$$

i.e., $z+k = s$. By virtue of $x+y+z+k = s$, we can get that there exist two possibilities: $k = s-1, z = 1, y = x = 0$ and $k = s, z = y = x = 0$. If $k = s-1, z = 1, y = x = 0$, then $h' = a_2a_3^{s-1}x_{s+1}\cdots x_{s+4}$ with $h'' = x_{s+1}x_{s+2}x_{s+3}x_{s+4} \in E_1^{5,4p^2q+4pq+3q,*} = 0$. If $k = s, z = y = x = 0$, then $h' = a_3^s x_{s+1}\cdots x_{s+4}$ with $h'' = x_{s+1}x_{s+2}x_{s+3}x_{s+4} \in E_1^{5,3p^2q+4pq+3q,*} = 0$. Thus in this case, h is impossible to exist.

When $m = s+6$, (3.5) can turn into

$$
\left\{
\begin{array}{ll}
\sum\limits_{i=s+1}^{s+5} e_i = 0, & \sum\limits_{i=s+1}^{s+5} c_{i,0} = s+3-y-z-k, \\
\sum\limits_{i=s+1}^{s+5} c_{i,1} = s+4-z-k, & \sum\limits_{i=s+1}^{s+5} c_{i,2} = s+3-k.
\end{array}
\right.
$$

Similarly, from $\sum\limits_{i=s+1}^{s+4} c_{i,2} = s+3-k$, we have

$$k = s+3-\sum_{i=s+1}^{s+5} c_{i,2} \geqslant s+3-5 = s-2$$

by virtue of $c_{i,2} = 0$ or 1. Meanwhile, from $\sum_{i=s+1}^{s+5} c_{i,1} = s + 4 - z - k$, we have that

$$z + k = s + 4 - \sum_{i=s+1}^{s+4} c_{i,1} \geqslant s + 4 - 5 = s - 1 .$$

Notice that $x+y+z+k = s$. It is easy to see that there exist seven possibilities satisfying the two conditions: $k \geqslant s - 2$ and $z + k \geqslant s - 1$. For the seven possibilities, we list a table as follows.

The possibility	k	z	y	x	$E_1^{5,t',*}$	The existence of $h'' = x_{s+1} \cdots x_{s+5}$
The 1st	$s-2$	1	1	0	$E_1^{5,q(5p^2+5p+3),*} = 0$	Nonexistence
The 2nd	$s-2$	1	0	1	$E_1^{5,q(5p^2+5p+4),*} = 0$	Nonexistence
The 3rd	$s-2$	2	0	0	$E_1^{5,q(5p^2+4p+3),*} = 0$	Nonexistence
The 4th	$s-1$	0	0	1	$E_1^{5,q(4p^2+5p+4),*} = 0$	Nonexistence
The 5th	$s-1$	0	1	0	$E_1^{5,q(4p^2+5p+3),*} = 0$	Nonexistence
The 6th	$s-1$	1	0	0	$E_1^{5,q(4p^2+4p+3),*} = 0$	Nonexistence
The 7th	s	0	0	0	$E_1^{5,q(3p^2+4p+3),*} = 0$	Nonexistence

From the above table, it follows that in this case h can not exist either.

Subcase 1.2. If $h = x_1 x_2 \cdots x_{m-1} b_{1,n-1}$, then $h' = a_0^x a_1^y a_2^z a_3^k x_{s+1} \cdots x_{m-1} \in E_1^{s+4,t-p^n q,*}$, $h'' = x_{s+1} \cdots x_{m-1} \in E_1^{4,t',*}$.

When $m = s + 6$, $h'' = x_{s+1} \cdots x_{s+5}$. Notice that $\dim x_i = 1$ or 2. Then we easily have that

$$\dim h'' = \sum_{i=s+1}^{s+5} \dim x_i \geqslant 5 > 4 = \dim h'' .$$

This is a contradiction. Thus in this case h'' cannot exist.

When $m = s + 5$, $h = x_1 x_2 \cdots x_{s+4} b_{1,n-1}$, then $h' = a_0^x a_1^y a_2^z a_3^k x_{s+1} \cdots x_{s+4} \in E_1^{s+4,t-p^n q,*}$, $h'' = x_{s+1} \cdots x_{s+4} \in E_1^{4,t',*}$. (3.5) can turn into

$$\begin{cases} \sum_{i=s+1}^{s+4} e_i = 0, & \sum_{i=s+1}^{s+4} c_{i,0} = s + 3 - y - z - k, \\ \sum_{i=s+1}^{s+4} c_{i,1} = s + 4 - z - k, & \sum_{i=s+1}^{s+4} c_{i,2} = s + 3 - k. \end{cases}$$

As in subcase 1.1, we can get that $k \geqslant s-1$ and $z+k = s$. By $x+y+z+k = s$, we can know that there also exist two possibilities: $k = s-1, z = 1, y = x = 0$ and $k = s, z = y = x = 0$. If $k = s-1, z = 1, y = x = 0$, then $h' = a_2 a_3^{s-1} x_{s+1} \cdots x_{s+4}$ with $h'' = x_{s+1} x_{s+2} x_{s+3} x_{s+4} \in E_1^{4,4p^2q+4pq+3q,*} = 0$. If $k = s, z = y = x = 0$, then $h' = a_3^s x_{s+1} \cdots x_{s+4}$ with $h'' = x_{s+1} x_{s+2} x_{s+3} x_{s+4} \in E_1^{4,3p^2q+4pq+3q,*} = 0$. Thus in this case h is impossible to exist either.

From Subcases 1.1 and 1.2, we have that when $0 \leqslant s < p - 6$, h cannot exist. Thus $E_1^{s+6,t,*} = 0$.

Case 2. $s = p - 6$.

From $\dim h = \sum_{i=1}^{m} \dim x_i = s + 6 = p - 6 + 6 = p$, we have $m \leqslant p$ by $\dim x_i = 1$ or 2. Since $0 \leqslant \sum_{i=1}^{m} e_i, \sum_{i=1}^{m} c_{i,j} \leqslant m \leqslant p$, $0 \leqslant s = p - 6 < p < q$, and $0 < s+3, s+4 < p$, by the knowledge of p-adic expression in number theory it is easy to get that the number sequence $(\lambda_{-1}, \lambda_0, \lambda_1, \lambda_2)$ must equal the sequence $(0, 0, 0, 0)$. From (3.3) we have $\sum_{i=1}^{m} c_{i,3} = \lambda_3 p$. By virtue of $0 \leqslant \sum_{i=1}^{m} c_{i,3} \leqslant m \leqslant p$, we have that λ_3 may equal 0 or 1.

Subcase 2.1 If $\lambda_3 = 0$, then $\sum_{i=1}^{m} c_{i,3} = 0$.

When $n = 4$, we have $\sum_{i=1}^{m} c_{i,4} = 1$. From the above results, it follows that there exists a factor $h_{1,4}$ or $b_{1,3}$ among h.

When $n > 4$, we can similarly discuss and obtain that λ_4 may equal 0 or 1. We claim that

$$\lambda_4 = 0\,.$$

Otherwise, we would have that $\lambda_4 = 1$ and $\sum_{i=1}^{m} c_{i,4} = p$. Then $m = p$. For each $1 \leqslant i \leqslant p$, $\deg x_i =$ higher terms $+p^4q+$lower terms. Since $\sum_{i=1}^{p} e_i = p - 6$, $\deg b_{i,j} \equiv 0 (\mathrm{mod}\ q)$ $(i > 0, j \geqslant 0)$, $\deg a_i \equiv 1 (\mathrm{mod}\ q)$ $(i \geqslant 0)$ and $\deg h_{i,j} \equiv 0 (\mathrm{mod}\ q)$ $(i > 0, j \geqslant 0)$, then by the graded commutativity of $E_1^{*,*,*}$, there would exist a factor $a_{j_1} a_{j_2} \cdots a_{j_{p-6}}$ $(0 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_{p-6} \leqslant n + 1)$ among x_i's such that for each $1 \leqslant i \leqslant p - 6$, $j_i \geqslant 5$ and $\deg a_{j_i} =$ higher terms $p^4q+p^3q+p^2q+pq+q+1$. Obviously $\sum_{i=1}^{m} c_{i,3} = \sum_{i=1}^{p} c_{i,3} \geqslant p-6$ which contradicts $\sum_{i=1}^{m} c_{i,3} = 0$, thus the claim is proved. By induction on j we can get

$$\lambda_j = 0 \qquad (4 \leqslant j \leqslant n - 1)\,.$$

It follows that

$$\sum_{i=1}^{m} c_{i,n} = 1\,,$$

that is to say, there is a factor $h_{1,n}$ or $b_{1,n-1}$ among h.

In all, for $n \geqslant 4$, there exists a factor $h_{1,n}$ or $b_{1,n-1}$ among h. By virtue of the graded commutativity of $E_1^{*,*,*}$, we can denote the factor $h_{1,n}$ or $b_{1,n-1}$ by x_m. By an argument similar to that used in the proof in Case 1, we can show that h cannot exist.

Subcase 2.2. If $\lambda_3 = 1$, then $\sum_{i=1}^{m} c_{i,3} = p$.

By virtue of $c_{i,3} = 0$ or 1 and $m \leqslant p$, we can get $m = p$. By $\dim h = p$, we can easily show that for each i, $\dim x_i = 1$ and

$$h = x_1 x_2 \cdots x_p \in E(h_{m,i} | m > 0, i \geqslant 0) \bigotimes P(a_n | n \geqslant 0)\,.$$

When $n = 4$, it is easy to see that $\sum_{i=1}^{p} c_{i,4} = \cdots = \sum_{i=1}^{p} c_{i,n} = 0$.

When $n > 4$, from (3.3) we have

$$\sum_{i=1}^{p} c_{i,4} + 1 = 0 + \lambda_4 p\,.$$

From $c_{1,4} = 0$ or 1, we have

$$\lambda_4 = 1\,.$$

By induction on j, we have $\lambda_j = 1$, $4 \leqslant j \leqslant n-1$. And then we have that $\sum_{i=1}^{p} c_{i,3} = p$, $\sum_{i=1}^{p} c_{i,4} = \cdots = \sum_{i=1}^{p} c_{i,n-1} = p-1$, and $\sum_{i=1}^{p} c_{i,n} = 0$.

When $n = 4$, From $\sum_{i=1}^{p} e_i = p-6$, $\sum_{i=1}^{p} c_{i,0} = p-3$, $\sum_{i=1}^{p} c_{i,1} = p-2$, $\sum_{i=1}^{p} c_{i,2} = p-3$, $\sum_{i=1}^{p} c_{i,3} = p$ we can show that $h = x_1 x_2 \cdots x_p$ cannot exist by Theorem (2.4).

When $n > 4$, From $\sum_{i=1}^{p} e_i = p-6$, $\sum_{i=1}^{p} c_{i,0} = p-3$, $\sum_{i=1}^{p} c_{i,1} = p-2$, $\sum_{i=1}^{p} c_{i,2} = p-3$, $\sum_{i=1}^{p} c_{i,3} = p$, $\sum_{i=1}^{p} c_{i,4} = \cdots = \sum_{i=1}^{p} c_{i,n-1} = p-1$, we also can show that $h = x_1 x_2 \cdots x_p$ cannot exist by Theorem (2.4) either.

From Subcases 2.1 and 2.2, we get that when $s = p-6$, $E_1^{s+6,t,*} = 0$.

From Cases 1 and 2, the lemma follows. □

The following lemma will be used in the proofs of Theorems 3.8 and 4.1. It was given in [5] and called the representation theorem.

LEMMA (3.7) ([5], Theorem 1.1). *Let $p \geqslant 7$, $0 \leqslant s < p-3$, then the permanent cycle*

$$a_3^s h_{3,0} h_{2,1} h_{1,2} \in E_r^{s+3,t,*}$$

converges to the third Greek letter family element

$$\tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+3,t}(\mathbb{Z}_p, \mathbb{Z}_p)$$

in the May spectral sequence, where $r \geqslant 1$, $t = (s+3)p^2q + (s+2)pq + (s+1)q + s$ and $\tilde{\gamma}_{s+3}$ converges to the γ-element

$$\gamma_{s+3} \in \pi_{(s+3)p^2q+(s+2)pq+(s+1)q-3} S$$

in the Adams spectral sequence, where $\gamma_{s+3} = jj'\bar{j}\gamma^{s+3}\bar{i}i'i \in \pi_{t-s-3}S$. □

THEOREM (3.8). *Let $p \geqslant 7$, $n \geqslant 4$, $0 \leqslant s < p-5$, then the product*

$$k_0 h_0 h_n \tilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+7, p^n q + (s+3)p^2 q + (s+4)pq + (s+3)q + s}(\mathbb{Z}_p, \mathbb{Z}_p)\,.$$

Proof. Since $h_{2,0}h_{1,1}$, $h_{1,n}$ and $a_3^s h_{3,0} h_{2,1} h_{1,2} \in E_1^{*,*,*}$ are permanent cycles in the MSS and converge nontrivially to k_0, h_n, $\tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{*,*}(Z_p, Z_p)$ for $n \geqslant 0$ respectively (see Lemma (3.7)),

$$h_{2,0}h_{1,1}h_{1,0}h_{1,n}a_3^s h_{3,0}h_{2,1}h_{1,2} \in E_1^{s+7, p^n q + (s+3)p^2 q + (s+4)pq + (s+3)q + s, *}$$

is a permanent cycle in the MSS and converges to $k_0 h_0 h_n \tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+7,*}(\mathbb{Z}_p, \mathbb{Z}_p)$.

From Lemma (3.1), we see that

$$E_1^{s+6, p^n q + (s+3)p^2 q + (s+4)pq + (s+3)q + s, *} = 0\,,$$

then for $r \geqslant 1$,

$$E_r^{s+6,p^nq+(s+3)p^2q+(s+4)pq+(s+3)q+s,*} = 0 .$$

Thus the permanent cycle $h_{2,0}h_{1,1}h_{1,0}h_{1,n}a_3^s h_{3,0}h_{2,1}h_{1,2} \in E_r^{s+7,*,*}$ does not bound. That is to say, $h_{2,0}h_{1,1}h_{1,0}h_{1,n}a_3^s h_{3,0}h_{2,1}h_{1,2} \in E_r^{s+7,*,*}$ cannot be hit by any differential in the MSS. It follows that $h_{2,0}h_{1,1}h_{1,0}h_{1,n}a_3^s h_{3,0}h_{2,1}h_{1,2} \in E_r^{s+7,*,*}$ is a permanent cycle in the May spectral sequence and converges non-trivially to $k_0h_0h_n\tilde{\gamma}_{s+3} \in \mathrm{Ext}_A^{s+7,p^nq+(s+3)p^2q+(s+4)pq+(s+3)q+s}(\mathbb{Z}_p, \mathbb{Z}_p)$. It follows that

$$k_0h_0h_n\tilde{\gamma}_{s+3} \neq 0 \in \mathrm{Ext}_A^{s+7,p^nq+(s+3)p^2q+(s+4)pq+(s+3)q+s}(\mathbb{Z}_p, \mathbb{Z}_p) .$$

The lemma is proved. $\square$

LEMMA (3.9). *Let $p \geqslant 7$, $n \geqslant 4$, $0 \leqslant s < p-5$, $2 \leqslant r \leqslant s+7$, then in the MSS,*

$$E_1^{s+7-r,q(p^n+(s+3)p^2+(s+4)p+(s+3))+(s-r+1),*} = 0 .$$

Proof. We divide the proof into two cases.

Case 1. $r = s+6, s+7$.

When $r = s+6$ and $r = s+7$, we easily get

$$E_1^{s+7-r,q(p^n+(s+3)p^2+(s+4)p+(s+3))+(s-r+1),*} = 0$$

by (2.2).

Case 2. $2 \leqslant r < s+6$.

For convenience, we let $t'' = q(p^n+(s+3)p^2+(s+4)p+(s+3))+(s-r+1)$. Consider $h = x_1x_2\cdots x_m \in E_1^{s+7-r,t'',*}$, where x_i is one of a_k, $h_{l,j}$ or $b_{u,z}$, $0 \leqslant k \leqslant n+1$, $0 \leqslant l+j \leqslant n+1$, $0 \leqslant u+z \leqslant n$, $l > 0$, $j \geqslant 0$, $u > 0$, $z \geqslant 0$. Assume $\deg x_i = q(c_{i,n}p^n + c_{i,n-1}p^{n-1} + \cdots + c_{i,0}) + e_i$, where $c_{i,j} = 0$ or 1, $e_i = 1$ if $x_i = a_{k_i}$, or $e_i = 0$. We have

$$\begin{aligned}
\deg h &= \sum_{i=1}^{m} \deg x_i \\
&= q((\sum_{i=1}^{m} c_{i,n})p^n + \cdots + (\sum_{i=1}^{m} c_{i,2})p^2 + (\sum_{i=1}^{m} c_{i,1})p + (\sum_{i=1}^{m} c_{i,0})) + (\sum_{i=1}^{m} e_i) \\
&= q(p^n + (s+3)p^2 + (s+4)p + (s+3)) + (s-r+1),
\end{aligned}$$
$$\dim h = \sum_{i=1}^{m} \dim x_i = s+7-r.$$

By $\dim x_i = 1$ or 2 and $2 \leqslant r < s+6$, we can get that $m \leqslant s+7-r \leqslant s+7-2 = s+5 < p$ from $\dim h = \sum_{i=1}^{m} \dim x_i = s+7-r$.

We claim that $s-r-1 \geqslant 0$. Otherwise, we would have $p > \sum_{i=1}^{m} e_i = q+(s+1-r) > q-5 = 2p-2-5 = 2p-7 \geqslant p$ by $2 \leqslant r < s+6$ and $p \geqslant 7$. That is a contradiction. The claim follows.

By an argument similar to that used in Case 1 of Lemma (3.1), we can get

$$\begin{cases} \sum_{i=1}^{m} e_i = s - r + 1, & \sum_{i=1}^{m} c_{i,0} = s + 3, \\ \sum_{i=1}^{m} c_{i,1} = s + 4, & \sum_{i=1}^{m} c_{i,2} = s + 3, \\ \sum_{i=1}^{m} c_{i,3} = \cdots = \sum_{i=1}^{m} c_{i,n-1} = 0, & \sum_{i=1}^{m} c_{i,n} = 1. \end{cases}$$

By (2.2), it is easy to see that there exists a factor $h_{1,n}$ or $b_{1,n-1}$ among h. We can denote the factor $h_{1,n}$ or $b_{1,n-1}$ by x_m, then $h' = x_1 \cdots x_{m-1} \in E_1^{l,t''-p^n q,*}$, where $l = s + 6 - r$ (when $x_m = h_{1,n}$) or $s + 5 - r$ (when $x_m = b_{1,n-1}$). And we have

$$\begin{cases} \sum_{i=1}^{m-1} e_i = s - r + 1, & \sum_{i=1}^{m-1} c_{i,0} = s + 3, \\ \sum_{i=1}^{m-1} c_{i,1} = s + 4, & \sum_{i=1}^{m-1} c_{i,2} = s + 3. \end{cases}$$

Since

$$\sum_{i=1}^{m-1} c_{i,0} - \sum_{i=1}^{m-1} e_i = s + 3 - (s - r + 1) = r + 2 > 2 + 1$$

by $r \geqslant 2$, we can get that

$$E_1^{s+6-r,t''-p^n q,*} = 0 \text{ and } E_1^{s+5-r,t''-p^n q,*} = 0$$

by Theorem (2.3), i.e., h' cannot exist. Thus h cannot exist

From Cases 1 and 2, it follows that $E_1^{s+7-r,t'',*} = 0$. This completes the proof of Lemma (3.9) □

THEOREM (3.10). *Let $p \geqslant 7$, $n \geqslant 4$, $0 \leqslant s < p - 5$, $2 \leqslant r \leqslant s + 7$, then*

$$\mathrm{Ext}_A^{s+7-r,q(p^n+(s+3)p^2+(s+4)p+(s+3))+(s-r+1)}(\mathbb{Z}_p, \mathbb{Z}_p) = 0\,.$$

Proof. By Lemma (3.9) and the MSS, Theorem (3.10) easily follows. □

4. Proofs of the main theorems

Proof of Theorem (1.2). It is known that $h_0, k_0 \in \mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ are permanent cycles in the ASS and converge nontrivially to the α-element $\alpha_1 = j\alpha i$ and β-element $\beta_2 = jj'\beta^2 i' i$, respectively.

Meanwhile, from Theorem (1.1), we know that

$$(i'i)_*(h_n) \in \mathrm{Ext}_A^{1,p^n q}(H^*V(1), \mathbb{Z}_p)\,,$$

the reduction of $h_n \in \mathrm{Ext}_A^{1,p^n q}(\mathbb{Z}_p, \mathbb{Z}_p)$, is a permanent cycle in the ASS and converges to a nontrivial element

$$\varpi_n \in \pi_{p^n q-1} V(1)\,.$$

Now consider the following composition of maps

$$\chi = \alpha_1 jj'\beta^2 \varpi_n \colon \Sigma^{p^n q-1} S \xrightarrow{\varpi_n} V(1) \xrightarrow{jj'\beta^2} \Sigma^{-2pq-q+2} S \xrightarrow{\alpha_1} \Sigma^{-2pq-2q+3} S\,.$$

Since $\varpi_n \in \pi_{p^n q-1} V(1)$ is represented up to nonzero scalar by

$$(i'i)_*(h_n) \in \mathrm{Ext}_A^{1,p^n q}(H^*V(1), \mathbb{Z}_p)$$

in the ASS, the map χ is represented up to nonzero scalar by

$$j_*\alpha_*i_*j_*j'_*\beta_*\beta_*i'_*i_*(h_n) \in \mathrm{Ext}_A^{4,p^nq+2pq+2q}(\mathbb{Z}_p, \mathbb{Z}_p)$$

in the ASS.

From the knowledge of Yoneda products we know that the composition

$$j_*j'_*\beta_*\beta_*i'_*i_*\colon \mathrm{Ext}_A^{0,*}(\mathbb{Z}_p, \mathbb{Z}_p) \xrightarrow{(i'i)_*} \mathrm{Ext}_A^{0,*}(H^*V(1), \mathbb{Z}_p) \xrightarrow{\beta_*}$$

$$\mathrm{Ext}_A^{s+1,*+(p+1)q+1}(H^*V(1), \mathbb{Z}_p) \xrightarrow{(jj'\beta)_*} \mathrm{Ext}_A^{s+2,*+2pq+q}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a multiplication up to nonzero scalar by

$$k_0 \in \mathrm{Ext}_A^{2,2pq+q}(\mathbb{Z}_p, \mathbb{Z}_p).$$

Hence, χ is represented up to nonzero scalar by

$$(j\alpha i)_*(k_0h_n) \in \mathrm{Ext}_A^{4,p^nq+2pq+2q}(\mathbb{Z}_p, \mathbb{Z}_p)$$

in the ASS.

By virtue of the fact that the homomorphism

$$(j\alpha i)_* = (\alpha_1)_*\colon \mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p) \to \mathrm{Ext}_A^{s+1,t+q}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a multiplication by $h_0 \in \mathrm{Ext}_A^{1,q}(\mathbb{Z}_p, \mathbb{Z}_p)$ (cf. [10]), we have that χ is represented up to nonzero scalar by

$$h_0k_0h_n \in \mathrm{Ext}_A^{4,p^nq+2pq+2q}(\mathbb{Z}_p, \mathbb{Z}_p)$$

in the ASS.

Just as the proofs of Lemma (3.1) and Theorem (3.8), we can easily show that

$$h_0k_0h_n \neq 0 \in \mathrm{Ext}_A^{4,p^nq+2pq+2q}(\mathbb{Z}_p, \mathbb{Z}_p).$$

Meanwhile, from [7] we have that

$$\mathrm{Ext}_A^{4-r,p^nq+2pq+2q-r+1}(\mathbb{Z}_p, \mathbb{Z}_p) = 0$$

for $r \geqslant 2$. Thus it follows that $h_0k_0h_n \neq 0 \in \mathrm{Ext}_A^{4,p^nq+2pq+2q}(\mathbb{Z}_p, \mathbb{Z}_p)$ cannot be hit by any differential in the ASS, and its corresponding homotopy element χ is nontrivial and of order p. This completes the proof of Theorem (1.2). □

To prove Theorem (1.3), it is equivalent to proving the following.

THEOREM (4.1). *Let $p \geqslant 7$, $n \geqslant 4$ and $0 \leqslant s < p-5$, then the product*

$$\tilde{\gamma}_{s+3}h_0k_0h_n \neq 0 \in \mathrm{Ext}_A^{s+7,q(p^n+(s+3)p^2+(s+4)p+(s+3))+s}(\mathbb{Z}_p, \mathbb{Z}_p)$$

is a permanent cycle in the ASS and converges to a nontrivial element $\gamma_{s+3}\alpha_1jj'\beta^2\varpi_n \in \pi_{q(p^n+(s+3)p^2+(s+4)p+(s+3))-7}S$ of order p, where $\tilde{\gamma}_{s+3}$ was given in [5] *and $\alpha_1 = i\alpha j$.*

Proof. From Theorem (1.2), the product $h_0k_0h_n \neq 0 \in \mathrm{Ext}_A^{4,q(p^n+2p+2)}(\mathbb{Z}_p, \mathbb{Z}_p)$ is a permanent cycle in the ASS and converges to a nontrivial element $\chi = \alpha_1jj'\beta^2\varpi_n \in \pi_{q(p^n+2p+2)-4}S$ of order p.

Now consider the following composition of maps

$$\lambda = \gamma_{s+3}\chi\colon \quad \Sigma^{q(p^n+2p+2)-4}S \xrightarrow{\chi} S \xrightarrow{\gamma_{s+3}} \Sigma^{-q((s+3)p^2+(s+2)p+(s+1))+3}S.$$

Since up to nonzero scalar the homotopy element χ is represented by $h_0k_0h_n \in \mathrm{Ext}_A^{4,q(p^2+2p+2)}(\mathbb{Z}_p,\mathbb{Z}_p)$ in the ASS respectively, then the above λ is represented up to nonzero scalar by the product

$$\tilde{\gamma}_{s+3}h_0k_0h_n \neq 0 \in \mathrm{Ext}_A^{s+7,p^nq+(s+3)p^2q+(s+4)pq+(s+3)q+s}(\mathbb{Z}_p,\mathbb{Z}_p)$$

in the ASS (see Theorem (3.8)).

Moreover, from Theorem (3.10) we have that $\tilde{\gamma}_{s+3}h_0k_0h_n$ cannot be hit by any differential in the ASS, and its corresponding homotopy element λ is nontrivial and of order p. This finishes the proof of Theorem (1.3). □

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NEW IMMERSIONS OF GRASSMANN MANIFOLDS

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ABSTRACT. Let $G_{k,n}$ denote the kn-dimensional Grassmann manifold of unoriented k-planes in $\mathbb{R}^{n+k}$. Monks [3] obtained a simple description of $H^*(G_{2,2^i-3};\mathbb{Z}_2)$ and used modified Postnikov towers to prove that for $i \geq 3$, $G_{2,2^i-3}$ immerses in $\mathbb{R}^{2^{i+2}-15}$. We give a finer modified Postnikov tower argument to prove that for $i \geq 4$, $G_{2,2^i-3}$ immerses in $\mathbb{R}^{2^{i+2}-17}$.

1. Statement of Results

Throughout this paper, $\mathbb{Z}_2$ will be the coefficient group of all cohomology groups. Let $G_{k,n}$ denote the kn-dimensional Grassmann manifold of unoriented k-planes in $\mathbb{R}^{n+k}$, w_i denote the i^{th} Stiefel-Whitney class of the canonical k-bundle γ over $BO(k) = G_{k,\infty}$, and $\overline{w}(G_{k,n})$ denote the total dual Stiefel-Whitney class of $G_{k,n}$.

THEOREM (1.1). *For $i \geq 4$, $G_{2,2^i-3}$ immerses in $\mathbb{R}^{2^{i+2}-17}$.*

This result is proved using modified Postnikov towers (MPTs), which were described by Gitler-Mahowald [2] and extended to $BO(k) \longrightarrow BO$ for odd k by Nussbaum [5], and improves by two dimensions the corresponding MPT immersion result of [3]. Cohen [1] claims to prove that any d-dimensional manifold immerses in $\mathbb{R}^{2d-\alpha(d)}$ where $\alpha(d)$ denotes the number of ones in the binary expansion of d. Cohen's claim implies that $G_{2,2^i-3}$ immerses in $\mathbb{R}^{2^{i+2}-i-11}$. Theorem (1.1) improves upon this for $i = 4$ and 5 by showing that $G_{2,13}$ and $G_{2,29}$ immerse in $\mathbb{R}^{47}$ and $\mathbb{R}^{111}$, respectively, whereas Cohen asserts that $G_{2,13}$ and $G_{2,29}$ immerse in $\mathbb{R}^{49}$ and $\mathbb{R}^{112}$, respectively. The best corresponding nonimmersion result is

THEOREM (1.2) (Tang). *For $i \geq 4$, $G_{2,2^i-3}$ does not immerse in $\mathbb{R}^{2^{i+2}-2i-17+\varepsilon}$ where*

$$\varepsilon = \begin{cases} 0 & \text{if } i \equiv 0 \mod 4, \\ 1 & \text{if } i \equiv 1 \mod 4, \\ 2 & \text{if } i \equiv 2, 3 \mod 4. \end{cases}$$

2. Proofs

We begin by recalling key results from [3].

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THEOREM (2.1) (Monks). *For $i \geq 3$,*

(1) *A vector space basis for $H^*\left(G_{2,2^i-3}\right)$ is the set of all monomials $w_2^b w_1^a$ such that $a < 2^i - 1$ and $b < 2^{i-1} - 1$. The product structure is completely determined by the relations $w_1^{2^i-1} = 0$ and $w_2^{2^{i-1}-1} = \sum_{j=0}^{i-2} w_2^{2^j-1} w_1^{2^i-2^{j+1}}$.*

(2) *The total dual Stiefel-Whitney class of $G_{2,2^i-3}$ is*

$$\overline{w}\left(G_{2,2^i-3}\right) = 1 + w_1 + w_2 + w_1^2 + w_1^3 + w_2 w_1^2 .$$

We also extend a technical lemma from [3] by expanding it to include formulas (5) through (8) below. Each of the eight formulas is a direct consequence of the Cartan formula and Wu's formula for the action of the Steenrod algebra on $H^*\left(BO\right) = \mathbb{Z}_2[w_1, w_2, \ldots]$, namely,

$$Sq^k w_m = \sum_{i=0}^{k} \binom{m+i-k-1}{i} w_{k-i} w_{m+i} .$$

LEMMA (2.2). *Let $m \geq 0$. Then in $H^*\left(BO(2)\right) = \mathbb{Z}_2[w_1, w_2]$ we have*

(1) $Sq^1 w_1^m = \begin{cases} 0 & \textit{if } m \textit{ is even}, \\ w_1^{m+1} & \textit{if } m \textit{ is odd}. \end{cases}$

(2) $Sq^1 w_2^m = \begin{cases} 0 & \textit{if } m \textit{ is even}, \\ w_2^m w_1 & \textit{if } m \textit{ is odd}. \end{cases}$

(3) $Sq^2 w_1^m = \begin{cases} 0 & \textit{if } m \equiv 0, 1 \mod 4, \\ w_1^{m+2} & \textit{if } m \equiv 2, 3 \mod 4. \end{cases}$

(4) $Sq^2 w_2^m = \begin{cases} 0 & \textit{if } m \equiv 0 \mod 4, \\ w_2^{m+1} & \textit{if } m \equiv 1 \mod 4, \\ w_2^m w_1^2 & \textit{if } m \equiv 2 \mod 4, \\ w_2^{m+1} + w_2^m w_1^2 & \textit{if } m \equiv 3 \mod 4. \end{cases}$

(5) $Sq^3 w_1^m = \begin{cases} 0 & \textit{if } m \equiv 0, 1, 2 \mod 4, \\ w_1^{m+3} & \textit{if } m \equiv 3 \mod 4. \end{cases}$

(6) $Sq^3 w_2^m = \begin{cases} 0 & \textit{if } m \equiv 0, 1, 2 \mod 4, \\ w_2^m w_1^3 & \textit{if } m \equiv 3 \mod 4. \end{cases}$

(7) $Sq^4 w_1^m = \begin{cases} 0 & \textit{if } m \equiv 0, 1, 2, 3 \mod 8, \\ w_1^{m+4} & \textit{if } m \equiv 4, 5, 6, 7 \mod 8. \end{cases}$

(8) $Sq^4 w_2^m = \begin{cases} 0 & \textit{if } m \equiv 0, 1 \mod 8, \\ w_2^{m+2} & \textit{if } m \equiv 2 \mod 8, \\ w_2^{m+2} + w_2^{m+1} w_1^2 & \textit{if } m \equiv 3 \mod 8, \\ w_2^m w_1^4 & \textit{if } m \equiv 4, 5 \mod 8, \\ w_2^{m+2} + w_2^m w_1^4 & \textit{if } m \equiv 6 \mod 8, \\ w_2^{m+2} + w_2^{m+1} w_1^2 + w_2^m w_1^4 & \textit{if } m \equiv 7 \mod 8. \end{cases}$

LEMMA (2.3). *For $i \geq 3$, $Sq^4(w_2^{2^{i-1}-3} w_1^{2^i-5}) = w_2^{2^{i-1}-2} w_1^{2^i-3}$ and $Sq^4(w_2^{2^{i-1}-4} w_1^{2^i-3}) = 0$ in $H^*(G_{2,2^i-3})$.*

Proof. Applying the Cartan formula, the previous lemma, and the relation $w_1^{2^i-1}=0$ yields

$$\begin{aligned} Sq^4(w_2^{2^{i-1}-3}w_1^{2^i-5}) &= Sq^4w_2^{2^{i-1}-3}\cdot w_1^{2^i-5} + Sq^3w_2^{2^{i-1}-3}\cdot Sq^1w_1^{2^i-5} \\ &\quad + Sq^2w_2^{2^{i-1}-3}\cdot Sq^2w_1^{2^i-5} + Sq^1w_2^{2^{i-1}-3}\cdot Sq^3w_1^{2^i-5} \\ &\quad + w_2^{2^{i-1}-3}\cdot Sq^4w_1^{2^i-5} \\ &= 0+0+w_2^{2^{i-1}-2}w_1^{2^i-3}+0+0 = w_2^{2^{i-1}-2}w_1^{2^i-3}. \end{aligned}$$

Likewise, applying the Cartan formula, the previous lemma, and the relation $w_1^{2^i-1}=0$ yields

$$\begin{aligned} Sq^4(w_2^{2^{i-1}-4}w_1^{2^i-3}) &= Sq^4w_2^{2^{i-1}-4}\cdot w_1^{2^i-3} + Sq^3w_2^{2^{i-1}-4}\cdot Sq^1w_1^{2^i-3} \\ &\quad + Sq^2w_2^{2^{i-1}-4}\cdot Sq^2w_1^{2^i-3} + Sq^1w_2^{2^{i-1}-4}\cdot Sq^3w_1^{2^i-3} \\ &\quad + w_2^{2^{i-1}-4}\cdot Sq^4w_1^{2^i-3} \\ &= 0+0+0+0+0 \\ &= 0. \end{aligned}$$

□

THEOREM (1.1). *For $i\geq 4$, $G_{2,2^i-3}$ immerses in $\mathbb{R}^{2^{i+2}-17}$.*

Proof. To simplify the notation we let $m=2^{i-2}-2$. We will use a $(8m+10)$-MPT

$$E_3 \longrightarrow E_2 \longrightarrow E_1 \longrightarrow BO$$

of $BO(8m+5)\longrightarrow BO$ and show that the stable normal bundle $\nu\colon G_{2,4m+5}\longrightarrow BO$ lifts to E_3 and, hence, lifts to $BO(8m+5)$. We note that since $i\geq 4$, $8m+10\leq 2(8m+5)-2$ as required by Nussbaum [5].

Our $(8m+10)$-MPT is

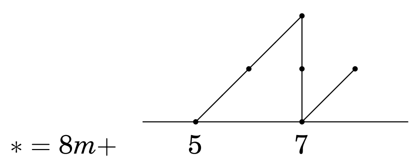
$$\begin{tikzcd} K_2 \arrow[dr] & & \\ & E_3 \arrow[d, "p_3"] & \\ K_1 \arrow[dr] & & \\ & E_2 \arrow[r, "k^2"] \arrow[d, "p_2"] & BK_2 \\ K_0 \arrow[dr] & & \\ & E_1 \arrow[r, "k^1"] \arrow[d, "p_1"] & BK_1 \\ & BO \arrow[r, "k^0"] & BK_0 \end{tikzcd}$$

where

$$
\begin{aligned}
K_0 &= K(\mathbb{Z}_2, 8m+5) \times K(\mathbb{Z}_2, 8m+7),\\
K_1 &= K(\mathbb{Z}_2, 8m+6) \times K(\mathbb{Z}_2, 8m+7) \times K(\mathbb{Z}_2, 8m+8),\\
K_2 &= K(\mathbb{Z}_2, 8m+7),\\
BK_s &= \prod_{i_s=1}^{m_s} K(\mathbb{Z}_2, c_{i_s}+1) \text{ if } K_s = \prod_{i_s=1}^{m_s} K(\mathbb{Z}_2, c_{i_s}),
\end{aligned}
$$

and k^0_{8m+6} and k^0_{8m+8} are the Stiefel-Whitney classes w_{8m+6} and w_{8m+8}, respectively.

Each vertical map in the above diagram is part of a fiber sequence preceded by the map from the fiber represented by a diagonal arrow, and followed by the classifying map represented by a horizontal arrow. The information of the diagram can be obtained from the Adams spectral sequence (ASS) of the stunted real projective space P_{8m+5}, which is, in the stable range, the fiber of $BO(8m+5) \longrightarrow BO$. The following is the ASS chart for $\pi_*(P_{8m+5})$ in dimensions $\leq 8m+9$, (see [4]):



ASS chart for $\pi_*(P_{8m+5})$

From Theorem (2.1), $w(\nu) = 1+w_1+w_2+w_1^2+w_1^3+w_2w_1^2$. Thus, $\nu^*: H^*(BO) \to H^*(G_{2,4m+5})$ sends w_{8m+6} and w_{8m+8} to 0 and the map $\nu\colon G_{2,4m+5} \to BO$ lifts to a map $l_1\colon G_{2,4m+5} \to E_1$. The lifting l_1 is not unique: given any map $\alpha\colon G_{2,4m+5} \to K_0$, we obtain another lifting, $l_1'\colon G_{2,4m+5} \to E_1$, as the composite

$$G_{2,4m+5} \xrightarrow{d} G_{2,4m+5} \times G_{2,4m+5} \xrightarrow{\alpha\times l_1} K_0 \times E_1 \xrightarrow{\mu_1} E_1,$$

where d is the diagonal map. However, all liftings $l_1'\colon G_{2,4m+5} \longrightarrow E_1$ can be obtained from l_1 as we vary α through the homotopy classes of maps $G_{2,4m+5} \longrightarrow K_0$ (see [2, p 95]). This process is referred to as "varying l_1 through the fiber K_0." Since a lifting $l_1\colon G_{2,4m+5} \longrightarrow E_1$ exists, we search for a lifting $l_2\colon G_{2,4m+5} \longrightarrow E_2$ by testing whether or not there is a lifting $G_{2,4m+5} \longrightarrow E_1$ whose induced map sends k^1_{8m+7}, k^1_{8m+8}, and k^1_{8m+9} to 0. We note that from Theorem (2.1) it follows that $l_1^*k^1_{8m+7} \in \text{span}\left\{w_2^{2^{i-1}-3}w_1^{2^i-3}, w_2^{2^{i-1}-2}w_1^{2^i-5}\right\}$, $l_1^*k^1_{8m+8} \in \text{span}\left\{w_2^{2^{i-1}-3}w_1^{2^i-2}, w_2^{2^{i-1}-2}w_1^{2^i-4}\right\}$, $l_1^*k^1_{8m+9} \in \text{span}\left\{w_2^{2^{i-1}-2}w_1^{2^i-3}\right\}$. We will show that there does exist a lifting l_2, and we will consider its variations through the fiber K_1 of $p_2\colon E_2 \longrightarrow E_1$ in searching for our desired lifting $l_3\colon G_{2,4m+5} \longrightarrow E_3$. We note that $l_2^*k^2_{8m+8} \in \text{span}\left\{w_2^{2^{i-1}-3}w_1^{2^i-2}, w_2^{2^{i-1}-2}w_1^{2^i-4}\right\}$.
$\square$

In order to determine the indeterminacy for lifting $G_{2,4m+5}$ in this MPT, we must know the relations which give rise to the k-invariants. These are computed by the method initiated in [2] and utilized in many subsequent papers by D. M. Davis and also in papers of K. Y. Lam and/or D. Randall. It is a matter of building a minimal resolution using Massey-Peterson algebras. The relations for our $(8m+10)$-MPT are given in the table below.

w_{8m+6}	
w_{8m+8}	
k^1_{8m+7}:	$(Sq^2+w_2)w_{8m+6}$
k^1_{8m+8}:	$Sq^1 w_{8m+8} + (Sq^2+w_2+w_1^2)Sq^1 w_{8m+6}$
k^1_{8m+9}:	$Sq^2 w_{8m+8} + (Sq^4+w_4)w_{8m+6}$
k^2_{8m+8}:	$Sq^1 k^1_{8m+8} + (Sq^2+w_2)k^1_{8m+7}$

Since $w(\nu) = 1 + w_1 + w_2 + w_1^2 + w_1^3 + w_2 w_1^2$ we note that $w_1(\nu) = w_1$, $w_2(\nu) = w_2 + w_1^2$, and $w_4(\nu) = w_2 w_1^2$.

LEMMA (2.4). *The induced map of any lifting $l_1\colon G_{2,4m+5} \longrightarrow E_1$ must send k^1_{8m+7} to either 0 or $w_2^{2^{i-1}-3} w_1^{2^i-3}$.*

Proof. The defining relation $Sq^1 k^1_{8m+8} + (Sq^2+w_2)k^1_{8m+7} = 0$ for k^2_{8m+8} implies that $Sq^1(l_1^* k^1_{8m+8}) + (Sq^2+w_2+w_1^2)(l_1^* k^1_{8m+7}) = 0$. Since

$$Sq^1\left(w_2^{2^{i-1}-3} w_1^{2^i-2}\right) = w_2^{2^{i-1}-3} w_1^{2^i-1} + 0 = 0$$

and

$$Sq^1\left(w_2^{2^{i-1}-2} w_1^{2^i-4}\right) = 0 + 0 = 0\,,$$

$Sq^1(l_1^* k^1_{8m+8}) = 0$ and, hence, $(Sq^2+w_2+w_1^2)(l_1^* k^1_{8m+7}) = 0$.

Then, since

$$\begin{aligned}(Sq^2+w_2+w_1^2)(w_2^{2^{i-1}-3} w_1^{2^i-3}) &= w_2^{2^{i-1}-2} w_1^{2^i-3} + w_2^{2^{i-1}-3} w_1^{2^i-1} + 0 \\ &\quad + w_2^{2^{i-1}-2} w_1^{2^i-3} + w_2^{2^{i-1}-3} w_1^{2^i-1} \\ &= 0\end{aligned}$$

and

$$\begin{aligned}(Sq^2+w_2+w_1^2)(w_2^{2^{i-1}-2} w_1^{2^i-5}) &= w_2^{2^{i-1}-2} w_1^{2^i-3} + 0 + w_2^{2^{i-1}-2} w_1^{2^i-3} \\ &\quad + w_2^{2^{i-1}-1} w_1^{2^i-5} + w_2^{2^{i-1}-2} w_1^{2^i-3} \\ &= w_2^{2^{i-1}-2} w_1^{2^i-3} + w_2^{2^{i-1}-1} w_1^{2^i-5} \\ &= w_2^{2^{i-1}-2} w_1^{2^i-3} + \left(\sum_{j=0}^{i-2} w_2^{2^j-1} w_1^{2^i-2^{j+1}}\right) w_1^{2^i-5} \\ &= w_2^{2^{i-1}-2} w_1^{2^i-3} + \left(w_1^{2^i-1}\right) \sum_{j=0}^{i-2} w_2^{2^j-1} w_1^{2^i-2^{j+1}-4} \\ &= w_2^{2^{i-1}-2} w_1^{2^i-3}\,,\end{aligned}$$

in order for $(Sq^2+w_2+w_1^2)(l_1^* k^1_{8m+7})=0$ to hold, $l_1^* k^1_{8m+7}$ must be 0 or $w_2^{2^{i-1}-3}w_1^{2^i-3}$. □

Based on the correspondence

$$[G_{2,4m+5}, K_0] \longleftrightarrow [G_{2,4m+5}, K(\mathbb{Z}_2, 8m+5)] \times [G_{2,4m+5}, K(\mathbb{Z}_2, 8m+7)],$$

we begin by considering (homotopy classes of) maps

$$G_{2,4m+5} \longrightarrow K(\mathbb{Z}_2, 8m+j)$$

that correspond to basis elements of $H^{8m+j}(G_{2,4m+5})$, for $j=5$ and 7.

Varying l_1 by the map $\alpha\colon G_{2,4m+5} \longrightarrow K_0$ with

$$\alpha^*(\pi_1^*(\iota_{8m+5})) = w_2^{2^{i-1}-3}w_1^{2^i-5} \in H^{8m+5}(G_{2,4m+5})$$

and $\alpha^*(\pi_2^*(\iota_{8m+7}))=0$, we have a lifting l_1' such that

$$\begin{aligned}
(l_1')^*(k^1_{8m+7}) &= (\alpha\times l_1)^*\mu_1^*(k^1_{8m+7})\\
&= (\alpha\times l_1)^*(1\otimes k^1_{8m+7} + Sq^2\iota_{8m+5}\otimes 1 + \iota_{8m+5}\otimes w_2)\\
&= l_1^*(k^1_{8m+7}) + Sq^2(w_2^{2^{i-1}-3}w_1^{2^i-5}) + w_2^{2^{i-1}-3}w_1^{2^i-5}\cdot(w_2+w_1^2)\\
&= l_1^*(k^1_{8m+7}) + w_2^{2^{i-1}-2}w_1^{2^i-5} + w_2^{2^{i-1}-3}w_1^{2^i-3} + w_2^{2^{i-1}-3}w_1^{2^i-3}\\
&\quad + w_2^{2^{i-1}-2}w_1^{2^i-5} + w_2^{2^{i-1}-3}w_1^{2^i-3}\\
&= l_1^*(k^1_{8m+7}) + w_2^{2^{i-1}-3}w_1^{2^i-3},
\end{aligned}$$

$$\begin{aligned}
(l_1')^*(k^1_{8m+8}) &= (\alpha\times l_1)^*\mu_1^*(k^1_{8m+8})\\
&= (\alpha\times l_1)^*(1\otimes k^1_{8m+8} + Sq^2Sq^1\iota_{8m+5}\otimes 1 + Sq^1\iota_{8m+5}\otimes(w_2+w_1^2)\\
&\quad + Sq^1\iota_{8m+7}\otimes 1)\\
&= l_1^*(k^1_{8m+8}) + Sq^2Sq^1(w_2^{2^{i-1}-3}w_1^{2^i-5}) + Sq^1(w_2^{2^{i-1}-3}w_1^{2^i-5})\cdot w_2\\
&= l_1^*(k^1_{8m+8}) + Sq^2(w_2^{2^{i-1}-3}w_1^{2^i-4} + w_2^{2^{i-1}-3}w_1^{2^i-4})\\
&\quad + (w_2^{2^{i-1}-3}w_1^{2^i-4} + w_2^{2^{i-1}-3}w_1^{2^i-4})\cdot w_2\\
&= l_1^*(k^1_{8m+8}),
\end{aligned}$$

and

$$\begin{aligned}
(l_1')^*(k^1_{8m+9}) &= (\alpha\times l_1)^*\mu_1^*(k^1_{8m+9})\\
&= (\alpha\times l_1)^*(1\otimes k^1_{8m+9} + Sq^4\iota_{8m+5}\otimes 1 + \iota_{8m+5}\otimes w_4 + Sq^2\iota_{8m+7}\otimes 1)\\
&= l_1^*(k^1_{8m+9}) + Sq^4(w_2^{2^{i-1}-3}w_1^{2^i-5}) + w_2^{2^{i-1}-3}w_1^{2^i-5}\cdot w_2w_1^2\\
&= l_1^*(k^1_{8m+9}) + w_2^{2^{i-1}-2}w_1^{2^i-3} + w_2^{2^{i-1}-2}w_1^{2^i-3}\\
&= l_1^*(k^1_{8m+9}).
\end{aligned}$$

Varying l_1 by the map $\alpha\colon G_{2,4m+5} \longrightarrow K_0$ with

$$\alpha^*(\pi_1^*(\iota_{8m+5})) = w_2^{2^{i-1}-4}w_1^{2^i-3} \in H^{8m+5}(G_{2,4m+5})$$

and $\alpha^*(\pi_2^*(\iota_{8m+7}))=0$, we have a lifting l_1' such that

$$\begin{aligned}(l_1')^*(k^1_{8m+7}) &= (\alpha\times l_1)^*\mu_1^*(k^1_{8m+7})\\ &= (\alpha\times l_1)^*(1\otimes k^1_{8m+7}+Sq^2\iota_{8m+5}\otimes 1+\iota_{8m+5}\otimes w_2)\\ &= l_1^*(k^1_{8m+7})+Sq^2(w_2^{2^{i-1}-4}w_1^{2^i-3})+w_2^{2^{i-1}-4}w_1^{2^i-3}\cdot(w_2+w_1^2)\\ &= l_1^*(k^1_{8m+7})+0+0+0+w_2^{2^{i-1}-3}w_1^{2^i-3}+w_2^{2^{i-1}-4}w_1^{2^i-1}\\ &= l_1^*(k^1_{8m+7})+w_2^{2^{i-1}-3}w_1^{2^i-3},\end{aligned}$$

$$\begin{aligned}(l_1')^*(k^1_{8m+8}) &= (\alpha\times l_1)^*\mu_1^*(k^1_{8m+8})\\ &= (\alpha\times l_1)^*(1\otimes k^1_{8m+8}+Sq^2Sq^1\iota_{8m+5}\otimes 1+Sq^1\iota_{8m+5}\otimes(w_2+w_1^2)\\ &\quad +Sq^1\iota_{8m+7}\otimes 1)\\ &= l_1^*(k^1_{8m+8})+Sq^2Sq^1(w_2^{2^{i-1}-4}w_1^{2^i-3})+Sq^1(w_2^{2^{i-1}-4}w_1^{2^i-3})\cdot w_2\\ &= l_1^*(k^1_{8m+8})+Sq^2(0+w_2^{2^{i-1}-4}w_1^{2^i-2})+(0+w_2^{2^{i-1}-4}w_1^{2^i-2})\cdot w_2\\ &= l_1^*(k^1_{8m+8})+0+0+w_2^{2^{i-1}-4}w_1^{2^i}+w_2^{2^{i-1}-3}w_1^{2^i-2}\\ &= l_1^*(k^1_{8m+8})+w_2^{2^{i-1}-3}w_1^{2^i-2},\end{aligned}$$

and

$$\begin{aligned}(l_1')^*(k^1_{8m+9}) &= (\alpha\times l_1)^*\mu_1^*(k^1_{8m+9})\\ &= (\alpha\times l_1)^*(1\otimes k^1_{8m+9}+Sq^4\iota_{8m+5}\otimes 1+\iota_{8m+5}\otimes w_4+Sq^2\iota_{8m+7}\otimes 1)\\ &= l_1^*(k^1_{8m+9})+Sq^4(w_2^{2^{i-1}-4}w_1^{2^i-3})+w_2^{2^{i-1}-4}w_1^{2^i-3}\cdot w_2w_1^2\\ &= l_1^*(k^1_{8m+9})+0+w_2^{2^{i-1}-3}w_1^{2^i-1}\\ &= l_1^*(k^1_{8m+9}).\end{aligned}$$

Varying l_1 by the map $\alpha\colon G_{2,4m+5}\longrightarrow K_0$ with

$$\alpha^*(\pi_2^*(\iota_{8m+7}))=w_2^{2^{i-1}-3}w_1^{2^i-3}\in H^{8m+7}(G_{2,4m+5})$$

and $\alpha^*(\pi_1^*(\iota_{8m+5}))=0$, we have a lifting l_1' such that

$$\begin{aligned}(l_1')^*(k^1_{8m+7}) &= (\alpha\times l_1)^*\mu_1^*(k^1_{8m+7})\\ &= (\alpha\times l_1)^*(1\otimes k^1_{8m+7}+Sq^2\iota_{8m+5}\otimes 1+\iota_{8m+5}\otimes w_2)\\ &= l_1^*(k^1_{8m+7}),\end{aligned}$$

$$\begin{aligned}(l_1')^*(k^1_{8m+8}) &= (\alpha\times l_1)^*\mu_1^*(k^1_{8m+8})\\ &= (\alpha\times l_1)^*(1\otimes k^1_{8m+8}+Sq^2Sq^1\iota_{8m+5}\otimes 1+Sq^1\iota_{8m+5}\otimes(w_2+w_1^2)\\ &\qquad +Sq^1\iota_{8m+7}\otimes 1)\\ &= l_1^*(k^1_{8m+8})+Sq^1(w_2^{2^{i-1}-3}w_1^{2^i-3})\\ &= l_1^*(k^1_{8m+8})+w_2^{2^{i-1}-3}w_1^{2^i-2}+w_2^{2^{i-1}-3}w_1^{2^i-2}\\ &= l_1^*(k^1_{8m+8}),\end{aligned}$$

and

$$\begin{aligned}(l_1')^*(k^1_{8m+9}) &= (\alpha\times l_1)^*\mu_1^*(k^1_{8m+9})\\ &= (\alpha\times l_1)^*(1\otimes k^1_{8m+9} + Sq^4\iota_{8m+5}\otimes 1 + \iota_{8m+5}\otimes w_4 + Sq^2\iota_{8m+7}\otimes 1)\\ &= l_1^*(k^1_{8m+9}) + Sq^2(w_2^{2^{i-1}-3}w_1^{2^i-3})\\ &= l_1^*(k^1_{8m+9}) + w_2^{2^{i-1}-2}w_1^{2^i-3} + w_2^{2^{i-1}-3}w_1^{2^i-1} + 0\\ &= l_1^*(k^1_{8m+9}) + w_2^{2^{i-1}-2}w_1^{2^i-3}\,.\end{aligned}$$

Varying l_1 by the map $\alpha\colon G_{2,4m+5}\longrightarrow K_0$ with

$$\alpha^*(\pi_2^*(\iota_{8m+7})) = w_2^{2^{i-1}-2}w_1^{2^i-5}\in H^{8m+7}(G_{2,4m+5})$$

and $\alpha^*(\pi_1^*(\iota_{8m+5}))=0$, we have a lifting l_1' such that

$$\begin{aligned}(l_1')^*(k^1_{8m+7}) &= (\alpha\times l_1)^*\mu_1^*(k^1_{8m+7})\\ &= (\alpha\times l_1)^*(1\otimes k^1_{8m+7} + Sq^2\iota_{8m+5}\otimes 1 + \iota_{8m+5}\otimes w_2)\\ &= l_1^*(k^1_{8m+7}),\end{aligned}$$

$$\begin{aligned}(l_1')^*(k^1_{8m+8}) &= (\alpha\times l_1)^*\mu_1^*(k^1_{8m+8})\\ &= (\alpha\times l_1)^*(1\otimes k^1_{8m+8} + Sq^2Sq^1\iota_{8m+5}\otimes 1 + Sq^1\iota_{8m+5}\otimes(w_2+w_1^2)\\ &\quad + Sq^1\iota_{8m+7}\otimes 1)\\ &= l_1^*(k^1_{8m+8}) + Sq^1(w_2^{2^{i-1}-2}w_1^{2^i-5})\\ &= l_1^*(k^1_{8m+8}) + 0 + w_2^{2^{i-1}-2}w_1^{2^i-4}\\ &= l_1^*(k^1_{8m+8}) + w_2^{2^{i-1}-2}w_1^{2^i-4},\end{aligned}$$

and

$$\begin{aligned}(l_1')^*(k^1_{8m+9}) &= (\alpha\times l_1)^*\mu_1^*(k^1_{8m+9})\\ &= (\alpha\times l_1)^*(1\otimes k^1_{8m+9} + Sq^4\iota_{8m+5}\otimes 1 + \iota_{8m+5}\otimes w_4 + Sq^2\iota_{8m+7}\otimes 1)\\ &= l_1^*(k^1_{8m+9}) + Sq^2(w_2^{2^{i-1}-2}w_1^{2^i-5})\\ &= l_1^*(k^1_{8m+9}) + w_2^{2^{i-1}-2}w_1^{2^i-3} + 0 + w_2^{2^{i-1}-2}w_1^{2^i-3}\\ &= l_1^*(k^1_{8m+9})\,.\end{aligned}$$

Coupled with Lemma (2.4) the above analysis of indeterminancies demonstrates that there exists a lifting $G_{2,4m+5}\longrightarrow E_1$ whose induced map sends k^1_{8m+7}, k^1_{8m+8}, and k^1_{8m+9} to 0. Hence, there is a lifting $l_2\colon G_{2,4m+5}\longrightarrow E_2$ and we now vary it through the fiber K_1 of $p_2\colon E_2\longrightarrow E_1$.

We first vary l_2 by the map $\alpha\colon G_{2,4m+5}\longrightarrow K_1$ characterized by the property that

$$\alpha^*(\pi_1^*(\iota_{8m+6})) = w_2^{2^{i-1}-4}w_1^{2^i-2}\in H^{8m+6}(G_{2,4m+5})$$

and $\alpha^*(\pi_2^*(\iota_{8m+7}))$ and $\alpha^*(\pi_3^*(\iota_{8m+8}))$ equal 0. With $\mu_2\colon K_1\times E_2\longrightarrow E_2$ denoting the action map of the fiber K_1 on the total space E_2 of the principal fibration

p_2, we then have a lifting $l_2'\colon G_{2,4m+5} \longrightarrow E_2$ such that

$$\begin{aligned}(l_2')^*(k^2_{8m+8}) &= (\alpha\times l_2)^*\mu_2^*(k^2_{8m+8})\\ &= (\alpha\times l_2)^*(1\otimes k^2_{8m+8} + Sq^1\iota_{8m+7}\otimes 1 + Sq^2\iota_{8m+6}\otimes 1\\ &+ \iota_{8m+6}\otimes w_2)\\ &= l_2^*(k^2_{8m+8}) + Sq^2(w_2^{2^{i-1}-4}w_1^{2^i-2}) + w_2^{2^{i-1}-4}w_1^{2^i-2}\cdot(w_2+w_1^2)\\ &= l_2^*(k^2_{8m+8}) + 0 + 0 + w_2^{2^{i-1}-4}w_1^{2^i} + w_2^{2^{i-1}-3}w_1^{2^i-2} + w_2^{2^{i-1}-4}w_1^{2^i}\\ &= l_2^*(k^2_{8m+8}) + w_2^{2^{i-1}-3}w_1^{2^i-2}.\end{aligned}$$

Varying l_2 by the map $\alpha\colon G_{2,4m+5} \longrightarrow K_1$ with

$$\alpha^*(\pi_2^*(\iota_{8m+7})) = w_2^{2^{i-1}-2}w_1^{2^i-5} \in H^{8m+7}(G_{2,4m+5})$$

and $\alpha^*(\pi_1^*(\iota_{8m+6})) = \alpha^*(\pi_3^*(\iota_{8m+8})) = 0$, we have a lifting l_2' such that

$$\begin{aligned}(l_2')^*(k^2_{8m+8}) &= (\alpha\times l_2)^*\mu_2^*(k^2_{8m+8})\\ &= (\alpha\times l_2)^*(1\otimes k^2_{8m+8} + Sq^1\iota_{8m+7}\otimes 1 + Sq^2\iota_{8m+6}\otimes 1 + \iota_{8m+6}\otimes w_2)\\ &= l_2^*(k^2_{8m+8}) + Sq^1(w_2^{2^{i-1}-2}w_1^{2^i-5})\\ &= l_2^*(k^2_{8m+8}) + 0 + w_2^{2^{i-1}-2}w_1^{2^i-4}\\ &= l_2^*(k^2_{8m+8}) + w_2^{2^{i-1}-2}w_1^{2^i-4}.\end{aligned}$$

Thus there exists a lifting $G_{2,4m+5} \longrightarrow BO(8m+5)$ and we have our desired immersion. □

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