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ON POSITIVE INTEGERS n SUCH THAT $\phi(1) + \phi(2) + \cdots + \phi(n)$ IS A SQUARE

FLORIAN LUCA AND AYYADURAI SANKARANARAYANAN

ABSTRACT. In this paper, we show that the set of positive integers n such that the sum of the first n values of the Euler function is a square is of asymptotic density zero.

1. Introduction

Let $\phi(n)$ be the Euler function. Let

$$a_n = \sum_{1 \le m \le n} \phi(m)$$

and consider the set $\mathcal{S} = \{n : a_n \text{ is a perfect square}\}$. One checks that

$$\mathcal{S} = \{1, 3, 14, 32, 54, 1458, 3765, 5343, 10342, 57918, 72432, 134072, \ldots\}.$$

We conjecture that $\mathcal{S}$ is infinite but we do not know how to prove this. In this note, we show that $\mathcal{S}$ is of asymptotic density zero. For a set $\mathcal{A}$ of positive integers and a real number x we put $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$. We then have the following result.

THEOREM (1.1). *The estimate* $\#\mathcal{S}(x) \le c_0 x/(\log x)^{.0003}$ *holds for all* $x > 2$ *with some positive constant* c_0.

A result with a similar flavor was proved in [2], where it was shown that the set of n such that the sum of the first n primes is a square is of asymptotic density zero. In what follows, we use the Landau symbols O and o as well as the Vinogradov symbols $\ll$ and $\gg$ with their usual meaning. Recall that for positive functions $f(x)$ and $g(x)$ the notations $f(x) = O(g(x))$, $f(x) \ll g(x)$ and $g(x) \gg f(x)$ are all equivalent to the fact that $f(x) < Kg(x)$ holds for some constant K and for all sufficiently large x, while $f(x) = o(g(x))$ means that $f(x)/g(x) \to 0$ as $x \to \infty$. We use p and q with or without subscripts to denote prime numbers.

2. Proof of the Theorem

Let x be a large positive real number and consider the set of $n \in \mathcal{S}(x)$. We also assume that our number n, if it exists, exceeds $x/\log x$, since there are at most $O(x/\log x)$ positive integers n failing this condition. Let $\varepsilon > 0$ be some small number to be fixed later. We now discard some integers.

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Let $P(n)$ be the largest prime factor of n. Put $y = \exp(\log x / \log\log x)$ and

$$\mathcal{S}_0 = \{n \le x : P(n) \le y\}.$$

From known results concerning smooth numbers (see, for example, Section III.5.4 in [3]), we have that

$$\#\mathcal{S}_0 = \Psi(x, y) \ll x \exp(-u/2), \quad \text{where } u = \frac{\log x}{\log y}.$$

Since for us $u = \log\log x$, we get that

$$\#\mathcal{S}_0 \ll \frac{x}{(\log x)^{\alpha_0}}, \tag{2.1}$$

where $\alpha_0 = 1/2$.

Let $\omega(n)$ be the number of distinct prime factors of n. Let

$$\mathcal{S}_1 = \{n \le x : \omega(n)/\log\log x \notin [1-\varepsilon, 2]\}.$$

It follows easily from the well-known Hardy-Ramanujan inequalities (see [1]) that

$$\#\mathcal{S}_1 \ll \frac{x(\log\log x)^2}{(\log x)^{\alpha_1}}, \tag{2.2}$$

where

$$\alpha_1 = \min\{1 - (1-\varepsilon)\log\big(e/(1-\varepsilon)\big), 1 - 2\log\big(e/2\big)\}. \tag{2.3}$$

We shall choose $\varepsilon > 0$ small enough so that $\alpha_1 = 1 - (1-\varepsilon)\log(e/(1-\varepsilon))$.

Let $\mathcal{P}_i = \{p : p \equiv 1 \pmod{2^i}\}$. Clearly, $\mathcal{P}_1$ is the set of all primes except 2. For $i \in \{2, 3, 4, 5\}$ we put $\omega_i(n)$ for the number of distinct prime factors of n in $\mathcal{P}_i$ and we define

$$\mathcal{S}_i = \{n \le x : \omega_i(n) \le 2^{-(i-1)}(1-\varepsilon)\log\log x\}.$$

We show that

$$\#\mathcal{S}_i \ll \frac{x(\log\log x)^3}{(\log x)^{\alpha_i}},$$

where $\alpha_i = 2^{-(i-1)}(1-(1-\varepsilon)\log(e/(1-\varepsilon)))$. Indeed, fix $i \in \{2, 3, 4, 5\}$. Assume that $n \in \mathcal{S}_i \backslash (\mathcal{S}_0 \cup \mathcal{S}_1)$ and write $n = Pm$, where $P = P(n)$. Fix m. Then the number of $n \le x$ with this property is

$$\le \pi\left(\frac{x}{m}\right) \ll \frac{x}{m\log(x/m)} \ll \frac{x}{m\log y} = \frac{x\log\log x}{m\log x},$$

where we used the fact that $x/m \ge P(n) \ge y$ because $n \notin \mathcal{S}_0$. Since $n \in \mathcal{S}_i \backslash \mathcal{S}_1$, it follows that $m = a_i b_i$, where a_i, b_i are coprime, a_i has $k_i \le K_i$, where $K_i = \lfloor 2^{-(i-1)}(1-\varepsilon)\log\log x\rfloor$, prime factors all congruent to 1 modulo 2^i, and b_i has $\ell_i \le L = \lfloor 2\log\log x\rfloor$ prime factors none of them congruent to $1 \pmod{2^i}$.

Fixing k_i and ℓ_i, we get that the number of n in this category is

$$\begin{aligned}
&\le \frac{x(\log\log x)}{\log x}\left(\sum_{\substack{a_i\le x\\ \omega(a_i)=\omega_i(a_i)=k_i}}\frac{1}{a_i}\right)\left(\sum_{\substack{b_i\le x\\ \omega(b_i)=\ell_i,\ \omega_i(b_i)=0}}\frac{1}{b_i}\right)\\
&\le \frac{x\log\log x}{\log x}\frac{1}{k_i!}\left(\sum_{\substack{p\le x\\ p\in\mathcal{P}_i}}\sum_{t\ge 1}\frac{1}{p^t}\right)^{k_i}\frac{1}{\ell_i!}\left(\sum_{\substack{p\le x\\ p\notin\mathcal{P}_i}}\sum_{t\ge 1}\frac{1}{p^t}\right)^{\ell_i}\\
&\le \frac{x\log\log x}{\log x}\left(\frac{e2^{-(i-1)}\log\log x+O(1)}{k_i}\right)^{k_i}\left(\frac{e(1-2^{-(i-1)})\log\log x+O(1)}{\ell_i}\right)^{\ell_i},
\end{aligned}$$

where we used Stirling's inequality $k!>(k/e)^k$, and the known fact that if a and b are fixed coprime positive integers then the estimate

$$\sum_{\substack{p\le z\\ p\equiv a \pmod b}}\frac{1}{p}=\frac{\log\log z}{\phi(b)}+O_b(1)$$

holds for all $z\ge 2$ with $a=1,\ b=2^i$. Since the pair (k_i,ℓ_i) can be chosen in at most $K_iL=O((\log\log x)^2)$ ways, we get that

$$\begin{aligned}
\#\mathcal{S}_i \ll \frac{x(\log\log x)^3}{\log x}&\max_{k_i\le K_i}\left(\frac{2^{-(i-1)}e\log\log x+O(1)}{k_i}\right)^{k_i}\\
&\times\max_{\ell_i\le L}\left(\frac{e(1-2^{-(i-1)})\log\log x+O(1)}{\ell_i}\right)^{\ell_i}.
\end{aligned}$$

It is well known that if A is fixed, then the function

$$t\mapsto (eA/t)^t$$

is increasing for $t<A$ and decreasing for $t>A$. Its maximum is at $t=A$ and equals e^A. For us, $k_i\le K_i<2^{-(i-1)}\log\log x$, therefore

$$\begin{aligned}
\#\mathcal{S}_i &\ll \frac{x(\log\log x)^3}{\log x}\left(\frac{2^{-(i-1)}e\log\log x+O(1)}{K_i}\right)^{K_i}\exp\left((1-2^{-(i-1)})\log\log x\right)\\
&\ll \frac{x(\log\log x)^3}{(\log x)^{\alpha_i}},
\end{aligned}$$

with the desired exponent α_i. Thus,

$$\#\bigcup_{i=0}^{5}\mathcal{S}_i \ll \frac{x(\log\log x)^3}{(\log x)^{\alpha_5}}. \tag{2.4}$$

We now choose $K=\lfloor(\log x)^{\beta}\rfloor$, where $\beta\in(0,\alpha_5)$ will be determined later, and let $\mathcal{S}_6$ be the set of $n\in\mathcal{S}$ such that $n+i\in\bigcup_{j=0}^{5}\mathcal{S}_j$ for some $i=1,\ldots,K$. By estimate (2.4), we get that

$$\#\mathcal{S}_6 \ll \frac{Kx(\log\log x)^3}{(\log x)^{\alpha_5}}\le\frac{x(\log\log x)^3}{(\log x)^{\alpha_5-\beta}}. \tag{2.5}$$

We now order all elements of $\mathcal{S}\setminus\left(\bigcup_{j=0}^{6}\mathcal{S}_j\right)$ as $x/\log x < n_1 < n_2 < \cdots < n_T < x$, and we denote by $\mathcal{S}_7$ the subset formed by the n_i's such that $n_{i+1}-n_i \ge K$. Clearly,

$$\#\mathcal{S}_7 \ll \frac{x}{K} \ll \frac{x}{(\log x)^{\beta}}. \tag{2.6}$$

We let $\mathcal{S}_8$ be the set of remaining numbers in $\mathcal{S}$. To estimate their number, let one of such numbers be n. Then $n \in \mathcal{S}$ and $n+k \in \mathcal{S}$ for some $k \in \{1,\dots,K\}$. We fix the number $k \le K$. Write $a_n = B^2$ and $a_{n+k} = C^2$. By an estimate of Walfisz (see [4]), we have that

$$\sum_{m\le T}\phi(m) = \frac{3}{\pi^2}T^2 + O(T(\log T)^{2/3}(\log\log T)^{4/3}) \tag{2.7}$$

holds for all T, therefore

$$B^2 = a_n = \sum_{m\le n}\phi(m) = \frac{3}{\pi^2}n^2 + O(n(\log n)^{2/3}(\log\log n)^{4/3}),$$

so that

$$\begin{aligned} B &= \frac{3^{1/2}}{\pi}n\left(1+O\left(\frac{(\log x)^{2/3}(\log\log x)^{4/3}}{n}\right)\right)^{1/2} \\ &= \frac{3^{1/2}}{\pi}n + O((\log x)^{2/3}\log\log x)^{4/3}). \end{aligned}$$

Let $M = c(\log x)^{2/3}(\log\log x)^{4/3}$ be an upper bound for the above error, where c is some absolute positive constant. Then

$$B = \lfloor \tau n\rfloor + m_1$$

for some integer $m_1 \in [-M, M]$, where we put $\tau = 3^{1/2}\pi^{-1}$. Similarly,

$$C = \lfloor \tau(n+k)\rfloor + m_2,$$

for some other integer $m_2 \in [-M, M]$. We now fix m_1, m_2. Then

$$\phi(n+1)+\cdots+\phi(n+k) = a_{n+k}-a_n = C^2-B^2 = (C-B)(C+B). \tag{2.8}$$

Note that

$$C-B = \lfloor \tau(n+k)\rfloor - \lfloor \tau n\rfloor + m_2 - m_1 = \lfloor \tau k\rfloor + m_2 - m_1 + \delta,$$

where $\delta \in \{0,1\}$. Fix also δ. Then $C-B$ is fixed, and so $C+B = 2B+C-B = 2\lfloor \tau n\rfloor + D$, where $D = m_1+m_2+\lfloor \tau k\rfloor + \delta$ is also fixed. Since $n+j \notin \bigcup_{j=0}^{6}\mathcal{S}_j$, it follows that $\omega_i(n+j) \ge 2^{-(i-1)}(1-\varepsilon)\log\log x$ holds for all $i = 1,2,3,4,5$, therefore

$$\nu_2(\phi(n+j)) \ge \sum_{i=1}^{5}\omega_i(n) \ge (1-\varepsilon)(\log\log x)\sum_{i=1}^{5}\frac{1}{2^{i-1}} \ge \frac{31}{16}(1-\varepsilon)\log\log x,$$

where for a positive integer s we put $\nu_2(s)$ for the exact exponent at which 2 appears in the prime factorization of s. Thus, putting $u = \lfloor (31/16)(\log 2)(1-\varepsilon)\log\log x\rfloor$, we get that 2^u divides the left hand side of equation (2.8). Note that

$$2^u \gg (\log x)^{(31/16)(1-\varepsilon)\log 2} > (\log x)^{4/3}(\log\log x)^{8/3} \gg C^2$$

once ε is sufficiently small because $(31\log 2)/16 > 4/3$. Write $C - B = 2^v E$, where $v < u/2$ and E is odd. Assuming that $\beta < 2/3$, we also get that $D = \lfloor \tau k \rfloor + m_1 + m_2 + \delta < 3C + 1 \ll (\log x)^{2/3}(\log\log x) = o(2^{u-v})$ as x tends to infinity. Now equation (2.8) gives that

$$2^{u-v} \mid \lfloor 2\tau n \rfloor + D,$$

therefore

$$\lfloor \tau n \rfloor = 2^{u-v}\Lambda + D, \tag{2.9}$$

must hold for some positive integer Λ. For a fixed D, the number of values of the right hand side is $\ll x/2^{u-v}$. Since $\tau > 1/2$, it follows that, given m, the equation $\lfloor \tau n \rfloor = m$ has at most two solutions n. Thus, when k, m_1, m_2, δ, v are fixed, equation (2.9) shows that the number of solutions for n is

$$\ll 2\left(\frac{x}{2^{u-v}} + 1\right) \ll \frac{x}{2^{u-v}}.$$

It remains to sum up over the number of choices. We note that m_1, m_2, δ and k determine v. We find it easier to count when we first fix v. Then k can be fixed in at most K ways, m_1 can be fixed in at most $2M$ ways, and δ in at most 2 ways. Once these parameters are fixed, then $C - B = \lfloor \tau k \rfloor + m_2 - m_1 + \delta$ is a translation of m_2, where $m_2 \in [-M, M]$. Thus, in order for $C - B$ to be divisible by 2^v, the parameter m_2 can be fixed in only $2(M/2^v + 1) \le 4M/2^v$ ways. Since now $v < u$, we get that the totality of such values for n is

$$\#\mathcal{S}_8 \ll \frac{xKM^2u}{2^u} \ll \frac{x(\log\log x)^{11/3}}{(\log x)^{\gamma}}, \tag{2.10}$$

where $\gamma = (31/16)(\log 2)(1-\varepsilon) - 4/3 - \beta$. It remains to optimize between estimates (2.5), (2.6) and (2.10), which leads to $\beta = \alpha_5/2$ and

$$(1-\varepsilon)\frac{31\log 2}{16} - \frac{4}{3} = \frac{3}{32}\left(1 - (1-\varepsilon)\log\left(\frac{e}{1-\varepsilon}\right)\right),$$

whose solution is $\varepsilon = .0067063\ldots$, for which $\beta = \alpha_5/2 = .000316476\ldots$. Since $(\log\log x)^{11/3} = (\log x)^{o(1)}$, we get that indeed

$$\#\mathcal{S}(x) \ll \frac{x}{(\log x)^{.0003}},$$

which is what we wanted to prove.

3. Comments and Remarks

As we mentioned in the Introduction, we conjecture that $\mathcal{S}$ is an infinite set. Here we give some heuristic in support of this conjecture. The expectation that a random integer of size n is a square is $1/n^{1/2}$. Assuming that a_n behaves like a random integer of its size, it would follow, via estimate (2.7), that the expectation that it is a square is about $(\tau^{-1} + o(1))/n$. Since

$$\sum_{n\le x}(\tau^{-1} + o(1))\frac{1}{n} = (\tau^{-1} + o(1))\log x$$

as $x \to \infty$, we expect that $\mathcal{S}$ is infinite and, in fact, that $\#\mathcal{S}(x) \gg \log x$.

Note that while our theorem shows that $\mathcal{S}$ is of asymptotic density zero, the estimate given there on $\#\mathcal{S}(x)$ is too weak to allow us to decide whether the series

$$\sum_{n\in\mathcal{S}} \frac{1}{n}$$

is convergent. We leave this as a challenge to the reader.

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PROBABLE PRIME TESTS FOR GENERALIZED MERSENNE NUMBERS

R. S. MELHAM

ABSTRACT. The classical Lucas-Lehmer test gives necessary and sufficient conditions for the primality of $2^p - 1$, p an odd prime. Such primes are called Mersenne primes. Here, taking $b \geq 2$ and $a \geq 3$ to be integers, and p to be an odd prime, we give probable prime tests for $(b^p+1)/(b+1)$ and $(a^p-1)/(a-1)$ that are analogous to the classical Lucas-Lehmer test.

1. Introduction

The celebrated Lucas-Lehmer test gives necessary and sufficient conditions for the primality of $2^p - 1$, where p is an odd prime. Define

$$N(p,b) = \frac{b^p+1}{b+1}, \quad \text{and} \quad M(p,a) = \frac{a^p-1}{a-1},$$

in which $b \geq 2$ and $a \geq 3$ are integers, and p is an odd prime. In this paper we construct probable prime tests for the numbers $N(p,b)$ and $M(p,a)$ that are analogous to the classical Lucas-Lehmer test. Our motivation arose from a perceived gap in the literature, since the classical Lucas-Lehmer test applies to the numbers $M(p,2)$. For example, we have proved the following two theorems. They are special cases of Theorem (3.7) and Theorem (4.2), respectively.

THEOREM (1.1). *Define $R_0 = 76$ and $R_{n+1} = R_n^3 + 3R_n$ for $n \geq 0$. Let $p \geq 3$ be prime and suppose $N(p,3)$ is prime. Then*
- $R_{p-1} - 76 \equiv 0 \mod N(p,3)$ *if* $p \equiv 1 \mod 4$;
- $R_{p-1} + 1364 \equiv 0 \mod N(p,3)$ *if* $p \equiv 3 \mod 4$.

THEOREM (1.2). *Define $S_0 = 198$ and $S_{n+1} = S_n^6 - 6S_n^4 + 9S_n^2 - 2$ for $n \geq 0$. Let $p \geq 3$ be prime and suppose $M(p,6)$ is prime. Then $S_{p-1} + 34 \equiv 0 \mod M(p,6)$.*

In Section 2 we consider $N(p,b)$ with b even, while in Section 3 we consider $N(p,b)$ with b odd. In Section 4 we consider the numbers $M(p,a)$, and in Section 5 we discuss the results obtained when our tests were used to search for probable primes of the relevant types. Finally, in Section 6 we indicate some possible directions for future research.

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2. The case of $N(p,b)$ with b even

Throughout this section we assume that $b=2c$, where c is a positive integer. Let $\alpha=1+\sqrt{2}$ and $\beta=1-\sqrt{2}$ so that $\alpha+\beta=2$, $\alpha\beta=-1$, and $\alpha-\beta=2\sqrt{2}$. Define two sequences of integers $\{U_n\}$ and $\{V_n\}$, for all integers n, by

$$U_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}, \quad \text{and} \quad V_n=\alpha^n+\beta^n .$$

We require the following, in which k and $n\geq 0$ are integers, and q is an odd prime.

$$V_{k-1}=-V_k+4U_k . \tag{2.1}$$

$$V_{k+1}=V_k+4U_k . \tag{2.2}$$

$$V_{kq}\equiv V_k \mod q,\ \ k\geq 1 . \tag{2.3}$$

$$U_{kq}\equiv \big(8/q\big)U_k=\big(2/q\big)U_k \mod q,\ \ k\geq 1 . \tag{2.4}$$

$$V_{bn}=V_{2cn}=\sum_{i=0}^{c}(-1)^{c+i}\frac{2c}{c+i}\binom{c+i}{2i}V_n^{2i},\ \ n\ \text{ even} . \tag{2.5}$$

Identities (2.1) and (2.2) follow from the definitions of U_n and V_n, while (2.3) and (2.4) are specific instances of such properties for Lucas sequences. For formulations more general than (2.3) and (2.4) see, for example, [5]. Identity (2.5) can be proved in the same manner as Theorem 5 in [6], which states that for the Lucas sequence L_n,

$$L_{2cn}=\sum_{i=0}^{c}(-1)^{(n+1)(c+i)}\frac{2c}{c+i}\binom{c+i}{2i}L_n^{2i},\ \ n,c\geq 1 .$$

We also require the following lemma.

LEMMA (2.6). *Let $p\geq 3$ be prime and suppose $N(p,b)$ is prime. If $b\equiv 0$ or $6 \bmod 8$ then $\big(2/N(p,b)\big)=1$. If $b\equiv 2$ or $4 \bmod 8$ then $\big(2/N(p,b)\big)=-1$.*

Proof. It is convenient to write

$$N(p,b)=b^{p-1}-b^{p-2}+\ldots+b^2-b+1 . \tag{2.7}$$

Since b is even we see from (2.7) that $N(p,b)\equiv b^2-b+1 \bmod 8$. Hence, if $b\equiv 0 \bmod 8$, then $N(p,b)\equiv 1 \bmod 8$, and if $b\equiv 6 \bmod 8$, then $N(p,b)\equiv 7 \bmod 8$. In either case, $\big(2/N(p,b)\big)=1$. The remainder of the proof follows similarly. □

Define the sequence $\{S_n\}=\big\{S_n^{(c)}\big\}$, $n\geq 0$, by

$$\begin{aligned} S_0 &= V_b=V_{2c},\\ S_{n+1} &= \sum_{i=0}^{c}(-1)^{c+i}\frac{2c}{c+i}\binom{c+i}{2i}S_n^{2i},\ \ n\geq 0 . \end{aligned}$$

We can now state the main result of this section.

THEOREM (2.8). *Let $p \geq 3$ be prime and suppose $N(p,b)$ is prime. Then*
- $S_{p-1} - V_b \equiv 0 \bmod N(p,b)$ *if* $b \equiv 0$ *or* $6 \bmod 8$;
- $S_{p-1} + V_{b+2} \equiv 0 \bmod N(p,b)$ *if* $b \equiv 2$ *or* $4 \bmod 8$.

Proof. From (2.1)-(2.4) and Lemma (2.6),

$$\begin{aligned} V_{(b+1)N(p,b)-1} = V_{b^p} &= -V_{(b+1)N(p,b)} + 4U_{(b+1)N(p,b)} \\ &\equiv -V_{b+1} + 4\left(2/N(p,b)\right) U_{b+1} \bmod N(p,b) \\ &= \begin{cases} -V_{b+1} + 4U_{b+1} \bmod N(p,b), & \text{if } b \equiv 0 \text{ or } 6 \bmod 8 \\ -V_{b+1} - 4U_{b+1} \bmod N(p,b), & \text{if } b \equiv 2 \text{ or } 4 \bmod 8 \end{cases} \\ &= \begin{cases} V_b \bmod N(p,b), & \text{if } b \equiv 0 \text{ or } 6 \bmod 8 \\ -V_{b+2} \bmod N(p,b), & \text{if } b \equiv 2 \text{ or } 4 \bmod 8 \end{cases}. \end{aligned}$$

By (2.5) the sequences $\{V_{b^n}\}, n \geq 1$, and $\{S_n\}, n \geq 0$, satisfy the same recurrence. Since $S_0 = V_b$ it follows by induction that $S_n = V_{b^{n+1}}$, $n \geq 0$, and this completes the proof of Theorem (2.8). □

3. The Case of $N(p,b)$ with b odd

Throughout this section we assume that $b = 2c+1$, where c is a positive integer. We require two additional sequences. To this end, let $\gamma = 2 + \sqrt{5}$ and $\delta = 2 - \sqrt{5}$ so that $\gamma + \delta = 4$, $\gamma\delta = -1$, and $\gamma - \delta = 2\sqrt{5}$. Define the two integer sequences $\{W_n\}$ and $\{X_n\}$, for all integers n, by

$$W_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}, \quad \text{and} \quad X_n = \gamma^n + \delta^n .$$

The terms W_n and X_n satisfy identities that parallel (2.1)-(2.5). With the same assumptions on k, n, and q we have

(3.1) $$X_{k-1} = -2X_k + 10W_k .$$

(3.2) $$X_{k+1} = 2X_k + 10W_k .$$

(3.3) $$X_{kq} \equiv X_k \bmod q, \ \ k \geq 1 .$$

(3.4) $$W_{kq} \equiv \left(20/q\right) W_k = \left(5/q\right) W_k \quad \bmod q, \ \ q \neq 5, \ \ k \geq 1 .$$

(3.5) $$X_{bn} = X_{(2c+1)n} = \sum_{i=0}^{c} \frac{2c+1}{c+i+1} \binom{c+i+1}{2i+1} X_n^{2i+1}, \ \ n \text{ odd} .$$

The same comments apply to (3.1)-(3.4) as for their counterparts in Section 2, and (3.5) can be proved in the same manner as Theorem 6 in [6], which states that for the Lucas sequence L_n,

$$L_{(2c+1)n} = \sum_{i=0}^{c} (-1)^{(n+1)(c+i)} \frac{2c+1}{c+i+1} \binom{c+i+1}{2i+1} L_n^{2i+1}, \ \ n, c \geq 0 .$$

LEMMA (3.6). *Let $p \geq 3$ be prime and suppose $N(p,b)$ is prime.*
- *Suppose $b \equiv 0$ or $1 \bmod 5$. Then $\left(5/N(p,b)\right) = 1$.*
- *Suppose $b \equiv 2$ or $3 \bmod 5$. If $p \equiv 1 \bmod 4$ then $\left(5/N(p,b)\right) = 1$. If $p \equiv 3 \bmod 4$ then $\left(5/N(p,b)\right) = -1$.*

• *Suppose* $b \equiv 4 \bmod 5$. *If* $p \equiv 1$ *or* $4 \bmod 5$ *then* $\big(5/N(p,b)\big) = 1$. *If* $p \equiv 2$ *or* $3 \bmod 5$ *then* $\big(5/N(p,b)\big) = -1$.

Proof. The proof of the first case of Lemma (3.6) is trivial and is left for the reader. Suppose $b \equiv 2$ or $3 \bmod 5$. If $p \equiv 1 \bmod 4$, we see from (2.7) that $N(p,b) \equiv 1 \bmod (b^2+1)$. This implies that $N(p,b) \equiv 1 \bmod 5$, and the result follows from the quadratic reciprocity law. Similarly, if $p \equiv 3 \bmod 4$, $N(p,b) \equiv b^2 - b + 1 \bmod (b^2+1)$, which implies that $N(p,b) \equiv 2$ or $3 \bmod 5$. Once again, the result follows from the quadratic reciprocity law.

Next, suppose $b \equiv 4 \bmod 5$. If $p \equiv 1 \bmod 5$, we see from (2.7) that $N(p,b) \equiv 1 \bmod (b^4 - b^3 + b^2 - b + 1)$. This implies that $N(p,b) \equiv 1 \bmod 5$, and by the quadratic reciprocity law, $\big(5/N(p,b)\big) = 1$. Similarly, if $p \equiv 4 \bmod 5$ it follows that $N(p,b) \equiv -b^3 + b^2 - b + 1 \bmod (b^4 - b^3 + b^2 - b + 1)$, which implies that $N(p,b) \equiv 4 \bmod 5$. Once again, $\big(5/N(p,b)\big) = 1$. The remainder of the proof follows similarly. □

Define the sequence $\{R_n\}=\{R_n^{(c)}\}$, $n \geq 0$, by

$$\begin{aligned} R_0 &= X_b = X_{2c+1}, \\ R_{n+1} &= \sum_{i=0}^{c} \frac{2c+1}{c+i+1} \binom{c+i+1}{2i+1} R_n^{2i+1}, \ n \geq 0 \,. \end{aligned}$$

Next we state two cases that are required for the statement of the main result of this section. These cases are drawn from the conditions in Lemma (3.6) and, accordingly, we take $p \geq 3$ to be prime.

CASE (1). $b \equiv 0$ *or* $1 \bmod 5$, *or* $b \equiv 2$ *or* $3 \bmod 5$ *and* $p \equiv 1 \bmod 4$, *or* $b \equiv 4 \bmod 5$ *and* $p \equiv 1$ *or* $4 \bmod 5$.

CASE (2). $b \equiv 2$ *or* $3 \bmod 5$ *and* $p \equiv 3 \bmod 4$, *or* $b \equiv 4 \bmod 5$ *and* $p \equiv 2$ *or* $3 \bmod 5$.

We then have the following theorem.

THEOREM (3.7). *Let* $p \geq 3$ *be prime and suppose* $N(p,b)$ *is prime. Then*
• $R_{p-1} - X_b \equiv 0 \bmod N(p,b)$ *if Case* (1) *holds;*
• $R_{p-1} + X_{b+2} \equiv 0 \bmod N(p,b)$ *if Case* (2) *holds.*

Proof. From (3.1)-(3.4) and Lemma (3.6),

$$\begin{aligned} X_{(b+1)N(p,b)-1} = X_{b^p} &= -2X_{(b+1)N(p,b)} + 10W_{(b+1)N(p,b)} \\ &\equiv -2X_{b+1} + 10\big(5/N(p,b)\big) W_{b+1} \mod N(p,b) \\ &= \begin{cases} -2X_{b+1} + 10W_{b+1} \bmod N(p,b), & \text{for Case (1)} \\ -2X_{b+1} - 10W_{b+1} \bmod N(p,b), & \text{for Case (2)} \end{cases} \\ &= \begin{cases} X_b \bmod N(p,b), & \text{for Case (1)} \\ -X_{b+2} \bmod N(p,b), & \text{for Case (2)} \end{cases} . \end{aligned}$$

By (3.5) the sequences $\{X_{b^n}\}$, $n \geq 1$, and $\{R_n\}$, $n \geq 0$, satisfy the same recurrence. Since $R_0 = X_b$ it follows by induction that $R_n = X_{b^{n+1}}$, $n \geq 0$, and this completes the proof of Theorem (3.7). □

Theorem (1.1) is a special case of Theorem (3.7) in which $b = 2c+1 = 3$. The cubic recurrence in Theorem (1.1) is therefore obtained from the recurrence for $\{R_n\}$ by taking $c = 1$.

4. The Case of $M(p, a)$

In this section we treat the numbers $M(p, a)$. The sequences $\{V_n\}$ and $\{X_n\}$ are the same as those used previously. In order to highlight analogies with the results involving $N(p, b)$, we take the sequences $\{S_n\} = \{S_n^{(c)}\}$, and $\{R_n\} = \{R_n^{(c)}\}$ to be generated by the same recurrences given earlier. Here, however, their initial terms are taken to be $S_0 = V_a = V_{2c}$, when a is even, and $R_0 = X_a = X_{2c+1}$, when a is odd. We state our results without proof since the proofs are entirely analogous to those given previously.

First let $a \geq 2$ be even. Then Lemma (4.1) and Theorem (4.2) are analogous to Lemma (2.6) and Theorem (2.8), respectively.

LEMMA (4.1). *Let $p \geq 3$ be prime and suppose $M(p, a)$ is prime. If $a \equiv 0$ or $2 \bmod 8$ then $\big(2/M(p, a)\big) = 1$. If $a \equiv 4$ or $6 \bmod 8$ then $\big(2/M(p, a)\big) = -1$.*

THEOREM (4.2). *Let $p \geq 3$ be prime and suppose $M(p, a)$ is prime. Then*
- $S_{p-1} - V_a \equiv 0 \bmod M(p, a)$ *if $a \equiv 0$ or $2 \bmod 8$;*
- $S_{p-1} + V_{a-2} \equiv 0 \bmod M(p, a)$ *if $a \equiv 4$ or $6 \bmod 8$.*

Theorem (1.2) is a special case of Theorem (4.2) in which $a = 2c = 6$. The recurrence in Theorem (1.2) is therefore obtained from the recurrence for $\{S_n\}$ by taking $c = 3$.

Next let $a \geq 3$ be odd. Then Lemma (4.3) and Theorem (4.4) are analogous to Lemma (3.6) and Theorem (3.7), respectively.

LEMMA (4.3). *Let $p \geq 3$ be prime and suppose $M(p, a)$ is prime.*
- *Suppose $a \equiv 0$ or $4 \bmod 5$. Then $\big(5/M(p, a)\big) = 1$.*
- *Suppose $a \equiv 2$ or $3 \bmod 5$. If $p \equiv 1 \bmod 4$ then $\big(5/M(p, a)\big) = 1$. If $p \equiv 3 \bmod 4$ then $\big(5/M(p, a)\big) = -1$.*
- *Suppose $a \equiv 1 \bmod 5$. If $p \equiv 1$ or $4 \bmod 5$ then $\big(5/M(p, a)\big) = 1$. If $p \equiv 2$ or $3 \bmod 5$ then $\big(5/M(p, a)\big) = -1$.*

For the statement of the next theorem, we require the following two cases which are drawn from Lemma (4.3).

CASE (3). *$a \equiv 0$ or $4 \bmod 5$, or $a \equiv 2$ or $3 \bmod 5$ and $p \equiv 1 \bmod 4$, or $a \equiv 1 \bmod 5$ and $p \equiv 1$ or $4 \bmod 5$.*

CASE (4). *$a \equiv 2$ or $3 \bmod 5$ and $p \equiv 3 \bmod 4$, or $a \equiv 1 \bmod 5$ and $p \equiv 2$ or $3 \bmod 5$.*

Recalling that a is taken to be odd, we then have

THEOREM (4.4). *Let $p \geq 3$ be prime and suppose $M(p, a)$ is prime. Then*
- $R_{p-1} - X_a \equiv 0 \bmod M(p, a)$ *if Case (3) holds;*
- $R_{p-1} + X_{a-2} \equiv 0 \bmod M(p, a)$ *if Case (4) holds.*

5. Results of Some Testing

The following theorem, which is an instance of Theorem (2.8), is worthy of special mention. It was discovered independently by the present author and Richard McIntosh.

THEOREM (5.1). *Define* $S_0 = 6$ *and* $S_{n+1} = S_n^2 - 2$ *for* $n \geq 0$. *Let* $p \geq 3$ *be prime and suppose* $N(p, 2)$ *is prime. Then* $S_{p-1} + 34 \equiv 0 \bmod N(p, 2)$.

Here the testing sequence is generated by the same recurrence used in the classical Lucas-Lehmer test. Furthermore, the test described in Theorem (5.1) is independent of Fermat tests (see Section 6). Our interest in the numbers $N(p, 2)$ was aroused when we became aware of a conjecture (the New Mersenne Conjecture) of Bateman, Selfridge, and Wagstaff, Jr. [2], that linked these numbers with the Mersenne primes. Although Theorem (5.1) is a compositeness test, we decided to test its performance in detecting the primes/probable primes $N(p, 2)$. In particular, we were interested to see if it turned up any false positives. We applied it to all odd primes $p < 100000$, and detected only the 32 known values of $p < 100000$ for which N_p is prime/probably prime. We then applied it to each of the 7 known values of $p > 100000$ for which N_p is prime/probably prime, and it detected each of these as well. For a list of the known primes/probable primes $N(p, 2)$, see [4].

We also tested the theorems corresponding to b =3, 5, 6, 7, 9, 10, 11, and 12 (of course only $p = 3$ produces a prime when b =4, and no primes arise when b =8). Our results are summarized in Table (1). Similar testing of $M(p, a)$ for a =3, 5, 6, 7, 10, 11, and 12, produced the outcomes summarized in Table (2).

In Table (1), except for $b = 3$, our limits for p (right column), at the time of writing, were higher than those that we could find in the available literature. For example, see [3], [4], and [7]. This resulted in one more outcome for each of $b = 5, 9, 11$, and 12, and two more outcomes for $b = 6$. In Table (2) our limits for p were higher for $a = 6, 7, 11$, and 12. This resulted in extra outcomes for $a = 7$, and 12. Each of the extra outcomes in Tables (1) and (2) passed a strong Fermat probable prime test with 20 random bases, and it is to this extent that our tests have produced no false positives.

6. Final Comments and Future Directions

When used to search for probable primes of the relevant types, our tests become slower as a and b become larger. Therefore, except for small a and b, they would not prove practical in conducting such searches.

To obtain the results in Tables (1) and (2) we programmed our tests in *Mathematica 5.1* and used a laboratory of Intel Pentium4 3.0GHz based personal computers over several months. For curiosity, we took a selection of cases from Tables (1) and (2) and compared the speed of our tests with the speed of *Mathematica's* PrimeQ test. The current version of PrimeQ first tests for divisibility using small primes, then uses the Miller-Rabin strong pseudoprime test base 2 and base 3, and then uses a Lucas test.

From Table (1) we selected the numbers $N(3, 26633)$, $N(5, 30133)$, $N(6, 38447)$, $N(7, 18251)$, $N(9, 3803)$, $N(10, 3011)$, $N(11, 6113)$, and $N(12, 11353)$. For each of these numbers we give, in the form of an ordered pair, the time (in

Table 1

base b	value of p that yields primes/probable primes $N(p, b)$	$p <$
3	3, 5, 7, 13, 23, 43, 281, 359, 487, 577, 1579, 1663, 1741, 3191, 9209, 11257, 12743, 13093, 17027, 26633	50000
5	5, 67, 101, 103, 229, 347, 4013, 23297, 30133	40000
6	3, 11, 31, 43, 47, 59, 107, 811, 2819, 4817, 9601, 33581, 38447	40000
7	3, 17, 23, 29, 47, 61, 1619, 18251	40000
9	3, 59, 223, 547, 773, 1009, 1823, 3803	30000
10	5, 7, 19, 31, 53, 67, 293, 641, 2137, 3011	30000
11	5, 7, 179, 229, 439, 557, 6113	20000
12	5, 11, 109, 193, 1483, 11353	20000

Table 2

base a	value of p that yields primes/probable primes $M(p, a)$	$p <$
3	3, 7, 13, 71, 103, 541, 1091, 1367, 1627, 4177, 9011, 9551, 36913, 43063, 49681	50000
5	3, 7, 11, 13, 47, 127, 149, 181, 619, 929, 3407, 10949, 13241, 13873, 16519	40000
6	3, 7, 29, 71, 127, 271, 509, 1049, 6389, 6883, 10613, 19889	40000
7	5, 13, 131, 149, 1699, 14221, 35201	40000
10	19, 23, 317, 1031	30000
11	17, 19, 73, 139, 907, 1907, 2029, 4801, 5153, 10867	20000
12	3, 5, 19, 97, 109, 317, 353, 701, 9739, 14951	20000

seconds) taken by our relevant test (first element), and *Mathematica's* PrimeQ test (second element), to verify probable primality. The times were (452, 789), (2540, 3111), (6391, 7606), (1632, 1385), (54, 30), (32, 21), (327, 146), and (1429, 774), respectively. For the seven numbers $M(p, a)$ corresponding to the primes in similar positions in Table (2) the times were (1655, 3495), (508, 666), (1124, 1395), (8850, 7432), (2, 1), (1408, 639), and (2761, 1561), respectively.

In Sections 2 and 3 our tests are independent of Fermat tests precisely when $\big(2/N(p, b)\big) = -1$ and $\big(5/N(p, b)\big) = -1$, respectively. The situation is similar for the tests in Section 4. This observation was made in [1], p 1401; in relation to the types of Lucas tests described there. Therefore, we have achieved independence from Fermat tests about half the time. In these instances, even for larger a and b, our tests could prove useful when used alongside other probable prime tests to verify probable primality.

Our objective has been achieved with the use of only two pairs of Lucas sequences, and independence from Fermat tests occurs about half the time. We pose the following questions. Can our objective be achieved with a finite number of pairs of Lucas sequences, and in a manner where independence from Fermat tests occurs *all* the time? If so, what is the minimum number of such pairs?

Other avenues of research could involve estimates of the probability of error, and comparisons between the speed of these tests and the speed of the standard Lucas tests. Finally, we have made no progress in proving that any of our tests are deterministic, and so there remains this possibility.

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A CHARACTERIZATION OF K-ANALYTICITY OF GROUPS OF CONTINUOUS HOMOMORPHISMS

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ABSTRACT. For an abelian locally compact group X let $X_p^\wedge$ be the group of continuous homomorphisms from X into the unit circle $\mathbb{T}$ of the complex plane endowed with the pointwise convergence topology. It is proved that X is metrizable iff $X_p^\wedge$ is K-analytic iff X endowed with its Bohr topology $\sigma(X, X^\wedge)$ has countable tightness. Using this result, we establish a large class of topological groups with countable tightness which are not sequential, so neither Fréchet-Urysohn.

1. Introduction

For abelian topological groups X and Y we denote by $\mathrm{Hom}_p(X, Y)$ and $\mathrm{Hom}_c(X, Y)$ the set $\mathrm{Hom}(X, Y)$ of all continuous homomorphisms from X into Y endowed with the pointwise and compact-open topology, respectively. Set $X_p^\wedge =: \mathrm{Hom}_p(X, \mathbb{T})$, $X_c^\wedge =: \mathrm{Hom}_c(X, \mathbb{T})$, where $\mathbb{T}$ denotes the unit circle of the complex plane. For every $x \in X$ the function $x^\wedge : X^\wedge \to \mathbb{T}$, defined by $x^\wedge(f) := f(x)$ for $f \in X^\wedge$, is a continuous homomorphism on $X_c^\wedge$ and $\{x^\wedge : x \in X\} \subset (X_c^\wedge)^\wedge$. By Pontryagin-van Kampen's Theorem (see [10], Theorem 24.8), *if X is a locally compact abelian group, the mapping $\alpha : x \mapsto x^\wedge$ is a topological isomorphism between X and $(X_c^\wedge)_c^\wedge$*. If X is an abelian locally compact group, $X_c^\wedge$ is also locally compact and abelian, and by Peter-Weyl-van Kampen's Theorem $X_c^\wedge$ is *dual separating*, i.e. for different $x, y \in X$, there exists $f \in X^\wedge$ such that $f(x) \neq f(y)$, see [14], Theorem 21. For an abelian group X the set of all homomorphisms from X into $\mathbb{T}$ endowed with the pointwise convergence topology is a compact abelian group, as it is a closed subgroup of the product $\mathbb{T}^X$, see [11], Proposition 1.16. For a metrizable abelian topological group X the group $X_c^\wedge$ is always an abelian complete Hausdorff hemicompact group which is also a k-space, see [1], Corollary 4.7 and [5].

The main result of this paper is the following theorem:

THEOREM (1.1). *Let X be a locally compact abelian group. The following assertions are equivalent: (1) X is metrizable. (2) $X_p^\wedge$ is σ-compact. (3) $X_p^\wedge$ is K-analytic. (4) $(X, \sigma(X, X^\wedge))$ has countable tightness. Moreover, if X is*

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Lindelöf, then each of the above conditions is equivalent to (5) $X_c^\wedge$ is metric complete and separable.

We believe this theorem is interesting since it relates two different fields of research. It was motivated by several similar results concerning the spaces $C_p(X)$ of all continuous real-valued functions on a completely regular space X endowed with the pointwise convergence topology. (Calbrix [4], Theorem 2.3.1) showed that if for a Tychonoff space X the space $C_p(X)$ is analytic, then X must be σ-compact and analytic (cf. also [7], Theorem 3.7). If X is locally compact, then X is a polish space iff $C_p(X)$ is analytic, [13], Corollary 5.7.6. In [9] Corson proved that a locally compact topological group X is metrizable iff the Banach space $C_0(X)$ of continuous, complex valued functions which vanish at infinity is weakly Lindelöf. On the other hand, $C_p(X)$ is Lindelöf provided X is second countable. The converse fails in general but very recently we have shown [12] that for a locally compact topological group X the space $C_p(X)$ *is Lindelöf* iff X is metrizable and σ-compact; in particular, $C_p(X)$ is Lindelöf iff X is second countable. Recall that a (Hausdorff) topological space X is said to be K-analytic [16] if there is an upper semi-continuous set-valued map from the polish space $\mathbb{N}^\mathbb{N}$ with compact values in X whose union is X. Notice that analytic $\Rightarrow$ K-analytic $\Rightarrow$ Lindelöf. A topological space X is said to have *countable tightness* if for each set $A \subset X$ and any $x \in \overline{A}$ (the closure of A) there exists a countable subset $B \subset A$ whose closure contains x. For a topological abelian group X the coarsest group topology on X for which all elements of $X^\wedge$ are continuous is called the *Bohr topology*; we denote this topology by $\sigma(X, X^\wedge)$. A topological space X is said to be *hemicompact* if X is covered by a fundamental sequence of compact sets, i.e. there is a sequence $(K_n)_n$ of compact subsets of X such that each compact set in X is contained in some K_n. Finally by $\mathbb{R}$ and $\mathbb{C}$ we denote the sets of real and complex numbers, respectively.

2. Proof of the Theorem

The proof of the theorem is derived from the following facts:

FACT (1). *A locally compact Lindelöf topological group X is hemicompact.*

Proof. Take an open neighbourhood of the neutral element U whose closure $\overline{U}$ is compact. Since $X = \bigcup_{x\in X} xU$ and X is Lindelöf, there exists a sequence $(x_n)_n$ such that $X = \bigcup_n x_n\overline{U}$. Set $K_n := \bigcup_{i=1}^n x_i\overline{U}$. Then $(K_n)_n$ is a fundamental sequence of compact subsets of X, so X is hemicompact. □

FACT (2). *If X is a metrizable abelian group, $X_c^\wedge$ is a hemicompact k-space.*

Proof. Let $(U_n)_{n\in\mathbb{N}}$ be a decreasing basis of neighbourhoods of zero in X. Then $U_n^\triangleright := \{\phi \in X_c^\wedge : \phi(U_n) \subseteq \mathbb{T}_+\}$ is compact in the compact-open topology, where $\mathbb{T}_+ := \{z \in \mathbb{C} : |z| = 1, \operatorname{Re} z \geqslant 0\}$. But $X^\wedge = \bigcup_n U_n^\triangleright$. In fact, if $\phi \in X^\wedge$, then $\phi^{-1}(\mathbb{T}_+)$ is a neighbourhood of zero, so there exists $m \in \mathbb{N}$ such that $U_m \subseteq \phi^{-1}(\mathbb{T}_+)$. Therefore $\phi \in U_m^\triangleright$. If K is a compact set in $X_c^\wedge$, then $K^\triangleright \subset (X_c^\wedge)^\wedge$ is a neighbourhood of zero. The canonical mapping α is continuous, therefore $\alpha^{-1}(K^\triangleright) = \{x \in X : \operatorname{Re} \phi(x) \geqslant 0, \forall \phi \in K\} =: K^\triangleleft$, is a neighborhood of zero in X. Thus, $K^\triangleleft \supseteq U_m$ for some $m \in \mathbb{N}$, and we have $K \subseteq K^{\triangleright\triangleleft} \subseteq U_m^\triangleright$.

The proof of the fact that $X_c^\wedge$ is a k-space is harder. It was given independently in [1], Corollary 4.7 and [5]. □

FACT (3) (See [1], Proposition 2.8). *If a topological abelian group X is hemicompact, then $X_c^\wedge$ is metrizable.*

FACT (4). *Let X be a Tychonoff space and assume that $C_p(X, \mathbb{R})$ has countable tightness. Then $C_p(X, Y)$ also has countable tightness for any metric space (Y, d).*

Proof. Let $A \subset C_p(X, Y)$ and assume that $f \in \overline{A}$ (the closure in $C_p(X, Y)$). Define a continuous map $T: C_p(X, Y) \to C_p(X, \mathbb{R})$ by $T(g)(x) := d(g(x), f(x))$, where $g \in C_p(X, Y)$ and $x \in X$. Note that $0 = T(f) \in T(\overline{A}) \subset \overline{T(A)}$. By assumption there exists a countable subset $B \subset A$ such that $T(f) \in \overline{T(B)}$; hence, as easily seen from the definition of the pointwise convergence topology in $C(X, \mathbb{R})$ and in $C(X, Y)$, $f \in \overline{B}$. □

Proof of the Theorem. (1) $\Rightarrow$ (2): By Fact (2) the group $X_c^\wedge$ is hemicompact. So $X_p^\wedge$ is σ-compact.

(2) $\Rightarrow$ (3): If $(B_n)_n$ is an increasing sequence of compact sets covering $X_p^\wedge$, set $T(\alpha) := B_{n_1}$ for $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$. It is clear that T is upper semi-continuous, with compact values which cover $X_p^\wedge$.

(3) $\Rightarrow$ (4): Since $X_p^\wedge$ is K-analytic, then any finite product $(X_p^\wedge)^n$ is Lindelöf. By [2], Theorem II.1.1, the space $C_p(X_p^\wedge, \mathbb{R})$ has countable tightness. Now Fact (4) applies to deduce that the space $C_p(X_p^\wedge, \mathbb{C})$ also has countable tightness. Therefore $(X, \sigma(X, X^\wedge))$ (as a subspace of $C_p(X_p^\wedge, \mathbb{C})$) has countable tightness.

(4) $\Rightarrow$ (1): Since X is a locally compact group, there exist a compact subgroup G of X, $n \in \mathbb{N} \cup \{0\}$, and a discrete subset $D \subset X$ such that X is homeomorphic to the product $\mathbb{R}^n \times D \times G$, see [8], Theorem 1, Remark(ii). Therefore the induced topology $\sigma(X, X^\wedge)|G$ coincides with the original one of G. Hence G has countable tightness. Since a compact group with countable tightness is metrizable, (see e.g. [12], Theorem 2); so X is metrizable as well.

For the last statement observe that under the assumption that X is metrizable and Lindelöf, Facts (1) and (3) apply to deduce that $X_c^\wedge$ is a separable metric space. It is complete since the dual group of a locally compact abelian group is also locally compact. The result follows now from the fact that $X_p^\wedge$ is the continuous image of $X_c^\wedge$ under the identity mapping. □

Recall that a topological space is sequential if every sequentially closed subset of X is closed. In [6], Theorem 2.1, it was shown that for a metrizable topological group X the dual group $X_c^\wedge$ is Fréchet-Urysohn iff $X_c^\wedge$ is locally compact and metrizable. This result provides a large class of complete strictly angelic hemicompact sequential non Fréchet-Urysohn groups [6], Theorem 2.3. We supplement this result by the following

COROLLARY (2.1). *If X is a metrizable locally compact non-compact abelian group, then the group $(X, \sigma(X, X^\wedge))$ has countable tightness but it is not sequential, nor Fréchet-Urysohn.*

Proof. From Glicksberg's Theorem it follows that $(X, \sigma(X, X^\wedge))$ has the same compact subsets as X. Since X is metrizable, it is already a k-space, and X

does not admit a weaker k-space topology with the same compact subsets. Therefore $(X, \sigma(X, X^\wedge))$ is not a k-space, and in particular it is not sequential neither Fréchet-Urysohn, which are stronger properties. □

Remark (2.2). If X is a hereditarily separable topological group (in particular if it is metrizable and separable), then $(X, \sigma(X, X^\wedge))$ has countable tightness.

This derives from the more general fact, brought to our attention by L. Aussenhofer: *If (X, τ) is a hereditarily separable topological space, then any weaker topology ξ on X is hereditarily separable and hence (as easily seen) has countable tightness.*

We complete the note with the following observation about continuous functions defined on a topological abelian group which are not homomorphisms.

PROPOSITION (2.3). *Let $f : X \to \mathbb{T}$ be a continuous functions defined on an abelian topological group X which is not a homomorphism. Then there exist a finite collection $\{z_1, \ldots, z_n\}$ of integer numbers and a finite set $\{x_1, \ldots, x_n\}$ in X such that for all $\phi \in X^\wedge$ one has $\phi(x_1)^{z_1}\phi(x_2)^{z_2}\cdots\phi(x_n)^{z_n} = 1$ and* $\mathrm{Re}(f(x_1)^{z_1}\cdots f(x_n)^{z_n}) < 0$, *where* Re *stands for the real part.*

Proof. Clearly $X_p^\wedge$ is a closed subgroup of $C_p(X, \mathbb{T})$ which can be considered as a subgroup of the topological group $\mathbb{T}^X$. Thus, $C_p(X, \mathbb{T})$ is precompact and so $X_p^\wedge$ is dually closed as a subgroup of $C_p(X, \mathbb{T})$, see [3], (8.6). This means that every element of $C_p(X, \mathbb{T}) \setminus X^\wedge$ can be separated from $X^\wedge$ by means of a continuous character of $C_p(X, \mathbb{T})$, which in particular is the restriction of a character on $\mathbb{T}^X$. On the other hand, it is well-known that the character group of a product of topological abelian groups is the direct sum of the corresponding dual groups. So the direct sum $\mathbb{Z}^{(X)}$ is the character group of $\mathbb{T}^X$. Hence for $f \in C_p(X, \mathbb{T})\setminus X^\wedge$ there exists a character ξ which can be written as $z_{x_1} + z_{x_2} + \cdots + z_{x_n}$ with $z_{x_i} \in \mathbb{Z}^{(X)}$ such that $\xi(\phi) = \phi(x_1)^{z_{x_1}}\phi(x_2)^{z_{x_2}}.\cdots.\phi(x_n)^{z_{x_n}} = 1$ for all $\phi \in X^\wedge$ and $\mathrm{Re}(\xi(f)) = f(x_1)^{z_{x_1}}f(x_2)^{z_{x_2}}.\cdots.f(x_n)^{z_{x_n}} < 0$. □

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GROUP TOPOLOGIES ON VECTOR SPACES AND CHARACTER LIFTING PROPERTIES

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ABSTRACT. It is known that every continuous character on a topological vector space can be lifted to a continuous linear functional and, moreover, these liftings give rise to a topological isomorphism between the dual group and the dual space, when both are endowed with the compact-open topology. We investigate the presence of these properties in more general topologized real vector spaces.

1. Preliminaries

In Functional Analysis, the study of not necessarily linear, additive group topologies on vector spaces can be traced back at least to the fifties. For instance, topological vector groups (as defined below) were introduced by Raĭkov ([19]), but the notion had been implicitly used in earlier references.

The first insights on the group duality of topological vector spaces are from about the same time (the words "group duality" meaning the use of the one-dimensional torus, rather than the real line, as the dualizing object). M. F. Smith proved in [20] that every continuous character on a topological vector space can be lifted to a continuous linear functional and, moreover, these liftings result in a topological isomorphism between the dual group and the dual space, both endowed with the compact-open topology. (The first part of this assertion was proved independently by Hewitt and Zuckermann in [14].)

These definitions and results were put together and given a new prominence in W. Banaszczyk's theory of nuclear groups ([2]). His work paved the way to proving significant generalizations of several basic theorems from abstract harmonic analysis and duality theory of topological Abelian groups, which were carried out by himself and other authors. The theory makes essential use of locally convex vector groups and other classes of topologized vector spaces.

In this paper we first give a survey of different topological properties that may be present in a topological Abelian group which algebraically is a real vector space. Next we explore the way these properties are related to the possibility of lifting continuous characters to continuous linear functionals on these spaces. Note that in the topological group framework, Nickolas ([18]) showed that the related problem of lifting continuous characters to continuous real valued group homomorphims is relevant in the study of the structure of dual groups.

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For any Abelian group G, any set $U \subset G$ and any $n \in \mathbb{N}$ we define

$$U_{(n)} = \bigcap_{k=1}^{n} \{x \in G : kx \in U\}$$

Note that if G is a topological Abelian group and U is a neighborhood of zero in G, the sets $U_{(n)},\ n \in \mathbb{N}$ are neighborhoods of zero as well.

The symbol $\mathbb{T}$ will denote the multiplicative group of all complex numbers with modulus 1, endowed with the topology induced by the usual one on $\mathbb{C}$. The canonical covering projection $[t \in \mathbb{R} \mapsto \exp(2\pi it) \in \mathbb{T}]$ will be denoted by p. We will also use a special notation for the following distinguished neighborhood of 1 in $\mathbb{T}$:

$$\mathbb{T}_+ = p([-1/4, 1/4]) = \{t \in \mathbb{T} : \mathrm{Re}\, t \geq 0\}$$

The following fact is standard (for a proof see e. g. [2], 1.2):

PROPOSITION (1.1). *For every $n \in \mathbb{N}$,* $\quad (\mathbb{T}_+)_{(n)} = p([-\frac{1}{4n}, \frac{1}{4n}])$.

For a given topological Abelian group G, let us denote by $\mathcal{N}_0(G)$ the family of all neighborhoods of the unit element $0 \in G$, in the case of additive notation. From Proposition (1.1) it is not difficult to derive the following known result:

PROPOSITION (1.2). *Let G be a topological Abelian group and $\chi\colon G \to \mathbb{T}$ a group homomorphism. Then χ is continuous if and only if there exists $U \in \mathcal{N}_0(G)$ such that $\chi(U) \subset \mathbb{T}_+$.*

Group homomorphisms from G to $\mathbb{T}$ are usually called *characters* of G. We shall denote by $G^\wedge$ the set of all continuous characters of the topological Abelian group G. Note that $G^\wedge$ is an Abelian group under pointwise product of characters.

Let G be a topological Abelian group, and U a subset of G. The set of all continuous characters $\chi \in G^\wedge$ such that $\chi(U) \subset \mathbb{T}_+$ is called the *polar* of U and denoted by $U^\triangleright$. Dually, if V is a subset of the Abelian group $G^\wedge$, the set of all $x \in G$ such that $\chi(x) \in \mathbb{T}_+$ is called the *inverse polar* of V and denoted by $V^\triangleleft$. A set $U \subset G$ is said to be *quasi-convex* if $U = U^{\triangleright\triangleleft}$. Keeping in mind Hahn-Banach separation theorems, topological Abelian groups with a basis of quasi-convex neighborhoods of zero (the so-called *locally quasi-convex groups*) are expected to be a convenient generalization of locally convex spaces. (We shall make more concrete this statement below.)

Let G be a topological Abelian group. Denote by $\sigma(G, G^\wedge)$ the coarsest topology for which all characters in $G^\wedge$ are continuous; in other words, the initial topology on G with respect to all homomorphisms in $G^\wedge$. $\sigma(G, G^\wedge)$, usually known as *Bohr topology* of G, is a precompact group topology, and it admits as a subbasis of neighborhoods of zero the family of sets $\{\chi\}^\triangleleft$ with χ running over $G^\wedge$.

It is common to consider group topologies on the dual group $G^\wedge$. The most usual one is the *compact-open topology* (topology of uniform convergence on compact subsets of G), which admits as a basis of neighborhoods at 1 the family of all polars of compact subsets of G (this is also an easy consequence of Proposition (1.1)). We will denote by $G^\wedge_c$ the group $G^\wedge$ equipped with the compact-open topology.

The following computation of the dual of $\mathbb{R}$ is classical; we single it out here for later use. For any fixed real number t, let $p_t : \mathbb{R} \to \mathbb{T}$ be the mapping defined by $p_t(\lambda) = p(\lambda t)$ for every $\lambda \in \mathbb{R}$.

PROPOSITION (1.3). *For the additive group $\mathbb{R}$ endowed with its usual topology we have*

$$\mathbb{R}^\wedge = \{p_t : t \in \mathbb{R}\}$$

and the mapping $t \mapsto p_t$ is a topological group isomorphism between $\mathbb{R}$ and $\mathbb{R}^\wedge_c$.

For an elementary proof of this result see for instance [21], Lem. 21.5, Prop. 21.8(a).

In what follows we will be mainly working in the context of a real vector space E endowed with a topology $\mathcal{T}$ such that $(E, \mathcal{T})$ is a topological Abelian group with respect to addition (we will abbreviate this to "an additive group topology" in what follows). A real vector space endowed with an additive group topology will be called *a group-topologized vector space* over $\mathbb{R}$.

Let E be a group-topologized vector space over $\mathbb{R}$. We denote by E^* the set of all continuous linear functionals $f : E \to \mathbb{R}$. The set E^* carries a natural vector space structure; this vector space, when equipped with the topology of uniform convergence on all compact subsets of E, will be denoted by E^*_c.

We want to study the relationship between continuous linear functionals and continuous characters defined on a group-topologized vector space E over $\mathbb{R}$. It is clear that if $f \in E^*$, then $p \circ f \in E^\wedge$.

The following notion arises naturally:

Definition (1.4). A group-topologized vector space E over $\mathbb{R}$ is said to have the *character lifting property* if for every continuous character $\chi : E \to \mathbb{T}$ there exists a continuous linear functional $f : E \to \mathbb{R}$ for which $p \circ f = \chi$.

Proposition (1.3) says, in particular, that $\mathbb{R}$ has the character lifting property. The above quoted result from [20] and [14] can be formulated as the fact that any real topological vector space has the character lifting property. This is our Corollary (3.11) below; an alternative proof of this result can be found in [13], (23.32).

We will show that in some cases the following question has a positive answer: let E be a group-topologized vector space over $\mathbb{R}$ with the character lifting property; is E necessarily a topological vector space?

2. Group topologies on vector spaces

Recall that a group-topologized vector space $(E, \mathcal{T})$ over $\mathbb{R}$ is said to be a *topological vector space* if the map

$$\begin{array}{ccc} \mathbb{R} \times E & \longrightarrow & E \\ (\lambda, x) & \mapsto & \lambda x \end{array}$$

is continuous when considering the usual topology on $\mathbb{R}$, the topology $\mathcal{T}$ on E and the product topology on $\mathbb{R} \times E$.

Next we weaken this requirement in several different ways. Recall that a subset U of a real vector space E is said to be *balanced* if $[-1, 1]U = U$, and *absorbing* if for every $x \in E$ there exists $\alpha > 0$ such that $x \in \lambda U$ whenever $|\lambda| > \alpha$.

Definition (2.1). A group-topologized vector space $(E, \mathcal{T})$ over $\mathbb{R}$ is said to be

(a) *locally absorbing* if every neighborhood of zero in E is absorbing,
(b) *locally balanced* if E has a basis of neighborhoods of zero formed by balanced sets,
(c) a *topological vector group* if for every $\lambda \in \mathbb{R}$, the map

$$\begin{array}{ccc} E & \longrightarrow & E \\ x & \mapsto & \lambda x \end{array}$$

is continuous.

Note that a group-topologized vector space $(E, \mathcal{T})$ over $\mathbb{R}$ is locally absorbing if and only if for every $x \in E$ the mapping $\left[\lambda \in \mathbb{R} \mapsto \lambda x \in E\right]$ is continuous (this simple fact was pointed out in [16]).

Topological vector groups[1] where first explicitly defined and studied in [19], but the condition defining this class had been separately considered before (see e. g. [3], §1).

In the following results and examples we give some information about the relationships among these classes of spaces.

PROPOSITION (2.2) ([3], §1.5, Proposition 4). *Let $(E, \mathcal{T})$ be a group-topologized vector space over $\mathbb{R}$. Then $(E, \mathcal{T})$ is a topological vector space if and only if it is locally balanced and locally absorbing.*

An analogue of Proposition (2.2) is true in the complex case, but not for a general locally compact valued base field. For simplicity we consider only the real case in this paper, but most of the results are easily generalizable to complex spaces.

PROPOSITION (2.3). *Let $(E, \mathcal{T})$ be a group-topologized vector space over $\mathbb{R}$. If $(E, \mathcal{T})$ is locally balanced, then $(E, \mathcal{T})$ is a topological vector group.*

Proof. We must show that $\frac{1}{\lambda}U$ is a neighborhood of zero for any fixed $\lambda \neq 0$ and $U \in \mathcal{N}_0(E)$.

Fix $n \in \mathbb{N}$ such that $|\lambda| \leq n$. Since $\mathcal{T}$ is a group topology, there exists a neighborhood of zero W such that $W + \overset{n}{\ldots} + W \subset U$. In particular $W \subset \frac{1}{n}U$. We may assume that U is balanced, and thus

$$W \subset \frac{1}{n}U \subset \frac{1}{|\lambda|}U = \frac{1}{\lambda}U$$

so $\frac{1}{\lambda}U$ is a neighborhood of zero, as required. □

PROPOSITION (2.4). *Let $(E, \mathcal{T})$ be a locally absorbing topological vector group. If $(E, \mathcal{T})$ is either metrizable or a Baire space then it is a topological vector space.*

Proof. This follows from the fact that a separately continuous biadditive map defined on the product of a Baire group and a metrizable group is continuous ([22], Theorem 11.15). □

[1]Note that the term "vector group" has been used in algebraic Abelian group theory for more general groups than those underlying real vector spaces, and "topological vector group" has been chosen in some references to designate the object we have called "group-topologized vector space".

Example (2.5). A nontrivial vector space endowed with the discrete topology is a locally balanced topological vector group which is not locally absorbing.

Example (2.6). (This example uses some basic facts about group topologies defined by characters; details not provided here can be found in [6], §2.3 or [4], §3). Let H be a dense subgroup of $\mathbb{R}$. Consider the initial topology on $\mathbb{R}$ with respect to the family of characters $\Gamma_H = \{p_h : h \in H\} \subset \mathbb{R}^\wedge$. We will denote this topology by τ_H. Clearly $\tau_{\mathbb{R}} = \sigma(\mathbb{R}, \mathbb{R}^\wedge)$ is the Bohr topology of $\mathbb{R}$.

(a) $(\mathbb{R}, \tau_H)$ is a Hausdorff precompact noncompact topological group. Moreover, τ_H is strictly coarser than the usual topology of $\mathbb{R}$.

b) $(\mathbb{R}, \tau_H)$ is a locally absorbing connected group-topologized vector space over $\mathbb{R}$. Moreover, $(\mathbb{R}, \tau_H)$ is not a topological vector space over $\mathbb{R}$.

(c) $(\mathbb{R}, \tau_H)$ is not locally balanced.

(d) $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ is a locally absorbing precompact topological vector group. Moreover, the scalar multiplication

$$\begin{array}{ccc} \mathbb{R} \times (\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge)) & \longrightarrow & (\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge)) \\ (\lambda, x) & \mapsto & \lambda x \end{array}$$

is sequentially continuous, but not continuous.

(e) If $(\mathbb{R}, \tau_H)$ is a topological vector group, then $H = \mathbb{R}$.

Proof. (a). It is well known that $(\mathbb{R}, \tau_H)$ is a topological group. It is Hausdorff because Γ_H separates points of $\mathbb{R}$. $(\mathbb{R}, \tau_H)$ is a precompact group, because it is topologically isomorphic to a subgroup of the compact group $\mathbb{T}^H$. Clearly, τ_H is coarser than the usual topology $\mathcal{T}_u$ of $\mathbb{R}$. Since $\mathbb{R}$ is not precompact, we get that τ_H is strictly coarser than $\mathcal{T}_u$. (For a proof of this statement which does not use this precompactness argument, see [5].) Assume that $(\mathbb{R}, \tau_H)$ is compact; from this, according to the Open Mapping Theorem (see e. g. [15], Theorem 3) we would deduce that the continuous identity mapping from $\mathbb{R}$ to $(\mathbb{R}, \tau_H)$ is open as well; hence we would get a contradiction: $\mathcal{T}_u = \tau_H$. (However, there exist nontrivial compact topological vector groups, see Remark (5.2).)

(b) Since τ_H is coarser than the usual topology of $\mathbb{R}$, we deduce that $(\mathbb{R}, \tau_H)$ is locally absorbing and connected. $(\mathbb{R}, \tau_H)$ is not a topological vector space over $\mathbb{R}$ because it is precompact and nontrivial.

(c) $(\mathbb{R}, \tau_H)$ is not locally balanced, because otherwise from the first part of (b) and Proposition (2.2) we would get that $(\mathbb{R}, \tau_H)$ is a topological vector space over $\mathbb{R}$, which would contradict the second part of (b). (Actually the only balanced neighborhood of zero in $(\mathbb{R}, \tau_H)$ is the whole group; this follows easily from the direct proof given in [5] of the fact that any τ_H-neighborhood of zero contains arbitrarily large real numbers).

d) Fix $\lambda \in \mathbb{R}$. We need to show that the mapping $x \mapsto m_\lambda(x) := \lambda x$ is $(\sigma(\mathbb{R}, \mathbb{R}^\wedge), \sigma(\mathbb{R}, \mathbb{R}^\wedge))$-continuous. This is true because $p_h \circ m_\lambda = p_{h\lambda}$ for any $h \in \mathbb{R}$. Consequently, the first part is proved. The scalar multiplication is sequentially continuous because $\mathbb{R}$ and $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ have the same convergent sequences (this is a classical result, see for instance [9]). It is not continuous because $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ is not a topological vector space over $\mathbb{R}$.

(e) Fix $\lambda \in \mathbb{R}$. Assume that the mapping $x \mapsto m_\lambda(x) = \lambda x$ is (τ_H, τ_H)-continuous. This implies that for every $h \in H$ the composition $p_h \circ m_\lambda = p_{h\lambda}$ is τ_H-continuous. From this (see [4], Theor. 3.7 or [6], Theor. 2.3.4.) we get that $p_{h\lambda} \in \Gamma_H$ for every $h \in H$. Hence, $h\lambda \in H$ for every $h \in H$. If $(\mathbb{R}, \tau_H)$ is a topological vector group, then for every $\lambda \in \mathbb{R}$ the mapping $x \mapsto m_\lambda(x) = \lambda x$ is (τ_H, τ_H)-continuous. Consequently we get $h\lambda \in H, \forall h \in H, \forall \lambda \in \mathbb{R}$ so $H = \mathbb{R}$. □

Example (2.7). Let E be as in Example (2.5), and $F = (\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ (see Example (2.6)). The space $E \times F$, with the product topology, is a Hausdorff topological vector group which is neither locally absorbing nor locally balanced.

In the remaining of this section we will discuss local convexity and local quasi-convexity of additive group topologies on a real vector space.

Definition (2.8) (cf. [2], 9.1). A group-topologized vector space over $\mathbb{R}$ is said to be a *locally convex vector group* if it has a basis of neighborhoods of zero formed by convex sets.

Locally convex vector groups are clearly locally balanced. From Proposition (2.3) it easily follows that any locally convex vector group is a (locally convex) topological vector group.

Locally convex vector groups were first defined in [19]. They became an interesting object of study in view of their good duality properties (see [11]) and the fundamental role they play in the theory of nuclear groups ([1], [2], [8]).

It is known (see [2], 2.4) that a topological vector space is locally quasi-convex as a group if and only if it is a locally convex topological vector space. It is *not* true that a topological vector group is locally quasi-convex if and only if it is a locally convex vector group; indeed, it is easy to show that the non locally balanced topological vector group $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ (see Example (2.6)) is locally quasi-convex.

In spite of this, something can be said about the relationship between local convexity and local quasi-convexity in topological vector group setting. We first state a Lemma:

LEMMA (2.9) (cf. [7], §18). *Let $(E, \mathcal{T})$ be a group-topologized vector space over $\mathbb{R}$. For every balanced subset B of E, the convex envelope* $\operatorname{co} B$ *of B is contained in $B^{\triangleright\triangleleft}$.*

Proof. Fix $b_1, \dots, b_n$ in B and $t_1, \dots, t_n$ nonnegative numbers such that $\sum_{k=1}^n t_k = 1$. We must show that

$$b = \sum_{k=1}^n t_k b_k \in B^{\triangleright\triangleleft},$$

i.e. that $\chi(b) \in \mathbb{T}_+$ for every $\chi \in B^{\triangleright}$. Fix such a χ and define the characters

$$\chi_k \colon \mathbb{R} \to \mathbb{T}, \quad \chi_k(\alpha) = \chi(\alpha b_k), \quad k \in \{1, \dots, n\}.$$

Since B is balanced and $\chi \in B^{\triangleright}$, we have $\chi_k([-1, 1]) \subset \mathbb{T}_+$ for every $k \in \{1, \dots, n\}$. Hence the characters χ_k are continuous (see Proposition (1.2)). By Proposition (1.3), for every $k \in \{1, \dots, n\}$ there exists $\lambda_k \in \mathbb{R}$ with $\chi_k(\alpha) =$

$p(\lambda_k\alpha)$ for every $\alpha \in \mathbb{R}$. From the inclusion $\lambda_k[-1,1] \subset [-1/4, 1/4] + \mathbb{Z}$ we deduce $\lambda_k \in [-1/4, 1/4]$, for every $k \in \{1, \dots, n\}$. Hence

$$\chi(b) = \chi(\sum_{k=1}^{n} t_k b_k) = \prod_{k=1}^{n} \chi(t_k b_k) \prod_{k=1}^{n} \chi_k(t_k) = p(\sum_{k=1}^{n} \lambda_k t_k) \in \mathbb{T}_+$$

since $\sum_{k=1}^{n} \lambda_k t_k \in \sum_{k=1}^{n} t_k[-1/4, 1/4] \subset [-1/4, 1/4]$. □

PROPOSITION (2.10). *Let E be a group-topologized vector space over $\mathbb{R}$.*

(a) *(Banaszczyk) If E is a locally convex vector group, then it is locally quasi-convex.*

(b) *If E is locally balanced and locally quasi-convex, then it is a locally convex vector group.*

Proof. (a). This statement is contained in the proof of [2], Theorem 15.7; it explicitly coincides with [1], Corollary 9.9.

(b). Fix an arbitrary $U \in \mathcal{N}_0(E)$; we shall find a convex zero neighborhood contained in U. Find a quasi-convex $W_1 \in \mathcal{N}_0(E)$ contained in U. Next find a balanced $W_2 \in \mathcal{N}_0(E)$ contained in W_1. By Lemma (2.9), $\operatorname{co} W_2 \subset W_2^{\triangleright\triangleleft}$. Now, since W_1 is quasi-convex and $W_2 \subset W_1$, it is immediate that $W_2^{\triangleright\triangleleft} \subset W_1^{\triangleright\triangleleft} = W_1$. Hence $\operatorname{co} W_2$ is a convex neighborhood of zero contained in U. □

3. Lifting continuous characters

In this section we shall consider conditions under which a continuous character on a group-topologized vector space E over $\mathbb{R}$ can be lifted (i.e. written as $p \circ f$) to a continuous linear functional $f : E \to \mathbb{R}$. First we prove some related algebraic results:

LEMMA (3.1). *Let E be a real vector space and f, g group homomorphisms from E to $\mathbb{R}$ with $p \circ f = p \circ g$. Then $f = g$.*

Proof. (Suggested by the referee.) Let $h = f - g$. Then $p \circ h = 0$, so $h(E) \subset \ker p = \mathbb{Z}$. Since $h(E)$ is divisible, this is possible only if $h = 0$, i. e. $f = g$. □

PROPOSITION (3.2). *Let E be a real vector space, $\chi : E \to \mathbb{T}$ a character. The following are equivalent:*

(a) *There exists a linear functional $f : E \to \mathbb{R}$ such that $p \circ f = \chi$*

(b) *For every $x \in E$ there exists $\mu_x \in \mathbb{R}$ with*

$$\chi(\lambda x) = p(\lambda \mu_x) \quad \forall \lambda \in \mathbb{R}\,.$$

(c) *For every $x \in E$, the function $[\lambda \in \mathbb{R} \mapsto \chi_x(\lambda) := \chi(\lambda x) \in \mathbb{T}]$ is continuous.*

Proof. (a)⇒(c): Fix $x \in E$; the function $\chi_x : \mathbb{R} \to \mathbb{T}$ is continuous because for any $\lambda \in \mathbb{R}$, $\chi_x(\lambda) = \chi(\lambda x) = p(f(\lambda x)) = p(\lambda f(x))$ and $p : \mathbb{R} \to \mathbb{T}$ is continuous.

(c)⇒(b): Fix $x \in E$; since $\chi : E \to \mathbb{T}$ is a character, the function $\chi_x : \mathbb{R} \to \mathbb{T}$ is a continuous character and (b) is true by Proposition (1.3).

(b)⇒(a): By Lemma (3.1), μ_x is unique. Define thus $f(x) = \mu_x$. To show that f is linear simply use that $\chi(\lambda(x+y)) = p(\lambda(\mu_x + \mu_y))$ for every $\lambda \in \mathbb{R}$ and every $x, y \in E$ and that $\chi(\lambda(\eta x)) = p(\lambda\eta\mu_x)$ for every $\lambda, \eta \in \mathbb{R}$ and $x \in E$. □

Remark (3.3). The following "real character version" of Proposition (3.2) is true: Let E be a real vector space, $\chi\colon E \to \mathbb{T}$ a character. The following are equivalent:

(i). There exists a group homomorphism $f\colon E \to \mathbb{R}$ such that $p \circ f = \chi$.

(ii). For every $x \in E$, the function $[\lambda \in \mathbb{Q} \mapsto \chi_x(\lambda) := \chi(\lambda x) \in \mathbb{T}]$ is continuous.

Using this result one can find a character $\chi\colon E \to \mathbb{T}$ for which (i) does not hold, even for $E = \mathbb{R}$.

COROLLARY (3.4). *Let $(E, \mathcal{T})$ be a locally absorbing group-topologized vector space over $\mathbb{R}$. Then for any $\mathcal{T}$-continuous character $\chi\colon E \to \mathbb{T}$ there exists a unique (not necessarily $\mathcal{T}$-continuous) linear functional $f\colon E \to \mathbb{R}$ such that $p \circ f = \chi$.*

Proof. The existence of such an f follows from implication (c)⇒(a) of Proposition (3.2); the uniqueness follows from Lemma (3.1). □

THEOREM (3.5). *Let $(E, \mathcal{T})$ be a group-topologized vector space over $\mathbb{R}$ and $\chi\colon E \to \mathbb{T}$ be a $\mathcal{T}$-continuous character. The following are equivalent:*

(a) *There exists a $\mathcal{T}$-continuous linear functional $f\colon E \to \mathbb{R}$ with $p \circ f = \chi$.*

(b) (b1) *For every $x \in E$, $\big[\lambda \in \mathbb{R} \mapsto \chi(\lambda x) \in \mathbb{T}\big]$ is continuous, and*

(b2) *The mapping $\big[(\lambda, x) \in \mathbb{R} \times (E, \mathcal{T}) \mapsto \chi(\lambda x) \in \mathbb{T}\big]$ is "locally not onto" at $(0,0)$, i.e. there exist $U \in \mathcal{N}_0(E, \mathcal{T})$ and $\delta > 0$ with $\chi([-\delta, \delta]U) \neq \mathbb{T}$.*

(c) *The mapping $\big[(\lambda, x) \in \mathbb{R} \times (E, \mathcal{T}) \mapsto \chi(\lambda x) \in \mathbb{T}\big]$ is continuous.*

Proof. (a)⇒(c): Note that $\chi(\lambda x) = p(f(\lambda x)) = p(\lambda f(x))$ and thus the biadditive map $\big[(\lambda, x) \mapsto \chi(\lambda x)\big]$ may be written as a composition of continuous maps:

$$(\lambda, x) \in \mathbb{R} \times E \mapsto (\lambda, f(x)) \in \mathbb{R} \times \mathbb{R} \mapsto \lambda f(x) \in \mathbb{R} \mapsto p(\lambda f(x)) \in \mathbb{T}\,.$$

(c)⇒(b) is clear.

(b)⇒(a): Since (b1) is satisfied, by Proposition (3.2) there exists a linear functional $f\colon E \to \mathbb{R}$ such that $p \circ f = \chi$. For f to be continuous it is sufficient its being bounded on a neighborhood of zero. Since (b2) is satisfied there exist $\delta > 0$, $U \in \mathcal{N}_0(E)$ and $\theta \in \mathbb{R}$ such that $p(\theta) \notin \chi([-\delta, \delta]U) = p([-\delta, \delta]f(U))$. Consequently, the sets $[-\delta, \delta]f(U)$ and $\theta + \mathbb{Z}$ are disjoint. From this, since $[-\delta, \delta]f(U)$ is balanced, we get that $[-\delta, \delta]f(U)$ is bounded. Hence, the set $f(U)$ is bounded as well. □

Example (3.6). There are nonliftable characters $\chi \in E^\wedge$ for which $[\lambda \in \mathbb{R} \mapsto \chi(\lambda x) \in \mathbb{T}]$ $(x \in E)$ and $[x \in E \mapsto \chi(\lambda x) \in \mathbb{T}]$ $(\lambda \in \mathbb{R})$ are both continuous, e. g. any nontrivial continuous character defined on the locally absorbing topological vector group $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ (see Example (2.6)).

Example (3.7). There are nonliftable characters $\chi \in E^\wedge$ for which the maps $[x \in E \mapsto \chi(\lambda x) \in \mathbb{T}]$ $(\lambda \in \mathbb{R})$ are continuous and $[(\lambda, x) \in \mathbb{R} \times E \mapsto \chi(\lambda x) \in \mathbb{T}]$ is continuous at zero: actually, if $E = \mathbb{R}$ with the discrete topology, the map $[(\lambda, x) \in \mathbb{R} \times E \mapsto \lambda x \in E]$ is continuous at zero; and E has nonliftable characters (those which are discontinuous with respect to the ordinary topology; see e. g. [13], 25.6).

Next we collect some particular cases of additional properties of the space E which are useful in lifting characters:

PROPOSITION (3.8). *Let $(E, \mathcal{T})$ be a locally balanced group-topologized vector space over $\mathbb{R}$ and let $\chi\colon E \to \mathbb{T}$ be a $\mathcal{T}$-continuous character. The following are equivalent:*

(i) *There exists a $\mathcal{T}$-continuous linear functional $f\colon E \to \mathbb{R}$ with $p \circ f = \chi$.*

(ii) *For every $x \in E$, the function $\left[\lambda \in \mathbb{R} \mapsto \chi(\lambda x) \in \mathbb{T}\right]$ is continuous.*

Proof. In view of Theorem (3.5), only the implication (ii)⇒(i) needs a proof. This implication follows from (b)⇒(a) in Theorem (3.5). Actually, since χ satisfies hypothesis (b1) in this result, we need to show only that χ satisfies (b2), too. Since χ is continuous and E is locally balanced, there exists a balanced neighborhood of zero U in E for which $\chi(U) \subset \mathbb{T}_+ \neq \mathbb{T}$. Then $\chi([-1, 1]U) = \chi(U) \neq \mathbb{T}$. □

PROPOSITION (3.9). *Let E be a group-topologized vector space over $\mathbb{R}$. If E is either metrizable or Baire and $\chi \in E^\wedge$ is such that the biadditive map $[(\lambda, x) \in \mathbb{R} \times E \mapsto \chi(\lambda x) \in \mathbb{T}]$ is separately continuous, then there exists a continuous linear functional $f\colon E \to \mathbb{R}$ with $p \circ f = \chi$.*

Proof. This is another consequence of [22], Theorem 11.15; in view of the implication (c)⇒(a) of Theorem (3.5). □

Recall that E is said to have the character lifting property if every continuous character on E can be lifted to a continuous linear functional. Using this terminology, the equivalence (a)⇔(c) of Theorem (3.5) can be written as follows:

THEOREM (3.10). *Let E be a group-topologized vector space over $\mathbb{R}$. The following conditions are equivalent:*

(i) *E has the character lifting property.*

(ii) *The mapping*

$$\begin{array}{ccc} \mathbb{R} \times E & \longrightarrow & (E, \sigma(E, E^\wedge)) \\ (\lambda, x) & \mapsto & \lambda x \end{array}$$

is jointly continuous.

COROLLARY (3.11) ([14], [20]). *If E is a real topological vector space, then E has the character lifting property.*

Proof. As E is a topological vector space and the topology $\sigma(E, E^\wedge)$ is coarser than the topology of E, the conclusion follows from (ii)⇒(i) in Theorem (3.10). □

In connection with Corollary (3.11) it is natural to pose the following question: *If $(E, \mathcal{T})$ is a group-topologized vector space over $\mathbb{R}$ which has the character lifting property, is then $(E, \mathcal{T})$ a topological vector space?*

If we do not impose on E some extra conditions making the dual group rich enough, the answer to the above question is negative, as the following example shows.

Example (3.12). Let $\mathcal{T}$ be a group topology on $\mathbb{R}$ such that $(\mathbb{R},\mathcal{T})^\wedge = \{1\}$ (such a topology exists, even a metrizable one; see [17]). Then the group-topologized vector space $(\mathbb{R},\mathcal{T})$ trivially has the character lifting property, but it is not a topological vector space.

There is an important class of topological vector groups for which having the character lifting property and being a topological vector space are equivalent conditions:

THEOREM (3.13). *Let E be a locally convex vector group over $\mathbb{R}$. If E has the character lifting property then E is a locally convex topological vector space.*

Proof. By Proposition (2.2), it suffices to show that E is locally absorbing. Suppose this is not so. Since E is locally convex, we can find an $x \in E$ and a balanced convex $U \in \mathcal{N}_0(E)$ such that x not absorbed by U. Since U is balanced and convex, we have $\operatorname{sp} U = \bigcup_{t\in\mathbb{R}} tU$. Hence,

$$\operatorname{sp}\{x\} \cap \operatorname{sp} U = \{0\}\,.$$

Let $E = \operatorname{sp}\{x\} \oplus \operatorname{sp} U \oplus F$ be an algebraic decomposition of E. Fix some discontinuous character $\kappa\colon \mathbb{R} \to \mathbb{T}$. Define now

$$\chi\colon E \to \mathbb{T}, \quad \chi(\mu x + u + f) = \kappa(\mu)$$

for every $u \in \operatorname{sp} U$ and $f \in F$. χ is clearly a character; moreover, it is continuous since $\chi(U) = \{1\}$ (Proposition (1.2)). Let us prove that χ is not liftable. Suppose that there exists $f \in E^*$ with $p \circ f = \chi$. Then, for every $\mu \in \mathbb{R}$,

$$\kappa(\mu) = \chi(\mu x) = p(f(\mu x)) = p(\mu f(x))$$

and κ would be continuous, as a composition of continuous maps:

$$\mu \in \mathbb{R} \mapsto \mu f(x) \in \mathbb{R} \mapsto p(\mu f(x)) \in \mathbb{T}\,.$$

□

Remark (3.14). This proof actually shows that the following variant of Theorem (3.13) is true: Let E be a group-topologized vector space over $\mathbb{R}$. Suppose that E has a basis $\mathcal{B}$ of neighborhoods of zero such that for every $U \in \mathcal{B}$, U is balanced and $\operatorname{sp} U = \bigcup_{t\in\mathbb{R}} tU$. If for every continuous character χ on E there is a linear functional $f\colon E \to \mathbb{R}$ with $p \circ f = \chi$, then E is a topological vector space.

COROLLARY (3.15). *A real vector space endowed with the discrete topology has not the character lifting property unless it is trivial.*

An application of Proposition (2.10) yields the following reformulation of Theorem (3.13) :

PROPOSITION (3.16). *Let E be a group-topologized vector space over $\mathbb{R}$. If E is locally balanced, locally quasi-convex and has the character lifting property, then it is a locally convex topological vector space.*

The following question arises: Is it possible to eliminate the assumption of local balancedness from Proposition (3.16)?

4. Nickolas' theorem and liftable characters

The following result can be extracted from the proof of the main theorem in [18]:

THEOREM (4.1). *Let G be a topological Abelian group which is a k-space. Then for every character χ in the path-connected component of 1 in $G_c^\wedge$ there exists a continuous homomorphism $f\colon G \to \mathbb{R}$ with $\chi = p \circ f$.*

THEOREM (4.2). *Let E be a group-topologized vector space over $\mathbb{R}$ and $\chi\colon E \to \mathbb{R}$ a continuous character. Define for each $t \in \mathbb{R}$ the character $[x \in E \mapsto \chi_t(x) := \chi(tx)]$. Consider the following properties:*

(i) *There exists a continuous linear functional $f\colon E \to \mathbb{R}$ with $p \circ f = \chi$.*

(ii) *For every $x \in E$ the function $[t \in \mathbb{R} \mapsto \chi(tx)]$ is continuous and moreover, the mapping $[t \in \mathbb{R} \mapsto \chi_t \in E_c^\wedge]$ is continuous.*

(iii) *For every $x \in E$ the character $[t \in \mathbb{R} \mapsto \chi(tx)]$ is continuous and moreover, χ is in the path-connected component of 1 in $E_c^\wedge$.*

Then (i)⇒(ii)⇒(iii) and, if E is a k-space, (i)⇔(ii)⇔(iii).

Proof. (i)⇒(ii): Suppose that there exists $f \in E^*$ with $p \circ f = \chi$. Then clearly $[t \in \mathbb{R} \mapsto \chi(tx)]$ is continuous. On the other hand, in order to prove that $[t \in \mathbb{R} \mapsto \chi_t \in E_c^\wedge]$ is continuous we must show that for every compact $K \subset E$ there exists $\varepsilon > 0$ with

$$|t| \leq \varepsilon,\ x \in K \Rightarrow \chi(tx) \in \mathbb{T}_+ .$$

Given $t \in \mathbb{R}$ and $x \in E$, we have $\chi(tx) = p(tf(x))$. Since K is compact, $f(K)$ is a bounded subset of $\mathbb{R}$. From this the existence of such an $\varepsilon > 0$ follows at once.

(ii)⇒(iii) is immediate.

(iii)⇒(i) if E is a k-space: By Proposition (3.2), χ can be lifted to a linear functional f. By Theorem (4.1), χ can be lifted to a continuous homomorphism $g\colon E \to \mathbb{R}$. By Lemma (3.1), $f = g$. □

Example (4.3). Consider the locally balanced topological vector group $E = (\mathbb{R}, \mathcal{T}_d)$, where $\mathcal{T}_d$ denotes the discrete topology. E is metrizable, hence a k-space. By Hamel's classical result ([12]), there exist many nonhomogeneous group homomorphisms $f\colon \mathbb{R} \to \mathbb{R}$. Let f be such a homomorphism. The character $p \circ f\colon E \to \mathbb{T}$ can be lifted to a continuous homomorphism $f\colon E \to \mathbb{R}$ but not to a linear functional. This shows that the condition concerning continuity of $[t \in \mathbb{R} \mapsto \chi(tx)]$ $(x \in E)$ cannot be dispensed with in (iii).

Example (4.4). Let E be the topological vector group $(\mathbb{R}, \sigma(\mathbb{R}, \mathbb{R}^\wedge))$ (Example (2.6)). Consider the (nonliftable) continuous character χ on E defined by $\chi(x) = p(x)$. The space $E_c^\wedge$ is topologically isomorphic with $\mathbb{R}$ with the usual topology; this is a consequence of the classical Glicksberg theorem ([10]), and the fact that $\mathbb{R}$ is topologically self-dual (Proposition (1.3) above). This shows that (ii)⇒(i) does not hold in general.

5. The space of liftable characters

Let E be a group-topologized vector space over $\mathbb{R}$.

For E two kinds of dual objects can be considered: the set E^* of all continuous linear functionals $f\colon E \to \mathbb{R}$ and the set $E^\wedge$ of all continuous characters $\chi\colon E \to \mathbb{T}$.

The set E^* is a vector space over $\mathbb{R}$ and it is usually called *the dual space* of E. The dual space E^* endowed with the compact-open topology will be denoted by E_c^*. (A basis of neighborhoods of zero for the compact-open topology on E^* is given by the sets $K^\circ = \{f \in E^* : f(K) \subset [-1, 1]\}$, where K is an arbitrary compact subset of E.) Note that E_c^* is a Hausdorff locally convex topological vector space.

As we have already mentioned, the set $E^\wedge$ is an Abelian group under pointwise product of characters; it is called *the dual group* of E. We denote by $E_c^\wedge$ the dual group $E^\wedge$ endowed with the compact-open topology. Note that $E_c^\wedge$ is a Hausdorff locally quasi-convex topological Abelian group.

Suppose that E is a topological vector group over $\mathbb{R}$. Then the Abelian group $E^\wedge$ carries a natural vector space structure; namely, if $\chi \in E^\wedge$ and $\lambda \in \mathbb{R}$ we may define $\lambda\chi\colon E \to \mathbb{T}$ by the expression $(\lambda\chi)(x) := \chi(\lambda x)$, $x \in E$. Since the mapping $x \mapsto \lambda x$ is continuous, we get that $\lambda\chi \in E^\wedge$.

PROPOSITION (5.1). *Let E be a topological vector group over $\mathbb{R}$. Then $E_c^\wedge$ is also a topological vector group over $\mathbb{R}$.*

Proof. $E_c^\wedge$ is a topological vector group since λK is a compact subset of E for any $\lambda \in \mathbb{R}$ and any compact $K \subset E$. □

Remark (5.2). Let $E \neq \{0\}$ be a vector space over $\mathbb{R}$ endowed with the discrete topology. Then $E_c^\wedge$ presents an example of a nontrivial **compact** topological vector group over $\mathbb{R}$. Indeed, $E_c^\wedge$ is a topological vector group over $\mathbb{R}$ by Proposition (5.1); $E_c^\wedge$ is nontrivial and compact, because it is the Pontryagin dual group of a nontrivial discrete Abelian group E.

Let us introduce a mapping $P\colon E^* \to E^\wedge$ by the following equality: $P(f) = p \circ f$, $f \in E^*$. Clearly, $P(E^*)$ is the subgroup of $E^\wedge$ formed by those characters which are liftable to continuous linear forms. We denote by $P(E^*)_c$ the set $P(E^*)$ endowed with compact-open topology.

PROPOSITION (5.3). *Let E be a topological vector group. Then*

(a) $P\colon E_c^* \to E_c^\wedge$ *is an injective continuous linear mapping;*

(b) $P(E^*)_c$ *is a locally absorbing topological vector group.*

Proof. (a) is straightforward.

(b): Since by Proposition (5.1) $E_c^\wedge$ is a topological vector group, its subspace $P(E^*)_c$ is a topological vector group as well.

To show that $P(E^*)_c$ is locally absorbing, for any $\chi \in P(E^*)$ and any compact $K \subset E$ we must find $\varepsilon > 0$ with $\chi([-\varepsilon, \varepsilon]K) \subset \mathbb{T}_+$. It suffices to choose $\varepsilon > 0$ with $f(K) \subset \left[-\frac{1}{4\varepsilon}, \frac{1}{4\varepsilon}\right]$, where $f \in E^*$ is the lifting of χ. □

For a topological vector group E the mapping $P\colon E_c^* \to E_c^\wedge$ may fail to be a topological embedding (this is for instance the case if E is $\mathbb{R}$ endowed with the discrete topology). Below we present two non-pathological cases.

PROPOSITION (5.4). *Let E be a hemicompact topological vector group. Then $P\colon E_c^* \to E_c^\wedge$ is a topological embedding (equivalently, $P(E^*)_c$ is a topological vector space).*

Proof. Since E is hemicompact, $E_c^\wedge$ (and thus, $P(E^*)_c$) is a metrizable group. By Proposition (5.3), the biadditive map $[(\lambda,\chi) \in \mathbb{R} \times P(E^*)_c \mapsto \lambda\chi \in P(E^*)_c]$ is separately continuous. By [22], Theorem 11.15; it is jointly continuous. □

PROPOSITION (5.5). *Let E be a topological vector group. Suppose that every compact subset of E is contained in some connected (or balanced) compact set. Then $P\colon E_c^* \to E_c^\wedge$ is a topological embedding (equivalently, $P(E^*)_c$ is a topological vector space).*

Proof. We know that $P\colon E_c^* \to E_c^\wedge$ is a continuous monomorphism. To show that it is open onto its image we need to prove the following: For every compact $K \subset E$ there exists a compact $S \subset E$ such that $P(K^\circ) \supset S^\triangleright \cap P(E^*)$; equivalently,

$$f \in E^*,\ f(S) \subset [-1/4, 1/4] + \mathbb{Z} \ \Rightarrow\ f(K) \subset [-1, 1].$$

Given a compact $K \subset E$, choose a compact and connected (resp. a compact and balanced) S which contains K. Since $f(S)$ is a connected set for any $f \in E^*$, if $f(S) \subset [-1/4, 1, 4] + \mathbb{Z}$ then in fact $f(S) \subset [-1/4, 1, 4]$ and thus $f(K) \subset [-1/4, 1, 4]$. □

COROLLARY (5.6) ([20]). *If E is a topological vector space, then P is a topological isomorphism between E_c^* and $E_c^\wedge$.*

Proof. This follows from Proposition (5.5) together with Corollary (3.11). □

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HÖLDER ESTIMATES FOR THE $\overline{\partial}$-EQUATION ON SINGULAR QUOTIENT VARIETIES

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ABSTRACT. Let $\mathbb{C}^n/\mathcal{G}$ be a singular quotient variety, where $n \geq 1$ and $\mathcal{G}$ is a finite group of complex unitary matrices. The main objective of this paper is to present a general technique for calculating Hölder continuous solutions to the $\overline{\partial}$-equation on the singular variety $\mathbb{C}^n/\mathcal{G}$. We then calculate Hölder continuous solutions to the $\overline{\partial}$-equation on any surface with a simple (rational double point) isolated singularity.

1. Introduction

As it is well known, solving the $\overline{\partial}$-equation forms a main part of complex analysis, but also has deep consequences on algebraic geometry, partial differential equations and other areas. In general, it is not easy to solve the $\overline{\partial}$-equation. The existence of solutions depends mainly on the geometry of the variety on which the equation is considered. There is a vast literature about this subject on smooth manifolds, both in books and papers, but the theory on singular varieties has only been developed recently.

Given a $\overline{\partial}$-closed $(0,1)$-differential form λ well defined on the regular part of a singular subvariety Σ of $\mathbb{C}^m$, Gavosto, Fornæss and Ruppenthal have produced a general technique for calculating Hölder continuous solutions h to the $\overline{\partial}$-equation $\lambda = \overline{\partial}h$ on the regular part of Σ; see for example [6], [5] and [11]. Their basic idea was to analyse Σ as an analytic covering over $\mathbb{C}^n$, to solve the corresponding $\overline{\partial}$-equation on $\mathbb{C}^n$, and to *lift* the solution from $\mathbb{C}^n$ into Σ again. An alternative technique for solving the $\overline{\partial}$-equation on surfaces with an isolated simple (rational double point) singularity was later proposed by Acosta and Zeron in [1], [2].

The main objective of this paper is to extend Acosta and Zeron's technique in order to calculate Hölder continuous solutions to the $\overline{\partial}$-equation on singular quotient varieties. We define the quotient varieties $\mathbb{C}^n/\mathcal{G}$ via the natural action of groups of unitary matrices $\mathcal{G}$ on $\mathbb{C}^n$, for $n \geq 1$ and with the obvious matrix multiplication. Thus, the quotient variety $\mathbb{C}^n/\mathcal{G}$ is the space of different orbits $\{Hz \,|\, H\in\mathcal{G}\}$ with $z \in \mathbb{C}^n$. Recall that a square $[n\times n]$-complex matrix H is called unitary if the Euclidean norm $\|Hz\| = \|z\|$ is invariant for every point z in $\mathbb{C}^n$. We also need the following definition of critical spaces.

Definition (1.1)**.** The critical space Σ of a finite group of unitary matrices $\mathcal{G}$ is composed by all the vectors $z \in \mathbb{C}^n$ such that $Hz = z$ for some element $H \in \mathcal{G}$ different from the identity matrix. The critical space $\Sigma \subset \mathbb{C}^n$ is then the union

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of a finite number of proper linear subspaces of $\mathbb{C}^n$, and it is also an analytic space of codimension greater than or equal to one.

Let $\mathcal{G}$ be a finite group of unitary $[n{\times}n]$-matrices, and $\Sigma \subset \mathbb{C}^n$ be its critical space. The group $\mathcal{G}$ acts freely and properly discontinuous on the open set $\mathbb{C}^n \setminus \Sigma$, because Σ contains the fixed points of every element in $\mathcal{G}$ different from the identity matrix. Thus, the quotient mapping π defined from $\mathbb{C}^n$ onto the singular quotient variety $\mathbb{C}^n/\mathcal{G}$ is locally a biholomorphism in $\mathbb{C}^n \setminus \Sigma$. Moreover, the image $\pi(\Sigma)$ is the singular set of $\mathbb{C}^n/\mathcal{G}$ and π is an orbifold covering in $\mathbb{C}^n$. On the other hand, there exists a natural metric $D_{\mathcal{G}}(\cdot)$ induced on the quotient variety $\mathbb{C}^n/\mathcal{G}$. Given a pair of points x and y in $\mathbb{C}^n/\mathcal{G}$, we define $D_{\mathcal{G}}(x,y)$ as the Euclidean metric between the inverse fibres of the quotient mapping $\pi^{-1}(x)$ and $\pi^{-1}(y)$ in $\mathbb{C}^n$. It is easy to prove that $D_{\mathcal{G}}(\cdot)$ is a well defined metric function, because the norm $\|Hz\| = \|z\|$ is invariant for all $H \in \mathcal{G}$ and $z \in \mathbb{C}^n$.

Finally, a quotient variety $\mathbb{C}^n/\mathcal{G}$ is said to be **well embedded** in $\mathbb{C}^p$, with $p > n$, whenever the quotient mapping π defined from $\mathbb{C}^n$ onto $\mathbb{C}^n/\mathcal{G}$ may be seen as a holomorphic mapping from $\mathbb{C}^n$ into $\mathbb{C}^p$; a characterisation is presented in Lemma (2.1). In particular, given a variety $\mathbb{C}^n/\mathcal{G}$ well embedded in $\mathbb{C}^p$, we say that a $(0,1)$-differential form λ is defined and continuous on an open subset U of $\mathbb{C}^n/\mathcal{G}$ if and only if there exist continuous functions f_k on U such that λ is equal to the finite sum $\sum_k f_k d\overline{w}_k$ on the regular part of U, and where w_k are the axes of $\mathbb{C}^p$. The operator $\overline{\partial}$ must then be computed in terms of distributions. We may now restate Henkin and Leiterer results on the $\overline{\partial}$-equation [9].

THEOREM (1.2). *Let $\mathbb{C}^n/\mathcal{G}$ be a singular quotient variety well embedded in $\mathbb{C}^p$, where $\mathcal{G}$ is a finite group of unitary matrices and $p{>}n{>}0$. Consider the quotient mapping π from $\mathbb{C}^n$ onto $\mathbb{C}^n/\mathcal{G}$, and the natural metric $D_{\mathcal{G}}(\cdot)$ induced in $\mathbb{C}^n/\mathcal{G}$. Given any exponent $0{<}\delta{<}1$ and any open ball B_R in $\mathbb{C}^n$ of finite radius $R > 0$ and centre in the origin, there are finite positive constants $C_1(R)$ and $C_2(R,\delta)$ such that: For every continuous $(0,1)$-differential form λ defined on an open neighbourhood of $\pi(\overline{B_R})$, and $\overline{\partial}$-closed on the regular part of $\pi(B_R)$, there exists a continuous function h defined on $\pi(B_R)$ which satisfies both the equation $\overline{\partial}h = \lambda$ on the regular part of $\pi(B_R)$ and the pair of Hölder estimates:*

$$\|h\|_{\pi(B_R)} + \sup_{x,y\in\pi(B_R)} \frac{|h(x)-h(y)|}{D_{\mathcal{G}}(x,y)^{1/2}} \leq C_1(R)\|\lambda\|_{\pi(B_R)}, \tag{1.3}$$

$$\text{and} \quad \sup_{x,y\in\pi[B_{R/2}]} \frac{|h(x)-h(y)|}{D_{\mathcal{G}}(x,y)^{\delta}} \leq C_2(R,\delta)\|\lambda\|_{\pi(B_R)}. \tag{1.4}$$

This theorem is shown in the second section of this paper. Notice that the regular part of $\pi(B_R)$ is obtained by removing the singular set $\pi(\Sigma)$, and that the notation $\|h\|_E$ stands for the supremum of $|h|$ on the set E. However, the condition that the group of matrices $\mathcal{G}$ is finite cannot be relaxed in Theorem (1.2). Consider for example the infinite group $\mathcal{S}$ generated by the biholomorphism $z \mapsto e^{i}z$ from $\mathbb{C} \setminus \{0\}$ onto itself, we have that the quotient $\mathbb{C}/\mathcal{S}$ is not even a singular manifold, for there is no point in $\mathbb{C}/\mathcal{S}$ with an open neighbourhood biholomorphic to an open set of $\mathbb{C}$. Moreover, if we consider the infinite group $\mathcal{T}$ generated by the isometries $z \mapsto 1{+}z$ and $z \mapsto i{+}z$ of $\mathbb{C}$, we will have that the quotient $\mathbb{C}/\mathcal{T}$ is a torus (a non Stein manifold), and so the $\overline{\partial}$-equation cannot be solved in general.

On the other hand, one can easily generalise Theorem (1.2) in order to include (p, q)-differential forms λ. Nevertheless, we are not going to analyse these ideas here. Instead, we want to explain why the Hölder exponents $1/2$ and $0<\delta<1$ used in the pair of equations (1.3)-(1.4) are quite different from the original exponents calculated by Acosta and Zeron [1], [2]. We shall prove in the Hölder estimate (3.12) of Theorem (3.10) that this difference originates (and strongly depends) on the way $\mathbb{C}^n/\mathcal{G}$ is embedded in the space $\mathbb{C}^p$, where $p > n$.

The next section is completely devoted to prove Theorem (1.2). A detailed analysis on the embeddings of singular quotient varieties into some space $\mathbb{C}^m$ is done in the third chapter of this work. Finally, in the last section of this paper, we calculate Hölder continuous solution to the $\overline{\partial}$-equation on any surface with a simple (rational double point) isolated singularity.

2. Proof of Theorem (1.2).

The main ideas in the proof of Theorem (1.2) were already presented in [2]. Nevertheless, we include this proof for the sake of completeness. We begin by presenting the following characterisation of the quotient varieties which are well embedded in the space $\mathbb{C}^p$.

LEMMA (2.1). *Let $\mathbb{C}^n/\mathcal{G}$ be a singular quotient variety, where $\mathcal{G}$ is a finite group of unitary matrices and $n \geq 1$. Consider the quotient mapping π from $\mathbb{C}^n$ onto the variety $\mathbb{C}^n/\mathcal{G}$, and a topological embedding J from $\mathbb{C}^n/\mathcal{G}$ into the space $\mathbb{C}^p$, for $p > n$. The composition $J \circ \pi$ is a holomorphic mapping from $\mathbb{C}^n$ into $\mathbb{C}^p$ if and only if the topological embedding J is holomorphic in the regular part of $\mathbb{C}^n/\mathcal{G}$.*

Proof. Recall that the topological embedding J is an injective, continuous and closed mapping from $\mathbb{C}^n/\mathcal{G}$ into $\mathbb{C}^p$; so that J is a homeomorphism onto its image. Let $\Sigma \subset \mathbb{C}^n$ be the critical space of the finite group $\mathcal{G}$, given in Definition (1.1). We automatically have that Σ is an analytic set of codimension greater than or equal to one, and that the regular part of $\pi(B_R)$ is obtained by removing the singular set $\pi(\Sigma)$.

We easily have that J is holomorphic in the regular part of $\mathbb{C}^n/\mathcal{G}$ if and only if $J \circ \pi$ is also holomorphic in $\mathbb{C}^n \setminus \Sigma$, because π is locally a biholomorphism from $\mathbb{C}^n \setminus \Sigma$ onto the regular part of the variety $\mathbb{C}^n/\mathcal{G}$. On the other hand, notice that $J \circ \pi$ is a continuous function defined from $\mathbb{C}^n$ into $\mathbb{C}^p$. Whence, since Σ is an analytic set of codimension greater than or equal to one, the composition $J \circ \pi$ is holomorphic in $\mathbb{C}^n$ if and only if it is also holomorphic in $\mathbb{C}^n \setminus \Sigma$; recall the Riemann extension theorem for continuous functions. $\square$

We need as well the following simple lemma on the extension of $\overline{\partial}$-closed differential forms. This lemma is strongly inspired on Lemma (2.1) of Acosta and Zeron [2].

LEMMA (2.2). *Let Σ be a smooth manifold in $\mathbb{C}^n$ of complex codimension greater than or equal to one, with $n \geq 1$. Given a continuous $(0, 1)$-differential* *form $\aleph$ defined on an open set U of $\mathbb{C}^n$, and $\overline{\partial}$-closed in $U \setminus \Sigma$, the differential $\overline{\partial}\aleph$ vanishes everywhere in U.*

Proof. The result is trivial when $n = 1$, for every continuous $(0,1)$-differential form is $\overline{\partial}$-closed in $\mathbb{C}$; so we suppose from now on that $n \geq 2$. Moreover, since the differential $\overline{\partial}$ is calculated locally, we can fix U to be a small open ball. Finally, the differential $\overline{\partial}\aleph$ is calculated in terms of distributions, so the given hypotheses imply that $\int_U \aleph\wedge\overline{\partial}\eta$ vanishes for every smooth $(n, n-2)$-differential form η with compact support in $U \setminus \Sigma$. And we must prove that the same integral vanishes for differential forms with compact support in U.

Define $\rho(z)$ to be the Euclidean metric between the point $z \in \mathbb{C}^n$ and the compact set $\Sigma \cap \overline{U}$. We may easily find an open neighbourhood W of $\Sigma \cap \overline{U}$ such that ρ is smooth (and the differential $\overline{\partial}\rho$ is bounded) in the open set $W \setminus \Sigma$, because the intersection $\Sigma \cap U$ is a smooth manifold of complex codimension $d \geq 1$. Moreover, we also have that the $2n$-volume of the level sets $\{\rho \leq r\}$ is of order $O(r^{2d})$ for small real numbers $r > 0$. In other words, there is a finite bound $b > 0$ such that ρ satisfies the following conditions for all points z in $W \setminus \Sigma$ and indexes k :

$$\left|\frac{\partial\rho(z)}{\partial\bar{z}_k}\right| \leq b, \quad \Sigma \cap \overline{U} = \{\rho \leq 0\} \quad \text{and} \quad \lim_{r\to 0+} \frac{\mathrm{Vol}\{\rho \leq r\}}{r} = 0\,;$$

where $\mathrm{Vol}(E)$ denotes the $2n$-volume of the measurable set $E \subset \mathbb{C}^n$. Consider now a smooth partition of the unit $\xi(r)$ on the real line with :

$$0 \leq \xi(r) \leq 1, \quad \xi(r) = \begin{cases} 0 \text{ if } r \leq 1/5, \\ 1 \text{ if } r \geq 1; \end{cases} \quad \text{and} \quad |\xi'(r)| \leq 5\,.$$

Notice that $\int_U \aleph\wedge\overline{\partial}\big[\xi(\rho/r)\omega\big]$ vanishes for all smooth $(n, n-2)$-differential forms ω with compact support on U, and all real numbers $r > 0$ small enough such that the level set $\{\rho \leq r\}$ is contained in W. We just need to recall that $\aleph$ is a $(0,1)$-differential form continuous on U and $\overline{\partial}$-closed in $U \setminus \Sigma$. Moreover, the form $\xi(\rho/r)\omega$ has compact support in $U \setminus \Sigma$. Calculating by parts the differential $\overline{\partial}$ of the product $\xi(\rho/r)\omega$ yields that,

$$\left|\int_U \xi(\rho/r)\aleph\wedge\overline{\partial}\omega\right| = \left|\int_U \aleph\wedge\omega\wedge\overline{\partial}\xi(\rho/r)\right| \leq \|\aleph\wedge\omega\|_U \frac{5bn}{r}\,\mathrm{Vol}\{\rho \leq r\},$$

where $\overline{\partial}\xi(\rho/r)$ vanishes outside of the level set $\{\rho \leq r\}$. Moreover, the form $\aleph\wedge\omega$ has finite norm because it is continuous and has compact support on U. On the other hand, we also have that,

$$\begin{aligned}\left|\int_U \aleph\wedge\overline{\partial}\omega\right| \quad &\leq \quad \left|\int_U \big[1 - \xi(\rho/r)\big]\aleph\wedge\overline{\partial}\omega\right| + \left|\int_U \xi(\rho/r)\,\aleph\wedge\overline{\partial}\omega\right| \\ &\leq \quad \Big(\|\aleph\wedge\overline{\partial}\omega\|_U + \frac{5bn}{r}\|\aleph\wedge\omega\|_U\Big)\,\mathrm{Vol}\{\rho \leq r\} < \infty.\end{aligned}$$

Finally, when $r > 0$ converges to zero, we obtain that $\int_U \aleph\wedge\overline{\partial}\omega$ vanishes for every smooth $(n, n-2)$-differential form ω with compact support on U, and so the form $\aleph$ is $\overline{\partial}$-closed everywhere in U. □

We have the same result for analytic spaces.

LEMMA (2.3). *Let Σ be an analytic set in $\mathbb{C}^n$ of complex codimension greater than or equal to one, with $n \geq 1$. Given a continuous $(0,1)$-differential form $\aleph$ defined on an open set U of $\mathbb{C}^n$, and $\overline{\partial}$-closed in $U \setminus \Sigma$, the differential $\overline{\partial}\aleph$ vanishes everywhere in U.*

Proof. Define $\Sigma_1 := \Sigma$ to be an analytic set of codimension $d \geq 1$. Recall that $\overline{\partial}$ is calculated locally, and that the regular part Σ_1 is locally a smooth manifold. Since $\aleph$ is $\overline{\partial}$-closed in $U \setminus \Sigma_1$, a direct application of Lemma (2.2) yields that $\overline{\partial}\aleph$ vanishes in $U \setminus \Sigma_2$, where Σ_2 is the singular part of Σ_1. The new analytic set Σ_2 has codimension $d \geq 2$, so can we apply Lemma (2.2) in an inductive process until we conclude that $\overline{\partial}\aleph$ vanishes everywhere in U. Recall that the empty set has codimension $n{+}1$. □

We need as well the following Henkin and Leiterer estimates, deduced from Theorems 2.1.5 and 2.2.2 of [9]. The proof involves reformulating some parts of Lemma 2.2.1 in [9], and the details may be found in [1] or [2].

THEOREM (2.4). *Given an exponent $0<\delta<1$ and an open ball B_R in $\mathbb{C}^n$ of radius $R > 0$ and centre in the origin, for $n \geq 1$, there are finite positive constants $C_3(R)$ and $C_4(R,\delta)$ such that: For every continuous $(0,1)$-differential form $\aleph$ defined on the compact set $\overline{B_R}$, and $\overline{\partial}$-closed on the interior B_R, there exists a continuous function g defined on B_R which satisfies both the equation $\overline{\partial}g = \aleph$ and the pair of Hölder estimates:*

$$\|g\|_{B_R} + \sup_{z,\zeta\in B_R} \frac{|g(z)-g(\zeta)|}{\|z-\zeta\|^{1/2}} \leq C_3(R)\|\aleph\|_{B_R}, \tag{2.5}$$

$$\text{and} \quad \sup_{z,\zeta\in B_{R/2}} \frac{|g(z)-g(\zeta)|}{\|z-\zeta\|^{\delta}} \leq C_4(R,\delta)\|\aleph\|_{B_R}. \tag{2.6}$$

We are now in position to prove Theorem (1.2), one of the main results of this paper.

Proof of Theorem (1.2). Let $\mathcal{G}$ be a finite group of unitary matrices, and $\mathbb{C}^n/\mathcal{G}$ be a singular quotient variety well embedded in $\mathbb{C}^p$, so that the hypotheses of Lemma (2.1) hold with $p>n>0$. Consider the quotient mapping π from $\mathbb{C}^n$ onto $\mathbb{C}^n/\mathcal{G}$, and the natural metric $D_{\mathcal{G}}(\cdot)$ defined as follows: given two points x and y in $\mathbb{C}^n/\mathcal{G}$, the value of $D_{\mathcal{G}}(x,y)$ is the Euclidean metric between the inverse fibres $\pi^{-1}(x)$ and $\pi^{-1}(y)$ in $\mathbb{C}^n$. We shall verify that $D_{\mathcal{G}}(\cdot)$ is indeed a metric function. Recall that the fibre $\pi^{-1}(x)$ is an orbit $\{Hz|H\in\mathcal{G}\}$ with $\pi(z) = x$. The finiteness of the fibres implies that $D_{\mathcal{G}}(x,y)$ vanishes if and only if $\pi^{-1}(x)$ and $\pi^{-1}(y)$ are the same orbit, that is, if and only if $x = y$ in $\mathbb{C}^n/\mathcal{G}$. Moreover, the fact that the norm $\|Hz\| = \|z\|$ is invariant for all $H \in \mathcal{G}$ and $z \in \mathbb{C}^n$, allows us to deduce that for each $\widehat{z}$ in $\pi^{-1}(x)$ there exists a second point ζ in $\pi^{-1}(y)$ such that $D_{\mathcal{G}}(x,y)$ is equal to $\|\widehat{z}-\zeta\|$. Whence, given three points $\{x_k\}$ in $\mathbb{C}^n/\mathcal{G}$, we can find relate points z_k in $\pi^{-1}(x_k)$ such that $D_{\mathcal{G}}(x_1,x_2)$ and $D_{\mathcal{G}}(x_2,x_3)$ are respectively equal to $\|z_1-z_2\|$ and $\|z_2-z_3\|$. The metric $D_{\mathcal{G}}(x_1,x_3)$ is then less than or equal to $\|z_1-z_3\|$, by definition, and it is also less than or equal to the sum of $D_{\mathcal{G}}(x_1,x_2)$ plus $D_{\mathcal{G}}(x_2,x_3)$. This inequality proves that $D_{\mathcal{G}}(\cdot)$ is a well defined metric function.

On the other hand, we have that the quotient mapping π can be seen as a holomorphic function from $\mathbb{C}^n$ into $\mathbb{C}^p$, because Lemma (2.1) and since $\mathbb{C}^n/\mathcal{G}$ is well embedded in $\mathbb{C}^p$. Hence, given any open ball B_R in $\mathbb{C}^n$ of finite radius $R > 0$ and centre in the origin, the partial derivatives of π are all continuous and bounded mappings on the compact closure $\overline{B_R}$. Thus, there exists a finite positive constant $C_5(R)$ such that the following inequality holds for all continuous $(0,1)$-differential forms λ defined on an open neighbourhood of the compact set $\pi(\overline{B_R})$,

$$\|\pi^*\lambda\|_{B_R} \le C_5(R)\|\lambda\|_{\pi(B_R)}. \tag{2.7}$$

Let $\Sigma \subset \mathbb{C}^n$ be the critical space of the finite group of unitary matrices $\mathcal{G}$, given in Definition 1.1. We automatically have that Σ is an analytic set of codimension greater than or equal to one, and that the regular part of $\pi(B_R)$ is obtained by removing the singular set $\pi(\Sigma)$. Suppose that the continuous differential form λ is $\overline{\partial}$-closed on the regular part of $\pi(B_R)$. The pull-back $\aleph = \pi^*\lambda$ is then a continuous $(0,1)$-differential form well defined on an open neighbourhood of $\overline{B_R}$, and $\overline{\partial}$-closed in the open set $B_R \setminus \Sigma$, because π is locally a biholomorphism from $B_R \setminus \Sigma$ onto the regular part of $\pi(B_R)$. Whence, considering Lemma (2.3) and Theorem (2.4), we automatically have that the differential $\overline{\partial}(\pi^*\lambda)$ vanishes and the equation $\overline{\partial}g = \pi^*\lambda$ has a continuous solution g on B_R which satisfies the pair of Hölder estimates stated in (2.5) and (2.6) for a fixed exponent $0<\delta<1$.

Notice that the *pull-back* H^*g is also a continuous function well defined on B_R for every H in $\mathcal{G}$, because the norm $\|Hz\| = \|z\|$ is invariant. Thus, the finite sum $\sum_{\mathcal{G}} H^*g$, added over all matrices $H \in \mathcal{G}$, is both defined on B_R and constant in the fibres of π (it is invariant under every *pull back* H^*). Hence, if $|\mathcal{G}|$ is the cardinality of the group $\mathcal{G}$, there exists a continuous function h defined on $\pi(B_R)$ such that the pull-back $(\pi^*h)|\mathcal{G}|$ is equal to the sum $\sum_{\mathcal{G}} H^*g$. We assert that $\overline{\partial}h = \lambda$ on the regular part of $\pi(B_R)$. This result follows automatically, for:

$$\pi^*\overline{\partial}h = \sum_{H\in\mathcal{G}} \frac{\overline{\partial}\, H^*g}{|\mathcal{G}|} = \sum_{H\in\mathcal{G}} \frac{H^*\pi^*\lambda}{|\mathcal{G}|} = \pi^*\lambda.$$

Recall that the projection $\pi(Hz) = \pi(z)$ is invariant under the natural action of every matrix H in the finite group $\mathcal{G}$. Moreover,

$$\|h\|_{\pi(B_R)} = \left\| \frac{1}{|\mathcal{G}|} \sum_{H\in\mathcal{G}} H^*g \right\|_{B_R} \le \|g\|_{B_R}. \tag{2.8}$$

Now then, given two points x and y in the image $\pi(B_R)$, the natural metric $D_{\mathcal{G}}(x,y)$ is defined as the Euclidean metric between the finite fibres $\pi^{-1}(x)$ and $\pi^{-1}(y)$. Whence, we can choose two relate points z and ζ in the ball B_R such that the norm $\|z-\zeta\|$ is equal to $D_{\mathcal{G}}(x,y)$, and their respective projections satisfy $x = \pi(z)$ and $y = \pi(\zeta)$. Recall that the fibre $\pi^{-1}(x)$ is equal to the orbit $\{Hz|H\in\mathcal{G}\}$, and that this orbit is completely contained in B_R because the norm $\|Hz\| = \|z\|$ is invariant for every H in $\mathcal{G}$. We may now apply equation (2.5) of Theorem (2.4), with $\aleph = \pi^*\lambda$, in order to deduce that,

$$\frac{|h(x)-h(y)|}{D_{\mathcal{G}}(x,y)^{1/2}} \le \frac{1}{|\mathcal{G}|} \sum_{H\in\mathcal{G}} \frac{|g(Hz)-g(H\zeta)|}{\|Hz-H\zeta\|^{1/2}} \le C_3(R)\|\pi^*\lambda\|_{B_R}. \tag{2.9}$$

Besides, whenever x and y are both inside $\pi(B_{R/2})$, we may apply equation (2.6), in order to deduce the following inequality for a fixed exponent $0<\delta<1$,

$$\frac{|h(x) - h(y)|}{D_{\mathcal{G}}(x, y)^{\delta}} \leq C_4(R, \delta)\|\pi^*\lambda\|_{B_R}. \tag{2.10}$$

Finally, considering Theorem (2.4) and equations (2.7) to (2.10), we can deduce the existence of finite positive constants $C_1(R)$ and $C_2(R, \delta)$ such that equations (1.3) and (1.4) in Theorem (1.2) holds. □

3. Embeddings of quotient varieties

Let $\mathcal{G}$ be a finite group of $[n\times n]$-unitary matrices, and $\mathbb{C}^n/\mathcal{G}$ be a singular quotient variety well embedded in $\mathbb{C}^p$, so that the hypotheses of Lemma (2.1) hold with $p>n>0$. One automatically has that the quotient projection π can be seen as a holomorphic function from $\mathbb{C}^n$ into $\mathbb{C}^p$, and so it is interesting to determinate whether there exists a relation between $\|x-y\|$ and the natural metric $D_{\mathcal{G}}(x, y)$ for every pair of points x and y in $\pi(\mathbb{C}^n)$. This chapter is devoted to do a partial analysis of the possible relations between these metric functions.

We restrict ourselves to analyse those quotient varieties $\mathbb{C}^n/\mathcal{G}$ which are well embedded in $\mathbb{C}^p$ and whose singular set is the origin of $\mathbb{C}^p$. Moreover, we assume that the holomorphic quotient mapping π defined from $\mathbb{C}^n$ into $\mathbb{C}^p$ is polynomial, and that each entry π_k is a homogeneous polynomial of degree $\eta_k \geq 1$. That is, each entry $\pi_k(sz)$ is equal to $s^{\eta_k}\pi_k(z)$ for all $s \in \mathbb{C}$ and $z \in \mathbb{C}^n$; and so $\pi(0) = 0$. The surfaces with an isolate simple (rational double point) singularity are the perfect model of these quotient varieties, as we shall see in the next chapter.

The fact that the singular set of $\pi(\mathbb{C}^n) = \mathbb{C}^n/\mathcal{G}$ is the origin implies that the critical space of $\mathcal{G}$ satisfies $\Sigma = \{0\}$, according to (1.1). Notice that Σ is contained in $\pi^{-1}(0)$, and that $\pi^{-1}(0)$ is the orbit of $\mathcal{G}$ composed exclusively by the origin. Thus, whenever the equality $Hz = z$ holds for some vector $z \in \mathbb{C}^n$ and matrix $H \in \mathcal{G}$, then, either $z = 0$ is the origin or H is the identity matrix. Consider the following estimate.

LEMMA (3.1). *Let π be a mapping defined from $\mathbb{C}^n$ into $\mathbb{C}^p$ which is locally a biholomorphism from $\mathbb{C}^n \setminus \{0\}$ onto its image, for $p>n>0$, and such that each entry π_k is a homogeneous polynomial of degree $\eta_k \geq 2$. Then, there are two finite constants $C_6 > 0$ and $\Lambda \gg 1$ which satisfy the following inequality for all w and z in $\mathbb{C}^n$ with norms $\|w\| \leq 1$ and $\|z\| \geq \Lambda$,*

$$\sum_{k=1}^{p} \frac{|\pi_k(z + w) - \pi_k(z)|}{\|z\|^{\eta_k-1}\|w\|} \geq C_6. \tag{3.2}$$

Proof. Recall that each homogeneous polynomial π_k of degree η_k on $\mathbb{C}^n$ has an associate symmetric multilinear form Π_k which is defined on the Cartesian product $[\mathbb{C}^n]^{\eta_k}$, so it has $\eta_k \geq 2$ entries, and it satisfies the main relation: $\pi_k(z)$ is equal to $\Pi_k(z, ..., z)$ for every $z \in \mathbb{C}^n$. One may apply the polynomial algebra theory [7, p. 221–226], or use the following polarisation formula for all elements $s_j \in \mathbb{C}$ and $z_j \in \mathbb{C}^n$,

$$\Pi_k(z_1, z_2, ..., z_{\eta_k}) = \frac{1}{\eta_k!} \cdot \frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_{\eta_k}} \pi_k(s_1 z_1 + \cdots + s_{\eta_k} z_{\eta_k}).$$

Thus, every entry π_k has a simplified Taylor polynomial expansion for all vectors w and z in $\mathbb{C}^n$,

$$\pi_k(z+w) = \pi_k(z) + \sum_{j=1}^{\eta_k} \binom{\eta_k}{j} \Pi_k\big(\overbrace{w, ..., w}^{j}, \overbrace{z, ..., z}^{\eta_k - j}\big). \tag{3.3}$$

The j-term in the sum above has order $O\big(\|w\|^j \|z\|^{\eta_k - j}\big)$. Hence, there exists of a finite constant $C_7 > 0$ such that the following inequality holds for all w and z in $\mathbb{C}^n$ whose norms $\|w\| \leq 1$ and $\|z\| \geq 1$,

$$C_7 \frac{\|w\|}{\|z\|} \geq \sum_{k=1}^{p} \frac{|\pi_k(z+w) - \pi_k(z) - \eta_k \Pi_k(w, z, ..., z)|}{\|z\|^{\eta_k - 1} \|w\|}. \tag{3.4}$$

On the other hand, the linear term $\eta_k \Pi_k(w, z, ..., z)$ in the Taylor expansion (3.3) is equal to $[\nabla_z \pi_k] w$, where $\nabla_z \pi_k$ is the complex gradient with entries $\partial \pi_k(z)/\partial z_l$. Therefore, if we define Δ_z to be the $[p \times n]$-matrix whose p-rows are the horizontal vectors $\nabla_z \pi_k / \|z\|^{\eta_k - 1}$, with $z \neq 0$ fixed, we shall have that Δ_z is invariant under the multiplication of $z \in \mathbb{C}^n$ by any positive real number. Recall that Π_k is a multilinear form with η_k entries. Moreover, it is easy to verify that the following equation holds, and that the terms are all independent of the magnitude of $\|w\|$ and $\|z\|$,

$$\sum_{k=1}^{p} \frac{|\eta_k \Pi_k(w, z, ..., z)|}{\|z\|^{\eta_k - 1} \|w\|} = \sum_{k=1}^{p} \frac{|[\nabla_z \pi_k] w|}{\|z\|^{\eta_k - 1} \|w\|} \geq \frac{\|\Delta_z w\|}{\|w\|}. \tag{3.5}$$

The last inequality in (3.5) can be shown by applying the square power to both terms. The given hypotheses also imply that the set of all vectors $\nabla_z \pi_k$ has maximal rank for every fixed point $z \neq 0$ in $\mathbb{C}^n$, and so there are n linearly independent gradient vectors $\nabla_z \pi_k$, because π is locally a biholomorphism from $\mathbb{C}^n \setminus \{0\}$ onto its image. Whence, the $[p \times n]$-matrix Δ_z defined in the paragraph above has both maximal rank and pseudo-inverse, recall that $p > n > 0$. We can then deduce the following result for all point z and w in $\mathbb{C}^n$ with $z \neq 0$,

$$\|w\| = \|\Delta_z^+ \Delta_z w\| \leq \|\Delta_z^+\| \cdot \|\Delta_z w\|,$$
$$\text{where} \quad \Delta_z^+ := (\Delta_z^* \Delta_z)^{-1} \Delta_z^*.$$

Notice that the $[n \times n]$-matrix $\Delta_z^* \Delta_z$ is invertible when Δ_z has maximal rank; see [3, p. 20] or [8], p. 285. Thus, the matrix norm $\|\Delta_z^+\|$ is a continuous function bounded on the compact set given by $\|z\| = 1$. That is, there exists a finite constant $C_6 > 0$ such that the norm $\|\Delta_z w\|$ is greater than or equal to $2C_6 \|w\|$ whenever $\|z\| = 1$ and w is in $\mathbb{C}^n$. Finally, a direct application of equations (3.4) and (3.5) yields the following inequality for all elements w and z in $\mathbb{C}^n$ whose norms $\|w\| \leq 1$ and $\|z\| \geq 1$,

$$C_7 \frac{\|w\|}{\|z\|} + \sum_{k=1}^{p} \frac{|\pi_k(z+w) - \pi_k(z)|}{\|z\|^{\eta_k - 1} \|w\|} \geq \frac{\|\Delta_z w\|}{\|w\|} \geq 2C_6. \tag{3.6}$$

The last inequality in (3.6) holds because the matrix Δ_z is invariant under the multiplication of $z \in \mathbb{C}^n$ by any positive real number, as we have already said in the paragraph located between equations (3.4) and (3.5). Thus, we only need to interchange z by its normalisation $z/\|z\|$. Finally, can choose a constant $\Lambda \gg 1$ such that $C_6 \geq C_7/\Lambda$, and so equation (3.6) will imply the result (3.2) for $\|w\| \leq 1$ and $\|z\| \geq \Lambda$. □

We may now present the central result of this section: an estimate of the metric $\|\pi(z)-\pi(\zeta)\|$ against some power of $\|z-\zeta\|$. Notice that we can interchange $\|z\|+\|\zeta\|$ by the norm $\|z-\zeta\|$ in equation (3.9), because this norm is indeed less than or equal to $\|z\|+\|\zeta\|$.

THEOREM (3.7). *Let $\mathbb{C}^n/\mathcal{G}$ be a quotient variety well embedded in $\mathbb{C}^p$ and whose singular set is the origin of $\mathbb{C}^p$, where $\mathcal{G}$ is a finite group of unitary matrices and $p>n>0$. Suppose that the quotient mapping π defined from $\mathbb{C}^n$ into $\mathbb{C}^p$ is polynomial, and that each entry π_k is a homogeneous polynomial of degree $\eta_k \geq 2$. Then, there exists a finite constant $C_8 > 0$ which satisfies the following inequality for all points z and ζ in $\mathbb{C}^n$ such that $\|z-\zeta\|$ is less than or equal to $\|z-H\zeta\|$ for each element $H \in \mathcal{G}$,*

$$\text{(3.8)} \qquad \frac{1}{\|z-\zeta\|}\sum_{k=1}^{p}\frac{|\pi_k(z)-\pi_k(\zeta)|}{\big(\|z\|+\|\zeta\|\big)^{\eta_k-1}} \;\geq\; C_8.$$

Moreover, if the sum $\|z\|+\|\zeta\|$ is also less than or equal to a given finite constant $\theta > 0$; and we define N to be the maximum of all degrees $\eta_k \geq 2$. Then, we also have that,

$$\text{(3.9)} \qquad \|\pi(z)-\pi(\zeta)\| \;\geq\; \frac{C_8\|z-\zeta\|\big(\|z\|+\|\zeta\|\big)^{N-1}}{p\cdot\max\{1,\theta^{N-2}\}}.$$

Proof. The fact that the singular set of $\pi(\mathbb{C}^n) = \mathbb{C}^n/\mathcal{G}$ is the origin implies that the critical space of $\mathcal{G}$ is given by $\Sigma = \{0\}$, according to (1.1). Notice that Σ is contained in $\pi^{-1}(0)$, and that $\pi^{-1}(0)$ is the orbit of $\mathcal{G}$ composed exclusively by the origin, for $\pi(0) = 0$. Thus, the quotient mapping π is locally a biholomorphism from $\mathbb{C}^n \setminus \{0\}$ onto its image. We can then apply Lemma (3.1) and deduce the existence of a pair of constants $C_6 > 0$ and $\Lambda \gg 1$ such that the inequality (3.2) holds for all elements w and $\widehat{z}$ in $\mathbb{C}^n$ with norms $\|w\| \leq 1$ and $\|\widehat{z}\| \geq \Lambda$.

On the other hand, it is easy to see that equation (3.8) does not change if we multiply both points z and ζ in $\mathbb{C}^n$ by any complex number $s \neq 0$. Whence, we only need to show that inequality (3.8) holds on the compact set defined by $\|z\|+\|\zeta\|$ equal to 2Λ. We consider two principal cases in order to show that (3.8) holds.

Case I. Take the norm $\|z-\zeta\| \leq 1$. We may suppose, without loss of generality, that $\|z\|$ is greater than or equal to $\|\zeta\|$; and so,

$$\|z\| \;\geq\; \big(\|z\|+\|\zeta\|\big)/2 \;=\; \Lambda\,.$$

Inequality (3.8) can be easily deduce from (3.2) in Theorem (3.1), we just need to introduce the new variable w equal to $\zeta-z$. That is, fixing N to be the maximum of all degrees $\eta_k \geq 2$, we automatically have that,

$$\frac{2^{N-1}}{\|z-\zeta\|}\sum_{k=1}^{p}\frac{|\pi_k(\zeta)-\pi_k(z)|}{(2\Lambda)^{\eta_k-1}} \geq \sum_{k=1}^{p}\frac{|\pi_k(\zeta)-\pi_k(z)|}{\|z\|^{\eta_k-1}\|z-\zeta\|} \geq C_6\,.$$

Case II. Suppose that $\|z-\zeta\| \geq 1$. Define the compact set K composed by all the pairs (z,ζ) in the product $\mathbb{C}^n\times\mathbb{C}^n$ which satisfies the following conditions, $\|z\|+\|\zeta\|$ is equal to 2Λ and :

$$\|z-H\zeta\| \;\geq\; \|z-\zeta\| \;\geq\; 1 \quad \text{for every} \quad H\in\mathcal{G}\,.$$

It is easy to verify that the left term of equation (3.8) vanishes if and only if $z=H\zeta$ for some element H in $\mathcal{G}$, because the quotient mapping $\pi(Hz)=\pi(z)$ is invariant under the action of each element H. Hence, the left term of (3.8) is a continuous and non-vanishing function well defined for every pair (z,ζ) in the compact set K described in the paragraph above, and so this function is bounded from below by a finite positive constant $C_8>0$. In other words, inequality (3.8) holds in this case as well. □

Proof of (3.9). It is easy to show that inequality (3.9) holds by multiplying (3.8) times $\|z-\zeta\|$ and the $N-1$ power of $\|z\|+\|\zeta\|$. Thus, whenever the sum of norms $\|z\|+\|\zeta\|$ is less than or equal to $\theta>0$, we have the following inequalities, we introduce the maximum in order to deal with the cases whether θ is greater or smaller than one,

$$\begin{aligned} C_8\|z-\zeta\|\big(\|z\|+\|\zeta\|\big)^{N-1} &\leq \sum_{k=1}^{p}\big(\|z\|+\|\zeta\|\big)^{N-\eta_k}|\pi_k(z)-\pi_k(\zeta)| \\ &\leq\; p\cdot\max\{1,\theta^{N-2}\}\|\pi(z)-\pi(\zeta)\|. \end{aligned}$$

□

We are now in position of showing an alternative version of the main Theorem (1.2), using the Euclidean metric $\|\cdot\|$ of the ambiance space instead of the natural metric $D_{\mathcal{G}}(\cdot)$ of the quotient variety. The main difference between Theorems (1.2) and (3.10) is that we need to introduce a new upper bound for the main exponent $0<\delta<1$.

THEOREM (3.10). *Let $\mathbb{C}^n/\mathcal{G}$ be a singular quotient variety well embedded in $\mathbb{C}^p$ and whose singular set is the origin of $\mathbb{C}^p$, where $\mathcal{G}$ is a finite group of unitary matrices and $p>n>0$. Consider the natural metric $D_{\mathcal{G}}(\cdot)$ induced in $\mathbb{C}^n/\mathcal{G}$, and the quotient mapping π defined from $\mathbb{C}^n$ into $\mathbb{C}^p$. Besides, suppose that each entry π_k is a homogeneous polynomial of degree $\eta_k\geq 2$, and define $N\geq$ to be the maximum of all degrees η_k. Given any open ball B_R in $\mathbb{C}^n$ of finite radius $R>0$ and centre in the origin, the following inequality holds for the finite constant $C_8>0$ given in Theorem (3.7) and all points x and y inside the image $\pi(B_R)$,*

$$\|x-y\| \;\geq\; \frac{C_8\,D_{\mathcal{G}}^N(x,y)}{p\cdot\max\{1,[2R]^{N-2}\}}\,. \tag{3.11}$$

Moreover, given any exponent $0<\beta<1/N$, there is a finite positive constant $C_9(R,\beta)$ such that: For every continuous $(0,1)$-differential form λ defined on an open neighbourhood of $\pi(\overline{B_R})$, and $\overline{\partial}$-closed on the regular part of $\pi(B_R)$, *there exists a continuous function h defined on $\pi(B_R)$ which satisfies both the equation $\overline{\partial} h = \lambda$ on the regular part of $\pi(B_R)$ and the estimate :*

$$\|h\|_{\pi(B_R)} + \sup_{x,y\in\pi(B_R)} \frac{|h(x) - h(y)|}{\|x-y\|^\beta} \leq C_9(R,\beta)\|\lambda\|_{\pi(B_R)} . \tag{3.12}$$

Proof. Let x and y be two points in the image $\pi(B_R)$. The natural metric $D_{\mathcal{G}}(x,y)$ is defined as the Euclidean metric between the finite fibres $\pi^{-1}(x)$ and $\pi^{-1}(y)$. Therefore, we can choose two relate points z and ζ in the ball B_R such that the norm $\|z-\zeta\|$ is equal to $D_{\mathcal{G}}(x,y)$, and their respective projections satisfy $x = \pi(z)$ and $y = \pi(\zeta)$. Recall that the fibre $\pi^{-1}(y)$ is equal to the orbit $\{H\zeta | H\in\mathcal{G}\}$, and that this orbit is completely contained in the ball B_R because the norm $\|H\zeta\| = \|\zeta\|$ is invariant. We automatically have that $\|z-\zeta\|$ is less than or equal to $\|z-H\zeta\|$ for every matrix H in $\mathcal{G}$, because $\|z-\zeta\|$ is the minimal Euclidean metric between the point z and the inverse fibre $\pi^{-1}(y)$. Thus, we may apply Theorem (3.7) in order to prove that inequality (3.11) holds, we just need to analyse equation (3.9) and the following facts,

$$2R \geq \|z\| + \|\zeta\| \geq \|z-\zeta\| = D_{\mathcal{G}}(x,y) .$$

Moreover, whenever one of the points x or y is not contained in the *smaller* image $\pi(B_{R/2})$, we can easily deduce that the sum $\|z\|+\|\zeta\|$ is greater than or equal to $R/2$; and so equation (3.9) can be rewritten as follows for every exponent $0<\beta<1/2$, where M is the maximum of $\|\widehat{x}-\widehat{y}\|$ for all points $\widehat{x}$ and $\widehat{y}$ in the compact set $\pi(\overline{B_R})$,

$$\|x-y\|^\beta M^{1/2-\beta} \geq \|x-y\|^{1/2} \geq \left[\frac{C_8\, D_{\mathcal{G}}(x,y)\,[R/2]^{N-1}}{p\cdot \max\{1,[2R]^{N-2}\}}\right]^{1/2} . \tag{3.13}$$

On the other hand, Theorem (1.2) implies the existence of a continuous function h defined on $\pi(B_R)$ which satisfies both the differential equation $\overline{\partial} h = \lambda$ and the Hölder estimates (1.3) and (1.4). We can then deduce the existence of a finite positive constant $C_9(R,\beta)$ which satisfies our final result (3.12), we just need to apply equations (1.3) and (3.13) whenever one of the points x or y is not in $\pi(B_{R/2})$, or equations (1.4) and (3.11) whenever both points x and y are inside $\pi(B_{R/2})$. □

4. Surfaces with simple singularities

We have already said, in the previous chapter, that the surfaces with an isolate simple (rational double point) singularity are the perfect model of quotient varieties which satisfy all the hypotheses of Theorems (3.7) and (3.10). A deep analysis can be found in the works of Dimca [4], Klein [10] and Slodowy [12]. In particular, any surface with an isolate simple singularity X is a quotient space $\mathbb{C}^2/\mathcal{G}$ which can be identified with the zero locus of a holomorphic polynomial in $\mathbb{C}^3$, where $\mathcal{G}$ a finite subgroup of the special linear group $SL_2(\mathbb{C})$. Now then, we need to verify that the product of the transpose conjugate H^* times H is the identity matrix for every H in $\mathcal{G}$. Previous fact implies that $\mathcal{G}$ is a group of unitary matrices, for,

$$\|Hz\|^2 = (Hz)^*(Hz) = z^*H^*Hz = z^*z = \|z\|^2.$$

On the other hand, every finite subgroups of $SL_2(\mathbb{C})$ is generated by some combinations of the following matrices, see for example pages 73 and 74 of Slodowy [12],

$$H_1 := \tfrac{1}{1+i}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \quad H_2 := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$
$$H_3 := \frac{-i}{\sqrt{1+\mu^2}}\begin{bmatrix} \mu & 1 \\ 1 & -\mu \end{bmatrix}, \quad M_k := \begin{bmatrix} \rho_k & 0, \\ 0 & \overline{\rho_k} \end{bmatrix},$$

where $\mu = 2\cos(2\pi/5)$ and $\rho_k = \mathrm{e}^{2\pi i/k}$ is the k-root of the unity. It is easy to verify that the products $H_j^* H_j$ and $M_k^* M_k$ are all equal to the identity matrix, and so every finite subgroup $\mathcal{G}$ of $SL_2(\mathbb{C})$ is a group of unitary matrices. Finally, we only need to verify that every surface with an isolated simple singularity X is well embedded in $\mathbb{C}^3$, and that each entry of the corresponding quotient mapping π is a homogeneous polynomial, in order to apply Theorem (3.10) for calculating Hölder continuous solution to the $\overline{\partial}$-equation on surfaces $X = \mathbb{C}^2/\mathcal{G}$ with an isolate simple singularity. We present below a list of the different surfaces with isolated singularities of the cyclic, dihedral, tetrahedral, octahedral and icosahedral type. See [10] and [12] for details. We present the polynomial relation which defines the surface X in $\mathbb{C}^3$, the generators of the quotient group $\mathcal{G}$, the quotient mapping π (whose entries are homogeneous polynomials), and the corresponding main exponent β which is used in the Hölder estimate (3.12). It is interesting to compare the upper bounds for the exponent β presented below against the calculations done by Acosta and Zeron [1] and [2].

- Cyclic singularities A_{m-1}: $x_1 x_2 = x_3^m$, for $m \geq 2$.
- Generators of the quotient group: M_m.
- Main exponent in (3.12): $0 < \beta < 1/m$.
- Quotient mapping: $\pi(s,t) = \big[s^m, t^m, st\big]$.

- Dihedral singularities D_{m+2}: $x_1^2 x_3 = x_2^2 + 4x_3^{m+1}$, for $m \geq 2$.
- Generators of the quotient group: M_{2m} and H_2.
- Main exponent in (3.12): $0 < \beta < 1/(2m+2)$.
- Quotient mapping: $\pi(s,t) = \big[(s^{2m} + t^{2m}), st(s^{2m} - t^{2m}), s^2 t^2\big]$.

- Tetrahedral singularity E_6: $x_1^3 = x_2^2 + 108x_3^4$.
- Generators of the quotient group: M_4, H_1 and H_2.
- Main exponent in (3.12): $0 < \beta < 1/12$.
- Quotient mapping: $\pi(s,t) = [x_1, x_2, x_3]$, where:
  $x_1 := s^8 + 14s^4t^4 + t^8$, $\quad x_2 := (s^4 + t^4)(s^8 - 34s^4t^4 + t^8)$,
  $x_3 := st(s^4 - t^4)$.

- Octahedral singularity E_7: $x_1^3 x_3 = x_2^2 + 108x_3^3$.
- Generators of the quotient group: M_8, H_1 and H_2.
- Main exponent in (3.12): $0 < \beta < 1/18$.
- Quotient mapping: $\pi = [x_1, x_2, x_3]$, where:
  $x_1 := s^8 + 14s^4t^4 + t^8$, $\quad x_2 := st(s^8 - t^8)(s^8 - 34s^4t^4 + t^8)$,
  $x_3 := s^2t^2(s^4 - t^4)^2$.

- Icosahedral singularity E_8: $x_1^2 = x_2^3 + x_3^5$.

- Generators of the quotient group: M_{10} and H_3.
- Main exponent in (3.12): $0 < \beta < 1/30$.
- Quotient mapping: $\pi(s,t) = [x_1, x_2, x_3]$, where:

$$x_1 := s^{30} + t^{30} + 522(s^{25}t^5 - s^5t^{25}) - 10005(s^{20}t^{10} + s^{10}t^{20}),$$
$$x_2 := s^{20} - 228s^{15}t^5 + 494s^{10}t^{10} + 228s^5t^{15} + t^{20},$$
$$x_3 := (1728)^{1/5}(s^{11}t + 11s^6t^6 - st^{11}).$$

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GLOBAL QUALITATIVE ANALYSIS OF THE HARRISON'S PREDATOR-PREY MODEL

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ABSTRACT. In this paper we describe the global analysis of the solutions of the Harrison's predator-prey model, which has been established in terms of the dependence of the storage of nutrients in the rate of growth of the predator that generates a delay in the consumption of the prey population. Rather than just insert an ad hoc time lag, a mechanism has been proposed to produce the delay in the response. The model attempts to improve, from a theoretical point of view, the modelation of the predator-prey interaction. We show that the Harrison's predator-prey model has the point dissipative property and therefore has a unique global attractor. The main result in this research is that under certain feasible conditions on the parameters it is possible to guarantee the existence of a unique orbitally asymptotically stable periodic orbit which constitutes the global attractor when the system has a unique nontrivial hyperbolic unstable equilibrium point in the positive cone $\mathbb{R}^3_+$. Also, it is possible to give conditions on the parameters such that the equilibrium with positive coordinates is the global attractor for the solutions of the system.

1. Introduction

In this paper we analyze a predator-prey model proposed by Gary W. Harrison in [5]. One of the main improvements of this model is the idea introduced by the authors related to the predator growth rate which depends on the energy or the nutrient storage instead of the rate of consumption of prey. The ideas of Harrison were conceived by comparing how well predictions of the predator-prey models match both qualitative and quantitatively the data of an experiment of Leo Luckinbill (see [7]) with *Didinium nasutum* (predator) and *Paramecium aurelia* (prey). Harrison tested the quantitative data obtained from standard predator-prey models against the data of this experiment in order to determine modifications that are important to introduce in the predator-prey models to obtain a good match.

In order to introduce the Harrison's predator-prey model, we set x as the prey population density and y the predator population density. In the standard predator-prey models the prey usually grows logistically in the absence of predators with a carrying capacity k. The saturating (Holling Type II) predator functional response is widely accepted, and is given by

$$f(x) = \frac{x}{\phi + x}, \tag{1.1}$$

where ϕ is the half-saturation constant, i.e., the level of prey at which half of the maximum consumption rate occurs. In such a case, and assuming that

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there are not delays in the system, one of the most common predator-prey mathematical model is given by the following system of ordinary differential equations

$$x' = \rho\left(1 - \frac{x}{k}\right)x - \omega f(x)\, y \tag{1.2}$$

$$y' = \sigma f(x)\, y - \gamma y, \tag{1.3}$$

where the tilde ($'$) denotes the derivative with respect to time t. The parameter ρ denotes the specific growth rate of the prey. The term $-\gamma y$ represents the rate at which the predators would decrease in the absence of the prey. The parameter ω is the maximum rate of prey consumption by a single predator. The second equation for the predator population y in (1.3) holds the usual assumption of proportionality between the predator rate increase and the rate of prey consumption. The efficiency of converting prey into predator is given by $\frac{\sigma}{\omega}$. The model (1.2)-(1.3) has been analyzed extensively in [6] and certainly this model explains the experimental results reported by Luckinbill on a qualitative level, however, it cannot give a very good quantitative fit of the experimental population densities. So, G. Harrison has explored several modifications of the previous model by looking for a better fit of the Luckinbill's data and has tested some features that may be important to include in the predator-prey interactions. He has observed that adding either predator mutual interference or a sigmoid (or Holling Type III) functional response improves the fit dramatically (see [7]). Also, Harrison has pointed out that when populations are measured in terms of biomass, it makes sense that population growth rate would depend directly on the rate of nutrient intake. On the other hand, when the populations are measured in terms of number of individuals as in Luckinbill's study, the reproductive rate would depend on the nutrient storage and thus, in a delayed manner on the nutrient intake. The Harrison observations indicate that this point of view helps to fit better the data.

Here we analyze mathematically the behavior of the solutions of a predator-prey model of the Harrison type considering the Holling Type II functional response $f(x)$ that we have mentioned earlier in (1.1) combined with the delayed reproductive rate. According to the Harrison's reasoning, this model improves the model (1.2)-(1.3) where it is supposed that the predator reproduction rate responds instantly to prey consumption. In Harrison's model, the delay mentioned earlier is introduced assuming that: 1) prey consumption creates an inflow of energy and/or nutrients into the total energy and/or nutrients stored in the predators, and 2) the overall predator-specific growth rate is an increasing function of the average energy or nutrients per predator. So, the following system of three ordinary differential equations

$$\begin{aligned} x' &= \rho\left(1 - \frac{x}{k}\right)x - \omega f(x)\, y \\ Z' &= \sigma f(x)\, y - \delta Z \\ y' &= \left(\frac{\lambda Z}{y} - \gamma\right) y, \end{aligned} \tag{1.4}$$

is introduced, where Z denotes the total energy/nutrient storage of the predators and the term δZ in the second of the equations (1.4) denotes the rate of energy expenditure, $\frac{Z}{y}$ in the third equation of (1.4) is the average energy/nutrients per predator, γ is the specific death rate of the predators, σ and λ are constants of proportionality that label energy per prey and predators per energy per time, respectively. For sake of simplicity, we introduce the new variable $z = \lambda Z$ which simplifies the equations (1.4) as

$$\begin{aligned} x' &= \rho\left(1 - \frac{x}{k}\right)x - \frac{\omega xy}{\phi + x} \\ y' &= z - \gamma y \\ z' &= \frac{\sigma xy}{\phi + x} - \delta z, \end{aligned} \tag{1.5}$$

where we continue denoting by σ the product $\sigma\lambda$.

The variable z has units predators per time and represents the predator's reproductive rate. The third equation of the system (1.5) says that the predator's reproductive rate is not instantly a function of the rate of energy/nutrient intake, but the rate of change of the reproduction rate is a function of the rate of energy/nutrient intake. For given x and y the equilibrium level of z is given by

$$z^* = \frac{\sigma xy}{\delta(\phi + x)}$$

and the equation for z could be written as

$$z' = \delta(z^* - z),$$

the larger value of δ, the faster z will move toward z^*. Of course, z^* is nonconstant since it depends on x and y. Note also that, if we replace z by z^* in the second of the equations of the system (1.5) and identify σ with $\frac{\sigma}{\delta}$, the first and second equations of (1.5) would be equivalent to the equations (1.2)-(1.3). Thus, when δ is large, z will track z^* closely and the resulting trajectory will be very close to that given by equations (1.2)-(1.3) with a very little time delay in the numerical response. If δ is small, z will lag behind z^*, producing a significant delay in the predator's numerical response to the prey density.

The main purpose of this paper is to obtain some global properties of solutions for the model given by equations (1.5). We show the existence of orbitally asymptotically stable periodic orbits with the same technical background as in [1]. In our case, we apply the property of stability of periodic orbits as in [10] and also the result of uniqueness of a periodic orbit, given in the next section. Under certain conditions on the parameters we obtain the global attractor of the system as a unique periodic orbit or an equilibrium point of positive coordinates.

This paper has the following content: section 2 has some prelimiries definitions and theorems; in section 3 we reduce the equations of the model to an equivalent pointwise dissipative system in order to establish the existence of a global atractor; in section 4 we analize the parameter conditions for the stability and instability of the equilibria; in section 5 we show that the system is uniform persistent and the existence and stability of periodic orbits; finally, in section 6 we show parameters conditions for the existence of an unique globally asymptotically stable periodic orbit.

2. Preliminaries

In this section we will summarize the main facts related with our research. Let $D \subset \mathbb{R}^3$, and consider the system of differential equations

$$x' = F(x), \quad x \in D. \tag{2.1}$$

For this system we put the same notations and related concepts as in the section 2 of [1]. The system (2.1) is said to be competitive in D if the Jacobian matrix of F at x, $F'(x)$, has non-positive off-diagonal elements at each point of D. For the theory of competitive systems and the definitions of p-convex sets and irreducible systems we refer to [9].

The following two theorems for competitive systems in $\mathbb{R}^3$ are given in [9], the first one is a Poincaré-Bendixson like theorem and the second one provides sufficient conditions for the existence of periodic orbits.

THEOREM (2.2). *A compact limit set of a competitive system in $\mathbb{R}^3$ that contains no equilibrium points is a periodic orbit.*

THEOREM (2.3). *Let (2.1) be a competitive system in $D \subset \mathbb{R}^3$ and suppose that D contains a unique equilibrium point p which is hyperbolic and assume that $F'(p)$, the Jacobian matrix of F at p, is irreducible. Suppose further that $W^s(p)$, the stable manifold of p, is one dimensional. If $q \in D \setminus W^s(p)$ and $\varphi^+(q)$ has compact closure in D, then $\omega(q)$ is a nontrivial periodic orbit.*

Definition (2.4). The system (2.1) is said to be pointwise dissipative if there exists a bounded set $B \subset R^n$, such that for $x_0 \in R^n$ and $\epsilon > 0$, there exists $T = T(x_0, \epsilon) > 0$ such that $\varphi_t(x_0) \in V_\epsilon(B)$, for $t > T$, where $V_\epsilon(B) = \{x \in \mathbb{R}^n : \operatorname{dist}(x, B) < \epsilon\}$.

The following hypothesis are important for the next result given in [12].

(H1). System (2.1) is dissipative: For each $x \in D$, $\varphi^+(x)$ has compact closure in D. Moreover, there exists a compact subset B of D with the property that for each $x \in D$ there exists $T(x) > 0$ such that $x(t, x) \in B$ for $t \geq T(x)$.

(H2). System (2.1) is competitive and irreducible in D.

(H3). D is an open, p-convex subset of $\mathbb{R}^3$.

(H4). D contains a unique equilibrium point x^* and $\det(F'(x^*)) < 0$.

THEOREM (2.5). *Let (H1) through (H4) hold. Then either*

(a) *x^* is stable, or*

(b) *there exists a nontrivial orbitally stable periodic orbit in D.*

In addition, let us assume that F is analytic in D. If x^* is unstable then there is at least one but no more that finitely many periodic orbits for (2.1) and at least one of these is orbitally asymptotically stable.

It is well known that the dissipative hypothesis (H1) implies the existence of a maximal compact invariant set, A, for the flow of the system (2.1) which uniformly attracts compact subsets of D, see [4]. The following proposition of H. R. Zhu and H. L. Smith in [12] has to do with this set A.

PROPOSITION (2.6). *Let w be a positive unit vector and Π be the orthogonal projection onto the plane that is orthogonal to w. Then $\Pi|_A$, the restriction of Π to A, is injective and $(\Pi|_A)^{-l}$ is Lipschitzian. If this hyperplane is identified with $\mathbb{R}^2$, there exists a Lipschitz vector field, Y, on $\mathbb{R}^2$ such that Π maps orbits in A onto orbits of Y in $\mathbb{R}^2$, respecting the parametrization. Thus $\Pi(A)$ is a compact invariant set for the flow generated by Y.*

Now we introduce the following definition:

Definition (2.7). We say that the system (2.1) has the property of stability of periodic orbits if each periodic orbit of this system is orbitally asymptotically stable.

From the Proposition (2.6) we can establish the existence of a unique orbitally asymptotically periodic orbit for dissipative competitive and irreducible system in $\mathbb{R}^3$.

THEOREM (2.8). *Let (H1) through (H4) hold and assume that F is analytic in D. If x^* is unstable and if the system (2.1) has the property of stability of periodic orbits, then there exists exactly one periodic orbit. Moreover, the maximal compact positively invariant set, A, for the flow of the solutions of system (2.1) is this unique orbitally asymptotically stable periodic orbit.*

Proof. The condition (H4) implies that if the point x^* is unstable then either it is hyperbolic (i.e., no eigenvalues of the Jacobian matrix $F'(x^*)$ have zero real part) and the equilibrium point x^* has a two dimensional unstable manifold, or x^* is nonhyperbolic due to a nontrivial pair of purely imaginary eigenvalues (plus a negative real eigenvalue). As the equilibrium point x^* is unstable, it is clear from the proof of the Theorem 1.2 in [12] that there are no more that finitely many periodic orbits, $\{\gamma_i\}_{i=1}^n$. If each one of these periodic orbits is orbitally asymptotically stable, then according to the Proposition (2.6) we have a Lipschitz vector field, Y, in $\mathbb{R}^3$ having a finite number of orbitally asymptotically stable periodic orbits, say $\{\Pi(\gamma_i)\}_{i=1}^n$. Clearly these orbits must be concentric in the plane. But according to Corollary 1.4, pag. 55 in [3], two concentric periodic orbits cannot be both orbitally asymptotically stable. This contradiction implies that there must exist exactly one periodic orbit, i.e., the number n must be equal to 1. Therefore the system (2.1) must have exactly one orbitally asymptotically stable periodic orbit. By virtue of this and the unstability of the only equilibrium point x^*, it is clear that for $t > 0$, the periodic orbit is the maximal positively invariant set A of the system (2.1). □

We need some results about stability theory related with compound matrices, most of them can be found in [8].

Definition (2.9). Let A be a $n \times m$ matrix of real or complex numbers. Let $a_{i_1,\ldots,i_k j_1,\ldots,j_k}$ be the minor of A determined by the rows $(i_1,\ldots,i_k)$ and the columns $(j_1,\ldots,j_k)$, $1 \le i_1 < i_2 < \cdots < i_k \le n$, $1 \le j_1 < j_2 < \cdots < j_k \le m$. The kth multiplicative compound matrix $A^{(k)}$ of A is the $\binom{n}{k} \times \binom{m}{k}$ matrix whose entries, written in lexicographic order, are $a_{i_1,\ldots,i_k j_1,\ldots,j_k}$.

The kth additive compound of the $n \times n$ matrix A is defined as follows.

Definition (2.10). Let A an $n \times n$ matrix. The kth additive compound $A^{[k]}$ of A is the $\binom{n}{k} \times \binom{n}{k}$ matrix given by

$$A^{[k]} = \frac{d}{dh}\left(I + hA\right)^{(k)} |_{h=0},$$

where I is the $\binom{n}{k} \times \binom{n}{k}$ identity matrix .

For instance the second additive compound matrix $A^{[2]}$ for $n = 3$ is given by

$$A^{[2]} = \begin{pmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{pmatrix}.$$

Finally, we need the following lemma given in [2].

LEMMA (2.11). *Let $k\colon [0, \infty) \to \mathbb{R}$, a twice differentiable function such that $\lim_{t\to\infty} k(t)$ exists and $k''(t)$ is bounded for all $t \geq 0$. Then $\lim\limits_{t\to\infty} k'(t) = 0$.*

3. Dissipativity and equivalence to a competitive system

By introducing the scaling $z \to \frac{\omega z}{\sigma}$, $y \to \frac{\omega y}{\sigma}$ in the equations (1.5) we obtain the following system

$$\begin{aligned} x' &= \rho\left(1 - \frac{x}{k}\right)x - \frac{\sigma x y}{\phi + x} \\ y' &= z - \gamma y \\ z' &= \frac{\sigma x y}{\phi + x} - \delta z, \end{aligned} \tag{3.1}$$

where, for sake of simplicity, we continue denoting the variables and parameters with the same symbols. So, we continue the analysis of the model with the equivalent equations (3.1). For biological reasons all of the parameters $\rho, k, \sigma, \phi, \delta, \gamma$ are considered as positives.

The next theorem guarantees that (3.1) is biologically well behaved and that the dynamics of the system is concentrated in a bounded region of $\mathbb{R}^3$.

THEOREM (3.2). *Let $\mathbb{E} = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0,\ y \geq 0,\ z \geq 0\}$. Then, $\mathbb{E}$ is positively invariant under the flow induced by (3.1). Moreover, (3.1) is pointwise dissipative and there exists a constant M, independent of the initial data, such that the absorbing set (into which the solutions eventually enter and remain) is given by: $B = [0, k] \times [0, M] \times [0, M]$, where $M = \max\left\{\dfrac{\rho k}{\min\{\rho, \delta\}}, \dfrac{\rho k}{\gamma \min\{\rho, \delta\}}\right\}$.*

Proof. Clearly the system (3.1) is equivalent to the following integral equations

$$\begin{aligned} x(t) &= x_0 \exp \int_0^t \left\{\rho\left(1 - \frac{x(\tau)}{k}\right) - \frac{\sigma y(\tau)}{\phi + x(\tau)}\right\} d\tau, \\ y(t) &= y_0 e^{-\gamma t} + \int_0^t z(\tau) e^{\gamma(\tau - t)} d\tau, \\ z(t) &= z_0 e^{-\delta t} + \int_0^t \frac{\sigma x(\tau) y(\tau)}{\phi + x(\tau)} e^{\delta(\tau - t)} d\tau. \end{aligned} \tag{3.3}$$

These equalities certainly imply that solutions of (3.1) remain in $\mathbb{E}$ as long as they are defined. Now, let us prove that the solutions of the system (3.1) are bounded for $t \geq 0$. Taking into account the solution of the initial value problem

$$n' = \rho\left(1 - \frac{n}{k}\right) n, \; n(0) = x_0 > 0, \tag{3.4}$$

by comparison arguments, we get that the solution of the first equation of (3.1) satisfies

$$0 < x(t) \leq \frac{k}{1 + c_0 \exp(-\rho t)}, \quad c_0 = \frac{k - x_0}{x_0}, \; t \geq 0, \tag{3.5}$$

which implies the boundedness of the solution $x(t)$ on the interval $[0, \infty)$. The previous relation implies that for a given $\varepsilon > 0$, there exists $T > 0$ such that, $x(t) \leq k + \varepsilon$ for $t \geq T$. So, taking into account that

$$1 - \frac{x(t)}{k} < 1 - \frac{x(t)}{k + \varepsilon},$$

for $t \geq T$ we get

$$\left(x(t) + z(t)\right)' \leq \rho(k + \varepsilon) - \min\{\rho, \delta\}\left(x(t) + z(t)\right),$$

which implies,

$$0 < x(t) + z(t) \leq \frac{\rho k}{\min\{\rho, \delta\}} + \varepsilon, \quad t \geq T.$$

So, the positivity of $x(t)$ and $z(t)$ implies the boundedness of the solution $z(t)$. On the other hand, for $t \geq T$, it follows that

$$\begin{aligned} y'(t) &= z(t) - \gamma y(t) \\ &\leq \beta - \gamma y(t), \; \beta = \frac{\rho k}{\min\{\rho, \delta\}} + \varepsilon. \end{aligned}$$

This implies that

$$y(t) \leq \frac{\beta}{\gamma} + \left(y(T) - \frac{\beta}{\gamma}\right) \exp(T - t), \quad t \geq T,$$

and therefore $y(t)$ is a bounded function. This shows the theorem. □

As a consequence of the theory given in [4], page 39, we get the following corollary.

COROLLARY (3.6). *If the system (3.1) is pointwise dissipative then there exists a nonempty global attractor in E.*

Let D be the region in $\mathbb{R}^3$ given by

$$D = \{(x, y, z) \in \mathbb{R}^3 : x < 0, \; y < 0, \; z > 0\}. \tag{3.7}$$

The following lemma shows that system (3.1) is equivalent to a competitive and irreducible system in D.

LEMMA (3.8). *By setting $X = -x$ and $Y = -y$, the system (3.1) is equivalent to a competitive and irreducible system in D.*

Proof. In this case the equations (3.1) are equivalent to the

$$
\begin{aligned}
X' &= \rho\left(1+\frac{X}{k}\right)X + \frac{\sigma XY}{\phi - X}\\
Y' &= -z - \gamma Y \\
z' &= \frac{\sigma XY}{\phi - X} - \delta z,
\end{aligned} \tag{3.9}
$$

that has the following Jacobian matrix given by

$$
\begin{pmatrix}
a_{11} & \dfrac{\sigma X}{\phi - X} & 0 \\
0 & a_{22} & -1 \\
\dfrac{\sigma\phi Y}{(\phi - X)^2} & \dfrac{\sigma X}{\phi - X} & a_{33}
\end{pmatrix},
$$

and this matrix has non-positive off-diagonal elements because $X, Y < 0$. □

4. Stability of the equilibrium points

System (3.1) has three equilibrium points: $E_0 = (0,0,0)$, $E_k = \big(k,0,0\big)$ and the equilibrium with positive coordinates $E_* = (x_*, y_*, z_*)$, $x_*, y_*, z_* > 0$, where

$$
x_* = \frac{\phi\delta\gamma}{\sigma - \delta\gamma},\; y_* = \frac{\rho}{\delta\gamma}\left(1 - \frac{x_*}{k}\right)x_*,\; z_* = \gamma y_*. \tag{4.1}
$$

As we can see, the equilibrium $E_* = (x_*, y_*, y_*)$ of the system (3.1) is in the interior of the positive orthant, if and only if, the following constellation of parameters hold

$$
\delta\gamma < \frac{k\sigma}{\phi + k} \text{ and } \sigma > \delta\gamma. \tag{4.2}
$$

In order to study the local stability of the equilibria, note that the Jacobian matrix associated to system (3.1) is given by

$$
J = \begin{pmatrix}
\rho\left(1 - \dfrac{2x}{k}\right) - \dfrac{\sigma\phi y}{(\phi + x)^2} & -\dfrac{\sigma x}{\phi + x} & 0 \\
0 & -\gamma & 1 \\
\dfrac{\sigma\phi y}{(\phi + x)^2} & \dfrac{\sigma x}{\phi + x} & -\delta
\end{pmatrix}. \tag{4.3}
$$

The proof of the following two lemmas are strightforward.

LEMMA (4.4). *The equilibrium point E_0 of the system (3.1) is a saddle.*

LEMMA (4.5). *The solutions of the system (3.1) have the following property:* $\lim_{t\to\infty} z(t) = 0$ *if and only if* $\lim_{t\to\infty} y(t) = 0$.

In the following lemma we give a sufficient condition for the extinction of the predator population.

LEMMA (4.6). a) *If*

$$
\frac{k\sigma}{\phi + k} \le \delta\gamma, \tag{4.7}
$$

then the equilibrium point E_k is globally asymptotically stable.

Proof. There are two possibilities: *a)* $\sigma \leq \delta\gamma$ and *b)* $\sigma > \delta\gamma$.

a) From the last two equations of (3.1), we have

$$\begin{aligned} y' &= z - \gamma y \\ z' &\leq \sigma y - \delta z. \end{aligned}$$

If we consider the comparison equations

$$u' = v - \gamma u \tag{4.8}$$

$$v' = \sigma u - \delta v, \tag{4.9}$$

it is easy to show that if $\sigma < \delta\gamma$ for any solutions of the system (4.8)-(4.9) with nonnegative initial values we have $\lim_{t\to\infty} u(t) = 0$, $\lim_{t\to\infty} v(t) = 0$. Let $0 < y(0) \leq u(0)$, $0 < z(0) \leq v(0)$. If $(u(t), v(t))$ is a solution of system (4.8)-(4.9) with initial conditions $(u(0), v(0))$, then by the comparison theorem we have $y(t) \leq u(t)$, $z(t) \leq v(t)$ for all $t > 0$. Hence $\lim_{t\to\infty} y(t) = 0$, $\lim_{t\to\infty} z(t) = 0$. Now, for $\varepsilon \in (0, 1)$, there exists $t^* = t^*(\varepsilon)$ such that, for $t > t^*$,

$$\rho\left(1 - \varepsilon - \frac{x}{k}\right) x \leq x' \leq \rho\left(1 - \frac{x}{k}\right) x.$$

This clearly implies that $\lim_{t\to\infty} x(t) = k$. For this reason the equilibrium point $E_k = (k, 0, 0)$ is globally asymptotically stable.

b) First note that, if $x(t) > k$ then $x'(t) < 0$ and therefore $x(t)$ decreases and there exists some time t_1 at which $x(t_1) = k$. The dissipativity property implies that $x(t) < k$ for all $t > t_1$.

Now, by mean of the last two equations of system (3.1) we get

$$\begin{aligned} z'(t) + \delta y'(t) &= \frac{\sigma x(t) y(t)}{\phi + x(t)} - \gamma\delta y(t) \\ &= \frac{(\sigma - \gamma\delta) x(t) y(t) - \gamma\delta y(t)}{\phi + x(t)}. \end{aligned} \tag{4.10}$$

This implies that

$$z'(t) + \delta y'(t) = \frac{(\sigma - \gamma\delta) y(t)}{\phi + x(t)} \left(x(t) - \frac{\gamma\delta\phi}{(\sigma - \gamma\delta)}\right).$$

The relation (4.7) imply that,

$$z'(t) + \delta y'(t) \leq \frac{(\sigma - \gamma\delta) y(t)}{\phi + x(t)} \left(x(t) - k\right). \tag{4.11}$$

And this implies that $z'(t) + \delta y'(t) < 0$, for all $t > t_1$. If $\lim_{t\to\infty} (z(t) + \delta y(t)) = c$, $c > 0$, then taking into account Lemma (4.5) we get that $\lim_{t\to\infty} z(t) = c_1$ and $\lim_{t\to\infty} y(t) = c_2$, $c_1, c_2 > 0$. This and Lemma (4.4) imply the existence of the equilibrium of positive coordinates (k, c_1, c_2) wich is contradictory with the fact that under condition (4.7) no such equilibrium exists. This implies that

$$\lim_{t\to\infty} (z(t) + \delta y(t)) = 0,$$

thus, $\lim_{t\to\infty} z(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$.

Finally, if $k = \dfrac{\delta\gamma\phi}{\sigma - \delta\gamma}$, $\sigma - \delta\gamma > 0$, we have that in the region where $x(t) \geq k$; $x'(t) < 0$, but in the point $\big(k, 0, 0\big)$ where $x'(t) = 0$. So, $x(t)$ is strictly decreasing and therefore none of the points in the hyperplane $x = k$ can belong to the $\omega-limit$ set of a trajectory except the point $\big(k, 0, 0\big)$. Furthermore, from the equation for $(z(t) + \delta y(t))'$ in (4.10) we can see that the function $(z(t) + \delta y(t))'$ changes its sign at most once, this implies that in the future the function $(z + \delta y)$ is increasing or decreasing, in any case $\lim\limits_{t\to\infty} (z(t) + \delta y(t))$ there exists, since the function $(z + \delta y)$ is positive and bounded. Suppose that $\lim\limits_{t\to\infty} (z(t) + \delta y(t)) = c > 0$; as the function $(z + \delta y)''$ is bounded, according to Lemma (2.11), we have that $\lim\limits_{t\to\infty} (z(t) + \delta y(t))' = 0$. Taking into account the equations in (3.1) we get,

$$\lim_{t\to\infty} \left(\frac{\sigma x(t)\, y(t)}{\phi + x(t)} - \gamma\delta y(t) \right) = 0.$$

If $\lim\limits_{t\to\infty} y(t) = p > 0$ then $\lim\limits_{t\to\infty} z(t) = q > 0$. This implies that $\lim\limits_{t\to\infty} x(t) = \frac{\phi\gamma\delta}{\sigma-\gamma\delta} = k$; therefore point (k, p, q), $p, q > 0$ must be an equilibrium point. But such a form for an equilibrium point it is impossible to exist in this case. This contradiction implies that must be $p = 0$, that is

$$\lim_{t\to\infty} (z(t) + \delta y(t)) = 0,$$

and again $\lim\limits_{t\to\infty} y(t) = 0$, $\lim\limits_{t\to\infty} z(t) = 0$. □

Thus we have the following theorem about the stability of the equilibrium point E_k.

THEOREM (4.12). *The equilibrium point E_k is globally asymptotically stable, if and only if, inequality (4.7) holds. Otherwise, if relation (4.2) holds then the equilibrium point E_k is unstable.*

In the sequel we suppose that condition (4.2) hold. We are going to study the stability of the equilibrium point of positive coordinates E_*. The Jacobian matrix of system (3.1) evaluated at this equilibrium point is

$$J(E_*) = \begin{pmatrix} \rho\left(1 - \dfrac{2x_*}{k}\right) - \dfrac{\sigma\phi y_*}{(\phi + x_*)^2} & -\delta\gamma & 0 \\ 0 & -\gamma & 1 \\ \dfrac{\sigma\phi y_*}{(\phi + x_*)^2} & \delta\gamma & -\delta \end{pmatrix},$$

and the characteristic polynomial of $J(E_*)$ is

$$p(\lambda) = \det(J(E_*) - \lambda I) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3\,.$$

Is easy to see that

$$a_3 = -\det(J(E_*)) = \frac{\delta\gamma\sigma\phi y_*}{(\phi + x_*)^2} > 0\,,$$

and so we have the following lemma.

LEMMA (4.13).

$$\det J(E_*) = -a_3 < 0. \tag{4.14}$$

And exactly one of the following hold:

(a) *Re*$(\lambda) < 0$ *for all eigenvalues.*

(b) *There is one negative eigenvalue and a pair of nonzero purely imaginary eigenvalues (if and only if* $a_1 > 0,\ a_2 > 0$*, and* $a_1 a_2 = a_3$*).*

(c) There is one negative eigenvalue and a pair of eigenvalues with positive real part.

We are looking for sufficient conditions for E_* to be hyperbolic and unstable, because in this case Theorem (2.3) implies the existence of periodic orbits. The following lemma provides sufficient condition for the local stability and unstability of E_*

LEMMA (4.15). *Let* $k_0 = \dfrac{\phi(\sigma + \delta\gamma)}{\sigma - \delta\gamma}$, $a = \rho\left(1 - \dfrac{2x_*}{k}\right)$, $b = \dfrac{\phi\rho\left(1 - \dfrac{x_*}{k}\right)}{\phi + x_*}$, *then:*

(i) *If* $k \geq k_0$, E_* *is an unstable hyperbolic equilibrium with a one-dimensional stable manifold.*

(ii) *If* $k < k_0$, E_* *is locally asymptotically stable if and only if*

$$\left(\delta + \gamma + b - a\right)\left(b - a\right)(\delta + \gamma) > b\delta\gamma. \tag{4.16}$$

Specifically, there exists k_* *satisfying* $x_* < k_* < k_0$ *such that for each* k, $k_* < k < k_0$, E_* *is hyperbolic and unstable with a one-dimensional stable manifold. While for* k, $x_* < k < k_*$, E_* *is locally asymptotically stable.*

Proof. The matrix $J(E_*)$ can be written as

$$J(E_*) = \begin{pmatrix} a - b & -\delta\gamma & 0 \\ 0 & -\gamma & 1 \\ b & \delta\gamma & -\delta \end{pmatrix}$$

and its characteristic polynomial is given by

$$p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3. \tag{4.17}$$

where $a_1 = \gamma + b - a + \delta$, $a_2 = \left(b - a\right)(\delta + \gamma)$ and $a_3 = b\delta\gamma$. We can see that

$$b - a = \frac{\phi\rho}{\phi + x_*}\left(1 - \frac{x_*}{k}\right) - \rho\left(1 - \frac{2x_*}{k}\right) = \frac{\rho}{k}\frac{\delta\gamma}{\sigma}\left(k_0 - k\right).$$

So, if $k \geq k_0$ then $b - a \leq 0$, and therefore the coefficient a_2 of the polynomial p in (4.17) is non-positive, hence the Routh-Hurwitz criterion implies that the equilibrium point E_* is unstable and the case (c) of Lemma (4.13) holds, and this shows part (i). On the other hand, if $k < k_0$, then $b - a > 0$. In that case, the Routh-Hurwitz criterion implies that equilibrium point E_* is locally asymptotically stable if and only if the relation (4.16) holds. In order to analyze the conclusion of the part (ii) of the lemma, let

$$\begin{aligned}\Delta\left(k\right) &= \left(\delta+\gamma+\frac{\rho}{k}\frac{\delta\gamma}{\sigma}\left(k_0-k\right)\right)\left(\frac{\rho}{k}\frac{\delta\gamma}{\sigma}\left(k_0-k\right)\right)(\delta+\gamma)-b\delta\gamma\\ &= \left(\delta+\gamma+b-a\right)\left(b-a\right)(\delta+\gamma)-b\delta\gamma\\ &= (\gamma+\delta)\left(b-a\right)^2+(\gamma+\delta)^2\left(b-a\right)-\delta\gamma b.\end{aligned}$$

The function $\Delta\left(k\right)$ is decreasing for $0<k<k_0$, and as we can see

$$\Delta(x_*) = (\gamma+\delta+\rho)(\gamma+\delta)\rho > 0,$$

and $\Delta\left(k_0\right)=-\delta\gamma b<0$. So there exists a unique value k_*, $x_*<k_*<k_0$, such that $\Delta\left(k_*\right)=0$. Clearly, if $k_*<k<k_0$, then $\Delta\left(k\right)<0$, hence E_* is unstable and again case (c) of Lemma (4.13) holds; while if $x_*<k<k_*$, then $\Delta\left(k\right)>0$, so E_* is locally asymptotically stable. This shows part (ii) of the lemma. □

5. Uniform persistence and stability of periodic orbits

In this section we apply the theory of Uniform Persistence given by H. Thieme in [11] and we refer to the reader to this paper for the basic definitions.

We can easily determine the behavior of the solutions of (3.1) on the boundary of $\mathbb{E}$. The following lemma summarizes this situation and give us the two dimensional stable manifolds of the equilibria E_0 and E_k.

LEMMA (5.1). *Assume that relation (4.2) holds. Then*

(i) *The x-axis, the y-axis and the* (y,z)*−plane are invariants under the flow induced by system (3.1).*

(ii) *The intersection of the stable manifold of the equilibrium point* E_0 *with* $\mathbb{E}$ *consists of all of the points* $(0,y,z)$ *such that* $y\geq 0$ *and* $z\geq 0$.

(iii) *The intersection of the stable manifold of the equilibrium point* E_k *with* $\mathbb{E}$ *consists of all of the points* $(x,0,z)$ *such that* $x\geq 0$ *and* $z\geq 0$.

Now we will show that the predator and the prey populations are persistent (see [11]) when relation (4.2) holds.

LEMMA (5.2). *Suppose that relation (4.2) holds, then system (3.1) is uniformly persistent, i.e., there exist constants* $\varepsilon_i>0$, $i=1,2,3$, *independent of the initial conditions, such that*

$$\lim_{t\to\infty}\inf x(t)\geq\varepsilon_1\,,\ \lim_{t\to\infty}\inf y(t)\geq\varepsilon_2\ \text{ and }\ \lim_{t\to\infty}\inf z(t)\geq\varepsilon_3\,.$$

Proof. Let $Y=\{(x,0,0):0\leq x\leq k+1\}$. We apply Theorem 4.6 of [11], and notation of that result. Consider the following sets

$$\begin{aligned}X_1 &= \{(x,y,z):0<x\leq k+1,\ 0<y\leq M+1, 0<z\leq M+1\}-Y,\\ X_2 &= \{(0,y,z):0\leq y\leq M+1,\ 0\leq z\leq M+1\}\cup Y,\\ X &= X_1\cup X_2.\end{aligned}$$

where M is the constant given in Theorem (3.2). Now, X_1 and X_2 are disjoint subsets of $\mathbb{E}$, X_1 and X_2 are positively invariant sets for the solutions of the system (3.1), X_2 is compact and $X=X_1\cup X_2$ is also compact. By Theorem (3.2)

X is a global attractor in $\mathbb{E}$. We need to prove that solutions $u(t)$ of system (3.1) starting in X_1 are eventually bounded away from X_2, uniformly with respect to the initial data. The compactness assumption $(C_{4,2})$ of Theorem 4.6 of [11] holds with B as in Theorem (3.2) (for small positive δ as defined in [11]). Define $\Omega_2 = \cup_{p\in X_2}\omega(p)$, $p = (x, y, z)$. According to Lemma (5.1), Ω_2 consists of the equilibria E_0 and E_k and hence it has an acyclic isolated covering $M = M_0 \cup M_k$, where $M_l = \{E_l\}$ for $l = 0, k$ (for the definitions of acyclicity and isolatedness see [11]). The acyclicity comes out from Lemma (5.1) and the the isolatedness is consequence of the hiperbolicity of the points E_0 and E_k. We must also to show that each M_l is a weak repeller for X_1; for all $u(0) \in X_1$, $\limsup\limits_{t\to\infty} |u(t) - E_l| > 0$. Suppose, for contradiction, that a solution $u(t)$ with $u(0) \in X_1$ satisfies that $\lim\limits_{t\to\infty} u(t) = E_0$. Then $u(0)$ belongs to the stable manifold of E_0. But the intersection of the latter with $\mathbb{E}$ consists of the $(y, z)-$plane by Lemma (5.1) so we have a contradiction to $u(0) \in X_1$. An entirely similar contradiction is reached if $\lim\limits_{t\to\infty} u(t) = E_k$ since the intersection of the stable manifold of E_k with $\mathbb{E}$, as described in Lemma (5.1), contains no points of X_1. Hence, by Theorem 4.6 of [11], X_2 is a uniform strong repeller for X_1; that is, there are $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\varepsilon_3 > 0$ such that

$$\lim_{t\to\infty} \inf x(t) \geq \varepsilon_1,\ \lim_{t\to\infty} \inf y(t) \geq \varepsilon_2 \text{ and } \lim_{t\to\infty} \inf z(t) \geq \varepsilon_3,$$

where ε_1, ε_2 and ε_3 do not depend on the initial values in X_1. □

Now we show that under certain conditions the system (3.1) has the property of stability of periodic orbits.

THEOREM (5.3). *Let* $\delta\gamma < \dfrac{k\sigma}{\phi + k}$ *and suppose that the numbers* $\varepsilon_1, \varepsilon_2$ *given in the previous lemma satisfy*

$$\frac{2\rho\varepsilon_1}{k} > \rho - \frac{\sigma\varepsilon_2}{(\phi + \varepsilon_1)^2} > 0. \tag{5.4}$$

Then system (3.1) has the property of stability of periodic orbits.

Proof. Let $p(t) = (x(t), z(t), y(t))$ be a periodic solution, then his orbit Γ is contained in $intX_1$. In accordance with the criterion given by J. Muldowney in [8], in order to obtain periodic orbit of a autonomous system is orbitally asymptotically stable, it is sufficient to show that the linear non-autonomous system

$$w'(t) = J^{[2]}(p(t))w(t) \tag{5.5}$$

is asymptotically stable, where $J^{[2]}$ is the second additive compound matrix of the Jacobian J given by (4.3). For the solution $p(t)$, the system (5.5) becomes

$$\begin{aligned} w_1' &= -\left(-\rho + \frac{2\rho}{k}x + \frac{\sigma\phi y}{(\phi+x)^2} + \gamma\right) w_1 + w_2, \\ w_2' &= \frac{\sigma x}{\phi + x}w_1 + \left(\rho - \frac{2\rho}{k}x - \frac{\sigma y}{(\phi+x)^2} - \delta\right) w_2 - \frac{\sigma x}{\phi + x}w_3, \\ w_3' &= \frac{\sigma y}{(\phi+x)^2}w_1 - (\delta + \gamma) w_3. \end{aligned} \tag{5.6}$$

In order to show that the system (5.6) is asymptotically stable, we apply the following Lyapunov function:

$$V(w_1(t), w_2(t), w_3(t), x(t), y(t), z(t)) = \left\|\left(\tfrac{z(t)}{y(t)}w_1(t), w_2(t), \tfrac{z(t)}{y(t)}w_3(t)\right)\right\|,$$

where $\|\cdot\|$ is the norm in $\mathbb{R}^3$ defined by

$$\|(w_1, w_2, w_3)\| = \sup\{|w_2|, |w_1| + |w_3|\}.$$

From Lemma (5.2), we obtain that the orbit of $p(t)$ remains at a positive distance from the boundary of X_1; therefore

$$y(t) \geq \alpha,\ \ z(t) \geq \alpha,\ \ \alpha = \min\{\varepsilon_2, \varepsilon_3\} \text{ for all } t.$$

Hence the function V is well defined along $p(t)$ and

$$V(w_1, w_2, w_3; x, y, z) \geq \frac{\alpha}{\overline{M}} \|(w_1, w_2, w_3)\|, \tag{5.7}$$

where $\overline{M} = \max\{k+1, M+1\}$. Along a solution $(w_1(t), w_2(t), w_3(t))$ of system (5.6), V becomes

$$V(t) = \sup\left\{|w_2(t)|, \frac{z(t)}{y(t)}\left(|w_1(t)| + |w_3(t)|\right)\right\}.$$

Then we have the following inequalities:

$$D_+|w_1(t)| \leq -\left(-\rho + \frac{2\rho}{k}x + \frac{\sigma y}{(\phi+x)^2} + \gamma\right)|w_1(t)| + |w_2(t)|, \tag{5.8}$$

$$D_+|w_2(t)| \leq -\left(-\rho + \tfrac{2\rho}{k}x + \tfrac{\sigma y}{(\phi+x)^2} + \delta\right)|w_2(t)| + \tfrac{\sigma x}{\phi+x}\left(|w_1(t)| + |w_3(t)|\right), \tag{5.9}$$

$$D_+|w_3(t)| \leq -(\delta+\gamma)|w_3(t)| + \frac{\sigma y}{(\phi+x)^2}|w_1(t)|. \tag{5.10}$$

From (5.8) and (5.10) we get

$$D_+\left(|w_1(t)| + |w_3(t)|\right) \leq |w_2(t)| - L\left(|w_1(t)| + |w_3(t)|\right),$$

where $L = \min\left\{-\rho + \dfrac{2\rho}{k}x + \dfrac{\sigma y}{(\phi+x)^2} + \gamma, \delta+\gamma\right\}$. Therefore

$$\begin{aligned}
D_+(\tfrac{z}{y}(|w_1(t)| + |w_3(t)|)) &= (\frac{z'}{z} - \frac{y'}{y})\frac{z}{y}\left(|w_1(t)| + |w_3(t)|\right) \\
&\quad + \frac{z}{y}D_+\left(|w_1(t)| + |w_3(t)|\right) \\
&\leq (\frac{z'}{z} - \frac{y'}{y})\frac{z}{y}\left(|w_1(t)| + |w_3(t)|\right) \\
&\quad + \frac{z}{y}|w_2(t)| - L\frac{z}{y}\left(|w_1(t)| + |w_3(t)|\right) \\
&\leq \frac{z}{y}|w_2(t)| + \qquad\qquad\qquad\qquad (5.11)\\
&\quad (\frac{z'}{z} - \frac{y'}{y} - L)\frac{z}{y}(|w_1(t)| + |w_3(t)|).
\end{aligned}$$

From (5.9) and (5.11) we get

$$D_+V(t) \leq \sup\{h_1(t), h_2(t)\}V(t), \tag{5.12}$$

where

$$\begin{aligned} h_1(t) &= -(-\rho + \frac{2\rho}{k}x + \frac{\sigma y}{(\phi+x)^2} + \delta) + \frac{\sigma x y}{(\phi+x)z}, \\ h_2(t) &= \frac{z}{y} + \frac{z'}{z} - \frac{y'}{y} - L. \end{aligned}$$

and from the last two equations of (3.1)

$$\begin{aligned} h_1(t) &= \rho - \frac{2\rho}{k}x - \frac{\sigma y}{(\phi+x)^2} - \delta + \frac{z'}{z} + \delta \\ &= \rho - \frac{2\rho}{k}x - \frac{\sigma y}{(\phi+x)^2} + \frac{z'}{z} \end{aligned}$$

If $-\delta < \rho - \frac{2\rho}{k}x - \frac{\sigma y}{(\phi+x)^2} < 0$, then $L = -\rho + \frac{2\rho}{k}x + \frac{\sigma y}{(\phi+x)^2} + \gamma$; in that case

$$h_2(t) = \frac{z}{y} + \frac{z'}{z} - \frac{y'}{y} - \left(-\rho + \frac{2\rho}{k}x + \frac{\sigma y}{(\phi+x)^2} + \gamma\right) = \frac{z'}{z} + \rho - \frac{2\rho}{k}x - \frac{\sigma y}{(\phi+x)^2}.$$

Hence

$$\sup\{h_1(t), h_2(t)\} \le \frac{z'}{z} + \rho - \frac{2\rho}{k}x - \frac{\sigma y}{(\phi+x)^2} \le -\mu + \frac{z'}{z}, \tag{5.13}$$

where $\mu > 0$ satisfies that $\rho - \frac{2\rho}{k}x - \frac{\sigma y}{(\phi+x)^2} \le -\mu < 0$. If $\rho - \frac{2\rho}{k}x - \frac{\sigma y}{(\phi+x)^2} < -\delta$, then $L = \delta + \gamma$; and then we get

$$h_2(t) = \frac{z}{y} + \frac{z'}{z} - \frac{y'}{y} - (\delta + \gamma) = -\delta + \frac{z'}{z}.$$

Hence

$$\sup\{h_1(t), h_2(t)\} \le -\delta + \frac{z'}{z}. \tag{5.14}$$

Let $\nu = \min\{\mu, \delta\}$. Then from (5.13) and (5.14), we have

$$\sup\{h_1(t), h_2(t)\} \le -\nu + \frac{z'}{z}. \tag{5.15}$$

Therefore, from (5.12) and Gronwall inequality,

$$V(t) \le V(0)\, z(t)\, e^{-\nu t} \le V(0)\, \overline{M} e^{-\nu t},$$

which implies that $V(t) \to 0$ as $t \to \infty$. By the equation (5.7) we can see that

$$(w_1(t), w_2(t), w_3(t)) \to 0, \quad \text{as} \quad t \to \infty.$$

This implies that the linear system (5.6) is asymptotically stable and therefore according to the previos mentioned result of J. Muldowney [8], the periodic solution $p(t)$ is orbitally asymptotically stable. □

6. Existence of a stable periodic orbit and global analysis

In this section we give our main results concerning with the existence of periodic orbits and we describe the global attractor under the conditions of the Theorem (5.3). The following theorem has to do with the existence of orbitally asymptotically stable periodic orbits.

THEOREM (6.1). *Let* $\delta\gamma < \dfrac{\sigma k}{\phi + k}$*, and assume that the unique nontrivial equilibrium* E_* *is hyperbolic and unstable. Then* E_* *has a one-dimensional stable manifold* $W^s(E_*)$*. Furthermore, there exists an orbitally asymptotically stable periodic orbit, and the* $\omega -$ *limit set of every solution* $(x(t), y(t), z(t))$ *with* $x(0) > 0$*,* $y(0) > 0$*,* $z(0) > 0$ *and* $(x(0), y(0), z(0)) \notin W^s(E_*)$ *is a nonconstant periodic orbit.*

Proof. We apply Theorem (2.5) and Theorem (2.3) to system (3.1). From Lemma (4.13) we see that the stable manifold of E_* is one dimensional and from Lemma (3.8) we can see that the system (3.1) is equivalent to the competitive and irreducible system (3.9). The existence of an orbitally asymptotically stable periodic orbit follows from Theorem (2.5) and the analyticity of the vector field. Note that (H1) holds by Theorem (2.5) and Lemma (5.2) (the latter must be translated appropriately to system (3.9)), in particular, we take the domain D given in (3.7). Using Lemma (5.2), Theorem (2.3) implies the final assertion. □

In the following theorem we give conditions wich imply the existence of a unique globally orbitally asymptotically stable periodic orbit. Also, we give conditions for the global asymptotic stability of the equilibrium point E_*.

THEOREM (6.2). *Under the conditions of the Theorem* (5.3)*, the global attractors of the solutions of the system (3.1) are given as follows:*

(i) *If* $k \geq k_0$*, then there exists a unique periodic orbit that is globally orbitally asymptotically stable.*

(ii) *If* $k_* < k < k_0$*, then there exists a unique periodic orbit which is globally orbitally asymptotically stable.*

(iii) *If* $x_* < k < k_*$*, then the equilibrium point* E_* *is globally asymptotically stable.*

Proof. (i) According to the previous theorem there exists a periodic orbit. Also, under the conditions of the Theorem (5.3) each periodic solution is orbitally asymptotically stable and in that case Theorem (2.8) assures that there exists a unique periodic orbit. On the other hand, taking into account that the system (3.1) is equivalent to the competitive and irreducible system (3.9) the Poincaré-Bendixson Theorem (2.2) implies that the periodic orbit is globally orbitally asymptotically stable.

(ii) The proof is similar to the previous case.

(iii) Since the equilibrium point E_* is locally asymptotically stable and the system (3.1) has the property of stable periodic orbit by the Theorem (2.8), there exists a unique periodic orbit which is orbitally asymptotically stable. This situation is contradictory with the Poincaré-Bendixson Theorem (2.2) because system (3.1) is equivalent to a competitive system. For this reason, no such periodic orbit exists and the equilibrium point E_* is globally asymptotically stable. □

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FREE-DIMENSIONAL BOUNDEDNESS OF THE MAXIMAL OPERATOR

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ABSTRACT. In this work we study the behavior of the constant appearing in the weak type inequality (1, 1), for the centered maximal operator acting on radial functions and associatedwith monotone radial measures, with respect to the dimension of the underlying space euclidean. In the case of a monotone increasing radial measure, we will prove that this constant is indeed independent of the dimension. We will also see that the techniques used in this case cannot be extended to the case of the maximal operator associated with decreasing radial measures.

1. Introduction

Let μ be a non negative Borel measure defined over the Euclidean space $\mathbb{R}^n$. Given a function $f \in L^1_{\text{loc}}(d\mu)$, the Hardy-Littlewood maximal operator is defined as

$$\mathcal{M}_\mu f(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y),$$

where the supremum is taken over all balls B containing the point x, with $\mu(B) > 0$. Also the centered Hardy-Littlewood maximal operator $\mathcal{M}^c_\mu f$, is defined by taking the supremum only over balls centered on x.

The centered maximal operator maps $L^1(d\mu)$ into $L^{1,\infty}(d\mu)$. This can be proved using the Besicovitch covering lemma. An operator that satisfies this boundedness property is of weak type (1, 1),

$$\mu\left\{x \in \mathbb{R} : |\mathcal{M}_\mu f(x) > \lambda|\right\} \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)|\, d\mu(x),$$

for more details see E. Stein [7].

One of the problems that has had special attention in recent years is the study of the boundedness constants dependence of the underlying Euclidean space. This has special importance to develop harmonic analysis for the infinite dimensional case.

E. Stein [7] (see also [8]) proved that the centered Hardy-Littlewood maximal operator associated to the Lebesgue measure, $\mathcal{M}^c$, is bounded on $L^p(\mathbb{R}^n)$, $p > 1$, with a constant independent of dimension.

E. Stein's result was extended by J. Bourgain [1] who proved it for the maximal operator defined over symmetric bodies for $p \geq 2$. The range of p can be extended down to $p > \frac{3}{2}$ which was proved independently by J. Bourgain [2] and A. Carbery [3].

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E. Stein and O. Strömberg [8] proved it for euclidean balls using an argument involving the method of rotations, that the constant of weak type $(1, 1)$ grows no faster than $O(n)$.

Then, T. Menárguez and F. Soria [6] studied the behavior of the constants appearing on the weak type inequalities for the dyadic maximal operator associated to a convex body. Let B be a symmetric convex set in $\mathbb{R}^n$, $B_r(x)$ denotes dilation by r and translation by x of B. If $q = \{q_k\}_{k\in\mathbb{Z}}$ is a lacunary sequence of positive numbers, i.e. there is a positive number a_q such that $\frac{q_{k+1}}{q_k} \geq a_q > 1$ for all k, we define the maximal operator as

$$M_q f(x) = \sup_{k\in\mathbb{Z}} \frac{1}{|B_{q_k}|} \int_{B_{q_k}(x)} |f(y)|\, dy.$$

Then, there exists a constant C independent of dimension such that

$$|\{x \in \mathbb{R}^n \,:\, M_q f(x) > \lambda\}| \leq \frac{C}{\lambda}\left(1 + \frac{\log n}{\log a_q}\right) \|f\|_1,$$

for every function $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$.

T. Menárguez and F. Soria [6] have considered the special case in which the maximal operator associated to the Lebesgue measure is defined for radial functions of $\mathbb{R}^n$, these are functions of the form $f(x) = f_0(|x|)$, and they have proved that for this maximal operator the bound is independent of the dimension.

In this paper we study the size of the constant appearing on the weak type $(1, 1)$ inequality for the centered maximal operator associated with a monotone radial measure defined over radial functions. In the case of monotone increasing measures we will see that the constant obtained is independent of dimension from the associated Euclidean space. But in the case of monotone decreasing measures we see that the previous argument fails. Thus it is an open problem the study the behavior of the constant appearing on the weak type $(1, 1)$ inequality, for the centered maximal operator associated to a monotone decreasing radial measure defined over radial functions.

2. Case of monotone increasing measures

In this section we will study the size of the constant appearing on the weak type $(1, 1)$ inequality, for the centered Hardy-Littlewood maximal operator associated to a monotone increasing radial measure and defined on radial functions of $\mathbb{R}^n$. In this case we will prove that this inequality can be established with a constant independent of dimension.

THEOREM (2.1). *Let $\mathcal{M}_\mu^c$ be the centered maximal operator, defined over balls on $\mathbb{R}^n$ and associated to a measure $d\mu(x) = \gamma_0(|x|)dx$, with γ_0 a positive increasing function. Let f be a radial function over $\mathbb{R}^n$, then*

$$\mu\left\{x :\, \mathcal{M}_\mu^c f(x) > \lambda\right\} \leq \frac{4}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, d\mu(y) \qquad \text{for all } \lambda > 0.$$

In order to prove this result we need the next lemma in which the measure of any ball centered at x can be compared with the measure of a set that contains B with constants independent of dimension and such that it has a good description in polar coordinates.

LEMMA (2.2). *Let $x \in \mathbb{R}^n$ and B any ball centered at x. Then there exists a set $\Sigma_B \subset S^{n-1}$, and two functions $\epsilon_1,\ \epsilon_2 : \Sigma_B \to \mathbb{R}_+$, such that $\epsilon_1(\omega) \le |x| \le \epsilon_2(\omega)$ for every $\omega \in \Sigma_B$,*

$$B \subset D := \{t\omega :\ \omega \in \Sigma_B,\ \epsilon_1(\omega) \le t \le \epsilon_2(\omega)\}$$

and $\mu(D) \le 2\mu(B)$.

The radial projection of μ, denoted by μ_0, on the positive axis is defined for a measurable set $E \subset \mathbb{R}_+$ as

$$\omega_{n-1} \int_E d\mu_0(t) = \mu(\{y :\ |y| \in E\}).$$

If μ has density γ_0 then we clearly have

$$d\mu_0(t) = t^{n-1}\gamma_0(t)dt.$$

Proof of the Lemma (2.2). Let B be a ball centered at x and radius r. We consider two cases. We consider first the case of the balls B containing the origin $\overline{0}$. In this case we will set $\Sigma_B = S^{n-1}$ and $D = B \cup B_{|x|}(0)$. We define $\epsilon_1(\omega) \equiv 0$ and $\epsilon_2(\omega)$ as the distance between the origin and the intersection of the ray passing by the origin on the direction ω with the boundary of D, i.e.

$$\epsilon_2(\omega) = \left|\sup_{t>0} \{t\omega \in D\}\right|.$$

It is clear that $\mu(D) \le 2\mu(B)$, because $\mu(B_{|x|}(0)) \le \mu(B)$.

In the case of $\overline{0} \notin B$, we denote by Γ the interior of the smallest cone containing the ball B and with vertex at origin. We set $\Sigma_B = \Gamma \cap S^{n-1}$.

For each $\omega \in \Sigma_B$, we define $\epsilon_1(\omega)$ as the distance from origin to the first intersection point of the ray ω which passes by origin with the closure of B. We define the set D by

$$D = D_1 \cup D_2,$$

where

$$D_1 = \{t\omega :\ \omega \in \Sigma_B,\ \epsilon_1(\omega) \le t \le |x|\}$$

and

$$D_2 = B \setminus B_{|x|}(0).$$

We define $\epsilon_2(\omega)$ as the distance between the origin and the farthest point of the intersection of the ray ω with the closure of D.

Clearly $D_2 \subset B$. Also D_1 is contained in the ball $B_r(z)$, where z is the point on the segment $\overline{0x}$ such that $|z|$ equal to the distance from the set $\partial\Gamma \cap B$ to the origin. As $D_1 \subset B_r(z)$ and the measure μ is increasing, we have

$$\mu(D_1) \le \mu(B_r(z)) \le \mu(B),$$

where $B = B_r(x)$. □

Another tool that we will use in the proof of the theorem (2.1) is the following result for $n = 1$: if μ is a regular measure on $\mathbb{R}$, then we can obtain the weak type inequality using a simple geometric argument (see A. Garsia [4] and B. Muckenhoupt and E. Stein [5]). More precisely, it satisfies

$$\mu\{x :\ \mathcal{M}_\mu f(x) > \lambda\} \le \frac{2}{\lambda} \int |f(y)|\, d\mu(y), \tag{2.3}$$

for all $\lambda > 0$. Observe that the constant appearing on the inequality (2.3) is 2.

Proof of the Theorem (2.1). Given a ball B with center at x and radius r, $\Sigma_B \subset S^{n-1}$, $\epsilon_1,\ \epsilon_2 : \Sigma_B \to \mathbb{R}_+$ and D given as in the Lemma (2.2), then

$$\mu(D) = \int_{\Sigma_B} \left(\int_{\epsilon_1(\omega)}^{\epsilon_2(\omega)} t^{n-1} \gamma_0(t) dt \right) d\sigma(\omega) = \int_{\Sigma_B} \mu_0(I_\omega) d\sigma(\omega),$$

where μ_0 is the radial projection of μ, and $I_\omega = [\epsilon_1(\omega), \epsilon_2(\omega)]$. Observe that $\epsilon_1(\omega) \leq |x| \leq \epsilon_2(\omega)$ for every $\omega \in \Sigma_B$, and therefore $|x| \in I_\omega$.

Now by (2.3), considering

$$\mathcal{M}_{\mu_0} f_0(|x|) = \sup_{|x| \in I} \frac{1}{\mu_0(I)} \int_I |f_0(t)|\, \mu_0(t) dt,$$

we get, that

$$\mu_0 \left(\{ t \in \mathbb{R} :\ \mathcal{M}_{\mu_0} f_0(t) > \lambda \} \right) \leq \frac{2}{\lambda} \int_{\mathbb{R}} |f(t)|\, \mu_0(t) dt, \qquad \lambda > 0. \tag{2.4}$$

Now consider a radial function f with $f_0(|x|) = f(x)$, using the previous Lemma, we have

$$\begin{aligned}
\frac{1}{\mu(B)} \int_B |f(y)|\, d\mu(y) &\leq \frac{2}{\mu(D)} \int_D |f(y)|\, d\mu(y) \\
&= \frac{2}{\mu(D)} \int_{\Sigma_B} \mu_0(I_\omega) \left(\frac{1}{\mu_0(I_\omega)} \int_{I_\omega} |f_0(t)|\, \mu_0(t) dt \right) d\sigma(\omega) \\
&\leq 2 \sup_{|x| \in I} \frac{1}{\mu_0(I)} \int_I |f_0(t)|\, \mu_0(t) dt \\
&= 2 \mathcal{M}_{\mu_0} f_0(|x|).
\end{aligned}$$

In the previous inequalities we have used that

$$\int_{\Sigma_B} \mu_0(I_\omega) d\sigma(\omega) = \mu(D).$$

We have,

$$\begin{aligned}
\mu \left\{ x \in \mathbb{R}^n :\ \mathcal{M}_\mu^c f(x) > \lambda \right\} &\leq \int_{S^{n-1}} d\sigma(\omega) \int_{\left\{ t > 0:\ \mathcal{M}_{\mu_0} f_0(t) > \frac{\lambda}{2} \right\}} t^{n-1} \gamma_0(t) dt \\
&= \int_{S^{n-1}} \mu_0 \left(\left\{ t > 0 :\ \mathcal{M}_{\mu_0} f_0(t) > \frac{\lambda}{2} \right\} \right) d\sigma(\omega).
\end{aligned}$$

Using the estimate (2.4), we finally have

$$\begin{aligned}
\mu \left\{ x \in \mathbb{R}^n :\ \mathcal{M}_\mu^c f(x) > \lambda \right\} &\leq \frac{4}{\lambda} \int_{S^{n-1}} d\sigma(\omega) \int_{\mathbb{R}} |f_0(t)|\, \mu_0(t) dt \\
&= \frac{4}{\lambda} \int_{\mathbb{R}^n} |f|\, d\mu.
\end{aligned}$$

$\square$

3. Case of monotone decreasing measures

The previous argument can not be used in the case of radial monotone decreasing measure μ, since it is not known that

$$\mu(B) \sim \mu(D),$$

with constant independent of the dimension. Trying to retrieve part of the argument we will study the problem of constant independence for a modified maximal operator.

Let $B = B_r(x)$ be a ball with center x and radius r, with $|x| > \frac{r}{2}$. We define

$$\mathcal{C}_r(x) = B_r(x) \cap \Gamma_r(x)$$

where $\Gamma_r(x)$ is the smallest cone with vertex at origin and containing the set $(\partial B \cap \partial B_{|x|}(0))$. If $|x| \leq \frac{r}{2}$, we simply say that $C_r(x) = B_r(x)$.

Given a Borel measure μ defined on $\mathbb{R}^n$, we define the maximal operator

$$\widehat{M}_\mu f(x) = \sup \frac{1}{\mu(\mathcal{C}_r(x))} \int_{\mathcal{C}_r(x)} |f| \, d\mu,$$

where the supremum is taken over all values $r > 0$.

THEOREM (3.1). *If f is a radial function, then*

$$\mu \left\{ x \in \mathbb{R}^n : \widehat{M}_\mu f(x) > \lambda \right\} \leq \frac{2}{\lambda} \int_{\mathbb{R}^n} |f| \, d\mu \, .$$

Proof. The key of the proof is that each set $C_r(x)$ can be written in polar coordinates as

$$C_r(x) = \{(t, \theta) : \ \theta \in \Sigma_r(x), \ \epsilon_1(\theta) \leq t \leq \epsilon_2(\theta)\}$$

where $\Sigma_r(x) = \Gamma_r(x) \cap S^{n-1}$ if $|x| > \frac{r}{2}$ and $\Sigma_r(x) = S^{n-1}$, if $|x| \leq \frac{r}{2}$. Besides, and this is crucial in the argument, we have $\epsilon_1(\theta) \leq |x| \leq \epsilon_2(\theta)$ for all $\theta \in \Sigma_r(x)$. The remainder is similar to the argument given in the previous section.

If we set $d\mu(x) = d\mu_0(r)d\sigma(\theta)$, where (r, θ) are the polar coordinates of x and $d\mu_0$ the radial projection of $d\mu$, we have

$$\begin{aligned} \mu(\mathcal{C}_r(x)) &= \int_{\Sigma_r(x)} \int_{\epsilon_1(\theta)}^{\epsilon_2(\theta)} d\mu_0(r) d\sigma(\theta) \\ &= \int_{\Sigma_r(x)} \alpha_B(\theta) d\sigma(\theta), \end{aligned}$$

where

$$\alpha_B(\theta) = \int_{\epsilon_1(\theta)}^{\epsilon_2(\theta)} d\mu_0(r).$$

If $f(x) = f_0(|x|)$, we have

$$\begin{aligned} &\frac{1}{\mu(\mathcal{C}_r(x))} \int_{\mathcal{C}_r(x)} |f(y)| \, d\mu(y) \\ = \ &\frac{1}{\mu(C_r(x))} \int_{\Sigma_r(x)} \alpha_B(\theta) \frac{1}{\alpha_B(\theta)} \int_{\epsilon_1(\theta)}^{\epsilon_2(\theta)} |f_0(r)| \, d\mu_0(r) d\sigma(\theta) \\ \leq \ &M_{\mu_0} f_0(|x|). \end{aligned}$$

We have

$$\begin{aligned}\mu\left\{x:\ \widehat{M}_\mu f(x)>\lambda\right\} &= \int_{S^{n-1}} \mu_0\left(\{r:\ M_{\mu_0} f_0(r)>\lambda\}\right) d\sigma \\ &\leq \int_{S^{n-1}} \frac{2}{\lambda}\int_0^\infty |f_0|(r) d\mu_0(r) d\sigma = \frac{2}{\lambda}\int_{\mathbb{R}^n} |f|\, d\mu.\end{aligned}$$

□

As we have mentioned before, the reason for not using the set D associated to a ball $B = B_r(x)$ is that we do not know in general that

$$\mu(D) \leq C\mu(B),$$

with constant C independent of the dimension. It would be interesting to study this inequality on specific decreasing measures as the gaussian measure $d\mu_2(x) = e^{-|x|^2}dx$. For such measure one might expect for it to satisfy the following inequality,

$$\mu(D \setminus B_r(x)) \leq C\mu(B_r(x) \cap B_{|x|}(0)),$$

with constant C independent of the dimension.

However, this is not true even in the case of the Lebesgue measure. We will take $r = |x|$, so the origin is in the bound of the ball.

If it would be true, we should have

$$\int_0^{r/2}\left((r^2-h^2)^{\frac{n-1}{2}} - \left(2rh-h^2\right)^{\frac{n-1}{2}}\right) dh \leq C\int_0^{r/2}\left(2rh-h^2\right)^{\frac{n-1}{2}} dh,$$

with C independent of n. So

$$\int_0^{r/2}(r^2-h^2)^{\frac{n-1}{2}} dh \leq (C+1)\int_0^{r/2}\left(2rh-h^2\right)^{\frac{n-1}{2}} dh.$$

By homogeneity we can take $r = 1$. In this case, the left hand side is

$$\int_0^{1/2}\left(1-h^2\right)^{\frac{n-1}{2}} dh \geq \frac{3^{\frac{n-1}{2}}}{2^n}.$$

Therefore,

$$\begin{aligned}\frac{3^{\frac{n-1}{2}}}{2^n} &\leq (C+1)\int_0^{1/2}\left(2h-h^2\right)^{\frac{n-1}{2}} dh \\ &= \frac{(C+1)}{2^n}\int_0^1 (4y-y^2)^{\frac{n-1}{2}} dy.\end{aligned}$$

Equivalently, we have

$$0 < \frac{1}{C+1} \leq \int_0^1\left(\frac{4y-y^2}{3}\right)^{\frac{n-1}{2}} dy,$$

for every $n \in \mathbb{N}$, which is false because $0 < \frac{4y-y^2}{3} < 1$, for every $y \in (0,1)$ and by monotone convergence theorem we get

$$\lim_{n\to\infty}\int_0^1\left(\frac{4y-y^2}{3}\right)^{\frac{n-1}{2}} dy = 0.$$

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ON $\mathbb{Q}$-INVARIANT ADÈLE-VALUED FUNCTIONS ON $\mathbb{A}$

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ABSTRACT. We initiate the development of an adèle-valued function theory defined on the adèle group $\mathbb{A}$ of the rationals, parallel to the classical theory over the circle. By introducing a variation of the Bohr–von Neumann concept of almost periodic functions we characterize the adèle-valued $\mathbb{Q}$-invariant functions on $\mathbb{A}$. We also present a variation of the notion of invariant mean.

1. Introduction

In the real classical analysis over the circle, the basic object of study is the $\mathbb{Z}$-bundle structure $\mathbb{Z} \hookrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ and the first step in the development of the whole theory is the basic remark that there is a one to one correspondence between

$$\{\mathbb{Z}\text{-invariant functions } \mathbb{R} \longrightarrow \mathbb{R}\} \longleftrightarrow \{\text{Continuous functions } \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{R}\}.$$

In this paper we initiate a parallel theory by replacing this basic object with the $\mathbb{Q}$-bundle structure $\mathbb{Q} \hookrightarrow \mathbb{A} \longrightarrow \mathbb{A}/\mathbb{Q}$, where $\mathbb{A}$ is the adèle group of the rationals which is a locally compact abelian group, $\mathbb{Q}$ is a discrete subgroup of $\mathbb{A}$ and $\mathbb{A}/\mathbb{Q}$ is a compact abelian group (see [Mac] for a very nice historical and complete description of the adèle group of $\mathbb{Q}$). Since

$$\mathbb{A}/\mathbb{Q} \cong \varprojlim \mathbb{R}/n\mathbb{Z},$$

we see that $\mathbb{A}/\mathbb{Q}$ is a projective limit whose n-th component corresponds to the unique covering of degree $n \geq 1$ of $\mathbb{R}/\mathbb{Z}$. This is also called the *algebraic universal covering space* of the circle $\mathbb{R}/\mathbb{Z}$ with canonical projection $\mathbb{A}/\mathbb{Q} \longrightarrow \mathbb{R}/\mathbb{Z}$, determined by coordinates projection. The Galois group of the covering is $\widehat{\mathbb{Z}}$, the *algebraic fundamental group* of $\mathbb{R}/\mathbb{Z}$. So, in many senses, the group $\mathbb{A}/\mathbb{Q}$ is a "generalized" circle and it seems plausible to try to rewrite, on the one hand, the complete theory of Fourier analysis on it and, on the other, the complete adèle-valued covering function theory. We present the first steps of the adèle-valued theory on $\mathbb{A}$ keeping in mind both motivations. We note that the $\mathbb{C}$–valued Fourier analysis on $\mathbb{A}$ has been developed since the 1950's (see [Tate], and [RV] for a more recent account).

Similar to the classical case, we first prove that there exists a one to one correspondence between

$$\{\mathbb{Q}\text{-invariant functions } \mathbb{A} \longrightarrow \mathbb{A}\} \longleftrightarrow \{\text{Continuous functions } \mathbb{A}/\mathbb{Q} \longrightarrow \mathbb{A}\}.$$

This is done by presenting a variation of the theory of almost periodic functions, originally introduced by Harald Bohr in 1925 (see [Bohr]), and later on

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generalized by John von Neumann in 1934 (see [Neu]). In 1938, André Weil, in his important monograph [Weil], did a description of almost periodic functions from a very general point of view, for transformation groups acting on general topological spaces. Here we use these general ideas in the concrete case of the group $\mathbb{A}$ identified with the group of translations $\{s \longrightarrow L_s\}$ as a group of homeomorphisms acting on $C_u(\mathbb{A}, \mathbb{A})$, the space of adèle-valued uniformly continuous functions on $\mathbb{A}$.

By studying this $\mathbb{A}$-action, we extend part of the theory of Bohr-von Neumann to functions defined in the adèle group of $\mathbb{Q}$ that are adèle-valued. We use a characterization introduced by A. Weil of these kind of functions, by using the fact that any topological group admits a continuous representation into a compact topological group whose image is dense. This extended notion will then permit us to characterize the class of functions we are interested in: the adèle-valued functions that are invariant under the (additive) action of $\mathbb{Q}$.

After presenting this variation of the almost periodicity concept, the next important point to be considered (as in the classical case) is the notion of invariant mean.

As has been already said, the ideas developed in this paper are the starting point of the development of a complete theory of adèle-valued Fourier analysis on $\mathbb{A}$. Here we are only interested in invariant functions since they are also the basic ingredient for the development of a 'covering' function theory for the $\mathbb{Q}$-bundle structure $\mathbb{Q} \hookrightarrow \mathbb{A} \longrightarrow \mathbb{A}/\mathbb{Q}$. In forthcoming papers we will continue developing this theory.

Finally, it should be pointed out that the analysis on the adèle group of the rationals has been of interest due to its importance for the physical formulation of adelic quantum mechanics which treats real and p-adic quantum mechanics in a unified way (see [DRK] and [Dra]).

In section 2 we describe the adèle group $\mathbb{A}$ of the rationals and define an uniform structure on it. In section 3 we make the description of the adèle-valued almost periodic functions defined on $\mathbb{A}$, and in the last section we generalize the notion of invariant mean.

2. The adèle group of $\mathbb{Q}$

The adèle group of $\mathbb{Q}$. In this section we define our basic object, the adèle group $\mathbb{A}$ of the rationals $\mathbb{Q}$. Since this is a classical object, we will only establish the general definitions and properties without any comment. A complete description of $\mathbb{A}$ can be found in [RV] (Chap. 5).

If $p \neq \infty$ is any prime number, define the *absolute value* or *norm* $|\cdot|_p$ on $\mathbb{Q}$ by

$$|x|_p = \begin{cases} p^{-\mathrm{ord}_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases},$$

where $\mathrm{ord}_p(x)$ is an integer such that $x = p^{\mathrm{ord}_p(x)}\dfrac{a}{b}$ and p does not divide a and b. It can be seen that $|\cdot|_p$ satisfies the usual properties of an ordinary absolute value and is called a *non-Archimedian absolute value* or *non-Archimedian norm*. If $p = \infty$, then $|x|_\infty$ is the usual absolute value on $\mathbb{Q}$.

By Ostrowski's theorem, every nontrivial norm $|\cdot|$ on $\mathbb{Q}$ is equivalent to $|\cdot|_p$ for some prime number p or for $p=\infty$. The completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ is the field of p-adic numbers $\mathbb{Q}_p$, which is a locally compact complete topological field. The ring of integers in $\mathbb{Q}_p$ is the compact topological ring of p-adic integers $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. If $p=\infty$, then $\mathbb{Q}_\infty = \mathbb{R}$ and $\mathbb{Z}_\infty = \mathbb{Z}$.

From now on, we consider only the additive structure on $\mathbb{Q}_p$ which makes it a locally compact abelian group and $\mathbb{Z}_p$ a compact abelian group.

Definition (2.1). The *adèle group* $\mathbb{A}$ of $\mathbb{Q}$ is the restricted direct product of $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$. That is,

$$\mathbb{A} := \left\{ (x_p) \in \prod_p \mathbb{Q}_p : x_p \in \mathbb{Q}_p \text{ with } x_p \in \mathbb{Z}_p \text{ for all but finitely many } p \right\},$$

where p is varying in the set of all prime numbers including $p=\infty$.

Clearly, $\mathbb{A}$ is a subset of the ordinary set-theoretic direct product of the $\mathbb{Q}_p$ and, moreover, a subgroup of the group-theoretic direct product.

We define a topology on $\mathbb{A}$ by specifying a neighborhood base of the identity consisting of the sets of the form $\prod_p N_p$, where N_p is a neighborhood of 0 in $\mathbb{Q}_p$ and $N_p := \mathbb{Z}_p$ for all but finitely many p. It is not difficult to prove that $\mathbb{A}$ is a locally compact abelian topological group. Also, by Haar's theorem we know that there exists a (unique) left invariant Haar measure defined on $\mathbb{A}$.

It is also possible to describe the adèle group of $\mathbb{Q}$ in the following way: If $\mathbb{A}_f$ denote the set of *finite adèles*, that is, $\mathbb{A}_f$ is the restricted direct product of $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$, for all $p<\infty$, then $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$.

A very important fact is that $\mathbb{Q}$ is a discrete cocompact subgroup of $\mathbb{A}$; i.e., $\mathbb{A}/\mathbb{Q}$ is a compact abelian topological group with canonical projection $\Pi_{\mathbb{Q}} : \mathbb{A} \longrightarrow \mathbb{A}/\mathbb{Q}$. The group $\mathbb{A}/\mathbb{Q}$ is called the *adèle class group* of $\mathbb{Q}$ and the following theorem gives a nice description of the adèle class group of $\mathbb{Q}$ (see [RV]):

THEOREM (2.2). *There is an isomorphism of topological groups* $\mathbb{A}/\mathbb{Q} \longrightarrow \varprojlim \mathbb{R}/n\mathbb{Z}$.

Uniform structure on $\mathbb{A}$. An *uniform structure* on $\mathbb{A}$ is determined by specifying a family of subsets of $\mathbb{A}\times\mathbb{A}$, called *entourages*, that satisfy certain axioms. In the concrete case in which we are interested in, we exhibit a *fundamental system of entourages* of such structure. To construct this system, we use the sets U defined in the following way. To each neighborhood U_0 of 0 in $\mathbb{A}$ we associate the set $U := \{(x,y) \in \mathbb{A}\times\mathbb{A} : y-x \in U_0\}$. Let $\mathfrak{U}$ be the family which consists of all such sets U, where U_0 varies in a fundamental system of neighborhoods of 0.

We verify directly that this family satisfies the axioms of a fundamental system of entourages of a uniformity (i.e., a uniform structure) on $\mathbb{A}$. That is,

- Each U contains the diagonal $\Delta \subset \mathbb{A}\times\mathbb{A}$ since $0 = x-x \in U_0$.
- Since $y-x \in U_0$ if and only if $x-y \in -U_0$, we conclude that $-U \in \mathfrak{U}$.
- If $z-x \in U_0$ and $y-z \in U_0$, then, $y-x = (y-z)+(z-x) \in U_0+U_0$ and therefore, $U+U \in \mathfrak{U}$.

We then have that $\mathfrak{U}$ is a fundamental system of entourages of a uniformity on $\mathbb{A}$. This uniformity is also compatible with the topology on $\mathbb{A}$, since, if $U(x) = \{y \in \mathbb{A} : y - x \in U_0\}$, then $y \in U(x)$ is equivalent to $y \in x + U_0$. That is, $U(x) = x + U_0$, which is a neighborhood of $x \in \mathbb{A}$. We also have that, with this uniform structure, $\mathbb{A}$ is a complete topological group.

We say that $x, y \in \mathbb{A}$ are 'close' if there exists an entourage U of $\mathbb{A}$ such that $y \in U(x)$. In this case, we will simply write $x \sim_U y$ or, $x \sim y$.

3. Adèle-valued almost periodic functions on $\mathbb{A}$

Function theory on $\mathbb{A}$. We will use the previously defined uniform structure on $\mathbb{A}$ to define an uniform structure on $\mathrm{Map}(\mathbb{A}, \mathbb{A})$, the set which consists of all maps of $\mathbb{A}$ into itself. As before, we will denote by U any entourage of the uniformity on $\mathbb{A}$ and by U_0 any neighborhood of the identity in $\mathbb{A}$.

For each entourage U of $\mathbb{A}$, we define $W(U)$ by:

$$\begin{aligned} W(U) &= \{(\Phi, \Psi) : (\Phi(x), \Psi(x)) \in U \text{ for every } x \in \mathbb{A}\} \\ &= \{(\Phi, \Psi) : \Psi(x) - \Phi(x) \in U_0 \text{ for every } x \in \mathbb{A}\}. \end{aligned}$$

If U varies in the family of entourages of $\mathbb{A}$, the family of sets $\mathfrak{W} := \{W(U) : U \in \mathfrak{U}\}$ form a fundamental system of entourages of a uniformity on $\mathrm{Map}(\mathbb{A}, \mathbb{A})$:

- Every $W(U)$ contains the diagonal in $\mathrm{Map}(\mathbb{A}, \mathbb{A}) \times \mathrm{Map}(\mathbb{A}, \mathbb{A})$, since U contains the diagonal in $\mathbb{A} \times \mathbb{A}$.
- $-W(U) = W(-U) \in \mathfrak{W}$, since $\Psi(x) - \Phi(x) \in U_0$ if and only if $\Phi(x) - \Psi(x) \in -U_0$.
- $W(U) + W(U) \subset W(U + U) \in \mathfrak{W}$, since, if $\Psi(x) - \Phi(x) \in U_0$ and $\Sigma(x) - \Psi(x) \in U_0$, then $\Sigma(x) - \Phi(x) \in U_0 + U_0$.

If $W_U(\Phi) := \{\Psi \in \mathrm{Map}(\mathbb{A}, \mathbb{A}) : \Psi(x) - \Phi(x) \in U_0\}$, then $\Psi \in W_U(\Phi)$ implies that $\Psi(x) \in \Phi(x) + U_0$, for every $x \in \mathbb{A}$.

Definition (3.1). The uniformity on $\mathrm{Map}(\mathbb{A}, \mathbb{A})$ that has as a fundamental system of entourages the set $\mathfrak{W}$ is called the *uniformity of uniform convergence*. The topology that induces is called the *topology of uniform convergence*. If a filter $\mathcal{F}$ in $\mathrm{Map}(\mathbb{A}, \mathbb{A})$ converges to an element Φ with respect to this topology, we say that $\mathcal{F}$ converges uniformly to Φ.

The uniform space that we obtain when providing $\mathrm{Map}(\mathbb{A}, \mathbb{A})$ with the uniformity of uniform convergence will be denoted by $\mathrm{Map}_u(\mathbb{A}, \mathbb{A})$. $\mathrm{Map}_u(\mathbb{A}, \mathbb{A})$ is a topological group and the uniformity just defined is the natural uniform structure it has as a topological group. Since $\mathbb{A}$ is complete, the uniform space $\mathrm{Map}_u(\mathbb{A}, \mathbb{A})$ is also complete.

Denote by $\mathrm{C}(\mathbb{A}, \mathbb{A})$ the set which consists of all continuous functions from $\mathbb{A}$ into itself and by $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$ the set $\mathrm{C}(\mathbb{A}, \mathbb{A})$ provided with the topology of uniform convergence. In what follows we will only consider $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$.

Remark (3.2). 1. $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$ is a closed subset of $\mathrm{Map}_u(\mathbb{A}, \mathbb{A})$.
2. $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$ is complete since it is a closed uniform subspace of the closed uniform space $\mathrm{Map}_u(\mathbb{A}, \mathbb{A})$.

Definition (3.3). a). We say that a subset H of $\mathrm{Map}_u(\mathbb{A}, \mathbb{A})$ is *equicontinuous at the point* $x_0 \in \mathbb{A}$ if, for each entourage U of $\mathbb{A}$, there exists a neighborhood

U_0 of x_0 in $\mathbb{A}$ such that $(\Phi(x), \Phi(x_0)) \in U$ for every $x \in U_0$ and every $\Phi \in H$. We say that H is *equicontinuous* if it is equicontinuous at every point of $\mathbb{A}$.

b). We say that a subset H of $\mathrm{Map}_u(\mathbb{A}, \mathbb{A})$ is *uniformly equicontinuous* if, for every entourage V of $\mathbb{A}$, there exists an entourage U of $\mathbb{A}$ such that $(\Phi(x), \Phi(y)) \in V$ provided that $(x, y) \in U$ and $\Phi \in H$.

Example (3.4). Recall that if $\Phi : \mathbb{A} \longrightarrow \mathbb{A}$ is any uniformly continuous function, then the *translation* of Φ by an element $s \in \mathbb{A}$ is defined by

$$L_s\Phi(x) = \Phi_s(x) := \Phi(s + x).$$

If $\Phi : \mathbb{A} \longrightarrow \mathbb{A}$ is uniformly continuous, then the family of translates $\{\Phi_s : s \in \mathbb{A}\}$ of Φ is uniformly equicontinuous since the relation $y - x \in U_0$ is equivalent to $(s + y) - (s + x) \in U_0$. That is, for every entourage V of $\mathbb{A}$, there exists an entourage U of $\mathbb{A}$ such that $(\Phi_s(x), \Phi_s(y)) \in V$ for every $(x, y) \in U$ and every $s \in \mathbb{A}$.

Almost periodic functions on $\mathbb{A}$. According with the last example, if $\Phi : \mathbb{A} \longrightarrow \mathbb{A}$ is uniformly continuous, then the family of translates of Φ, $\{L_s\Phi : s \in \mathbb{A}\}$ is uniformly equicontinuous and therefore $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$ is a translation–invariant subspace. Then, identifying $\mathbb{A}$ with the group of translations $\{s \longrightarrow L_s\}$ acting as a group of homeomorphisms of $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$, we can consider the $\mathbb{A}$-action by translations in $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$:

$$\mathbb{A} \times \mathrm{C}_u(\mathbb{A}, \mathbb{A}) \longrightarrow \mathrm{C}_u(\mathbb{A}, \mathbb{A}), \qquad (s, \Phi) \longmapsto L_s\Phi = \Phi_s.$$

The representation $s \longrightarrow L_s$ is continuous, since, for each neighborhood V_0 of the identity in $\mathbb{A}$, there exists, by uniform continuity of Φ, a neighborhood U_0 of 0 such that, if $t - s \in U_0$, then $\Phi(t) - \Phi(s) \in V_0$. Then, $(t+x) - (s+x) \in U_0$, and therefore $\Phi_t(x) - \Phi_s(x) \in V_0$, for all $x \in \mathbb{A}$. That is, if $s \sim t$, then $\Phi_s \sim \Phi_t$, for all $s, t \in \mathbb{A}$.

Also, if $\Psi \in W_U(\Phi)$, then $\Psi(y) \in \Phi(y) + U_0$, for every $y \in \mathbb{A}$. That is, $\Psi(y) - \Phi(y) \in U_0$, for each $y \in \mathbb{A}$. Letting $y = s + x$, we see that, if $s \in \mathbb{A}$, then $\Psi_s(x) - \Phi_s(x) \in U_0$, for all $x \in \mathbb{A}$. Then, $\Psi_s \in W_U(\Phi_s)$, for all $s \in \mathbb{A}$, and thus the family $\{s \longrightarrow L_s\}$ is uniformly equicontinuous.

If $\Phi \in \mathrm{C}_u(\mathbb{A}, \mathbb{A})$, the *orbit* of Φ under the $\mathbb{A}$-action is the set

$$\mathcal{O}(\Phi) := \{\Phi_s : s \in \mathbb{A}\} \subset \mathrm{C}_u(\mathbb{A}, \mathbb{A}).$$

Definition (3.5). We say that $\Phi \in \mathrm{C}_u(\mathbb{A}, \mathbb{A})$ is *almost periodic* on $\mathbb{A}$ if $\mathcal{O}(\Phi)$ is relatively compact in $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$; i.e., if $\overline{\mathcal{O}(\Phi)} \subset \mathrm{C}_u(\mathbb{A}, \mathbb{A})$ is compact. Denote by $\mathrm{AP}(\mathbb{A}, \mathbb{A})$ the set of all adèle-valued almost periodic functions on $\mathbb{A}$.

The next proposition gives a characterization of this class of functions:

PROPOSITION (3.6). *$\Phi \in \mathrm{C}_u(\mathbb{A}, \mathbb{A})$ is almost periodic on $\mathbb{A}$ if and only if for every neighborhood $W(\Phi)$ of Φ in $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$, there exist $s_1, ..., s_n \in \mathbb{A}$ such that, for any $s \in \mathbb{A}$, $L_{-s_j}(L_s\Phi) \in W(\Phi)$, for at least one $1 \leq j \leq n$.*

Proof. Suppose first that $\mathcal{O}(\Phi)$ is relatively compact in $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$. Let $W_U(\Phi) \subset \mathrm{C}_u(\mathbb{A}, \mathbb{A})$ be a (uniform) neighborhood of Φ. By compactness of $\overline{\mathcal{O}(\Phi)}$, there exist a finite number of elements $s_1, \ldots, s_n \in \mathbb{A}$, such that

$$\overline{\mathcal{O}(\Phi)} \subset \bigcup_{j=1}^{n} L_{s_j}(W_U(\Phi)).$$

That is, for each $s \in \mathbb{A}$, we have that $L_s\Phi \in L_{s_j}(W_U(\Phi))$, for some j. Hence, for every $s \in \mathbb{A}$, $L_{-s_j}(L_s\Phi) \in W_U(\Phi)$, for some j.

Reciprocally, suppose that for each neighborhood $W_U(\Phi)$ of $\Phi \in \mathrm{C}_u(\mathbb{A}, \mathbb{A})$, there exist $s_1, ..., s_n \in \mathbb{A}$ such that, for every $s \in \mathbb{A}$, $L_{-s_j}(L_s\Phi) \in W(\Phi)$, for at least one j, $1 \leq j \leq n$. That is, for every $s \in \mathbb{A}$ we have that

$$L_s\Phi(x) \in \bigcup_{j=1}^{n}(L_{s_j}\Phi(x) + U_0), \qquad \text{for all } x \in \mathbb{A}.$$

This means that the set $\mathcal{O}(\Phi)(x) = \{L_s\Phi(x) : s \in \mathbb{A}\}$ is a relatively compact set in $\mathbb{A}$ for every $x \in \mathbb{A}$. Since $\mathcal{O}(\Phi)$ is also a uniformly equicontinuous set and $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$ is complete, by Ascoli's theorem we conclude that $\mathcal{O}(\Phi)$ is relatively compact in $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$. □

Remark (3.7). The above proposition is equivalent to: $\Phi \in \mathrm{C}_u(\mathbb{A}, \mathbb{A})$ is almost periodic on $\mathbb{A}$ if and only if for every neighborhood V of the identity $0 \in \mathbb{A}$, there exist $s_1, ..., s_n \in \mathbb{A}$ such that for any $s \in \mathbb{A}$, there exists at leat one j, $1 \leq j \leq n$ such that

$$\Phi_s(x) - \Phi_{s_j}(x) \in V, \qquad \text{for all } x \in \mathbb{A}.$$

According with A. Weil (see [Weil]), it is possible to associate to any topological group G, a compact group Γ and a continuous representation $\rho : G \longrightarrow \Gamma$ such that the image of G under ρ is a dense subgroup of Γ. The next proposition also gives a characterization of almost periodic functions on $\mathbb{A}$:

PROPOSITION (3.8) (Weil's Criterium). $\Phi \in \mathrm{C}_u(\mathbb{A}, \mathbb{A})$ *is almost periodic on* $\mathbb{A}$ *if and only if there exists a continuous extension* $\Phi_0 : \Gamma \longrightarrow \mathbb{A}$ *of* Φ *such that* $\Phi = \Phi_0 \circ \rho$, *where* Γ *is a compact group and* $\rho : \mathbb{A} \longrightarrow \Gamma$ *is a continuous representation whose image is dense.*

Proof. Suppose first that $\Phi \in \mathrm{C}_u(\mathbb{A}, \mathbb{A})$ is an almost periodic function. Let $K := \overline{\mathcal{O}(\Phi)}$ denote the compact closure of $\mathcal{O}(\Phi)$ in $\mathrm{C}_u(\mathbb{A}, \mathbb{A})$. Clearly, K is invariant under the $\mathbb{A}$-action. Denote by L^s the restriction of L_s to K in the space $\mathrm{C}_u(K, K)$. Then, L^s is a homeomorphism of K. Let Γ be the closure in $C_u(\mathbb{A}, \mathbb{A})$, of the image of $\mathbb{A}$, under the map $\mathbb{A} \longrightarrow \mathrm{Homeo}(K)$ given by $s \mapsto L^s$. Then, Γ is a compact group of homeomorphisms which is transitive over K. Define $\Phi_0 : \Gamma \longrightarrow \mathbb{A}$ by $\Phi_0(\Phi_x) := \Phi(x)$. Then, $\Phi_0 : \Gamma \longrightarrow \mathbb{A}$ is continuous and $\Phi = \Phi_0 \circ \rho$.

Reciprocally, suppose that $\Phi_0 : \Gamma \longrightarrow \mathbb{A}$ is an adèle-valued continuous function such that $\Phi = \Phi_0 \circ \rho$. Since $s \longrightarrow L_s\Phi_0$ is continuous and Γ is compact, then $\mathcal{O}(\Phi_0) = \{L_s\Phi_0 : s \in \Gamma\}$ is compact. If $s \in \mathbb{A}$, then

$$L_s(\Phi_0 \circ \rho)(x) = \Phi_0 \circ \rho(s + x) = \Phi_0(\rho(s) + \rho(x)) = (L_{\rho(s)}\Phi_0 \circ \rho)(x).$$

Hence, the orbit of $\Phi_0 \circ \rho$ is mapped into the orbit of Φ_0, which is compact. □

Invariant functions on $\mathbb{A}$. First of all, observe that any continuous function with compact support $\varphi : \mathbb{A} \longrightarrow \mathbb{R}$ determines a continuous invariant ($\mathbb{Q}$-periodic) function

$$\widetilde{\varphi}(x) := \sum_{\gamma\in\mathbb{Q}} \varphi(\gamma + x),$$

and, any real-valued continuous $\mathbb{Q}$-periodic function on $\mathbb{A}$ can be written in the above form with φ continuous and of compact support.

The set $C_{\mathbb{Q}}(\mathbb{A})$ of all these real-valued continuous invariant functions is the totality of real-valued continuous functions defined on $\mathbb{A}/\mathbb{Q}$. Now, if $\Phi \in C_u(\mathbb{A}, \mathbb{A})$, then $\varphi \circ \Phi : \mathbb{A} \longrightarrow \mathbb{R}$ is continuous for any $\varphi \in C_{\mathbb{Q}}(\mathbb{A})$. Then, Φ determines a (vector space)-homomorphism $\Phi^* : C_{\mathbb{Q}}(\mathbb{A}) \longrightarrow C_{\mathbb{Q}}(\Phi^{-1}\mathbb{A})$, given by $\varphi \longmapsto \Phi^*\varphi = \varphi \circ \Phi$.

Definition (3.9). We say that an adèle-valued uniformly continuous function $\Phi : \mathbb{A} \longrightarrow \mathbb{A}$ is *invariant* under $\mathbb{Q}$ or $\mathbb{Q}$-*invariant* or, simply *invariant* if

$$\Phi(\gamma + x) = \Phi(x), \quad \gamma \in \mathbb{Q},\ x \in \mathbb{A}.$$

Denote by $C_{inv}(\mathbb{A}, \mathbb{A})$ the subspace of $C_u(\mathbb{A}, \mathbb{A})$ which consist of all adèle-valued invariant functions. If $\Phi \in C_{inv}(\mathbb{A}, \mathbb{A})$, then $\Phi^*\varphi$ is also invariant, for every $\varphi \in C_{\mathbb{Q}}(\mathbb{A})$.

Example (3.10). Recall that $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ and consider the natural inclusion $\sigma : \mathbb{R} \hookrightarrow \mathbb{A}$. If φ is any real-valued invariant function on $\mathbb{A}$, then $\Phi := \sigma \circ \varphi : \mathbb{A} \longrightarrow \mathbb{A}$ is an adèle-valued invariant function on $\mathbb{A}$.

Example (3.11). Let $T : \mathbb{R} \times \mathbb{A} \longrightarrow \mathbb{A}$ be the continuous translation flow on $\mathbb{A}$ given by $T(t, x) := \sigma(t) + x$. So, for any $x \in \mathbb{A}$ we have a continuous translation map $T_x : \mathbb{R} \longrightarrow \mathbb{A}$ given by $T_x(t) := \sigma(t) + x$.

If $\varphi \in C_{\mathbb{Q}}(\mathbb{A})$ we get an adèle-valued continuous function with compact support $\Phi := T_x \circ \varphi : \mathbb{A} \longrightarrow \mathbb{A}$ given by

$$\Phi(y) := (T_x \circ \varphi)(y) = \sigma(\varphi(y)) + x.$$

Then, Φ is $\mathbb{Q}$-invariant: $\Phi(\gamma + y) = \Phi(y)$ for all $\gamma \in \mathbb{Q}$ and $y \in \mathbb{A}$.

According with the theory developed in the last section, we have the following characterization (compare Chap. VIII, Sec. 33, [HR]):

THEOREM (3.12). *Every invariant function on $\mathbb{A}$ is almost periodic. That is,* $C_{inv}(\mathbb{A}, \mathbb{A}) \subset AP(\mathbb{A}, \mathbb{A})$. *Moreover, if $\Phi : \mathbb{A} \longrightarrow \mathbb{A}$ is almost periodic, then $\mathcal{O}(\Phi)$ is compact if and only if there exist a uniform compact abelian group Γ, a continuous surjective homomorphism $\rho : \mathbb{A} \longrightarrow \Gamma$, and a continuous function $\Phi_0 : \Gamma \longrightarrow \mathbb{A}$ such that $\Phi = \Phi_0 \circ \rho$.*

Proof. The first statement is clear since for every $s \in \mathbb{A}$ we have that $\Phi_s = \Phi_{s+\gamma}$, for all $\gamma \in \mathbb{Q}$. By proposition (3.6) we conclude that Φ is almost periodic.

In order to prove the second part, first suppose that the orbit of Φ is compact. We can define a group structure on $\mathcal{O}(\Phi)$ in the following way: If $\Phi_s, \Phi_t \in \mathcal{O}(\Phi)$ define

$$\Phi_s \oplus \Phi_t := \Phi_{s+t}.$$

Then,

- $\oplus$ is well-defined: If $\Phi_s = \Phi_{s'}$ and $\Phi_t = \Phi_{t'}$, then
$$\Phi_{s+t}(x) = \Phi(s+t+x) = \Phi(s'+t+x) = \Phi(t'+s'+x) = \Phi_{s'+t'}(x).$$
Hence, $\Phi_{s+t} = \Phi_{s'+t'}$.
- $\oplus$ is associative, commutative. The neutral element is Φ and the inverse of Φ_s is Φ_{-s}.
- $\oplus$ is continuous in the uniform topology of $\mathcal{O}(\Phi)$.

Therefore, $\mathcal{O}(\Phi)$ is a compact uniform abelian group. The mapping $\rho : \mathbb{A} \longrightarrow \mathcal{O}(\Phi)$ given by $s \longmapsto L_s\Phi$ is a continuous surjective homomorphism. Define $\Phi_0 : \mathcal{O}(\Phi) \longrightarrow \mathbb{A}$ by
$$\Phi_0(\Phi_x) := \Phi(x).$$
Then, $\Phi_0 : \mathcal{O}(\Phi) \longrightarrow \mathbb{A}$ is continuous and $\Phi = \Phi_0 \circ \rho$.

Reciprocally, suppose that $\Phi_0 : \Gamma \longrightarrow \mathbb{A}$ is an adèle-valued continuous function such that $\Phi = \Phi_0 \circ \rho$. Since $s \longrightarrow L_s\Phi_0$ is continuous and Γ is compact, then $\mathcal{O}(\Phi_0) = \{L_s\Phi_0 : s \in \Gamma\}$ is compact. If $s \in \mathbb{A}$, then
$$L_s(\Phi_0 \circ \rho)(x) = \Phi_0 \circ \rho(s+x) = \Phi_0(\rho(s) + \rho(x)) = (L_{\rho(s)}\Phi_0 \circ \rho)(x).$$
Hence, the orbit of $\Phi_0 \circ \rho$ is send it onto the orbit of Φ_0, which is compact. Therefore $\mathcal{O}(\Phi)$ is compact. $\square$

If $\Phi \in \mathrm{C}_{inv}(\mathbb{A}, \mathbb{A})$, the map $\rho : \mathbb{A} \longrightarrow \mathcal{O}(\Phi)$ given by $s \longmapsto L_s\Phi$ is a continuous surjective homomorphism. Since Φ is invariant, the kernel of this map is $\mathbb{Q}$ and therefore $\mathcal{O}(\Phi) \cong \mathbb{A}/\mathbb{Q}$. Consequently, the map ρ coincide with the canonical projection $\Pi_{\mathbb{Q}} : \mathbb{A} \longrightarrow \mathbb{A}/\mathbb{Q}$. We conclude that there exist a one to one correspondence between
$$\{\mathbb{Q}\text{-invariant functions } \mathbb{A} \longrightarrow \mathbb{A}\} \longleftrightarrow \{\text{Continuous functions } \mathbb{A}/\mathbb{Q} \longrightarrow \mathbb{A}\}.$$

4. Invariant means

Recall that we can also describe $\mathbb{A}$ as $\mathbb{R} \times \mathbb{A}_f$, where $\mathbb{A}_f$ is the restricted direct product of $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$, for all $p < \infty$. Denote by $0 = (0, 0)$ the neutral element in $\mathbb{A}$. The 'leaf' through 0, $L_0 := \mathbb{R} \times \{0\}$ is homeomorphic to $\mathbb{R}$ and, if $\Phi : \mathbb{A} \longrightarrow \mathbb{A}$ is any almost periodic function that sends L_0 into itself, then
$$\Phi_{|L_0} := \Phi_0$$
is an almost periodic function on $\mathbb{R}$. That is, $\Phi_{|L_0}$ can be identified in a canonical way with an element of $\mathrm{AP}(\mathbb{R})$, the set of real-valued almost periodic functions. In fact, the *mean value* $M(\Phi_0)$ of Φ_0 coincides with the usual mean value:
$$M(\Phi_0) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \Phi_0(t)dt.$$

Suppose now that $\Phi : \mathbb{A} \longrightarrow \mathbb{A}$ is any almost periodic function on $\mathbb{A}$ and consider the function space $\mathrm{AP}(\mathbb{A})$, which consists of all real-valued almost periodic functions defined on $\mathbb{A}$. Then, $\Phi^*\varphi = \varphi \circ \Phi$ is also an almost periodic function for any $\varphi \in \mathrm{AP}(\mathbb{A})$. The *invariant mean* of $\Psi = \Phi^*\varphi$ is given by
$$M(\Psi) := \int \Phi^*\varphi \, d\mu = \int (\varphi \circ \Phi) \, d\mu,$$
where μ is the normalized Haar measure on $\mathbb{A}$ described in section 2.

Invariant means. If $\mathbb{A} = \mathbb{R}\times\mathbb{A}_f$, then we have a natural inclusion $\sigma : \mathbb{R} \hookrightarrow \mathbb{A}$. If $\varphi : \mathbb{A} \longrightarrow \mathbb{R}$ is any uniformly continuous function on $\mathbb{A}$, then the function $\Phi_\sigma : \mathbb{A} \longrightarrow \mathbb{A}$ given by

$$\Phi_\sigma(x) := \sigma(\varphi(x)),$$

is uniformly continuous on $\mathbb{A}$. The set which consists of all adèle-valued functions defined in this way is invariant under the $\mathbb{A}$-action: for any $s \in \mathbb{A}$, $L_s\Phi_\sigma$ is uniformly continuous. If $\varphi : \mathbb{A} \longrightarrow \mathbb{R}$ is almost periodic, then $L_s\Phi_\sigma(x) = \sigma \circ \varphi_s(x)$. Then, $\overline{\mathcal{O}(\Phi_\sigma)} \subset \sigma(\overline{\mathcal{O}(\varphi)})$ is compact and therefore Φ_σ is almost periodic. If φ is an invariant function, then Φ_σ is also invariant.

Denote by $\mathrm{AP}_\sigma(\mathbb{A}, \mathbb{A})$ the set which consists of all such almost periodic functions defined on $\mathbb{A}$. That is,

$$\mathrm{AP}_\sigma(\mathbb{A}, \mathbb{A}) := \{\Phi_\sigma \in \mathrm{AP}(\mathbb{A}, \mathbb{A}) : \Phi_\sigma = \sigma \circ \varphi,\ \ \varphi \in \mathrm{AP}(\mathbb{A})\}.$$

THEOREM (4.1). *If $M : \mathrm{AP}(\mathbb{A}) \longrightarrow \mathbb{R}$ is the invariant mean on $\mathbb{A}$, then there exists an adèle-valued invariant mean on $\mathbb{A}$*

$$\mathbb{M} : \mathrm{AP}_\sigma(\mathbb{A}, \mathbb{A}) \longrightarrow \mathbb{A}$$

given by

$$\mathbb{M}(\Phi_\sigma) := \sigma \circ M(\varphi),$$

where $\Phi_\sigma := \sigma \circ \varphi$ and $\varphi \in \mathrm{AP}(\mathbb{A})$.

Proof. For each $\varphi \in \mathrm{AP}(\mathbb{A})$, there exists a unique invariant mean $M(\varphi)$. If $\Phi_\sigma := \sigma\circ\varphi$ is any almost periodic function in $\mathrm{AP}_\sigma(\mathbb{A}, \mathbb{A})$, then we have a unique adèle $\mathbb{M}(\Phi_\sigma)$ given by

$$\mathbb{M}(\Phi_\sigma) := \sigma \circ M(\varphi).$$

This value $\mathbb{M}(\Phi_\sigma)$ satisfies:

- $\mathbb{M}(\Phi_\sigma) = \sigma(M(\varphi)) \in \mathbb{A}$ for every $\Phi_\sigma \in \mathrm{AP}_\sigma(\mathbb{A}, \mathbb{A})$.
- $\mathbb{M}$ is linear: $\mathbb{M}(\Phi_\sigma + \Psi_\sigma) = \sigma(M(\varphi+\psi)) = \sigma(M(\varphi)) + \sigma(M(\psi)) = \mathbb{M}(\Phi_\sigma) + \mathbb{M}(\Psi_\sigma)$, for any $\Phi_\sigma = \sigma \circ \varphi$, $\Psi_\sigma = \sigma \circ \psi \in \mathrm{AP}_\sigma(\mathbb{A}, \mathbb{A})$.
- $\mathbb{M}$ is left translation–invariant: $\mathbb{M}(L_s\Phi_\sigma) = \sigma \circ M(L_s\varphi) = \sigma \circ M(\varphi) = \mathbb{M}(\Phi_\sigma)$, for any $\Phi_\sigma \in \mathrm{AP}_\sigma(\mathbb{A}, \mathbb{A})$ and $s \in \mathbb{A}$.
- $\mathbb{M}(1_\sigma) = 1_\sigma$: If 1_σ is the function identically equal to $\sigma(1) \in \mathbb{A}$, then $\mathbb{M}(1_\sigma) = \sigma \circ M(1) = \sigma(1) = 1_\sigma$.

Therefore $\mathbb{M}$ is an invariant mean with adelic values defined on $\mathbb{A}$. □

Remark (4.2). The existence of an invariant mean open the ways for the further development of the theory. The next important things to do are, among other things, to define the notions of characters and character group as well as to analyze the possibility of the existence of a Bohr compactification. This will be the subject of future work.

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A NOTE ON TRACES OF HÖRMANDER SPACES

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ABSTRACT. Our aim in this note is to extend a classical result on trace operators and extension operators to Hörmander spaces in the sense of Beurling-Björck, by using exponential weights and Beurling ultradistributions. The vector-valued case is also considered.

1. Introduction

In the theory of trace operators, the following restriction theorem plays an important role (see [7] and [8]): "Let k be a Hörmander weight ($k(x+y) \leq (1+c\,|x|)^N k(y),\ x, y \in \mathbb{R}^n$, c and N positive constants) and $1 \leq p < \infty$. Let also Σ be the hyperplane in $\mathbb{R}^n$ of equation $x_n = 0$. If there is an open set $\Omega \subset \mathbb{R}^n$ with $\Omega \cap \Sigma \neq \emptyset$ and the map

$$\left(\mathcal{D}(\Omega), \|\cdot\|_{p,k}\right) \to \mathcal{D}'(\mathbb{R}^{n-1}) : u \mapsto u_{\Sigma} (u_{\Sigma}(x') = u(x', 0), x' \in \mathbb{R}^{n-1})$$

is continuous, then the function

$$k'(x') = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dx_n}{k(x', x_n)^{p'}} \right)^{-1/p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

is again a Hörmander weight. Reciprocally, if the function k' as defined by the above formula is $\neq 0$ for some x', the map $u \mapsto u_{\Sigma}$ can be extended from $\mathcal{S}$ to a homomorphism of $\mathcal{B}_{p,k}$ onto $\mathcal{B}_{p,k'}$". As it is well known (see, e.g. [7], [8], [12], [16] and [21]) this kind of results is important in function spaces theory (embedding theorems, trace theorems, extension operators, structure theory, linear partial differential operators,...). In this paper we consider more general weights than in Hörmander's book (exponential weights; e.g. $k(x) = e^{|x|^{\beta}}, 0 < \beta < 1$, see Section 2) and extend the above result to Hörmander spaces in the sense of Beurling-Björck (see [3], [6], [9], [13], [14], [15], and [22]) by using Beurling ultradistributions (the theory of the usual tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ is inadequate for such general weights and one needs the more general theory of Beurling ultradistributions). In Section 2 we collect some results on (scalar and vector-valued) Beurling ultradistributions and in Section 3, following the Hörmander's approach, we prove our main theorem and a corollary for the scale of Sobolev spaces. Our methods are also valid in the vector-valued case.

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2. Notation and preliminaries

In this section we collect some basic facts about vector-valued ultradistributions. Comprehensive treatments of the theory of (scalar or vector-valued) ultradistributions can be found in [3], [4], [5] and [10].

The linear spaces we use are defined over $\mathbb{C}$. Let E and F be locally convex spaces. Then $\mathcal{L}_b(E, F)$ is the locally convex space of all continuous linear operators equipped with the bounded convergence topology. We write $E \hookrightarrow F$ if E is a linear subspace of F and the canonical injection is continuous. We replace $\hookrightarrow$ by $\overset{d}{\hookrightarrow}$ if E is also dense in F. The topological dual of E is denoted by E' and is given the strong topology so that $E' = \mathcal{L}_b(E, \mathbb{C})$. $\mathcal{C}^m$, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{D}'$ and $\mathcal{S}'$ have the usual meaning (see [18]). In the vector valued case we write $\mathcal{C}^m(E)$, $\mathcal{D}(E)$, $\mathcal{S}(E)$, $\mathcal{D}'(E)$ and $\mathcal{S}'(E)$ (see [19]).

Let $1 \leq p \leq \infty$, $k\colon \mathbb{R}^n \to]0, \infty[$ a Lebesgue measurable function, and E a Banach space. Then $L_p(E)$ is the set of all Bochner measurable functions $f\colon \mathbb{R}^n \to E$ for which $\|f\|_p = \left(\int_{\mathbb{R}^n} \|f(x)\|_E^p\, dx\right)^{1/p}$ is finite (if $p = \infty$ we assume $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \|f\|_E < \infty$). $L_{p,k}(E)$ denotes the set of all measurable Bochner functions $f\colon \mathbb{R}^n \to E$ such that $kf \in L_p(E)$. Putting $\|f\|_{L_{p,k}(E)} = \|kf\|_p$ for $f \in L_{p,k}(E)$, $L_{p,k}(E)$ becomes a Banach space isometrically isomorphic to $L_p(E)$. When E is the field $\mathbb{C}$, we simply write L_p and $L_{p,k}$. If $f \in L_1(E)$ the Fourier transform of f, $\hat{f}$ or $\mathcal{F}f$, is defined by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\, e^{-i\xi x} dx$.

Let $\mathcal{M}$ (or $\mathcal{M}_n$) be the set of all continuous real-valued functions $\omega(x)$ on $\mathbb{R}^n$ such that $\omega(x) = \sigma(|x|)$ where $\sigma(t)$ is an increasing continuous concave function on $[0, \infty[$ with the following properties: (i) $\sigma(0) = 0$, (ii) $\int_0^\infty \frac{\sigma(t)}{1+t^2} dt < \infty$ (Beurling's condition), (iii) there exists a real number a and a positive number b such that $\sigma(t) \geq a + b\log(1+t)$ for $t \geq 0$. The main assumption is (ii), which is essentially the Denjoy-Carleman non-quasi-analyticity condition (see [3], Sec.1.5).

If $\omega \in \mathcal{M}$, then $\mathcal{K}_\omega$ is the set of all positive functions k on $\mathbb{R}^n$ for which there exists a constant $\lambda > 0$ such that $k(x+y) \leq e^{\lambda\omega(x)} k(y)$ for all x and y in $\mathbb{R}^n$. If $k, k_1, k_2 \in \mathcal{K}_\omega$ and s is a real number then $\log k$ is uniformly continuous, $k^s \in \mathcal{K}_\omega$, $k_1 k_2 \in \mathcal{K}_\omega$ and $M_k(x) = \sup_{y\in\mathbb{R}^n} \frac{k(x+y)}{k(y)} \in \mathcal{K}_\omega$ (see [3], Th.2.1.3).

If $\omega \in \mathcal{M}$ and E is a Banach space, we denote by $\mathcal{D}_\omega(E)$ ($\mathcal{D}_\omega$ if E is $\mathbb{C}$) the set of all functions $f \in L_1(E)$ with compact support, such that $\|f\|_\lambda := \int_{\mathbb{R}^n} \|\hat{f}(x)\|_E e^{\lambda\omega(x)} dx < \infty$, for all $\lambda > 0$. For each compact subset K of $\mathbb{R}^n$, $\mathcal{D}_\omega(K, E) = \{f \in \mathcal{D}_\omega(E) : \operatorname{supp} f \subset K\}$ ($\mathcal{D}_\omega(K)$ if E is $\mathbb{C}$), equipped with the topology induced by the family of norms $\left\{\|\cdot\|_\lambda : \lambda > 0\right\}$, is a Fréchet space and $\mathcal{D}_\omega(E)$ with the inductive limit topology, $\mathcal{D}_\omega(E) = \underset{\overrightarrow{K}}{\operatorname{ind}} \mathcal{D}_\omega(K, E)$, becomes a strict (LF)-space. Let $S_\omega(E)$ ($\mathcal{S}_\omega$ if E is $\mathbb{C}$) be the set of all functions $f \in L_1(E)$ such that both f and $\hat{f}$ are infinitely differentiable functions on $\mathbb{R}^n$ with $\vec{p}_{\alpha,\lambda}(f) = \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha f(x)\|_E < \infty$ and $\vec{q}_{\alpha,\lambda}(f) = \sup_{x\in\mathbb{R}^n} e^{\lambda\omega(x)} \|\partial^\alpha \hat{f}(x)\|_E < \infty$ for all multi-indices α and all positive numbers λ. $\mathcal{S}_\omega(E)$ with the topology induced by the family of seminorms $\{\vec{p}_{\alpha,\lambda}, \vec{q}_{\alpha,\lambda}\}$ is a Fréchet space and the Fourier transformation $\mathcal{F}$ is an automorphism of $\mathcal{S}_\omega(E)$. A continuous linear operator from $\mathcal{D}_\omega$ into E is said to be a ultradistribution with values in E. We write $\mathcal{D}'_\omega(E)$ ($\mathcal{D}'_\omega$ if E is $\mathbb{C}$) for the space of all E-valued ultradistributions endowed with the bounded convergence topology. If Ω is any open set in $\mathbb{R}^n$, $\mathcal{D}_\omega(\Omega, E)$ ($\mathcal{D}_\omega(\Omega)$ if E is $\mathbb{C}$) is the subspace of $\mathcal{D}_\omega(E)$ consisting of all functions f with $\operatorname{supp} f \subset \Omega$. $\mathcal{D}_\omega(\Omega, E)$ is endowed with the corresponding inductive limit topology. $\mathcal{D}'_\omega(\Omega, E) = \mathcal{L}_b(\mathcal{D}_\omega(\Omega), E)$ ($\mathcal{D}'_\omega(\Omega)$ if E is $\mathbb{C}$) is the space of all ultadistributions on Ω with values in E. A continuous linear operator from $\mathcal{S}_\omega$ into E is said to be an E-valued tempered ultradistribution. $\mathcal{S}'_\omega(E)$ ($\mathcal{S}'_\omega$ if E is $\mathbb{C}$) is the space of all E-valued tempered ultradistributions equipped with the bounded convergence topology.

Finally, if $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \le p \le \infty$ and E is a Banach space, we denote by $\mathcal{B}_{p,k}(E)$ (see [3], [6], [9], [13], [14], [16] and [22]) the set of all E-valued tempered ultradistributions T for which there exists a function $f \in L_{p,k}(E)$ such that $\langle u, \widehat{T}\rangle = \int_{\mathbb{R}^n} u(x) f(x)\, dx$, $u \in \mathcal{S}_\omega$. $\mathcal{B}_{p,k}(E)$ with the norm

$$\|T\|_{p,k} = \begin{cases} \left((2\pi)^{-n} \int_{\mathbb{R}^n} \|k(x)\widehat{T}(x)\|_E^p dx\right)^{1/p} & \text{if } p < \infty \\ \\ \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \|k(x)\widehat{T}(x)\|_E & \text{if } p = \infty \end{cases}$$

becomes a Banach space isometrically isomorphic to $L_{p,k}(E)$ and, therefore, to $L_p(E)$. (In the previous formulae we have written $\widehat{T}(x)$ instead of $f(x)$, we shall frequently commit this abuse of notation.) Spaces $\mathcal{B}_{p,k}(E)$ are called Hörmander-Beurling spaces with values in E.

In the proof of our theorem we will need the following proposition.

PROPOSITION (2.1). *Let $\omega \in \mathcal{M}$, $k \in \mathcal{K}_\omega$, $1 \le p \le \infty$ and E a Banach space. Then $\mathcal{S}_\omega(E) \hookrightarrow \mathcal{B}_{p,k}(E) \hookrightarrow \mathcal{S}'_\omega(E)$. Moreover, $\mathcal{S}_\omega(E)$ is dense in $\mathcal{B}_{p,k}(E)$ if $p < \infty$.*

Proof. See [22], Prop. 3.3. □

3. Main results

In this section we extend the restriction theorem of Hörmander to the case of weights in the class $\mathcal{K}_\omega$ and so we work in the context of (scalar and vector-valued) Hörmander-Beurling spaces.

THEOREM (3.1). *Let $\omega \in \mathcal{M}_n$ and $\omega' \in \mathcal{M}_{n-1}$ be such that $\omega(x', 0) \le c\,\omega'(x')$ ($x' \in \mathbb{R}^{n-1}$ and c some constant > 0). Let also $p \in [1, \infty)$, $k \in \mathcal{K}_\omega$ and k' the function*

$$(*) \qquad k'(\xi') = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\xi_n}{k(\xi', \xi_n)^{p'}}\right)^{-1/p'}, \ \frac{1}{p} + \frac{1}{p'} = 1,$$

($k'(\xi') = \inf_{\xi_n} k(\xi', \xi_n)$ if $p = 1$). Let Σ be the hyperplane $x_n = 0$ in $\mathbb{R}^n$ and E a Banach space. Then we have

1. *$k' \in \mathcal{K}_{\omega'}$ if there is an open set Ω in $\mathbb{R}^n$ such that $\Omega \cap \Sigma \ne \emptyset$ and the map*

$$\left(\mathcal{D}_\omega(\Omega), \|\cdot\|_{p,k}\right) \to \mathcal{D}'_{\omega'}(\mathbb{R}^{n-1}) : u \mapsto u_{\Sigma} (u_{\Sigma}(x') = u(x', 0),\ x' \in \mathbb{R}^{n-1})$$

is continuous.

2. *If the function k' defined by $(*)$ is $\ne 0$ for some ξ', the map $u \mapsto u_\Sigma$ extends from $\mathcal{S}_\omega(E)$ to a topological homomorphism $\bar\sigma$ from $\mathcal{B}_{p,k}(E)$ onto $\mathcal{B}_{p,k'}(E)$ ($\bar\sigma$ is the trace operator). Moreover, it admits a continuous linear right inverse R (and*

hence R is an extension operator), therefore $\ker\bar\sigma$ *is a complemented subspace of $\mathcal{B}_{p,k}(E)$. Finally, if $u\in\mathcal{B}_{p,k}(E)$ has compact support then $\bar\sigma u\in\mathcal{B}_{p,k'}(E)$ also has compact support and* $\operatorname{supp}\bar\sigma\, u\subset \operatorname{supp}\, u\cap\mathbb{R}^{n-1}$.

Proof. (1) Case $p>1$.Since $k\in\mathcal{K}_\omega$, by hypothesis we have

$$k(\xi_0',\xi_n)\le e^{\lambda c\omega'(\xi_0'-\xi')}k(\xi',\xi_n)$$

for all $\xi',\xi_0'\in\mathbb{R}^{n-1}$ and every $\xi_n\in\mathbb{R}$ (λ, c positive constants) and then

$$\int_{-\infty}^{\infty}\frac{d\xi_n}{k(\xi_0',\xi_n)^{p'}}\ge e^{-\lambda cp'\omega'(\xi_0'-\xi')}\int_{-\infty}^{\infty}\frac{d\xi_n}{k(\xi',\xi_n)^{p'}}\,.$$

Therefore, k' is well defined over all $\mathbb{R}^{n-1}$ if $1/k'<\infty$ at some point. In this case we see that

$$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\xi_n}{k(\xi_0',\xi_n)^{p'}}\right)^{-1/p'}\le e^{\lambda\, c\,\omega'(\xi_0'-\xi')}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\xi_n}{k(\xi',\xi_n)^{p'}}\right)^{-1/p'},$$

that is to say

$$k'(\xi_0')\le e^{\lambda\, c\,\omega'(\xi_0'-\xi')}k'(\xi')$$

and so $k'\in\mathcal{K}_{\omega'}$. Consequently to prove (1) it will suffice to show that the hypothesis on Ω imply that integral $\int_{-\infty}^{\infty}\frac{d\xi_n}{k(\xi',\xi_n)^{p'}}$ converges for some ξ'. So we follow Hörmander's argument in the classical case: Let φ be any element of $\mathcal{D}_{\omega'}(\Omega\cap\Sigma)$ ($\varphi\not\equiv 0$), $\chi\in\mathcal{D}_\omega(\Omega)$ such that $\chi\equiv 1$ in a neighborhood of $\operatorname{supp}\varphi$ contained in Ω (here we use [3], Th.1.3.7) and consider the linear form

$$\begin{aligned}(\mathcal{S}_\omega,\|\cdot\|_{p,k})\to\mathbb{C}:u\quad &\mapsto \int_{\mathbb{R}^{n-1}}u_{\Sigma}(x')\overline{\varphi(x')}\,dx'\\ &= \int_{\mathbb{R}^{n-1}}(\chi\, u)_{\Sigma}(x')\overline{\varphi(x')}\,dx'.\end{aligned}$$

By hypothesis and Theorem 2.2.7 in [3] we have

$$\begin{aligned}\left|\int_{\mathbb{R}^{n-1}}u_{\Sigma}(x')\overline{\varphi(x')}\,dx'\right| &= |\langle\bar\varphi,(\chi u)_\Sigma\rangle|\le C\|\chi u\|_{p,k}\\ &\le C\|\chi\|_{1,M_k}\|u\|_{p,k},\quad u\in\mathcal{S}_\omega,\end{aligned}$$

or equivalently (by Parseval's identity and the definition of norm $\|\cdot\|_{p,k}$)

$$\left|(2\pi)^{-n+1}\int_{\mathbb{R}^{n-1}}\widehat{u}_{\Sigma}(\xi')\overline{\widehat{\varphi}}(\xi')\,d\xi'\right|\le C\|\chi\|_{1,M_k}(2\pi)^{-n/p}\left\|k\widehat{u}\right\|_p,\quad u\in\mathcal{S}_\omega\,.$$

Therefore, by virtue of Fourier's inversion formula and Fubini's theorem, $\widehat{u}_{\Sigma}(\xi')$ can be written as

$$\widehat{u}_{\Sigma}(\xi')=\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{u}(\xi',\xi_n)d\xi_n$$

and then it follows that

$$(2\pi)^{-n+1}\int_{\mathbb{R}^{n-1}}\hat{u}_{\Sigma}(\xi')\bar{\hat\varphi}(\xi')d\xi'=\quad(2\pi)^{-n}\textstyle\int_{\mathbb{R}^n}\hat{u}(\xi)\bar{\hat\varphi}(\xi')d\xi$$

which yields the estimate

$$\left|\int_{\mathbb{R}^n}\hat{u}(\xi)\bar{\hat\varphi}(\xi')d\xi\right|\le C\|\chi\|_{1,M_k}(2\pi)^{n/p'}\|\hat{u}\|_{L_{p,k}},\quad u\in\mathcal{S}_\omega\,.$$

Since the Fourier transform is an automorphism of $\mathcal{S}_\omega$, $\mathcal{S}_\omega$ is dense in $L_{p,k}$ (note that $\mathcal{D}_\omega \overset{d}{\hookrightarrow} \mathcal{D} \overset{d}{\hookrightarrow} L_{p,k}$ [3] and $\mathcal{D}_\omega \subset \mathcal{S}_\omega \subset L_{p,k}$) we deduce $\hat{\varphi}(\xi') \in L_{p',1/k}$, that is,

$$\int_{\mathbb{R}^n} \left| \frac{\hat{\varphi}(\xi')}{k(\xi)} \right|^{p'} d\xi = \int_{\mathbb{R}^{n-1}} |\hat{\varphi}(\xi')|^{p'} \left(\int_{-\infty}^{\infty} \frac{d\xi_n}{k(\xi', \xi_n)^{p'}} \right) d\xi' < \infty \,,$$

showing that there are points ξ' such that

$$\int_{-\infty}^{\infty} \frac{d\xi_n}{k(\xi', \xi_n)^{p'}} < \infty \,.$$

In case $p = 1$ we obtain

$$\sup_{\xi} |\hat{\varphi}(\xi')| \frac{1}{k(\xi)} = \sup_{\xi'} \{ |\hat{\varphi}(\xi')| \sup_{\xi_n} \frac{1}{k(\xi', \xi_n)} \} < \infty$$

and there are points ξ' for which

$$\sup_{\xi_n} \frac{1}{k(\xi', \xi_n)} < \infty.$$

The proof of (1) is complete.

(2) From the relationship between ω and ω' we cannot conclude that $u_{\Sigma} \in \mathcal{S}_{\omega'}(E)$ when $u \in \mathcal{S}_\omega(E)$, in principle one can only assert that

$$\begin{array}{ccc} \mathcal{S}_\omega(E) & \to & \mathcal{S}\left(\mathbb{R}^{n-1}, E\right) (\hookrightarrow \mathcal{S}'\left(\mathbb{R}^{n-1}, E\right) \hookrightarrow \mathcal{S}'_{\omega'}(E)). \\ u & \mapsto & u_{\Sigma} \end{array}$$

Let us see that this restriction operator σ is well defined and bounded from

$$\begin{array}{ccc} \left(\mathcal{S}_\omega(E), \|\cdot\|_{p,k} \right) & \xrightarrow{\sigma} & \mathcal{B}_{p,k'}(E) \\ u & \longmapsto & \sigma u = u_{\Sigma}. \end{array}$$

Since $e' \circ u \in \mathcal{S}_\omega$ for all $e' \in E'$ we have (taking into account the scalar case)

$$\begin{aligned} \langle \hat{u}_{\Sigma}(\xi'), e' \rangle = \widehat{e' \circ u_{\Sigma}}(\xi') &= (\widehat{e' \circ u})_{\Sigma}(\xi') \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{e' \circ u}(\xi', \xi_n) d\xi_n \\ &= \left\langle \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\xi', \xi_n) d\xi_n, e' \right\rangle \end{aligned}$$

and using Hahn-Banach theorem we get,

$$\hat{u}_{\Sigma}(\xi') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\xi', \xi_n) d\xi_n$$

for all $u \in \mathcal{S}_\omega(E)$. Taking norms and using Hölder's inequality we see that

$$\begin{aligned} \left\| \hat{u}_{\Sigma}(\xi') \right\|_E &\leq \frac{1}{2\pi} \left(\int_{-\infty}^{\infty} \|\hat{u}(\xi', \xi_n)\|_E^p \, k^p(\xi', \xi_n) d\xi_n \right)^{1/p} \\ &\qquad \left(\int_{-\infty}^{\infty} \frac{d\xi_n}{k(\xi', \xi_n)^{p'}} \right)^{1/p'} \\ &= (2\pi)^{\frac{1}{p'}-1} \frac{1}{k'(\xi')} \left(\int_{-\infty}^{\infty} \|\hat{u}(\xi', \xi_n)\|_E^p \, k^p(\xi', \xi_n) d\xi_n \right)^{1/p}; \end{aligned}$$

multiplying both members by $k'(\xi')$, taking $p-$th power, integrating over $\mathbb{R}^{n-1}$ and using Fubini's theorem we obtain

$$\left(\int_{\mathbb{R}^{n-1}} \left\|\hat{u}_{\Sigma}(\xi')\right\|_E^p k'(\xi')^p d\xi'\right)^{1/p} \leq (2\pi)^{-\frac{1}{p}} \left(\int_{\mathbb{R}^n} \|\hat{u}(\xi)\|_E^p k^p(\xi) d\xi\right)^{1/p},$$

that is, $\|\sigma u\|_{p,k'} \leq \|u\|_{p,k}$, $u \in \mathcal{S}_\omega(E)$. Since, by virtue of Proposition (2.1), $\mathcal{S}_\omega(E)$ is dense in $\mathcal{B}_{p,k}(E)$ we can extend σ to all $\mathcal{B}_{p,k}(E)$ obtaining a bounded operator $\bar{\sigma} : \mathcal{B}_{p,k}(E) \to \mathcal{B}_{p,k'}(E)$. Observe now that, for each $u \in \mathcal{B}_{p,k}(E)$, $\bar{\sigma}u$ coincides with $\mathcal{F}^{-1}\vartheta$ where ϑ is the function

$$\vartheta(\xi') := \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\xi', \xi_n) d\xi_n.$$

In fact, if $u \in \mathcal{B}_{p,k}(E)$ then $\hat{u} \in L_{p,k}(E)$ and so $\hat{u}(\xi', \cdot)$ is Bochner measurable ξ'-a.e and $k^p(\xi', \cdot) \|\hat{u}(\xi', \cdot)\|_E^p \in L_1(\mathbb{R})$ ξ'-a.e. Reasoning as before with $\hat{u}_\Sigma$ (when $u \in \mathcal{S}_\omega(E)$), we see that ϑ is well defined ξ'-a.e, belongs to $L_{p,k'}(E)$ and satisfies $\left\|\mathcal{F}^{-1}\vartheta\right\|_{p,k'} \leq \|u\|_{p,k}$. Now, since $\bar{\sigma}u = \mathcal{F}^{-1}\vartheta$ when $u \in \mathcal{S}_\omega(E)$ and $\mathcal{S}_\omega(E)$ is dense in $\mathcal{B}_{p,k}(E)$, then follows from this and from what was said before that $\bar{\sigma}u$ also coincides with $\mathcal{F}^{-1}\vartheta$ if $u \in \mathcal{B}_{p,k}(E)$. To conclude the proof we will show that $\bar{\sigma}$ is onto. Let $z \in \mathcal{B}_{p,k'}(E)$. Suppose first that $p = 1$. Given any $\epsilon > 0$ we can find a function $g \in \mathcal{C}^\infty(\mathbb{R}^n)$ such $0 < g(\xi) \leq 1$ on $\{\xi : k(\xi) < k'(\xi')(1+\epsilon)\}$ and $g(\xi) = 0$ on $\{\xi : k(\xi) \geq k'(\xi')(1+\epsilon)\}$ (see [11], Th.1.1.4). Put

$$U(\xi) := \frac{\hat{z}(\xi')g(\xi)h(\xi_n)}{c(\xi')}$$

where $h \in L_1(\mathbb{R})$ is strictly positive and

$$c(\xi') = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\xi', \xi_n) h(\xi_n) d\xi_n\,.$$

U is of course Bochner measurable and

$$\begin{aligned}
\int_{\mathbb{R}^n} k(\xi)\|U(\xi)\|_E d\xi &= \int_{\{\xi : k(\xi) < k'(\xi')(1+\epsilon)\}} k(\xi)\|U(\xi)\|_E\, d\xi \\
&\leq (1+\epsilon) \int_{\mathbb{R}^{n-1}} k'(\xi') \left(\int_{-\infty}^{\infty} \|U(\xi', \xi_n)\|_E\, d\xi_n\right) d\xi' \\
&= (2\pi)(1+\epsilon) \int_{\mathbb{R}^{n-1}} k'(\xi')\|\hat{z}(\xi')\|_E d\xi' < \infty.
\end{aligned}$$

So $U \in L_{1,k}(E)$ and $u = \mathcal{F}^{-1}U \in \mathcal{B}_{1,k}(E)$. Finally, it is easy to see that $\bar{\sigma}u = z$ (remember that $\bar{\sigma}u = \mathcal{F}^{-1}\vartheta$ where $\vartheta(\xi') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\xi', \xi_n) d\xi_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\xi', \xi_n) d\xi_n = \hat{z}(\xi')$). In case $p > 1$ we take

$$U(\xi) := \frac{\hat{z}(\xi') k'(\xi')^{p'}}{k(\xi)^{p'}}$$

then also U is Bochner measurable and

$$\int_{\mathbb{R}^n} \frac{\|\hat{z}(\xi')\|_E^p \, k(\xi)^p k'(\xi')^{pp'}}{k(\xi)^{pp'}} d\xi = \int_{\mathbb{R}^{n-1}} \|\hat{z}(\xi')\|_E^p k'(\xi')^{pp'} \left(\int_{-\infty}^{\infty} \frac{d\xi_n}{k(\xi', \xi_n)^{p(p'-1)}} \right) d\xi'$$
$$= 2\pi \int_{\mathbb{R}^{n-1}} \|\hat{z}(\xi')\|_E^p k'(\xi')^p d\xi' < \infty.$$

Therefore $U \in L_{p,k}(E)$ and $u = \mathcal{F}^{-1}U \in \mathcal{B}_{p,k}(E)$. In the same way as before, we check that $\bar{\sigma}u = z$. As a consequence, $\bar{\sigma}$ is a topological homomorphism from $\mathcal{B}_{p,k}(E)$ onto $\mathcal{B}_{p,k'}(E)$. Examining the above proof, we see that the operator

$$R : \mathcal{B}_{p,k'}(E) \to \mathcal{B}_{p,k}(E) : z \mapsto u = \mathcal{F}^{-1}U$$

is linear and satisfies

$$\begin{aligned} \|Rz\|_{1,k} &\leq (1+\epsilon)\|z\|_{1,k'} \\ \|Rz\|_{p,k} &\leq \|z\|_{p,k'}, \quad p > 1. \end{aligned}$$

Consequently R is a continuous linear right inverse of $\bar{\sigma}$ and as it is well known (see, [20], p.543) this implies that $\ker \bar{\sigma}$ is a topologically complemented subspace of $\mathcal{B}_{p,k}(E)$.

Finally, let $u \in B_{p,k}(E)$ be with $\operatorname{supp} u = K$ compact in $\mathbb{R}^n$ and let $(\varphi_j)_1^\infty$ be a regularizing sequence in D_ω ($\varphi_j \in D_\omega$, $\int \varphi_j = 1$, $\operatorname{supp}\varphi_j \subset B_{\varepsilon_j}$ and $\varepsilon_j \downarrow 0$). By [15], Prop. 3.4; $u_j (= \varphi_j * u) \in D_\omega(K_{\varepsilon_j})$ and $u_j \to u$ in $B_{p,k}(E)$. Since the trace operator $\bar{\sigma}$ is continuous and $\bar{\sigma}u_j(x') = u_j(x', 0)$ it follows that $\bar{\sigma}u_j \to \bar{\sigma}u$ in $B_{p,k'}(E)$ and $\operatorname{supp}\bar{\sigma}u_j \subset K_{\varepsilon_j} \cap \mathbb{R}^{n-1}$. Thus $\operatorname{supp} \bar{\sigma}u \subset K \cap \mathbb{R}^{n-1}$ (recall that $B_{p,k'}(E) \hookrightarrow S'_{\omega'}(E)$ by Proposition (2.1)). □

Remark (3.2). If $p = \infty$ part (1) of the theorem is still valid. In fact, using the estimate (notation as in the proof of the theorem)

$$\left| \int_{\mathbb{R}^n} \hat{u}(\xi)\bar{\hat{\varphi}}(\xi') \right| d\xi \leq c\|\hat{u}\|_{L_{\infty,k}}, \qquad u \in \mathcal{S}_\omega,$$

(c positive constant) it follows that the map $\xi \mapsto \hat{\varphi}(\xi')$ is in $L_{1,1/k}$ since by virtue of the above estimate, the density of $\mathcal{D}_\omega$ in $\mathcal{S}_\omega$ and the fact that for each $h \in \mathcal{C}_{00}$ (continuous functions of compact support) there is a sequence $(\varphi_j)_1^\infty \subset \mathcal{D}_\omega$ such that $\operatorname{supp} k\varphi_j$ all lie in a fixed compact and $k\varphi_j \to h$ uniformly on $\mathbb{R}^n$ (see [3]), we have

$$\sup_{h \in \mathcal{C}_{00}, \|h\|_\infty \leq 1} \left| \int_{\mathbb{R}^n} \frac{\bar{\hat{\varphi}}(\xi')}{k(\xi)} h(\xi) d\xi \right| \leq c$$

which in turn implies that for each compact K of $\mathbb{R}^n$

$$\sup_{h \in \mathcal{C}(K), \|h\|_\infty \leq 1} \left| \int_{\mathbb{R}^n} \frac{\bar{\hat{\varphi}}(\xi')}{k(\xi)} h(\xi) d\xi \right| \leq c,$$

that is, $\|\frac{\bar{\hat{\varphi}}}{k}\|_{L_1(K)} \leq c$ ($L_1(K)$ is isometrically imbedded in $\mathcal{C}(K)'$). Since the constant c is independent of K, we obtain $\hat{\varphi}(\xi')$ belongs to $L_{1,1/k}$. Therefore, there exist points ξ' such that

$$\int_{-\infty}^{\infty} \frac{d\xi_n}{k(\xi', \xi_n)} < \infty\,.$$

Let $m \in \mathbb{R}$, $1 < p < \infty$, E be a Banach space and let I_m be the isomorphism of $\mathcal{S}'(E)$ onto itself given by

$$I_m(f) = \mathcal{F}^{-1}\Big(\big(1 + |x|^2\big)^{m/2}\hat{f}\Big).$$

Then $W_p^m(\mathbb{R}^n, E)$, $m > 0$, are the *classical Sobolev spaces* (*Slobodeckij spaces* if m not integer) and

$$H_p^m(\mathbb{R}^n, E) = I_{-m}L_p(\mathbb{R}^n, E), \quad m \in \mathbb{R},$$

are the *Sobolev spaces*. For definitions, notation and basic results about vector-valued *classical Sobolev spaces* and *Sobolev spaces* see [1], [2], [17] and [23]. Finally, let us recall that a Banach space $E \in$UMD (cf. [1]) provided that for $1 < p < \infty$ martingale difference sequences $d = (d_1, d_2, \dots)$ in $L_p([0, 1], E)$ are unconditional, i.e.

$$\|\epsilon_1 d_1 + \epsilon_2 d_2 + \cdots \|_p \le C_p \|d_1 + d_2 + \cdots \|_p$$

whenever $\epsilon_1, \epsilon_2, \dots$ are numbers in $\{-1, 1\}$.

COROLLARY (3.3). *Let $1 < p < \infty$, $m > \frac{1}{p'}$ and let E be a Banach space.*

1. *The short exact sequence*

$$0 \to \ker\bar{\sigma} \xrightarrow{j} H_p^m(\mathbb{R}^n, E) \xrightarrow{\bar{\sigma}} H_p^{m-\frac{1}{p'}}(\mathbb{R}^{n-1}, E) \to 0$$

($\bar{\sigma}$ is the trace operator and j the canonical injection) splits, i.e., $\bar{\sigma}$ has a continuous linear right inverse R (= an extension operator) and j has a continuous linear left inverse.

2. *If m is a positive integer and $E \in$UMD then the short exact sequence*

$$0 \to \ker\bar{\sigma} \xrightarrow{j} W_p^m(\mathbb{R}^n, E) \xrightarrow{\bar{\sigma}} H_p^{m-\frac{1}{p'}}(\mathbb{R}^{n-1}, E) \to 0$$

splits.

3. *If $p = 2$ and E is hilbertian then the short exact sequence*

$$0 \to \ker\bar{\sigma} \xrightarrow{j} W_2^m(\mathbb{R}^n, E) \xrightarrow{\bar{\sigma}} W_2^{m-\frac{1}{2}}(\mathbb{R}^{n-1}, E) \to 0$$

splits.

Proof. (1) If we take the weight $k(\xi) = \big(1 + |\xi|^2\big)^{m/2}$ then it is easily seen that

$$k'(\xi) = C(m, p')\big(1 + |\xi'|^2\big)^{(m-\frac{1}{p'})/2}$$

where $C(m, p')$ is a positive constant. Thus $\mathcal{B}_{p,k}(E)$ coincides with $H_p^m(\mathbb{R}^n, E)$ and $\mathcal{B}_{p,k'}(E)$ with $H_p^{m-\frac{1}{p'}}(\mathbb{R}^{n-1}, E)$ and so (1) is a direct consequence of Theorem 3.1.

(2) If m is a positive integer and $E \in$UMD then

$$W_p^m(\mathbb{R}^n, E) = H_p^m(\mathbb{R}^n, E)$$

(equivalent norms) (see [1], [2] and [17]) and so we obtain (2).

(3) It is well-known that

$$W_2^m(\mathbb{R}^n, E) = H_2^m(\mathbb{R}^n, E)$$

and

$$W_2^{m-\frac{1}{2}}(\mathbb{R}^{n-1}, E) = H_2^{m-\frac{1}{2}}(\mathbb{R}^{n-1}, E)$$

(equivalent norms) when E is hilbertian. □

The above corollary extends well-known scalar results (see, f.i., [12]).

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IDEALS OF INTEGRAL AND r-FACTORABLE POLYNOMIALS

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Abstract. We study the ideals of vector-valued m-homogeneous polynomials between Banach spaces factoring through an $L_r(\mu)$ space as a composition of an operator T and a polynomial Q, either in the form $Q \circ T$ (right factorization) or in the form $T \circ Q$ (left factorization).

We prove that these ideals are maximal and, as a consequence, we show that, if $r < m$, in most cases, the ideal of m-homogeneous right r-factorable polynomials is not normed. This answers a question raised by K. Floret.

In order to obtain these results, we first study the ideals of (Pietsch) integral polynomials. We show that these polynomials factor through the canonical inclusion of a $C(K)$ space into $L_1(K, \mu)$ for some measure μ. We also prove that the ideal of integral polynomials is the smallest maximal normed ideal of polynomials.

Some general results on ideals of polynomials are also obtained. Namely, we prove that, for $0 < \lambda \leq 1$, a λ-normed ideal of vector-valued m-homogeneous polynomials is maximal if and only if it is ultrastable and regular.

1. Introduction

Given a real number $r \geq 1$ and an integer $m \geq 2$, we introduce the ideal $\left(\mathcal{P}_r^{m,\text{left}}, \gamma_r^{\text{left}}\right)$ of all m-homogeneous polynomials that are factorable through an $L_r(\mu)$ space as a composition of a (linear bounded) operator T and a (continuous) polynomial Q in the form $T \circ Q$ (left factorization), and the ideal $\left(\mathcal{P}_r^{m,\text{right}}, \gamma_r^{\text{right}}\right)$ of all m-homogeneous polynomials that are factorable through an $L_r(\mu)$ space in the form $Q \circ T$ (right factorization).

We prove that these ideals are maximal and, as a consequence, we show that, for $1 \leq r < m$, in most cases, the ideal $\left(\mathcal{P}_r^{m,\text{right}}, \gamma_r^{\text{right}}\right)$ is not normed. This proves a conjecture of K. Floret (see comment after [F2], Proposition 3.1).

To achieve this, we establish some properties of the ideals of (Pietsch) integral and nuclear polynomials.

More precisely, in Section 2, we prove that the (Pietsch) integral polynomials factor through the canonical inclusion of a $C(K)$ space into $L_1(K, \mu)$ for some measure μ. This result, of independent interest, is used in Section 4 to prove that the ideal of Pietsch integral polynomials is ultrastable.

In Section 3, we show that, for $0 < \lambda \leq 1$, a λ-normed ideal of m-homogeneous polynomials is maximal if and only if it is ultrastable and regular. It

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is also shown that maximality implies completeness. These results extend to the vector-valued case what had been obtained for scalar-valued polynomials in [F2], and the extension works as expected, so that the Section has a rather expository nature.

In Section 4, we show that the ideal of nuclear polynomials is the smallest Banach ideal of polynomials, and that the ideal of integral polynomials is the smallest maximal normed ideal of polynomials.

Section 5 deals specifically with the ideals of left and right r-factorable polynomials, giving the results mentioned above.

Throughout, E, F, G, X, Y denote Banach spaces, E^* is the dual of E, and B_E stands for its closed unit ball. The closed unit ball B_{E^*} will always be endowed with the weak-star topology. By $\mathbb{N}$ we represent the set of all natural numbers, and by $\mathbb{K}$ the scalar field (real or complex). By $E \equiv F$, we mean that the Banach spaces E and F are isometrically isomorphic. We use the symbol $\mathcal{L}(E, F)$ for the space of all (linear bounded) operators from E into F endowed with the operator norm. We say that an operator $h \in \mathcal{L}(E, F)$ is an *embedding* if it is an isomorphism onto its image $h(E)$. Given a space F, we shall denote by I_F the identity operator on F, and by k_F the natural isometric embedding of F into its bidual F^{**}.

Given $m \in \mathbb{N}$, we denote by $\mathcal{L}(^mE_1, \ldots, E_m; F)$ the space of m-linear (continuous) mappings from $E_1 \times \cdots \times E_m$ into F, while $\mathcal{P}(^mE, F)$ denotes the space of all m-homogeneous (continuous) polynomials from E into F endowed with the supremum norm. When F is omitted, it is understood to be the scalar field. Recall that with each $P \in \mathcal{P}(^mE, F)$ we can associate a unique symmetric m-linear mapping $\widehat{P} \colon E \times \overset{(m)}{\cdots} \times E \to F$ so that

$$P(x) = \widehat{P}\left(x, \overset{(m)}{\ldots}, x\right) \quad (x \in E).$$

With each polynomial $P \in \mathcal{P}(^mE, F)$ we associate an operator $T_P \colon E \to \mathcal{P}(^{m-1}E, F)$ given by

$$T_P(x)(y) := \widehat{P}\left(x, y, \overset{(m-1)}{\ldots}, y\right) \quad (x, y \in E).$$

For the general theory of multilinear mappings and polynomials on Banach spaces, we refer the reader to [Di] and [Mu]. The notion of ideal of multilinear mappings goes back to Pietsch [Pi2]; further developments can be found, for instance, in [F2] and [FH].

We use the notation $\otimes^m E := E \otimes \overset{(m)}{\cdots} \otimes E$ for the m-fold tensor product of E, and $E \otimes_\pi F$ for the completed projective tensor product of E and F (see [DU] for the theory of tensor products). By $\otimes_s^m E := E \otimes_s \overset{(m)}{\cdots} \otimes_s E$ we denote the m-fold symmetric tensor product of E, that is, the set of all elements $u \in \otimes^m E$ of the form

$$u = \sum_{j=1}^{n} \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j \qquad (n \in \mathbb{N}, \lambda_j \in \mathbb{K}, x_j \in E, 1 \leq j \leq n).$$

We write $\otimes_{\epsilon_s, s}^m E$ for the completion of the space $\otimes_s^m E$ endowed with the *injective symmetric tensor norm* ϵ_s defined by

$$\left\| \sum_{j=1}^{n} \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j \right\|_{\epsilon_s} := \sup_{x^* \in B_{E^*}} \left| \sum_{j=1}^{n} \lambda_j (x^*(x_j))^m \right| .$$

On the space $\otimes_s^m E$, we can also define the *projective symmetric tensor norm* π_s by

$$\pi_s(u) := \inf \sum_{j=1}^{n} |\lambda_j| \|x_j\|^m ,$$

where the infimum is taken over all representations of $u \in \otimes_s^m E$ in the form

$$u = \sum_{j=1}^{n} \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j ,$$

for $x_j \in E$, $\lambda_j \in \mathbb{K}$ $(1 \leq j \leq n)$. We write $\otimes_{\pi_s,s}^m E$ for the completion of $\otimes_s^m E$ endowed with the π_s norm.

For symmetric tensor products, the reader is referred to [F1].

If $A \in \mathcal{L}(^m E_1, \ldots, E_m; F)$, the *linearization* of A is the operator

$$\overline{A} \colon E_1 \otimes_\pi \cdots \otimes_\pi E_m \longrightarrow F$$

given by

$$\overline{A} \left(\sum_{j=1}^{n} x_{1,j} \otimes \cdots \otimes x_{m,j} \right) = \sum_{j=1}^{n} A(x_{1,j}, \ldots, x_{m,j})$$

for all $x_{k,j} \in E_k$ $(1 \leq k \leq m,\ 1 \leq j \leq n)$ [Ry], page 24. For a polynomial $P \in \mathcal{P}(^m E, F)$, its *linearization*

$$\overline{P} \colon \otimes_{\pi_s,s}^m E \longrightarrow F$$

is the operator given by

$$\overline{P} \left(\sum_{j=1}^{n} \lambda_j x_j \otimes \overset{(m)}{\cdots} \otimes x_j \right) = \sum_{j=1}^{n} \lambda_j P(x_j)$$

for all $x_j \in E$ and $\lambda_j \in \mathbb{K}$ $(1 \leq j \leq n)$.

The operator

$$\mathcal{P}(^m E, F) \longrightarrow \mathcal{L}\left(\otimes_{\pi_s,s}^m E, F\right)$$

given by

$$P \longmapsto \overline{P}$$

is an isometric isomorphism [F1], Proposition 2.2.

Given $1 \leq r < \infty$, a polynomial $P \in \mathcal{P}(^m E, F)$ is *r-dominated* (see, e.g., [M, MT]) if there exists a constant $k > 0$ such that, for all $n \in \mathbb{N}$ and $(x_i)_{i=1}^n \subset E$, we have

$$\left(\sum_{i=1}^{n} \|P(x_i)\|^{\frac{r}{m}} \right)^{\frac{m}{r}} \leq k \sup_{x^* \in B_{E^*}} \left(\sum_{i=1}^{n} |x^*(x_i)|^r \right)^{\frac{m}{r}} .$$

Note that, for $m = 1$, we obtain the (absolutely) r-summing operators. If $m = 1$ and $r = 1$, we have an absolutely summing operator, whose norm (the infimum of the constants k that verify the definition) is denoted by $\|\cdot\|_{\mathrm{as}}$.

It is well known that a polynomial $P \in \mathcal{P}(^m E, F)$ is r-dominated if and only if there are a Banach space G, an absolutely r-summing operator $T \in \mathcal{L}(E, G)$, and a polynomial $Q \in \mathcal{P}(^m G, F)$ such that $P = Q \circ T$. A proof of this fact may be seen in [CDG], Theorem 5.

A polynomial $P \in \mathcal{P}(^m E, F)$ is *nuclear* [A1] if it can be written in the form

$$P(x) = \sum_{i=1}^{\infty} x_i^*(x)^m y_i \qquad (x \in E) \tag{1.1}$$

where $(x_i^*) \subset E^*$ and $(y_i) \subset F$ are bounded sequences such that

$$\sum_{i=1}^{\infty} \|x_i^*\|^m \|y_i\| < \infty. \tag{1.2}$$

We denote by $\mathcal{P}_{\mathrm{N}}(^m E, F)$ the space of all m-homogeneous nuclear polynomials from E into F, endowed with the nuclear norm

$$\|P\|_{\mathrm{N}} := \inf \sum_{i=1}^{\infty} \|x_i^*\|^m \|y_i\|,$$

where the infimum is taken over all bounded sequences $(x_i^*) \subset E^*$ and $(y_i) \subset F$ which satisfy (1.1) and (1.2).

A polynomial $P \in \mathcal{P}(^m E, F)$ is *Pietsch integral* (resp., *(Grothendieck) integral*) [A1] if there exists a regular countably additive, F-valued (resp., F^{**}-valued) Borel measure $\mathcal{G}$ of bounded variation on B_{E^*} such that

$$P(x) = \int_{B_{E^*}} [x^*(x)]^m \, d\mathcal{G}(x^*) \qquad (x \in E). \tag{1.3}$$

The symbol $\mathcal{P}_{\mathrm{PI}}(^m E, F)$ (resp., $\mathcal{P}_{\mathrm{I}}(^m E, F)$) denotes the space of all Pietsch integral (resp., integral) m-homogeneous polynomials from E into F, endowed with the *Pietsch integral norm* (resp., *integral norm*)

$$\|P\|_{\mathrm{PI}} := \inf |\mathcal{G}|\,(B_{E^*}) \qquad \big(\text{resp.,} \quad \|P\|_{\mathrm{I}} := \inf |\mathcal{G}|\,(B_{E^*})\big),$$

where $|\mathcal{G}|$ denotes the variation of $\mathcal{G}$ and the infimum is taken over all vector measures $\mathcal{G}$ satisfying the definition.

It is well-known and easy to verify that

$$\mathcal{P}_{\mathrm{N}}(^m E, F) \subseteq \mathcal{P}_{\mathrm{PI}}(^m E, F) \subseteq \mathcal{P}_{\mathrm{I}}(^m E, F)$$

with

$$\|P\|_{\mathrm{I}} \leq \|P\|_{\mathrm{PI}} \leq \|P\|_{\mathrm{N}} \quad \text{for every} \quad P \in \mathcal{P}_{\mathrm{N}}(^m E, F).$$

For $\phi \in E^*$ and $y \in F$, let $\phi^m \underline{\otimes} y \in \mathcal{P}(^m E, F)$ be the polynomial given by

$$\left(\phi^m \underline{\otimes} y\right)(x) := [\phi(x)]^m y \qquad (x \in E).$$

A polynomial in the linear span of the set

$$\{\phi^m \underline{\otimes} y : \ \phi \in E^*, \ y \in F\}$$

is called a *polynomial of finite type.*

2. Factorization of Pietsch integral polynomials

In this Section, we give a result on factorization of integral and Pietsch integral polynomials through the canonical inclusion from a $C(K)$ space into $L_1(K,\mu)$ for some measure μ. This result, interesting in itself, will be used in Section 4 (Theorem (4.2)).

We start by two preparatory lemmas, the first of which is well-known and immediate.

LEMMA (2.1). *For $m \in \mathbb{N}$, the operator*

$$j\colon \otimes^m_{\epsilon_s,s} E \longrightarrow C(B_{E^*})$$

given by

$$j\left(\sum_{i=1}^{n} \lambda_i x_i \otimes \overset{(m)}{\cdots} \otimes x_i\right)(x^*) := \sum_{i=1}^{n} \lambda_i (x^*(x_i))^m,$$

where $\lambda_i \in \mathbb{K}$, $x_i \in E$ $(1 \le i \le n)$, $x^* \in E^*$, is an isometric embedding.

Given a compact Hausdorff space K, let $\delta\colon K \to C(K)^*$ be the evaluation mapping defined by $\delta(w) = \delta_w$ for all $w \in K$, where

$$\delta_w(f) := f(w) \quad \text{for all} \quad f \in C(K).$$

It is easy to check that $\|\delta_w\| = 1$ for every $w \in K$.

The following result was essentially noticed by Villanueva [V2], Corollary 2.4.

LEMMA (2.2). *Let K be a compact Hausdorff space, and let $h \in \mathcal{L}(E, C(K))$ be an operator. Define the polynomial $R \in \mathcal{P}(^mE, C(K))$ by $R(x) = [h(x)]^m$. Then the linearization*

$$\overline{R}\colon \otimes^m_{\epsilon_s,s} E \longrightarrow C(K)$$

is a well-defined operator, with norm

$$\left\|\overline{R}\right\| = \|R\| = \|h\|^m.$$

Proof. Clearly, $\|R\| = \|h\|^m$. For $\lambda_i \in \mathbb{K}$ and $x_i \in E$ $(1 \le i \le n)$, we have

$$\begin{aligned}
\left\|\overline{R}\left(\sum_{i=1}^{n} \lambda_i x_i \otimes \overset{(m)}{\cdots} \otimes x_i\right)\right\| &= \left\|\sum_{i=1}^{n} \lambda_i R(x_i)\right\| = \sup_{w\in K}\left|\sum_{i=1}^{n} \lambda_i R(x_i)(w)\right| \\
&= \sup_{w\in K}\left|\sum_{i=1}^{n} \lambda_i [h(x_i)(w)]^m\right| \\
&= \sup_{w\in K}\left|\sum_{i=1}^{n} \lambda_i [\delta_w(h(x_i))]^m\right| \\
&\le \|h\|^m \sup_{x^*\in B_{E^*}}\left|\sum_{i=1}^{n} \lambda_i [x^*(x_i)]^m\right| \\
&= \|h\|^m \left\|\sum_{i=1}^{n} \lambda_i x_i \otimes \overset{(m)}{\cdots} \otimes x_i\right\|_{\epsilon_s},
\end{aligned}$$

so $\left\|\overline{R}\right\| \leq \|h\|^m$. On the other hand, for every $x \in B_E$, we have

$$\left\|\overline{R}\right\| \geq \left\|\overline{R}\left(x\otimes \overset{(m)}{\cdots} \otimes x\right)\right\| = \|R(x)\| = \|h(x)\|^m ,$$

hence

$$\left\|\overline{R}\right\| \geq \sup_{x\in B_E} \|h(x)\|^m = \|h\|^m .$$

Combining both inequalities yields $\left\|\overline{R}\right\| = \|h\|^m$. □

THEOREM (2.3). *Given a polynomial $P \in \mathcal{P}(^mE, F)$, the following assertions are equivalent:*

(a) *P is Pietsch integral.*

(b) *There are a compact Hausdorff space K, an embedding $h \in \mathcal{L}(E, C(K))$, and a regular countably additive, F-valued Borel measure $\mathcal{G}$ of bounded variation on K such that*

$$P(x) = \int_K [h(x)(w)]^m \, d\mathcal{G}(w) \qquad (x \in E) .$$

(c) *There are a compact Hausdorff space K, an embedding $h \in \mathcal{L}(E, C(K))$, a finite nonnegative countably additive, Borel measure μ on K, and an operator $S \in \mathcal{L}(L_1(K, \mu), F)$ such that the following diagram is commutative*

$$\begin{array}{ccc} E & \xrightarrow{P} & F \\ {\scriptstyle R}\downarrow & & \uparrow{\scriptstyle S} \\ C(K) & \xrightarrow[J]{} & L_1(K, \mu) \end{array}$$

where J is the natural inclusion, and R is the polynomial given by

$$R(x) := [h(x)]^m \qquad \textit{for all} \quad x \in E .$$

(d) *There are a finite measure space (Ω, Σ, μ), an operator $S \in \mathcal{L}(L_1(\Omega, \mu), F)$, and an embedding $h \in \mathcal{L}(E, L_\infty(\Omega, \mu))$ such that the following diagram is commutative*

$$\begin{array}{ccc} E & \xrightarrow{P} & F \\ {\scriptstyle R}\downarrow & & \uparrow{\scriptstyle S} \\ L_\infty(\Omega, \mu) & \xrightarrow[J]{} & L_1(\Omega, \mu) \end{array}$$

where J is the natural inclusion, and R is the polynomial given by $R(x) = [h(x)]^m$, for all $x \in E$.

(e) *There are a finite measure space (Ω, Σ, μ) and operators*

$$T\colon \otimes^m_{\epsilon_s,s} E \longrightarrow L_\infty(\Omega, \mu)$$

and $S \in \mathcal{L}(L_1(\Omega, \mu), F)$ such that the following diagram is commutative

$$\begin{array}{ccc} \otimes^m_{\epsilon_s,s} E & \xrightarrow{\overline{P}} & F \\ {\scriptstyle T}\downarrow & & \uparrow{\scriptstyle S} \\ L_\infty(\Omega, \mu) & \xrightarrow[J]{} & L_1(\Omega, \mu) \end{array}$$

where J is the natural inclusion. In this case, the operator $\overline{P}$ is Pietsch integral and, for every $\epsilon > 0$, we can choose the measure space and the operators S and T so that

$$\|S\| \leq 1 \qquad \textit{and} \qquad \left\|\overline{P}\right\|_{\mathrm{PI}} \leq \|T\|\mu(\Omega) \leq \left\|\overline{P}\right\|_{\mathrm{PI}} + \epsilon\,.$$

If one (and then all) of these assertions holds, then we have

$$\|P\|_{\mathrm{PI}} = \left\|\overline{P}\right\|_{\mathrm{PI}}\,.$$

Moreover, if P is Pietsch integral, then

$$\|P\|_{\mathrm{PI}} = \inf \|h\|^m |\mathcal{G}|(K)$$

where the infimum is taken over all K, h, and $\mathcal{G}$ as in (b). *Also, we can choose S in* (c) *with $\|S\| \leq 1$, and*

$$\|P\|_{\mathrm{PI}} = \inf \|R\|\mu(K)$$

where the infimum is taken over all R, K, S, and μ as in (c). *A similar equality holds for the factorization of* (d).

Proof. (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c). By [DU], Theorem VI.2.1, $\mathcal{G}$ may be seen as the representing measure of an operator $T\colon C(K) \to F$ given by

$$T(f) := \int_K f(w)\, d\mathcal{G}(w)\,.$$

Since $\mathcal{G}$ has bounded variation, T is absolutely summing, and $\|T\|_{\mathrm{as}} = |\mathcal{G}|(K)$ [DU], Theorem VI.3.3. By [DU], Corollary VI.3.7, there are a nonnegative countably additive, Borel measure μ on K, with $\mu = |\mathcal{G}|$, and an operator $S \in \mathcal{L}(L_1(K,\mu), F)$, with $\|S\| \leq 1$, such that $T = S \circ J$, where J is the natural inclusion.

$$\begin{tikzcd} C(K) \arrow[rr, "T"] \arrow[dr, "J"'] & & F \\ & L_1(K,\mu) \arrow[ur, "S"'] & \end{tikzcd}$$

Let $R \in \mathcal{P}(^mE, C(K))$ be the polynomial given by

$$R(x) := [h(x)]^m \qquad \text{for all} \quad x \in E\,.$$

Then

$$S \circ J \circ R(x) = T(R(x)) = \int_K R(x)(w)\, d\mathcal{G}(w) = \int_K [h(x)(w)]^m\, d\mathcal{G}(w) = P(x),$$

so the diagram of (c) is commutative.

(c) $\Rightarrow$ (d). The operator J of the diagram of (c) can be naturally factored in the form $J = i_1 \circ i_\infty$, where i_∞ and i_1 are natural inclusions:

$$\begin{tikzcd}
E \arrow[rr, "P"] \arrow[d, "Q"'] & & F \arrow[d, "S"] \\
C(K) \arrow[rr, "J"] \arrow[dr, "i_\infty"'] & & L_1(K, \mu) \\
& L_\infty(K, \mu) \arrow[ur, "i_1"'] &
\end{tikzcd}$$

If $Q(x) := [h(x)]^m$ for all $x \in E$, we obtain the diagram of (d) with $R := i_\infty \circ Q = (i_\infty \circ h)^m$.

(d) $\Rightarrow$ (e). Given the diagram of (d), the operator $T := \overline{R}$ is well-defined and continuous (Lemma (2.2)) and we obtain the diagram of (e).

Therefore, $\overline{P}$ is a Pietsch integral operator (see [DU], Theorem VI.3.12 and [DJT], Examples 2.9). By the argument of [DU], pages 167–168, given $\epsilon > 0$, it is easy to see that we can obtain a diagram as in (e) with $\|S\| \leq 1$ and

$$\left\|\overline{P}\right\|_{\mathrm{PI}} \leq \|T\| \mu(\Omega) \leq \left\|\overline{P}\right\|_{\mathrm{PI}} + \epsilon.$$

(e) $\Rightarrow$ (a). Assume that $\overline{P}$ satisfies the diagram of (e). Let

$$j\colon \otimes^m_{\epsilon_s,s} E \longrightarrow C\left(B_{E^*}\right)$$

be the isometric embedding of Lemma (2.1). Since the space $L_\infty(\Omega, \mu)$ is injective, T admits an extension

$$\widetilde{T}\colon C\left(B_{E^*}\right) \longrightarrow L_\infty(\Omega, \mu)$$

such that $T = \widetilde{T} \circ j$ and $\left\|\widetilde{T}\right\| = \|T\|$.

$$\begin{tikzcd}
& \otimes^m_{\epsilon_s,s} E \arrow[r, "\overline{P}"] \arrow[dd, "T"] & F \arrow[dd, "S"] \\
C(B_{E^*}) \arrow[ur, "j"] \arrow[dr, "\widetilde{T}"'] & & \\
& L_\infty(\Omega, \mu) \arrow[r, "J"'] & L_1(\Omega, \mu)
\end{tikzcd}$$

The operator $S \circ J \circ \widetilde{T}$ has a representing measure $\mathcal{G}$ defined on the Borel subsets of B_{E^*}, with values in F^{**} [DU], Theorem VI.2.1. Since $S \circ J \circ \widetilde{T}$ is absolutely summing (and hence weakly compact), the measure $\mathcal{G}$ is regular, countably additive with bounded variation, and its range is contained in F [DU], Theorem VI.2.5, Corollary VI.2.14, Theorem VI.3.3. Moreover,

$$S \circ J \circ \widetilde{T}(f) = \int_{B_{E^*}} f(x^*)\, d\mathcal{G}(x^*) \qquad \left(f \in C\left(B_{E^*}\right)\right),$$

and

$$|\mathcal{G}|\left(B_{E^*}\right) = \left\|S \circ J \circ \widetilde{T}\right\|_{\mathrm{as}}$$

(this equality will be used later on). We have

$$\begin{aligned} P(x) &= \overline{P}\left(x\otimes \overset{(m)}{\cdots} \otimes x\right) \\ &= S\circ J\circ \widetilde{T}\left(j\left(x\otimes \overset{(m)}{\cdots} \otimes x\right)\right) \\ &= \int_{B_{E^*}} j\left(x\otimes \overset{(m)}{\cdots} \otimes x\right)(x^*)\,d\mathcal{G}(x^*) \\ &= \int_{B_{E^*}} [x^*(x)]^m\,d\mathcal{G}(x^*), \end{aligned}$$

so P is Pietsch integral, and (a) is proved.

Suppose now that P is Pietsch integral. Given $\epsilon > 0$, choosing $K := B_{E^*}$ in (b) and letting $h \in \mathcal{L}(E, C(K))$ be the natural isometric embedding, it is clear that there is a measure $\mathcal{G}$ such that

$$\|P\|_{\mathrm{PI}} \leq \|h\|^m |\mathcal{G}|(K) \leq \|P\|_{\mathrm{PI}} + \epsilon\,.$$

From the proof of (b) $\Rightarrow$ (c), we obtain $\|S\| \leq 1$, $\mu = |\mathcal{G}|$, and $\|R\| = \|h\|^m$, so

$$\|P\|_{\mathrm{PI}} \leq \|R\|\mu(K) \leq \|P\|_{\mathrm{PI}} + \epsilon\,.$$

Similarly, there are Ω, μ, S with $\|S\| \leq 1$, and an embedding $h \in \mathcal{L}(E, L_\infty(\Omega, \mu))$ such that the diagram of (d) holds, and

$$\|P\|_{\mathrm{PI}} \leq \|h\|^m \mu(\Omega) \leq \|P\|_{\mathrm{PI}} + \epsilon\,.$$

From the proof of (d) $\Rightarrow$ (e), we have

$$\begin{aligned} \left\|\overline{P}\right\|_{PI} &\leq \|T\|\|J\|_{\mathrm{PI}}\|S\| \\ &\leq \|T\|\|J\|_{\mathrm{as}} && \text{[DU], Theorem VI.3.12} \\ &= \left\|\overline{R}\right\|\mu(\Omega) && \text{[DJT], Examples 2.9} \\ &= \|h\|^m\mu(\Omega) && \text{(by Lemma (2.2))} \\ &\leq \|P\|_{\mathrm{PI}} + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get

$$\left\|\overline{P}\right\|_{\mathrm{PI}} \leq \|P\|_{\mathrm{PI}}.$$

Moreover, since $\overline{P}$ is Pietsch integral, we can factor it as in (e), with

$$\|S\| \leq 1 \quad \text{and} \quad \left\|\overline{P}\right\|_{\mathrm{PI}} \leq \|T\|\mu(\Omega) \leq \left\|\overline{P}\right\|_{\mathrm{PI}} + \epsilon\,.$$

Following the proof of (e) $\Rightarrow$ (a), we obtain

$$
\begin{aligned}
\|P\|_{\mathrm{PI}} &\leq |\mathcal{G}|\,(B_{E^*}) \\
&= \left\| S \circ J \circ \widetilde{T} \right\|_{\mathrm{as}} \\
&\leq \|J\|_{\mathrm{as}} \left\| \widetilde{T} \right\| \|S\| \\
&\leq \|J\|_{\mathrm{as}} \|T\| \\
&= \|T\| \mu(\Omega) \qquad \text{[DJT], Examples 2.9} \\
&\leq \left\| \overline{P} \right\|_{\mathrm{PI}} + \epsilon\,.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, we get

$$\|P\|_{\mathrm{PI}} \leq \left\| \overline{P} \right\|_{\mathrm{PI}},$$

and we conclude that

$$\|P\|_{\mathrm{PI}} = \left\| \overline{P} \right\|_{\mathrm{PI}}.$$

Let $P \in \mathcal{P}_{\mathrm{PI}}(^mE, F)$. Assume that we have found K, h, and $\mathcal{G}$ so that the formula of (b) holds, with

$$\|P\|_{\mathrm{PI}} > \|h\|^m |\mathcal{G}|(K)\,.$$

Then, going through the proofs of (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e), we have

$$
\begin{aligned}
\|P\|_{\mathrm{PI}} &> \|h\|^m |\mathcal{G}|(K) \\
&= \|R\| \mu(K) \\
&= \left\| \overline{R} \right\| \mu(\Omega) \qquad \text{(by Lemma (2.2))} \\
&= \|T\| \mu(\Omega) \\
&\geq \left\| \overline{P} \right\|_{\mathrm{PI}} \\
&= \|P\|_{\mathrm{PI}}\,,
\end{aligned}
$$

a contradiction. Therefore,

$$\|P\|_{\mathrm{PI}} = \inf \left\{ \|h\|^m |\mathcal{G}|(K) : \; K, h, \mathcal{G} \text{ as in (b)} \right\}.$$

Recalling that $\|S\| \leq 1$, the above chain of inequalities also proves that

$$
\begin{aligned}
\|P\|_{\mathrm{PI}} &= \inf \left\{ \|R\| \mu(K) : \; K, R, S, \mu \text{ as in (c)} \right\} \\
\|P\|_{\mathrm{PI}} &= \inf \left\{ \|R\| \mu(\Omega) : \; \Omega, R, S, \mu \text{ as in (d)} \right\},
\end{aligned}
$$

and the Theorem is proved. □

In (b) $\Rightarrow$ (c), we chose $\mu = |\mathcal{G}|$. Since $\mathcal{G}$ is regular, μ is also regular [D], §15.6, Corollary 2 to Proposition 21, so the measure μ in (c), (d), and (e) may be supposed to be regular.

In order to see that Theorem (2.3) is also true for integral polynomials, we need the following result whose proof is standard.

THEOREM (2.4). *A polynomial $P \in \mathcal{P}(^mE, F)$ is integral if and only if the polynomial $k_F \circ P \in \mathcal{P}(^mE, F^{**})$ is Pietsch integral. Moreover, in this case we have*

$$\|P\|_{\mathrm{I}} = \|k_F \circ P\|_{\mathrm{PI}}\,.$$

Proof. Let $P \in \mathcal{P}_{\mathrm{I}}(^mE, F)$ and $\epsilon > 0$. Then there is an F^{**}-valued measure $\mathcal{G}$ on B_{E^*} as in the definition of the integral polynomials such that the equality (1.3) holds, and

$$\|P\|_{\mathrm{I}} \le |\mathcal{G}|\,(B_{E^*}) \le \|P\|_{\mathrm{I}} + \epsilon\,,$$

so

$$k_F \circ P(x) = \int_{B_{E^*}} [x^*(x)]^m \, d\mathcal{G}(x^*) \qquad (x \in E)\,, \tag{2.5}$$

and $k_F \circ P$ is Pietsch integral, with

$$\|k_F \circ P\|_{\mathrm{PI}} \le |\mathcal{G}|\,(B_{E^*}) \le \|P\|_{\mathrm{I}} + \epsilon\,.$$

Since $\epsilon > 0$ was arbitrary, we obtain

$$\|k_F \circ P\|_{\mathrm{PI}} \le \|P\|_{\mathrm{I}}\,.$$

Given now a polynomial $P \in \mathcal{P}(^mE, F)$ such that $k_F \circ P$ is Pietsch integral, there is an F^{**}-valued measure $\mathcal{G}$ on B_{E^*} as in the definition of the Pietsch integral polynomials such that the equality (2.5) holds, and

$$\|k_F \circ P\|_{\mathrm{PI}} \le |\mathcal{G}|\,(B_{E^*}) \le \|k_F \circ P\|_{\mathrm{PI}} + \epsilon\,.$$

Identifying $y \in F$ with $k_F(y) \in F^{**}$, since $k_F \circ P(x) = P(x)$ for all $x \in E$, we obtain the equality (1.3), so P is integral, and

$$\|P\|_{\mathrm{I}} \le |\mathcal{G}|\,(B_{E^*}) \le \|k_F \circ P\|_{\mathrm{PI}} + \epsilon\,.$$

Since $\epsilon > 0$ was arbitrary, we have

$$\|P\|_{\mathrm{I}} \le \|k_F \circ P\|_{\mathrm{PI}}\,.$$

Combining both inequalities yields

$$\|k_F \circ P\|_{\mathrm{PI}} = \|P\|_{\mathrm{I}}.$$

□

Therefore, the statement of Theorem (2.3) is true for integral polynomials, with the following modifications:

Part	Theorem (2.3)	new Theorem
(a)	P is Pietsch integral	P is integral
(b)	$\mathcal{G}$ is F-valued	$\mathcal{G}$ is F^{**}-valued
(c) and (d)	factorization of P	factorization of $k_F \circ P$
(e)	factorization of $\overline{P}$	factorization of $k_F \circ \overline{P}$
	$\overline{P}$ is Pietsch integral	$\overline{P}$ is integral

In the case of multilinear mappings, Theorem (2.3) is essentially contained in [V1], Corollary 2.7. The proof in the multilinear setting is easier, and our proof is not an adaptation of it, since a simple adaptation would yield factorization through $C(K^m)$, while we are interested in factoring the polynomials through $C(K)$. The equivalence

$$P \text{ is a Pietsch integral polynomial} \quad \Leftrightarrow \quad \overline{P} \text{ is a Pietsch integral operator}$$

is given in [V1], Corollary 2.8, without the equality of the norms. The isometric version was established in [CL1], Proposition 2.10. The (isometric) equivalence of (a) and (e) for the (Grothendieck) integral version of Theorem (2.3) was obtained in [CL2], Proposition 1.

Remark (2.6). We might ask whether, given a compact Hausdorff space K and an embedding $h \in \mathcal{L}(E, C(K))$, it is always possible to factor every $P \in \mathcal{P}_{\mathrm{PI}}(^mE, F)$ in the form

$$\begin{array}{ccc} E & \xrightarrow{P} & F \\ {\scriptstyle R}\big\downarrow & & \big\uparrow{\scriptstyle S} \\ C(K) & \xrightarrow[J]{} & L_1(K, \mu) \end{array} \tag{2.7}$$

with $R(x) = [h(x)]^m$ for all $x \in E$. This is true in the linear case ($m = 1$) [V1], page 59; it is also true in a certain sense for m-linear mappings [V1], Corollary 2.7, but it is false in general for polynomials of degree $m > 1$ as the following example shows.

Let $E := C[0, 1]$, $K := [0, 1]$, let $h \in \mathcal{L}(E, C(K))$ be the identity map, and let $R \in \mathcal{P}(^2E, C(K))$ be the polynomial given by $R(x) = [h(x)]^2$ ($x \in E$). Consider the functional $\phi \in C[0, 1]^*$ given by

$$\phi(f) := \int_0^1 f(t)\,dt \qquad (f \in C[0, 1]),$$

and the polynomial of finite type (so Pietsch integral) $P \in \mathcal{P}(^2C[0, 1])$ defined by $P := \phi^2$.

Let $f \in C[0, 1]$ be given by

$$f(t) := \begin{cases} 4(1 - 2t), & \text{if } \ 0 \le t \le \dfrac{1}{2} \\ 0, & \text{if } \ \dfrac{1}{2} \le t \le 1. \end{cases}$$

Let $g(t) := f(1 - t)$ for all $t \in [0, 1]$.

Consider the symmetric tensor

$$u := f \otimes g + g \otimes f \in \otimes_s^2 C[0, 1].$$

Then

$$\begin{aligned} \overline{P}(u) &= \overline{\widehat{P}}(f \otimes g) + \overline{\widehat{P}}(g \otimes f) \\ &= \widehat{P}(f, g) + \widehat{P}(g, f) \\ &= 2\phi(f)\phi(g) = 2. \end{aligned}$$

On the other hand,

$$\overline{R}(u) = \overline{\widehat{R}}(f \otimes g) + \overline{\widehat{R}}(g \otimes f) = 2fg = 0\,.$$

If the diagram (2.7) is commutative for $F = \mathbb{K}$ and $S = \Psi \in L_\infty([0,1],\mu)$, so is also the following one for the functional $\overline{P}$:

$$\begin{array}{ccc} \otimes^2_{\epsilon_s,s} C[0,1] & \xrightarrow{\overline{P}} & \mathbb{K} \\ {\scriptstyle \overline{R}}\big\downarrow & & \big\uparrow{\scriptstyle \Psi} \\ C[0,1] & \xrightarrow[J]{} & L_1([0,1],\mu) \end{array}$$

so

$$2 = \overline{P}(u) = \Psi \circ J \circ \overline{R}(u) = 0\,,$$

a contradiction.

3. Ideals of vector-valued polynomials

In this Section, we recall the notion of ideal of vector-valued polynomials, and we show that, for $0 < \lambda \leq 1$, a λ-normed ideal of polynomials is maximal if and only if it is ultrastable and regular. It is also noted that maximality implies completeness. The Section extends to the vector-valued setting the results given in [F2], §2, for ideals of scalar-valued polynomials.

The notion of λ-norm may be seen in [DF], 9.2. The following definitions are given in [FH], 4.1 (see also [F2], 1.2).

Definition (3.1). A subclass $\mathcal{Q}$ of the class $\mathcal{P}^m$ of m-homogeneous polynomials between Banach spaces is called an *ideal of m-homogeneous polynomials* if the following three conditions are satisfied:

(a) The component $\mathcal{Q}(E,F) := \mathcal{P}(^mE,F) \cap \mathcal{Q}$ is a vector subspace of $\mathcal{P}(^mE,F)$ for all Banach spaces E and F;

(b) for all $P \in \mathcal{Q}(E,F)$, $T \in \mathcal{L}(X,E)$, and $S \in \mathcal{L}(F,Y)$, we have $S \circ P \circ T \in \mathcal{Q}(X,Y)$;

(c) the polynomial $\underline{\otimes}^m 1 \colon \mathbb{K} \to \mathbb{K}$ given by $\underline{\otimes}^m 1(z) = z^m$ belongs to $\mathcal{Q}$.

Definition (3.2). Given an ideal of m-homogeneous polynomials $\mathcal{Q}$, a function $\boldsymbol{Q} \colon \mathcal{Q} \to [0,\infty[$, and a number $0 < \lambda \leq 1$, the pair $(\mathcal{Q},\boldsymbol{Q})$ is called a *λ-normed* (resp., *normed*) ideal of m-homogeneous polynomials if:

(a$'$) $\boldsymbol{Q}$ is a λ-norm (resp., a norm) on $\mathcal{Q}(E,F)$ for all Banach spaces E and F;

(b$'$) under the hypotheses of (b), we have

$$\boldsymbol{Q}(S \circ P \circ T) \leq \|S\|\boldsymbol{Q}(P)\|T\|^m\ ;$$

(c$'$) $\boldsymbol{Q}(\underline{\otimes}^m 1) = 1$.

Every ideal of polynomials contains all the polynomials of finite type of the same degree.

Easy arguments show that, if $(\mathcal{Q},\boldsymbol{Q})$ is a λ-normed ideal of m-homogeneous polynomials, and $P \in \mathcal{Q}(E,F)$, then

$$\|P\| \leq \boldsymbol{Q}(P),$$

and, for all $\phi \in E^*$ and $y \in F$, we have

$$\boldsymbol{Q}\left(\phi^m \underline{\otimes} y\right) = \|\phi\|^m\|y\|\,.$$

If E is finite-dimensional, then

$$\mathcal{Q}(E, F) = \mathcal{P}(^mE, F).$$

In this case, all λ-norms $\boldsymbol{Q}$ generate on the space $\mathcal{P}(^mE, F)$ the same topology of pointwise convergence on E.

In what follows we shall need some basic results of the theory of ultraproducts. We give the definitions and results that we shall use, and we refer the reader to [H] or [Si] for more details.

Let I be an index set and let $\mathcal{U}$ be an ultrafilter on I. Let $(E_i)_{i\in I}$ be a family of Banach spaces. Consider the Banach space $\ell_\infty(I, E_i)$ of all bounded families $(x_i)_{i\in I}$ with $x_i \in E_i$ for every $i \in I$. For each $(x_i) \in \ell_\infty(I, E_i)$, $\lim_{\mathcal{U}}\|x_i\|$ exists. Consider the subspace of $\ell_\infty(I, E_i)$ given by

$$N_{\mathcal{U}} := \left\{(x_i) \in \ell_\infty(I, E_i) : \lim_{\mathcal{U}}\|x_i\| = 0\right\}.$$

The quotient space $(E_i)_{\mathcal{U}} := \ell_\infty(I, E_i)/N_{\mathcal{U}}$ is called the *ultraproduct* of the family $(E_i)_{i\in I}$ with respect to the ultrafilter $\mathcal{U}$. It is a Banach space with the norm $\|x\| := \lim_{\mathcal{U}}\|x_i\|$ where $x = (x_i)_{\mathcal{U}}$ is the element admitting the family $(x_i)_{i\in I}$ as a representative.

Now, for Banach spaces E_i, F_i $(i \in I)$, let a bounded family of polynomials $P_i \in \mathcal{P}^m(E_i, F_i)$ $(i \in I)$ be given. Then the mapping

$$(P_i)_{\mathcal{U}} \colon (E_i)_{\mathcal{U}} \longrightarrow (F_i)_{\mathcal{U}},$$

given by

$$(P_i)_{\mathcal{U}}\left((x_i)_{\mathcal{U}}\right) := (P_i(x_i))_{\mathcal{U}},$$

is an m-homogeneous polynomial called the *ultraproduct of the polynomials* (P_i) with respect to the ultrafilter $\mathcal{U}$ and its norm is given by $\|(P_i)_{\mathcal{U}}\| := \lim_{\mathcal{U}}\|P_i\|$.

By $\mathrm{Dim}(E)$ and $\mathrm{Cod}(F)$ we denote, respectively, the collection of all finite-dimensional subspaces of E, and the collection of all finite-codimensional subspaces of F. If $M \in \mathrm{Dim}(E)$ and $N \in \mathrm{Cod}(F)$, J_M^E and S_N^F stand for the natural maps from M into E and from F onto F/N, respectively. Consider the index set

$$I := \{i = (M, N) : M \in \mathrm{Dim}(E),\ N \in \mathrm{Cod}(F)\}.$$

Given the indices $i = (M, N)$, $i' = (M', N')$ in I, we shall write $i \leq i'$ if $M \subseteq M'$ and $N' \subseteq N$. Let $\mathcal{U}$ be an ultrafilter on I containing all subsets $\{i \in I : i \geq i_0\}$ for $i_0 \in I$ fixed. For every index $i = (M, N)$, write $E_i := M$, $F_i := F/N$. Given a polynomial $P \in \mathcal{P}(^mE, F)$, the polynomials $P_i := S_N^F P J_M^E$ are called the *elementary parts of* P.

The following Lemma shows that a polynomial may be reconstructed from its elementary parts. Its proof is an easy adaptation of the proof of [B], Lemma 12.2, and follows the lines of [Pi1], Lemma 8.8.4.

LEMMA (3.3). *Given Banach spaces E and F, let $P \in \mathcal{P}(^mE, F)$ be a polynomial. With the above notation, there exist operators $J \colon E \to (E_i)_{\mathcal{U}}$ with $\|J\| \leq 1$ and $H \colon (F_i)_{\mathcal{U}} \to F^{**}$ with $\|H\| \leq 1$ such that*

$$k_F \circ P = H \circ (P_i)_{\mathcal{U}} \circ J.$$

Definition (3.4). [F2]

(a) We say that a λ-normed ideal of polynomials $(\mathcal{Q}, \boldsymbol{Q})$ is *ultrastable* if, given a set I and an ultrafilter $\mathcal{U}$ on I, for every family $(P_i)_{i\in I} \subset \mathcal{Q}(E_i, F_i)$ with

$$\sup_{i\in I} \boldsymbol{Q}(P_i) < \infty\,,$$

we have

$$(P_i)_{\mathcal{U}} \in \mathcal{Q} \quad \text{and} \quad \boldsymbol{Q}((P_i)_{\mathcal{U}}) \leq \lim_{\mathcal{U}} \boldsymbol{Q}(P_i)\,.$$

(b) We say that a λ-normed ideal $(\mathcal{Q}, \boldsymbol{Q})$ is *regular* if $k_F \circ P \in \mathcal{Q}(E, F^{**})$ implies that $P \in \mathcal{Q}(E, F)$ and $\boldsymbol{Q}(k_F \circ P) = \boldsymbol{Q}(P)$.

(c) Given a λ-normed ideal $(\mathcal{Q}, \boldsymbol{Q})$, we say that a polynomial $P \in \mathcal{P}({}^mE, F)$ is in $\mathcal{Q}^{\max}$ if

$$\boldsymbol{Q}^{\max}(P) := \sup\left\{\boldsymbol{Q}(S_N^F P J_M^E):\ M \in \mathrm{Dim}(E),\ N \in \mathrm{Cod}(F)\right\} < +\infty\,.$$

A λ-normed ideal $(\mathcal{Q}, \boldsymbol{Q})$ is called *maximal* if $(\mathcal{Q}, \boldsymbol{Q}) = (\mathcal{Q}^{\max}, \boldsymbol{Q}^{\max})$.

The following result is a version of [H], Theorem 8.1, for polynomials. For completeness, we shall give a proof by modifying that of [H], Theorem 8.1.

THEOREM (3.5). *For a λ-normed ideal of m-homogeneous polynomials $(\mathcal{Q}, \boldsymbol{Q})$, the following assertions are equivalent:*

(a) *$(\mathcal{Q}, \boldsymbol{Q})$ is ultrastable and regular;*

(b) *$(\mathcal{Q}, \boldsymbol{Q})$ is regular and, for every family of polynomials $P_i \in \mathcal{P}({}^mE_i, F_i)$ $(i \in I)$, where E_i and F_i are finite-dimensional Banach spaces, such that $\sup_{i\in I} \boldsymbol{Q}(P_i) < \infty$, and for every ultrafilter $\mathcal{U}$ on I, we have $(P_i)_{\mathcal{U}} \in \mathcal{Q}$ and*

$$\boldsymbol{Q}((P_i)_{\mathcal{U}}) \leq \lim_{\mathcal{U}} \boldsymbol{Q}(P_i)\,.$$

(c) *$(\mathcal{Q}, \boldsymbol{Q})$ is maximal.*

The proof of this Theorem needs two preparatory results that we give in the following Propositions.

PROPOSITION (3.6). *Let $\mathcal{U}$ be an ultrafilter on a set I, and let $(E_i)_{i\in I}$ be a family of Banach spaces. Given a subspace $M \in \mathrm{Dim}((E_i)_{\mathcal{U}})$, let $J_M: M \to (E_i)_{\mathcal{U}}$ be the natural embedding. Then there are operators $R_i \in \mathcal{L}(M, E_i)$ $(i \in I)$ with $\|R_i\| \leq 1$ such that $J_M(x) = (R_i(x))_{\mathcal{U}}$ for all $x \in M$.*

Proof. Let $\{x^1, \ldots, x^n\}$ be a unit vector basis of M, and choose representations $x^k = \left(x_i^k\right)_{\mathcal{U}}$ $(1 \leq k \leq n)$, with $\left\|x_i^k\right\| \leq 2$ $(1 \leq k \leq n,\ i \in I)$. For each $i \in I$, define a linear map

$$\begin{aligned} T_i\colon M &\longrightarrow E_i \\ x^k &\longmapsto x_i^k \quad (1 \leq k \leq n)\,. \end{aligned}$$

For $x = \sum_{k=1}^{n} \lambda_k x^k \in M$ ($x \neq 0$), we have

$$
\begin{aligned}
\|x\| &= \left\| \sum_{k=1}^{n} \lambda_k x^k \right\| \\
&= \left\| \sum_{k=1}^{n} \lambda_k \left(x_i^k \right)_{\mathcal{U}} \right\| \\
&= \lim_{\mathcal{U}} \left\| \sum_{k=1}^{n} \lambda_k x_i^k \right\| \\
&= \lim_{\mathcal{U}} \|T_i(x)\| \ . \\
\Rightarrow \quad 1 &= \lim_{\mathcal{U}} \frac{\|T_i(x)\|}{\|x\|} \ .
\end{aligned}
$$

So, given $0 < \epsilon < 1/4$, there is $I_x \in \mathcal{U}$ such that

$$(1-\epsilon)\|x\| < \|T_i(x)\| < (1+\epsilon)\|x\| \qquad (i \in I_x) \ .$$

Let

$$K := \max \left\{ \sum_{k=1}^{n} |\lambda_k| : \ \left\| \sum_{k=1}^{n} \lambda_k x^k \right\| = 1 \right\} \ .$$

Then, for every $i \in I$,

$$
\begin{aligned}
\|T_i\| &= \sup_{\|x\|=1} \|T_i(x)\| \\
&= \sup \left\{ \left\| \sum_{k=1}^{n} \lambda_k T_i(x^k) \right\| : \ \left\| \sum_{k=1}^{n} \lambda_k x^k \right\| = 1 \right\} \\
&= \sup \left\{ \left\| \sum_{k=1}^{n} \lambda_k x_i^k \right\| : \ \left\| \sum_{k=1}^{n} \lambda_k x^k \right\| = 1 \right\} \\
&\leq 2 \sup \left\{ \sum_{k=1}^{n} |\lambda_k| : \ \left\| \sum_{k=1}^{n} \lambda_k x^k \right\| = 1 \right\} \\
&= 2K \ .
\end{aligned}
$$

Let $\delta := \epsilon/(2K)$. Choose a δ-net $\{y^1, \ldots, y^m\}$ in the unit sphere S_M of M, and set

$$I_0 := \bigcap_{k=1}^{m} I_{y^k} \ .$$

Fix $i \in I_0$, $x \in S_M$. There is $k \in \{1, \ldots, m\}$ such that $\|x - y^k\| < \delta$. Then

$$\|T_i(x)\| \leq \|T_i(x) - T_i(y^k)\| + \|T_i(y^k)\| < 2K\delta + 1 + \epsilon = 1 + 2\epsilon \ .$$

$$\|T_i(x)\| \geq \|T_i(y^k)\| - \|T_i(y^k) - T_i(x)\| > 1 - \epsilon - 2K\delta = 1 - 2\epsilon \ .$$

So

$$1 - 2\epsilon < \|T_i\| < 1 + 2\epsilon \qquad (i \in I_0)$$

$$\Rightarrow \quad \left| 1 - \frac{1}{\|T_i\|} \right| = \left| \frac{\|T_i\| - 1}{\|T_i\|} \right| < \frac{2\epsilon}{1 - 2\epsilon} < 4\epsilon \qquad (i \in I_0) \ .$$

Let

$$R_i := \begin{cases} \dfrac{1}{\|T_i\|} \, T_i \,, & \text{if } i \in I_0 \\ 0, & \text{if } i \notin I_0. \end{cases}$$

Then $\|R_i\| \le 1$. For $x = \sum_{k=1}^{n} \lambda_k x^k$, we have

$$J_M(x) = \sum_{k=1}^{n} \lambda_k \left(x_i^k \right)_{\mathcal{U}} = \left(\sum_{k=1}^{n} \lambda_k x_i^k \right)_{\mathcal{U}}$$

and

$$R_i(x) = \frac{1}{\|T_i\|} \sum_{k=1}^{n} \lambda_k x_i^k \qquad (i \in I_0)\,.$$

Since

$$\begin{aligned} \lim_{\mathcal{U}} \left\| \sum_{k=1}^{n} \lambda_k x_i^k - \frac{1}{\|T_i\|} \sum_{k=1}^{n} \lambda_k x_i^k \right\| &= \lim_{\mathcal{U}} \left\| T_i(x) - \frac{1}{\|T_i\|} \, T_i(x) \right\| \\ &= \lim_{\mathcal{U}} \|T_i(x)\| \left| 1 - \frac{1}{\|T_i\|} \right| \\ &< 8\epsilon K \|x\| \qquad (i \in I_0), \end{aligned}$$

and $\epsilon > 0$ may be made arbitrarily small, we obtain

$$J_M(x) = \big(R_i(x) \big)_{\mathcal{U}} \,.$$

□

Given two isomorphic Banach spaces E and F, and $\epsilon > 0$, we say that an isomorphism $T \in \mathcal{L}(E, F)$ is a $(1+\epsilon)$-*isomorphism* if

$$\|T\| \le 1 + \epsilon \quad \text{and} \quad \|T^{-1}\| \le 1 + \epsilon \,.$$

PROPOSITION (3.7). *Given an ultrafilter $\mathcal{U}$ on a set I and a family $(F_i)_{i\in I}$ of Banach spaces, let $Y \in \mathrm{Dim}((F_i)_{\mathcal{U}})$ be a subspace with embedding $J_Y \colon Y \to (F_i)_{\mathcal{U}}$. Let $N \in \mathrm{Cod}((F_i)_{\mathcal{U}})$ with quotient map $S_N \colon (F_i)_{\mathcal{U}} \to (F_i)_{\mathcal{U}}/N$. Then there are operators $S_i \in \mathcal{L}(F_i, (F_i)_{\mathcal{U}}/N)$ with $\|S_i\| \le 1$ such that*

$$S_N(J_Y(y)) = \lim_{\mathcal{U}} S_i(y_i)$$

whenever $y \in Y$ and $y_i \in F_i$ satisfy $J_Y(y) = (y_i)_{\mathcal{U}}$.

Proof. Let $N^{\perp}$ be the subspace of $(F_i)_{\mathcal{U}}^*$ orthogonal to N. Recall that there is an isometric isomorphism $N^{\perp} \equiv \big((F_i)_{\mathcal{U}}/N \big)^*$.

Given $\epsilon > 0$, by the local duality of ultraproducts [H], Theorem 7.3, there is a $(1+\epsilon)$-isomorphism T from $N^{\perp}$ onto a subspace H of $(F_i^*)_{\mathcal{U}}$ such that

$$\langle L J_H T(y^*), J_Y(y) \rangle = \langle y^*, J_Y(y) \rangle \qquad (y^* \in N^{\perp},\ y \in Y)\,,$$

where $J_H \colon H \to (F_i^*)_{\mathcal{U}}$ is the natural embedding, and $L \colon (F_i^*)_{\mathcal{U}} \to (F_i)_{\mathcal{U}}^*$ is the canonical isometric embedding [H], page 87. Note that, for $y^* \in N^{\perp}$ and $y \in Y$,

$$\begin{aligned} \langle y^*, S_N(J_Y(y)) \rangle &= \langle y^*, J_Y(y) + N \rangle \\ &= \langle y^*, J_Y(y) \rangle \\ &= \langle L J_H T(y^*), J_Y(y) \rangle \,. \end{aligned}$$

Since $\dim(H) < \infty$, Proposition (3.6) gives us operators $R_i \in \mathcal{L}(H, F_i^*)\,(i \in I)$ with $\|R_i\| \leq 1$ such that $J_H(h) = (R_i(h))_{\mathcal{U}}$ for all $h \in H$. Moreover, there are operators $W_i\colon F_i \to H^*$ such that $R_i = W_i^*$ $(i \in I)$. Now, for $y \in Y$ fixed, choose $y_i \in F_i$ $(i \in I)$ such that $J_Y(y) = (y_i)_{\mathcal{U}}$. We obtain

$$\begin{aligned}\langle y^*, S_N(J_Y(y))\rangle &= \langle LJ_HT(y^*), J_Y(y)\rangle \\ &= \langle L((R_i(T(y^*)))_{\mathcal{U}}), (y_i)_{\mathcal{U}}\rangle \\ &= \lim_{\mathcal{U}} \langle R_iT(y^*), y_i\rangle \qquad \text{[H], page 87} \\ &= \lim_{\mathcal{U}} \langle W_i^*T(y^*), y_i\rangle \\ &= \lim_{\mathcal{U}} \langle y^*, T^*W_i(y_i)\rangle\end{aligned}$$

for $y^* \in N^{\perp}$. It follows that

$$S_N(J_Y(y)) = \text{weak-}\lim_{\mathcal{U}} T^*W_i(y_i) = \lim_{\mathcal{U}} T^*W_i(y_i)\,.$$

Let

$$S_i := \frac{1}{\|T\|}\, T^*W_i\,.$$

Since T is a $(1+\epsilon)$-isomorphism, we easily obtain

$$\begin{aligned}\|S_i(y_i) - T^*W_i(y_i)\| &= \left\| \frac{1}{\|T\|}\, T^*W_i(y_i) - T^*W_i(y_i)\right\| \\ &= \left| \frac{1}{\|T\|} - 1\right| \|T^*W_i(y_i)\| \\ &< \epsilon\|y_i\|\,.\end{aligned}$$

Therefore,

$$S_N(J_Y(y)) = \lim_{\mathcal{U}} S_i(y_i)\,.$$

□

Proof of Theorem (3.5). (a) ⇒ (b) is obvious.

(b) ⇒ (c). Suppose that the ideal of polynomials $(\mathcal{Q}, \boldsymbol{Q})$ satisfies (b). Let $P \in \mathcal{P}(^mE, F)$ be a polynomial with $\boldsymbol{Q}^{\max}(P) < +\infty$. By Lemma (3.3), we may write $k_F \circ P = H \circ (P_i)_{\mathcal{U}} \circ J$ with $\|H\| \leq 1$ and $\|J\| \leq 1$, where $P_i := S_N^F P J_M^E$ are the elementary parts of P. Since P_i is a polynomial between finite-dimensional spaces, it belongs to $\mathcal{Q}$. Since $\boldsymbol{Q}(P_i) \leq \boldsymbol{Q}^{\max}(P)$, assertion (b) implies that $(P_i)_{\mathcal{U}} \in \mathcal{Q}$, and $\boldsymbol{Q}((P_i)_{\mathcal{U}}) \leq \lim_{\mathcal{U}} \boldsymbol{Q}(P_i)$. By the ideal property (Definition (3.1),(b)), we obtain

$$k_F \circ P = H \circ (P_i)_{\mathcal{U}} \circ J \in \mathcal{Q}\,.$$

By the regularity, $P \in \mathcal{Q}$ and $\boldsymbol{Q}(k_F \circ P) = \boldsymbol{Q}(P)$. Hence,

$$\begin{aligned}\boldsymbol{Q}(P) = \boldsymbol{Q}(k_F \circ P) &= \boldsymbol{Q}(H \circ (P_i)_{\mathcal{U}} \circ J) \\ &\leq \|H\|\|J\|^m \boldsymbol{Q}((P_i)_{\mathcal{U}}) \leq \lim_{\mathcal{U}} \boldsymbol{Q}(P_i) \leq \boldsymbol{Q}^{\max}(P)\,.\end{aligned}$$

On the other hand, since $P \in \mathcal{Q}$, we have $\boldsymbol{Q}^{\max}(P) \leq \boldsymbol{Q}(P)$, and we conclude that $\boldsymbol{Q}^{\max}(P) = \boldsymbol{Q}(P)$.

(c) ⇒ (a). Let $(\mathcal{Q}, \boldsymbol{Q})$ be a maximal ideal of polynomials. We first show that $(\mathcal{Q}, \boldsymbol{Q})$ is ultrastable. Let I be a set and $\mathcal{U}$ an ultrafilter on I. Given polynomials

$P_i \in \mathcal{Q}(E_i, F_i)$ with $\boldsymbol{Q}(P_i) \leq 1$ $(i \in I)$, let $P := (P_i)_{\mathcal{U}} \in \mathcal{P}(^m(E_i)_{\mathcal{U}}, (F_i)_{\mathcal{U}})$. We have to prove that $P \in \mathcal{Q}$ and $\boldsymbol{Q}(P) \leq 1$. By the maximality, it is enough to show that $\boldsymbol{Q}(S_N P J_M) \leq 1$ for all $M \in \mathrm{Dim}((E_i)_{\mathcal{U}})$ and $N \in \mathrm{Cod}((F_i)_{\mathcal{U}})$. By Proposition (3.6), there are operators $R_i \in L(M, E_i)$ $(i \in I)$ with $\|R_i\| \leq 1$ such that $J_M(x) = (R_i(x))_{\mathcal{U}}$ for all $x \in M$. There are operators $S_i \in \mathcal{L}(F_i, (F_i)_{\mathcal{U}}/N)$ with $\|S_i\| \leq 1$ such that the statement of Proposition (3.7) holds for the subspace generated by $PJ_M(M)$ in $(F_i)_{\mathcal{U}}$. For every $x \in M$ we have

$$S_N P(J_M(x)) = S_N P\left((R_i(x))_{\mathcal{U}}\right) = S_N\left((P_i R_i(x))_{\mathcal{U}}\right) = \lim_{\mathcal{U}} S_i P_i R_i(x)\,.$$

Therefore, $S_N P J_M$ is the pointwise limit of $(S_i P_i R_i)_{i \in I}$ along $\mathcal{U}$ in the finite-dimensional space $\mathcal{P}(^m M, (F_i)_{\mathcal{U}}/N)$ and so it is its limit for the topology generated by $\boldsymbol{Q}$. Now, since

$$\boldsymbol{Q}(S_i P_i R_i) \leq \|S_i\| \boldsymbol{Q}(P_i) \|R_i\| \leq 1,$$

and the λ-norm $\boldsymbol{Q}$ is continuous for the topology generated by it [DF], page 109, it follows that $\boldsymbol{Q}(S_N P J_M) \leq 1$, and we conclude that $(\mathcal{Q}, \boldsymbol{Q})$ is ultrastable.

We now prove that $(\mathcal{Q}, \boldsymbol{Q})$ is regular. Let $P \in \mathcal{P}(^m E, F)$ with $k_F \circ P \in \mathcal{Q}(E, F^{**})$. Given $M \in \mathrm{Dim}(E)$ and $N \in \mathrm{Cod}(F)$, we have

$$\boldsymbol{Q}\left(S_N^F P J_M^E\right) = \boldsymbol{Q}\left(\left(S_N^F\right)^{**} k_F P J_M^E\right) \leq \left\|S_N^F\right\| \boldsymbol{Q}(k_F P) \left\|J_M^E\right\| = \boldsymbol{Q}(k_F P)\,.$$

Then $\boldsymbol{Q}^{\max}(P) \leq \boldsymbol{Q}(k_F P) < \infty$, so $P \in \mathcal{Q}^{\max} = \mathcal{Q}$ and $\boldsymbol{Q}(P) = \boldsymbol{Q}^{\max}(P) \leq \boldsymbol{Q}(k_F P)$. It is obvious that $\boldsymbol{Q}(k_F P) \leq \boldsymbol{Q}(P)$, so $\boldsymbol{Q}(P) = \boldsymbol{Q}(k_F P)$. □

THEOREM (3.8). *If $(\mathcal{Q}, \boldsymbol{Q})$ is a maximal λ-normed ideal of m-homogeneous polynomials, then $\mathcal{Q}(E, F)$ is complete for all Banach spaces E and F.*

Proof. The proof is as in [F2], 1.5. □

4. Ideals of nuclear and integral polynomials

In this Section, we show that the ideal of nuclear polynomials is the smallest Banach ideal of polynomials, and that the ideal of integral polynomials is the smallest maximal normed ideal of polynomials.

We also prove that the ideal of Pietsch integral polynomials is ultrastable.

These results were known for scalar-valued polynomials, but the extension to the vector-valued setting in the case of (Pietsch) integral polynomials is not trivial.

Given a λ-normed ideal of polynomials $(\mathcal{A}, \boldsymbol{A})$ and a λ'-normed ideal of polynomials $(\mathcal{B}, \boldsymbol{B})$, where $0 < \lambda, \lambda' \leq 1$, we say that $(\mathcal{A}, \boldsymbol{A})$ is *contained* in $(\mathcal{B}, \boldsymbol{B})$ if $\mathcal{A} \subseteq \mathcal{B}$ and $\boldsymbol{B}(P) \leq \boldsymbol{A}(P)$ for all $P \in \mathcal{A}$ [Pi1], 6.7.1.

The following result was proved in [F2], 1.6, for scalar-valued polynomials. The proof in the vector-valued case is essentially the same.

THEOREM (4.1). *$(\mathcal{P}_{\mathrm{N}}, \|\cdot\|_{\mathrm{N}})$ is the smallest Banach ideal of polynomials.*

Proof. It is easy to see that $(\mathcal{P}_{\mathrm{N}}, \|\cdot\|_{\mathrm{N}})$ is a Banach ideal of polynomials, so we have to prove that, if $(\mathcal{A}, \boldsymbol{A})$ is a Banach ideal of m-homogeneous polynomials and $P \in \mathcal{P}_{\mathrm{N}}(^m E, F)$, then $P \in \mathcal{A}$ and $\boldsymbol{A}(P) \leq \|P\|_{\mathrm{N}}$.

There are bounded sequences $(\phi_i) \subset E^*$ and $(y_i) \subset F$ such that

$$P(x) = \sum_{i=1}^{\infty} [\phi_i(x)]^m y_i$$

for every $x \in E$, and $\sum_{i=1}^{\infty} \|\phi_i\|^m \|y_i\| < \infty$. For each $r \in \mathbb{N}$, let $P_r := \sum_{i=1}^{r} \phi_i^m \underline{\otimes} y_i$. Clearly, $P_r \in \mathcal{A}$. For $r, s \in \mathbb{N}$ $(r < s)$, we have

$$A(P_s - P_r) = A\left(\sum_{i=r+1}^{s} \phi_i^m \underline{\otimes} y_i\right) \leq \sum_{i=r+1}^{s} A\left(\phi_i^m \underline{\otimes} y_i\right) = \sum_{i=r+1}^{s} \|\phi_i\|^m \|y_i\| \xrightarrow[r,s\to\infty]{} 0\,,$$

so $(P_r)_{r=1}^{\infty}$ is a Cauchy sequence in the Banach space $\mathcal{A}(E, F)$, which is therefore convergent to some $R \in \mathcal{A}(E, F)$. Obviously, $P_r \to R$ also in the supremum norm, so $R = P$, and $P \in \mathcal{A}(E, F)$. Moreover

$$A(P) = \lim_{r\to\infty} A(P_r) \leq \lim_{r\to\infty} \sum_{i=1}^{r} A\left(\phi_i^m \underline{\otimes} y_i\right) = \lim_{r\to\infty} \sum_{i=1}^{r} \|\phi_i\|^m \|y_i\| = \sum_{i=1}^{\infty} \|\phi_i\|^m \|y_i\|\,.$$

It follows that $A(P) \leq \|P\|_{\mathrm{N}}$. □

The following two results were proved in [FH], Theorem 3.2, for scalar-valued polynomials, using the fact that the space $\mathcal{P}_{\mathrm{PI}}(^mE) = \mathcal{P}_{\mathrm{I}}(^mE)$ is isometrically isomorphic to the dual space $\left(\otimes_{\epsilon_s,s}^m E\right)^*$, a property which is, obviously, not true in the vector-valued case.

THEOREM (4.2). *The ideal of polynomials* $\left(\mathcal{P}_{\mathrm{PI}}, \|\cdot\|_{\mathrm{PI}}\right)$ *is ultrastable.*

Proof. Take a set I and an ultrafilter $\mathcal{U}$ on I. For each $i \in I$, let $P_i \in \mathcal{P}_{\mathrm{PI}}(^mE_i, F_i)$ be a polynomial with $\sup_i \|P_i\|_{\mathrm{PI}} < \infty$. We have to prove that $(P_i)_{\mathcal{U}}$ is a Pietsch integral polynomial and that $\|(P_i)_{\mathcal{U}}\|_{\mathrm{PI}} \leq \lim_{\mathcal{U}} \|P_i\|_{\mathrm{PI}}$. Let $\epsilon > 0$. For every $i \in I$, by Theorem (2.3), we can find a finite nonnegative countably additive, Borel measure μ_i on $B_{E_i^*}$ and an operator $S_i \in \mathcal{L}\left(L_1\left(B_{E_i^*}, \mu_i\right), F_i\right)$ such that the following diagram is commutative

$$\begin{array}{ccc} E_i & \xrightarrow{P_i} & F_i \\ {\scriptstyle R_i}\big\downarrow & & \big\uparrow{\scriptstyle S_i} \\ C\left(B_{E_i^*}\right) & \xrightarrow[J_i]{} & L_1\left(B_{E_i^*}, \mu_i\right) \end{array}$$

where J_i is the canonical inclusion, and R_i is the polynomial defined by

$$R_i(x)(x^*) = [x^*(x)]^m \quad \text{for all} \quad x \in E_i\,,\ x^* \in B_{E_i^*}\,.$$

Moreover, we can assume that

$$\|S_i\| \leq 1 \quad \text{and} \quad \mu_i(B_{E_i^*}) = \|J_i\|_{\mathrm{as}} \leq \|P_i\|_{\mathrm{PI}} + \epsilon\,.$$

Hence, $\sup_{i\in I} \|J_i\|_{\mathrm{as}} < \infty$. Since the ideal of absolutely summing operators is ultrastable [Pi1], 8.8.11, the operator $(J_i)_{\mathcal{U}}$ is absolutely summing, and

$$\|(J_i)_{\mathcal{U}}\|_{\mathrm{as}} \leq \lim_{\mathcal{U}} \|J_i\|_{\mathrm{as}}\,.$$

Taking ultraproducts, the upper half of the diagram of Figure (1) is commutative.

By [H], Theorem 3.3, there are a compact Hausdorff space K, and a multiplicative surjective isometric isomorphism

$$\Phi\colon C(K) \longrightarrow \left(C(B_{E_i^*})\right)_{\mathcal{U}} .$$

Since $(J_i)_{\mathcal{U}} \circ \Phi$ is an absolutely summing operator, there exist a nonnegative countably additive, Borel measure μ on K and an operator

$$b \in \mathcal{L}\left(L_1(K,\mu), \left(L_1\left(B_{E_i^*}, \mu_i\right)\right)_{\mathcal{U}}\right)$$

such that $(J_i)_{\mathcal{U}} \circ \Phi = b \circ J'$, where $J' \in \mathcal{L}(C(K), L_1(\mu))$ is the natural inclusion [DU], Corollary VI.3.7. Therefore, the diagram of Figure (1) is commutative. Moreover, we can suppose that $\|b\| \leq 1$ and

$$\mu(K) = \|(J_i)_{\mathcal{U}} \circ \Phi\|_{\text{as}} \leq \|(J_i)_{\mathcal{U}}\|_{\text{as}} \leq \lim_{\mathcal{U}} \|P_i\|_{\text{PI}} + \epsilon \,.$$

$$\begin{tikzcd}
(E_i)_{\mathcal{U}} \arrow[r, "(P_i)_{\mathcal{U}}"] \arrow[d, "(R_i)_{\mathcal{U}}"'] & (F_i)_{\mathcal{U}} \\
(C(B_{E_i^*}))_{\mathcal{U}} \arrow[r, "(J_i)_{\mathcal{U}}"] \arrow[d, shift left, "\Phi"] & (L_1(B_{E_i^*}, \mu_i))_{\mathcal{U}} \arrow[u, "(S_i)_{\mathcal{U}}"'] \\
C(K) \arrow[r, "J'"'] \arrow[u, shift left, "\Phi^{-1}"] & L_1(K, \mu) \arrow[u, "b"']
\end{tikzcd}$$

Figure 1. Ultrastability of the ideal of Pietsch integral polynomials

Let

$$k_i \in \mathcal{L}\left(E_i, C\left(B_{E_i^*}\right)\right)$$

be the canonical isometric embedding given by $k_i(x_i)(x_i^*) := x_i^*(x_i)$ for all $x_i \in E_i$ and $x_i^* \in B_{E_i^*}$. For every $x = (x_i)_{\mathcal{U}} \in (E_i)_{\mathcal{U}}$, we have

$$\begin{aligned}
\left(\Phi^{-1} \circ (R_i)_{\mathcal{U}}\right)(x) &= \Phi^{-1}\left((R_i(x_i))_{\mathcal{U}}\right) \\
&= \Phi^{-1}\left(\left([k_i(x_i)]^m\right)_{\mathcal{U}}\right) \\
&= \Phi^{-1}\left(\left[(k_i(x_i))_{\mathcal{U}}\right]^m\right) \quad \text{(see the proof of [H], Proposition 3.1)} \\
&= \left[\Phi^{-1}\left((k_i(x_i))_{\mathcal{U}}\right)\right]^m \quad \text{(since } \Phi^{-1} \text{ is also multiplicative)} \\
&= \left[\Phi^{-1}\left((k_i)_{\mathcal{U}}(x)\right)\right]^m \\
&= \left[\left(\Phi^{-1} \circ (k_i)_{\mathcal{U}}\right)(x)\right]^m .
\end{aligned}$$

Since

$$\Phi^{-1} \circ (k_i)_{\mathcal{U}} \colon (E_i)_{\mathcal{U}} \longrightarrow C(K)$$

is an isometric embedding, we conclude by Theorem (2.3) that $(P_i)_{\mathcal{U}}$ is Pietsch integral. Moreover,

$$\|(P_i)_{\mathcal{U}}\|_{\text{PI}} \leq \left\|\Phi^{-1} \circ (R_i)_{\mathcal{U}}\right\| \mu(K) = \mu(K) \leq \lim_{\mathcal{U}} \|P_i\|_{\text{PI}} + \epsilon \,.$$

Since $\epsilon > 0$ is arbitrary, we obtain

$$\|(P_i)_{\mathcal{U}}\|_{\mathrm{PI}} \leq \lim_{\mathcal{U}} \|P_i\|_{\mathrm{PI}}\,,$$

and the result is proved. □

THEOREM (4.3). *The ideal of polynomials $\left(\mathcal{P}_{\mathrm{I}}, \|\cdot\|_{\mathrm{I}}\right)$ is maximal.*

Proof. Let $P \in \mathcal{P}(^m E, F)$ with

$$\|P\|_{\mathrm{I}}^{\max} = \sup\left\{\left\|S_N^F \circ P \circ J_M^E\right\|_{\mathrm{I}} :\ M \in \mathrm{Dim}(E),\ N \in \mathrm{Cod}(F)\right\} < \infty\,.$$

We have to prove that P is integral and that

$$\|P\|_{\mathrm{I}}^{\max} = \|P\|_{\mathrm{I}}\,.$$

By Lemma (3.3), with its notation, there exist operators $J\colon E \to (E_i)_{\mathcal{U}}$ and $H\colon (F_i)_{\mathcal{U}} \to F^{**}$ with $\|J\| \leq 1$ and $\|H\| \leq 1$, such that $k_F \circ P = H \circ (P_i)_{\mathcal{U}} \circ J$, where $P_i := S_N^F \circ P \circ J_M^E$ are the elementary parts of P, for $i \in I$. Since P_i is a polynomial between finite-dimensional spaces, it is Pietsch integral. Moreover, we have

$$\|P_i\|_{\mathrm{PI}} = \|P_i\|_{\mathrm{I}} = \left\|S_N^F \circ P \circ J_M^E\right\|_{\mathrm{I}} \leq \|P\|_{\mathrm{I}}^{\max}$$

so $\left(\|P_i\|_{\mathrm{PI}}\right)_{i\in I}$ is bounded. By Theorem (4.2), the polynomial $(P_i)_{\mathcal{U}}$ is Pietsch integral and

$$\|(P_i)_{\mathcal{U}}\|_{\mathrm{PI}} \leq \lim_{\mathcal{U}} \|P_i\|_{\mathrm{PI}} \leq \|P\|_{\mathrm{I}}^{\max}\,.$$

¿From the equality $k_F \circ P = H \circ (P_i)_{\mathcal{U}} \circ J$, we obtain that $k_F \circ P$ is Pietsch integral. By Theorem (2.4), P is integral and

$$\|P\|_{\mathrm{I}} = \|k_F \circ P\|_{\mathrm{PI}} \leq \|(P_i)_{\mathcal{U}}\|_{\mathrm{PI}} \leq \|P\|_{\mathrm{I}}^{\max}\,.$$

On the other hand,

$$\left\|S_N^F \circ P \circ J_M^E\right\|_{\mathrm{I}} \leq \left\|S_N^F\right\| \|P\|_{\mathrm{I}} \left\|J_M^E\right\| \leq \|P\|_{\mathrm{I}}\,,$$

so

$$\|P\|_{\mathrm{I}}^{\max} \leq \|P\|_{\mathrm{I}}\,,$$

and the proof is complete. □

The following result is given in [F2], 1.6.(c), for scalar-valued polynomials, using the representation

$$\mathcal{P}_{\mathrm{I}}(^m E) \equiv \left(\otimes_{\epsilon_s, s}^m E\right)^*\,.$$

THEOREM (4.4). *The ideal $\left(\mathcal{P}_{\mathrm{I}}, \|\cdot\|_{\mathrm{I}}\right)$ is the smallest maximal normed ideal of polynomials.*

Proof. Let $(\mathcal{A}, \boldsymbol{A})$ be a maximal normed ideal of m-homogeneous polynomials, and let $P \in \mathcal{P}_{\mathrm{I}}(^m E, F)$. By Lemma (3.3), we may write $k_F \circ P = H \circ (P_i)_{\mathcal{U}} \circ J$ with $\|H\| \leq 1$, $\|J\| \leq 1$, and $P_i := S_N^F \circ P \circ J_M^E$, for $i \in I$. Since P_i is a polynomial between finite-dimensional spaces, it belongs to $\mathcal{A}$ and $\boldsymbol{A}(P_i) \leq \|P_i\|_{\mathrm{N}}$, for $i \in I$ (Theorem (4.1)). Moreover, since P is integral,

$$\|P_i\|_{\mathrm{N}} = \left\|S_N^F \circ P \circ J_M^E\right\|_{\mathrm{N}} = \left\|S_N^F \circ P \circ J_M^E\right\|_{\mathrm{I}} \leq \|P\|_{\mathrm{I}}$$

for all $i \in I$, so $A(P_i) \leq \|P\|_{\mathrm{I}}$ for all $i \in I$ (for the equality of norms, see [CD], Theorem 1.4). Since $(\mathcal{A}, A)$ is maximal, it is also ultrastable (Theorem (3.5)), so $(P_i)_{\mathcal{U}} \in \mathcal{A}$ and

$$A((P_i)_{\mathcal{U}}) \leq \lim_{\mathcal{U}} A(P_i) \leq \|P\|_{\mathrm{I}} .$$

By the ideal property (Definition (3.2),(b′)), $k_F \circ P \in \mathcal{A}$ and

$$A(k_F \circ P) = A(H \circ (P_i)_{\mathcal{U}} \circ J) \leq \|H\| A\left((P_i)_{\mathcal{U}}\right) \|J\| \leq A\left((P_i)_{\mathcal{U}}\right) \leq \|P\|_{\mathrm{I}} .$$

Since $(\mathcal{A}, A)$ is also regular (Theorem (3.5)), it follows that $P \in \mathcal{A}$ and $A(P) = A(k_F \circ P) \leq \|P\|_{\mathrm{I}}$, which finishes the proof. □

5. Left and right r-factorable polynomials

In this Section, we introduce the ideals of left and right r-factorable polynomials, showing that they are λ-normed and maximal. As a consequence, we prove that, for $1 \leq r < m$, in most cases, the ideal $\left(\mathcal{P}_r^{m,\text{right}}, \gamma_r^{\text{right}}\right)$ is not normed. This proves a conjecture of K. Floret [F2], 3.1.

Definition (5.1). Given a polynomial $P \in \mathcal{P}(^mE, F)$ and $1 \leq r \leq \infty$, we say that P is *left r-factorable* if there exist a positive measure space (Ω, Σ, μ), a polynomial $Q \in \mathcal{P}(^mE, L_r(\mu))$, and an operator $T \in \mathcal{L}(L_r(\mu), F^{**})$ such that $k_F \circ P = T \circ Q$.

$$\begin{array}{ccc} E & \xrightarrow{P} & F \\ {\scriptstyle Q}\big\downarrow & & \big\downarrow{\scriptstyle k_F} \\ L_r(\mu) & \xrightarrow[T]{} & F^{**} \end{array}$$

In this case we set

$$\gamma_r^{\text{left}}(P) := \inf\{\|Q\|\|T\| \text{ for } Q, T \text{ as above}\} .$$

We denote by $\mathcal{P}_r^{m,\text{left}}(E, F)$ the subspace of all $P \in \mathcal{P}(^mE, F)$ which are left r-factorable.

Definition (5.2). Given a polynomial $P \in \mathcal{P}(^mE, F)$ and $1 \leq r \leq \infty$, we say that P is *right r-factorable* if there exist a positive measure space (Ω, Σ, μ), a polynomial $Q \in \mathcal{P}(^mL_r(\mu), F^{**})$, and an operator $T \in \mathcal{L}(E, L_r(\mu))$ such that $k_F \circ P = Q \circ T$.

$$\begin{array}{ccc} E & \xrightarrow{P} & F \\ {\scriptstyle T}\big\downarrow & & \big\downarrow{\scriptstyle k_F} \\ L_r(\mu) & \xrightarrow[Q]{} & F^{**} \end{array}$$

In this case we set

$$\gamma_r^{\text{right}}(P) := \inf\{\|Q\|\|T\|^m \text{ for } Q, T \text{ as above}\} .$$

We denote by $\mathcal{P}_r^{m,\text{right}}(E, F)$ the subspace of all $P \in \mathcal{P}(^mE, F)$ which are right r-factorable.

The proof of the following result is as in [F2], 3.1.

Proposition (5.3). *The ideal of polynomials* $\left(\mathcal{P}_r^{m,\text{right}}, \gamma_r^{\text{right}}\right)$ *is* r/m*-normed if* $r < m$ *and is normed if* $r \geq m$.

Proposition (5.4). *The ideal of polynomials* $\left(\mathcal{P}_r^{m,\text{right}}, \gamma_r^{\text{right}}\right)$ *is maximal.*

Proof. It is enough to prove that $\left(\mathcal{P}_r^{m,\text{right}}, \gamma_r^{\text{right}}\right)$ satisfies condition (b) of Theorem (3.5).

Let I be a set and let $\mathcal{U}$ be an ultrafilter on I. For every $i \in I$, let $P_i \in \mathcal{P}(^m E_i, F_i)$ be a right r-factorable polynomial, where E_i and F_i are finite-dimensional Banach spaces, and suppose that $\sup_i \gamma_r^{\text{right}}(P_i) < \infty$. Given $\epsilon > 0$, we can find positive measure spaces $(\Omega_i, \Sigma_i, \mu_i)$, operators $T_i \in \mathcal{L}(E_i, L_r(\mu_i))$, and polynomials $R_i \in \mathcal{P}(^m L_r(\mu_i), F_i)$ $(i \in I)$ such that the following diagrams commute

$$\begin{tikzcd} E_i \arrow[rr, "P_i"] \arrow[dr, "T_i"'] & & F_i \\ & L_r(\mu_i) \arrow[ur, "R_i"'] & \end{tikzcd}$$

and $\|R_i\|\|T_i\|^m \leq \gamma_r^{\text{right}}(P_i) + \epsilon$. We can assume that the families $(\|R_i\|)_{i\in I}$ and $(\|T_i\|)_{i\in I}$ are bounded, and then so is the family $(\|P_i\|)_{i\in I}$. Hence, taking ultraproducts, we obtain the following commutative diagram

$$\begin{tikzcd} (E_i)_{\mathcal{U}} \arrow[rr, "(P_i)_{\mathcal{U}}"] \arrow[dr, "(T_i)_{\mathcal{U}}"'] & & (F_i)_{\mathcal{U}} \\ & (L_r(\mu_i))_{\mathcal{U}} \arrow[ur, "(R_i)_{\mathcal{U}}"'] & \end{tikzcd}$$

Since $(L_r(\mu_i))_{\mathcal{U}}$ is isometrically isomorphic to an $L_r(\nu)$ space for some measure ν [Pi1], Lemma 19.3.4, which can be assumed to be positive [DS], IV.8, it follows that $(P_i)_{\mathcal{U}}$ is right r-factorable. Moreover

$$\gamma_r^{\text{right}}((P_i)_{\mathcal{U}}) \leq \|(R_i)_{\mathcal{U}}\|\|(T_i)_{\mathcal{U}}\|^m = \lim_{\mathcal{U}}\|R_i\|\|T_i\|^m \leq \lim_{\mathcal{U}} \gamma_r^{\text{right}}(P_i) + \epsilon\,.$$

Therefore,

$$\gamma_r^{\text{right}}((P_i)_{\mathcal{U}}) \leq \lim_{\mathcal{U}} \gamma_r^{\text{right}}(P_i),$$

and $\left(\mathcal{P}_r^{m,\text{right}}, \gamma_r^{\text{right}}\right)$ satisfies the second condition of Theorem (3.5),(b).

We now prove that $\left(\mathcal{P}_r^{m,\text{right}}, \gamma_r^{\text{right}}\right)$ is regular. Let $P \in \mathcal{P}(^m E, F)$ be a polynomial such that

$$k_F \circ P \in \mathcal{P}_r^{m,\text{right}}(E, F^{**})\,.$$

Then there exist a positive measure space (Ω, Σ, μ), an operator $T \in \mathcal{L}(E, L_r(\mu))$, and a polynomial

$$R \in \mathcal{P}(^m L_r(\mu), F^{****})$$

such that $k_{F^{**}} \circ k_F \circ P = R \circ T$ (Figure (2)). Since $(k_{F^*})^* \circ k_{F^{**}} = I_{F^{**}}$ [Me], Example 3.1.8, we have

$$k_F \circ P = (k_{F^*})^* \circ k_{F^{**}} \circ k_F \circ P = (k_{F^*})^* \circ R \circ T\,.$$

Therefore, P is right r-factorable

$$
\begin{tikzcd}
E \arrow[r, "P"] \arrow[d, "T"'] & F \arrow[r, "k_F"] & F^{**} \arrow[d, shift right, "k_{F^{**}}"'] \\
L_r(\mu) \arrow[rr, "R"] & & F^{****} \arrow[u, shift right, "(k_{F^*})^*"']
\end{tikzcd}
$$

Figure 2. Regularity of the ideal of right r-factorable polynomials

By the ideal property,

$$\gamma_r^{\text{right}}(k_F \circ P) \leq \|k_F\| \gamma_r^{\text{right}}(P) = \gamma_r^{\text{right}}(P) \,.$$

On the other hand, we have

$$\gamma_r^{\text{right}}(P) \leq \|(k_{F^*})^* \circ R\| \|T\|^m \leq \|R\| \|T\|^m$$

and then

$$\gamma_r^{\text{right}}(P) \leq \gamma_r^{\text{right}}(k_F \circ P) \,.$$

So we obtain the equality $\gamma_r^{\text{right}}(P) = \gamma_r^{\text{right}}(k_F \circ P)$, and we are done. □

PROPOSITION (5.5). *The ideal of polynomials* $\left(\mathcal{P}_r^{m,\text{left}}, \gamma_r^{\text{left}}\right)$ *is a maximal Banach ideal.*

Proof. Clearly, a polynomial $P \in \mathcal{P}(^mE, F)$ is left r-factorable if and only if its linearization

$$\overline{P} \in \mathcal{L}\left(\otimes_{\pi_s, s}^m E, F\right)$$

is an r-factorable operator. Since, for every polynomial Q on E, we have $\|Q\| = \left\|\overline{Q}\right\|$ (see Section 1), we obtain that $\gamma_r^{\text{left}}(P) = \gamma_r\left(\overline{P}\right)$, where γ_r is the r-factorable norm of an operator (see [DJT], Chapter 7). Since the r-factorable operators, with the γ_r norm, form a Banach ideal [DJT], Theorem 7.1, the ideal of polynomials $\left(\mathcal{P}_r^{m,\text{left}}, \gamma_r^{\text{left}}\right)$ is also a Banach ideal.

With obvious modifications in the arguments of Proposition (5.4), we see that $\left(\mathcal{P}_r^{m,\text{left}}, \gamma_r^{\text{left}}\right)$ is maximal. □

We will see (Theorem (5.10)) that $\left(\mathcal{P}_r^{m,\text{right}}, \gamma_r^{\text{right}}\right)$ is not normed in general if $r < m$. In order to prove this fact we need some preparatory results.

PROPOSITION (5.6). *Let E be an $\mathcal{L}_\infty$-space and let $P \in \mathcal{P}(^mE, F)$ be a right r-factorable polynomial. Then, if $1 \leq r \leq 2$, P is 2-dominated while, if $r > 2$, P is s-dominated for every $s > r$.*

Proof. Assume first that $1 \leq r \leq 2$. Then there are a positive measure space (Ω, Σ, μ), a polynomial $Q \in \mathcal{P}(^mL_r(\mu), F^{**})$ and an operator $T \in \mathcal{L}(E, L_r(\mu))$ such that $k_F \circ P = Q \circ T$. Since $L_r(\mu)$ has cotype 2 [DJT], Corollary 11.7, T is absolutely 2-summing [DJT], Theorem 11.14, so $k_F \circ P$ is 2-dominated (see, for instance, [MT], Theorem 10) and P is 2-dominated too. Suppose now that $r > 2$ and factor $k_F \circ P$ as above. Since $L_r(\mu)$ has cotype $r > 2$ [DJT], Corollary 11.7, it follows that T is s-summing for every $s > r$ [DJT], Theorem 11.14, and then P is s-dominated. □

PROPOSITION (5.7). *Given* $\lambda = (\lambda_n)_{n=1}^\infty \in \ell_1$, *let* $M_\lambda \in \mathcal{P}({}^m\ell_\infty, \ell_1)$ *be the polynomial defined by*

$$M_\lambda(x) := (\lambda_n x_n^m)_{n=1}^\infty \qquad \text{for all} \quad x = (x_n)_{n=1}^\infty \in \ell_\infty \, ,$$

and let $1 \le s \le m$. *Then* M_λ *is* s*-dominated if and only if* $\lambda \in \ell_{s/m}$.

Proof. The proof is a slight modification of the proof of [CG1], Theorem 11. We include it for completeness.

Suppose that $\lambda \in \ell_{s/m}$. If the field is complex, write $\lambda_n = |\lambda_n| e^{i\theta_n}$. Define an operator $T \in \mathcal{L}(\ell_\infty, \ell_s)$ by

$$T(x) = \left(|\lambda_n|^{1/m} e^{i\theta_n/m} x_n \right)_{n=1}^\infty \qquad \text{for all} \quad x = (x_n)_{n=1}^\infty \in \ell_\infty \, .$$

Then T is absolutely s-summing [DJT], Examples 2.9.

Denoting by $i\colon \ell_s \to \ell_m$ the natural inclusion, let $P \in \mathcal{P}({}^m\ell_m, \ell_1)$ be a polynomial given by

$$P(y) := \left(y_n^m \right)_{n=1}^\infty \quad \text{for} \quad y = (y_n)_{n=1}^\infty \in \ell_m \, .$$

Since $M_\lambda = P \circ i \circ T$, M_λ is s-dominated [MT], Theorem 10.

If the field is real, we write $M_\lambda = M_\mu - M_\nu$, with $\mu = (\mu_n)$ and $\nu = (\nu_n)$, where $\mu_n \ge 0$ and $\nu_n \ge 0$ for all n. By the above argument, M_μ and M_ν are s-dominated, and so is M_λ [MT], page 196.

Conversely, suppose that M_λ is s-dominated. Let

$$e_i = \Bigl(0, \dots, 0, \underset{(i)}{1}, 0, \dots \Bigr) \, .$$

Since the sequence $(e_i)_{i=1}^\infty$ is weakly s-summable in ℓ_∞, then the sequence

$$\left(M_\lambda(e_i) \right)_{i=1}^\infty$$

is absolutely s/m-summable in ℓ_1 [CG2], page 422, so

$$\sum_{i=1}^\infty |\lambda_i|^{s/m} = \sum_{i=1}^\infty \|M_\lambda(e_i)\|^{s/m} < \infty \, ,$$

and $\lambda \in \ell_{s/m}$. □

PROPOSITION (5.8). *For every integer* $m > 2$ *there is a polynomial* $P \in \mathcal{P}_{\mathrm{N}}({}^m\ell_\infty, \ell_1)$ *which is not right* r*-factorable for any* $1 \le r < m$.

Proof. Let $1 \le r < m$. Choose $\max(r, 2) < s < m$ and define a polynomial $P \in \mathcal{P}({}^m\ell_\infty, \ell_1)$ by $P(x) = (\lambda_n x_n^m)_{n=1}^\infty$ for all $x = (x_n)_{n=1}^\infty \in \ell_\infty$, where $(\lambda_n)_n \in \ell_1 \backslash \ell_{s/m}$. Since $(\lambda_n)_n \in \ell_1$, P is nuclear [CG1], Proposition 6. If P were right r-factorable, it would be s-dominated (by Proposition (5.6)) and then, by Proposition (5.7), $(\lambda_n)_n \in \ell_{s/m}$, a contradiction. □

It is even possible to find scalar-valued polynomials satisfying the requirements of Proposition (5.8), as the following result shows.

PROPOSITION (5.9). *For any integer* $m > 2$, *there is a polynomial* $P \in \mathcal{P}_{\mathrm{N}}({}^m\ell_\infty)$ *which is not right* r*-factorable for any* $1 \le r < m$.

Proof. Given $1 \le r < m$, choose s with

$$\max(r, 2) < s < m\,.$$

Let $(a_k)_{k=1}^\infty \in \ell_1 \backslash \ell_{s/m}$. Define a polynomial $P \in \mathcal{P}(^m\ell_\infty)$ by

$$P(x) = \sum_{k=1}^\infty a_k x_k^m \qquad \text{for} \quad x = (x_k)_{k=1}^\infty \in \ell_\infty\,.$$

By [CG1], Proposition 6, P is nuclear. Assume that P is right r-factorable. By Proposition (5.6), P is s-dominated. Since the sequence $(e_k)_{k=1}^\infty$ is weakly s-summable in ℓ_∞, the sequence

$$(a_k)_{k=1}^\infty = \big(P(e_k)\big)_{k=1}^\infty$$

is s/m-summable, which is in contradiction with the hypothesis. □

THEOREM (5.10).

(a) *For every integer $m > 2$ and every real number r with $1 \le r < m$, the ideal of polynomials $\left(\mathcal{P}_r^{m,\text{right}}, \gamma_r^{\text{right}}\right)$ is not normed.*

(b) *The ideal of polynomials $\left(\mathcal{P}_1^{2,\text{right}}, \gamma_1^{\text{right}}\right)$ is not normed.*

Proof. (a) Suppose $m > 2$ and $1 \le r < m$. If $\left(\mathcal{P}_r^{m,\text{right}}, \gamma_r^{\text{right}}\right)$ were normed, it would contain all the nuclear m-homogeneous polynomials (see Theorem (3.8), Proposition (5.4), and Theorem (4.1)), and this is false by Proposition (5.8) (or (5.9)).

(b) Suppose now $m = 2$. Consider the scalar-valued polynomial $P \in \mathcal{P}(^2C[0,1])$ given by

$$P(f) = \int_0^1 f(t)^2\,dt\,.$$

This polynomial is integral [A2], Remark 2.4. If it were right 1-factorable, we could find a positive measure space (Ω, Σ, μ), an operator $A \in \mathcal{L}(C[0,1], L_1(\mu))$ and a polynomial $Q \in \mathcal{P}(^2L_1(\mu))$ such that $P = Q \circ A$. In this case, the following diagram commutes

$$\begin{tikzcd} C[0,1] \arrow[r, "T_P"] \arrow[d, "A"'] & C[0,1]^* \\ L_1(\mu) \arrow[r, "T_Q"'] & L_\infty(\mu) \arrow[u, "A^*"'] \end{tikzcd}$$

where T_P and T_Q are the operators associated to P and Q, respectively (see definition in Section 1). Since A and A^* are absolutely 2-summing [DJT], Theorem 3.7, the operator T_P is nuclear [DJT], Theorem 5.31, and hence compact, which is in contradiction with [A2], Remark 2.4. □

Remarks (5.11).

(a) The polynomials of Proposition (5.8) are m-dominated [CG2], Lemma 1, but they are not 2-dominated, unlike the linear operators with range in a space with cotype 2 [DJT], Theorem 11.13.

(b) The polynomials of Proposition (5.8) are left r-factorable for every $r \geq 1$ by Proposition (5.5) and Theorem (4.4), and right r-factorable for every $r \geq m$ by Proposition (5.3) and Theorem (4.4).

(c) The polynomial given in the proof of Theorem (5.10),(b) is left r-factorable for every $r \geq 1$ by Proposition (5.5) and Theorem (4.4). It is also right r-factorable for every $r \geq 2$ by Proposition (5.3) and Theorem (4.4), and then it is also right r-factorable for every $r > 1$ by [DJT], Corollary 9.2.

(d) Since

$$\mathcal{P}_{\mathrm{I}}(^2E, F) \subseteq \mathcal{P}_2^{2,\mathrm{right}}(E, F)$$

by Propositions (5.3) and (5.4) and Theorem (4.4), and

$$\mathcal{P}_2^{2,\mathrm{right}}(E, F) \subseteq \mathcal{P}_r^{2,\mathrm{right}}(E, F)$$

for all $1 < r < \infty$, by [DJT], Corollary 9.2, we have that every 2-homogeneous integral polynomial is right r-factorable for all $1 < r < \infty$, so the argument of Theorem (5.10) does not allow us to know whether the ideal $\left(\mathcal{P}_r^{2,\mathrm{right}}, \gamma_r^{\mathrm{right}}\right)$ is normed or not for $1 < r < 2$.

Added in proof

(1) In the proof of Theorem (2.3), part (d) $\Rightarrow$ (e), we use Lemma (2.2). In fact what we need is a version of Lemma (2.2) valid for $L_\infty(\Omega, \mu)$ spaces instead of $C(K)$. A proof of such a result will be given in a forthcoming paper by the authors dealing with p-integral polynomials.

(2) Theorem (4.1) is proved in 1.7, Proposition of K. Floret, Minimal ideals of n-homogeneous polynomials on Banach spaces, *Results Math.* **39** (2001), 201–217.

(3) The equality of the Pietsch integral norms of P and $\overline{P}$ in Theorem (2.3) is given in [CL1, Proposition 2.10].

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EXTENDED AFFINE SURFACE AREA AND ZONOTOPES

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ABSTRACT. The L_p-polar curvature image is introduced and studied. An equivalent relationship between extended mixed affine surface areas and mixed volumes of arbitrary convex bodies is presented, so is its L_p analog. For the inequalities whose conditions involve the areas of the projections of convex bodies, an approach to the extensions from the ordinary forms to their L_p versions is presented. As applications of this approach, we establish Aleksandrov's projection theorem and Petty-Schneider theorem for the L_p-mixed Quermassintegral version.

1. Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies, $\mathcal{F}^n$ denote the set of convex bodies having a positive continuous curvature functions, and Ω denote the affine surface area (see Section 2). In [11], Leichtweiß presented a definition of extended affine surface area which extends the domain of affine surface area from $\mathcal{F}^n$ to $\mathcal{K}^n$. Since then, a number of geometric inequalities and various isoperimetric inequalities involving affine surface area have been extended to arbitrary convex bodies (see, for example, [17]). In addition, Lutwak [20] showed that there are natural extensions of affine and geominimal surface areas in the Brunn-Minkowski-Firey theory. Surprisingly, it turns out that there are extensions of all of the known inequalities involving affine and geominimal surface areas to the p-affine and p-geominimal surface areas (see, for example, [14]).

One of the aims of this article is to introduce and study the p-polar curvature images, another main one is to show that for a special class $\mathcal{W}^n$(or $\mathcal{W}_p^n$) which is a subset of the set of convex bodies, there exists an equivalent relationship between the extended mixed affine surface area, of an arbitrary convex body and a body in $\mathcal{W}^n$(or $\mathcal{W}_p^n$), and the mixed volume, of an arbitrary convex body and a line segment. The approach to the later is closely related to the properties of zonotopes, and in fact, with this approach most of the problems involving the area of projections of convex bodies will be remarkably promoted.

A result of Winterniz (see [2], page 200) states that if $K \in \mathcal{F}^n$ (actually a somewhat more restricted class) and E is an ellipsoid such that $K \subset E$, then $\Omega(K) \leq \Omega(E)$. There are many generalization of this result, for example, in [15], Lutwak extended it as: if $K \in \mathcal{K}^n$, and E is an ellipsoid, then if the areas of the projections of K do not exceed those of E, it follows that $\Omega(K) \leq \Omega(E)$. It was shown that this is a special case of a strong result due to Lutwak [17]. Here we will use the equivalence between mixed volumes and extended affine

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surface areas, obtained in this article, to reprove Lutwak's generalization. It turns out that Lutwak's generalizaiton can be extended to an L_p version by using the equivalence of this kind.

The organization of this paper is as follows.

For quick reference, a number of elementary definitions(Minkowski and Firey addition, ordinary and L_p-mixed volumes, classical and extended affine surface area, zonotopes) and well-known results(some integral representations) which are essential, are stated in Section 2. Lukwak [15]-[20], Leichtweiß [10]-[12], Petty [22]-[24], Schütt [27] are recommended as references.

In Section 3, corresponding to the work of Lukwak [17], we introduce L_p-polar curvature images of the symmetric star bodies. An equivalent definition of the L_p-extended affine surface area, and various results which are required later, regarding the L_p-polar curvature images, are obtained in this section.

Some main results of ours in the classical Brunn-Minkowski theory are presented in Section 4. It will be demonstrated that for the special class $\mathcal{W}^n$, there is a surprising equivalence between extended mixed affine surface areas and mixed volumes. In this section, we will reprove Lutwak's strong result, which holds for arbitrary convex bodies, by using the equivalence result abovementioned that may simplify the original proof of Lutwak's.

Section 5 is the L_p analogs of Section 4. In this section, we will introduce the concept of L_p-zonotopes and their limits, L_p-zonoids. A new class, $\mathcal{W}_p^n$, is also to be defined. It will be shown that the L_p version of Lutwak's monotonicity result holds for arbitrary convex bodies containing the origin in their interiors.

As applications of the approaches taken here, we establish Aleksandrov's projection theorem and Petty-Schneider theorem for the L_p-mixed Quermassintegral version in Section 6.

In this paper, our main results are the following theorems:

THEOREM (1.1). *If $n \neq p > 1$, $K \in \mathcal{K}_0^n$, $L \in \mathcal{W}_p^n$, and for all $u \in S^{n-1}$,*

$$V_p(K, \bar{u}) \leq V_p(L, \bar{u}), \tag{1.1a}$$

then

$$\Omega_p(K) \leq \Omega_p(L). \tag{1.1b}$$

THEOREM (1.2). *If $p \geq 1$, $K \in \mathcal{K}_0^n$, $L \in \mathcal{K}_s^n$, $i \in \{0, 1, \ldots, n-1\}$, and for all $u \in S^{n-1}$,*

$$W_{p,i}(K, \bar{u}) = W_{p,i}(L, \bar{u}), \tag{1.2a}$$

then

$$\begin{cases} W_i(K) \leq W_i(L), & \text{for } 1 \leq p < n-i. \\ W_i(K) \geq W_i(L), & \text{for } p > n-i. \end{cases} \tag{1.2b}$$

with equality if and only if $K = L$.

THEOREM (1.3). *If $p \geq 1$, $K \in \mathcal{K}_0^n$, $L \in \mathcal{Z}_p^n$, $i \in \{0, 1, \ldots, n-2\}$, and for all $u \in S^{n-1}$,*

$$W_{p,i}(K, \bar{u}) < W_{p,i}(L, \bar{u}), \tag{1.3a}$$

then

$$\begin{cases} W_i(K) < W_i(L), & \text{for } 1 \le p < n-i. \\ W_i(K) > W_i(L), & \text{for } p > n-i. \end{cases} \tag{1.3b}$$

2. Background and notation

Let $\mathcal{C}^n$ denote the set of compact convex subsets of Euclidean space $\mathbb{R}^n$. The subset of $\mathcal{C}^n$ consisting of convex bodies (compact, convex subsets with non-empty interiors) will be denoted by $\mathcal{K}^n$. For the set of compact convex sets containing the origin in their relative interiors, write $\mathcal{C}_0^n$. And for the set of convex bodies containing the origin in their interiors, write $\mathcal{K}_0^n$, and let $\mathcal{K}_c^n$ denote the set of convex bodies whose centroids lie at the origin. Then, let $\mathcal{C}_s^n$, $\mathcal{K}_s^n$ denote set of all bodies in $\mathcal{C}^n$, $\mathcal{K}^n$, respectively, that are symmetric about the origin. We reserve u for unit vectors, and B the unit ball centered at the origin in $\mathbb{R}^n$. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u, and let $K|E_u$ denote the image of the orthogonal projection of $K \in \mathcal{K}^n$ onto E_u. Let $V(K)$ denote the volume of $K \in \mathcal{K}^n$, $v(K|E_u)$ denote the area ($(n-1)$-dimensional volume) of $K|E_u$, and ω_n the volume of B. For $K \in \mathcal{C}^n$, let $h_K = h(K,\cdot)\colon S^{n-1} \to \mathbb{R}$ denote the support function of K; i.e., for $u \in S^{n-1}$, $h(K,u) = \max\{u \cdot x : x \in K\}$, where $u \cdot x$ denotes the standard inner product of u and x in $\mathbb{R}^n$. The set $\mathcal{C}^n$ will be viewed as equipped with the Hausdorff metric, d, defined by $d(K,L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ is the sup (or max) norm on the space of continuous functions on the unit sphere, $C(S^{n-1})$. For $K \in \mathcal{K}_0^n$, let K^* denote the polar of K; i.e., $K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1, \text{ for all } y \in K\}$.

(2.1) Minkowski, Firey combination, and mixed volumes. For $K, L \in \mathcal{K}^n$ and $\lambda, \mu \ge 0$ (not both zero), the Minkowski linear combination $\lambda K + \mu L \in \mathcal{K}^n$ is defined by

$$h(\lambda K + \mu L, \cdot) = \lambda h(K,\cdot) + \mu h(L,\cdot). \tag{2.2}$$

For real $p \ge 1$, $K, L \in \mathcal{K}_0^n$, and $\lambda, \mu \ge 0$ (not both zero), the Firey linear combination $\lambda \cdot K +_p \mu \cdot L$, is defined (see [6]) by

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K,\cdot)^p + \mu h(L,\cdot)^p. \tag{2.3}$$

The mixed volume $V_1(K,L)$ of $K, L \in \mathcal{K}^n$ is defined by

$$nV_1(K,L) = \lim_{\varepsilon \to 0} \frac{V(K+\varepsilon L) - V(K)}{\varepsilon}. \tag{2.4}$$

For $p \ge 1$, the L_p-mixed volume $V_p(K,L)$ of $K, L \in \mathcal{K}_0^n$ was defined in [19] by

$$\frac{n}{p} V_p(K,L) = \lim_{\varepsilon \to 0} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.5}$$

Aleksandrov [1] and Fenchel and Jessen [5] have shown that for each $K \in \mathcal{K}^n$, there is a positive Borel measure, $S(K,\cdot)$ on S^{n-1}, called the surface area measure of K, such that

$$V_1(K,Q) = \frac{1}{n} \int_{S^{n-1}} h(Q,u)\, dS(K,u), \tag{2.6}$$

for all $Q \in \mathcal{K}^n$. For $K \in \mathcal{K}^n$, $V_1(K, \cdot) : \mathcal{K}^n \to (0, \infty)$ is Minkowski linear; and $V_1(\cdot, \cdot)$ is continuous with respect to each argument. In addition, by the definition of support function, the extension of $V_1(\cdot, \cdot)$ from $\mathcal{K}^n \times \mathcal{K}^n$ to $\mathcal{K}^n \times \mathcal{C}^n \to (0, \infty)$ is practicable.

It was shown in [19], that for each $K \in \mathcal{K}_0^n$, there is a positive Borel measure, $S_p(K, \cdot)$ on S^{n-1}, such that

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h(Q, u)^p dS_p(K, u), \tag{2.7}$$

for all $Q \in \mathcal{K}_0^n$. Similarly, $V_p(K, \cdot) : \mathcal{K}_0^n \to (0, \infty)$ is Firey linear and $V_p(\cdot, \cdot)$ is continuous with respect to each argument. Also, the extension of the L_p-mixed volume $V_p(\cdot, \cdot)$ from $\mathcal{K}_0^n \times \mathcal{K}_0^n$ to $\mathcal{K}_0^n \times \mathcal{C}_0^n$ is practicable.

(2.8) Dual mixed volumes and Firey combinations. The radial function $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \backslash \{0\} \to [0, \infty)$, of a compact star-shaped (about the origin) $K \subset \mathbb{R}^n$, is defined by $\rho(K, x) = \max\{\lambda \geq 0 : \lambda x \in K\}$. If ρ_K is positive and continuous, call K a star body (about the origin). Write φ_0^n for the set of star bodies in $\mathbb{R}^n$. For $K \in \mathcal{K}_0^n$, it is easily seen that $\rho(K^*, \cdot) = 1/h(K, \cdot)$ and $h(K^*, \cdot) = 1/\rho(K, \cdot)$.

If $K, L \in \varphi_0^n$, and $\lambda, \mu \geq 0$ (not both zero), then for $p \geq 1$, the harmonic p-combination, $\lambda \diamond K \hat{+}_p \mu \diamond L \in \varphi_0^n$ is defined by:

$$\rho(\lambda \diamond K \hat{+}_p \mu \diamond L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}. \tag{2.9}$$

For $p \geq 1$, and $K, L \in \varphi_0^n$, the L_p-dual mixed volume, $\tilde{V}_{-p}(K, L)$, is defined by

$$-\frac{n}{p} \tilde{V}_{-p}(K, L) = \lim_{\varepsilon \to 0} \frac{V(K \hat{+}_p \varepsilon \diamond L) - V(K)}{\varepsilon}. \tag{2.10}$$

From the polar coordinate formula for volume, and (2.9), (2.10), one obtains: If $p \geq 1$, and $K, L \in \varphi_0^n$, then

$$\tilde{V}_{-p}(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^{n+p} \rho(L, u)^{-p} dS(u). \tag{2.11}$$

Obviously,

$$\tilde{V}_{-p}(K, K) = V(K). \tag{2.12}$$

For $K \in \mathcal{K}_0^n$, $L \in \varphi_0^n$, and $p \geq 1$, define $V_p(K, L^*)$ by:

$$V_p(K, L^*) = \frac{1}{n} \int_{S^{n-1}} \rho(L, u)^{-p} dS_p(K, u). \tag{2.13}$$

(2.14) Affine surface area. A convex body $K \in \mathcal{K}^n$ is said to have a positive continuous curvature function, $f(K, \cdot)$: $S^{n-1} \to \mathbb{R}$, provided the integral representation

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u) f(K, u) dS(u), \tag{2.15}$$

holds for all $L \in \mathcal{K}^n$. Let $\mathcal{F}^n, \mathcal{F}_0^n, \mathcal{F}_c^n$ denote set of all bodies in $\mathcal{K}^n, \mathcal{K}_0^n, \mathcal{K}_c^n$, respectively, that have positive continuous curvature functions. From (2.15) and (2.6), it follows that, for $K \in \mathcal{F}^n$, $f(K, u) = dS(K, u)/dS$.

A convex body $K \in \mathcal{K}_0^n$ will be said to have a positive continuous p-curvature function, $f_p(K, \cdot)\colon S^{n-1} \to \mathbb{R}$, if the integral representation

$$V_p(K, Q) = \frac{1}{n} \int_{S^{n-1}} h(Q, u)^p f_p(K, u) dS(u), \tag{2.16}$$

holds for all $Q \in \mathcal{K}_0^n$. From (2.16) and (2.7), it follows that, for $K \in \mathcal{F}_0^n$,

$$f_p(K, u) = dS_p(K, u)/dS. \tag{2.17}$$

The affine surface area, $\Omega(K)$, of $K \in \mathcal{F}^n$ can be defined by:

$$\Omega(K) = \int_{S^{n-1}} f(K, u)^{n/(n+1)} dS(u). \tag{2.18}$$

It is natural to define p-affine surface area, $\Omega_p(K)$, of $K \in \mathcal{F}_0^n$ by

$$\Omega_p(K) = \int_{S^{n-1}} f_p(K, u)^{n/(n+p)} dS(u). \tag{2.19}$$

The mixed affine surface area, $\Omega_{-1}(K, L)$, of $K, L \in \mathcal{F}^n$, was defined in [16] by

$$\Omega_{-1}(K, L) = \int_{S^{n-1}} f(K, u) f(L, u)^{-1/(n+1)} dS(u). \tag{2.20}$$

From (2.18), it follows that for $K \in \mathcal{F}^n$,

$$\Omega_{-1}(K, K) = \Omega(K). \tag{2.21}$$

Similarly, the p-mixed affine surface area, $\Omega_{p,-p}(K, L)$, of $K, L \in \mathcal{F}_0^n$, will be defined by

$$\Omega_{p,-p}(K, L) = \int_{S^{n-1}} f_p(K, u) f_p(L, u)^{-p/(n+p)} dS(u). \tag{2.22}$$

From (2.19), it follows that for $K \in \mathcal{F}_0^n$,

$$\Omega_{p,-p}(K, K) = \Omega_p(K). \tag{2.23}$$

(2.24) Extended affine surface area. In [11], Leichtweiß defined the affine surface area, $\Omega(K)$ of a body $K \in \mathcal{K}^n$, by:

$$n^{-1/n} \Omega(K)^{(n+1)/n} = inf\{nV_1(K, Q^*)V(Q)^{1/n} : Q \in \varphi_0^n\}. \tag{2.25}$$

For $p \geq 1$, we define the p-affine surface area, $\Omega_p(K)$, of $K \in \mathcal{K}_0^n$, by

$$n^{-p/n} \Omega_p(K)^{(n+p)/n} = inf\{nV_p(K, Q^*)V(Q)^{p/n} : Q \in \varphi_0^n\}. \tag{2.26}$$

It was shown in [20] that for $K \in \mathcal{F}_0^n$, $p \geq 1$, we can get (2.19) from (2.26).

There is a natural extension of definition (2.20) of the mixed affine surface area Ω_{-1} from $\mathcal{F}^n \times \mathcal{F}^n$ to $\mathcal{K}^n \times \mathcal{F}^n$. Specifically, for $K \in \mathcal{K}^n$, and $L \in \mathcal{F}^n$, let

$$\Omega_{-1}(K, L) = \int_{S^{n-1}} f(L, u)^{-1/(n+1)} dS(K, u). \tag{2.27}$$

Similarly, an alternate definition of p-extended mixed affine surface area, $\Omega_{p,-p}$, can be given as follows: For $K \in \mathcal{K}_0^n$, and $L \in \mathcal{F}_0^n$,

$$\Omega_{p,-p}(K, L) = \int_{S^{n-1}} f_p(L, u)^{-p/(n+p)} dS_p(K, u). \tag{2.28}$$

(2.29) Zonotopes. A zonotope is a Minkowski combination of line segments; i.e., $Z \in \mathcal{C}^n$ is a zonotope if there exist $u_1, \ldots, u_m \in S^{n-1}$ and $\lambda_1, \ldots, \lambda_m \geq 0$, such that

$$Z = \lambda_1 \bar{u}_1 + \cdots + \lambda_m \bar{u}_m, \tag{2.30}$$

where $\bar{u}$ denotes the closed line segment joining u and $-u$. Thus zonotopes are in $\mathcal{C}_s^n$. From the definition of support functions, it follows that $h(Z, u) = \sum_{i=1}^{m} \lambda_i |u_i \cdot u|$. In fact, for $Z \in \mathcal{C}^n$ with such a support function is a zonotope with centre at the origin (see [25], page 183). A body in $\mathcal{C}^n$ that is the limit (with respect to the Hausdorff metric) of zonotopes is called a zonoid, and $\mathcal{Z}^n$ will be used to denote the class of zonoids. Thus each zonoid has its centre (at the origin) of symmetry (see [8], [26]).

A body $K \in \mathcal{F}^n$ is defined to be in $\mathcal{W}^n$ if $f(K, \cdot)^{-1/(n+1)}$ is the support function of a zonoid. Then, $\mathcal{W}^n$ is an affine invariant class (see [15]).

3. L_p-polar curvature images

The concept of polar curvature images was posed by Lutwak in [17]. To determine the polar curvature image of a star body, the solution of the Minkowski problem (see, for example, [21]) is needed. So before we give the definition of L_p-polar curvature images, it is necessary to investigate the L_p-Minkowski problem.

In [19], Lutwak gave a weak solution to the L_p-Minkowski problem with even data: Suppose $n \neq p > 1$. If μ is an even positive Borel measure on S^{n-1} which is not concentrated on a great sphere of S^{n-1}, then there exists a unique centered L, symmetric about the origin, such that $S_p(L, \cdot) = \mu$. Thus, given a continuous function $f\colon S^{n-1} \to (0, \infty)$ such that $f(\cdot)S(\cdot)$ is an even positive Borel measure on S^{n-1}, then there exists a body $L \in \mathcal{F}_s^n$, such that

$$f_p(L, \cdot) = f(\cdot). \tag{3.1}$$

Use φ_s^n to denote the class of star bodies (about the origin) which are symmetric about the origin, and $\mathcal{F}_s^n$ the class of convex bodies in $\mathcal{F}^n$ which are symmetric about the origin. We define the L_p-polar curvature image, $\Lambda_p^\circ K$, of a star body $K \in \varphi_s^n$ as follows:

Definition (3.2). Suppose $n \neq p > 1$, and $K \in \varphi_s^n$. Define the mapping $\Lambda_p^\circ\colon \varphi_s^n \to \mathcal{F}_s^n$ as follows: Let $f(\cdot) = \omega_n \rho(K, \cdot)^{n+p}/V(K)$, then $f(\cdot)S(\cdot)$ is an even positive Borel measure on S^{n-1}, by (3.1), there must exist a unique convex body $\Lambda_p^\circ K \in \mathcal{F}_s^n$, such that

$$f_p(\Lambda_p^\circ K, \cdot) = \omega_n \rho(K, \cdot)^{n+p}/V(K). \tag{3.3}$$

From (2.11), (2.17), (2.13) and (3.3), we have

PROPOSITION (3.4). *If $n \neq p > 1$, and $K \in \varphi_s^n$, $L \in \varphi_0^n$, then*

$$V_p(\Lambda_p^\circ K, L^*) = \omega_n \tilde{V}_{-p}(K, L)/V(K). \tag{3.5}$$

PROPOSITION (3.6). *For $n \neq p > 1$, the mapping $\Lambda_p^\circ\colon \varphi_s^n \to \mathcal{F}_s^n$ is bijective.*

Proof. The integral representation (2.11), together with the Hölder inequality (see [9], page 140), and the polar coordinate formula, immediately gives: If $p \geq 1$, and $K, L \in \varphi_0^n$, then

$$\tilde{V}_{-p}(K, L)^n \geq V(K)^{n+p} V(L)^{-p}, \tag{3.7}$$

with equality if and only if K and L are dilates.

From (3.5), it follows that for $K, L \in \varphi_s^n$, $\Lambda_p^\circ K = \Lambda_p^\circ L$ implies that

$$\tilde{V}_{-p}(K, Q)/V(K) = \tilde{V}_{-p}(L, Q)/V(L),$$

for all $Q \in \varphi_0^n$. Taking $Q = L$ gives $\tilde{V}_{-p}(K, L)/V(K) = \tilde{V}_{-p}(L, L)/V(L) = 1$. Now (3.7) gives $V(L) \geq V(K)$, with equality if and only if K and L are dilates. Taking $Q = K$, and get $V(K) \geq V(L)$. Hence, $V(K) = V(L)$, and K and L must be dilates. Thus $K = L$. This proves the injectivity of Λ_p°.

To see that Λ_p° is surjective, suppose $K \in \mathcal{F}_s^n$. Then the measure μ determined by $dS_p(K, \cdot) = f_p(K, \cdot) dS(\cdot)$ is an even positive Borel measure. Define the star body L by

$$\rho(L, \cdot) = c f_p(K, \cdot)^{1/(n+p)},$$

where

$$c^p = \frac{1}{n\omega_n} \int_{S^{n-1}} f_p(K, u)^{n/(n+p)} dS(u).$$

From the fact that the measure $d\mu = dS_p(K, \cdot)$ is an even positive one, it follows that $L \in \varphi_s^n$, and it is trivial to verify, from (3.3) and the polar coordinate formula of volume, that $\Lambda_p^\circ L = K$. □

From definition (2.28), (3.3), and (2.13), it follows immediately that for $n \neq p > 1$, and $K \in \mathcal{K}_0^n$, $L \in \varphi_s^n$,

$$\omega_n^p \Omega_{p,-p}(K, \Lambda_p^\circ L)^{n+p} = n^{n+p} V(L)^p V_p(K, L^*)^{n+p}. \tag{3.8}$$

Take $\Lambda_p^\circ L$ for K in (3.8), use (2.23), and from (3.5) get

$$\begin{aligned} \omega_n^p \Omega_p(\Lambda_p^\circ L)^{n+p} &= n^{n+p} V(L)^p V_p(\Lambda_p^\circ L, L^*)^{n+p} \\ &= n^{n+p} V(L)^p \omega_n^{n+p} \tilde{V}_{-p}(L, L)^{n+p} / V(L)^{n+p}. \end{aligned}$$

Thus, by (2.12),

$$\Omega_p(\Lambda_p^\circ L)^{n+p} = n^{n+p} \omega_n^n V(L)^p. \tag{3.9}$$

Combine (3.8) and (3.9), and the result is:

PROPOSITION (3.10). *If $n \neq p > 1$, and $K \in \mathcal{K}_0^n$, $L \in \varphi_s^n$, then*

$$n^{-p/n} \Omega_{p,-p}(K, \Lambda_p^\circ L) \Omega_p(\Lambda_p^\circ L)^{p/n} = n V(L)^{p/n} V_p(K, L^*). \tag{3.11}$$

From Proposition (3.6) and (3.11), (2.26), we immediately obtain

THEOREM (3.12). *If $n \neq p > 1$, and $K \in \mathcal{K}_0^n$, then*

$$\Omega_p(K)^{(n+p)/n} = \inf\{\, \Omega_{p,-p}(K, L) \Omega_p(L)^{p/n} : L \in \mathcal{F}_s^n \}. \tag{3.13}$$

Obviously, Theorem (3.12) could have been used to define the p-affine surface area ($n \neq p > 1$) of arbitrary convex bodies containing the origin in their interiors.

Furthermore, the results for the case $p = 1$ corresponding to this section, which are more general, were obtained in [17], while the case $p = n$ remains open.

4. Extended affine surface areas and mixed volumes

The background for this section is the classical Brunn-Minkowski theory (see, for example, [25]). Just as mentioned formerly, it is natural that the definition (2.27) extends (2.20) of the mixed affine surface area Ω_{-1} from $\mathcal{F}^n \times \mathcal{F}^n$ to $\mathcal{K}^n \times \mathcal{F}^n$. In addition, we can get the following identically strong results. The techniques used in the proof are from the work of Chakerian and Lutwak [4].

LEMMA (4.1). *If $K, L \in \mathcal{K}^n$, then*

$$V_1(K, \bar{u}) \leq V_1(L, \bar{u}), \tag{4.1a}$$

for all $u \in S^{n-1}$, if and only if

$$\Omega_{-1}(K, Q) \leq \Omega_{-1}(L, Q), \tag{4.1b}$$

for all $Q \in \mathcal{W}^n$.

Proof. ¿From the definition of a zonotope (2.29), and the Minkowski linearity, in its second argument, of mixed volume (2.6), it follows that (4.1a) implies

$$V_1(K, Z) \leq V_1(L, Z), \tag{4.2}$$

for all zonotopes Z. The continuity of mixed volumes and the definition of a zonoid Z show that (4.2) must hold for all $Z \in \mathcal{Z}^n$.

For any $Q \in \mathcal{W}^n$, there exists a $Z \in \mathcal{Z}^n$ such that

$$f(Q, u)^{-1/(n+1)} = h(Z, u).$$

Hence by (2.6) and (2.27), we get

$$\begin{aligned} V_1(K, Z) &= \frac{1}{n} \int_{S^{n-1}} h(Z, u) dS(K, u) \\ &= \frac{1}{n} \int_{S^{n-1}} f(Q, u)^{-1/(n+1)} dS(K, u) \\ &= \frac{1}{n}\, \Omega_{-1}(K, Q). \end{aligned}$$

Similarly, $V_1(L, Z) = 1/n\ \Omega_{-1}(L, Q)$. Thus (4.1b) is now an immediate consequence of (4.2), where $Z \in \mathcal{Z}^n$.

That (4.1b) implies (4.1a) is trivial: Choose a sequence $Z_i \in \mathcal{Z}^n$, such that $\lim_{i\to\infty} Z_i = \bar{u}$, and use the continuity of mixed volumes. □

The following inequality which will be needed is due to Lutwak.

LEMMA (4.3). [17] *If $K \in \mathcal{K}^n$, $L \in \mathcal{F}^n$, then*

$$\Omega_{-1}(K, L)^n \geq \Omega(K)^{n+1} \Omega(L)^{-1}. \tag{4.4}$$

Obviously, if $K, L \in \mathcal{F}^n$, then (4.4) is a consequence of the Hölder inequality, with equality if and only if K and L are homothetic (see [16]).

As mentioned before, Winternitz showed that if $K \in \mathcal{F}^n$ and E is an ellipsoid such that $K \subset E$, then $\Omega(K) \leq \Omega(E)$. A striking generalization of Winternitz' result was obtained by Lutwak [17]. Recall that all ellipsoids are members of $\mathcal{W}^n$. Lutwak proved the following extension of Winternitz' result:

THEOREM (4.5). [17] *If $K \in \mathcal{K}^n$, $L \in \mathcal{W}^n$, and for all $u \in S^{n-1}$,*

$$v(K|E_u) \leq v(L|E_u), \tag{4.5a}$$

then

$$\Omega(K) \leq \Omega(L). \tag{4.5b}$$

Lutwak proved Theorem (4.5) by using projection operator. Here we use the equivalence obtained in Lemma (4.1) to reprove Theorem (4.5), which may partly simplify the original proof of Lutwak's. Furthermore, this approach can be used to generalize Theorem (4.5) to an L_p version. And the L_p analog of Theorem (4.5) will be demonstrated in Section 5.

Theorem (4.5), for $K \in \mathcal{F}^n$, $L \in \mathcal{W}^n$, was also established by Lutwak [15].

Proof. The area of $K|E_u$ is related to an n-dimensional mixed volume (see [3], page 45; or [7], page 359) by

$$2v(K|E_u) = nV_1(K, \bar{u}).$$

Thus, (4.5a) implies (4.1a), and from Lemma (4.1), then implies (4.1b).

Since $L \in \mathcal{W}^n \subset \mathcal{F}^n$, take L for Q in (4.1b), from (2.21) and (4.5), we get

$$\Omega(L) = \Omega_{-1}(L, L) \geq \Omega_{-1}(K, L) \geq \Omega(K)^{(n+1)/n}\Omega(L)^{-1/n}.$$

That is, $\Omega(L) \geq \Omega(K)$. □

5. L_p-zonotopes, L_p-extended affine surface areas, and L_p-mixed volumes

Definition (5.1). A p-zonotope is defined as a Firey combination of line segments; i.e., $Z_p \in \mathcal{C}_0^n$ is a p-zonotope if there exist $u_1, \ldots, u_m \in S^{n-1}$ and $\lambda_1, \ldots, \lambda_m \geq 0$, such that

$$Z_p = \lambda_1 \cdot \bar{u}_1 +_p \cdots +_p \lambda_m \cdot \bar{u}_m, \tag{5.2}$$

where as before $\bar{u}$ denotes the closed line segment joining u and $-u$.

From the definition of Firey combination, it follows immediately that $h(Z_p, u)^p = \sum_{i=1}^{m} \lambda_i |u_i, u|^p$. Similarly, for a body $Z_p \in \mathcal{C}_0^n$ with such a support function is a p-zonotope with centre at the origin. A body in $\mathcal{C}_0^n$ that is the limit (with respect to the Hausdorff metric) of p-zonotopes is called a p-zonoid, and $\mathcal{Z}_p^n$ will be used to denote the class of p-zonoids. Thus, by Definition (5.1), each p-zonoid has its centre (at the origin) of symmetry.

Define

$$\mathcal{W}_p^n = \{Q_p \in \mathcal{F}_0^n : \text{ there exists a } Z_p \in \mathcal{Z}_p^n \text{ with } f_p(Q_p, \cdot) = h(Z_p, \cdot)^{-(n+p)}\}.$$

It can be easily shown that $\mathcal{W}_p^n$ is a centro-affine invariant class (c.f. [20]). And from the symmetry of $Z_p \in \mathcal{Z}_p^n$, it follows that $\mathcal{W}_p^n \subset \mathcal{F}_s^n$.

The following Lemma is the L_p version of Lemma (4.1):

LEMMA (5.3). *If $p \geq 1$, and $K, L \in \mathcal{K}_0^n$, then*

$$V_p(K, \bar{u}) \leq V_p(L, \bar{u}), \tag{5.3a}$$

for all $u \in S^{n-1}$, if and only if

$$\Omega_{p,-p}(K, Q_p) \leq \Omega_{p,-p}(L, Q_p), \tag{5.3b}$$

for all $Q_p \in \mathcal{W}_p^n$.

Proof. From Definition (5.1), and the Firey linearity, in its second argument, of L_p-mixed volumes (2.7), it follows that (5.3a) implies

$$V_p(K, Z_p) \leq V_p(L, Z_p), \tag{5.4}$$

for all p-zonotopes Z_p. The continuity of L_p-mixed volumes and the definition of a p-zonoid show that (5.4) must hold for all $Z_p \in \mathcal{Z}_p^n$.

From the definition of $\mathcal{W}_p^n$, for any $Q_p \in \mathcal{W}_p^n$, there is a $Z_p \in \mathcal{Z}_p^n$ such that

$$f_p(Q_p, u)^{-p/(n+p)} = h(Z_p, u)^p.$$

Hence, by (2.7) and (2.28), we have

$$\begin{aligned} V_p(K, Z_p) &= \frac{1}{n} \int_{S^{n-1}} h(Z_p, u)^p dS_p(K, u) \\ &= \frac{1}{n} \int_{S^{n-1}} f_p(Q_p, u)^{-p/(n+p)} dS_p(K, u) \\ &= \frac{1}{n}\, \Omega_{p,-p}(K, Q_p). \end{aligned}$$

Similarly, we can get

$$V_p(L, Z_p) = \frac{1}{n}\, \Omega_{p,-p}(L, Q_p).$$

Thus (5.3b) is now an immediate consequence of (5.4), where $Z_p \in \mathcal{Z}_p^n$.

That (5.3b) implies (5.3a) is also trivial: Choose a sequence $Z_{p,i} \in \mathcal{Z}_p^n$, such that $\lim_{i\to\infty} Z_{p,i} = \bar{u}$, and use the continuity of L_p-mixed volumes. $\square$

The following Lemma is an immediate consequence of (3.13):

LEMMA (5.5). *If $n \neq p > 1$, and $K \in \mathcal{K}_0^n$, $L \in \mathcal{F}_s^n$, then*

$$\Omega_{p,-p}(K, L)^n \geq \Omega_p(K)^{n+p}\, \Omega_p(L)^{-p}. \tag{5.6}$$

Obviously, if $K, L \in \mathcal{F}_0^n$, then we can get (5.6) immediately from the Hölder inequality, with equality if and only if K and L are dilates.

Now we are in the position to prove Theorem (1.1), it is the L_p analog of Lutwak's generalization(Theorem (4.5)).

Proof of Theorem (1.1). From Lemma (5.3), it follows that (1.1a) implies (5.3b). Since $L \in \mathcal{W}_p^n \subset \mathcal{F}_s^n$, take L for Q_p in (5.3b), and from (2.23) and (5.6), we obtain:

$$\Omega_p(L) = \Omega_{p,-p}(L, L) \geq \Omega_{p,-p}(K, L) \geq \Omega_p(K)^{(n+p)/n}\, \Omega_p(L)^{-p/n}.$$

That is (1.1b). $\square$

6. Aleksandrov's projection theorem and Petty-Schneider theorem for L_p-mixed Quermassintegrals version

Before demonstrating the results of this section, we firstly make some preparations. For $Q \in \mathcal{K}^n$, let $W_i(Q)$ $(0 \le i \le n)$ denote the Quermassintegrals of Q. Thus, $W_0(Q) = V(Q)$, $nW_1(Q) = S(Q)$, the surface area of Q, $W_n(Q) = \omega_n$. The mixed Quermassintegrals $W_i(K, L)$ $(0 \le i < n)$ of $K, L \in \mathcal{K}^n$ are defined by

$$(n-i)W_i(K, L) = \lim_{\varepsilon \to 0} \frac{W_i(K + \varepsilon L) - W_i(K)}{\varepsilon}. \tag{6.1}$$

Obviously, $W_i(K, K) = W_i(K)$, and $W_0(K, L)$ is usually written as $V_1(K, L)$; i.e., when $i = 0$, (6.1) is just (2.4). Furthermore, $W_i(\cdot, \cdot)$ is continuous in each argument and Minkowski linearity in its second argument.

The Minkowski inequality for mixed Quermassintegrals states that: For $K, L \in \mathcal{K}^n$, and $0 \le i < n-1$,

$$W_i(K, L)^{n-i} \ge W_i(K)^{n-1-i} W_i(L), \tag{6.2}$$

with equality if and only if K and L are homothetic. For $i = n-1$, the quantities on both sides of inequality (6.2) are equal.

For $K, L \in \mathcal{K}_0^n$, and real $p \ge 1$, define the mixed p-Quermassintegrals $W_{p,i}(K, L)(0 \le i < n)$ as

$$\frac{n-i}{p} W_{p,i}(K, L) = \lim_{\varepsilon \to 0} \frac{W_i(K +_p \varepsilon \cdot L) - W_i(K)}{\varepsilon}. \tag{6.3}$$

Obviously, $W_{p,i}(K, K) = W_i(K)$ for all $p \ge 1$, $W_{p,0}(K, L) = V_p(K, L)$. And $W_{p,i}(\cdot, \cdot)$ is continuous in each argument and Firey linearity in its second argument.

An extension of (6.2) is: For $p \ge 1$, $K, L \in \mathcal{K}_0^n$, and $0 \le i < n-1$,

$$W_{p,i}(K, L)^{n-i} \ge W_i(K)^{n-p-i} W_i(L)^p, \tag{6.4}$$

with equality if and only if K and L are dilates.

Similar to (2.7), the mixed p-Quermassintegral $W_{p,i}$ has the following integral representation: For $p \ge 1$, $i \in \{0, 1, \dots, n-1\}$, and $K \in \mathcal{K}_0^n$, there exists a regular Borel measure $S_{p,i}(K, \cdot)$ on S^{n-1}, such that

$$W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_{p,i}(K, u), \tag{6.5}$$

for all $L \in \mathcal{K}_0^n$.

The well-known Aleksandrov's projection theorem states that if $K, L \in \mathcal{K}_s^n$ and if

$$v(K|E_u) = v(L|E_u) \text{ for all } u \in S^{n-1},$$

then K is a translate of L.

A generalization of Aleksandrov's projection theorem was established by G.S. Leng and L.S. Zhang as follows:

THEOREM (6.6). [13] *If $K \in \mathcal{K}^n$, $L \in \mathcal{K}_s^n$, $i \in \{0, 1, \dots, n-1\}$, and for all $u \in S^{n-1}$,*

$$W_i(K, \bar{u}) = W_i(L, \bar{u}),$$

then

$$W_i(K) \le W_i(L),$$

with equality if and only if K is a translate of L.

Another famous result involving the areas of projections of convex bodies is Petty-Schneider theorem which states that if $K \in \mathcal{K}^n$, $L \in \mathcal{Z}^n$, and if

$$v(K|E_u) < v(L|E_u) \text{ for all } u \in S^{n-1},$$

then $V(K) < V(L)$.

It will be shown that there is a similar generalization of Petty-Schneider theorem:

THEOREM (6.7). *If $K \in \mathcal{K}^n$, $L \in \mathcal{Z}^n$, $i \in \{0, 1, \dots, n-2\}$, and for all $u \in S^{n-1}$,*

$$W_i(K, \bar{u}) < W_i(L, \bar{u}), \tag{6.7a}$$

then

$$W_i(K) < W_i(L). \tag{6.7b}$$

Proof. By discussion similar to (4.2), from (6.7a) we have

$$W_i(K, Z) < W_i(L, Z), \tag{6.8}$$

holds for all $Z \in \mathcal{Z}^n$.

Now take L for Z in (6.8), and from (6.7) get

$$W_i(L) = W_i(L, L) > W_i(K, L) \geq W_i(K)^{(n-1-i)/(n-i)} W_i(L)^{1/(n-i)}.$$

That is (6.7b). □

In the following component of this section, we will establish the L_p versions of Theorem (6.6) and Theorem (6.7). At first, we note that a body $K \in \mathcal{C}_0^n$ is said to be a generalized p-zonoid if there exists a $Z_p \in \mathcal{Z}_p^n$, such that $K +_p Z \in \mathcal{Z}_p^n$. While the class of generalized p-zonoids, denoted by $\mathcal{Z}_{p,g}^n$, is a proper subset of $\mathcal{C}_s^n$, it turns out that $\mathcal{Z}_{p,g}^n$ is dense in $\mathcal{C}_s^n$.

Proof of Theorem (1.2). By discussion similar to (5.4), from (1.2a) we have

$$W_{p,i}(K, Z_p) = W_{p,i}(L, Z_p), \tag{6.9}$$

holds for all $Z_p \in \mathcal{Z}_p^n$.

Suppose $Q_p \in \mathcal{Z}_{p,g}^n$, then there exists a $Z_p \in \mathcal{Z}_p^n$ such that $Q_p +_p Z_p \in \mathcal{Z}_p^n$, it follows from (6.9), and the Firey linearity, in their second arguments, of mixed p-Quermassintegrals (6.5), that

$$W_{p,i}(K, Q_p) + W_{p,i}(K, Z_p) = W_{p,i}(K, Q_p +_p Z_p) = W_{p,i}(L, Q_p) + W_{p,i}(L, Z_p).$$

From (6.9) again, it follows that $W_{p,i}(K, Q_p) = W_{p,i}(L, Q_p)$. Since every member of $\mathcal{K}_s^n$ is the limit of generalized p-zonoids, it follows from the continuity of mixed p-Quermassintegrals, that

$$W_{p,i}(K, Q) = W_{p,i}(L, Q), \tag{6.10}$$

must hold for all $Q \in \mathcal{K}_s^n$.

Take L for Q in (6.10), and from (6.4) get

$$W_i(L) = W_{p,i}(L, L) = W_{p,i}(K, L) \geq W_i(K)^{(n-p-i)/(n-i)} W_i(L)^{p/(n-i)}.$$

That is (1.2b).

The equality condition can be obtained from that of (6.4), and from (6.5), (6.10). □

Proof of Theorem (1.3). Also by discussion similar to (5.4), from (1.3a) we have

$$W_{p,i}(K, Z_p) < W_{p,i}(L, Z_p), \tag{6.11}$$

holds for all $Z_p \in \mathcal{Z}_p^n$.

Take L for Z_p in (6.11), and from (6.4) get

$$W_i(L) = W_{p,i}(L, L) > W_{p,i}(K, L) \geq W_i(K)^{(n-p-i)/(n-i)} W_i(L)^{p/(n-i)}.$$

That is (1.3b). □

We note, as an aside, that according to Lutwak's Brunn-Minkowski-Firey theory, the conditions (1.2a) and (1.3a) could be viewed as the quantitative relationships for L_p-mixed projection operator.

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ORBIFOLD VIRTUAL COHOMOLOGY OF THE SYMMETRIC PRODUCT

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ABSTRACT. The virtual cohomology of an orbifold is a ring structure on the cohomology of the inertia orbifold whose product is defined via the pull-push formalism and the Euler class of the excess intersection bundle. In this paper we calculate the virtual cohomology of a large family of orbifolds, including the symmetric product.

1. Introduction

It was noticed in [LUX07] that the ring structure defined in the homology of the loop space of the symmetric product orbifold (see [LUX]) induces a ring structure on the cohomology of the inertia orbifold, by restricting the structure to the constant loops. This led the authors of [LUX07] to define a ring structure on the inertia orbifold of any orbifold that the authors coined *virtual cohomology*. This cohomology is defined via the pull-push formalism in as much as the same way that the Chen-Ruan product for orbifolds is defined (see [CR04, FG03]). In [GLS+07] the relation between the virtual and the Chen-Ruan cohomology was clarified, namely, that for an almost complex orbifold, its virtual cohomology is isomorphic to the Chen-Ruan cohomology of its cotangent orbifold.

In this paper we give an algorithm to calculate the virtual cohomology of a large family of orbifolds. We first show that for any global quotient orbifold $[Y/G]$, the virtual cohomology $H^*_{\mathrm{virt}}(Y, G; \mathbb{Z})$ maps to the group ring $H^*(Y; \mathbb{Z})[G]$, and therefore, when this map is injective we can see the virtual cohomology as a subring of the group ring. This for example is the case when the inclusions of the fixed point sets $Y^g \to Y$ induce a monomorphism in homology. We calculate the virtual cohomology of these orbifolds by describing a set of generators in the group ring. In the case of the symmetric product we reduce the set of generators to the lower degree cohomology classes of the fixed point sets of the transpositions.

This paper was motivated by the master's thesis of the first author [Riv] where the virtual cohomology of the symmetric product of spheres was calculated. Lastly, we would like to thank A. Cardona, E. Lupercio and M. Xicotencatl for useful conversations and the anonymous referee for pointing out some redundant relations on the presentation of the last example.

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2. Virtual Cohomology

Let $[Y/G]$ be an orbifold with Y differentiable, compact, oriented and closed, and G a finite group acting smoothly on Y preserving the orientation. The inertia orbifold $I[M/G]$ is defined as the orbifold

$$I[Y/G] = \left[\left(\bigsqcup_{g \in G} Y^g \times \{g\} \right) /G \right]$$

where Y^g denotes the fixed point set of the element g, we label the components with the elements of the group and G acts in the following way:

$$\begin{array}{rcl} \left(\bigsqcup_{g \in G} Y^g \times \{g\} \right) \times G & \to & \bigsqcup_{g \in G} Y^g \times \{g\} \\ ((x,g),h) & \mapsto & (xh, h^{-1}gh). \end{array} \tag{2.1}$$

From [LUX07] we know that the virtual intersection product defines a ring structure on the cohomology of the inertia orbifold $I[Y/G]$, this ring is what the authors in [LUX07] have called *virtual cohomology*. Let's recall its definition.

Consider the groups

$$H^*(Y,G;\mathbb{Z}) := \bigoplus_{g \in G} H^*(Y^g;\mathbb{Z}) \times \{g\}$$

and for $g, h \in G$ define the maps

$$\begin{array}{rcl} \times \colon H^*(Y^g;\mathbb{Z}) \times H^*(Y^h;\mathbb{Z}) & \to & H^*(Y^{gh};\mathbb{Z}) \\ (\alpha, \beta) & \mapsto & i_{gh!}(i_g^*\alpha \cdot i_h^*\beta \cdot e(Y, Y^g, Y^h)) \end{array}$$

where $i_g \colon Y^g \cap Y^h \to Y^g$, $i_h \colon Y^g \cap Y^h \to Y^h$ and $i_{gh} \colon Y^g \cap Y^h \to Y^{gh}$ are the inclusion maps, $e(Y, Y^g, Y^h)$ is the Euler class of the excess bundle of the inclusions $Y^g \to Y \leftarrow Y^h$ (see [Qui71]) and $i_{gh!}$ is the pushforward map in cohomology.

In [LUX07] it was required that the orbifold be almost complex with a compatible G action, but for the product to be well defined it is only necessary that the Euler classes of the excess bundles be of even degree. This can be achieved if for all $g_i \in G$ the fixed point sets

$$Y^{g_1,\dots,g_n} := Y^{g_1} \cap \cdots \cap Y^{g_n}$$

are of even dimension.

The group G acts on $H^*(Y,G;\mathbb{Z})$ in the following way: for $g, h \in G$ and $\alpha \in H^*(Y^g;\mathbb{Z})$ we have

$$(\alpha, g) \cdot h := ((h^{-1})^*\alpha, h^{-1}gh).$$

Definition (2.2). Let $[Y/G]$ be an orbifold such that for all $g_i \in G$ the fixed point sets $Y^{g_1,\dots,g_n}$ are even dimensional. Then, the group $H^*(Y,G;\mathbb{Z})$ together with the ring structure

$$\begin{array}{rcl} \bullet \colon H^*(Y,G;\mathbb{Z}) \times H^*(Y,G;\mathbb{Z}) & \to & H^*(Y,G;\mathbb{Z}) \\ ((\alpha,g),(\beta,h)) & \mapsto & (\alpha \times \beta, gh) \end{array}$$

is what is called the *virtual cohomology* of the pair (Y, G); we will denote it by $H^*_{\mathrm{virt}}(Y, G; \mathbb{Z})$. Moreover, as the ring structure is G-equivariant with respect to the action of G on $H^*(Y, G; \mathbb{Z})$, we define the virtual cohomology of the orbifold $[Y/G]$ as the G invariant part of the ring $H^*_{\mathrm{virt}}(Y, G; \mathbb{R})$, i.e.

$$H^*_{\mathrm{virt}}([Y/G]; \mathbb{R}) := H^*(Y, G; \mathbb{R})^G .$$

In what follows we will show how to calculate the virtual cohomology for a large family of orbifolds.

For all $g \in G$ let $f_g \colon Y^g \to Y$ be the inclusion of manifolds and

$$f_{g!} \colon H^*(Y^g; \mathbb{Z}) \to H^*(Y; \mathbb{Z})$$

be the pushforward in cohomology. Consider the group ring $H^*(Y; \mathbb{Z})[G]$ of the group G with coefficients in the ring $H^*(Y; \mathbb{Z})$, together with the G action defined by

$$\left(\sum_i \alpha_i g_i\right) \cdot h := \sum_i ((h^{-1})^* \alpha_i) h^{-1} g_i h .$$

THEOREM (2.3). *The inclusions $f_g \colon Y^g \to Y$ induce an equivariant ring homomorphism from the virtual cohomology to the group ring*

$$\begin{array}{rcl} f \colon H^*_{\mathrm{virt}}(Y, G; \mathbb{Z}) & \to & H^*(Y; \mathbb{Z})[G] \\ (\alpha, g) & \mapsto & (f_{g!}\alpha) g. \end{array}$$

Proof. To show that the map f is a ring homomorphism we only need to check the commutativity of the following diagram

$$\begin{tikzcd} H^*(Y^g; \mathbb{Z}) \times H^*(Y^h; \mathbb{Z}) \arrow[r, "f_{g!} \times f_{h!}"] \arrow[d, "\times"] & H^*(Y; \mathbb{Z}) \times H^*(Y; \mathbb{Z}) \arrow[d, "\cdot"] \\ H^*(Y^{gh}; \mathbb{Z}) \arrow[r, "f_{hg!}"'] & H^*(Y; \mathbb{Z}) . \end{tikzcd}$$

Consider the diagram of inclusions

$$\begin{tikzcd} Y & Y^g \arrow[l, "f_g"'] \\ Y^h \arrow[u, "f_h"] & Y^{g,h} \arrow[l, "i_h"] \arrow[u, "i_g"'] \arrow[ul, "s"'] \end{tikzcd} .$$

It was proven in [LUX07, Lemma 16] by an application of Quillen's excess intersection formula [Qui71, Prop. 3.3] that for $\alpha \in H^*(Y^g; \mathbb{Z})$ and $\beta \in H^*(Y^h; \mathbb{Z})$ one has

$$f_{g!}\alpha \cdot f_{h!}\beta = s_!(i_g^*\alpha \cdot i_h^*\beta \cdot e(Y, Y^g, Y^h)) .$$

Now, as $s = f_{gh} \circ i_{gh}$ we have that $s_! = f_{gh!} \circ i_{gh!}$ and therefore

$$\begin{array}{rcl} f_{g!}\alpha \cdot f_{h!}\beta & = & f_{gh!}(i_{gh!}(i_g^*\alpha \cdot i_h^*\beta \cdot e(Y, Y^g, Y^h))) \\ & = & f_{gh!}(\alpha \times \beta). \end{array}$$

To check that the map f is G-equivariant we simply consider the inclusion

$$\begin{array}{rcl} \psi\colon \bigsqcup_{g\in G} Y^g \times \{g\} & \to & \bigsqcup_{g\in G} Y^g \times \{g\} \\ (x,g) & \mapsto & (f_g x, g). \end{array}$$

If we endow the space $\bigsqcup_{g\in G} Y^g \times \{g\}$ with the same G-action as in 2.1 then the map ψ becomes G-equivariant and we have the commutativity of the following square:

$$\begin{tikzcd} Y^g \times \{g\} \arrow[r, "f_g"] \arrow[d, "h"] & Y \times \{g\} \arrow[d, "h"] \\ Y^{h^{-1}gh} \times \{h^{-1}gh\} \arrow[r, "f_{h^{-1}gh}"'] & Y \times \{h^{-1}gh\}. \end{tikzcd}$$

Therefore we have that for $g, h \in G$ and $\alpha \in H^*(Y^g;\mathbb{Z})$

$$\begin{aligned} (f_{g!}\alpha, g)\cdot h &= \big((h^{-1})^* f_{g!}\alpha, h^{-1}gh\big) \\ &= \big(f_{h^{-1}gh!}(h^{-1})^*\alpha, h^{-1}gh\big) \end{aligned}$$

and this implies that the map f is G-equivariant. □

COROLLARY (2.4). *If the inclusion maps in homology $f_{g*}\colon H_*(Y^g;\mathbb{Z}) \to H_*(Y;\mathbb{Z})$ are injective for all $g \in G$, then the map*

$$f\colon H^*_{\mathrm{virt}}(Y,G;\mathbb{Z}) \to H^*(Y;\mathbb{Z})[G]$$

is an injective homomorphism of rings. Then the ring $H^*(Y,G;\mathbb{Z})$ can be calculated as the subring $f(H^*_{\mathrm{virt}}(Y,G;\mathbb{Z}))$ of $H^*(Y;\mathbb{Z})[G]$.

Proof. The pushforward $f_{g!}\colon H^*(Y^g;\mathbb{Z}) \to H^*(Y;\mathbb{Z})$ in cohomology can be defined as the composition of the maps $PD \circ f_{g*} \circ PD_g^{-1}$ where $PD\colon H_*(Y;\mathbb{Z}) \xrightarrow{\cong} H^*(Y;\mathbb{Z})$ and $PD_g\colon H_*(Y^g;\mathbb{Z}) \xrightarrow{\cong} H^*(Y^g;\mathbb{Z})$ are the Poincaré duality isomorphisms. It follows that the maps $f_{g!}$ are injective. □

The above corollary will allow us to calculate the virtual cohomology of a large family of orbifolds, as in the following example.

Example (2.5). Consider the action of $\mathbb{Z}/p$ on the complex projective space $\mathbb{C}P^n$

$$\begin{array}{rcl} \mathbb{C}P^n \times \mathbb{Z}/p & \to & \mathbb{C}P^n \\ ([z_0 : \cdots : z_n], \lambda^i) & \mapsto & [z_0 : \cdots : z_{n-1} : \lambda^i z_n] \end{array}$$

where the elements of $\mathbb{Z}/p$ are taken as p-th roots of unity. For $i \neq 0$ one has that the fixed point set of λ^i is

$$(\mathbb{C}P^n)^{\lambda^i} \cong \mathbb{C}P^{n-1} \cup \{*\}$$

and therefore if we only consider the connected component of $\mathbb{C}P^{n-1}$ then the maps $f_{\lambda^i *}$ are all injective. The pushforward of the inclusions are

$$\begin{array}{rcl} f_{\lambda^i !}\colon H^*(\mathbb{C}P^{n-1};\mathbb{Z}) = \mathbb{Z}[y]/\langle y^n\rangle & \to & H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[x]/\langle x^{n+1}\rangle \\ y^j & \mapsto & x^{j+1}, \end{array}$$

and

$$\begin{array}{rcl} f_{\lambda^i !}\colon H^*(\{*\};\mathbb{Z}) = \mathbb{Z}\langle z\rangle & \to & H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[x]/\langle x^{n+1}\rangle \\ z & \mapsto & x^n. \end{array}$$

Therefore we have

$$H^*_{\text{virt}}(\mathbb{C}P^n, \mathbb{Z}/p\,;\mathbb{Z}) \cong \left(\mathbb{Z}[x]/\langle x^{n+1}\rangle[1] \oplus \bigoplus_{i=1}^{p-1} \left(x\mathbb{Z}[x]/\langle x^{n+1}\rangle \oplus \mathbb{Z}\langle z\rangle\right)[\lambda^i]\right),$$

where $zx = 0$ and $z^2 = 0$.

If we take a closer look at the virtual cohomology generated by the inclusions of the $\mathbb{C}P^{n-1}$'s, we can see that its elements are truncated polynomials of maximum degree n, whose coefficients are elements in the group ring $\mathbb{Z}[\mathbb{Z}/p]$ except for the constant term that it should be an integer, i.e.

$$\{P(x) \in \mathbb{Z}[\mathbb{Z}/p][x]/\langle x^{n+1}\rangle | P(0) \in \mathbb{Z}\}$$

where the ring structure is given by multiplication of polynomials. If we add the classes coming from the inclusions of the points $*$ we obtain that

$$H^*_{\text{virt}}(\mathbb{C}P^n, \mathbb{Z}/p;\mathbb{Z}) \cong \{P(x) \in \mathbb{Z}[\mathbb{Z}/p][x]/\langle x^{n+1}\rangle | P(0) \in \mathbb{Z}\} \oplus \oplus_{i=1}^{p} \mathbb{Z}\langle z\rangle[\lambda^i]/(xz, z^2).$$

Now, as the group $\mathbb{Z}/p$ is abelian and its action can be factored through an action of S^1, we have that

$$H^*_{\text{virt}}(\mathbb{C}P^n, \mathbb{Z}/p\,;\mathbb{R})^{\mathbb{Z}/p} = H^*_{\text{virt}}(\mathbb{C}P^n, \mathbb{Z}/p\,;\mathbb{R}).$$

Then

$$H^*_{\text{virt}}([\mathbb{C}P^n/\mathbb{Z}/p];\mathbb{R}) \cong \{P(x) \in \mathbb{R}[\mathbb{Z}/p][x]/\langle x^{n+1}\rangle | P(0) \in \mathbb{R}\} \oplus \oplus_{i=1}^{p} \mathbb{R}\langle z\rangle[\lambda^i]/(xz, z^2).$$

We have seen that for the case in which the homomorphisms f_{g*} are injective, the virtual cohomology is isomorphic to the subring $f(H^*_{\text{virt}}(Y, G;\mathbb{Z}))$ of the group ring $H^*(Y;\mathbb{Z})[G]$. In what follows we will find a set of generators for $f(H^*_{\text{virt}}(Y, G;\mathbb{Z}))$.

Let $H^*(Y;\mathbb{Z})[1_G]$ be the set of elements of the group ring whose label is the identity 1_G of the group G.

PROPOSITION (2.6). *Suppose that all the homomorphisms f_{g*} are injective and all the f_g^* are surjective. Denote by $1_g \in H^0(Y^g;\mathbb{Z})$ the identity of the ring $H^*(Y^g;\mathbb{Z})$. Then the set*

$$W := H^*(Y;\mathbb{Z})[1_G] \cup \{(f_{g!}1_g)g \mid g \in G\}$$

generates the ring $f(H^*_{\text{virt}}(Y, G;\mathbb{Z}))$.

Proof. It is clear that $W \subset f(H^*_{\text{virt}}(Y, G;\mathbb{Z}))$; we need to prove that for any $a \in H^*(Y^g;\mathbb{Z})$ the element $(f_{g!}a)g$ can be generated with elements in W.

We know that the pullback f_g^* is surjective. Therefore there exists $b \in H^*(Y;\mathbb{Z})$ such that $f_g^* b = a$. By the module structure of the pushforward we have

$$f_{g!}(a) = f_{g!}(1_g a) = f_{g!}(1_g f_g^* b) = (f_{g!}1_g)b$$

which implies that in the group ring

$$(1_g\, g)(b\, 1_G) = \left((f_{g!}1_g)b\right) g = (f_{g!}a)g.$$

$\square$

3. Symmetric Product

It was shown in [LUX07] that for an even dimensional compact and closed manifold M, the virtual cohomology of the orbifold $[M^n/\mathfrak{S}_n]$ is a subring of the string homology of the loop orbifold of the symmetric product (see [LUX]).

In this section we will calculate the virtual cohomology of the pair $(M^n, \mathfrak{S}_n)$ in terms of the cohomology of M. As we will make use of the Kunneth isomorphism we will restrict to real coefficients. We would like to remark that if the manifold has torsion free homology, all the calculations that follow can be done with integer coefficients.

So, abusing the notation, we will talk indistinctly of the rings $H^*(M^k;\mathbb{R})$ and $H^*(M;\mathbb{R})^{\otimes k}$.

We know that the diagonal inclusion $\Delta\colon M \to M \times M$ induces an injection $\Delta_*\colon H_*(M;\mathbb{R}) \to H_*(M\times M;\mathbb{R})$, and as $\Delta^*(a\otimes 1) = a$ we have that the pullback Δ^* is surjective. For $\tau \in \mathfrak{S}_n$ the map $f_\tau\colon (M^n)^\tau \to M^n$ is a composition of diagonal maps, so we have that $f_{\tau *}$ is injective and that f_τ^* is surjective. We can therefore apply proposition (2.6) to the pair $(M^n, \mathfrak{S}_n)$ to get a set of generators. In what follows we will show that we can reduce the set of generators by only considering the transpositions.

LEMMA (3.1). *Let $\delta\colon M \to M^k$ be the diagonal inclusion and let*

$$\begin{array}{rcl} \sigma_i^k\colon M^{k-1} & \to & M^k \\ (x_1,\dots,x_{k-1}) & \mapsto & (x_1,\dots,x_{i-1},x_i,x_i,x_{i+1},\dots x_{k-1}) \end{array}$$

be the inclusion that repeats the i-th coordinate. Then in cohomology

$$\delta_!1 = \prod_{j=1}^{k-1} (\sigma^k_{j\,!}1)\,.$$

Proof. We will proceed by induction on k. When $k=2$ the formula is true because $\delta = \sigma_1^2$. Assume that we have shown the formula for $k=n$ and let's try to show it for $k=n+1$. Consider the following diagram of inclusions

$$\begin{tikzcd} M^n \arrow[r, "\sigma_n^{n+1}"] & M^{n+1} \\ M \arrow[u, "\delta"] \arrow[ur, "\delta'"'] & \end{tikzcd}$$

and using the properties of the pushfoward we have,

$$\begin{aligned} \delta'_!1 &= \sigma_n^{n+1}{}_!(\delta_!1) \\ &= \sigma_n^{n+1}{}_!\left((\sigma_n^{n+1})^*((\delta_!1)\otimes 1)\right) \\ &= (\sigma_n^{n+1}{}_!1)((\delta_!1)\otimes 1) \\ &= (\sigma_n^{n+1}{}_!1)\left(\prod_{j=1}^{n}[(\sigma^n_{j\,!}1)\otimes 1]\right) \end{aligned}$$

$$= \quad (\sigma_n^{n+1}{}_!1)\prod_{j=1}^{n-1}(\sigma_j^{n+1}{}_!1)$$

$$= \quad \prod_{j=1}^{n}(\sigma_j^{n+1}{}_!1).$$

By the induction hypothesis the lemma follows. □

If we take the cycle $\alpha = (k, k-1, \ldots, 2, 1) \in \mathfrak{S}_n$ and the transpositions $\tau_i = (i, i+1)$ we have that $f_{\alpha !}1 = \delta_! 1$ and $f_{\tau_i !}1 = \sigma^k_i{}_! 1$. By lemma (3.1) we can conclude that for the cycle α of size k which is the composition of $k-1$ transpositions $\tau_1 \ldots \tau_{k-1}$ we have that

$$f_{\alpha !}1_\alpha = \prod_{i=1}^{k-1}(f_{\tau_i !}1_{\tau_i}).$$

Therefore we can reduce the set of generators of the virtual cohomology of the pair $(M^n, \mathfrak{S}_n)$ by considering only the transpositions.

PROPOSITION (3.2). *The ring $f(H^*_{\text{virt}}(M^n, \mathfrak{S}_n; \mathbb{R}))$ is generated by the set*

$$W = H^*(M^n;\mathbb{R})[1_{\mathfrak{S}_n}] \cup \{(f_{\tau !}1_\tau)\tau \mid \tau \in \mathfrak{S}_n \text{ is a transposition}\}$$

as a subring of $H^*(M^n;\mathbb{R})[\mathfrak{S}_n]$.

If $H^*(M;\mathbb{Z})$ is torsion free, the same result holds but with integer coefficients.

Example (3.3). Let's consider the pair $(M^n, \mathfrak{S}_n)$ with $M = \mathbb{C}P^m$. We therefore have that

$$H^*(M^n;\mathbb{Z}) \cong \mathbb{Z}[x_1, \ldots, x_n]/\langle x_1^{m+1}, \ldots, x_n^{m+1}\rangle$$

and we only need to calculate the pushforward of the diagonal inclusion $\Delta : M \to M \times M$.

By the Kunneth isomorphism we have that

$$H_*(\mathbb{C}P^m \times \mathbb{C}P^m;\mathbb{Z}) \cong H_*(\mathbb{C}P^m;\mathbb{Z}) \otimes H_*(\mathbb{C}P^m;\mathbb{Z})$$

and $H^*(\mathbb{C}P^m \times \mathbb{C}P^m;\mathbb{Z}) \cong \mathbb{Z}[x_1, x_2]/\langle x_1^{m+1}, x_2^{m+1}\rangle$. Let's take $H^*(\mathbb{C}P^m;\mathbb{Z}) \cong \mathbb{Z}[y]/\langle y^{m+1}\rangle$ and denote by $[\mathbb{C}P^i] \in H_{2i}(\mathbb{C}P^m;\mathbb{Z})$ the generator of the homology in degree $2i$ given by the inclusion $\mathbb{C}P^i \to \mathbb{C}P^m$, $[z_0 : \cdots : z_i] \mapsto [z_0 : \cdots : z_i : 0 : \cdots : 0]$.

By Poincaré duality (see [Hat02, page 213]) we know that the homology class $[\mathbb{C}P^i]$ is dual to the cohomology class y^{m-i}. Now, $\Delta^* x_1 = \Delta^* x_2 = y$ implies that $\Delta^* x_1^j x_2^{m-j} = y^m$, and as the classes $\{x_1^j x_2^{m-j} | 0 \leq j \leq m\}$ generate the degree $2m$ cohomology of $\mathbb{C}P^m \times \mathbb{C}P^m$, we have by Poincaré duality that

$$\Delta_*[\mathbb{C}P^m] = \sum_{j=0}^{m}[\mathbb{C}P^j] \otimes [\mathbb{C}P^{m-j}].$$

Therefore we can conclude that $\Delta_! 1 = \sum_{j=0}^m x_1^j x_2^{m-j}$.

Then, for the transposition $\tau = (k, l)$ we get that

$$f_{\tau !}1 = \sum_{j=0}^{m} x_k^j x_l^{m-j}$$

and we can conclude that $H^*_{\mathrm{virt}}((\mathbb{C}P^m)^n, \mathfrak{S}_n, \mathbb{Z})$ is isomorphic to the subring of $\mathbb{Z}[x_1, \ldots, x_n]/\langle x_1^{m+1}, \ldots, x_n^{m+1}\rangle[\mathfrak{S}_n]$ generated by

$$W = \{(1\,1_{\mathfrak{S}_n})\} \cup \{(x_i\,1_{\mathfrak{S}_n}) \mid 1 \leq i \leq n\} \cup \left\{ \left(\sum_{j=0}^{m} x_k^j x_l^{m-j}\,(k,l) \right) \mid 1 \leq k < l \leq n \right\}$$

Example (3.4). Let's pay particular attention to the case on which $n = 2$ and M is a connected, differentiable, compact and closed manifold of dimension d. Denote by $\Omega \in H^d(M;\mathbb{R})$ the generator of the top cohomology, then we have that

$$(\Delta_!1)(\Delta_!1) = \chi(M)\Omega \otimes \Omega$$

where $\chi(M)$ is the Euler number of M (see [BT82], Pro. 11.24). Moreover, by the properties of the pushforward in cohomology we have that for any $\alpha, \beta \in H^*(M;\mathbb{R})$

$$(\Delta_!1)(\alpha \otimes \beta) = \Delta_!(\Delta^*(\alpha \otimes \beta)) = \Delta_!(\alpha\beta),$$

hence, if $\alpha_1\beta_1 = \alpha_2\beta_2$ we have that

$$(\Delta_!1)(\alpha_1 \otimes \beta_1) = (\Delta_!1)(\alpha_2 \otimes \beta_2)$$

and if $deg(\alpha) + deg(\beta) > d$

$$(\Delta_!1)(\alpha \otimes \beta) = 0\,.$$

If we consider the ring

$$H^*(M;\mathbb{R})^{\otimes 2}[u]$$

where u represents the element $(\Delta_!1)$ then we can se that $u^3 = 0$ and $u^2 - \chi(M)\Omega \otimes \Omega$. The annihilator ideal of u is generated by the elements $\alpha \otimes \beta$ where $deg(\alpha) + deg(\beta) > d$, and by the elements $(\alpha_1 \otimes \beta_1 - \alpha_2 \otimes \beta_2)$ where $\alpha_1\beta_1 = \alpha_2\beta_2$.

In the case that $M = \mathbb{C}P^m$ we can see that $H^*_{\mathrm{virt}}((\mathbb{C}P^m)^2, \mathfrak{S}_2;\mathbb{Z})$ is isomorphic to

$$\mathbb{Z}[x, y, u]/\langle x^{m+1}, y^{m+1}, u^2 - (m+1)x^m y^m, u(x-y)\rangle$$

where $u^3 = 0$ because $u^3 = (m+1)x^m y^m u = (m+1)x^{m-1}y^{m+1}u = 0$.

In the case that $m = 1$ we have that $H^*_{\mathrm{virt}}((\mathbb{C}P^1)^2, \mathfrak{S}_2;\mathbb{Z})$ is

$$\mathbb{Z}[x, y, u]/\langle x^2, y^2, u^2 - 2xy, u(x-y)\rangle\,.$$

The $\mathbb{Z}/2$ invariants are generated as an $\mathbb{R}$-module by $x+y$, u, xy and ux. Therefore, if we take $w = x + y$, then $w^2 = 2xy = u^2$, $2ux = uw$ and $w^3 = 0$. So we can conclude that

$$H^*_{\mathrm{virt}}([(\mathbb{C}P^1)^2/\mathfrak{S}_2];\mathbb{R}) \cong \mathbb{R}[w, u]/\langle w^3, u^3, u^2 - w^2\rangle\,.$$

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ADAPTIVE POLICIES FOR STOCHASTIC SYSTEMS UNDER A RANDOMIZED DISCOUNTED COST CRITERION

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ABSTRACT. The paper deals with a class of discrete-time stochastic control processes under a discounted optimality criterion with random discount rate, and possibly unbounded costs. The state process $\{x_t\}$ and the discount process $\{\alpha_t\}$ evolve according to the coupled difference equations $x_{t+1} = F(x_t, \alpha_t, a_t, \xi_t)$, $\alpha_{t+1} = G(\alpha_t, \eta_t)$ where the state and discount disturbance processes $\{\xi_t\}$ and $\{\eta_t\}$ are sequences of i.i.d. random variables with *unknown* distributions θ^{ξ} and θ^{η} respectively. Assuming observability of the process $\{(\xi_t, \eta_t)\}$, we use the empirical estimator of its distribution to construct asymptotically discounted optimal policie.

1. Introduction

Among the main motivations to study a discounted optimality criterion in stochastic control problems are 1) the mathematical convenience (the discounted criterion is the best understood of all performance indices), and 2) its natural economic or financial interpretation (see, for instance, [35]). In both cases, the discount factor is typically assumed to be fixed or constant on the course of the process, which simplifies the mathematical analysis. However, from the point of view of applications, this assumption might be too strong or unrealistic. Indeed, in economic and financial models (see e.g., [2], [9], [14], [21], [28], [32], [35]), the discount factor is typically a function of interest rates, which in turn are random variables. In these cases, we have a time-varying random discount factor that can be represented as a stochastic process.

In this paper we consider a class of discrete-time stochastic control processes under a discounted optimality criterion with random discount rate. The state and discount processes evolve according to the difference equations:

$$x_{t+1} = F(x_t, \alpha_t, a_t, \xi_t), \tag{1.1}$$

$$\alpha_{t+1} = G(\alpha_t, \eta_t), \tag{1.2}$$

for $t = 0, 1, \ldots,$ where F and G are known continuous functions, x_t, α_t, and a_t are the state, the discount rate, and the control at time t, respectively. Moreover, the state and discount disturbance process $\{\xi_t\}$ and $\{\eta_t\}$ are observable sequences of independent and identically distributed (i.i.d.) random variables with *unknown* distributions θ^{ξ} and θ^{η}, respectively.

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The actions applied by the controller at the decision times are selected according to rules known as control policies. The role of such policies is to minimize a discounted performance index with possibly unbounded one-stage cost and a random discount rate that varies as in (1.2). Clearly, this performance index depends on the unknown distributions θ^{ξ} and θ^{η}. Thus, to construct "minimizing" policies, the controller must combine control tasks with suitable statistical estimation methods of the joint distribution θ of the random variables ξ and η. The resulting policy of this procedure is called *adaptive*.

Our approach consists in estimating θ by means of the empirical distribution θ_t of the process $\{(\xi_t, \eta_t)\}$. This method is very general in the sense that θ can be arbitrary. However, since the discounted cost criterion depends strongly on the decision selected at first stages (precisely when the information about θ is deficient) we can not ensure, in general, the existence of an optimal adaptive policy. Thus, the discounted optimality will be analyzed in an asymptotic sense (see [33], [16]).

The discounted cost criterion in stochastic control problems has been widely studied under different approaches: dynamic programming (see, e.g., [16], [18], [19], [20], [24], [27], [29]); convex analysis (see, e.g., [5], [6], [7], [26]); linear programming (see, e.g., [15]); Lagrange multipliers (see, e.g., [25]); adaptive procedures (see, e.g., [3], [12], [16], [17], [22], [23]); see also [5], [6], [7], [34] for other variants. In these references, a fixed (non-random) discount factor is assumed. Recently, in [13], the discount criterion with a random discount factor was studied under the assumption that the components of the corresponding control model are known by the controller. In contrast, the main feature of this paper is that the distribution of the state and the discount disturbances are unknown.

There are other models related with our problem, the so–called models with imperfect information. For instance, suppose that the joint distribution of the random vectors ξ and η is $\theta(z)$ where z is a time invariant parameter which is unknown by the controller. Of course, since $\theta(z)$ is unknown, we can apply our approach to solve the corresponding optimal control problem. However, considering a parametric model allows us to implement a Bayesian approach. Indeed, starting with a prior distribution of z and using a Bayesian filtering process, the distribution for this unknown parameter is updated over time. Then following standard techniques on partially observable Markov control processes [see, e.g., [4], [16]], we transform this problem to an equivalent and fully observable problem, in which is possible to obtain optimal control policies instead of asymptotically optimal policies. However, from the point of view of the applications and the implementation of such policies, this procedure has a disadvantage. While a finite dimensional space suffice to accommodate the system state (1.1)–(1.2) in our approach, an infinite dimensional space is required in the Bayesian approach. On the other hand, let us suppose that we have no way of making any estimation to the unknown join distribution θ, then we can think the problem we have as a game against nature and look for a minmax policy. This is a work that is in process.

The paper is organized as follows. In Section 2 we introduce the Markov control model we are concerned with. Next, in Section 3, we present the discounted optimality criterion with random discount rates. Section 4 contains the basic assumptions and some preliminary results on the discounted criterion and the estimation process. The construction of adaptive control policies together with our main results are introduced in Section 5 and proved in Section 6. Finally, in Section 7 we present a consumption-investment example to illustrate our assumptions and results.

Notación. Given a Borel space X (that is, a Borel subset of a complete and separable metric space) its Borel sigma-algebra is denoted by $\mathcal{B}(X)$, and "measurable", for either sets or functions, means "Borel measurable". Given a Borel space X, we denote by $\mathbb{P}(X)$ the family of probability measures on X, endowed with the weak topology. Let X and Y be Borel spaces. Then a stochastic kernel $\gamma(dx \mid y)$ on X given Y is a function such that $\gamma(\cdot \mid y)$ is a probability measure on X for each fixed $y \in Y$, and $\gamma(B \mid \cdot)$ is a measurable function on Y for each fixed $B \in \mathcal{B}(X)$.

2. Markov control model

The Markov control process associated to the system (1.1)-(1.2) is specified by the elements

$$\mathcal{M} := \left(X, \Gamma, A, S_1, S_2, P_1, P_2, c\right) \tag{2.1}$$

satisfying the following conditions. The state space X, the action space A, and the state and discount disturbance spaces S_1 and S_2, respectively, are Borel spaces. The set $\Gamma := [\alpha^*, \infty)$, $\alpha^* > 0$, is the discount rate space. For each pair $(x, \alpha) \in X \times \Gamma$, $A(x, \alpha)$ is a nonempty Borel subset of A denoting the set of admissible controls when the system is in state x and a discount rate α is imposed. The set

$$\mathbb{K} = \{(x, \alpha, a) : x \in X, \alpha \in \Gamma, a \in A(x, \alpha)\} \tag{2.2}$$

of admissible state-discount-action triplets is assumed to be a Borel subset of the Cartesian product of X, Γ, and A. In addition, the transition law P_1, corresponding to (1.1), is a stochastic kernel on X given $\mathbb{K}$, that is, for all $t \geq 0$, $(x, \alpha, a) \in \mathbb{K}$ and $B \in \mathcal{B}(X)$,

$$\begin{aligned} P_1(B|x, \alpha, a) :&= \text{Prob}\left[F(x_t, \alpha_t, a_t, \xi_t) \in B | x_t = x, \alpha_t = \alpha, a_t = a\right] \\ &= \int_{S_1} 1_B\left(F(x, \alpha, a, s)\right) \theta^{\xi}(ds), \end{aligned} \tag{2.3}$$

where $F : X \times \Gamma \times A \times S_1 \to X$, the function in (1.1), is continuous, $1_B(\cdot)$ denotes the indicator function of the set B, and $\{\xi_t\}$ is a sequence of i.i.d. random variables in S_1 and common *unknown* distribution $\theta^{\xi} \in \mathbb{P}(S_1)$. Similarly, for all $t \geq 0$, $\alpha \in \Gamma$ and $D \in \mathcal{B}(\Gamma)$, the transition law P_2, corresponding to (1.2), is defined as:

$$\begin{aligned} P_2(D|\alpha) &:= \text{Prob}\left[G(\alpha_t, \eta_t) \in D | \alpha_t = \alpha\right] \\ &= \int_{S_2} 1_D\left(G(\alpha, s)\right) \theta^{\eta}(ds), \end{aligned} \tag{2.4}$$

where $G\colon \Gamma\times S_2 \to \Gamma$, the function in (1.2), is continuous, and $\{\eta_t\}$ is a sequence of i.i.d. random variables in S_2 (independent of the process $\{\xi_t\}$) with unknown distribution $\theta^\eta \in \mathbb{P}(S_2)$. Finally, the cost-per-stage $c(x, \alpha, a)$ is a measurable real-valued function on $\mathbb{K}$, possibly unbounded.

The control model $\mathcal{M}$ has the following interpretation. At stage t, the system is in the state $x_t = x \in X$ and the discount factor $\alpha_t = \alpha \in \Gamma$ is imposed. Then, the controller gets estimates θ_t^ξ and θ_t^η of the unknown distributions θ^ξ and θ^η, respectively, and combines these estimates with the history of the system to select a control $a = a_t(\theta_t^\xi, \theta_t^\eta) \in A(x, \alpha)$. As a consequence of this the following happens: 1) a cost $c(x, \alpha, a)$ is incurred, and 2) the system moves to a new state $x_{t+1} = x'$ and a new discount factor $\alpha_{t+1} = \alpha'$ is imposed according to the transition laws (2.3) and (2.4). Once the transition to state x' occurs, the process is repeated.

Control policies. We define the space of admissible histories up to time t by $\mathbb{H}_0 := X \times \Gamma$ and $\mathbb{H}_t := (\mathbb{K} \times S_1 \times S_2)^t \times X \times \Gamma,\ t \geq 1$. A generic element of $\mathbb{H}_t$ is written as $h_t = (x_0, \alpha_0, a_0, \xi_0, \eta_0, \ldots, x_{t-1}, \alpha_{t-1}, a_{t-1}, \xi_{t-1}, \eta_{t-1}, x_t, \alpha_t)$. A (randomized, history-dependent) control policy is a sequence $\pi = \{\pi_t\}$ of stochastic kernels π_t on A given $\mathbb{H}_t$ such that $\pi_t(A(x_t, \alpha_t) \mid h_t) = 1$, for all $h_t \in \mathbb{H}_t,\ t \geq 0$. We denote by Π the set of all control policies and by $\mathbb{F} \subset \Pi$ the set of all (deterministic) stationary policies. As usual, every stationary policy $\pi \in \mathbb{F}$ is identified with some measurable function $f\colon X \times \Gamma \to A$ such that $f(x, \alpha) \in A(x, \alpha)$ for every $(x, \alpha) \in X \times \Gamma$, taking the form $\pi = \{f, f, f, \ldots\} =: f$. In this case we use the notation

$$c(x, \alpha, f) := c(x, \alpha, f(x, \alpha)) \quad \text{and} \quad F(x, \alpha, f, s) := F(x, \alpha, f(x, \alpha), s)$$

for all $x \in X,\ \alpha \in \Gamma$ and $s \in S$.

3. Discounted criterion

We assume that the costs are exponentially discounted with accumulative random discounted rates. That is, a cost C incurred at stage t is equivalent to a cost $C\exp(-S_t)$ at time 0, where $S_t = \sum_{i=0}^{t-1} \alpha_i$ if $t \geq 1$, $S_0 = 0$. In this sense, when using a policy $\pi \in \Pi$, given the initial state $x_0 = x$ and the initial discount factor $\alpha_0 = \alpha$, we define the total expected discounted cost (with random discount rates) as

$$V(\pi, x, \alpha) := E_{(x,\alpha)}^{\pi} \left[\sum_{t=0}^{\infty} \exp(-S_t) c(x_t, \alpha_t, a_t) \right], \tag{3.1}$$

where $E_{(x,\alpha)}^{\pi}$ denotes the expectation operator with respect to the probability measure $P_{(x,\alpha)}^{\pi}$ induced by the policy π, given $x_0 = x$ and $\alpha_0 = \alpha$. (see, e.g., [4] for the construction of $P_{(x,\alpha)}^{\pi}$).

The optimal control problem associated to the control model $\mathcal{M}$, is then to find an optimal policy $\pi^* \in \Pi$ such that $V(\pi^*, x, \alpha) = V^*(x, \alpha)$ for all $(x, \alpha) \in X \times \Gamma$, where

$$V^*(x, \alpha) := \inf_{\pi \in \Pi} V(\pi, x, \alpha) \tag{3.2}$$

is the optimal value function.

Remark (3.3). (a) From (1.2), observe that $\{\exp(-S_t)\}$ is a sequence of random variables (not necessarily independent) representing the rate of discount at each stage t. Moreover, if $\alpha_t = \alpha$ for all $t \geq 0$ and some $\alpha \in (0, \infty)$, the performance index (3.1) reduces to the usual $\beta-$discounted cost criterion with $\beta = \exp(-\alpha)$.

(b) From (3.1) and (3.2), observe that V depends on the unknown distributions θ^ξ and θ^η), therefore V^* is truly unknown.

In the context of our work (unknown distributions θ^ξ and θ^η) we must combine suitable statistical estimation methods of θ^ξ and θ^η with control procedures in order to construct optimal policies. However, as the performance index (3.1) depends heavily on the controls selected at the first stages (precisely when the information about the distributions θ^ξ and θ^η is deficient), we can not ensure, in general, the existence of such policies. Thus, the optimality of policies constructed in this paper will be studied in the following asymptotic sense.

Definition (3.4). A policy $\pi \in \Pi$ is said to be asymptotically discounted optimal for the control model $\mathcal{M}$ if

$$\left| V^{(n)}(\pi, x, \alpha) - E^{\pi}_{(x,\alpha)} \left[V^*(x_n, \alpha_n) \right] \right| \to 0 \ \ as \ \ n \to \infty, \ \text{for all } (x, \alpha) \in X \times \Gamma,$$

where

$$V^{(n)}(\pi, x, \alpha) := E^{\pi}_{(x,\alpha)} \left[\sum_{t=n}^{\infty} \exp(-S_{n,t}) c(x_t, \alpha_t, a_t) \right] \tag{3.5}$$

is the total expected discounted cost from stage n onward, and

$$S_{n,t} = \sum_{k=n}^{t-1} \alpha_k \ \text{ for } \ t > n, \quad S_{n,n} = 0. \tag{3.6}$$

Clearly, discounted optimality implies asymptotic discounted optimality. The notion of asymptotic optimality was introduced by Schall [33] to study a problem of estimation and control in dynamic programming (see also [12], [16], [22]).

4. Assumptions and preliminary results

Observe that we can write the system (1.1)-(1.2) as

$$y_{t+1} = H(y_t, a_t, \chi_t), \quad t = 0, 1, \ldots,$$

where, letting $Y := X \times \Gamma$, $S := S_1 \times S_2$, $y_t^T := (x_t, \alpha_t)$, and $\chi_t := (\xi_t, \eta_t)$, $H\colon Y \times A \times S \to Y$ is a continuous function defined as

$$H(y_t, a_t, \chi_t) := (F(x_t, \alpha_t, a_t, \xi_t), G(\alpha_t, \eta_t))^T,$$

and $\{\chi_t\}$ is a sequence of i.i.d. $S-$valued random variables, defined on an underlying probability space $(\Omega, \mathcal{F}, P)$, with unknown common distribution $\theta(\cdot) = \theta^\xi(\cdot)\theta^\eta(\cdot)$. Thus

$$\theta(B) = P(\chi_t \in B), \ t \geq 0, \ B \in \mathcal{B}(S).$$

In the remainder, the probability space $(\Omega, \mathcal{F}, P)$ is fixed and $a.s.$ means *almost surely with respect to* P.

Now, for notational convenience, we put the control model $\mathcal{M}$ in the form

$$\left(Y, A, \{A(y) \subset A | y \in Y\}, Q, c\right),$$

where Q is the stochastic kernel on Y given $\mathbb{K} = \{(y, a) : y \in Y, a \in A(y)\}$ (see (2.2)) defined as

$$\begin{aligned} Q(B|y, a) &:= \text{Prob}\left[y_{t+1} \in B | y_t = y, a_t = a\right] \\ &= \int_S 1_B\left(H(y, a, s)\right) \theta(ds) \\ &= \theta\left(\{s \in S \ : \ H(y, a, s) \in B\}\right), \quad B \in \mathcal{B}(Y). \end{aligned}$$

We shall require two sets of assumptions. In the first one, Assumption (4.1), we impose continuity and compactness conditions to ensure the existence of minimizers and a solution to the optimality equation, while Assumption (4.3) are technical requirements to get a suitable estimation process of the distribution θ (see Remark (4.5)(c) below). Note that Assumption (4.1)(a) allows a unbounded one-stage cost function $c(y, a)$ provided that it is majorized by a "bounding" function W.

Assumption (4.1). a) For each $y \in Y$, the set $A(y)$ is compact.
b) For all $y \in Y$ the function $a \to c(y, a)$ is lower semi-continuous (l.s.c.) on $A(y)$. Moreover, there exists a continuous function $W \colon Y \to [1, \infty)$ and a constant M such that

$$\sup_{a \in A(y)} c(y, a) \leq MW(y), \quad y \in Y.$$

c) The function

$$\bar{v}(y, a) := \int_S v(H(y, a, s))\theta(ds)$$

is continuous and bounded on $\mathbb{K}$ for every measurable bounded function v on Y.
d) There exist constants $p > 1$ and $\beta_0 < \infty$ satisfying $1 \leq \beta_0 < \exp(\alpha^*)$ such that for all $y \in Y$ and $a \in A(y)$,

$$W^p\left(H(y, a, \chi_0)\right) \leq \beta_0 W^p(y) \quad \text{a.s.} \tag{4.2}$$

In addition, Assumption c) holds when v is replaced with W.

An equivalent condition to relation (4.2) is that for all $y \in Y$ and $a \in A(y)$

$$W\left(H(y, a, \chi_0)\right) \leq \beta_0' W(y) \quad \text{a.s.},$$

for some $1 \leq \beta_0' < \exp(\alpha^*)$. However, for convenience we use Assumption (4.1) (d).

Assumption (4.3). a) The family of functions

$$\mathcal{V}_W := \left\{ \frac{V^*\left(H(y, a, .)\right)}{W(y)} : (y, a) \in \mathbf{K} \right\}$$

is equicontinuous on S, where V^* is the optimal value function (see (3.2)).

b) The function

$$\varphi(s) := \sup_{(y,a)} \left[W(y)\right]^{-1} W\left(H(y,a,s)\right)$$

is continuous on S.

Remark (4.4). Clearly Assumption (4.3) (a) holds if S is a countable set. In addition, the function φ in Assumption (4.3) might be non continuous. In such case we replace Assumption (4.3) (b) by supposing the existence of a continuous majorant $\bar{\varphi}$ of φ such that $E\left[\bar{\varphi}(\chi_0)\right]^p < \infty$ (see (4.5) (c) below).

To estimate θ we use the empirical distribution $\{\theta_t\} \subset \mathbb{P}(S)$ of the disturbance process $\{\chi_t\}$, defined as follows. Let $\nu \in \mathbb{P}(S)$ be a given arbitrary probability measure. Then

$$\theta_0 := \nu,$$

$$\theta_t(B) := \frac{1}{t}\sum_{i=0}^{t-1} 1_B(\chi_i), \quad \text{for all } t \geq 1 \text{ and } B \in \mathcal{B}(S).$$

Remark (4.5). a) Observe that the inequality ((4.2)) implies, for all $(y,a) \in \mathbb{K}$,

$$\int_S W^p\left(H(y,a,s)\right)\theta_t(ds) = \frac{1}{t}\sum_{i=0}^{t-1} W^p\left(H(y,a,\chi_i)\right) \leq \beta_0 W^p(y) \quad \text{a.s.,} \tag{4.6}$$

which in turn yields [see Lemma (6.1) below]

$$\int_S W^p\left(H(y,a,s)\right)\theta(ds) \leq \beta_0 W^p(y).$$

b) It is well-known (See, e.g., [8]) the fact that θ_t converges weakly to θ *a.s.*, that is,

$$\int u d\theta_t \to \int u d\theta \quad \text{a.s.} \quad \text{as } t \to \infty,$$

for every real-valued, continuous and bounded function u on S.
c) Furthermore, from Assumption (4.1) (d)

$$E\left[\varphi(\chi_0)\right]^p < \infty\,.$$

Thus, from Assumption (4.3), using the fact that $V^*(y) \leq CW(y)$ (see Proposition (4.10) below), and applying Theorem 6.4 in [30], we get

$$D_t \to 0 \ \text{ a.s., as } \ t \to \infty, \tag{4.7}$$

where

$$D_t := \sup_{(y,a)\in\mathbb{K}} \left| \int_S \frac{V^*(H(y,a,s))}{W(y)}\theta_t(ds) - \int_S \frac{V^*(H(y,a,s))}{W(y)}\theta(ds)\right|. \tag{4.8}$$

We denote by $\mathbb{B}_W$ the normed linear space of all measurable functions $u\colon Y \to \Re$ with a finite norm $\|u\|_W$ defined as

$$\|u\|_W := \sup_{y\in Y} \frac{|u(y)|}{W(y)}. \tag{4.9}$$

A first consequence of Assumption (4.1), which is stated in [13], is the following proposition.

PROPOSITION (4.10). *Suppose that Assumption* (4.1) *holds. Then* $V^* \in \mathbb{B}_W$, *that is, there exists a constant* $C > 0$ *such that*

$$V^*(y) \leq CW(y) \quad \text{for all } \; y \in Y. \tag{4.11}$$

In addition, V^* *satisfies the optimality equation*

$$V^*(y) = \inf_{a \in A(y)} \left(c(y,a) + \exp(-\alpha) \int_S V^*(H(y,a,s))\theta(ds) \right), \quad \forall\, y \in Y. \tag{4.12}$$

5. Main results

Let $\{V_t\}$ be a sequence of functions in $\mathbb{B}_W$ defined as $V_0 \equiv 0$, and for $t \geq 1$,

$$V_t(y) = \inf_{a \in A(y)} \left(c(y,a) + \exp(-\alpha) \int_S V_{t-1}(H(y,a,s))\theta_t(ds) \right), \quad y \in Y. \tag{5.1}$$

A straightforward calculation shows that for some constant C',

$$V_t(y) \leq C'W(y) \quad \text{a.s,} \quad y \in Y, \;\; t \geq 0. \tag{5.2}$$

Now, applying standard arguments on the existence of minimizers (see, e.g., [31]), under Assumption (4.1) and the continuity of H, for each $t > 0$ and $\delta_t > 0$, there exists $f_t \in \mathbb{F}$ such that

$$c(y,f_t) + \exp(-\alpha) \int_S V_{t-1}(H(y,f_t,s))\theta_t(ds) \leq V_t(y) + \delta_t \quad \text{a.s.} \;\; y \in Y\,. \tag{5.3}$$

Definition (5.4). Let $\{\delta_t\}$ be an arbitrary sequence of positive numbers such that $\delta_t \to 0$ as $t \to \infty$, and let $\{f_t\}$ be a sequence of functions in $\mathbb{F}$ satisfying (5.3). We define the policy $\hat{\pi} = \{\hat{\pi}_t\}$ as

$$\hat{\pi}_t(h_t) = \hat{\pi}_t(h_t;\theta_t) := f_t(y_t), \;\; t > 0,$$

and $\hat{\pi}_0(y)$ is any fixed action in $A(y)$.

We can state our main results as follows:

THEOREM (5.5). *Under Assumptions* (4.1) *and* (4.3), *we have*
a) $\|V_t - V^*\|_W \to 0$ *a.s., as* $t \to \infty$;
b) *The policy* $\hat{\pi}$ *is asymptotically discount optimal.*

6. Proofs

The proof of Theorem (5.5) is based on the following results.

LEMMA (6.1). *Suppose that Assumption* (4.1) *holds. Then:*
a) *For all* $y \in Y$ *and* $a \in A(y)$,

$$\int_S W^p \left(H(y,a,s)\right) \theta(ds) \leq \beta_0 W^p(y)\,. \tag{6.2}$$

b) *For all* $y \in Y$, $a \in A(y)$, *and* $t > 0$,

$$\int_S W \left(H(y,a,s)\right) \theta_t(ds) \leq \beta W(y) \quad \textit{a.s.} \tag{6.3}$$

and

$$\int_S W \left(H(y,a,s)\right) \theta(ds) \leq \beta W(y), \tag{6.4}$$

where $\beta := \beta_0^{1/p}$.

c) *For all* $y \in Y$ *and* $\pi \in \Pi$, *we have*

$$\sup_{t>0} E_y^\pi \left[W^p(y_t)\right] < \infty \ \ and \ \ \sup_{t>0} E_y^\pi \left[W(y_t)\right] < \infty\,.$$

Proof. It is clear that the part (a) follows from Assumption (4.1) (d). Next, the part (b) follows from the relations (4.6) and (6.2), and applying Jensen's inequality, while part (c) is a consequence of (6.2) and (6.4) (see details in [10], [11], [12], [19], [22]). □

We also need the following characterization of asymptotic optimality (see Definition (3.4)) which is an adaptation of Theorem 4.6.2 in [18] (see also [33]) to our context (randomized discounted cost criterion).

LEMMA (6.5). *A policy* $\pi \in \Pi$ *is asymptotically discount optimal for the control model* $\mathcal{M}$ *if, for all* $y \in Y$,

$$E_y^\pi \left[\Phi(y_t, a_t)\right] \to 0 \ as\ t \to \infty,$$

where

$$\Phi(y,a) := c(y,a) + \exp(-\alpha) \int_S V^*\left(H(y,a,s)\right)\theta(ds) - V^*(y), \ (y,a) \in \mathbb{K}\,. \tag{6.6}$$

(Note that, by (4.12), Φ *is nonnegative.)*

Proof. Observe that for each $\pi \in \Pi$, $y \in Y$, and $t \geq 0$,

$$\Phi(y_t, a_t) = E_y^\pi \left[c(y_t,a_t) + \exp(-\alpha_t)V^*(y_{t+1}) - V^*(y_t)|h_t, a_t\right]\,,$$

where h_t represent the history of the system up to time t (see definition of control policies). Hence, from definition (3.5) and (3.6), using the fact $\exp(-S_{n,t})$ $\exp(-\alpha_t) = \exp(-S_{n,t+1})$, and applying the properties of conditional expectation, we have, for each $n \geq t$, $\pi \in \Pi$, and $y \in Y$

$$\begin{aligned}
&\sum_{t=n}^{\infty} E_y^\pi \left[\exp(-S_{n,t})\Phi(y_t,a_t)\right] \\
&= \sum_{t=n}^{\infty} E_y^\pi \left[\exp(-S_{n,t}) E_y^\pi \left[c(y_t,a_t) + \exp(-\alpha_t)V^*(y_{t+1}) - V^*(y_t)|h_t,a_t\right]\right] \\
&= \sum_{t=n}^{\infty} E_y^\pi \left[\exp(-S_{n,t})c(y_t,a_t)\right] + \sum_{t=n}^{\infty} E_y^\pi \left[\exp(-S_{n,t+1})V^*(y_{t+1})\right. \\
&\qquad \left. - \exp(-S_{n,t})V^*(y_t)\right] \\
&= V^{(n)}(\pi,y) - E_y^\pi\left[V^*(y_n)\right] + \lim_{m\to\infty} E_y^\pi\left[\exp(-S_{n,m})V^*(y_m)\right] \qquad (6.7)\\
&= V^{(n)}(\pi,y) - E_y^\pi\left[V^*(y_n)\right],
\end{aligned}$$

where the last equality follows from Holder's inequality, Lemma (6.1) (c), (4.11), the fact $\alpha^* \leq \alpha_t$, $t \geq 0$, and the following relation

$$\begin{aligned}
\lim_{m\to\infty} E_y^\pi\left[\exp(-S_{n,m})V^*(y_m)\right] &\leq \lim_{m\to\infty}\left(E_y^\pi\left[\exp(-p'S_{n,m})\right]\right)^{1/p'}(E_y^\pi[V^*(y_m)]^p)^{1/p}\\
&\leq \lim_{m\to\infty} C(E_y^\pi[W(y_m)]^p)^{1/p}(\exp(-p'\alpha^*(m-n)))^{1/p'}\\
&\leq CM\lim_{m\to\infty}(\exp(-p'\alpha^*(m-n)))^{1/p'}\\
&= 0\,.
\end{aligned}$$

(See Lemma (6.1) (c) for constant M).

Finally, since the limit

$$\lim_{t\to\infty} E_y^\pi\left[\Phi(y_t,a_t)\right] = 0$$

implies

$$\lim_{n\to\infty}\sum_{t=n}^{\infty} E_y^\pi\left[\exp(-S_{n,t})\Phi(y_t,a_t)\right] = 0$$

then, the relation (6.7) yields the desired result. □

We define the operators

$$Tu(y) := \inf_{a\in A(y)}\left\{c(y,a) + \exp(-\alpha)\int_S u\left(H(y,a,s)\right)\theta(ds)\right\},$$

$$T_t u(y) := \inf_{a\in A(y)}\left\{c(y,a) + \exp(-\alpha)\int_S u\left(H(y,a,s)\right)\theta_t(ds)\right\},$$

for all $y \in Y$ and $u \in \mathbb{B}_W$. Observe that from Assumption (4.1) and Lemma (6.1), T and T_t map $\mathbb{B}_W$ to itself. In addition, following similar ideas of Proposition 8.3.9 in [19], we have that T and T_t are contraction operators with modulus $\gamma := \beta_0 \exp‡í(-\alpha^*) < 1$ (see Assumption (4.1) (d)), respect to the norm $\|\cdot\|_W$. That is, for all $u, v \in \mathbb{B}_W$,

$$\|Tv - Tu\|_W \leq \gamma\,\|v-u\|_W$$

and

$$\|T_t v - T_t u\|_W \leq \gamma\,\|v-u\|_W \text{ a.s.}$$

(6.8) Proof of Theorem (5.5)**.** a) From (4.11)-(5.2), $V^*, V_t \in \mathbb{B}_W,\ t > 0,$

$$TV^* = V^* \qquad \text{and} \qquad T_tV_{t-1} = V_t \text{ a.s. } \forall\, t > 0. \tag{6.9}$$

Hence

$$\begin{aligned}
\|V^*-V_t\|_W &\leq \|TV^*-T_tV^*\|_W + \|T_tV^*-T_tV_{t-1}\|_W\\
&\leq \|TV^*-T_tV^*\|_W + \gamma\,\|V^*-V_{t-1}\|_W \quad \text{a.s.}
\end{aligned}\tag{6.10}$$

Now, from definition of T and T_t, and (4.8),

$$\begin{aligned}
\|TV^*-T_tV^*\|_W &\leq \sup_{(y,a)\in I\!K}\left|\int_S \frac{V^*\left(H(y,a,s)\right)}{W(y)}\theta_t(ds) - \int_S \frac{V^*\left(H(y,a,s)\right)}{W(y)}\theta(ds)\right|\\
&= D_t \quad \text{a.s.}
\end{aligned}\tag{6.11}$$

Combining (6.10) and (6.11), we have,

$$\|V^*-V_t\|_W \leq D_t + \gamma\,\|V^*-V_{t-1}\|_W \quad a.s. \tag{6.12}$$

Finally, denoting $l := \limsup_{t\to\infty} \|V^* - V_t\|_W < \infty$ (see (4.11) and (5.2)) and taking limsup on both sides of (6.12), from (4.7) we obtain $l \le \gamma l$, which implies (since $0 < \gamma < 1$) that $l = 0$. This proves the part (a).

b) For each $t > 0$, we define the function $\Phi_t \colon \mathbb{K} \to \mathbb{R}$ (see (6.6)) as

$$\Phi_t(y, a) := c(y, a) + \exp(-\alpha) \int_S V_{t-1}\left(H(y, a, s)\right) \theta_t(ds) - V_t(y).$$

We also define, for each $t > 0$,

$$\Psi_t := \sup_{y\in Y}[W(y)]^{-1} \sup_{a\in A(y)} |\Phi(y, a) - \Phi_t(y, a)| \,. \tag{6.13}$$

Observe that from definition of the policy $\hat{\pi}$, we have (see (5.3)) $\Phi_t\left(., \hat{\pi}_t(.)\right) \le \delta_t$, for each $t > 0$. Hence,

$$\begin{aligned}\Phi\left(y_t, \hat{\pi}_t(h_t)\right) &\le \left|\Phi\left(y_t, \hat{\pi}_t(h_t)\right) - \Phi_t\left(y_t, \hat{\pi}_t(h_t)\right) + \delta_t\right| \\ &\le \sup_{a\in A(y_t)} |\Phi(y_t, a) - \Phi_t(y_t, a)| + \delta_t \\ &\le W(y_t)\Psi_t + \delta_t \quad \text{a.s.}\end{aligned}$$

Therefore, according to Lemma (6.5), to prove asymptotic optimality of $\hat{\pi}$, it is sufficient to show that

$$E_y^{\hat{\pi}}\left(W(y_t)\Psi_t\right) \to 0 \quad \text{as } t \to \infty. \tag{6.14}$$

By adding and subtracting the term $\exp(-\alpha)\int_S V^*\left(H(y, a, s)\right)\theta_t(ds)$, we have, for each $(y, a) \in \mathbb{K}$ and $t > 0$,

$$\begin{aligned}|\Phi_t(y, a) - \Phi(y, a)| &\le |V^*(y) - V_t(y)| \\ &\quad + \exp(-\alpha)\int_S \left|V_{t-1}\left(H(y, a, s)\right) - V^*\left(H(y, a, s)\right)\right| \theta_t(ds) \\ &\quad + \exp(-\alpha)\left|\int_S V^*(H(y, a, s))\theta_t(ds) - \int_S V^*\left(H(y, a, s)\right)\theta(ds)\right|,\end{aligned}$$

which, from Lemma (6.1) (a), (b), and definitions of the norm $\|\cdot\|_W$ and D_t (see (4.8)), implies

$$\begin{aligned}\frac{|\Phi_t(y, a,) - \phi(y, a)|}{W(y)} &\le \|V^* - V_t\|_W \\ &\quad + \beta\exp(-\alpha)\|V^* - V_{t-1}\|_W + D_t \quad \text{a.s.}\end{aligned} \tag{6.15}$$

Thus, from Theorem (5.5) (a) and (4.7),

$$\Psi_t \to 0 \ \text{ a.s., as } t \to \infty. \tag{6.16}$$

Now observe that from (6.15), (4.7), (4.11), and (5.2), $\sup_{t>0}\Psi_t \le M_1 < \infty$ for some constant M_1. In addition, from (6.16) we have the convergence in probability

$$\Psi_t \xrightarrow{P_y^{\hat{\pi}}} 0 \quad \text{as } t \to \infty, \tag{6.17}$$

whereas from Lemma (6.1) (c)

$$\sup_{t>0} E_y^{\hat{\pi}}\left(W(y_t)\Psi_t\right)^p \le M_1^p \sup_{t>0} E_y^{\hat{\pi}}\left(W^p(y_t)\right) < \infty.$$

This implies (see, for instance, Lemma 7.6.9 in [1]) that $\{W(y_t)\Psi_t\}$ is $P_y^{\hat{\pi}}$-uniformly integrable.

On the other hand, for arbitrary positive numbers l_1 and l_2, we have, for $t > 0$,

$$P_y^{\hat{\pi}}\left(W(y_t)\Psi_t > l_1\right) \le P_y^{\hat{\pi}}\left(\Psi_t > \frac{l_1}{l_2}\right) + P_y^{\hat{\pi}}\left(W(y_t) > l_2\right),$$

which, applying Chebyshev's inequality, yields

$$P_y^{\hat{\pi}}\left(W(y_t)\Psi_t > l_1\right) \le P_y^{\hat{\pi}}\left(\Psi_t > \frac{l_1}{l_2}\right) + \frac{E_y^{\hat{\pi}}\left(W(y_t)\right)}{l_2}. \tag{6.18}$$

Now, (6.18) together with Lemma (4.5) (c) and (6.17), implies the convergence in probability

$$W(y_t)\Psi_t \xrightarrow{P_y^{\hat{\pi}}} 0 \quad \text{as } t \to \infty\,. \tag{6.19}$$

Finally, (6.14) holds from (6.19) and the fact that $\{W(y_t)\Psi_t\}$ is $P_y^{\hat{\pi}}$-uniformly integrable. □

7. Example

We consider an infinite horizon consumption-investment problem where an investor must allocate his/her current wealth x_t between investment (a_t) and consumption $(x_t - a_t)$, in each stage $t = 0, 1, 2, \ldots$. In addition, in each stage t, a discount factor $\exp(-\alpha_t)$ is imposed, which depends upon the current bank interest rate.

The state and action spaces are $X = A = [0, \infty)$, and assuming that borrowing is not allowed, the investment constraint set (i.e., the set of admissible controls) takes the form $A(x, \alpha) = [0, x]$. Moreover, we suppose that the bank receives at least an interest rate of $\exp(\alpha^*) - 1$, for some $\alpha^* > 0$. In this sense, the discount rate space is $\Gamma = [\alpha^*, \infty)$.

The state process $\{x_t\}$ and the discount process $\{\alpha_t\}$ evolve according to the coupled difference equations

$$x_{t+1} = a_t\rho(\xi_t), \qquad \alpha_{t+1} = h\alpha_t + \eta_t, \quad t = 0, 1, 2, \ldots,$$

(x_0, α_0) given, where $h > 0$, $\{\xi_t\}$ and $\{\eta_t\}$ are independent sequences of i.i.d. random variables, and independent of (x_0, α_0), having a discrete distribution with values in S_1 and S_2, respectively. In addition, $\rho\colon S_1 \to (0, \gamma]$ is a given measurable function with $1 \le \gamma < \exp(\alpha^*)$.

The one-stage cost $c(x, \alpha, a)$ is an arbitrary nonnegative measurable function, which is l.s.c. in a, and satisfying

$$\sup_{a\in A(x,\alpha)} c(x, \alpha, a) \le M\left(\bar{b}x + 1\right)^{1/p}, \quad (x, \alpha) \in X \times \Gamma, \tag{7.1}$$

for some $\bar{b} > 0$, $M > 0$, and $p > 1$.

Clearly, the Assumptions (4.1) (a), (b) and (4.3) are satisfied, by taking $W(y) = W(x, \alpha) = \left(\bar{b}x + 1\right)^{1/p}$ and from Remark (4.4). We get Assumption

(4.1) (d) from the following relations: for all $y = (x, \alpha) \in Y = X \times \Gamma$, and $a \in A(y) = [0, x]$,

$$
\begin{aligned}
W^p[H(y, a, \chi_0)] &= \bar{b} a \rho(\xi_0) + 1 \\
&\leq \bar{b} x \gamma + 1 \leq \bar{b} x \gamma + \gamma \\
&= \gamma(\bar{b} x + 1) \\
&= \beta_0 W^p(y),
\end{aligned}
$$

where $\beta_0 := \gamma$.

Finally, Assumption (4.1) (c) follows from Example C.6, Appendix C in [18].

Remark (7.2). Usually, in a consumption-investment problem where the objective is to maximize a randomized discounted reward criterion, a utility function r is considered as the one-stage return. In particular, if we take

$$r(x, \alpha, a) = b\sqrt{x - a}, \quad (x, \alpha) \in Y = X \times \Gamma, \; a \in A(x, \alpha),$$

the relation (7.1) is satisfied with the function r instead of c.

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