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# REPRESENTATIONS OF RESIDUE CLASSES BY PRODUCT OF FACTORIALS, BINOMIAL COEFFICIENTS AND SUM OF HARMONIC SUMS MODULO A PRIME

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ABSTRACT. We study various problems on distribution properties of factorials, binomial coefficients and harmonic sums modulo a large prime.

## 1. Introduction

It is not known much about the distribution properties of factorials modulo a large prime $p$. In [8], F11; it is conjectured that about $p/e$ of the residue classes modulo $p$ are missed by the sequence $n!$. If this conjecture were true, the sequence $n!$ modulo $p$ should assume about $(1-1/e)p$ distinct values, see [1] for some results of this spirit. This in turn would imply the representability of every residue class modulo $p$ as a product of two factorials. Unconditionally, the Wilson theorem implies that

$$\lambda! \cdot (p-\lambda)! \equiv \lambda \pmod p$$

holds for any even $\lambda \in \{0, 2, \ldots, p-1\}$. From this one derives that every nonzero residue class modulo $p$ can be represented as a product of three factorials modulo $p$.

However, this argument does not apply to proving the existence of representations involving factorials of integers of restricted size and do not provide with asymptotic formulas for the number of representations. Some progress in this direction has been made in [4]–[7]. In particular, multiplicative character sums and exponential sums involving these functions have been estimated. These estimates have been then applied to study various additive and multiplicative congruences with factorials of integers in short intervals.

For example, in [4] it is shown that for any nonprincipal character $\chi$ modulo $p$ we have

$$\sum_{n=L+1}^{L+N} \chi(n!) \ll N^{3/4} p^{1/8} \log^{3/4} p,$$

where $L, N$ denote nonnegative integer numbers. Combining this result with upper bound estimates for the number of solutions of special congruences (see, Lemma (2.2) below) in [4] it is shown that for a given $\lambda \not\equiv 0 \pmod p$ the number of solutions of the congruence

$$\prod_{i=1}^{7} n_i! \equiv \lambda \pmod p; \quad L+1 \le n_1, \cdots, n_7 \le L+N, \tag{1.1}$$

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asymptotically behaves like $N^7/p$, for $0 < L+1 \le L+N \le p$ and

$$Np^{-11/12}\log^{-1/2} p \to 0 \quad \text{when} \quad p \to \infty.$$

Moreover, Theorem 8 of [4] gives an asymptotic formula for the number of solutions of the congruence

$$\prod_{i=1}^{k} n_i! \equiv \lambda \pmod p; \quad L+1 \le n_1, \cdots, n_k \le L+N.$$

The number 7 of factors in (1.1) is the smallest integer $k$ for which this asymptotic formula is effective with $N = o(p)$.

In [6] additive analogies of the above results have been obtained for harmonic sums

$$H_s(n) = \sum_{i=1}^{n} \frac{1}{i^s},$$

where $s$ is a fixed positive integer and $H_s(n)$ is computed modulo $p$. That is, a nontrivial upper bound for exponential sums with $H_s(n)$ has been obtained; it has been shown that for any integer $\lambda$ the number of solutions of the congruence

$$\sum_{i=1}^{7} H_s(n_i) \equiv \lambda \pmod p; \quad L+1 \le n_1, \cdots, n_7 \le L+N,$$

asymptotically behaves like $N^7/p$, for $0 \le L < L+N < p$

$$Np^{-11/12}\log^{-1/2} p \to 0 \quad \text{when} \quad p \to \infty.$$

In [7] distribution properties of middle binomial coefficients

$$b_n = \binom{2n}{n}, \qquad n = 0, 1, \dots,$$

and Catalan numbers

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n = 0, 1, \dots,$$

have also been investigated. It has been shown in [7] that for all sufficiently large primes $p$ and every integer $\lambda$ there exist positive integers $r, s \ll p^{13/2}\log^6 p$ such that

$$b_r \equiv \lambda \pmod p \quad \text{and} \quad c_s \equiv \lambda \pmod p.$$

This substantially improved the previously known result from [2], requiring $r, s$ to be of the size $p^{O(p)}$.

Although the numbers that appear on the exponent of $p$ in the above mentioned results seem to be the barrier points, the logarithmic factors in some sense can be removed. The aim of the present paper is to study this question.

The method we use is the combination of methods of [4]–[7] with the argument described in [3].

Throughout the paper the letters $L, N, M$ are used to denote integers.

## 2. Lemmas

The following lemma is taken from [3]. It gives a sharp upper bound for the average value of a product of modulus of two linear rational sums.

LEMMA (2.1). *Let $L_1$, $L_2$, $A$, $B$ be any integers, $1 \le A, B \le p$. Then the following estimate holds:*

$$\sum_{a=0}^{p-1} \left| \sum_{x=L_1+1}^{L_1+A} e^{2\pi i \frac{ax}{p}} \right| \left| \sum_{y=L_2+1}^{L_2+B} e^{2\pi i \frac{ay}{p}} \right| \ll pA \log(BA^{-1} + 2).$$

The following result is obtained in [4].

LEMMA (2.2). *Let $0 \le L < L + N \le p$ and $k$ be a fixed positive integer. Then the number of solutions of the congruence*

$$n_1! \cdots n_k! \equiv n_{k+1}! \cdots n_{2k}! \pmod{p}; \quad L < n_1, \ldots, n_{2k} \le L + N$$

*is $\ll N^{2k-1+2^{-k}}$, where the implied constant may depend only on $k$.*

An additive analogy of Lemma (2.2) also holds for harmonic sums

$$H_s(n) = \sum_{i=1}^{n} \frac{1}{i^s},$$

where $s$ is a fixed positive integer and $H_s(n)$ is computed modulo $p$.

LEMMA (2.3). *Let $0 \le L < L + N \le p$ and $s$, $k$ be fixed positive integers. The number of solutions of the congruence*

$$\sum_{i=1}^{k} H_s(n_i) \equiv \sum_{i=k+1}^{2k} H_s(n_i) \pmod{p}; \quad L < n_1, \ldots, n_{2k} \le L + N$$

*is $\ll N^{2k-1+2^{-k}}$, where the implied constant may depend only on $k$ and $s$.*

For the proof, see [6].

## 3. Theorems and corollaries

The following theorem extends one of the results from [4].

THEOREM (3.1). *Let*

$$N \ge 1, \quad M \ge 1, \quad 0 \le L < L + N + M \le p.$$

*Then for any nonprincipal character $\chi \pmod{p}$ the following inequality holds:*

$$\sum_{x=1}^{N} \sum_{y=1}^{M} \chi((x + y + L)!) \ll MN^{3/4} p^{1/8} \log^{1/4}(NM^{-1} + 2).$$

*Furthermore, if $L - M \ge 0$, then we also have*

$$\sum_{x=1}^{N} \sum_{y=1}^{M} \chi((x - y + L)!) \ll MN^{3/4} p^{1/8} \log^{1/4}(NM^{-1} + 2).$$

Combining Theorem (3.1) and Lemma (2.2) with the method of [3], we will derive the following result.

THEOREM (3.2). *Let $1 \le N \le p$. Let $\lambda$ be an integer, $\lambda \not\equiv 0 \pmod p$. If $J$ denotes the number of solutions of the congruence*

$$n_1! \cdots n_7! \equiv \lambda \pmod p; \quad 1 \le n_1, \ldots, n_7 \le N,$$

*then*

$$J = \frac{N^7}{p-1} + O\left(N^{11/2} p^{3/8} \log^{3/4}(Np^{-11/12} + 2)\right).$$

¿From Theorem (3.2) we obtain the following consequence.

COROLLARY (3.3). *For any residue class $\lambda \not\equiv 0 \pmod p$ the congruence*

$$n_1! \cdots n_7! \equiv \lambda \pmod p$$

*holds with some positive integers $n_1, n_2, \ldots, n_7$ satisfying*

$$\max_{1 \le i \le 7} n_i \ll p^{11/12}.$$

Corollary (3.3) slightly relaxes the condition that has been posed on $n_i$ in [4].

THEOREM (3.4). *Let*

$$N \ge 1, \quad M \ge 1, \quad 0 \le L < L + N + M < p.$$

*Then the following inequality holds:*

$$\max_{\gcd(a,p)=1} \left| \sum_{x=1}^{N} \sum_{y=1}^{M} e^{2\pi i a H_s(x+y+L)/p} \right| \ll M N^{3/4} p^{1/8} \log^{1/4}(NM^{-1} + 2).$$

*Furthermore, if $L - M \ge 0$, then we also have*

$$\max_{\gcd(a,p)=1} \left| \sum_{x=1}^{N} \sum_{y=1}^{M} e^{2\pi i a H_s(x-y+L)/p} \right| \ll M N^{3/4} p^{1/8} \log^{1/4}(NM^{-1} + 2).$$

THEOREM (3.5). *Let $1 \le N \le p$. Let $\lambda$ be an integer, if $J_1$ denotes the number of solutions of the congruence*

$$H_s(n_1) + \cdots + H_s(n_7) \equiv \lambda \pmod p; \quad 1 \le n_1, \ldots, n_7 \le N,$$

*then*

$$J_1 = \frac{N^7}{p} + O\left(N^{11/2} p^{3/8} \log^{3/4}(Np^{-11/12} + 2)\right).$$

¿From Theorem (3.5) we obtain the following consequence:

COROLLARY (3.6). *For any residue class $\lambda \not\equiv 0 \pmod p$ the congruence*

$$H_s(n_1) + \cdots + H_s(n_7) \equiv \lambda \pmod p$$

*is solvable in positive integers $n_1, n_2, \ldots, n_7$ satisfying*

$$\max_{1 \le i \le 7} n_i \ll p^{11/12}.$$

Corollary (3.6) slightly relaxes the condition on $n_1, \dots, n_7$ given in [6].

Consider now the middle binomial coefficients

$$b_n = \binom{2n}{n}, \qquad n = 0, 1, \dots,$$

and Catalan numbers

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \qquad n = 0, 1, \dots,$$

where as usual we define $0! = 1$.

THEOREM (3.7). *For all sufficiently large primes p and every integer $\lambda$ there exist positive integers r, s $\ll p^{13/2}$ such that $b_r \equiv c_s \equiv \lambda \pmod p$.*

This result removes the logarithmic factor of the corresponding bound given in [7].

In the next sections we prove Theorems (3.1) and (3.2). The proofs of the other results we omit, since they follow the same lines as those of Theorems (3.1) and (3.2).

## 4. Proof of Theorem (3.1)

Denote

$$F_1 = \sum_{x=1}^{N}\sum_{y=1}^{M} \chi((x+y+L)!)$$

and

$$F_2 = \sum_{x=1}^{N}\sum_{y=1}^{M} \chi((x-y+L)!).$$

The proofs of the required estimates for $F_1$ and $F_2$ are similar, so we deal only with $F_1$.

If

$$\frac{N^{1/2}}{p^{1/4}\log^{1/2}(NM^{-1}+2)} < 10,$$

then the required inequality becomes trivial. Therefore, we can assume that

$$K := \left[\frac{N^{1/2}}{p^{1/4}\log^{1/2}(NM^{-1}+2)}\right] > 9.$$

The shifting argument gives

$$F_1 = \frac{1}{K}\sum_{k=1}^{K}\sum_{x=1}^{N}\sum_{y=1}^{M} \chi((x+y+k+L)!) + O(KM). \tag{4.1}$$

Squaring the modulus and using the Cauchy-Schwartz inequality, we obtain

$$F_1^2 \ll \frac{NM}{K^2}\sum_{x=1}^{N}\sum_{y=1}^{M}\left|\sum_{k=1}^{K}\chi\left((x+y+k+L)!\right)\right|^2 + K^2M^2. \tag{4.2}$$

Hence

$$F_1^2 \ll \frac{NM}{K^2} \sum_{k_1=1}^{K} \sum_{k_2=1}^{K} W(k_1, k_2) + K^2 M^2, \tag{4.3}$$

where

$$W(k_1, k_2) = \sum_{x=1}^{N} \sum_{y=1}^{M} \chi\left((x + y + k_1 + L)!\right) \overline{\chi\left((x + y + k_2 + L)!\right)}.$$

Substituting $z = x + y + L$, we obtain

$$|W(k_1, k_2)| = \frac{1}{p} \left| \sum_{z=0}^{p-1} \chi((z + k_1)!) \overline{\chi((z + k_2)!)} \sum_{a=0}^{p-1} \sum_{x=1}^{N} \sum_{y=1}^{M} e^{2\pi i \frac{a}{p}(z-(x+y+L))} \right|$$

$$\leq \frac{1}{p} \sum_{a=0}^{p-1} \left| \sum_{x=1}^{N} e^{2\pi i \frac{a}{p} x} \right| \left| \sum_{y=1}^{M} e^{2\pi i \frac{a}{p} y} \right| \left| \sum_{z=0}^{p-1} \chi((z + k_1)!) \overline{\chi((z + k_2)!)}\, e^{2\pi i \frac{a}{p} z} \right|.$$

By definition of $W(k_1, k_2)$ we have

$$|W(k, k)| \leq NM.$$

If $k_1 \neq k_2$, then according to the classical Weil estimate,

$$\left| \sum_{z=0}^{p-1} \chi((z + k_1)!) \overline{\chi((z + k_2)!)}\, e^{2\pi i \frac{a}{p} z} \right| \leq K p^{1/2}.$$

Indeed, if $k_1 > k_2$, then we have

$$\begin{aligned}
&\left| \sum_{z=0}^{p-1} \chi((z + k_1)!) \overline{\chi((z + k_2)!)}\, e^{2\pi i \frac{a}{p} z} \right| \leq \\
&\left| \sum_{z=0}^{p-k_1} \chi((z + k_1)!) \overline{\chi((z + k_2)!)}\, e^{2\pi i \frac{a}{p} z} \right| \leq \\
&\left| \sum_{z=0}^{p-k_1} \chi((z + k_2 + 1) \cdots (z + k_1))\, e^{2\pi i \frac{a}{p} z} \right| \leq \\
&\left| \sum_{z=0}^{p-1} \chi((z + k_2 + 1) \cdots (z + k_1))\, e^{2\pi i \frac{a}{p} z} \right| + K \leq K p^{1/2}.
\end{aligned}$$

Therefore, when $k_1 \neq k_2$ we have

$$|W(k_1, k_2)| \leq K p^{-1/2} \sum_{a=0}^{p-1} \left| \sum_{x=1}^{N} e^{2\pi i \frac{a}{p} x} \right| \left| \sum_{y=1}^{M} e^{2\pi i \frac{a}{p} y} \right| \ll K p^{1/2} M \log(NM^{-1} + 2).$$

Substituting the obtained inequalities for $W(k_1, k_2)$ in (4.3), we deduce

$$F_1^2 \ll \frac{NM}{K^2}\left(\sum_{k=1}^{K} NM + \sum_{k_1=1}^{K}\sum_{k_2=1}^{K} Kp^{1/2}M\log(NM^{-1}+2)\right) + K^2M^2 \ll$$
$$\frac{N^2M^2}{K} + Kp^{1/2}NM^2\log(NM^{-1}+2) + K^2M^2.$$

Recalling the definition of $K$ and observing that the last term never dominates and the first two terms are of the order that is needed, we conclude the proof.

## 5. Proof of Theorem (3.2)

We can assume that $N \le p/2$ since for $N > p/2$ the corresponding result from [4] gives a better formula than Theorem (3.2).

Let $\lambda \not\equiv 0 \pmod p$ and let $J = J(\lambda, N)$ be the number of solutions of the congruence

$$n_1!\cdots n_7! \equiv \lambda \pmod p; \quad 1 \le n_1, \dots, n_7 \le N.$$

Denote $r = \left[\frac{\log N}{\log 2}\right]$. We divide the interval $[1, N]$ into disjoint subintervals

$$[1, N] = \left[1, N/2^r\right] \cup \left(N/2^r, N/2^{r-1}\right] \cup \cdots \cup \left(N/4, N/2\right] \cup (N/2, N].$$

Given $1 \le j_1, j_2, j_3 \le r-2$, denote by $J(j_1, j_2, j_3)$ the number of solutions of the congruence

$$n_1!\cdots n_7! \equiv \lambda \pmod p$$

subject to the conditions

$$\frac{N}{2^{j_i}} < n_i \le \frac{N}{2^{j_i-1}}, \quad i = 1, 2, 3; \qquad 1 \le n_4, n_5, n_6, n_7 \le N.$$

Then

$$J = J_1 + O(J_2),$$

where

$$J_1 = \sum_{j_1=1}^{r-2}\sum_{j_2=1}^{r-2}\sum_{j_3=1}^{r-2} J(j_1, j_2, j_3) \tag{5.1}$$

and $J_2$ is the number of solutions of the congruence

$$n_1!\cdots n_7! \equiv \lambda \pmod p; \quad 1 \le n_1 \le 8, \qquad 1 \le n_2, \dots, n_7 \le N.$$

Note that

$$J_2 = \frac{1}{p-1}\sum_{\chi}\left(\sum_{n_1\le 8}\chi(n_1!)\right)\left(\sum_{n\le N}\chi(n!)\right)^6 \overline{\chi(\lambda)}$$
$$\ll \frac{1}{p-1}\sum_{\chi}\left|\sum_{n\le N}\chi(n!)\right|^6.$$

The last term is equal to the number of solutions of the congruence

$$n_1!n_2!n_3! \equiv n_4!n_5!n_6! \pmod p; \qquad 1 \le n_1, \dots, n_6 \le N.$$

Therefore, from Lemma (2.2) (taking with $k=3$), we have

$$J_2 \ll N^{5+1/8}.$$

Thus,

$$J = J_1 + O\left(N^{5+1/8}\right). \tag{5.2}$$

Following the argument from [3], we will establish a suitable asymptotic formula for $J_1$. Let $1 \le j_1, j_2, j_3 \le r-2$ be fixed and let

$$M_1 = M_1(j_1,j_2,j_3), \quad M_2 = M_2(j_1,j_2,j_3), \quad M_3 = M_3(j_1,j_2,j_3)$$

be some positive integers to be chosen later with

$$2 \le M_1 < N2^{-j_1} - 1, \quad 2 \le M_2 < N2^{-j_2} - 1, \quad 2 \le M_3 < N2^{-j_3} - 1.$$

Let $J'(j_1,j_2,j_3)$ denote the number of solutions of the congruence

$$(n_1+m_1)!(n_2+m_2)!(n_3+m_3)!n_4!n_5!n_6!n_7! \equiv \lambda \pmod p$$

subject to the conditions

$$1 \le m_i \le M_i, \quad \frac{N}{2^{j_i}} - M_i < n_i \le \frac{N}{2^{j_i-1}}, \quad i=1,2,3; \quad 1 \le n_4,n_5,n_6,n_7 \le N.$$

For fixed $m_1, m_2, m_3$ the number of solutions of the preceding congruence is $\ge J(j_1,j_2,j_3)$. Therefore,

$$J(j_1,j_2,j_3) \le \frac{J'(j_1,j_2,j_3)}{M_1M_2M_3}.$$

Analogously, define $J''(j_1,j_2,j_3)$ to be the number of solutions of the congruence

$$(n_1-m_1)!(n_2-m_2)!(n_3-m_3)!n_4!n_5!n_6!n_7! \equiv \lambda \pmod p$$

subject to the conditions

$$1 \le m_i \le M, \quad \frac{N}{2^{j_i}} + M_i < n_i \le \frac{N}{2^{j_i-1}}, \quad i=1,2,3, \quad 1 \le n_4,n_5,n_6,n_7 \le N.$$

For fixed $m_1, m_2, m_3$ the number of solutions of the preceding congruence is $\le J(j_1,j_2,j_3)$. Hence,

$$\frac{J''(j_1,j_2,j_3)}{M_1M_2M_3} \le J(j_1,j_2,j_3).$$

Thus,

$$\frac{J''(j_1,j_2,j_3)}{M_1M_2M_3} \le J(j_1,j_2,j_3) \le \frac{J'(j_1,j_2,j_3)}{M_1M_2M_3}. \tag{5.3}$$

Our goal is to use this inequality to prove that

$$J(j_1,j_2,j_3) = \frac{N^7}{p-1}2^{-(j_1+j_2+j_3)} + O\Big(\frac{p^{3/8}N^{11/2}}{2^{3(j_1+j_2+j_3)/4}}\big(\log(Np^{-11/12}+2)\big)^{3/4}\Big).$$

Let

$$L_i = [N2^{-j_i}] - M_i, \quad N_i = [N2^{-j_i+1}] - L_i, \quad i=1,2,3.$$

We express $J'(j_1,j_2,j_3)$ in terms of character sums and obtain

$$J'(j_1,j_2,j_3) = \frac{1}{p-1}\sum_{\chi}\left(\prod_{i=1}^{3}\mathcal{F}(\chi;M_i,N_i,L_i)\right)\left(\sum_{n=1}^{N}\chi(n!)\right)^4\overline{\chi(\lambda)},$$

where $\chi$ runs through the set of multiplicative characters modulo $p$ and

$$\mathcal{F}(\chi; M_i, N_i, L_i) = \sum_{m_i=1}^{M_i} \sum_{L_i<n_i\le L_i+N_i} \chi((n_i+m_i)!).$$

Separating the term corresponding to the principal character $\chi = \chi_0$, we obtain

$$J'(j_1, j_2, j_3) = \frac{N^4}{p-1}\prod_{i=1}^{3}\left(\frac{N}{2^{j_i}} + M_i + \theta_i 2\right) M_i +$$

$$+ O\left(\left(\prod_{i=1}^{3}\max_{\chi\ne\chi_0}|\mathcal{F}(\chi; M_i, N_i, L_i)|\right)\frac{1}{p-1}\sum_{\chi}\left|\sum_{n=1}^{N}\chi(n!)\right|^4\right),$$

where $|\theta_i| \le 1$, $i = 1, 2, 3$. We observe that

$$\frac{1}{p-1}\sum_{\chi}\left|\sum_{n=1}^{N}\chi(n!)\right|^4$$

is equal to the number of solutions of the congruence

$$n_1!n_2! \equiv n_3!n_4! \pmod p; \qquad 1 \le n_1, n_2, n_3, n_4 \le N.$$

Hence, from Lemma (2.2) with $k = 2$, we have

$$\frac{1}{p-1}\sum_{\chi}\left|\sum_{n=1}^{N}\chi(n!)\right|^4 \ll N^{3+1/4}.$$

Therefore, using this estimation, Theorem (3.1) to estimate $\mathcal{F}(\chi; M_i, N_i, L_i)$ and also taking into account that $2^{j_i}M_i/N \ll 1$ and $N_i \ll N2^{-j_i} + M_i$, we deduce

$$\begin{aligned}\frac{J'(j_1, j_2, j_3)}{M_1M_2M_3} &= \frac{N^4}{p-1}\prod_{i=1}^{3}\left(\frac{N}{2^{j_i}} + M_i + \theta_i 2\right)\\ &\quad + O\left(p^{3/8}N^{13/4}\prod_{i=1}^{3}N_i^{3/4}\left(\log\left(N_iM_i^{-1}+2\right)\right)^{1/4}\right)\\ &= \frac{N^7}{p-1}2^{-(j_1+j_2+j_3)} + O\left(\frac{N^7}{p}2^{-(j_1+j_2+j_3)}\left(\frac{2^{j_1}}{N}M_1 + \frac{2^{j_2}}{N}M_2 + \frac{2^{j_3}}{N}M_3\right)\right.\\ &\quad \left. + \frac{p^{3/8}N^{11/2}}{2^{3(j_1+j_2+j_3)/4}}\prod_{i=1}^{3}\left(\log\left(N2^{-j_i}M_i^{-1}+2\right)\right)^{1/4}\right).\end{aligned}$$

Next, we will prove that for a suitable choice of parameters $M_1, M_2, M_3$ we have the asymptotic formula

$$\frac{J'(j_1, j_2, j_3)}{M_1M_2M_3} = \frac{N^7}{p-1}2^{-(j_1+j_2+j_3)} + O\left(\frac{p^{3/8}N^{11/2}}{2^{3(j_1+j_2+j_3)/4}}\log^{3/4}(Np^{-11/12}+2)\right). \tag{5.4}$$

If $j_1, j_2, j_3$ are such that $N2^{-(j_1+j_2+j_3)/6} < 100p^{11/12}$, then we choose

$$M_i = [N2^{-j_i-1}], \quad i = 1, 2, 3$$

to obtain
$$\frac{J'(j_1,j_2,j_3)}{M_1M_2M_3} = O(Np^{9/2}\log^{3/4}(Np^{-11/12}+2)),$$
and the bound (5.4) is proved in this case because the error term in (5.4) dominates.

If $j_1, j_2, j_3$ are such that $N2^{-(j_1+j_2+j_3)/6} \geq 100p^{11/12}$, then define
$$V = \left[\left(\frac{N^2 2^{-(j_1+j_2+j_3)/3} p^{-11/6}}{\log(N2^{-(j_1+j_2+j_3)/6} p^{-11/12})}\right)^{3/4}\right].$$
Clearly, $V \geq 2$. Since $\max\{2^{j_1}, 2^{j_2}, 2^{j_3}\} < N < p$, it is also verified that
$$V < 0.5N^{3/2}2^{-(j_1+j_2+j_3)/4}p^{-11/8} < 0.5\min\{N2^{-j_1}, N2^{-j_2}, N2^{-j_3}\}.$$
Hence, in this case we can choose
$$M_i = \left[\frac{N2^{-j_i}}{V}\right], \quad i = 1, 2, 3,$$
and obtain
$$\begin{aligned}\frac{J'(j_1,j_2,j_3)}{M_1M_2M_3} &= \frac{N^7}{p-1}2^{-(j_1+j_2+j_3)} + \\ &+ O\Big(\frac{p^{3/8}N^{11/2}}{2^{3(j_1+j_2+j_3)/4}}\log^{3/4}(N2^{-(j_1+j_2+j_3)/6}p^{-11/12}+2)\Big) \\ &= \frac{N^7}{p-1}2^{-(j_1+j_2+j_3)} + O\Big(\frac{p^{3/8}N^{11/2}\log^{3/4}(Np^{-11/12}+2)}{2^{3(j_1+j_2+j_3)/4}}\Big).\end{aligned}$$

Thus, the required bound (5.4) holds in both cases. Now combining this with (5.3), we get
$$J(j_1,j_2,j_3) \leq \frac{N^7}{p-1}2^{-(j_1+j_2+j_3)} + O\Bigg(\frac{p^{3/8}N^{11/2}\log^{3/4}(Np^{-11/12}+2)}{2^{3(j_1+j_2+j_3)/4}}\Bigg).$$
Analogously, we obtain the same asymptotic formula for $J''(j_1,j_2,j_3)$ and using (5.3) deduce that
$$J(j_1,j_2,j_3) \geq \frac{N^7}{p-1}2^{-(j_1+j_2+j_3)} + O\Bigg(\frac{p^{3/8}N^{11/2}\log^{3/4}(Np^{-11/12}+2)}{2^{3(j_1+j_2+j_3)/4}}\Bigg).$$
Therefore,
$$J(j_1,j_2,j_3) = \frac{N^7}{p-1}2^{-(j_1+j_2+j_3)} + O\Big(\frac{p^{3/8}N^{11/2}\log^{3/4}(Np^{-11/12}+2)}{2^{3(j_1+j_2+j_3)/4}}\Big).$$

In view of (5.1) and (5.2), we obtain
$$\begin{aligned}J &= \sum_{j_1=1}^{r-2}\sum_{j_2=1}^{r-2}\sum_{j_3=1}^{r-2} J(j_1,j_2,j_3) + O(N^{5+1/8}) \\ &= \frac{N^7}{p-1}\left(\sum_{j=1}^{\infty}\frac{1}{2^j} - \sum_{j>r-2}\frac{1}{2^j}\right)^3 + R(N) + O(N^{5+1/8}) \\ &= \frac{N^7}{p-1} + R(N) + O\left(\frac{1}{p}N^6 + N^{5+1/8}\right)\end{aligned}$$

where

$$R(N) \ll p^{3/8} N^{11/2} \log^{3/4}(Np^{-11/12}+2) \sum_{j_1=1}^{r-2} \sum_{j_2=1}^{r-2} \sum_{j_3=1}^{r-2} \frac{1}{2^{3(j_1+j_2+j_3)/4}}$$
$$\ll p^{3/8} N^{11/2} \log^{3/4}(Np^{-11/12}+2).$$

Since

$$\frac{1}{p} N^6 + N^{5+1/8} \ll p^{3/8} N^{11/2} \log^{3/4}(Np^{-11/12}+2),$$

we conclude

$$J = \frac{N^7}{p-1} + O\left(p^{3/8} N^{11/2} \log^{3/4}(Np^{-11/12}+2)\right).$$

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# DERIVED CLASSIFICATION OF GENTLE ALGEBRAS WITH TWO CYCLES

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ABSTRACT. We classify gentle algebras defined by quivers with two cycles under derived equivalence in a non degenerate case, by using some combinatorial invariants constructed from the quiver with relations defining these algebras. We also present a list of normal forms; any such algebra is derived equivalent to one of the algebras in the list. The article includes an Appendix presenting a slightly modified and extended version of a technical result in the unpublished manuscript [HSZ01] by Holm, Schröer and Zimmermann, describing some essential elementary transformations over the quiver with relations defining the algebra.

## 1. Introduction

Let $A$ be a finite-dimensional connected k-algebra $A$ over an algebraically closed field k. Denote by $D^b(A)$ the bounded derived category of the module category of finite-dimensional left $A$-modules, $A$-mod. It is an interesting problem to classify such algebras up to derived equivalence.

In particular, the family of gentle algebras is closed under derived equivalence [SZ03]. The problem of classifying gentle algebras up to derived equivalence is well understood in the case where the associated quiver has one cycle, see [AH81], [AS87], [Vo01], [GP99] and [BGS04]. In this paper we focus our attention on gentle algebras with two cycles.

We use combinatorial invariants $\phi_A : \mathbb{N}^2 \to \mathbb{N}$ defined in [AG08] in order to classify them under derived equivalence. Roughly speaking $\phi_A$ is obtained as follows: Start with a maximal directed path in $Q$ which contains no relations. Then continue in opposite direction as long as possible with zero relations. Repeat this until the first path appears again, say after $n$ steps. Then we obtain a pair $(n, m)$ where $m$ is the number of arrows which appeared in a zero relation. Repeat this procedure until all maximal paths without a zero relation have been used; $\phi_A$ counts then how often each pair $(n, m) \in \mathbb{N}^2$ occurred. Recall $\phi_A$ has always a finite support. Let $\{(n_1, m_1), (n_2, m_2), \ldots, (n_k, m_k)\}$ be the support of $\phi_A$ , denote $\phi_A$ by $[(n_1, m_1), (n_2, m_2), \ldots, (n_k, m_k)]$ where each $(n_j, m_j)$ is written $\phi_A(n_j, m_j)$ times and the order in which they are written is arbitrary. Define also $\#\phi_A := \sum_{1\leq j\leq k} \phi_A(n_j, m_j)$. See [AG08, 3,5] for a precise description.

We can show by induction over the number of vertices that:

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# HYPERBOLIC WEIGHTED BERGMAN CLASSES

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ABSTRACT. In this paper we present hyperbolic weighted Bergman classes in terms of the Green function of the unit complex disk as well as Möbius transformations. We also study its different representations and inclusions

## 1. Introduction

Let $r > 0$. Define $\mathbb{D}_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$ and $\mathbb{D}_r = \mathbb{D}_r(0)$. We denote by $\mathbb{D} = \mathbb{D}_1$ the open unit disk in the complex plane $\mathbb{C}$ and by $T$ its boundary. Let $\phi_a : \mathbb{C} \to \mathbb{C}$ be the Möbius transformation,

$$\phi_a(z) = \frac{a - z}{1 - \overline{a}z}, \qquad |a| < 1,$$

with pole at $z = 1/\overline{a}$ that satisfies $\phi_a^{-1} = \phi_a$. Further, we denote the pseudo-hyperbolic disk by $U(a, r) = \{z \in \mathbb{D} : |\phi_a(z)| < R\}$. We observe that

$$1 - |\phi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \overline{a}z|^2} = (1 - |z|^2)|\phi_a'(z)|\,. \tag{1.1}$$

For $z,\ a \in \mathbb{D}$, we denote a Green's function of $\mathbb{D}$, with logarithmic singularity at $a$, by

$$g(z, a) = \ln \frac{|1 - \overline{a}z|}{|z - a|} = \ln \frac{1}{|\phi_a(z)|}\,. \tag{1.2}$$

In 2005 Xianon Li [Li] introduced, for $0 < s < \infty$, the so called hyperbolic $\mathcal{Q}_s^*$ class as, the set of analytic functions $f : \mathbb{D} \to \mathbb{D}$ such that

$$\sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} f^*(z)^2 g^s(z, a)\, dx\, dy < \infty\,,$$

where

$$f^*(z) = \frac{|f'(z)|}{1 - |f(z)|^2}$$

is the hyperbolic derivative [Ya]. Recently Reséndis and Tovar [ReTo1] introduced the weighted Bergman spaces for $0 < p < \infty$, $-2 < q < \infty$, $0 \leq s < \infty$, as the set of analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^q g^s(z, a)\, dx\, dy < \infty \quad .$$

Suppose that $f : \mathbb{D} \to \mathbb{C}$ is an injective analytic function.

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Then, by change of variable formula with $z = f(w)$,

$$\iint_{f(\mathbb{D})} \frac{w^p}{|f'(f^{-1}(w))|^2}(1-|f^{-1}(w)|^2)^q g^s(f^{-1}(w), a)\, du\, dv$$
$$= \iint_{\mathbb{D}} |f(z)|^p (1-|z|^2)^q g^s(z, a)\, dx\, dy\,.$$

Now let $f\colon (\mathbb{D}, |\ |_e) \to (\mathbb{D}, |\ |_{hyp})$ where $|\ |_e$ and $|\ |_{hyp}$ means Euclidean and hyperbolic measures, respectively. Again we suppose that $f\colon \mathbb{D} \to \mathbb{D}$ is an injective analytic function. Then, by change of variable formula with $z = f(w)$,

$$\iint_{f(\mathbb{D})} \frac{w^p}{|f'(f^{-1}(w))|^2}(1-|f^{-1}(w)|^2)^q g^s(f^{-1}(w), a)\, \frac{du\, dv}{(1-|w|^2)^2}$$
$$= \iint_{\mathbb{D}} \frac{|f(z)|^p}{(1-|f(z)|^2)^2}(1-|z|^2)^q g^s(z, a)\, dx\, dy$$

With this motivation we introduce then the following classes.

We denote by $B(\mathbb{D})$, the set of analytic functions $f\colon \mathbb{D} \to \mathbb{D}$. For $0 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$, consider those functions $f \in B(\mathbb{D})$, that satisfy

$$h^*_{p,q,s}(f)(a) = \iint_{\mathbb{D}} \frac{|f(z)|^p}{(1-|f(z)|^2)^2}(1-|z|^2)^q g^s(z, a)\, dx\, dy \ < \infty$$

We define the hyperbolic $q, s$-weighted $p$-Bergman class as

$$A^*(p, q, s) = \{f \in B(\mathbb{D}) : \sup_{a\in\mathbb{D}} h^*_{p,q,s}(f)(a) < \infty\}$$

and for $0 < s < \infty$, the little hyperbolic $q, s$-weighted $p$-Bergman class as

$$A^*_0(p, q, s) = \{f \in B(\mathbb{D}) : \lim_{|a|\to 1^-} h^*_{p,q,s}(f)(a) = 0\}\,.$$

In a similar way, for $0 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$, consider those functions $f \in B(\mathbb{D})$, that satisfy

$$l^*_{p,q,s}(f)(a) = \iint_{\mathbb{D}} \frac{|f(z)|^p}{(1-|f(z)|^2)^2}(1-|z|^2)^q (1-|\phi_a(z)|^2)^s dx\, dy < \infty\,.$$

We will use $l^*_{q,s}$ to denote $l^*_{0,q,s}$. Thus define

$$L^*(p, q, s) = \{f \in B(\mathbb{D}) : \sup_{a\in\mathbb{D}} l^*_{p,q,s}(f)(a) < \infty\}$$

and for $0 < s < \infty$,

$$L^*_0(p, q, s) = \{f \in B(\mathbb{D}) : \lim_{|a|\to 1^-} l^*_{p,q,s}(f)(a) = 0\}\,.$$

We write $L^*_p = L^*(p, 0, 0)$ and observe that $A^*(2, 0, 0) = L^*(2, 0, 0) = L^*_2$, is the hyperbolic Bergman class of analytic functions.

We say that $f \in B(\mathbb{D})$ belongs to the hyperbolic Bloch Bergman class $B^*(A)$ if

$$\sup_{z\in\mathbb{D}} (1-|z|^2)\frac{|f(z)|}{1-|f(z)|^2} < \infty$$

and to the little hyperbolic Bloch Bergman class $B^*_0(A)$ if

$$\lim_{|z|\to 1^-} (1-|z|^2)\frac{|f(z)|}{1-|f(z)|^2} = 0.$$

In a similar way, for $-1 < q < \infty$, we say that $f \in B(\mathbb{D})$ belongs to the hyperbolic $q$-Dirichlet $p$-Bergman class $L^*(p, q, 0)$ if

$$l^*_{p,q,0}(f) = \iint_{\mathbb{D}} \frac{|f(z)|^p}{(1-|f(z)|^2)^2}(1-|z|^2)^q \, dx\, dy < \infty \, .$$

The aim of this paper is to obtain explicitely the basic properties of the hyperbolic weighted Bergman classes.

The main references for this work are, R. Aulaskari et al [AuStXi], X. Li [Li], Jie Xiao [Xi],[Xi1], Ruhan Zhao [Zha], [Zha1] and Reséndis, Tovar [ReTo1]. It is remarkable the big differences in methods and results between the classes introduced in this paper and -for instance- the $\mathcal{Q}^*$ classes introduced by Xianon Li, [Li]. In a forthcoming paper we will study the properties of the convex metric space $L^*(p, q, s)$ and its representations and characterizations in terms of series expansions.

## 2. Properties of $L^*(p, q, s)$

In this part we clarify some elementary aspects of our function classes. We require the next result.

LEMMA (2.1) ([Zh], Chapter 4). *Let $t > -1$, $c \in \mathbb{R}$ and define $I_{t,c} : \mathbb{D} \to \mathbb{R}$ by*

$$I_{t,c}(a) = \iint_{\mathbb{D}} \frac{(1-|z|^2)^t}{|1-\overline{a}z|^{2+t+c}} \, dxdy.$$

*Then*

(a) *If $c < 0$ then $I_{t,c}(a)$ is bounded in $a$.*

(b) *If $c = 0$, then*

$$I_{t,c}(a) \approx \ln \frac{1}{1-|a|^2}, \qquad (\, |a| \to 1) \, .$$

(c) *If $c > 0$, then*

$$I_{t,c}(a) \approx \frac{1}{(1-|a|^2)^c}, \qquad (\, |a| \to 1) \, .$$

From the following result we see that the parameter $p$ does not have any significant role.

PROPOSITION (2.2). *Let $f \in B(\mathbb{D})$ and $0 < p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$, satisfying $q + s > -1$ . Then $f \in L^*(p, q, s)$ or $L^*_0(p, q, s)$ if and only if*

$$\sup_{a\in\Delta} \iint_{\mathbb{D}} \frac{1}{(1-|f(z)|^2)^2}(1-|z|^2)^q(1-|\phi_a(z)|^2)^s dx\, dy < \infty \quad , \tag{2.3}$$

*or*

$$\lim_{|a|\to 1^-} \iint_{\mathbb{D}} \frac{1}{(1-|f(z)|^2)^2}(1-|z|^2)^q(1-|\phi_a(z)|^2)^s dx\, dy = 0 \, ,$$

*respectively.*

*Proof.* Let $f \in B(\mathbb{D})$ and suppose that $f$ satisfy (2.3). Since $|f(z)| < 1$, we have $f \in L^*(p, q, s)$. Conversely for $f \in L^*(p, q, s)$ or $L^*_0(p, q, s)$, define

$$B = \{ \, z \in \mathbb{D} \, : \, |f(z)| \ge \frac{1}{2} \, \} \, .$$

Then

$$\iint_{\mathbb{D}} \frac{1}{(1-|f(z)|^2)^2}(1-|z|^2)^q(1-|\phi_a(z)|^2)^s dx\,dy$$
$$\le \iint_B \frac{2^p|f(z)|^p}{(1-|f(z)|^2)^2}(1-|z|^2)^q(1-|\phi_a(z)|^2)^s dx\,dy$$
$$+\frac{16}{9\cdot 2^p}\iint_{\mathbb{D}-B}(1-|z|^2)^q(1-|\phi_a(z)|^2)^s dx\,dy\,.$$

The result follows from the hypothesis and Lemma (2.1) (c). □

Then, by Proposition (2.2), $L^*(p,q,s) = L^*(0,q,s) := L^*(q,s)$ and $L_0^*(p,q,s) = L_0^*(0,q,s) := L_0^*(q,s)$. Thus we will write $l^*(0,q,s) = l^*(q,s)$. In a similar way, we have

PROPOSITION (2.4). *For $f \in B(\mathbb{D})$, $f \in B^*(A)$ or $B_0^*(A)$ if and only if*

$$\sup_{z\in\mathbb{D}} \frac{1-|z|^2}{1-|f(z)|^2} < \infty$$

*or*

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|f(z)|^2} = 0\,,$$

*respectively.*

LEMMA (2.5) (Yamashita, [Ya]). *Let $f \in B(\mathbb{D})$. Then the function $g\colon \mathbb{D} \to [0,\infty)$ defined by*

$$g(z) = \ln[-\ln(1-|f(z)|^2)]$$

*is subharmonic.*

We need the following result.

COROLLARY (2.6). *Let $f \in B(\mathbb{D})$. Then the function $g\colon \mathbb{D} \to [0,\infty)$ defined by*

$$g(z) = \frac{1}{(1-|f(z)|^2)^2}$$

*is subharmonic.*

*Proof.* The result follows from the convexity of $e^x$. □

By definition of hyperbolic Bergman classes, it is clear the usefulness of the following result (See Exercise 1, pag. 128 of [Co]).

THEOREM (2.7). *If $f \in B(\mathbb{D})$ then*

$$\frac{|f(0)|-|z|}{1-|f(0)||z|} \le |f(z)| \le \frac{|f(0)|+|z|}{1+|f(0)||z|} \qquad \textit{for all } z\in\mathbb{D}.$$

*In particular,*

$$\frac{1}{1-|f(z)|} \le \frac{1+|f(0)||z|}{(1-|f(0)|)(1-|z|)} \qquad \textit{for all } z\in\mathbb{D}. \tag{2.8}$$

From the previous theorem it follows immediately

COROLLARY (2.9). $B(\mathbb{D}) = B^*(A)$.

We study the region of values for which $L^*(q, s)$ is not trivial or is different from $B(\mathbb{D})$.

PROPOSITION (2.10). *Let $-2 < q < \infty$ and $0 \leq s < \infty$, with $q + s \leq -1$. Then the class $L^*(q, s) = \{0\}$.*

*Proof.* Let $f \in \mathcal{A}$ be different from the zero function.

Let $0 < b < 1$ be fixed. Since $\dfrac{1}{(1-|f(z)|^2)^2}$ is a subharmonic function

$$\begin{aligned} l^*_{q,s}(f)(0) &= \iint_{\mathbb{D}} \frac{(1-|z|^2)^{q+s}}{(1-|f(z)|^2)^2}\, dx\, dy \geq \int_b^1 \int_0^{2\pi} \frac{(1-r^2)^{q+s} r}{(1-|f(re^{i\theta})|^2)^2}\, d\theta\, dr \\ &\geq \int_0^{2\pi} \frac{1}{(1-|f(be^{i\theta})|^2)^2}\, d\theta \int_b^1 (1-r^2)^{q+s} r\, dr = \infty\,, \end{aligned}$$

hence we get a contradiction. □

Lemmas (2.1), (2.8) and the previous result permit us to continue our study about the parameters $q$ and $s$. Define

$$\Omega_1 = \{\, (q, s)\ :\ 0 \leq s \leq 1,\ 1-s < q < \infty \,\} \cup \{\, (q, s)\ :\ 1 < s < \infty,\ 0 < q < \infty \,\}\,.$$

THEOREM (2.11). *Let $(q, s) \in \Omega_1$. Then $L^*_0(q, s) = B(\mathbb{D})$. Moreover, $L^*(0, s) = B(\mathbb{D})$ for $1 < s < \infty$.*

*Proof.* Let $f \in B(\mathbb{D})$. Then, by (2.8) and (1.1) there exists $0 < c$ such that

$$\iint_{\mathbb{D}} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2} (1-|\phi_a(z)|^2)^s dx\, dy \leq c(1-|a|^2)^s \iint_{\mathbb{D}} \frac{(1-|z|^2)^{q+s-2}}{|1-\overline{a}z|^{2s}}\, dx\, dy\,.$$

The case $s = 0$ is obvious. Suppose now $s > 0$.

Applying Lemma (2.1) with $t = q + s - 2$ and $2 + t + c = 2s$, we obtain the result. □

We wish to include constant functions in our hyperbolic classes, but it is equivalent to include bounded functions in the hyperbolic sense, that is, $f \in B(\mathbb{D})$ such that $\overline{f(\mathbb{D})} \subset \mathbb{D}$. We denote this class by $\overline{B}(\mathbb{D})$.

Define

$$\Omega_2 = \{\, (q, s) : 0 < s \leq 1,\ -1-s < q < \infty \,\} \cup \{\, (q, s) : -2 < q < \infty,\ 1 < s < \infty \,\}\,.$$

THEOREM (2.12). *$\overline{B}(\mathbb{D}) \subset L^*(q, s)$ if and only if $(q, s) \in \Omega_2$.*

*Proof.* By Proposition (2.10), we may suppose $-1 < q + s$. It is enough to prove that $\overline{B}(\mathbb{D}) \subset L^*(q, s)$. Let $f \in \overline{B}(\mathbb{D})$. Then $f \in L^*(q, s)$ if and only if

$$(1-|a|^2)^s \iint_{\mathbb{D}} \frac{(1-|z|^2)^{q+s}}{|1-\overline{a}z|^{2s}}\, dx\, dy < \infty\,.$$

Applying Lemma (2.1) with $t = q + s$, $2 + t + c = 2s$, we get $c = s - q - 2$. In particular (c) of the same lemma implies that $q > -2$ if $s - 2 > q$. □

Define

$$\Omega = \{\, (q, s) : 0 \leq s < 1,\ -1-s < q \leq 1-s \,\} \cup \{\, (q, s) : 1 \leq s < \infty,\ -2 < q \leq 0 \,\}\,.$$

We need the following elementary estimations.

LEMMA (2.13). *Let $q \in \mathbb{R}$ and $a \in \mathbb{D}$. Then, for all $z \in \mathbb{D}$,*

$$\frac{1}{\rho(a,q)}(1-|z|^2)^q \le (1-|\phi_a(z)|^2)^q \le \rho(a,q)(1-|z|^2)^q \tag{2.14}$$

*where*

$$\rho(a,q) = \left(\frac{1+|a|}{1-|a|}\right)^{|q|}.$$

*Proof.* For $a \in \mathbb{D}$, we have

$$1-|a| \le |1-\bar{a}z| \le 1+|a|.$$

Then (2.14) follows from (1.1) and the fact that $x \to x^q$ is nondecreasing if $0 \le q$ and is nonincreasing if $q < 0$. □

COROLLARY (2.15). *Let $q \in \mathbb{R}$ and $a \in \mathbb{D}$. Then, for all $z \in \mathbb{D}$,*

$$\frac{1}{\rho(a,q)} \le |\phi_a'(z)|^q \le \rho(a,q). \tag{2.16}$$

The hyperbolic Dirichlet Bergman class is included in the little hyperbolic Bloch Bergman class.

PROPOSITION (2.17). $L^*(0,0) \subset B_0^*(A)$.

*Proof.* Let $f \in L^*(0,0)$. By Corollary (2.6), we have

$$\frac{1}{(1-|f(0)|^2)^2} \le \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{(1-|f(re^{i\theta})|^2)^2}\, d\theta.$$

Multiplying by $r$ and integrating from 0 to $R < 1$, we get

$$\frac{1}{(1-|f(0)|^2)^2} \le \frac{1}{\pi R^2}\iint_{\mathbb{D}_R} \frac{1}{(1-|f(z)|^2)^2}\, dx\, dy.$$

Because the composed mapping $f \circ \phi_a$ is subharmonic under the change of variable $w = \phi_a(z)$, we obtain by (2.16)

$$\begin{aligned}\frac{1}{(1-|f(a)|^2)^2} &\le \frac{1}{\pi R^2}\iint_{U(a,R)} \frac{1}{(1-|f(w)|^2)^2}|\phi_a'(w)|^2\, du\, dw \\ &\le \frac{1}{\pi R^2}\frac{(1+|a|)^2}{(1-|a|)^2}\iint_{U(a,R)} \frac{1}{(1-|f(w)|^2)^2}\, du\, dw\end{aligned}$$

therefore

$$(1-|a|^2)^2\frac{1}{(1-|f(a)|^2)^2} \le \frac{(1+|a|)^4}{\pi R^2}\iint_{U(a,R)} \frac{1}{(1-|f(w)|^2)^2}\, du\, dw.$$

If $|a| \to 1^-$ then the Euclidean area $|U(a,R)| \to 0$ and so we obtain that $f \in B^*(A)$. □

We will need the following result.

LEMMA (2.18). *Let $-2 < q < \infty$, $0 < s < \infty$ and $|a| < 1$. Then*

$$\frac{1}{\rho(a,|q|+2)} l_{q,s}^*(f)(a) \le l_{q+s,0}^*(f\circ\phi_a)(0) \le \rho(a,|q|+2) l_{q,s}^*(f)(a).$$

*Proof.* Suppose that $l^*_{q,s}(f)(a) < \infty$. Then, by the change of variable formula with $z = \phi_a(w)$ and ((2.14)), it is sufficient to consider

$$\begin{aligned}
&\iint_{\mathbb{D}} \frac{1}{(1-|f(\phi_a(z))|^2)^2}(1-|z|^2)^{q+s}\,dx\,dy \\
&\quad = \iint_{\mathbb{D}} \frac{1}{(1-|f(w)|^2)^2}|\phi_a'(w)|^2(1-|\phi_a(w)|^2)^{q+s}\,du\,dv \\
&\quad \le \rho(a,|q|+2)\iint_{\mathbb{D}} \frac{1}{(1-|f(w)|^2)^2}(1-|w|^2)^q(1-|\phi_a(w)|^2)^s\,du\,dv
\end{aligned}$$

We proceed in the similar way to prove the left hand inequality. □

Let $0 < s < s' < \infty$ and $-1 < q < q' < \infty$. It is immediate that

$$L^*(q,s) \subset L^*(q,s'), \quad L^*_0(q,s) \subset L^*_0(q,s'), \quad L^*(q,s) \subset L^*(q',s)$$
$$\text{and} \quad L^*_0(q,s) \subset L^*_0(q',s)\,.$$

The following results clarify the relation between $L^*_0(q,s)$ and $L^*(q,s)$.

PROPOSITION (2.19). *Let $-2 < q < \infty$, $0 \le s < \infty$ and let $l^*_{q,s}(f)\colon \mathbb{D} \to \mathbb{R}$ be well defined. Then $l^*_{q,s}(f)$ is a continuous function on $\mathbb{D}$.*

*Proof.* If $f = 0$ on $\mathbb{D}$, it is clear that $l^*_{q,s}(f)$ is continuous. Therefore we suppose that $f \ne 0$, in particular, $l^*_{q,s}(f)(0) \ne 0$.

Let $a \in \mathbb{D}$ be fixed and let $\delta > 0$ be such that $\overline{\mathbb{D}}(a,\delta) \subset \mathbb{D}$. The function $l\colon \overline{\mathbb{D}} \times \overline{\mathbb{D}}(a,\delta) \to \mathbb{R}$, defined by

$$(z,\zeta) \to \frac{(1-|\zeta|^2)^s}{|1-\overline{\zeta}z|^{2s}}\,,$$

is uniformly continuous on $\overline{\mathbb{D}} \times \overline{\mathbb{D}}(a,\delta)$. Then, for given $\epsilon > 0$, there exists $\rho > 0$ such that if $|z'-z| < \rho$ and $|\zeta'-\zeta| < \rho$ then

$$|l(z',\zeta') - l(z,\zeta)| < \frac{\epsilon}{l^*_{q,s}(f)(0)}\,,$$

and therefore

$$\begin{aligned}
&|l^*_{q,s}(f)(a) - l^*_{q,s}(f)(b)| \le \\
&\qquad \le \iint_{\mathbb{D}} \frac{1}{(1-|f(z)|^2)^2}\left(1-|z|^2\right)^{q+s}|l(z,a)-l(z,b)|\,dx\,dy < \epsilon\,.
\end{aligned}$$

□

COROLLARY (2.20). *Let $-2 < q < \infty$ and $0 \le s < \infty$. Then $L^*_0(q,s) \subset L^*(q,s)$.*

*Proof.* The trivial case $f = 0$ is clear. Suppose $f \ne 0$ and $f \in L^*_0(q,s)$. Then there exists $0 < R < 1$ such that $l^*_{q,s}(f)(a) < l^*_{q,s}(f)(0)$ for all $R < |a| < 1$. By Proposition (2.19), $l^*_{q,s}(f)$ attains its finite maximum on $\mathbb{D}_R$ and therefore $f \in L^*(q,s)$. □

COROLLARY (2.21). *Let $-2 < q < \infty$ and $0 \le s < \infty$. If $f \in L^*_0(q,s)$ then $l^*_{q,s}(f)\colon \mathbb{D} \to [0,\infty)$ is uniformly continuous.*

## 3. The equality $A^*(p,q,s) = L^*(p,q,s) = L^*(q,s)$

In this section we obtain basic properties of the hyperbolic Bergman classes $A^*(p,q,s)$, in particular, $A^*(p,q,s) = A^*(0,q,s) := A^*(q,s) = L^*(q,s)$.

THEOREM (3.1). *Let $-2 < q < \infty$ and $f \in B(\mathbb{D})$. If $l^*_{q,s}(f)(0) < \infty$ then, for $0 < s < \infty$,*

$$\iint_{\mathbb{D}} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2} \ln^s \frac{1}{|z|}\, dx\, dy \le t \iint_{\mathbb{D}} \frac{(1-|z|^2)^{q+s}}{(1-|f(z)|^2)^2}\, dx\, dy \tag{3.2}$$

*where $t = t(q,s,R)$ for some fixed $0 < R < 1$.*

*If $l_{q,s}(f)(0) < \infty$ then, for $0 < s < 1$,*

$$\iint_{\mathbb{D}} \frac{(1-|z|^2)^{q+s}}{(1-|f(z)|^2)^2} |z|^{-2s}\, dx\, dy \le \tilde{t} \iint_{\mathbb{D}} \frac{(1-|z|^2)^{q+s}}{(1-|f(z)|^2)^2}\, dx\, dy \tag{3.3}$$

*where $\tilde{t} = \tilde{t}(q,s,R)$ for some fixed $0 < R < 1$.*

*Proof.* Let $c = .0183403\ldots$ be the root of $-\ln x = 4(1-x^2)$. Let $R$ be fixed with $c < R < 1$. Define

$$\begin{aligned} 0 < \frac{1}{\tau(q,s,R)} = \int_c^R (1-r^2)^{q+s} r\, dr &= \frac{1}{2(1+q+s)}((1-c^2)^{1+q+s} - (1-R^2)^{1+q+s}) \\ &= \frac{1}{2(1+q+s)}(.999664^{1+q+s} - (1-R^2)^{1+q+s}). \end{aligned}$$

Since $R$ is fix $\tau(q,s,R) = \tau(q,s)$. By Lemma (2.5), $\dfrac{1}{(1-|f(z)|^2)^2}$ is subharmonic. Then

$$\begin{aligned} \frac{1}{\tau(q,s,R)} \int_0^{2\pi} \frac{1}{(1-|f(ce^{i\theta})|^2)^2}\, d\theta &= \int_c^R (1-r^2)^{q+s} r\, dr \int_0^{2\pi} \frac{1}{(1-|f(ce^{i\theta})|^2)^2}\, d\theta \\ &\le \int_c^R (1-r^2)^{q+s} r\, dr \int_0^{2\pi} \frac{1}{(1-|f(re^{i\theta})|^2)^2}\, d\theta \\ &= \iint_{\mathbb{D}_R - \mathbb{D}_c} \frac{(1-|z|^2)^{q+s}}{(1-|f(z)|^2)^2}\, dx\, dy\,. \end{aligned}$$

Therefore

$$\int_0^{2\pi} \frac{1}{(1-|f(ce^{i\theta})|^2)^2}\, d\theta \le \tau(q,s,R) \iint_{\mathbb{D}_R} \frac{(1-|z|^2)^{q+s}}{(1-|f(z)|^2)^2}\, dx\, dy\,. \tag{3.4}$$

Define

$$0 < \tilde{\tau}(q,s) = \int_0^c r(1-r^2)^q \ln^s \frac{1}{r}\, dr\,.$$

By subharmonicity and (3.4) we have the estimation

$$\iint_{\mathbb{D}_c} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2} \ln^s \frac{1}{|z|}\, dx\, dy = \int_0^c \int_0^{2\pi} \frac{(1-r^2)^q}{(1-|f(re^{i\theta})|^2)^2} r \ln^s \frac{1}{r}\, d\theta\, dr$$

$$\leq \int_0^c \int_0^{2\pi} \frac{(1-r^2)^q}{(1-|f(ce^{i\theta})|^2)^2} r \ln^s \frac{1}{r}\, d\theta\, dr$$

$$\leq \int_0^c r(1-r^2)^q \ln^s \frac{1}{r}\, dr \int_0^{2\pi} \frac{1}{(1-|f(ce^{i\theta})|^2)^2}\, d\theta$$

$$\leq \tau(q,s,R)\tilde{\tau}(q,s) \iint_{\mathbb{D}_R} \frac{(1-|z|^2)^{q+s}}{(1-|f(z)|^2)^2}\, dx\, dy$$

$$\leq \tau(q,s,R)\tilde{\tau}(q,s) \iint_{\mathbb{D}} \frac{(1-|z|^2)^{q+s}}{(1-|f(z)|^2)^2}\, dx\, dy. \tag{3.5}$$

From the inequality

$$-\ln x \leq 4(1-x^2) \quad \text{for each } x \in (c, 1]$$

we have

$$\iint_{\mathbb{D}-\mathbb{D}_c} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2} \ln^s \frac{1}{|z|}\, dx\, dy \leq 4^s \iint_{\mathbb{D}-\mathbb{D}_c} \frac{(1-|z|^2)^{q+s}}{(1-|f(z)|^2)^2}\, dx\, dy$$

$$\leq 4^s \iint_{\mathbb{D}} \frac{(1-|z|^2)^{q+s}}{(1-|f(z)|^2)^2}\, dx\, dy\ . \tag{3.6}$$

Let $t(q,s,R) = \tau(q,s,R)\tilde{\tau}(q,s) + 4^s$.

Combining (3.5) and (3.6) we have

$$\iint_{\mathbb{D}} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2} \ln^s \frac{1}{|z|}\, dx\, dy \leq t(q,s,R) \iint_{\mathbb{D}} \frac{(1-|z|^2)^{q+s}}{(1-|f(z)|^2)^2}\, dx\, dy\ .$$

For $0 < s < 1$ we need to consider instead of (3.16) the following equality

$$0 < \int_0^c r^{1-2s}(1-r^2)^{q+s}\, dr = \frac{1}{2} B[c^2, 1-s, 1+q+s],$$

(where $B$ denotes the incomplete Beta function) then we prove the formula (3.3) in a similar way. □

THEOREM (3.7). *Let $-2 < q < \infty$ and $0 \leq s < \infty$. Then*

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2} g^s(z,a)\, dx\, dy < \infty \tag{3.8}$$

*if and only if*

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2} (1-|\phi_a(z)|^2)^s\, dx\, dy < \infty\ . \tag{3.9}$$

*Proof.* We have

$$1 - x^2 \leq -2\ln x \quad \text{for each } x \in (0, 1]\,.$$

Taking $x = |\phi_a(z)|$ we have $1 - |\phi_a(z)|^2 \leq 2g(z,a)$ hence

$$l^*_{q,s}(f)(a) \leq 2h^*_{q,s}(f)(a) \quad \text{for each } a \in \mathbb{D}\ . \tag{3.10}$$

Then (3.8) implies (3.9).

By hypothesis and Lemma (2.18), $l^*_{q,s}(f\circ\phi)_a(0) < \infty$. Because

$$\frac{1}{(1-|f(\phi_a(z))|^2)^2}$$

is a subharmonic function, the formula (3.2) can be written in the following form

$$\iint_{\mathbb{D}} \frac{(1-|z|^2)^q}{(1-|f(\phi_a(z))|^2)^2} \ln^s \frac{1}{|z|}\, dx\, dy \le t(q,s,R) \iint_{\mathbb{D}} \frac{(1-|z|^2)^{q+s}}{(1-|f(\phi_a(z))|^2)^2} dx\, dy\,.$$

Consider the change of variable $z = \phi_a(w)$ to obtain

$$\begin{aligned}&\iint_{\mathbb{D}} \frac{(1-|\phi_a(w)|^2)^q}{(1-|f(w)|^2)^2} |\phi_a'(w)|^2 \ln^s \frac{1}{|\phi_a(w)|}\, du\, dv \\ &\quad \le t(q,s,R) \iint_{\mathbb{D}} \frac{(1-|\phi_a(w)|^2)^{q+s}}{(1-|f(w)|^2)^2} |\phi_a'(w)|^2\, du\, dv\end{aligned}$$

or equivalently,

$$\begin{aligned}0 &\le \iint_{\mathbb{D}} \frac{(1-|\phi_a(w)|^2)^q}{(1-|f(w)|^2)^2} |\phi_a'(w)|^2 \left( t(q,s,R)(1-|\phi_a(w)|^2)^s - \ln^s \frac{1}{|\phi_a(w)|} \right) du\, dv \\ &\le \rho(a, |q|+2) \iint_{\mathbb{D}} \frac{(1-|w|^2)^q}{(1-|f(w)|^2)^2} \left( t(q,s,R)(1-|\phi_a(w)|^2)^s - \ln^s \frac{1}{|\phi_a(w)|} \right) du\, dv\,.\end{aligned}$$

Since $0 < \rho(a, |q|+2)$, we obtain

$$\begin{aligned}&\iint_{\mathbb{D}} \frac{(1-|w|^2)^q}{(1-|f(w)|^2)^2} \ln^s \frac{1}{|\phi_a(w)|}\, du\, dv \\ &\le \quad t(q,s,R) \iint_{\mathbb{D}} \frac{(1-|w|^2)^q}{(1-|f(w)|^2)^2} (1-|\phi_a(w)|^2)^s\, du\, dv \end{aligned} \tag{3.11}$$

and the result follows. □

COROLLARY (3.12). *Let $-2 < q < \infty$ and $0 < s < \infty$. Then*

$$\begin{aligned}\iint_{\mathbb{D}} (1-|z|^2)^q (1-|\phi_a(z)|^2)^s\, dx\, dy &\le 2 \iint_{\mathbb{D}} (1-|z|^2)^q g^s(z,a)\, dx\, dy \\ &\le 2t(q,s,R) \iint_{\mathbb{D}} (1-|z|^2)^q (1-|\phi_a(z)|^2)^s\, dx\, dy\end{aligned}$$

*Proof.* This is a consequence of the formulas (3.10), (3.11) and Lemma (2.1), by taking $f(z) = 0$ for all $z \in \mathbb{D}$. □

From the previous corollary, we can reproduce the proof of Proposition 2.2 and obtain

PROPOSITION (3.13). *Let $f \in B(\mathbb{D})$, $0 \le p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$. Then $f \in A^*(p,q,s)$ or $A_0^*(p,q,s)$ if and only if*

$$\sup_{a \in \Delta} \iint_{\mathbb{D}} \frac{1}{(1-|f(z)|^2)^2} (1-|z|^2)^q g^s(z,a) dx\, dy < \infty\ ,$$

*or*

$$\lim_{|a| \to 1^-} \iint_{\mathbb{D}} \frac{1}{(1-|f(z)|^2)^2} (1-|z|^2)^q g^s(z,a)\, dx\, dy = 0\,,$$

*respectively.*

As before, we write only $A^*(p,q,s) = A^*(q,s)$ and $A_0^*(p,q,s) = A_0^*(q,s)$. From Theorem (3.7) and Proposition (3.13) we have

COROLLARY (3.14). *Let* $-2 < q < \infty$ *and* $0 \leq s < \infty$. *Then* $A^*(q,s) = L^*(q,s)$

COROLLARY (3.15). *Let* $-2 < q < \infty$ *and* $0 \leq s < \infty$. *Then* $A_0^*(q,s) = L_0^*(q,s)$.

*Proof.* This is a consequence from the formulas (3.10) and (3.11), since

$$l_{q,s}^*(f)(a) \leq 2h_{q,s}^*(f)(a) \leq 2t(q,s,R)l_{q,s}^*(f)(a). \qquad \square$$

From now, we will denote the hyperbolic Bergman classes $A^*(q,s)$ by $L^*(q,s)$ instead of $A^*(q,s)$.

It is possible to replace the weight $(1-|\phi_a(z)|^2)^s$ by its reflection $(|\phi_a(z)|^{-2} - 1)^s$, as the following theorem shows.

THEOREM (3.16). *Let* $-2 < q < \infty$ *and* $0 < s < 1$. *Then* $f \in L^*(q,s)$ *if and only if*

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} \frac{(1-|z|^2)^q}{1-|f(z)|^2)^2}(|\phi_a(z)|^{-2} - 1)^s \, dx\, dy < \infty.$$

*Proof.* Imitate the proof of theorem (3.7) using (3.3) (See [ReTo]). $\square$

Recall that $U(a,r)$ denotes the hyperbolic disk.

The little hyperbolic Bloch Bergman class has the following characterizations.

THEOREM (3.17). *Let* $0 < R < 1$ *and* $1 < s < \infty$. *For an analytic function* $f \in B(\mathbb{D})$ *the following properties are mutually equivalent:*

(a)

$$\lim_{|z|\to 1^-} \frac{1-|z|^2}{1-|f(z)|^2} = 0.$$

(b)

$$\lim_{|a|\to 1^-} \frac{1}{|\, U(a,R)|^{\frac{1}{2}}} \iint_{U(a,R)} \frac{1}{1-|f(z)|^2} \, dx\, dy = 0.$$

(c)

$$\lim_{|a|\to 1^-} \iint_{U(a,R)} \frac{1}{(1-|z|^2)(1-|f(z)|^2)} \, dx\, dy = 0.$$

(d)

$$\lim_{|a|\to 1^-} \iint_{\mathbb{D}} \frac{1}{(1-|z|^2)(1-|f(z)|^2)}(1-|\phi_a(z)|^2)^s \, dx\, dy = 0.$$

(e)

$$\lim_{|a|\to 1^-} \iint_{\mathbb{D}} \frac{1}{(1-|z|^2)(1-|f(z)|^2)} g^s(z,a) \, dx\, dy = 0.$$

(f)

$$\lim_{|a|\to 1^-} \iint_{\mathbb{D}} \frac{|\phi_a'(z)|^2}{1-|f(z)|^2} \ln\frac{1}{|z|} \, dx\, dy = 0.$$

*Proof.* Again we use subharmonicity of $\frac{1}{1-|f(z)|^2}$ and continue exactly in the same way as in the proof of Theorem 1 by R. Zhao in [Zh]. $\square$

*Remark.* Observe that after Corollary (2.9), the previous theorem has no sense for the hyperbolic Bloch space.

Define for $f \in B(\mathbb{D})$, $a \in \mathbb{D}$ and $0 < t < 1$

$$\Phi(f, a, t) = \iint_{U(a,t)} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}\, dx\, dy\,.$$

THEOREM (3.18). *Let $f \in B(\mathbb{D})$, $-2 < q < \infty$ and $0 \le s < \infty$. Then $f \in L^*(q,s)$ (or $L_0^*(q,s)$ ) if and only if*

$$\sup_{a\in\mathbb{D}} \int_0^1 \Phi(f,a,t)(1-t)^{s-1}\, dt < \infty$$

$$(or,\ \lim_{|a|\to 1^-} \int_0^1 \Phi(f,a,t)(1-t)^{s-1}\, dt = 0\ )\,.$$

*Proof.* By definition of $\Phi(f,a,t)$ and Fubini's Theorem we have

$$\begin{aligned}\int_0^1 \Phi(f,a,t)(1-t)^{s-1}\, dt &= \int_0^1 \left( \iint_{U(a,t)} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}\, dx\, dy \right)(1-t)^{s-1}\, dt \\ &= \iint_{\mathbb{D}} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2} \left( \int_{|\phi_a(z)|}^1 (1-t)^{s-1} dt \right) dx\, dy\,.\end{aligned}$$

Since

$$\int_{|\phi_a(z)|}^1 (1-t)^{s-1} dt = \frac{1}{s}(1-|\phi_a(z)|)^s = \frac{1}{s}\frac{(1-|\phi_a(z)|^2)^s}{(1+|\phi_a(z)|)^s} \ge \frac{1}{2^s s}(1-|\phi_a(z)|^2)^s\,,$$

we have

$$\begin{aligned}\frac{1}{2^s s} \iint_{\mathbb{D}} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}(1-|\phi_a(z)|^2)^s\, dx\, dy &\le \int_0^1 \Phi(f,a,t)(1-t)^{s-1}\, dt \\ &\le \frac{1}{s} \iint_{\mathbb{D}} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}(1-|\phi_a(z)|^2)^s\, dx\, dy\end{aligned}$$

and the result follows from the previous estimation. □

## 4. Some inclusions

In this section we will show several important inclusions among $L^*(q,s)$ classes for different cases of $q$ and $s$ and its relationships with the classical Hardy Spaces.

THEOREM (4.1). *Let $0 \le s < \infty$. Then, for $-2 < q < \infty$,*

$$L^*(q,0) \subset \bigcap_{0<s} L_0^*(q,s)$$

*and for $0 < q < \infty$,*

$$L^*(0,0) \subset \bigcap_{0<q} L^*(q,0)\,.$$

*Proof.* Since

$$\frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}(1-|\phi_a(z)|^2)^s \le \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}$$

and $f \in L^*(q,0)$, it follows from the Lebesgue Dominated Convergence Theorem

$$\lim_{|a|\to 1^-} \iint_{\mathbb{D}} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}(1-|\phi_a(z)|^2)^s\,dx\,dy =$$

$$= \iint_{\mathbb{D}} \lim_{|a|\to 1^-} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}(1-|\phi_a(z)|^2)^s\,dx\,dy = 0\,.$$

The second inclusion is obvious from the fact that $(1-|z|^2)^q \le 1$. □

Li, in [Li], defined for $f \in B(\mathbb{D})$,

$$\lambda(f)(z) = \ln \frac{1}{1-|f(z)|^2}\,.$$

We say that $f \in H_\lambda$ if $e^{2\lambda(f)}$ has an harmonic majorant.

THEOREM (4.2). *Let $0 \le s < \infty$. Then for $-1 < q < \infty$*

$$H_\lambda \subset \bigcap_{-1<q,\ 0\le s} L_0^*(q,s)\,.$$

*Proof.* Let $0 < r < 1$ and $f \in H_\lambda$.
Then we have

$$\begin{aligned} l^*_{q,s}(f)(a) &= \iint_{\mathbb{D}_r} \frac{(1-|z|^2)^{q+s}}{(1-|f(z)|^2)^2}\frac{(1-|a|^2)^s}{|1-\overline{a}z|^{2s}}\,dx\,dy \\ &\quad + \iint_{\mathbb{D}-\mathbb{D}_r} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}(1-|\phi_a(z)|^2)^s\,dx\,dy \\ &\le 2^{q+s}(1-|a|^2)^s \iint_{\mathbb{D}_r} \frac{1}{(1-|f(z)|^2)^2}\frac{1}{(1-|z|)^{s-q}}\,dx\,dy \\ &\quad + \iint_{\mathbb{D}-\mathbb{D}_r} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}\,dx\,dy\,. \end{aligned}$$

Let

$$M = \sup_{0\le r<1} \int_0^{2\pi} \frac{1}{(1-|f(re^{i\theta})|^2)^2}\,d\theta\ .$$

For the second integral we have

$$\begin{aligned} \iint_{\mathbb{D}-\mathbb{D}_r} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}\,dx\,dy &= \int_r^1 (1-t^2)^q t \int_0^{2\pi} \frac{1}{(1-|f(te^{i\theta})|^2)^2}\,d\theta\,dt \\ &\le \frac{M}{2}\frac{(1-r^2)^{q+1}}{q+1}\,. \end{aligned}$$

Then, given $\epsilon > 0$, there exists $0 < R < 1$ such that for all $R \le r < 1$,

$$\iint_{\mathbb{D}-\mathbb{D}_r} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}\,dx\,dy < \frac{\epsilon}{2}\,.$$

Also, there exists $\delta > 0$ such that, for all $a \in \mathbb{D}$ with $0 < 1 - |a| < \delta$

$$2^{q+s}(1-|a|^2)^s \iint_{\mathbb{D}_R} \frac{1}{(1-|f(z)|^2)^2} \frac{1}{(1-|z|)^{s-q}}\, dx\, dy < \frac{\epsilon}{2}\,. \qquad \square$$

For $0 \le p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$, we say that $f \in F^*(p, q, s)$ if

$$\sup_{a\in\mathbb{D}} \iint \frac{|f'(z)|^p}{(1-|f(z)|^2)^2}(1-|z|^2)^q(1-|\phi_a(z)|^2)^s\, dx\, dy < \infty\,.$$

It follows from the subharmonicity of

$$\frac{|f'(z)|^p}{(1-|f(z)|^2)^2}$$

that if we change $(1-|\phi_a(z)|^2)^s$ by $g^s(z, a)$, the hyperbolic class $F^*(p, q, s)$ does not change, see Theorem 5.2 in [ReTo2].

THEOREM (4.3). *Let* $0 \le p < \infty$, $-2 < q < \infty$ *and* $0 \le s < \infty$. *Then*

$$F^*(p, q, s) \subset L^*(p, q, s) = L^*(q, s)$$

*and*

$$F_0^*(p, q, s) \subset L_0^*(p, q, s) = L_0^*(q, s)\,.$$

*Proof.* Let $f \in B(\mathbb{D})$ and $0 < k < 1$ be such that $0 < |f(0)| + k = l < 1$. Define

$$A = \{\, z \in \mathbb{D} \;:\; k \le |f'(z)| \,\}\,.$$

Then

(4.4)

$$\iint_A \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}(1-|\phi_a(z)|^2)^s dx\, dy \le \frac{1}{k^p}\iint_A \frac{|f'(z)|^p}{(1-|f(z)|^2)^2}(1-|z|^2)^q(1-|\phi_a(z)|^2)^s\, dx\, dy\,.$$

Now

$$|f(z)| \le |f(0)| + \int_0^z |f'(\zeta)||d\zeta|\,.$$

If we take now $z \in \mathbb{D}\setminus A$, we have

$$|f(z)| \le |f(0)| + k|z| < |f(0)| + k = l < 1\,.$$

Hence

$$\iint_{\mathbb{D}\setminus A} \frac{(1-|z|^2)^q}{(1-|f(z)|^2)^2}(1-|\phi_a(z)|^2)^s dx\, dy \le \frac{1}{(1-l^2)^2}\iint_{\mathbb{D}\setminus A}(1-|z|^2)^q(1-|\phi_a(z)|^2)^s dx\, dy\,.$$

Then by (4.4) and Lemma (2.1), we have the result. $\square$

In particular the hyperbolic classes introduced by Li [Li], satisfy $\mathcal{Q}_s^* = F^*(2, 0, s) \subset L^*(2, 0, s) = L^*(0, s)$.

## 5. Strict inclusions in the classes $L^*(q, s)$

CONJECTURE. Since $f(z) = z \notin L^*(q, s)$ for $(q, s) \in \Omega$ and the hyperbolic bounded functions belongs to any class $L^*(q, s)$ with $-1 < q + s$, it is possible that for $0 < s < s' < \infty$, $0 < q < q' < \infty$, we have strict inclusions in the following inclusions

$$L^*(q, s) \subset L^*(q, s'),\ L_0^*(q, s) \subset L_0^*(q, s'),\ L^*(q, s) \subset L^*(q', s)$$
$$\text{and} \quad L_0^*(q, s) \subset L_0^*(q', s)\,.$$

However, in this case it is not possible to repeat the argument that Li [Li] used in her dissertation.

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# TOPOLOGICAL GROUPS AND MACKEY FUNCTORS

MARCELO A. AGUILAR AND CARLOS PRIETO

ABSTRACT. Let $M$ be a Mackey functor for a finite group $G$ and let $X$ be a pointed $G$-space. We define a topological group $\overline{F}^G(X, M)$, whose homotopy groups are isomorphic to the Bredon-Illman equivariant homology of $X$ with coefficients in a coefficient system $\overline{M}_*$ associated to $M$. When $M$ is a homological Mackey functor, we define another topological group $\mathbb{F}^G(X, M)$, whose homotopy groups are isomorphic to the Bredon-Illman equivariant homology of $X$ with coefficients in the covariant part of $M$. These topological groups are defined using simplicial groups $\overline{F}^G(\mathcal{S}(X), M)$ and $F^G(\mathcal{S}(X), M)$, which have the same underlying groups, namely the groups of $G$-fixed points $F(\mathcal{S}_n(X), M)^G$, where $\mathcal{S}(X)$ is the singular simplicial set of $X$.

Furthermore, we study the transfer for finite covering $G$-maps and give its pullback property. We also analyze the composite of the transfer with the homomorphism induced by the projection map, in particular, in the case of $(G, \Gamma)$-bundles.

## 1. Introduction

Let $M$ be a Mackey functor for a finite group $G$ and $X$ a pointed $G$-space. In [2] we defined an abelian group $F^G(X, M)$ with a topology that made it into a topological group. This group is given as the geometric realization of a simplicial group $F^G(\mathcal{S}(X), M)$, where $\mathcal{S}(X)$ denotes the singular simplicial set of $X$. This simplicial group is a quotient of another simplicial group $F(\mathcal{S}(X), M)$, which has a simplicial action of $G$ via isomorphisms. The $n$th group $F^G(\mathcal{S}(X), M)_n$ is the fixed-point subgroup $F(\mathcal{S}(X), M)_n^G$. We can also define another simplicial group, which is a simplicial subgroup of $F(\mathcal{S}(X), M)$, denoted by $\overline{F}^G(\mathcal{S}(X), M)$, whose $n$th group is also $F(\mathcal{S}(X), M)_n^G$.

Therefore, with the same groups of fixed points $F(\mathcal{S}(X), M)_n^G$ we have defined two different simplicial groups. Their geometric realizations, in turn, define two different topological groups $F^G(X, M)$ (as above) and $\overline{F}^G(X, M)$. In [2] we showed that the homotopy groups of $F^G(X, M)$ are isomorphic to the Bredon-Illman $G$-equivariant homology of $X$ with coefficients in the covariant part of $M$. In this paper we show that the homotopy groups of $\overline{F}^G(X, M)$ are isomorphic to the Bredon-Illman $G$-equivariant homology of $X$ with coefficients in a covariant coefficient system $\overline{M}_*$ associated to $M$.

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In [2] we also introduced a continuous transfer $t_p^G : F^G(X, M) \longrightarrow F^G(E, M)$ for an $n$-fold covering $G$-map $p : E \longrightarrow X$. In this paper we prove that this transfer has the pullback property.

The elements of $F^G(X, M)$ are defined in terms of the singular simplexes of $X$. However, when $M$ is a homological Mackey functor, we can define another topological abelian group $\mathbb{F}^G(X, M)$, whose elements are given directly in terms of the points of $X$. We prove that if $X$ has the homotopy type of a $G$-CW-complex, then this group is homotopy equivalent to $F^G(X, M)$, and thus its homotopy groups also yield the same $G$-equivariant homology theory with coefficients in $M$. The homological Mackey functors are precisely those for which the composite of the transfer and the projection is given by the expected formula.

We also study the transfer for a class of covering $G$-maps, called $(\Gamma, G)$-bundles.

The paper is organized as follows. In Section 2, for any pointed $G$-set $C$, we recall the definition of the abelian group $F(C, M)$, which is indeed a functor on $C$. We show that $G$ acts on this group by isomorphisms, and use it to define the subgroup $F(C, M)^G$ of $G$-fixed elements and the two different functorial structures on it. In Section 3, for any $G$-function $p : A \longrightarrow C$ with finite fibers, we define a transfer homomorphism $t_p^G : F(X, M)^G \longrightarrow F(E, M)^G$ and study its properties, especially the pullback property. In Section 4, if $X$ is a pointed $G$-space, we define topological groups $F^G(X, M)$ and $\overline{F}^G(X, M)$ and we show that the functors $F(-, M)$ and $F^G(-, M)$ are characterized by certain universal properties. In Section 5, we construct a topological abelian group $\mathbb{F}^G(X, M)$, which has the abelian group $F^G(X^\delta, M)$ as underlying group, where $X^\delta$ denotes the underlying pointed $G$-set of $X$. We prove also a universal property that characterizes $\mathbb{F}^G(X, M)$ as a topological group. In Section 6, when $p : E \longrightarrow X$ is a covering $G$-map, we study the continuity of the transfers $t_p^G$ for the groups $F^G(X, M)$ and $\mathbb{F}^G(X, M)$.

The main part of the paper is Section 7, where we prove that the homotopy groups of the (functorial) topological group $\overline{F}^G(X, M)$ are isomorphic to the (reduced) Bredon-Illman equivariant homology groups of $X$ with coefficients in the coefficient system $\overline{M}_*$, given on orbits $G/H$ by $\overline{M}_*(G/H) = M(G/H)$ and on quotient functions $q : G/H \longrightarrow G/K$ by $\overline{M}_*(q) = [K : H]M_*(q)$. We also prove that, if $M$ is homological, the homotopy groups of $\mathbb{F}^G(X, M)$ realize the Bredon-Illman homology with coefficients in the covariant part $M_*$ of $M$.

Finally, in Section 8 we study the transfers for some special examples of covering $G$-maps $p : E \longrightarrow X$, namely for $(G, \Gamma)$-bundles. We show that for a homological Mackey functor, the transfers have particularly nice properties.

The topological setting of this paper is the category of k-*spaces* (see e.g. [9], [11]).

## 2. The equivariant function-group functors

Throughout the paper $G$ will denote a finite group and we shall write $H \subset G$ for a subgroup $H$ of $G$. Let $G$-$\mathcal{S}\mathrm{et}_{\mathrm{fin}}$ denote the category of finite $G$-sets and $G$-equivariant functions (*$G$-functions*). Recall that a *Mackey functor* (see [4], for instance) consists of two functors, one covariant and one contravariant, both with the same object function $M : G\text{-}\mathcal{S}\mathrm{et}_{\mathrm{fin}} \longrightarrow \mathcal{A}\mathrm{b}$. If $\alpha : S \longrightarrow T$ is a $G$-function between $G$-sets, we denote the covariant part in morphisms by $M_*(\alpha) : M(S) \longrightarrow M(T)$ and the contravariant part by $M^*(\alpha) : M(T) \longrightarrow M(S)$. The functor has to be additive in the sense that the two embeddings $S \hookrightarrow S \sqcup T \hookleftarrow T$ into the disjoint union of $G$-sets define an isomorphism $M(S \sqcup T) \cong M(S) \oplus M(T)$ and if one has a pullback diagram of $G$-sets

(2.1)
$$\begin{tikzcd} U \arrow[r, "\widetilde{\beta}"] \arrow[d, "\widetilde{\alpha}"'] & S \arrow[d, "\alpha"] \\ T \arrow[r, "\beta"'] & V, \end{tikzcd}$$

then

(2.2)
$$M_*(\widetilde{\beta}) \circ M^*(\widetilde{\alpha}) = M^*(\alpha) \circ M_*(\beta)$$

(see [4] for details).

By the additivity property, the Mackey functor $M$ is determined by its restriction $M : \mathcal{O}(G) \longrightarrow \mathcal{A}b$, where $\mathcal{O}(G)$ is the full subcategory of $G$-orbits $G/H$, $H \subset G$. A particular role will be played by the $G$-function $R_{g^{-1}} : G/H \longrightarrow G/gHg^{-1}$, given by *right translation* by $g^{-1} \in G$, namely

$$R_{g^{-1}}(g'H) = g'Hg^{-1} = g'g^{-1}(gHg^{-1}).$$

We shall often denote the coset $gH$ by $[g]_H$ or simply by $[g]$, if there is no danger of confusion. Observe that if $C$ is a $G$-set and $x \in C$, then the canonical bijection $G/G_x \longrightarrow G/G_{gx}$ is precisely $R_{g^{-1}}$, where as usual $G_x$ denotes the *isotropy subgroup* of $x$, namely the maximal subgroup of $G$ that leaves $x$ fixed.

*Definition* (2.3). Let $M$ be a Mackey functor. Define the set $\widehat{M}$ as the union

$$\widehat{M} = \bigcup_{H \subset G} M(G/H).$$

If $C$ is any pointed $G$-set (where the base point $x_0$ is fixed under the action of $G$), then we define the set

$$F(C, M) = \{u : C \longrightarrow \widehat{M} \mid u(x) \in M(G/G_x),\ \ u(x_0) = 0,\ \text{ and }\ u(x) = 0$$
$$\text{for almost all } x \in C\}.$$

One may write the elements $u \in F(C, M)$ as $u = \sum_{x \in C} l_x x$, where $l_x = u(x) \in M(G/G_x)$ (the sum is obviously finite). $F(C, M)$ is again a $G$-set with the left action of $G$ on $F(C, M)$ given by

$$(g \cdot u)(x) = M_*(R_{g^{-1}})(u(g^{-1}x)).$$

For simplicity, if $l \in \widehat{M}$ and $g \in G$, we shall denote by $gl$ the element $M_*(R_{g^{-1}})(l)$. Thus the action of $G$ on $F(C, M)$ can be written as

$$g\left(\sum_x l_x x\right) = \sum_x (gl_x)(gx) = \sum_x (gl_{g^{-1}x})x\,.$$

The $G$-set $F(C, M)$ is indeed an abelian group with the sum $u + v$ for $u, v \in F(C, M)$ given by $(u + v)(x) = u(x) + v(x) \in M(G/G_x)$. We shall denote by $F(C, M)^G$ the subgroup of fixed points of $F(C, M)$ under the action of $G$.

In what follows, we shall define two functors from the category of arbitrary pointed $G$-sets $G\text{-}\mathcal{S}\mathrm{et}_*$ to the category of abelian groups $\mathcal{A}\mathrm{b}$

$$G\text{-}\mathcal{S}\mathrm{et}_* \xrightarrow{F^G(-,M)} \mathcal{A}\mathrm{b} \qquad G\text{-}\mathcal{S}\mathrm{et}_* \xrightarrow{\overline{F}^G(-,M)} \mathcal{A}\mathrm{b}\,.$$

These two functors have the same value on objects, namely

$$F^G(C,M) = \overline{F}^G(C,M) = F(C,M)^G\,,$$

as defined above, but on morphisms, they are different. In order to define these functors on morphisms, we shall extend $F(C,M)$ to a functor $G\text{-}\mathcal{S}\mathrm{et}_* \longrightarrow \mathcal{A}\mathrm{b}$ as follows.

Let $\gamma_x : M(G/G_x) \longrightarrow F(C,M)$ be given by $\gamma_x(l) = lx$. Then we clearly have the following.

PROPOSITION (2.4). *Let $A$ be an abelian group and for each $x \in C$ let $\varphi_x : M(G/G_x) \longrightarrow A$ be a homomorphism, such that $\varphi_{x_0} = 0$, where $x_0 \in X$ is the base point. Then there exists a unique homomorphism $\varphi : F(X,M) \longrightarrow A$ such that $\varphi \circ \gamma_x = \varphi_x$. In a diagram*

$$\begin{tikzcd} M(G/G_x) \arrow[r, "\gamma_x"] \arrow[dr, "\varphi_x"'] & F(C,M) \arrow[d, dotted, "\varphi"] \\ & A\,. \end{tikzcd}$$

The previous proposition allows us to define a covariant functor structure on $F(-,M)$ and the functor $\overline{F}(-,M)^G$.

*Definition* (2.5). For any $G$-function $f : C \longrightarrow D$, we shall denote by $\widehat{f}_x : G/G_x \longrightarrow G/G_{f(x)}$ the canonical quotient $G$-function. Let $f$ be a pointed $G$-function. Define the family

$$f_x : M(G/G_x) \longrightarrow F(D,M) \quad \text{by} \quad f_x(l) = M_*(\widehat{f}_x)(l)f(x)\,.$$

By Proposition (2.4) this family determines a homomorphism

$$f_* : F(C,M) \longrightarrow F(D,M)$$

given by

$$f_*\left(\sum_x l_x x\right) = \sum_x M_*(\widehat{f}_x)(l_x)f(x)\,.$$

This turns $F(-,M)$ into a covariant functor. Moreover, since

$$gM_*(\widehat{f}_x)(l) = M_*(\widehat{f}_{gx})(gl)\,,$$

$f_*$ is $G$-equivariant, and so, by restriction, it defines a homomorphism

$$\overline{f}_*^G : F(C,M)^G \longrightarrow F(D,M)^G\,.$$

This defines the functor $\overline{F}^G(-,M)$.

*Remark* (2.6). We denote by $G\text{-}\mathcal{A}\mathrm{b}$ the category whose objects are abelian groups with a $G$-action by group isomorphisms, and whose morphisms are $G$-equivariant homomorphisms. Notice that the functor $F(-,M)$ is indeed a functor $G\text{-}\mathcal{S}\mathrm{et}_* \longrightarrow G\text{-}\mathcal{A}\mathrm{b}$.

To define the second covariant functor $F^G(-, M)$, take a pointed $G$-set $C$ and consider the abelian group $F(C, M)^G$ once more. Let $x_0$ be the base point of the $G$-set $C$ which remains fixed under the action of $G$ and for each $x \in C$, let $\gamma_x^G : M(G/G_x) \longrightarrow F(C, M)^G$ be given by $\gamma_x^G(l) = \sum_{i=1}^{n}(g_i l)(g_i x)$, where $\{[g_1], \dots [g_n]\} = G/G_x$. Then $\gamma_{x_0}^G = 0$ and $\gamma_x^G = \gamma_{gx}^G \circ M_*(R_{g^{-1}})$.

In order to define the functor $F^G(-M)$, we showed that the abelian group $F(X, M)^G$, together with the family $\{\gamma_x^G\}$, is characterized by the following property (see [2], 1.6).

PROPOSITION (2.7). *Let $A$ be an abelian group and for each $x \in C$ let $\varphi'_x : M(G/G_x) \longrightarrow A$ be a homomorphism, such that $\varphi'_{x_0} = 0$, where $x_0 \in C$ is the base point, and such that $\varphi'_x = \varphi'_{gx} \circ M_*(R_{g^{-1}})$. Then there exists a unique homomorphism $\varphi' : F(C, M)^G \longrightarrow A$ such that $\varphi' \circ \gamma_x^G = \varphi'_x$. In a diagram*

$$\begin{tikzcd} M(G/G_x) \arrow[r, "\gamma_x^G"] \arrow[dr, "\varphi'_x"'] & F(C, M)^G \arrow[d, dotted, "\varphi'"] \\ & A\,. \end{tikzcd}$$

Notice that this proposition is a "coordinate-free" description of the fact that algebraically

$$F(C, M)^G \cong \bigoplus_{[x] \in C/G - \{[x_0]\}} M(G/G_x)\,.$$

The previous proposition allows us to define the second covariant functor $F^G(-, M)$.

*Definition* (2.8). Let $f : C \longrightarrow D$ be a pointed $G$-function. Define the family

$$f'_x : M(G/G_x) \longrightarrow F(D, M)^G \quad \text{by} \quad f'_x(l) = \gamma_{f(x)}^G M_*(\widehat{f}_x)(l)\,.$$

By Proposition (2.7) this family determines a homomorphism

$$f_*^G : F(C, M)^G \longrightarrow F(D, M)^G\,.$$

Then, for any $u = \sum_{i=1}^{k} \gamma_{x_i}^G(l_i) \in F(C, M)^G$, one has

$$f_*^G(u) = \sum_{i=1}^{k} \gamma_{f(x_i)}^G M_*(\widehat{f}_{x_i})(l_i)\,.$$

We denote this functor by $F^G(-, M)$.

The following result puts the definition of the functor structures $\overline{f}_*^G$ and $f_*^G$ in a diagram.

PROPOSITION (2.9). *Let $C$ be a pointed $G$-set and let $\beta_C : F(C, M) \longrightarrow F(C, M)^G$ be the surjective homomorphism given on generators by $\beta_C(lx) =$*

*$\gamma_x^G(l)$. If $f : C \longrightarrow D$ is a pointed $G$-function, then one has the following commutative diagram.*

(2.10)
$$\begin{tikzcd} \overline{F}^G(C,M) \arrow[r, hook, "i_C"] \arrow[d, "\overline{f}_*^G"'] & F(C,M) \arrow[r, two heads, "\beta_C"] \arrow[d, "f_*"'] & F^G(C,M) \arrow[d, "f_*^G"] \\ \overline{F}^G(D,M) \arrow[r, hook, "i_D"'] & F(D,M) \arrow[r, two heads, "\beta_D"'] & F^G(D,M). \end{tikzcd}$$

*This means, in particular, that $\beta : F(-,M) \longrightarrow F^G(-,M)$ is a natural transformation.*

Notice that the horizontal composites in (2.10) are not the identity.

The following result measures the difference between $f_*^G$ and $\overline{f}_*^G$ in the canonical generators $\gamma_x^G(l) \in F(C,M)^G$.

PROPOSITION (2.11). *Let $f : C \longrightarrow D$ be a pointed $G$-function. Then*
$$\overline{f}_*^G(\gamma_x^G(l)) = [G_{f(x)} : G_x] f_*^G(\gamma_x^G(l)) \in F(C,M)^G .$$

*Proof*: Let $G/G_{f(x)} = \{[g_1], \dots, [g_m]\}$ and $G_{f(x)}/G_x = \{[h_1], \dots, [h_k]\}$. Then $G/G_x = \{[g_1h_1], [g_1h_2], \dots, [g_mh_{k-1}], [g_mh_k]\}$. First observe that by definition, $f_*^G(\gamma_x^G(l)) = \gamma_{f(x)}^G(M_*(\widehat{f}_x)(l))$. Therefore,

$$\begin{aligned} \overline{f}_*^G(\gamma_x^G(l)) &= \overline{f}_*^G\left(\sum_{(i,j)=(1,1)}^{(m,k)} M_*(R_{(g_ih_j)^{-1}})(l) g_ih_jx\right) \\ &= \sum_{(i,j)=(1,1)}^{(m,k)} M_*(\widehat{f}_{g_ih_jx}) M_*(R_{(g_ih_j)^{-1}})(l) g_ih_jf(x) \\ &= \sum_{(i,j)=(1,1)}^{(m,k)} M_*(\widehat{f}_{g_ix}) M_*(R_{g_i^{-1}})(l) g_if(x) \\ &= \sum_{j=1}^{k}\sum_{i=1}^{m} M_*(R_{g_i^{-1}}) M_*(\widehat{f}_x)(l) g_if(x) \\ &= [G_{f(x)} : G_x] \gamma_{f(x)}^G(M_*(\widehat{f}_x)(l)) \\ &= [G_{f(x)} : G_x] f_*^G(\gamma_x^G(l)). \end{aligned}$$
□

*Remark* (2.12). From the previous result it follows that both homomorphisms $\overline{f}_*^G$ and $f_*^G$ coincide if the $G$-map $f$ is isovariant (i.e. if $G_{f(x)} = G_x$ for all $x \in C$), for instance if $D$ is $G$-free or if $C$ and $D$ are $G$-trivial.

*Definition* (2.13). Let $M$ be a Mackey functor for the finite group $G$. We define the coefficient system $\overline{M}_* : \mathcal{O}(G) \longrightarrow \mathcal{A}b$ as follows. Put $\overline{M}_*(G/H) = M(G/H)$. Moreover, let $f : G/H \longrightarrow G/K$ be a $G$-function. If $f = R_g : G/H \longrightarrow G/g^{-1}Hg$, then $\overline{M}_*(f) = M_*(f)$, and if $f = q : G/H \longrightarrow G/K$, where $H \subset K$, is the quotient function, then $\overline{M}_*(f) = [K : H] M_*(f)$.

THEOREM (2.14). *The functors* $\overline{F}^G(-,M), F^G(-,M) : G\text{-}\mathcal{S}\mathrm{et}_* \longrightarrow \mathcal{A}\mathrm{b}$ *are characterized by properties* (a) *and* (b$_1$), *and* (a) *and* (b$_2$), *respectively, where:*

(a) *Let $A$ be an abelian group and for each $x \in C$ let $\varphi'_x : M(G/G_x) \longrightarrow A$ be a homomorphism, such that $\varphi'_{x_0} = 0$, where $x_0 \in C$ is the base point, and such that $\varphi'_x = \varphi'_{gx} \circ M_*(R_{g^{-1}})$. Then there exists unique homomorphism $\varphi' : F(C,M)^G \longrightarrow A$ such that $\varphi' \circ \gamma_x^G = \varphi'_x$. In a diagram*

$$\begin{tikzcd} M(G/G_x) \arrow[r, "\gamma_x^G"] \arrow[dr, "\varphi'_x"'] & F(C,M)^G \arrow[d, dotted, "\varphi'"] \\ & A\,. \end{tikzcd}$$

*Note here that $\overline{F}^G(C,M) = F(C,M)^G = F^G(C,M)$.*

(b) *Given a pointed $G$-function $f : C \longrightarrow D$, the following diagrams commute:*

$$(\mathrm{b}_1) \qquad \begin{tikzcd} M(G/G_x) \arrow[r, "\gamma_x^G"] \arrow[d, "\overline{M}_*(\widehat{f_x})"'] & \overline{F}^G(C,M) \arrow[d, "\overline{f}_*^G"] \\ M(G/G_{f(x)}) \arrow[r, "\gamma_{f(x)}^G"'] & \overline{F}^G(D,M)\,, \end{tikzcd}$$

$$(\mathrm{b}_2) \qquad \begin{tikzcd} M(G/G_x) \arrow[r, "\gamma_x^G"] \arrow[d, "M_*(\widehat{f_x})"'] & F^G(C,M) \arrow[d, "f_*^G"] \\ M(G/G_{f(x)}) \arrow[r, "\gamma_{f(x)}^G"'] & F^G(D,M)\,. \end{tikzcd}$$

*Proof.* Part (a) is Proposition (2.7). Part (b) follows from the definition and from Proposition (2.11).

To see that (a) and (b$_1$) characterize the functor $\overline{F}^G(-,M)$, assume that we have two functors $F(-)$ and $F'(-)$ that satisfy (a) and (b$_1$). Property (a) allows us to construct $\alpha_C : F(C) \longrightarrow F'(C)$ and $\alpha'_C : F'(C) \longrightarrow F(C)$ that are inverse to each other. Moreover, property (b$_1$) allows us to show that $\alpha$ and $\alpha'$ are natural transformations. Similarly, one proves that (a) and (b$_2$) characterize the functor $F^G(-,M)$. □

*Remark* (2.15). Notice that in the proof of the previous theorem one only needs the covariant part of $M$. Thus the result is equally valid for any covariant coefficient system.

## 3. The transfer for the functor $F^G(-;M)$

We use the property (2.4) to give the transfer. We start with the following definition, that was given in [2], 1.10; we put it now in terms of the property (2.4).

*Definition* (3.1). Let $M$ be a Mackey functor and $p : A \longrightarrow C$ a $G$-function with finite fibers, that is, a $G$-function such that for each $x \in C$, the fiber

$p^{-1}(x) \subset A$ is finite. For any $x \in C$, let $t_x : M(G/G_x) \longrightarrow F(A^+, M)$ be given by

$$t_x(l) = \sum_{a \in p^{-1}(x)} M^*(\widehat{p}_a)(l)a\,.$$

By (2.4) for $F(C^+, M)$, there is a unique homomorphism

$$t_p : F(C^+, M) \longrightarrow F(A^+, M),$$

such that $t_p \circ \gamma_x = t_x$. Explicitly, on generators,

$$t_p\left(lx\right) = \sum_{a \in p^{-1}(x)} M^*(\widehat{p}_a)(l)a\,.$$

Since $p$ is a $G$-function, $t_p$ is also a $G$-function, as we show in the lemma below, and thus it determines, by restriction, the *transfer*

$$t_p^G : F(C^+, M)^G \longrightarrow F(A^+, M)^G\,.$$

*Remark* (3.2). The homomorphism $t_p : F(C^+, M) \longrightarrow F(A^+, M)$ can also be described as follows:

$$t_p(u)(a) = M^*(\widehat{p}_a)(u(p(a)))$$

(and $t_p(u)(*) = 0$).

LEMMA (3.3). *$t_p : F(C^+, M) \longrightarrow F(A^+, M)$ is a $G$-homomorphism.*

*Proof.* We have on the one hand

$$t_p(g \cdot u)(a) = M^*(\widehat{p}_a)(g \cdot u(p(a))) = M^*(\widehat{p}_a)M_*(R_{g^{-1}})(u(g^{-1}p(a))),$$

while on the other hand we have

$$(g \cdot t_p(u))(a) = M_*(R_{g^{-1}})(t_p(u)(g^{-1}a)) = M_*(R_{g^{-1}})M^*(\widehat{p}_{g^{-1}a})(u(g^{-1}p(a))).$$

Both terms are equal, since $M^*(\widehat{p}_a)\circ M_*(R_{g^{-1}}) = M_*(R_{g^{-1}})\circ M^*(\widehat{p}_{g^{-1}a})$, and this follows from the fact that the following square is clearly a pullback diagram of $G$-sets:

$$\begin{tikzcd}
G/G_{g^{-1}a} \arrow[r, "R_{g^{-1}}", "\cong"'] \arrow[d, "\widehat{p}_{g^{-1}a}"'] & G/G_a \arrow[d, "\widehat{p}_a"] \\
G/G_{g^{-1}p(a)} \arrow[r, "\cong", "R_{g^{-1}}"'] & G/G_{p(a)}.
\end{tikzcd}$$

□

*Remark* (3.4). Assume that $p : A \longrightarrow C$ and $q : C \longrightarrow D$ are $G$-functions with finite fibers. Then one has that $\widehat{(q \circ p)}_a = \widehat{q}_{p(a)} \circ \widehat{p}_a$. Using this, one easily verifies that the transfer is functorial in the sense that $t_{q\circ p}^G = t_p^G \circ t_q^G$.

LEMMA (3.5). *Let $p : A \longrightarrow C$ be a $G$-function with finite fibers. Then*

(3.6) $$t_p^G(\gamma_x^G(l)) = \sum_{[a] \in p^{-1}(x)/G_x} \gamma_a^G(M^*(\widehat{p}_a)(l)),$$

*Proof.* The isotropy group $G_x$ acts on $p^{-1}(x)$ and the inclusion $j : p^{-1}(x) \hookrightarrow p^{-1}(Gx)$ clearly induces a bijection $\overline{j} : p^{-1}(x)/G_x \longrightarrow p^{-1}(Gx)/G$. Let $\gamma_x^G(l)$ be a generator of $F^G(C^+, M)$. Since the value of the function $\gamma_x^G(l)$ on points which do not belong to $Gx$ is zero, and $\gamma_x^G(l)(x) = l$, we have that

$$t_p^G(\gamma_x^G(l)) = \sum_{[a]\in p^{-1}(x)/G_x} \gamma_a^G(M^*(\widehat{p}_a)(l)). \qquad \square$$

We shall now prove that the transfer $t_p^G$ has the pullback property. We start with some preliminary results on groups. One can easily prove the following.

LEMMA (3.7). *Let $H, H' \subset K \subset G$ be subgroups of $G$ and consider the fibered product*

$$G/H \times_{G/K} G/H' = \{([g]_H, [g']_{H'}) \mid g, g' \in G \text{ and } g^{-1}g' \in K\}\,.$$

*Consider the set of double cosets $H\backslash K/H' = \{{}_H[g_r]_{H'} \mid r = 1, \ldots, k\}$, where $g_1, \ldots, g_k \in K$ are fixed representatives. If $H_r'' = H \cap g_r H' g_r^{-1}$, then there is an isomorphism of $G$-sets*

$$\varphi : \sqcup_{r=1}^k G/H_r'' \xrightarrow{\cong} G/H \times_{G/K} G/H'\,,$$

*given by $\varphi[g]_{H_r''} = ([g]_H, [gg_r]_{H'})$.*

LEMMA (3.8). *Let $H, H' \subset K \subset G$ be subgroups of $G$ and let $M$ be a Mackey functor. Consider the isomorphism*

$$\bigoplus_{r=1}^k M(G/H_r'') \longrightarrow M(\sqcup_{r=1}^k G/H_r'')$$

*given by the family $M_*(\kappa_r)$, where $\kappa_r : G/H_r'' \hookrightarrow \sqcup_{r=1}^k G/H_r''$ is the inclusion. Then its inverse is given by the homomorphism induced by the family $M^*(\kappa_r)$.*

*Proof.* The following are pullback digrams:

$$\begin{tikzcd} G/H_r'' \arrow[r, "="] \arrow[d, "="'] & G/H_r'' \arrow[d, "\kappa_r"] \\ G/H_r'' \arrow[r, "\kappa_r"'] & \sqcup G/H_r'' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & G/H_s'' \arrow[d, "\kappa_s"] \\ G/H_r'' \arrow[r, "\kappa_r"'] & \sqcup G/H_r'' \end{tikzcd}$$

where $r \neq s$. Therefore

$$M^*(\kappa_r) \circ M_*(\kappa_r) = 1_{M(G/H_r'')} \quad \text{and} \quad M^*(\kappa_s) \circ M_*(\kappa_r) = 0\,.$$

Thus the result follows. $\square$

LEMMA (3.9). *Let $H, H' \subset K \subset G$ be subgroups of $G$ and let $M$ be a Mackey functor. Take $w \in M(G/H \times_{G/K} G/H')$; then*

$$w = \sum_{r=1}^k M_*(\varphi_r) M^*(\varphi_r)(w)\,,$$

*where $\varphi_r = \varphi \circ \kappa_r$.*

*Proof.* By the previous lemma, for any $z \in M(\sqcup G/H''_r)$ we have

$$z = \sum_{r=1}^{k} M_*(\kappa_r)M^*(\kappa_r)(z)\,. \tag{3.10}$$

By Lemma (3.7), we have an isomorphism

$$M_*(\varphi) : M(\sqcup_{r=1}^k G/H''_r) \longrightarrow M(G/H \times_{G/K} G/H')\,.$$

Then for some $z \in M(\sqcup_{r=1}^k G/H''_r)$, $w = M_*(\varphi)(z)$. By (3.10), $M_*(\varphi)(z) = M_*(\varphi)(\sum_{r=1}^k M_*(\kappa_r)M^*(\kappa_r)(z)) = \sum_{r=1}^k M_*(\varphi_r)M^*(\varphi_r)(w)$. The last equality follows from the fact that $M_*(\varphi)^{-1} = M^*(\varphi)$, as one easily sees. $\square$

Let $p : A \longrightarrow C$ be a $G$-function with finite fibers and let $f : D \longrightarrow C$ be any $G$-function. Consider the pullback diagram

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "p'"'] & A \arrow[d, "p"] \\ D \arrow[r, "f"'] & C\,, \end{tikzcd} \tag{3.11}$$

where $A' = D \times_C A = \{(y, a) \mid f(y) = p(a)\}$. Consider the restriction of $f'$ from the fiber $(p')^{-1}(y)$ to the fiber $p^{-1}(f(y))$. This function induces a surjective function

$$q : (p')^{-1}(y)/G_y \longrightarrow p^{-1}(f(y))/G_{f(y)}\,.$$

In what follows we analyze the fibers of $q$.

LEMMA (3.12). *There is a bijection*

$$\overline{\delta} : G_y\backslash G_{f(y)}/G_{a_0} \longrightarrow q^{-1}(G_{f(y)}a_0)\,,$$

*where $a_0 \in p^{-1}(f(y))$, given by $\overline{\delta}({}_{G_y}[g]_{G_{a_0}}) = G_y(y, ga_0)$.*

*Proof.* The function $\overline{\delta}$ is induced by the surjection $\delta : G_{f(y)} \longrightarrow q^{-1}(G_{f(y)}a_0)$ given by $\delta(g) = G_y(y, ga_0)$. One easily checks that $\delta$ factors through the set of double cosets and that $\overline{\delta}$ is injective. $\square$

THEOREM (3.13). *Let $p : A \longrightarrow C$ be a $G$-function with finite fibers, and let $f : D \longrightarrow C$ be a $G$-function. Then*

$$t_p^G \circ f_*^G = (f')_*^G \circ t_{p'}^G : F^G(D^+, M) \longrightarrow F^G(A^+, M)\,,$$

*where $f'$ and $p'$ are as in the pullback diagram* (3.11).

*Proof.* Take a generator $\gamma_y^G(l)$, $y \in D$ and $l \in M(G/G_y)$, and consider $q : (p')^{-1}(y)/G_y \longrightarrow p^{-1}(f(y))/G_{f(y)}$ as in Lemma (3.12). Then, by Definition (2.8) and the formula (3.6), we have

$$t_p^G f_*^G(\gamma_y^G(l)) = \sum_{[a_\iota] \in p^{-1}(f(y))/G_{f(y)}} \gamma_{a_\iota}^G M^*(\widehat{p}_{a_\iota})M_*(\widehat{f}_y)(l)\,. \tag{3.14}$$

On the other hand, we have

$$(f')_*^G t_{p'}^G(\gamma_y^G(l)) = \sum_{[y,a] \in (p')^{-1}(y)/G_y} \gamma_a^G M_*(\widehat{f'}_{(y,a)})M^*(\widehat{p'}_{(y,a)})(l)\,. \tag{3.15}$$

We can write $(p')^{-1}(y)/G_y = \sqcup q^{-1}(G_{f(y)}a_\iota)$, where $G_{f(y)}a_\iota = [a_\iota]$. By Lemma (3.12), $p^{-1}(f(y))/G_{f(y)} = \{[y, g_r a_\iota]\}$, where the group-elements $g_r$ are such that $\{_{G_y}[g_r]_{G_{a_\iota}}\}_{r=1}^{k} = G_y\backslash G_{f(y)}/G_{a_\iota}$ (notice that the set $\{g_r\}_{r=1}^{k}$ depends on each $\iota$). Clearly we have

$$\gamma^G_{g_r a_\iota} M_*(\widehat{f'}_{(y,g_r a_\iota)}) M^*(\widehat{p'}_{(y,g_r a_\iota)})(l) = \gamma^G_{a_\iota} M_*(R_{g_r} \circ \widehat{f'}_{(y,g_r a_\iota)}) M^*(\widehat{p'}_{(y,g_r a_\iota)})(l)\,. \tag{3.16}$$

Consider the following pullback diagram

$$\begin{tikzcd}
G/G_y \cap G_{g_r a_\iota} \arrow[dr, dashed, "\varphi_r"] \arrow[drr, bend left, "R_{g_r}\circ \widehat{f'}_{(y,g_r a_\iota)}"] \arrow[ddr, bend right, "\widehat{p'}_{(y,g_r a_\iota)}"'] & & \\
& G_y \times_{G/G_{f(y)}} G/G_{a_\iota} \arrow[r, "\tau"] \arrow[d, "\pi"'] & G/G_{a_\iota} \arrow[d, "\widehat{p}_{a_\iota}"] \\
& G/G_y \arrow[r, "\widehat{f}_y"'] & G/G_{f(y)}\,.
\end{tikzcd}$$

Hence, $M^*(\widehat{p}_{a_\iota}) \circ M_*(\widehat{f}_y) = M_*(\tau) \circ M^*(\pi)$. Using Lemma (3.9), we can write

$$M^*(\pi)(l) = \sum_{r=1}^{k} M_*(\varphi_r) M^*(\varphi_r) M^*(\pi)(l) = \sum_{r=1}^{k} M_*(\varphi_r) M^*(\widehat{p'}_{(y,g_r a_\iota)})(l)\,.$$

Composing with $M_*(\tau)$ on the left, we obtain

$$\begin{aligned}
M_*(\tau) M^*(\pi)(l) &= \sum_{r=1}^{k} M_*(\tau) M_*(\varphi_r) M^*(\widehat{p'}_{(y,g_r a_\iota)})(l) \\
&= \sum_{r=1}^{k} M_*(R_{g_r} \circ \widehat{f'}_{(y,g_r a_\iota)}) M^*(\widehat{p'}_{(y,g_r a_\iota)})(l)\,.
\end{aligned}$$

Hence

$$M^*(\widehat{p}_{a_\iota}) M_*(\widehat{f}_y)(l) = \sum_{r=1}^{k} M_*(R_{g_r} \circ \widehat{f'}_{(y,g_r a_\iota)}) M^*(\widehat{p'}_{(y,g_r a_\iota)})(l)\,,$$

and the result follows. □

## 4. The topological function groups

We start this section extending the definitions given in the previous sections in the case of $G$-sets to the case of simplicial $G$-sets. We denote by $\Delta$ the category whose objects are the ordered sets $\mathbf{n} = \{0, 1, \ldots, n\}$ and whose morphisms are order-preserving functions between them. A *simplicial pointed $G$-set* is thus a contravariant functor $K : \Delta \longrightarrow G\text{-}\mathcal{S}\mathrm{et}_*$. We denote by $K_n$ the value of $K$ in $\mathbf{n}$, and given a morphism $\mu : \mathbf{m} \longrightarrow \mathbf{n}$, we denote by $\mu^K : K_n \longrightarrow K_m$ the corresponding pointed $G$-function.

*Definition* (4.1). Let $K$ be a simplicial pointed $G$-set and $M$ a Mackey funtor for $G$. We define the simplicial abelian groups $F^G(K, M)$ and $\overline{F}^G(K, M)$ as the following composites:

$$\Delta \xrightarrow{K} G\text{-}\mathcal{S}\mathrm{et}_* \xrightarrow{F^G(-,M)} \mathcal{A}\mathrm{b}\,, \qquad \Delta \xrightarrow{K} G\text{-}\mathcal{S}\mathrm{et}_* \xrightarrow{\overline{F}^G(-,M)} \mathcal{A}\mathrm{b}\,.$$

Therefore, for each $n$, the value of the functors $F^G(K, M)$ and $\overline{F}^G(K, M)$ at $n$ are given by $F^G(K_n, M)$ and $\overline{F}^G(K_n, M)$, respectively.

Notice that by Remark (2.6), there is also a simplicial abelian $G$-group defined by the composite

$$\Delta \xrightarrow{K} G\text{-}\mathcal{S}\mathrm{et}_* \xrightarrow{F(-,M)} G\text{-}\mathcal{A}\mathrm{b}\,.$$

PROPOSITION (4.2). *Let $K$ be a simplicial pointed $G$-set. Then*

(a) *$\overline{F}^G(K, M)$ is a simplicial subgroup of $F(K, M)$, and*
(b) *$F^G(K, M)$ is a simplicial quotient group of $F(K, M)$.*

*Proof.* This follows by applying Proposition (2.9) to $\mu^K : K_n \longrightarrow K_m$, where $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in $\Delta$. The inclusion of (a) is given by the natural transformation $i : \overline{F}^G(-, M) \hookrightarrow F(-, M)$, and the surjection of (b) is given by the natural transformation $\beta : F(-, M) \twoheadrightarrow F^G(-, M)$. □

In what follows, we shall use the previous definitions to associate topological abelian groups $F^G(X, M)$ and $\overline{F}^G(X, M)$ to a pointed $G$-space $X$. We shall work in the category of k-spaces. We understand by a k-*space* a topological space $X$ with the property that a set $W \subset X$ is closed if and only if $f^{-1}W \subset Z$ is closed for any continuous map $f : Z \longrightarrow X$, where $Z$ is any compact Hausdorff space (see [9], [11]).

If $S$ is a simplicial set ($G$-set, group, etc.), we denote by $|S|$ its *geometric realization*. This is a quotient space of

$$\sqcup_n S_n \times \Delta^n$$

(see [8] for details).

LEMMA (4.3). *Let $S$ be a simplicial pointed $G$-set. Then there is a canonical homeomorphism $|S^G| \longrightarrow |S|^G$.*

*Proof.* Let $i : S^G \hookrightarrow S$ be the inclusion. This morphism induces an embedding $|i| : |S^G| \longrightarrow |S|$. One easily sees that the image of $|i|$ is a subset of $|S|^G$. In order to see that $|S|^G$ is indeed the image of $|i|$, let $[\sigma, t] \in |S|^G$ be represented by a nondegenerate element $(\sigma, t)$. Then $g[\sigma, t] = [g\sigma, t]$ coincides with $[\sigma, t]$. Since $\sigma$ is nondegenerate, so is also $g\sigma$. Therefore, $g\sigma = \sigma$ and so $[\sigma, t]$ is in the image of $|i|$. □

*Definition* (4.4). Let $X$ be a pointed $G$-space and let $\mathcal{S}(X)$ be the associated singular simplicial pointed $G$-set, where the base point in each $\mathcal{S}_n(X)$ is the constant $n$-simplex with value $x_0$. We define the following topological spaces:

$$F^G(X, M) = |F^G(\mathcal{S}(X), M)|\,, \quad \overline{F}^G(X, M) = |\overline{F}^G(\mathcal{S}(X), M)|\,.$$

Notice that these two spaces have the structure of regular CW-complexes.

*Remark* (4.5). One may also define $F(X, M) = |F(\mathcal{S}(X), M)|$ and by Lemma (4.3), $\overline{F}^G(X, M) = |F(\mathcal{S}(X), M)|^G = F(X, M)^G$.

If $X$ is a $G$-space, then the underlying groups of $F^G(X,M)$ and $\overline{F}^G(X,M)$ differ from the (discrete) group $F(X^\delta,M)^G$, as defined in section 2, where $X^\delta$ denotes the underlying $G$-set of $X$. However, we have the following.

PROPOSITION (4.6). *If $X$ is a discrete pointed $G$-space, then the topological abelian groups $F^G(X,M)$ and $\overline{F}^G(X,M)$ are discrete and both are isomorphic to the abelian group $F(X^\delta,M)^G$.*

*Proof*: Notice that if $K$ is a simplicial set such that $K_n = C$ for all $n$, and $f^K = \mathrm{id}_C$ for all $f$ in $\Delta$, then $|K|$ is a discrete space homeomorphic to $C$, because $|K|$ is a CW-complex with one $n$-cell for each nondegenerate $n$-simplex of $K$. We call such a simplicial set *trivial*.

Now, if $X$ is discrete, then $\mathcal{S}_n(X) = X^\delta$ for all $n$ and $f^{\mathcal{S}(X)} = \mathrm{id}_X$ for all $f$, thus it is trivial. Therefore, the simplicial groups $F^G(\mathcal{S}(X),M)$ and $\overline{F}^G(\mathcal{S}(X),M)$ are trivial too. Hence

$$|F^G(\mathcal{S}(X),M)| \cong F(X^\delta,M)^G \cong |\overline{F}^G(\mathcal{S}(X),M)|. \qquad \square$$

*Remark* (4.7). The functors $F^G(-,M)$ and $\overline{F}^G(-,M)$, restricted to the category of discrete pointed $G$-spaces, are indeed naturally isomorphic to the functors $F^G((-)^\delta,M)$ and $\overline{F}^G((-)^\delta,M)$, respectively.

PROPOSITION (4.8). *Let $X$ be a pointed $G$-space. Then the spaces $F^G(X,M)$ and $\overline{F}^G(X,M)$ are topological abelian groups (in the category of* k-*spaces).*

*Proof.* Since $F^G(\mathcal{S}(X),M)$ and $\overline{F}^G(\mathcal{S}(X),M)$ are simplicial abelian groups, their geometric realizations $|F^G(\mathcal{S}(X),M)|$ and $|\overline{F}^G(\mathcal{S}(X),M)|$ are topological groups (in the category of k-spaces, see [9], [11]). $\square$

*Remark* (4.9). In a similar way to the previous proposition, we have that $F(X,M)$ is a topological abelian $G$-group. By Proposition (4.2) and [5], we have that

(a) $\overline{F}^G(X,M)$ is a topological subgroup of $F(X,M)$, and
(b) $F^G(X,M)$ is a topological quotient group of $F(X,M)$.

We have the following.

*Definition* (4.10). Let $K$ be a simplicial pointed $G$-set and $M$ a Mackey functor for $G$. Let $\Lambda$ be any simplicial abelian group. We shall say that a family of homomorphisms $\{\varphi_\sigma : M(G/G_\sigma) \longrightarrow \Lambda_n \mid \sigma \in K_n, n \geq 0\}$ is *simplicial* if the following conditions are satisfied:

(a) If $\sigma_0 \in K_n$ is the base point, then $\varphi_{\sigma_0} = 0$, and
(b) for each morphism $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ in $\Delta$, the following diagram commutes:

$$\begin{tikzcd} M(G/G_\sigma) \arrow[r, "\varphi_\sigma"] \arrow[d, "M_*(\widehat{\mu^K}_\sigma)"'] & \Lambda_n \arrow[d, "\mu^\Lambda"] \\ M(G/G_{\mu^K(\sigma)}) \arrow[r, "\varphi_{\mu^K(\sigma)}"'] & \Lambda_m . \end{tikzcd}$$

We say that the simplicial family is *G-invariant* if for all $\sigma \in K$ and all $g \in G$,

$$\varphi_{g\sigma} = \varphi_\sigma \circ M_*(R_g),$$

Corresponding to the property (2.4), we have the following.

PROPOSITION (4.11). *Let $K$ be a simplicial pointed $G$-set and $M$ a Mackey functor for $G$. Then*

(i) *the family $\{\gamma_\sigma : M(G/G_\sigma) \longrightarrow F(K_n, M) \mid \sigma \in K_n, n \geq 0\}$ is simplicial. Moreover*
(ii) *if $\Lambda$ is any simplicial abelian group and $\{\varphi_\sigma\ M(G/G_\sigma) \longrightarrow \Lambda_n \mid \sigma \in K_n, n \geq 0\}$ is a simplicial family of homomorphisms, then there is a unique simplicial homomorphism $\varphi : F(K, M) \longrightarrow \Lambda$, such that $\varphi_n \circ \gamma_\sigma = \varphi_\sigma$, where $\sigma \in K_n$, $n \geq 0$.*

*Proof.* Let $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ be a morphism in $\Delta$. To see (i), take $l \in M(G/G_\sigma)$. Then

$$\mu_*^K \gamma_\sigma(l) = \mu_*^K(l\sigma) = M_*(\widehat{\mu^K}_\sigma)(l)\mu^K(\sigma) = \gamma_{\mu^K(\sigma)} M_*(\widehat{\mu^K}_\sigma)(l).$$

We now prove (ii). By Proposition (2.4), for each $n$ there is a unique homomorphism $\varphi_n : F(K_n, M) \longrightarrow \Lambda_n$ such that $\varphi_n \circ \gamma_\sigma = \varphi_\sigma$. To check that the family $\{\varphi_n\}$ is a morphism of simplicial groups, take a generator $l\sigma \in F(K_n, M)$. Then

$$\begin{aligned}\varphi_m \mu_*^K(l\sigma) &= \varphi_m(M_*(\widehat{\mu^K}_\sigma)(l)\mu^K(\sigma)) = \varphi_{\mu^K(\sigma)}(M_*(\widehat{\mu^K}_\sigma)(l)) \\ &= \mu^\Lambda \varphi_\sigma(l) = \mu^\Lambda \varphi_n(l\sigma).\end{aligned}$$

□

We now have the following result, which is similar to the previous proposition.

PROPOSITION (4.12). *Let $K$ be a simplicial pointed $G$-set and $M$ a Mackey functor for $G$. Then*

(i) *the family $\{\gamma_\sigma^G : M(G/G_\sigma) \longrightarrow F^G(K_n, M) \mid \sigma \in K_n, n \geq 0\}$ is simplicial and $G$-invariant. Moreover*
(ii) *if $\Lambda$ is any simplicial abelian group and $\{\varphi_\sigma\ M(G/G_\sigma) \longrightarrow \Lambda_n \mid \sigma \in K_n, n \geq 0\}$ is a simplicial $G$-invariant family of homomorphisms, then there is a unique simplicial homomorphism $\varphi^G : F^G(K, M) \longrightarrow \Lambda$, such that $\varphi_n^G \circ \gamma_\sigma^G = \varphi_\sigma$, where $\sigma \in K_n$, $n \geq 0$.*

Before passing to the definition of the functorial structures of $F(X; M)$, $F^G(X; M)$, and $\overline{F}^G(X; M)$, recall that a morphism of simplicial pointed $G$-sets $\alpha : K \longrightarrow Q$ consists of a family of pointed $G$-functions $\alpha_n : K_n \longrightarrow Q_n$ such that, if $\mu : \mathbf{m} \longrightarrow \mathbf{n}$ is a morphism in $\Delta$, then one has a commutative diagram

$$\begin{tikzcd} K_n \arrow[r, "\alpha_n"] \arrow[d, "\mu^K"'] & Q_n \arrow[d, "\mu^Q"] \\ K_m \arrow[r, "\alpha_m"'] & Q_m . \end{tikzcd}$$

Since we have functors $F(-,M), F^G(-,M), \overline{F}^G(-,M) : G\text{-}\mathcal{S}\mathrm{et}_* \longrightarrow \mathcal{A}\mathrm{b}$, they yield commutative diagrams

$$\begin{tikzcd} F(K_n,M) \arrow[r, "\alpha_{n*}"] \arrow[d, "\mu^K_*"'] & F(Q_n,M) \arrow[d, "\mu^Q_*"] \\ F(K_m,M) \arrow[r, "\alpha_{m*}"'] & F(Q_m,M), \end{tikzcd} \qquad \begin{tikzcd} F^G(K_n,M) \arrow[r, "{\alpha_n}_*^G"] \arrow[d, "{\mu^K}_*^G"'] & F^G(Q_n,M) \arrow[d, "{\mu^Q}_*^G"] \\ F^G(K_m,M) \arrow[r, "{\alpha_m}_*^G"'] & F^G(Q_m,M), \end{tikzcd}$$

$$\begin{tikzcd} \overline{F}^G(K_n,M) \arrow[r, "\overline{\alpha_n}_*^G"] \arrow[d, "\overline{\mu^K}_*^G"'] & \overline{F}^G(Q_n,M) \arrow[d, "\overline{\mu^Q}_*^G"] \\ \overline{F}^G(K_m,M) \arrow[r, "\overline{\alpha_m}_*^G"'] & \overline{F}^G(Q_m,M). \end{tikzcd}$$

Hence the functors $F(-,M)$, $F^G(-,M)$, and $\overline{F}^G(-,M)$ extend to functors of simplicial pointed $G$-sets.

*Definition* (4.13). Let $f : X \longrightarrow Y$ be a continuous pointed $G$-map. The map $f$ induces a morphism of simplicial pointed $G$-sets $\mathcal{S}(f) : \mathcal{S}(X) \longrightarrow \mathcal{S}(Y)$, which defines homomorphisms of simplicial groups

$$\mathcal{S}(f)_* : F(\mathcal{S}(X),M) \longrightarrow F(\mathcal{S}(Y),M),$$
$$\mathcal{S}(f)^G_* : F^G(\mathcal{S}(X),M) \longrightarrow F^G(\mathcal{S}(Y),M),$$
$$\overline{\mathcal{S}(f)}^G_* : \overline{F}^G(\mathcal{S}(X),M) \longrightarrow \overline{F}^G(\mathcal{S}(Y),M).$$

Define the homomorphisms

$$f_* : F(X,M) \longrightarrow F(Y,M),$$
$$f^G_* : F^G(X,M) \longrightarrow F^G(Y,M),$$
$$\overline{f}^G_* : \overline{F}^G(X,M) \longrightarrow \overline{F}^G(Y,M),$$

by $f_* = |\mathcal{S}(f)_*|$, $f^G_* = |\mathcal{S}(f)^G_*|$, and $\overline{f}^G_* = |\overline{\mathcal{S}(f)}^G_*|$, respectively.

*Remark* (4.14). One may obtain the simplicial homomorphisms

$$\mathcal{S}(f)_* : F(\mathcal{S}(X),M) \longrightarrow F(\mathcal{S}(Y),M),$$
$$\mathcal{S}(f)^G_* : F^G(\mathcal{S}(X),M) \longrightarrow F^G(\mathcal{S}(Y),M),$$

using the properties (4.11) and (4.12) for the families $\{\varphi_\sigma\}$ and $\{\varphi^G_\sigma\}$ given by

$$\varphi_\sigma(l) = \gamma_{\mathcal{S}(f)(\sigma)}(M_*(\widehat{\mathcal{S}_n(f)}_\sigma)(l)) \in F(\mathcal{S}_n(X),M),$$

$$\varphi^G_\sigma(l) = \gamma^G_{\mathcal{S}(f)(\sigma)}(M_*(\widehat{\mathcal{S}_n(f)}_\sigma)(l)) \in F^G(\mathcal{S}_n(X),M).$$

They provide the following explicit expressions for them on generators:

$$\mathcal{S}(f)_*(\gamma_\sigma(l)) = \gamma_{\mathcal{S}(f)(\sigma)}(M_*(\widehat{\mathcal{S}_n(f)}_\sigma)(l)),$$

$$\mathcal{S}(f)^G_*(\gamma^G_\sigma(l)) = \gamma^G_{\mathcal{S}(f)(\sigma)}(M_*(\widehat{\mathcal{S}_n(f)}_\sigma)(l)).$$

Since $\overline{\mathcal{S}(f)}^G_*$ is the restriction of $\mathcal{S}(f)_*$, the first gives also an explicit expression in this case.

Clearly, we have the following result.

PROPOSITION (4.15). *If $f\colon X \to Y$ is a continuous pointed $G$-map, then $f_*\colon F(X,M) \to F(Y,M)$, $f_*^G\colon F^G(X,M) \to F^G(Y,M)$, and $\overline{f}_*^G\colon \overline{F}^G(X,M) \to \overline{F}^G(Y,M)$ are continuous homomorphisms. Thus $F(-,M)$, $F^G(-,M)$, and $\overline{F}^G(-,M)$ are covariant functors from the category of pointed $G$-spaces to the category of topological abelian groups. In particular, $F(X,M)$ is a topological abelian $G$-group.*

*Remark* (4.16). Let $f : X \longrightarrow Y$ be a pointed $G$-map. By (2.9), it follows that one has an epimorphism of simplicial groups $\beta_{\mathcal{S}(X)} : F(\mathcal{S}(X),M) \longrightarrow F^G(\mathcal{S}(X),M)$. Thus, by [5], its geometric realization

$$\beta_X : F(X,M) \longrightarrow F^G(X,M)$$

is an identification for any pointed $G$-space $X$. One can visualize both functor structures in an analogous way to the commutative diagram (2.10), namely,

(4.17)
$$\begin{tikzcd}
\overline{F}^G(X,M) \arrow[r, hook] \arrow[d, "\overline{f}_*^G"'] & F(X,M) \arrow[r, two heads, "\beta_X"] \arrow[d, "f_*"'] & F^G(X,M) \arrow[d, "f_*^G"] \\
\overline{F}^G(Y,M) \arrow[r, hook] & F(Y,M) \arrow[r, two heads, "\beta_Y"'] & F^G(Y,M),
\end{tikzcd}$$

where the groups are now topological and all the homomorphisms are continuous.

To finish this section, we prove that the functors $F(X,M)$, $F^G(X,M)$, and $\overline{F}^G(X,M)$ are homotopy invariant. For that, we need the following.

LEMMA (4.18). *Let $K$ and $Q$ be simplicial pointed $G$-sets and be $\alpha_0, \alpha_1 : K \longrightarrow Q$ be morphisms. If $\alpha_0$ and $\alpha_1$ are $G$-homotopic, then*

(a) $\alpha_{0*}, \alpha_{1*} : F(K,M) \longrightarrow F(Q,M)$ *are $G$-homotopic homomorphisms;*
(b) $\alpha_{0*}^G, \alpha_{1*}^G : F^G(K,M) \longrightarrow F^G(Q,M)$ *are homotopic homomorphisms;*
(c) $\overline{\alpha}_{0*}^G, \overline{\alpha}_{1*}^G : \overline{F}^G(K,M) \longrightarrow \overline{F}^G(Q,M)$ *are homotopic homomorphisms.*

*Proof.* Let $\mathcal{H} : K \times \Delta[1] \longrightarrow Q$ be a $G$-homotopy between $\alpha_0$ and $\alpha_1$, since $\mathcal{H}$ is $G$-equivariant (where $\Delta[1]$ has the trival action), it induces homomorphisms

$$\mathcal{H}_* : F(K \times \Delta[1], M) \longrightarrow F(Q,M),$$
$$\mathcal{H}_*^G : F^G(K \times \Delta[1], M) \longrightarrow F^G(Q,M),$$
$$\overline{\mathcal{H}}_*^G : \overline{F}^G(K \times \Delta[1], M) \longrightarrow \overline{F}^G(Q,M).$$

Let $\iota : F(K,M) \times \Delta[1] \longrightarrow F(K \times \Delta[1], M)$ be given by

$$\iota_n(u,a)(\sigma,b) = \begin{cases} u(\sigma) & \text{if } b = a, \\ 0 & \text{if } b \neq a, \end{cases}$$

where $(u,a) \in F(K_n,M) \times \Delta[1]_n$ and $(\sigma,b) \in K_n \times \Delta[1]_n$. We have that $\iota_n(u+u',a) = \iota_n(u,a) + \iota_n(u',a)$. Therefore

$$\iota_n(\sum_\sigma l_\sigma \sigma, a) = \sum_\sigma l_\sigma(\sigma, a).$$

One can easily check that $\iota$ is a morphism of simplicial pointed sets, (where the base point in $\Delta[1]_n$ is the constant function with value 0). Then $\mathcal{H}_* \circ \iota$ is a homotopy between $\alpha_{0*}$ and $\alpha_{1*}$.

Since $\iota$ and $\mathcal{H}_*$ are $G$-equivariant, the restriction of $\mathcal{H}_* \circ \iota$ to $\overline{F}^G(K, M) \times \Delta[1]$ is a homotopy between $\overline{\alpha}^G_{0*}$ and $\overline{\alpha}^G_{1*}$.

Now let $\iota^G : F^G(K, M) \times \Delta[1] \longrightarrow F^G(K \times \Delta[1], M)$ be given by

$$\iota^G_n(u, a)(\sigma, b) = \begin{cases} u(\sigma) & \text{if } b = a\,, \\ 0 & \text{if } b \neq a\,, \end{cases}$$

where $(u, a) \in F^G(K_n, M) \times \Delta[1]_n$ and $(\sigma, b) \in K_n \times \Delta[1]_n$. Since $u$ is a $G$-invariant element, it follows that $\iota^G_n(u, a)$ is also $G$-invariant. We also have that $\iota^G_n(u + u', a) = \iota^G_n(u, a) + \iota^G_n(u', a)$. Therefore $\iota^G_n(\sum_\sigma \gamma^G_\sigma(l_\sigma), a) = \sum_\sigma \gamma^G_{(\sigma,a)}(l_\sigma)$. One can easily see that $\iota^G$ is a morphism of simplicial pointed sets. The composite $\mathcal{H}^G_* \circ \iota^G$ is a homotopy between $\alpha^G_{0*}$ and $\alpha^G_{1*}$. □

PROPOSITION (4.19). *If $f_0, f_1 : X \longrightarrow Y$ are G-homotopic pointed maps, then*

(a) *$f_{0*}, f_{1*} : F(X, M) \longrightarrow F(Y, M)$ are G-homotopic homomorphisms,*
(b) *$\overline{f}^G_{0*}, \overline{f}^G_{1*} : \overline{F}^G(X, M) \longrightarrow \overline{F}^G(Y, M)$ are homotopic homomorphisms, and*
(c) *$f^G_{0*}, f^G_{1*} : F^G(X, M) \longrightarrow F^G(Y, M)$ are homotopic homomorphisms.*

*Proof.* For convenience, we shall take the standard 1-simplex $\Delta^1$ instead of the unit interval $I$. Thus let $H : X \times \Delta^1 \longrightarrow Y$ be a pointed $G$-homotopy from $f_0$ to $f_1$. Consider the morphism of simplicial $G$-sets $R : \mathcal{S}(X) \times \Delta[1] \longrightarrow \mathcal{S}(Y)$ given as follows. If $s \in \Delta^n$, define $R_n : \mathcal{S}_n(X) \times \Delta[1]_n \longrightarrow \mathcal{S}_n(Y)$ by

$$R_n(\sigma, a)(s) = H(\sigma(s), a_\#(s))\,,$$

where $a_\# : \Delta^n \longrightarrow \Delta^1$ is the affine map determined by $a$. Then $R$ is a $G$-equivariant homotopy between $\mathcal{S}(f_0)$ and $\mathcal{S}(f_1)$. Thus, by the previous lemma, there is a homotopy $T$ between the morphisms $\mathcal{S}(f_0)_*$ and $\mathcal{S}(f_1)_*$. Then

$$H' : |F(\mathcal{S}(X), M)| \times |\Delta[1]| \xleftarrow{\approx} |F(\mathcal{S}(X), M) \times \Delta[1]| \xrightarrow{|T|} |F(\mathcal{S}(Y), M)|\,,$$

where the homeomorphism is canonical, is a homotopy between $f_{0*} = |\mathcal{S}(f_0)_*|$ and $f_{1*} = |\mathcal{S}(f_1)_*|$, and thus we have (a). Similarly, also using the previous lemma, we obtain (b) and (c). □

## 5. The topological function group $\mathbb{F}^G(X, M)$

In this section we shall define a new topological abelian group $\mathbb{F}^G(X, M)$, whose description is simpler than that of $F^G(X, M)$. Here our pointed $G$-spaces will be pointed k-spaces.

Let $X$ be a pointed $G$-space and let $\mathcal{S}(X)$ be the associated singular simplicial pointed $G$-set, where the base point in each $\mathcal{S}_n(X)$ is the constant $n$-simplex with value $x_0$. Denote by $X^\delta$ the underlying pointed $G$-set of $X$. We shall define a topology on the abelian group $F(X^\delta, M)^G$ as follows. Take the surjective homomorphism

$$\pi^G_X : |F^G(\mathcal{S}(X), M)| \twoheadrightarrow F(X^\delta, M)^G$$

defined by

$$\pi_X^G\left(\left[\sum_\sigma \gamma_\sigma^G(l_\sigma),\ t\right]\right) = \sum_\sigma \gamma_{\sigma(t)}^G M_*(p_{\sigma,t})(l_\sigma)\,.$$

We give $F(X^\delta, M)^G$ the identification topology, where $p_{\sigma,t} : G/G_\sigma \longrightarrow G/G_{\sigma(t)}$ is the quotient map. We denote the resulting space by $\mathbb{F}^G(X, M)$.

PROPOSITION (5.1). *Let $X$ be a pointed $G$-space. Then $\mathbb{F}^G(X, M)$ is a topological group (in the category of* k*-spaces).*

*Proof.* Consider the following commutative diagram:

$$\begin{tikzcd}
|F^G(\mathcal{S}(X), M)| \times |F^G(\mathcal{S}(X), M)| \arrow[r] \arrow[d, "\pi_X^G \times \pi_X^G"'] & |F^G(\mathcal{S}(X), M)| \arrow[d, "\pi_X^G"] \\
\mathbb{F}^G(X, M) \times \mathbb{F}^G(X, M) \arrow[r] & \mathbb{F}^G(X, M)\,,
\end{tikzcd}$$

since the product $\pi_X^G \times \pi_X^G$ in the category of k-spaces is an identification, the result follows. □

Let $f : X \longrightarrow Y$ be a continuous pointed $G$-map. It induces a pointed $G$-function $f : X^\delta \longrightarrow Y^\delta$ which defines a homomorphism $f_*^G : F(X^\delta, M)^G \longrightarrow F(Y^\delta, M)^G$. We have the following result.

PROPOSITION (5.2). *If $f : X \longrightarrow Y$ is a continuous pointed $G$-map, then*

$$f_*^G : \mathbb{F}^G(X, M) \longrightarrow \mathbb{F}^G(Y, M)$$

*is a continuous homomorphism. Thus $\mathbb{F}^G(-, M)$ is a covariant functor from the category of pointed $G$-spaces to the category of topological abelian groups.*

*Proof.* The $G$-map $f$ induces a morphism of simplicial $G$-sets $\mathcal{S}(f) : \mathcal{S}(X) \longrightarrow \mathcal{S}(Y)$ which in turn defines a homomorphism of simplicial groups

$$\mathcal{S}(f)_*^G : F^G(\mathcal{S}(X), M) \longrightarrow F^G(\mathcal{S}(Y), M)\,.$$

Consider the following diagram, where the top map is continuous:

$$\begin{tikzcd}
|F^G(\mathcal{S}(X), M)| \arrow[r, "|\mathcal{S}(f)_*^G|"] \arrow[d, "\pi_X^G"'] & |F^G(\mathcal{S}(Y), M)| \arrow[d, "\pi_Y^G"] \\
\mathbb{F}^G(X, M) \arrow[r, "f_*^G"'] & \mathbb{F}^G(Y, M)\,.
\end{tikzcd}$$

It is a straightforward verification that it is commutative. Therefore, $f_*^G$ is continuous. □

*Remark* (5.3). Notice that in (4.13) we defined a continuous homomorphism

$$f_*^G : F^G(X, M) \longrightarrow F^G(Y, M)\,,$$

which should not be confused with

$$f_*^G : \mathbb{F}^G(X, M) \longrightarrow \mathbb{F}^G(Y, M)\,.$$

They are related by the commutativity of the diagram

$$\begin{tikzcd} F^G(X,M) \arrow[r, "f_*^G"] \arrow[d, "\pi_X^G"'] & F^G(Y,M) \arrow[d, "\pi_Y^G"] \\ \mathbb{F}^G(X,M) \arrow[r, "f_*^G"'] & \mathbb{F}^G(Y,M), \end{tikzcd}$$

which is just the diagram in the proof of (5.2).

We shall now give a topological characterization of the group $\mathbb{F}^G(X,M)$, similar to Proposition (2.4). In order to do this, we need the following.

*Definition* (5.4). Let $X$ be a pointed $G$-space. Let $A$ be a topological abelian group in the category of k-spaces, and for each $x \in X$ let $\varphi_x : M(G/G_x) \longrightarrow A$ be a homomorphism, such that $\varphi_{x_0} = 0$, where $x_0 \in X$ is the base point. We say that $\{\varphi_x\}$ is a *continuous family* if the homomorphism

$$\widetilde{\varphi} : |F(\mathcal{S}(X),M)| \longrightarrow A$$

given by

$$\widetilde{\varphi}\left[\sum_{\sigma\in\mathcal{S}_n(X)} l_\sigma \sigma, t\right] = \sum_{\sigma\in\mathcal{S}_n(X)} \varphi_{\sigma(t)} M_*(p_{\sigma,t})(l_\sigma),$$

is continuous, where $p_{\sigma,t} : G/G_\sigma = G/G_{(\sigma,t)} \twoheadrightarrow G/G_{\sigma(t)}$ is the quotient map. We say that the family is *G-invariant*, if $\varphi_x = \varphi_{gx} \circ M_*(R_{g^{-1}})$ for all $g \in G$.

The universal property that characterizes the topological abelian group $\mathbb{F}^G(X,M)$, together with the family $\{\gamma_x^G\}$, is the following.

PROPOSITION (5.5). (i) *$\{\gamma_x^G\}$ is an equivariant continuous family.*

(ii) *Let $A$ be a topological abelian group and let $\{\varphi_x\}$ be an equivariant continuous family. Then there exists a unique continuous homomorphism $\varphi : \mathbb{F}^G(X,M) \longrightarrow A$ such that $\varphi \circ \gamma_x^G = \varphi_x$.*

*Proof.* By definition, the family $\{\varphi_x\}$ induces a continuous homomorphism $\widetilde{\varphi} : |F(\mathcal{S}(X),M)| \longrightarrow A$ and since the family is $G$-invariant, then by (2.7) there exists a unique homomorphism $\varphi : \mathbb{F}^G(X,M) \longrightarrow A$ such that $\varphi\circ\gamma_x^G = \varphi_x$ which satisfies $\varphi \circ \pi_X^G \circ |\beta_{\mathcal{S}(X)}| = \widetilde{\varphi}$. The simplicial homomorphism $\beta_{\mathcal{S}(X)}$ is surjective, hence by [5], $|\beta_{\mathcal{S}(X)}|$ is an identification, and since $\pi_X^G$ is also an identification, $\varphi$ is continuous. □

Observe that the continuity of $f_*^G$ shown above follows also from this universal property in a similar manner as that of (4.15).

We now show that the functor $\mathbb{F}^G(-,M)$ is homotopy invariant.

PROPOSITION (5.6). *If $f_0, f_1 : X \longrightarrow Y$ are G-homotopic pointed maps, then*

$$f_{0*}^G, f_{1*}^G : \mathbb{F}^G(X,M) \longrightarrow \mathbb{F}^G(Y,M)$$

*are homotopic homomorphisms.*

*Proof.* By (4.19), we have a homotopy $H' : F^G(X,M) \times \Delta^1 \longrightarrow F^G(Y,M)$. It is straightforward to verify that the map $\pi_Y^G \circ H'$ is compatible with the identification $\pi_X^G \times 1$, so that the following diagram commutes:

$$\begin{tikzcd} F^G(X,M) \times \Delta^1 \arrow[r, "H'"] \arrow[d, "\pi_X^G \times 1"'] & F^G(Y,M) \arrow[d, "\pi_Y^G"] \\ \mathbb{F}^G(X,M) \times \Delta^1 \arrow[r, "H''"'] & \mathbb{F}^G(Y,M). \end{tikzcd}$$

Then $H''$ is the desired homotopy. □

To finish this section we shall show that the group-functor $\mathbb{F}^G(-,M)$ has the same properties of $F^G(-,M)$, when $M$ is a homological Mackey functor. Recall the following.

*Definition* (5.7). A Mackey functor $M$ for $G$ is said to be *homological* if whenever $K \subset H \subset G$ and $q : G/H \longrightarrow G/K$ is the quotient function, one has $M_*(q)M^*(q) = [H:K]$, that is, multiplication by the index of $K$ in $H$.

*Example* (5.8). Given a $G$-module $L$, one defines a homological Mackey functor $M_L$ as follows. Put $M_L(G/H) = L^H$ and define

$$M_{L*}(R_{g^{-1}}) : L^H \longrightarrow L^{gHg^{-1}}, \quad l \longmapsto gl,$$

$$M_L^*(R_{g^{-1}}) : L^{gHg^{-1}} \longrightarrow L^H, \quad l \longmapsto g^{-1}l,$$

and if $H \subset K$, $K/H = \{[k_i]_H\}$, and $q : G/H \longrightarrow G/K$ is the quotient function, then

$$M_{L*}(q) : L^H \longrightarrow L^K, \quad l \longmapsto \sum k_i l,$$

$$M_L^*(q) : L^K \longrightarrow L^H \quad \text{is the inclusion.}$$

*Definition* (5.9). Given a $G$-module $L$, we define the functors $F(-,L)$ and $F^G(-,L)$ form the category of pointed $G$-sets to the category of abelian groups as follows:

$$F(C,L) = \{u : C \longrightarrow L \mid u(*) = 0 \text{ and } u(x) = 0 \text{ for almost all } x \in C\},$$
$$F^G(C,L) = \{u \in F(C,L) \mid u(gx) = gu(x) \text{ for all } x \in X,\ g \in G\},$$

(see [1], Def. 1.1). Moreover, if $X$ is a topological pointed $G$-space, then we can define a topology on $F(X,L)$ and on $F^G(X,L)$ as follows. Take the surjection

$$\mu : \sqcup_q (L \times X)^q \twoheadrightarrow F(X,L),$$

where $\mu(l_1, x_1, \ldots, l_q, x_q) = l_1x_1 + \cdots + l_qx_q$, and give $F(X,L)$ the identification topology, then give $F^G(X,L)$ the relative topology. We now have that $F(-,L)$ and $F^G(-,L)$ are functors from the category of pointed $G$-spaces to the category of abelian topological groups.

LEMMA (5.10). *The functors $F^G(-,L)$ and $F^G(-,M_L)$ form the category of pointed $G$-sets to the category of abelian groups are equal.*

*Proof.* Notice first that $\widehat{M_L} = L$ and if $u \in F^G(C, L)$, then $u(x) \in L^{G_x} = M_L(G/G_x)$. Let $f : C \longrightarrow D$ be a pointed $G$-function. Consider the projection $G/G_x \twoheadrightarrow G/G_{f(x)}$ with fiber $G_{f(x)}/G_x$. One can describe the cosets in $G/G_x$ as products of the cosets in $G/G_{f(x)}$ and those in $G_{f(x)}/G_x$, in a similar way as in the proof of Lemma (5.16), below. Then we can write a generator $\gamma_x^G(l)$ as $\sum(g_i h_j l)(g_i h_j x)$. Now we can easily check that the value of the homomorphisms induced by the functors $F^G(-, L)$ and $F^G(-, M_L)$ are equal on this generator. $\square$

*Remark* (5.11). Observe that when $X$ is a topological pointed $G$-space and $L$ is a $G$-module, we have two different abelian groups, namely, $F^G(X, L)$ as defined above, and $F^G(X, M_L) = |F^G(\mathcal{S}(X), M_L)|$ as defined in (4.4). However, $F^G(X, L)$ and $\mathbb{F}^G(X, M_L)$ are equal as abelian groups. Furthermore, the identity $\mathbb{F}^G(X, M_L) \longrightarrow F^G(X, L)$ is always continuous, as proved in [3]. We prove below (5.17) that it is a homeomorphism if $X = |K|$.

The following result of Thevenaz and Webb [10], Thm. (16.5)(i), will be used in what follows.

THEOREM (5.12). *Given a homological Mackey functor $M$, there exists a $G$-module $L$ and an epimorphism of Mackey functors $\xi : M_L \twoheadrightarrow M$.*

*Definition* (5.13). We shall denote by $\xi_\Diamond : F^G(-, M_L) \longrightarrow F^G(-, M)$ the natural transformation determined by $\xi : M_L \twoheadrightarrow M$, namely, if $u \in F^G(C, M_L)$, then $\xi_\Diamond(u)(x) = \xi_{G/G_x}(u(x))$, where $x \in C$.

Notice that for each $C$, $\xi_\Diamond$ is surjective, because if $\gamma_x^G(l')$ is a generator of $F^G(C, M)$ and $\xi_{G/G_x}(l) = l'$, then $\xi_\Diamond(\gamma_x^G(l)) = \gamma_x^G(l')$.

*Definition* (5.14). For a simplicial pointed $G$-set $K$ and a $G$-module $L$, we gave in [1], Prop. 2.3, a $G$-isomorphism of topological groups $\psi : |F(K, L)| \longrightarrow F(|K|, L)$ given on generators by $\psi([l\sigma, t]) = l[\sigma, t]$. We shall denote its restriction to the fixed-point subgroup by

$$\psi_L^G : |F^G(K, L)| \longrightarrow F^G(|K|, L)\,.$$

On the other hand, for any Mackey functor $M$ for $G$ we defined in [2], Prop. 2.6, an isomorphism

$$\psi_M^G : |F^G(K, M)| \longrightarrow \mathbb{F}^G(|K|, M)$$

as discrete groups, given by

$$\psi_M^G([\gamma_\sigma^G(l), t]) = \gamma_{[\sigma,t]}^G M_*(q_{\sigma,t})(l),$$

where $q_{\sigma,t} : G/G_\sigma \longrightarrow G/G_{[\sigma,t]}$ is the quotient function.

*Remark* (5.15). Notice that the identification

$$\pi_X^G : |F^G(\mathcal{S}(X), M)| \twoheadrightarrow \mathbb{F}^G(X, M)$$

factors as the composite

$$\rho_{X*}^G \circ \psi_M^G : |F^G(\mathcal{S}(X), M)| \longrightarrow \mathbb{F}^G(|\mathcal{S}(X)|, M) \longrightarrow \mathbb{F}^G(X, M)\,.$$

LEMMA (5.16). *The following is a commutative diagram*

$$\begin{tikzcd} {|F^G(K,L)|} \arrow[r, "\mathrm{id}"] \arrow[d, "\psi_L^G"'] & {|F^G(K,M_L)|} \arrow[d, "\psi_{M_L}^G"] \\ F^G(|K|,L) \arrow[r, "\mathrm{id}"'] & \mathbb{F}^G(|K|,M_L)\,. \end{tikzcd}$$

*Proof.* If we assume that $G/G_{[\sigma,t]} = \{[g_i]\ |, i = 1,\dots,r\}$ and $G_{[\sigma,t]}/G_\sigma = \{[h_j] \mid j = 1,\dots,s\}$, then $G/G_\sigma = \{[g_ih_j] \mid (i,j) = (1,1),\dots,(r,s)\}$. Thus we can write

$$\gamma_\sigma^G(l) = \sum_{(i,j)=(1,1)}^{(r,s)} (g_ih_jl)(g_ih_j\sigma) \in F^G(K,L)\,.$$

Therefore, $\psi_L^G([\gamma_\sigma^G(l),t]) = \sum_{(i,j)=(1,1)}^{(r,s)}(g_ih_jl)[g_ih_j\sigma,t]$.

On the other hand, $M_{L*}(q_{\sigma,t})(l) = \sum_{j=1}^{s} h_jl$, hence

$$\begin{aligned}\psi_{M_L}^G([\gamma_\sigma^G(l),t]) &= \gamma_{[\sigma,t]}^G\left(\sum_{j=1}^{s} h_jl\right)\\ &= \sum_{i=1}^{r} g_i\left(\sum_{j=1}^{s}(h_jl)g_i[\sigma,t]\right)\\ &= \sum_{i=1}^{r} g_i\left(\sum_{j=1}^{s}(h_jl)g_ih_j[\sigma,t]\right)\\ &= \psi_L^G([\gamma_\sigma^G(l),t]), \quad \text{since} \quad h_j \in G_{[\sigma,t]}\,.\end{aligned}$$

□

PROPOSITION (5.17). *If $K$ is a simplicial pointed $G$-set, then*

$$\mathrm{id} : \mathbb{F}^G(|K|,M_L) \longrightarrow F^G(|K|,L)$$

*is a homeomorphism.*

*Proof.* To simplify the notation we put $Y = |K|$. Consider the following diagram.

$$\begin{tikzcd} \mathbb{F}^G(|\mathcal{S}(Y)|,M_L) \arrow[dr, "\rho_{Y*}^G"'] & {|F^G(\mathcal{S}(Y),M_L)|} \arrow[l, "\psi_{M_L}^G"'] \arrow[r, equal] \arrow[d, "\pi_Y^G"] & {|F^G(\mathcal{S}(Y),L)|} \arrow[r, "\psi_L^G", "\cong"'] \arrow[d, "\widetilde{\pi}_Y^G"] & F^G(|\mathcal{S}(Y)|,L) \arrow[dl, "\widetilde{\rho}_{Y*}^G"] \\ & \mathbb{F}^G(Y,M_L) \arrow[r, "\mathrm{id}"'] & F^G(Y,L)\,. & \end{tikzcd}$$

The triangles commute by Remark (5.15) and the commutativity of the square follows from Lemma (5.10) and Lemma (5.16). On the other hand, by [2], 3.5, $\rho_Y : |\mathcal{S}(Y)| \longrightarrow Y$ is a $G$-retraction and, therefore, $\widetilde{\rho}_{Y*}^G$ is a retraction too, moreover $\psi_L^G$ is a homeomorphism (see [1], Prop. 2.3) and hence $\widetilde{\pi}_Y^G$ is an identification. Since by definition $\pi_Y^G$ is an identification, it follows that the identity on the bottom is a homeomorphism. □

As a consequence, we have the following.

COROLLARY (5.18). *For any pointed $G$-space $X$,*

$$\mathrm{id} : \mathbb{F}^G(|\mathcal{S}(X)|, M_L) \longrightarrow F^G(|\mathcal{S}(X)|, L)$$

*is a homeomorphism.*

We have the next.

PROPOSITION (5.19). *Let $M$ be a homological Mackey functor. Then*

$$\psi_M^G : |F^G(\mathcal{S}(X), M)| \longrightarrow \mathbb{F}^G(|\mathcal{S}(X)|, M)$$

*is an isomorphism of topological groups.*

*Proof.* Consider the following diagram

$$\begin{tikzcd}
 & {|F^G(\mathcal{S}(|\mathcal{S}(X)|), M_L)|} \arrow[r, two heads, "|\xi_\diamond|"] \arrow[d, two heads, "\pi^G_{|\mathcal{S}(X)|}"'] & {|F^G(\mathcal{S}(|\mathcal{S}(X)|), M)|} \arrow[d, two heads, "\pi^G_{|\mathcal{S}(X)|}"] \\
F^G(|\mathcal{S}(X)|, L) & \mathbb{F}^G(|\mathcal{S}(X)|, M_L) \arrow[l, "\mathrm{id}"', "\cong"] \arrow[r, two heads, "\xi_\diamond"] & \mathbb{F}^G(|\mathcal{S}(X)|, M) \\
{|F^G(\mathcal{S}(X), L)|} \arrow[u, "\psi_L^G", "\cong"'] \arrow[r, leftrightarrow, "\cong", "\mathrm{id}"'] & {|F^G(\mathcal{S}(X), M_L)|} \arrow[u, "\psi_{M_L}^G"] \arrow[r, two heads, "|\xi_\diamond|"'] & {|F^G(\mathcal{S}(X), M)|\,.} \arrow[u, dashed, "\psi_M^G"']
\end{tikzcd}$$

The subdiagram on the left commutes by Lemma (5.16), and the identity on the top of it is a homeomorphism by Corollary (5.18). One easily verifies that the other two subdiagrams commute too. Since $\xi_\diamond$ is surjective, $|\xi_\diamond|$ on the top is an identification (see [5]), hence $\xi_\diamond$ in the middle is also an identification. Since $|\xi_\diamond|$ on the bottom is an identification too and $\psi_L^G$ is a homeomorphism, as mentioned in (5.14), $\psi_M^G$ is a homeomorphism as well. □

PROPOSITION (5.20). *Let $X$ be a pointed $G$-space of the homotopy type of a $G$-CW-complex, and let $M$ be a homological Mackey functor. Then $\pi_X^G : F^G(X;M) \to \mathbb{F}^G(X, M)$ is a natural homotopy equivalence of topological groups.*

*Proof.* By [1], Prop. 2.12, $\rho_X : |\mathcal{S}(X)| \longrightarrow X$ is a $G$-homotopy equivalence. On the other hand, by Proposition (5.19), $\psi_M^G$ is an isomorphism of topological groups, and by (5.6) the functor $\mathbb{F}^G(-, M)$ is homotopy invariant. Therefore, by Remark (5.15),

$$\pi_X^G : F^G(X, M) = |F^G(\mathcal{S}(X), M)| \xrightarrow{\psi_M^G} \mathbb{F}^G(|\mathcal{S}(X)|, M) \xrightarrow{\rho_{X*}^G} \mathbb{F}^G(X, M)$$

is a homotopy equivalence of topological groups.

It is easy to see that the homormorphisms $\pi_X^G$ are natural, namely that if $f : X \longrightarrow Y$ is a pointed $G$-map, then the following diagram commutes:

$$\begin{tikzcd}
{|F^G(\mathcal{S}(X), M)|} \arrow[r, two heads, "|\mathcal{S}(f)_*^G|"] \arrow[d, two heads, "\pi_X^G"'] & {|F^G(\mathcal{S}(Y), M)|} \arrow[d, two heads, "\pi_Y^G"] \\
\mathbb{F}^G(X, M) \arrow[r, "f_*^G"'] & \mathbb{F}^G(Y, M)\,.
\end{tikzcd}$$

□

## 6. Continuity of the transfers

In this section we study the continuity of the transfer for the topological-group functors $F^G(-, M)$ and $\mathbb{F}^G(-, M)$. The following is the topological counterpart of Definition (3.1). Let $p : E \longrightarrow X$ be an *n-fold covering G-map*, i.e., an ordinary $n$-fold covering map, such that $E$ and $X$ are $G$-spaces and $p$ is equivariant. Hence $\mathcal{S}(p) : \mathcal{S}(E) \longrightarrow \mathcal{S}(X)$ has finite fibers. We have the following.

PROPOSITION (6.1). *The transfers*

$$t^G_{\mathcal{S}_q(p)} : F^G(\mathcal{S}_q(X)^+, M) \longrightarrow F^G(\mathcal{S}_q(E)^+, M)$$

*determine a homomorphism of simplicial abelian groups*

$$t^G_{\mathcal{S}(p)} : F^G(\mathcal{S}(X)^+, M) \longrightarrow F^G(\mathcal{S}(E)^+, M)\,.$$

*Proof.* Let $f : \mathbf{r} \longrightarrow \mathbf{q}$ be a morphism in $\Delta$ and consider the diagram

$$\begin{tikzcd} \mathcal{S}_q(E) \arrow[r, "f^{\mathcal{S}(E)}"] \arrow[d, "\mathcal{S}_q(p)"'] & \mathcal{S}_r(E) \arrow[d, "\mathcal{S}_r(p)"] \\ \mathcal{S}_q(X) \arrow[r, "f^{\mathcal{S}(X)}"'] & \mathcal{S}_r(X)\,. \end{tikzcd}$$

Take $\sigma \in \mathcal{S}_q(X)$. If $\mathcal{S}_q(p)^{-1}(\sigma) = \{\widetilde{\sigma}_i \mid i = 1, \ldots, n\}$, then $\mathcal{S}_r(p)^{-1}(f^{\mathcal{S}(X)}(\sigma)) = \{\widetilde{\sigma}_i \circ f_\# \mid i = 1, \ldots, n\}$. Therefore this is a pullback diagram. By Theorem (3.13), the following is a commutative diagram:

$$\begin{tikzcd} F^G(\mathcal{S}_q(E)^+, M) \arrow[r, "(f^{\mathcal{S}(E)^+})^G_*"] & F^G(\mathcal{S}_r(E)^+, M) \\ F^G(\mathcal{S}_q(X)^+, M) \arrow[r, "(f^{\mathcal{S}(X)^+})^G_*"'] \arrow[u, "t^G_{\mathcal{S}_q(p)}"] & F^G(\mathcal{S}_r(X)^+, M) \arrow[u, "t^G_{\mathcal{S}_r(p)}"'] \end{tikzcd}$$

and thus $t^G_{\mathcal{S}(p)} : F^G(\mathcal{S}(X)^+, M) \longrightarrow F^G(\mathcal{S}(E)^+, M)$ is a homomorphism of simplicial groups. $\square$

Hence we have the following.

*Definition* (6.2). Let $p : E \longrightarrow X$ be an $n$-fold covering $G$-map. Define the transfer $t^G_p : F^G(X^+, M) \longrightarrow F^G(E^+, M)$ by

$$t^G_p = |t^G_{\mathcal{S}(p)}|\,.$$

(Notice that for any space $X$, one has $\mathcal{S}_n(X^+) = \mathcal{S}_n(X)^+$.)

Thus we have the next result.

THEOREM (6.3). *The transfer* $t^G_p : F^G(X^+, M) \longrightarrow F^G(E^+, M)$ *is a continuous homomorphism.*

Let now $M$ be a homological Mackey functor. We shall now give a description of the transfer for the functor $\mathbb{F}^G(-, M)$.

Let $p\colon E \to X$ be an $n$-fold covering $G$-map. By (3.1), we have a transfer $t_p^G\colon F^G((X^\delta)^+, M) \to F^G((E^\delta)^+, M)$, which is a homomorphism $t_p^G\colon \mathbb{F}^G(X^+, M) \to \mathbb{F}^G(E^+, M)$.

THEOREM (6.4). *The transfer* $t_p^G : \mathbb{F}^G(X^+, M) \longrightarrow \mathbb{F}^G(E^+, M)$ *is continuous.*

*Proof.* The continuity of $t_p^G$ follows from the commutativity of the next diagram:

$$\begin{tikzcd}
{|F^G(\mathcal{S}(X^+), M)|} \arrow[r, "|t^G_{\mathcal{S}(p)}|"] \arrow[d, "\psi^G_M"] \arrow[dd, bend right=60, "\pi^G_X"'] & {|F^G(\mathcal{S}(E^+), M)|} \arrow[d, "\psi^G_M"] \arrow[dd, bend left=60, "\pi^G_E"] \\
F^G(|\mathcal{S}(X^+)|, M) \arrow[r, "t^G_{|\mathcal{S}(p)|}"] \arrow[d, "\rho^G_{X^+*}"] & F^G(|\mathcal{S}(E^+)|, M) \arrow[d, "\rho^G_{E^+*}"] \\
\mathbb{F}^G(X^+, M) \arrow[r, "t^G_p"'] & \mathbb{F}^G(E^+, M).
\end{tikzcd}$$

The square at the bottom commutes by the pullback property (3.13) applied to the pullback diagram

$$\begin{tikzcd}
{|\mathcal{S}(E)|} \arrow[r, "\rho_E"] \arrow[d, "\mathcal{S}(p)"'] & E \arrow[d, "p"] \\
{|\mathcal{S}(X)|} \arrow[r, "\rho_X"'] & X.
\end{tikzcd}$$

To see that this is indeed a pullback square, we shall show that for each $[\tau, t] \in |\mathcal{S}(X)|$, the fiber $|\mathcal{S}(p)|^{-1}([\tau, t])$ is mapped bijectively by $\rho_E$ onto the fiber $p^{-1}(\tau(t))$. So, assume first that $(\sigma, t)$ is a nondegenerate representative of $[\sigma, t]$. Since $p$ is an $n$-fold covering map, the fiber $\mathcal{S}(p)^{-1}(\tau)$ has $n$ elements, namely $\{\widetilde{\tau}_1, \ldots, \widetilde{\tau}_n\}$. We have a bijection $\mathcal{S}(p)^{-1}(\tau) \approx |\mathcal{S}(p)|^{-1}([\tau, t])$ given by $\widetilde{\tau}_j \leftrightarrow [\widetilde{\tau}_j, t]$. On the other hand, since $p$ is a covering map, there is a bijection $\mathcal{S}(p)^{-1}(\tau) \approx p^{-1}(\tau(t))$ given by $\widetilde{\tau}_j \leftrightarrow \widetilde{\tau}_j(t)$.

To prove that the diagram at the top commutes, we consider the inverse isomorphisms $\varphi_M^G$ of $\psi_M^G$, given by $\varphi_M^G(\gamma^G_{[\sigma,t]}(l)) = [\gamma^G_\sigma(l), t]$ provided that $(\sigma, t)$ is a nondegenerate representative. We shall show that

$$|t^G_{\mathcal{S}(p)}| \circ \varphi^G_M = \varphi^G_M \circ t^G_{|\mathcal{S}(p)|}\,.$$

Take $\gamma^G_{[\sigma,t]}(l) \in F^G(|\mathcal{S}(X^+)|, M)$. Then

$$|t^G_{\mathcal{S}(p)}|\varphi^G_M(\gamma^G_{[\sigma,t]}(l)) = \left[t^G_{\mathcal{S}(p)}(\gamma^G_\sigma(l)), t\right] = \left[\sum_{i=1}^{k} \gamma^G_{\widetilde{\sigma}_i} M^*(\widehat{\mathcal{S}(p)}_{\widetilde{\sigma}_i})(l), t\right]$$

and

$$\begin{aligned}\varphi^G_M t^G_{|\mathcal{S}(p)|}(\gamma^G_{[\sigma,t]}(l)) &= \varphi^G_M \left( \sum_{i=1}^{k} \gamma^G_{[\widetilde{\sigma}_i,t]} M^*(\widehat{|\mathcal{S}(p)|}_{[\widetilde{\sigma}_i,t]})(l) \right) \\ &= \left[ \sum_{i=1}^{k} \gamma^G_{\widetilde{\sigma}_i} M^*(\widehat{|\mathcal{S}(p)|}_{[\widetilde{\sigma}_i,t]})(l), t \right],\end{aligned}$$

where $\{\widetilde{\sigma}_i \mid i = 1, \dots, k\}$ is a set of representatives of $\mathcal{S}(p)^{-1}(\sigma)/G_\sigma$. To prove that the sums are equal, observe that, as we already mentioned above, there is a bijection between $\mathcal{S}(p)^{-1}(\sigma)$ and $|\mathcal{S}(p)|^{-1}([\sigma, t])$. Since $(\sigma, t)$ is nondegenerate, by [2], Prop. 2.4, the isotropy groups $G_\sigma$ and $G_{[\sigma,t]}$ are equal. Hence, $\{[\widetilde{\sigma}_i, t] \mid i = 1, \dots, k\}$ is a set of representatives of $|\mathcal{S}(p)|^{-1}([\sigma, t])/G_{[\sigma,t]}$. Moreover, since $(\widetilde{\sigma}_i, t)$ is also nondegenerate, then $G_{[\widetilde{\sigma}_i,t]} = G_{\widetilde{\sigma}_i}$, and therefore $\widehat{|\mathcal{S}(p)|}_{[\widetilde{\sigma}_i,t]} = \widehat{\mathcal{S}(p)}_{\widetilde{\sigma}_i}$. □

## 7. Homotopical homology theories

In the definition of the functors $F(-, M)$, $F^G(-, M)$, and $\overline{F}^G(-, M)$, given in Section 2, the contravariant structure of the Mackey functor $M$ was not used. Therefore the same definitions are valid if instead of $M$, we take a covariant coefficient system $N_*$ for the finite group $G$. Hence we have functors $F(-, N_*)$, $F^G(-, N_*)$, and $\overline{F}^G(-, N_*)$. We shall prove the following.

THEOREM (7.1). *Let $N_*$ be a covariant coefficient system for $G$ and let $X$ be a pointed $G$-space. Then the homotopy groups $\pi_q\big(F^G(X, N_*)\big)$ are naturally isomorphic to the (reduced) Bredon-Illman $G$-equivariant homology groups $\widetilde{H}^G_q(X; N_*)$.*

For the proof of this theorem we need the following result.

THEOREM (7.2). ([2], Thm. 4.5) *There is an isomorphism between Illman's chain complex $S^G(X, *; N_*)$ (cf. [6], p. 15) and the chain complex $F^G(\mathcal{S}(X), N_*)$.*

*Proof* of Theorem (7.1). We shall give an isomorphism

$$\widetilde{H}^G_q(X; N_*) \cong H_q\big(F^G(\mathcal{S}(X), N_*)\big) \longrightarrow \pi_q\big(F^G(X, N_*)\big).$$

Here the left-hand side is the Bredon-Illman (reduced) homology of $X$ with coefficients in $N_*$, which by definition is the homology of the chain complex $S^G(X, *; N_*)$, and the first isomorphism follows from the natural isomorphism of Theorem (7.2).

To construct the arrow, we shall give several isomorphisms as depicted in the following diagram.

$$\begin{tikzcd}
H_q\big(F^G(\mathcal{S}(X), N_*)\big) \arrow[dr, dotted] & \pi_q\big(F^G(\mathcal{S}(X), N_*)\big) \arrow[l, "i_*"', "\cong"] \arrow[r, "\Psi", "\cong"'] & \pi_q\big(\mathcal{S}(|F^G(\mathcal{S}(X), N_*)|)\big) \arrow[d, "\Phi", "\cong"'] \\
 & \pi_q\big(F^G(X, N_*)\big) \arrow[r, equal] & \pi_q\big(|F^G(\mathcal{S}(X), N_*)|\big)
\end{tikzcd}$$

By [2], Prop. 4.2, $i_*$ is an isomorphism. In particular, this shows that every cycle in $\widetilde{H}^G(X;N_*)$ is represented by a chain $u$, all of whose faces are zero. We call this a *special chain*.

The homomorphism $\Psi$, which is given by $\Psi(u)[t]=[u,t]$, where $u$ is a special $q$-chain and $t\in\Delta^q$, is an isomorphism, as follows from [8], 16.6.

In order to define $\Phi$, we must express $\Psi(u)$ as a map $\gamma:(\Delta[q],\dot{\Delta}[q])\longrightarrow\big(\mathcal{S}\big(|F^G(\mathcal{S}(X),N_*)|\big),*\big)$. By the Yoneda lemma, $\gamma$ is the unique map such that $\gamma(\delta_q)=\Psi(u)$, where $\delta_q=\mathrm{id}:\mathbf{q}\longrightarrow\mathbf{q}$. The homomorphism $\Phi$, defined by $\Phi[\gamma][f,s]=\gamma(f)(s)$, for $f\in\Delta[q]_n$ and $s\in\Delta^n$, is given by the adjunction between the realization functor and the singular complex functor (see [8], 16.1). $\square$

Proposition (7.3). *The functors* $\overline{F}^G(-,M)$ *and* $F^G(-,\overline{M}_*)$ *from* $G$-$\mathcal{S}\mathrm{et}_*$ *to* $\mathcal{A}\mathrm{b}$ *are the same.*

*Proof.* Since the covariant functors $M_*$ and $\overline{M}_*$ are equal in objects, then the groups $\overline{F}^G(C,M)$ and $F^G(C,\overline{M}_*)$ are equal. We shall see that on morphisms, these functors are also equal. For this, let $f:C\longrightarrow D$ be a pointed $G$-function and take $x\in C$. Consider the canonical projection $G/G_x\longrightarrow G/G_{f(x)}$ with fiber $G_{f(x)}/G_x$. Let us write $G/G_{f(x)}=\{[g_i]\mid i=1,\dots,r\}$ and $G_{f(x)}/G_x=\{[h_j]\mid j=1,\dots,k\}$. Therefore, $G/G_x=\{[g_ih_j]\mid i=1,\dots,r,\ j=1,\dots,k\}$. Take a generator $\gamma_x^G(l)\in\overline{F}^G(C,M)=F^G(C,\overline{M}_*)$. Then on the one hand,

$$\begin{aligned}f_*^G(\gamma_x^G(l))&=\gamma_{f(x)}^G(\overline{M}_*(\widehat{f}_x)(l))\\&=\sum_i g_i\overline{M}_*(\widehat{f}_x)(l)(g_if(x))\\&=\sum_i[G_{f(x)}:G_x]g_iM_*(\widehat{f}_x)(l)(g_if(x))\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\overline{f}_*^G(\gamma_x^G(l))&=f_*(\gamma_x^G(l))\\&=f_*\left(\sum_{i,j}(g_ih_jl)(g_ih_jx)\right)\\&=\sum_{i,j}M_*(\widehat{f}_{g_ih_jx})(g_ih_jl)f(g_ih_jx)\,.\end{aligned}$$

Since $h_j\in G_{f(x)}$ and by the formula $gM_*(\widehat{f}_x)(l)=M_*(\widehat{f}_{gx})(gl)$ given in Definition (2.5), we have

$$\overline{f}_*^G(\gamma_x^G(l))=\sum_{i,j}g_iM_*(\widehat{f}_x)(l)g_if(x)=\sum_i[G_{f(x)}:G_x]g_iM_*(\widehat{f}_x)(l)(g_if(x)).\qquad\square$$

Corollary (7.4). $\overline{F}^G(X,M)=F^G(X,\overline{M}_*)$ *when* $X$ *is a pointed* $G$*-space.*

*Proof.* By the previous proposition, the simplicial groups $\overline{F}^G(\mathcal{S}(X),M)$ and $F^G(\mathcal{S}(X),\overline{M}_*)$ are equal. Therefore their geometric realizations are equal as topological groups, and thus the result follows. $\square$

By Theorem (7.1) and the previous proposition, we have the following result.

THEOREM (7.5). *Let $M$ be a Mackey functor and $X$ a pointed $G$-space. Then the homotopy groups $\pi_q\big(\overline{F}^G(X, M)\big)$ are naturally isomorphic to the (reduced) Bredon-Illman $G$-equivariant homology groups $\widetilde{H}^G_q(X; \overline{M}_*)$ with coefficients in the coefficient system $\overline{M}_*$.*

As a consequence of Proposition (5.20), the homotopy invariance (5.6), and Theorem (7.1), we have the following.

THEOREM (7.6). *Let $M$ be a homological Mackey functor and $X$ a pointed $G$-space of the homotopy type of a $G$-CW-complex. Then the homotopy groups $\pi_q(\mathbb{F}^G(X, M))$ are naturally isomorphic to the (reduced) Bredon-Illman $G$-equivariant homology groups $\widetilde{H}^G_q(X; M_*)$ with coefficients in the coefficient system $M_*$.*

## 8. Some applications

We shall consider in this section a special family of finite covering $G$-maps and study the transfer homomorphism for this family.

*Definition* (8.1). Let $G$ and $\Gamma$ be two finite groups. A $(G, \Gamma)$-*bundle* is a principal $\Gamma$-bundle $p : E \longrightarrow X$, such that $E$ and $X$ are $G$-spaces, $p$ is equivariant, and the actions satisfy

$$g(a\gamma) = (ga)\gamma \quad \text{for all} \quad g \in G, \ \ a \in E, \ \ \gamma \in \Gamma\,. \tag{8.2}$$

Two $(G, \Gamma)$-bundles over $X$ are $(G, \Gamma)$-*equivalent* if they are $\Gamma$-equivalent via a $G$-equivariant bundle map.

*Example* (8.3). Let $G$ and $\Gamma$ be two finite groups, let $\xi : G \longrightarrow \Gamma$ be a homomorphism, and let $X$ be a $G$-space. Then we may consider the first projection $X \times \Gamma \longrightarrow X$. Define a $G$-action on $X \times \Gamma$ by $g(x, \gamma) = (gx, \xi(g)\gamma)$. Then we obtain a $(G, \Gamma)$-bundle, which we denote by $p_\xi$.

Observe that in this case the isotropy group $G_{(x,\gamma)} = G_x \cap \ker \xi$ for all $\gamma \in \Gamma$. Note that for any finite covering $G$-map $p : E \longrightarrow X$, the inclusion $j : p^{-1}(x) \hookrightarrow p^{-1}(Gx)$ clearly induces a bijection $\overline{j} : p^{-1}(x)/G_x \longrightarrow p^{-1}(Gx)/G$.

LEMMA (8.4). *Let $N_x$ be the cardinality of $p_\xi^{-1}(Gx)/G \approx p^{-1}(x)/G_x$. Then the index $[G_x \!:\! G_x \cap \ker \xi] = |\Gamma|/N_x$.*

*Proof.* There is a $G$-bijection between $p_\xi^{-1}(Gx)$ and $G/G_x \times \Gamma$ given by the correspondence $(gx, \gamma) \leftrightarrow ([g], \gamma)$, where $G$ acts on $p_\xi^{-1}(Gx)$ by $g'(gx, \gamma) = (g'gx, \xi(g')\gamma)$ and on $G/G_x \times \Gamma$ by $g'([g], \gamma) = ([g'g], \xi(g')\gamma)$. Thus the orbit of $([g], \gamma)$ has $[G \!:\! G_x \cap \ker \xi]$ elements. Hence, the cardinality of $G/G_x \times \Gamma$ is

$$[G \!:\! G_x]|\Gamma| = N_x[G \!:\! G_x \cap \ker \xi]\,.$$

Therefore,

$$[G_x \!:\! G_x \cap \ker \xi] = [G : G_x \cap \ker \xi]/[G \!:\! G_x] = |\Gamma|/N_x\,.$$

$\square$

*Definition* (8.5). Let $G$ and $\Gamma$ be two finite groups. A $(G,\Gamma)$-bundle $p : E \longrightarrow X$ is said to be a *$(G,\Gamma)$-locally trivial bundle* if for each $x_0 \in X$ there is a $G_{x_0}$-invariant neighborhood $U_{x_0}$, such that the restricted bundle $p^{-1}U_{x_0} \longrightarrow U_{x_0}$ is $(G_{x_0},\Gamma)$-equivalent to $p_{\xi_{x_0}} : U_{x_0} \times \Gamma \longrightarrow U_{x_0}$, for some homomorphism $\xi_{x_0} : G_{x_0} \longrightarrow \Gamma$, as in Example (8.3).

*Remark* (8.6). Lashof [7] gave a different condition for $(G,\Gamma)$-local triviality. However, he showed that his condition implies the definition above. He also constructed a universal $(G,\Gamma)$-bundle to classify numerable $(G,\Gamma)$-locally trivial bundles.

On the other hand, any principal $(G,\Gamma)$-bundle over a completely regular base space is a $(G,\Gamma)$-locally trivial bundle (see [7]).

*Example* (8.7). Let $G$ be a finite group and let $X$ be a *bi-$G$-space*, namely a space with a left and a right $G$-action such that for any $x \in X$ and $g, g' \in G$, $(gx)g' = g(xg')$. Let $K \subset H \subset G$ be subgroups such that $K$ is normal in $H$, and assume that the right action of $H$ on $X$ is free. Put $\Gamma = H/K$. Then we can define a principal $(G,\Gamma)$-bundle as follows. Let $p : X/K \longrightarrow X/H$ be the canonical projection. One can easily verify that $G$ acts on the left on both $X/K$ and $X/H$ in the obvious way, and that there is a free right $\Gamma$-action on $X/K$ using the right action of $G$.

The bi-$G$-action on $X$ implies that condition (8.2) is satisfied. Assume now that $X$ is completely regular (and Hausdorff). One can show that $X/H$ is also completely regular. Therefore we have that $p : X/K \longrightarrow X/H$ is a $(G,\Gamma)$-locally trivial bundle.

LEMMA (8.8). *Let $p : E \longrightarrow X$ be a $(G,\Gamma)$-locally trivial bundle and take $x_0 \in X$. Then the index $[G_{x_0} : G_{x_0} \cap \ker \xi_{x_0}] = |\Gamma|/N_{x_0}$, where $N_{x_0}$ is the cardinality of $p^{-1}(x_0)/G_{x_0}$, as in Lemma* (8.4).

*Proof.* Let $U_{x_0}$ be a neighborhood of $x_0$ as in Definition (8.5). Then the restricted bundle $p^{-1}U_{x_0} \longrightarrow U_{x_0}$ is $(G_{x_0},\Gamma)$-equivalent to $p_{\xi_{x_0}} : U_{x_0} \times \Gamma \longrightarrow U_{x_0}$. Thus the desired formula follows from Lemma (8.4). □

THEOREM (8.9). *For any finite covering $G$-map $p : E \longrightarrow X$ and a homological Mackey functor $M$ one has the following formula*

$$p_*^G t_p^G(\gamma_x^G(l)) = \sum_{\kappa \in K} [G_x : G_{a_\kappa}] \gamma_x^G(l) \in \mathbb{F}^G(X, M), \tag{8.10}$$

*where $p^{-1}(x)/G_x = \{[a_\kappa] \mid \kappa \in K\}$.*

*Proof.* By equation (3.6), we can write

$$p_*^G t_p^G(\gamma_x^G(l)) = \sum_{\kappa \in K} p_*^G \gamma_{a_\kappa}^G M^*(\widehat{p}_{a_\kappa})(l) = \sum_{\kappa \in K} \gamma_x^G M_*(\widehat{p}_{a_\kappa}) M^*(\widehat{p}_{a_\kappa})(l).$$

Since the composite $M_*(\widehat{p}_{a_\kappa}) \circ M^*(\widehat{p}_{a_\kappa})$ is multiplication by $[G_x : G_{a_\kappa}]$, the result follows. □

We now have the following consequence of Theorem (8.9) and Lemma (8.8).

THEOREM (8.11). *Let $p : E \longrightarrow X$ be a $(G,\Gamma)$-locally trivial bundle and let $M$ be a homological Mackey functor. Then one has that each of the composites*

$$p_*^G \circ t_p^G : \mathbb{F}^G(X^+, M) \longrightarrow \mathbb{F}^G(X^+, M) \quad \text{and}$$

$$p_*^G \circ t_p^G : H_*^G(X, M) \cong \pi_q\big(\mathbb{F}^G(X^+, M)\big) \longrightarrow \pi_q\big(\mathbb{F}^G(X^+, M)\big) \cong H_*^G(X, M),$$

*is multiplication by $|\Gamma|$.*

*Proof.* We only have to prove the result for the composite on the top. By (8.10), if $v = \gamma_{x_0}^G(l) \in \mathbb{F}^G(X^+, M)$, then

$$p_*^G t_p^G(\gamma_{x_0}^G(l)) = \sum_{\kappa \in K} [G_{x_0} : G_{x_0} \cap \ker \xi_{x_0}] \gamma_x^G(l),$$

where $\{[a_\kappa] \mid \kappa \in K\} = p^{-1}(x_0)/G_{x_0}$. By Lemma (8.8), $[G_{x_0} : G_{x_0} \cap \ker \xi_{x_0}] = |\Gamma|/N_{x_0}$, and since $N_{x_0}$ is the cardinality of $K$, $p_*^G t_p^G(\gamma_{x_0}^G(l)) = |\Gamma| \gamma_{x_0}^G(l)$. Since any element $v \in \mathbb{F}^G(X^+, M)$ is a sum of terms of the form $\gamma_{x_0}^G(l)$, the result follows. □

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# THREE MANIFOLDS AS GEOMETRIC BRANCHED COVERINGS OF THE THREE SPHERE

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ABSTRACT. One method for obtaining every closed orientable 3-manifold is as a branched covering of $S^3$ branched over a link. There are topological results along these lines that cannot be improved upon in two respects: ($A$), The minimum possible number of sheets in the covering is three; ($B$), There are individual knots and links (universal knots and links) that can serve as branch set for every 3-manifold, $M^3$. Given the growing importance of geometry in 3-manifold theory it is of interest to obtain geometrical versions of topological results ($A$) and ($B$). Twenty years ago a geometric version of result ($B$) was obtained using universal groups. In this paper we obtain a geometric version of result ($A$), also by means of universal groups.

## 1. Introduction

Some time ago it was established that all closed orientable 3-manifolds are threefold irregular simple branched coverings of $S^3$ with branch set a knot or a link. ([3],[11],[14]). Both the statement and proof of this theorem were purely topological in character. Geometry played no role.

The central purpose of this paper is to introduce the idea of a geometric branched covering and then to show that every closed orientable 3-manifold $M^3$ is a threefold irregular simple branched covering of $S^3$ that is "geometric" in the following sense:

There is a universal group $U$; the orbifold group of the Borromean rings with singular angle ninety degrees. Thus $U$ is a finite covolume Kleinian group of hyperbolic isometries of $H^3$.

There are finite index subgroups of $U$, $G$, and $G_1$, such that $M^3 = H^3/G$, $S^3 = H^3/G_1$, and $G$ is a non normal index three subgroup of $G_1$.

The groups $U$, $G$, and $G_1$ induce hyperbolic orbifold structures on $S^3$, $M^3$, and $S^3$ respectively and the maps induced by group inclusions,

$$M^3 = H^3/G \longrightarrow S^3 = H^3/G_1 \longrightarrow S^3 = H^3/U,$$

are branched covering space maps with $M^3 \longrightarrow S^3$ being three to one.

The new theorem, Theorem (2.9) of this paper, can be considered to be the "geometrization" of the old theorem.

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This work is closely related to the concepts of universal group and universal link.

A knot or link is said to be universal if every closed orientable 3-manifold occurs as a finite branched covering space of $S^3$ with branch set the knot or link. W. Thurston introduced this concept in his paper [17], where he also exhibited some universal links, and asked if the Whitehead link and the Figure eight knot were universal. In [6], [7] we ansewed Thurston's questions in the affirmative and proved that every non toroidal rational knot or link is universal, and that the Borromean rings are universal. Subsequently many other universal knots and links were found: [18], [19], [16], [8], [4], [12], [9].

In a branched covering $p\colon M \longrightarrow N$ of 3-manifold branched over a link $L$ the preimage of a meridian disk $D$ is a set of meridian disks $D_1, ..., D_k$. For $1 \leq j \leq k$ the disk $D_j$ is mapped in $n_j$ to 1 fashion to $D$. If $A$ is a set of positive integers we say that the branched covering $p$ is of type $A$ if $n_j$ belongs to $A$ for $1 \leq j \leq k$. In particular, in the sequel branched coverings of type $1, 2$ and $1, 2, 4$ will be particularly important.

Closely related to the concept of universal knot or link is that of universal group. A finite covolume, discrete group of hyperbolic isometries $U$, acting on $H^3$, is said to be universal if every closed orientable 3-manifold $M^3$ occurs as a quotient space, $M^3 = H^3/G$, where $G$ is a finite index subgroup of $U$. Such groups $U$ must contain rotations, else all 3-manifolds including $S^3$ would have hyperbolic structure.

Also, given $M^3$, there are infinitely many finite index $G$'s with $M^3 = H^3/G$ so that this doesn't give anything like a classification of 3-manifolds. It can be considered analogous to Heegaard splittings or Kirby calculus presentations. We know that $S^3 = H^3/U$ so that $S^3$ inherits a hyperbolic orbifold structure from $U$.

The existence of a universal group was demonstrated in ([5]). The group defined in ([5]), which from now on we denote by $U$, is a orbifold group of the Borromean rings. This group $U$ thus induces a hyperbolic orbifold structure on $S^3$ with singular set the Borromean rings, and singular angle ninety degrees. Unfortunately the proof of the universality of $U$ in ([5]) cannot be adapted to prove the geometric branched covering space theorem referred to earlier.

There is a new proof that serves our purposes. It starts out following ([5]), then follows ideas in ([6]), then follows section 5 of ([12]) in which infinitely many 2-universal links are defined, and then uses a branched covering of $S^3$ by $S^3$ with branch set the Borromean rings and branching indexes $\{1, 2\}$ (branching of type $\{1, 2\}$ for short) such that the "doubled" Borromean rings (2-universal) occur as a sublink of the preimage of the Borromean rings.

Rather than put the reader through the difficult task of actually finding these references in some library we prefer to give a new, relatively self-contained proof of the universality of $U$.

In the next section we state and prove the geometric branched covering space theorem.

## 2. Geometric branched covering space theorem

Our point of departure is the following theorem ([3],[11],[14]).

THEOREM (2.1). *Let $M^3$ be a closed orientable 3-manifold. Then $M^3$ is a 3 to 1 irregular simple branched cover of $S^3$ with branch set a knot or link (as opposed to a graph).*

Such 3 to 1 coverings with branch set a link $L$ correspond to transitive representations $\rho\colon \pi_1(S^3 - L) \longrightarrow \Sigma_3$ in which meridians are sent to transpositions. Years ago Ralph Fox ([2]) had the genial idea of representing transpositions by colours. Throughout the paper we shall follow this idea; Red $= R = (12)$, Yellow $= Y = (23)$, Blue $= B = (13)$.

Then, given a classical knot or link diagram, simple transitive representations to $\Sigma_3$ (Simple means meridians go to transpositions), correspond to colourings of the bridges such that at each crossing either all three bridges are the same colour or all three have different colours and at least two colours are used. (This itself is equivalent to the Wirtinger relations, which have form $xy = yz$, being satisfied.)

Given a 3-1 simple branched covering $p\colon M^3 \to S^3$ branched over the coloured link $L$, there is a move, illustrated in Figure 1 and called a Montesinos move [15], which changes the coloured link $L$ to a different coloured link $L'$. The change takes place inside a ball. Although $L$ is changed, the topological type of $M^3$ is not. The reason is that the 3-fold simple cover of a ball branched over two unknotted, unlinked arcs is a ball. Thus doing a Montesinos move on a link in $S^3$ is equivalent to removing a ball from $M^3$ and sewing it in differently. We call a sequence of Montesinos moves a Montesinos transformation.

*Montesinos Move*

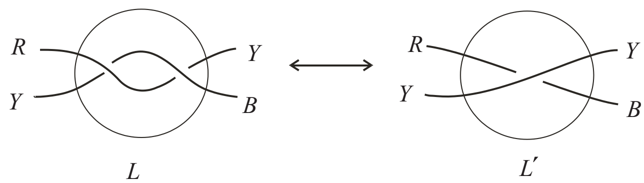


**Figure 1**

We give several examples of transformations of links using Montesinos moves and isotopies which will be useful to us in the new proof of the universality of $U$.

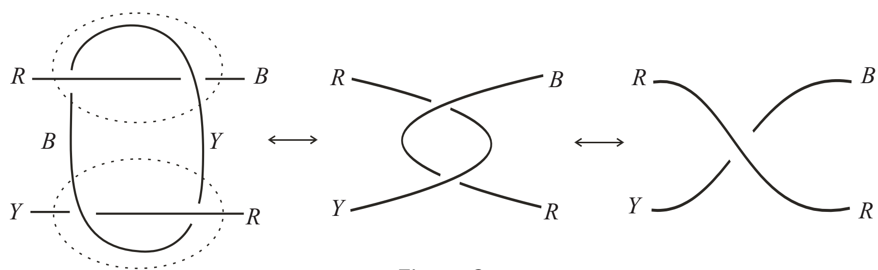


**Figure 2**

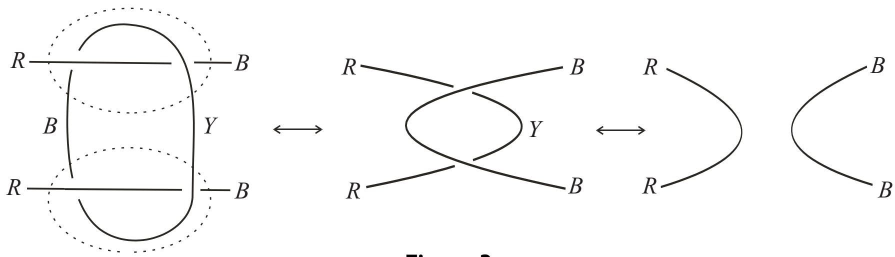


**Figure 3**

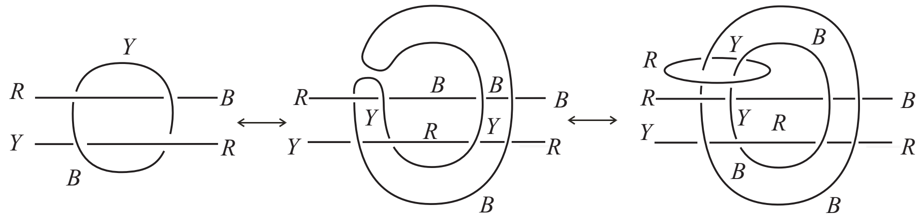


**Figure 4**

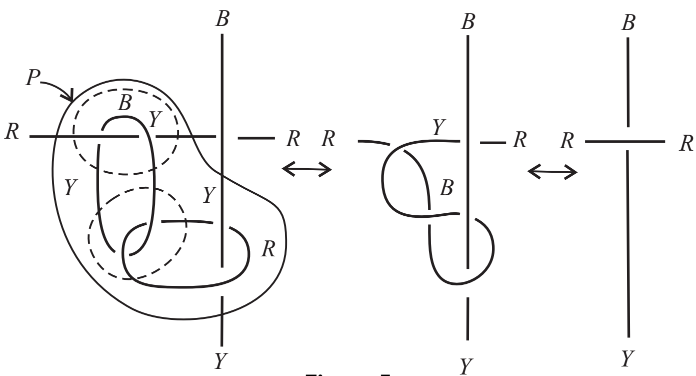


**Figure 5**

Now we are ready to prove the universality of $U$. Let $M^3$ be a closed orientable 3-manifold and let $p\colon M^3 \longrightarrow S^3$ be a simple 3-fold covering branched over the link $L$. We shall apply a Montesinos transformation to $L$ to obtain a link $L'$ which suits our purposes. For definitiveness we shall think of $S^3$ as $E^3 \cup \{\infty\}$, assume $L$ is contained in $E^3$ and use cylindrical coordinates in $E^3$.

As every link is a closed braid we can assume $L$ is a closed braid. This means that each component of $L$ has a parametrization $(r(t), \theta(t), z(t))$ in which $\theta(t)$ is strictly increasing and the projection on the plane $z = 0$ is "nice". (i.e., there are no triple points.)

Using an isotopy of the type illustrated in Figure 7 we can assume every crossing has 3 colours.

Then, using Montesinos moves as in Figure 1, we can assume all crossings are "positive". (See Figure 8.)

We replace each crossing with a new small circle component, as in the left hand side of Figure 2. After an isotopy we can assume that our link $L$ has two

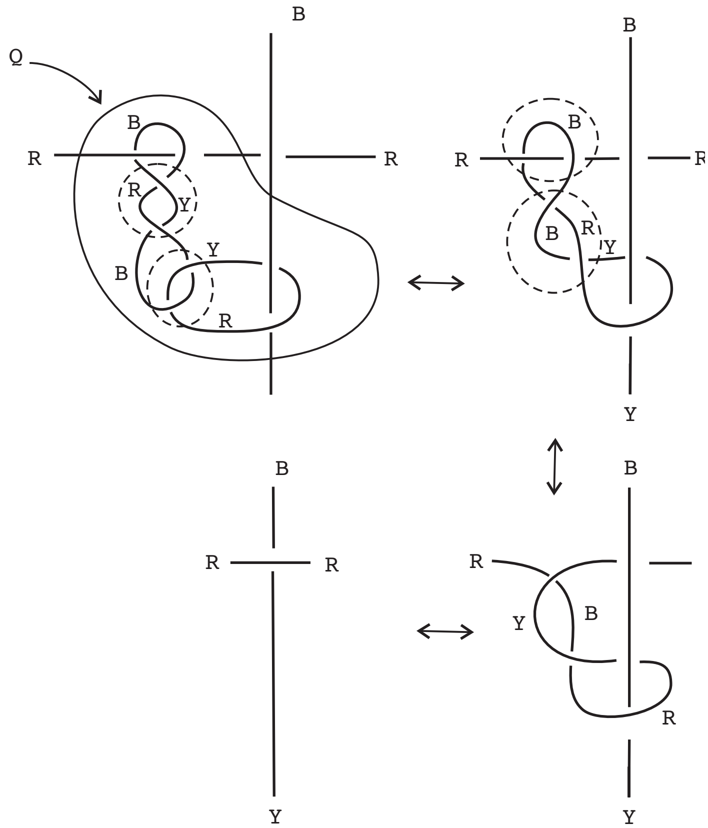


**Figure 6**

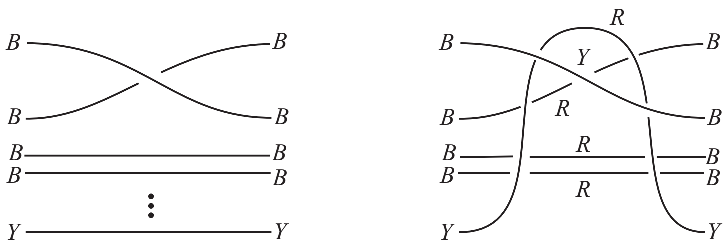


**Figure 7**

types of components; "braid" or "horizontal" components that lie in the plane $z = 0$ and have equations $r =$ constant; $\theta =$ arbitrary, $z = 0$, and "small circle" components, whose projection on the plane $z = 0$ appears as in the left side of Figure 2.

Next, using the Montesinos transformation of Figure 4, (which is validated by using the Montesinos transformation of Figure 3.), we replace each small circle component by three components as in the right hand side of Figure 4. Now our link $L$ has three types of components; horizontal components, big


**Figure 8**

circle components and small circle components as illustrated in the right hand side of Figure 4. Each small circle component links two big circle components.

We isotope out link $L$ so that each big circle component appears as in the left hand side of Figure 9, that is, it extends over the top and bottom of all the horizontal components.

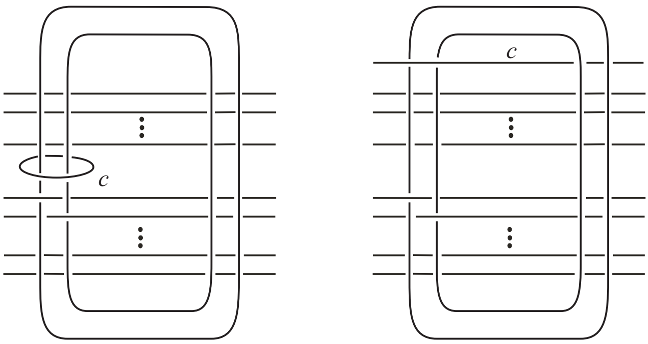


**Figure 9**

Then we isotope the small circle components, one at a time, so that they become braid or horizontal components. As we do this to a particular small circle component "$c$" as in the left hand side of Figure 9 it becomes, briefly, the topmost braid component.

Now our link $L$ has two types of components, horizontal components which lie in the plane $z = 0$ and have equation of form $r =$ constant, $\theta =$ arbitrary, and $z = 0$ and large circle components, which we now call vertical components, whose projections on the plane $z = 0$ are rectangles in the $(r, \theta)$ coordinate system. Crossings are called horizontal or vertical as indicated in Figure 10.

We observe, and this is crucial in what follows, that every horizontal crossing is 3-coloured. (The vertical crossings may not be.) Also, a particular vertical component links a particular horizontal component if and only if the left crossing on the right hand side of Figure 9 is horizontal. The next step will be to eliminate horizontal crossings. We refer the reader to Figures 5 and 6.

We use the Montesinos transformations illustrated in either Figures 5 or 6 (both are useful) to replace each horizontal crossing by a vertical crossing. In the course of doing this, new components, as indicated in the left sides of Figures 5 and 6, are introduced. These are contained in the "peanut shaped" balls indicated by a "$P$" or "$Q$" in Figures 5 and 6.

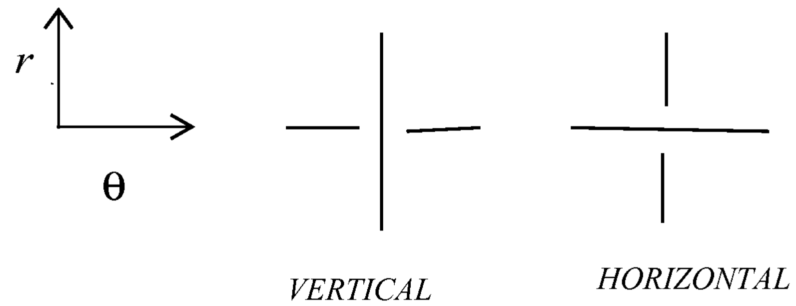


**Figure 10**

After a slight isotopy our link $L$ has three types of components; horizontal (with equation $r$ = constant, $z = 0$, $\theta$ = arbitrary); vertical (lying in the plane $z = \varepsilon > 0$, whose projection in the plane $z = 0$ is a rectangle in $(r, \theta)$ coordinates); and "special". Each special component is contained in a "peanut shaped" topological ball lying in the region $-\varepsilon \leqq z \leqq 2\varepsilon$ and having projection on the $z = 0$ plane as indicated in either the left or right hand sides of Figure 11. (We can use one or the other but never both in the same proof.)

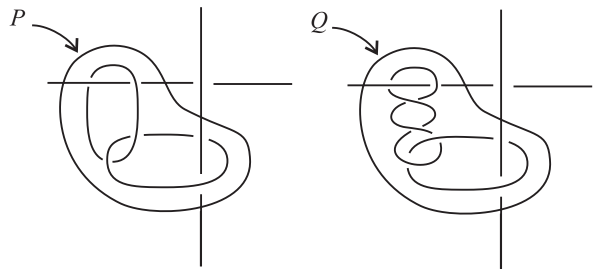


**Figure 11**

Thus we have shown that every closed orientable 3-manifold $M^3$ is a 3-fold simple branched covering of a link of very special type. We shall isotope the link some more and then state a theorem. We shall use cylindrical coordinates in $S^3 = E^3 \cup \{\infty\}$ to better describe the link. At the moment our link is contained in the region $[-\varepsilon \leqq z \leqq 2\varepsilon,\ 0 < r_0 < r \leqq R_0,\ \theta = \text{arbitrary}]$

We isotope the region $-\varepsilon \leqq z \leqq 2\varepsilon, r_0 \leqq r \leqq R_0$ without changing the $\theta$ coordinate, so that the link lies in the thickened toroidal region $1-\delta \leqq z^2+(r-2)^2 \leqq 1+\delta$ and so that the horizontal components lie in the torus $z^2+(r-2)^2 = 1$, have equations of form [$r$ = constant, $z$ = constant, $\theta$ = arbitrary], and are "evenly spaced". (This means that after the isotopy the images of the horizontal components intersect the circle $z^2 + (r-2)^2 = 1, \theta = \theta_0$, in $n$ evenly spaced points on the circle.)

Next we isotope the vertical components so that their images lie on the torus $z^2 + (r-2)^2 = 1 - \delta$, have equations $\theta$ = constant, $z^2 + (r-2)^2 = 1 - \delta$, and are "evenly spaced" which means that if there are $m$ vertical components the constants referred to are $\{2\pi j/m; 0 \leqq j \leqq m-1\}$.

It is convenient to introduce new "toroidal" coordinates $(\rho, \theta, \varphi)$ which are well defined in a neighborhood of the torus $z^2+(r-2)^2 = 1-\delta$. These are given by equations $\theta = \theta$; $\rho = \sqrt{z^2+(r-2)^2}$ = distance of a point from the circle $[z = 0, r = 2, \theta =$ arbitrary$]$; $\sin\varphi = z/\rho$ and $\cos\varphi = r-2/\rho$. Using these "toroidal" coordinates we can define projections of $E^3 - \{(z\,\text{axis}) \cup (\text{circle } [r = 2; z = 0]\}$ onto the torus $z^2+(r-2)^2 = 1$ or the torus $z^2+(r-2)^2 = 1-\delta$ by $(\rho, \theta, \varphi) \longrightarrow (\theta, \varphi)$.

We summarize the results of the preceding isotopies in a theorem.

THEOREM (2.2). *Let $M^3$ be a closed oriented 3-manifold. Then there is a 3-fold simple branched covering $p\colon M^3 \longrightarrow S^3$ branched over a link $L$.*

*The link $L$ has three types of components.*

*a. Horizontal. These lie in the torus $\rho = 1$ or $z^2+(r-2)^2 = 1$ and have equations $[\theta = \text{arbitrary}; \varphi = 2\pi j/n, 0 \leqq j \leqq n-1, \rho = 1]$*

*b. Vertical. These lie in the torus $\rho = .99$ or $z^2+(r-1)^2 = (.99)^2$ and have equations $\varphi = \text{arbitrary}, \theta = 2\pi j/m, 0 \leqq j \leqq m-1, \rho = .99$*

*c. Special. These have local projections on the torus $\rho = 1$ as in either the left or right hand side of Figure 11. The vertical coordinate is $\varphi$, the horizontal coordinate is $\theta$.*

In this proof we will use all left hand side or all right hand side of Figure 11. Both are useful.

Now we find it useful to define two rotations $T_1$ and $T_2$ of $S^3 = E^3 \cup \{\infty\}$. The rotation $T_1$ is simply the $m$-fold rotation about the $z$-axis given by $T_1\colon (r, \theta, z) \longrightarrow (r, \theta + 2\pi/m, z)$ the rotation $T_1$ leaves invariant the set of horizontal and the set of vertical components of the link $L$. The rotation $T_2$ is more difficult to describe in coordinates and we shall not attempt to do so. Instead we indicate its important properties. The rotation $T_2$ has as its axis the circle $z = 0, r = 2, \theta =$ arbitrary; it has order $n$ and leaves the $\theta$ coordinate unchanged. It leaves the set of horizontal and the set of vertical components of $L$ invariant. It cyclically permutes the horizontal components and it sends each vertical component to itself. Its restriction to a vertical component is just the usual $n$-fold rotation of a circle.

At this point we must decide whether to use the branch set in the left or right hand side of Figure 11. We choose the left for purposes of illustration.

Using the projection $(\rho, \theta, \varphi) \longrightarrow (\theta, \varphi)$ a portion of the image of the link $L$ appears as in Figure 12. Some of the "peanut shaped" balls contain two component links and arcs from a vertical and horizontal component, others contain only arcs from a vertical and horizontal component.

We can and shall assume that both rotations $T_1$ and $T_2$ leave the "expanded" link $L$ invariant.

Now consider the map $f_1\colon S^3 \longrightarrow S^3/T_1 = S^3$ which is an $m$-fold cyclic branched covering $S^3 \longrightarrow S^3$ with branch set the trivial knot or $z$-axis. The branch set for the composite map $f_1 \circ p\colon M^3 \longrightarrow S^3$ consists of the union of the branch set for $f_1$ and the image $f_1$ (branch set for $p$) = $z$-axis $\bigcup f_1(L)$.

The branching is of type $\{1, 2\}$ on $f_1(L)$ and of type $\{m\}$ on the $z$–axis$\cup\{\infty\}$. This means that any disc which belongs to the preimage of a meridian disk $D$ for $f_1(L)$ is either mapped homeomorphically $(1-1)$ to $D$ or mapped to $D$ as a $2-1$ branched covering. The preimage of a meridian disk $D$ for the $z$-axis

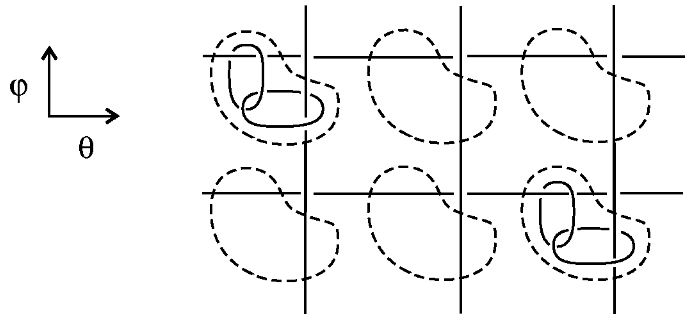


**Figure 12**

is mapped to $D$ as an $m$-fold branched covering. The overall branching is of type $\{1, 2, m\}$ for the map $f_1 \circ p$ and the "$m$" is bad for our purposes. Shortly we shall show how to change the map $f_1$ to a map $f$ so that the branching for $f \circ p$ is of type $\{1, 2\}$ but first we observe that the part of the branch set $f_1(L)$ for $f_1 \circ p$ contains $n$ horizontal components, $n$ "peanut" components but only one vertical component.

Consider the connected $k$-fold branched coverings of a disk $D^2$ with two branch points $A$ and $B$ in the interior of $D^2$. These are determined by transitive representations $\rho$ of $\pi_1(D^2 - \{A, B\})$ (which is free on the two meridian generators, call them $x$ and $y$, pictured in Figure 13) into $\Sigma_k$.

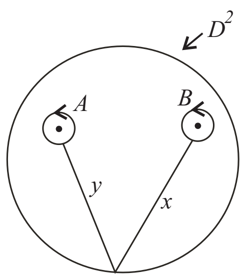


**Figure 13**

We are interested in a particular dihedral representation. Let $P$ be a regular $k$-gon with its vertices labeled 1 through $k$ as in Figure 14. We map $x$ to the reflection in the axis $l_1$ and $y$ to the reflection in the axis $l_2$ of Figure 14.

The reflections induce permutations of vertices and elements of $\Sigma_k$. Then $\rho(x) = (1, 2)(3, k)(4, k-1)\cdots$ and $\rho(y) = (2, k)(3, k-1)(4, k-2)\cdots$ are both products of disjoint transpositions and as $xy$ is a counterclockwise rotation by $2\pi/k$, $\rho(xy)$ is the $k$–cycle $(1, k, k-1, \cdots, 3, 2)$.

Let $q\colon X \longrightarrow D^2$ be the branched covering induced by $\rho$. Then $xy$ is a generator of $\pi_1(S^1) = Z$ where $S^1 =$ boundary $D^2$ and restricting $q$ to boundary $X$ we see that $q\colon \partial X \longrightarrow S^1$ is just the usual $k$–fold unbranched cover of $S^1$. Using the branching data we compute the Euler characteristic of $X$ which is one. Thus $\rho$ induces a $k$–fold branched cover $q\colon D^2 \longrightarrow D^2$ with branch set

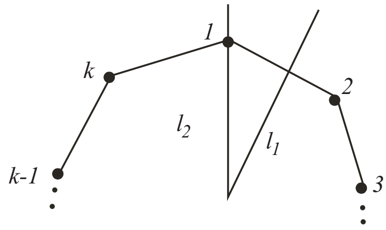


**Figure 14**

two points in interior $D^2$ and branching type $\{1, 2\}$. The restriction of $q$ to $S^1 = \partial X = \partial D^2$ is the usual $k$–fold unbranched covering of $S^1$ by $S^1$. It is convenient to summarize the preceding in a proposition.

PROPOSITION (2.3). *Let $q_1 \colon D^2 \longrightarrow D^2$ be the usual $k$–fold cyclic covering of $D^2$; the one given by $q_1 \colon z \longrightarrow z^n$ in complex coordinates if $D^2$ is the unit disc in C. Then there is another $k$–fold branched covering $q \colon D^2 \longrightarrow D^2$ such that the branch set is two points in interior $D^2$, the branching is of type $\{1, 2\}$ and $q = q_1$ on $S^1 =$* boundary $D^2$.

By "crossing" with $S^1$ the next proposition follows easily.

PROPOSITION (2.4). *Let $q_1 \colon S^1 \times D^2 \longrightarrow S^1 \times D^2$ be the usual $k$–fold cyclic covering of $S^1 \times D^2$ by $S^1 \times D^2$; the one given by $q_1 \colon (e^{i\theta}, z) \longrightarrow (e^{i\theta}, z^n)$ in natural coordinates for $S^1 \times D^2$. Then there is another $k$–fold branched covering $q : S^1 \times D^2 \longrightarrow S^1 \times D^2$ such that the branch set equals $S^1 \times \{A, B\}$ where $A$ and $B$ are points in the interior of $D^2$; the branching is of type $\{1, 2\}$, and $q = q_1$ on boundary $(S^1 \times D^2)$.*

We return to our map $f_1$ which is an $m$–fold cyclic covering of $S^3$ by $S^3$ with branch set and preimage of branched set the $z$–axis. We choose a natural solid torus neighborhood of the $z$–axis in $S^3$ and its preimage and we coordinatize this neighborhood so that $f_1$ is the usual $m$–fold cyclic covering of $S^1 \times D^2$ by $S^1 \times D^2$ as in Proposition (2.4). This neighborhood should be small enough so that it doesn't intersect the link $L$ which is the branch set of the map $p \colon M^3 \longrightarrow S^3$ defined earlier. Then we use Proposition (2.4) to define a new map $f : S^3 \longrightarrow S^3$ where outside the solid torus neighborhood $f = f_1$ and within the solid torus neighborhood $f$ is like $q$ of Proposition (2.4). Thus $f \circ p$ is a $3m$ to 1 branched covering of $S^3$ by $M^3$ with branch set $S^1 \times \{A, B\} \cup f(L)$.

The part of the branch set $f(L) = f_1(L)$ has one vertical component and $n$-horizontal components and $n$-"peanut" components. Via an isotopy, if necessary, we may assume that the rotation $T_2$ leaves $S^1 \times \{A\}$ and $S^1 \times \{B\}$ invariant and that its restriction to either component is just the usual $n$–fold rotation. The relevant part of the branch set for $f \circ p$ is depicted in Figure 15. The branching is type $\{1, 2\}$.

Next we let $g_1$ be the map $S^3 \longrightarrow S^3/T_2 = S^3$. Then $g_1 \circ f \circ p$ is a branched covering of $S^3$ by $M^3$ which is $3mn$ to 1 and has branch set equal to $g_1$(branch set $(f \circ p)) \cup \{$the circle $[z = 0,\ r = 2,\ \theta =$ any$]\}$. The branching is of type $\{1, 2\}$ on $g_1$(branch set $(f \circ p)$) and of type $\{m\}$ on the circle $[z = 0, r = z, \theta =$ any$]$.

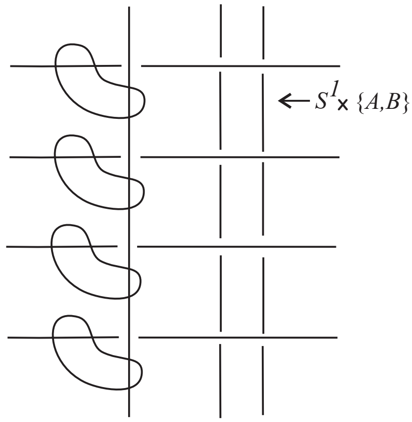


**Figure 15**

We choose a solid torus neighborhood of the circle $z = 0, r = z, \theta = any$, small enough so that it does not intersect $g_1$(branch set $(f \circ p)$), and so that it can be coordinatized as $S^1 \times D^2$ with $g_1 = q_1$ as in Proposition (2.4).

Then we define $g : S^3 \longrightarrow S^3$ so that $g = g_1$ outside the torus neighborhood and $g$ behaves like $q$ of Proposition (2.4) inside the torus neighborhood. The $3mn$ to 1 branched covering $g \circ f \circ p$: $M^3 \longrightarrow S^3$ has branch set with three vertical components and three horizontal components and one "peanut" component as pictured in Figure 16. All branching is of type $\{1, 2\}$.

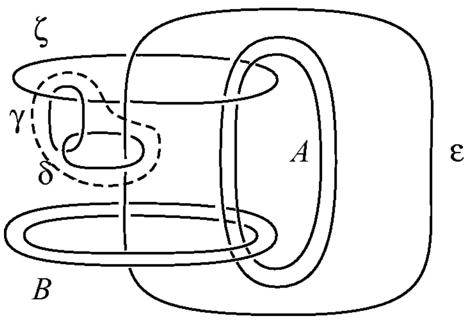


**Figure 16**

The link in Figure 16 can be isotoped to the link in Figure 17. To help the reader see this we have labeled the corresponding components in Figures 16 and 17. There are two obvious annuli in Figures 16 and 17 labeled $A$ and $B$. The other components are labeled $\gamma, \delta, \epsilon$, and $\zeta$.

We may add four components, $\gamma_1, \delta_1, \epsilon_1, \zeta_1$, to the link of Figure 17 so that there are annuli $C, D, E$, and $F$ with boundaries $\gamma \cup \gamma_1$, $\delta \cup \delta_1$, $\epsilon \cup \epsilon_1$ and $\zeta \cup \zeta_1$ respectively in such a way that the new link has a 3–fold rational symmetry. Let $T_3$ be this 3–fold rotation and let $h_1$: $S^3 \longrightarrow S^3 = S^3/T_3$ be the resulting branched covering. The map $h_1 \circ g \circ f \circ p$ is a $9mn$ to 1 branched covering of $S^3$

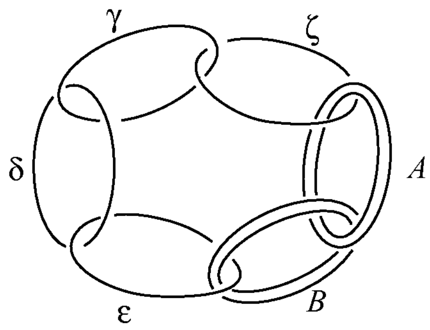


**Figure 17**

by $M^3$ with branching of type $\{1, 2\}$ on the part of the branch set $h_1$ (branch set $g \circ f \circ p$) and branching of type $\{3\}$ on $h_1(\text{axis}\, T_3)$.

This branch set is depicted in Figure 18.

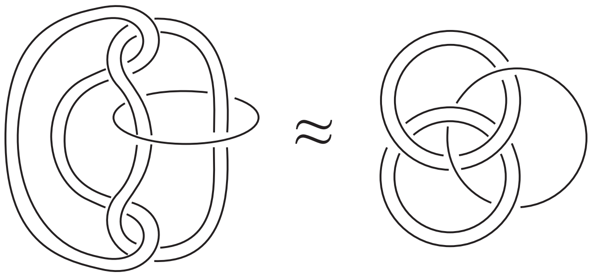

**Figure 18**

As before we replaced $f_1$ by $f$ and $g_1$ by $g$ we now replace $h_1$ by $h$ using Proposition (2.4) where $h_1 = h$ except in solid torus neighbourhood of the axis of rotation of $h$ but the branching of the map $h\colon S^3 \longrightarrow S^3$ is of type $\{1, 2\}$. The new branch set is displayed in Figure 19.

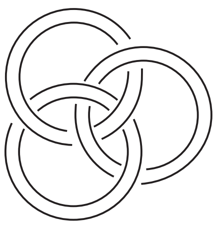

**Figure 19**

We call the branch set of Figure 19 the "doubled Borromean rings". We summarize this result in the form of a theorem [12].

THEOREM (2.5). *Let $M^3$ be a closed orientable 3–manifold. Then $M^3$ is a branched covering of $S^3$ with branch set the doubled Borromean rings, and with branching of type $\{1, 2\}$. That is, the doubled Borromean rings are 2-universal.*

If we use the right hand side of Figure 11 instead of the left hand side we obtain the following theorem [12].

THEOREM (2.6). *Let $M^3$ be a closed orientable 3–manifold. Then $M^3$ is a branched covering of $S^3$ with branch set the doubled Whitehead link and with all branching of type $\{1, 2\}$. That is, the doubled Whitehead link is 2-universal.*

The doubled Whitehead link is depicted in Figure 20.

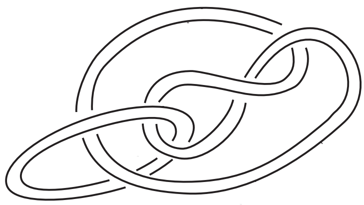

**Figure 20**

Theorem (2.5) and Theorem (2.6) show that the doubled Borromean rings and doubled Whitehead link are 2-universal. In ([12]) two co–authors of this paper prove this result and additionally give infinitely many examples of 2-universal links. The three component link in the right bottom part of Figure 5.10 of [12] is a minimal hyperbolic 2-universal link. In fact any proper sublink of it is either a split link or a toroidal link.

We note, in passing, that 2-universal knots are known to exist ([9]) but so far there are no easy examples.

Our next task, which is the new idea of this paper, will be to define a branched covering $k : S^3 \longrightarrow S^3$ with branch set the Borromean rings for which the doubled Borromean rings occur as a sublink of the preimage of the branch set.

We begin by tessellating $E^3$ by $2 \times 2 \times 2$ cubes all of whose vertices have odd integer coordinates. Let $\widehat{U}$ be the group generated by $180°$ rotations in the axes *a, b,* and *c* displayed in Figure 21. The cube there is centered at the origin.

The group $\widehat{U}$ is a well known Euclidean crystallographic group that preserves the tessellation. A fundamental domain for $\widehat{U}$ is the $2 \times 2 \times 2$ cube of Figure 21 centered at the origin. The map $E^3 \longrightarrow E^3/\widehat{U} \approx S^3$ is a branched cover of $S^3$ by $E^3$ with branch set the Borromean rings. This gives $S^3$ the structure of a Euclidean orbifold with singular set the Borromean rings and singular angle $180°$. We can see that $E^3/\widehat{U}$ equals $S^3$ with singular set the Borromean rings by making face identifications in the fundamental domain of Figure 21. We do this in Figure 22.

Next we consider a tessellation of $E^3$ by $6 \times 6 \times 6$ cubes with integer coordinates that are odd multiples of three. Let $\widetilde{U}$ be the group generated

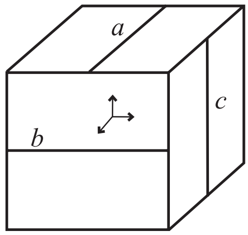


**Figure 21**

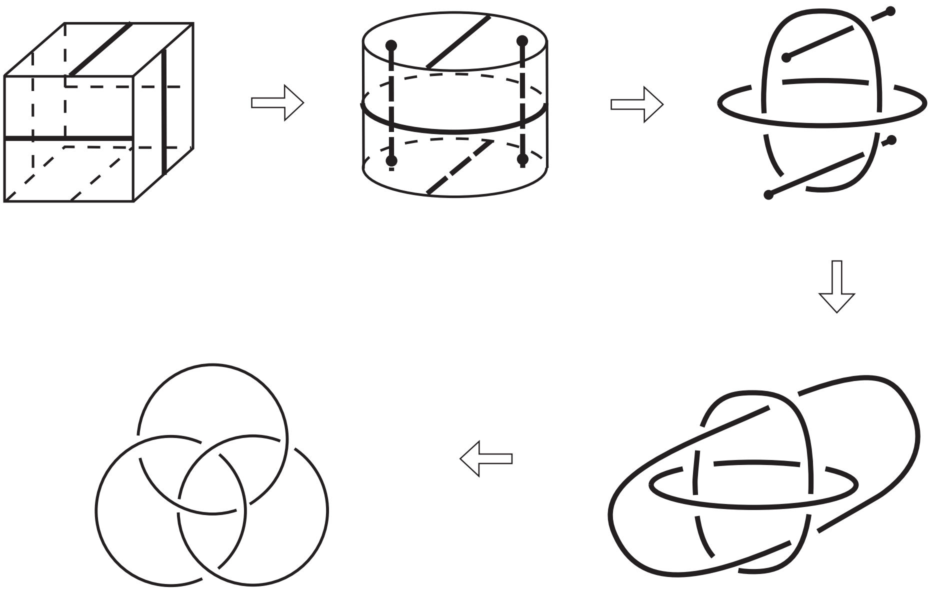

**Figure 22**

by 180° rotations in the axes $a'$, $b'$, $c'$ where $a' = (t, 0, 3)$, $b' = (3, t, 0)$ and $c' = (0, 3, t)$; $-\infty < t < \infty$. We can envision a fundamental domain for $\widetilde{U}$ by looking at Figure 21 and imagining prime accents over $a$, $b$, and $c$. Of course $E^3/\widetilde{U} = S^3$ and the map $E^3 \longrightarrow E^3/\widetilde{U}$ is a branched covering of $S^3$ by $E^3$. As the rotations about $a'$, $b'$ and $c'$ belong to $\widehat{U}$ we see that $\widetilde{U} \subset \widehat{U}$ and we see that $\left[\widehat{U} \colon \widetilde{U}\right] = 27$ by comparing the size of fundamental domains. $\widetilde{U}$ is not a normal subgroup of $\widehat{U}$. We are in fact really interested in the map $t : S^3 = E^3/\widetilde{U} \longrightarrow E^3/\widehat{U} = S^3$ induced by the inclusion of $\widetilde{U}$ in $U$.

Consider the following commutative diagram

$$\begin{array}{ccc} E^3 & \xrightarrow{\quad id \quad} & E^3 \\ \Big\downarrow & & \Big\downarrow \\ S^3 = E^3/\widetilde{U} & \longrightarrow & E^3/\widehat{U} = S^3\,. \end{array}$$

The maps $E^3 \longrightarrow E^3/\widetilde{U}$ and $E^3 \longrightarrow E^3/\widehat{U}$ are both branched covers of $S^3$ by $E^3$ with all branching of type $\{2\}$. (The map $E^3 \longrightarrow S^3$ is infinite to one but the notion of branched covering generalizes naturally from finite to one maps to infinite to one maps, at least in this special case.) We see, by considering images of meridian discs in $E^3$, that the map $S^3 = E^3/\widetilde{U} \longrightarrow E^3/\widehat{U} = S^3$ is also a branched covering space map with branching of type $\{1, 2\}$.

The set of points in $S^3 = E^3/\widetilde{U}$ in the preimage of the Borromean rings branch set in $E^3/\widehat{U} = S^3$ for which the branching of type $\{2\}$ is called the upper branch set. The set of points in $S^3 = E^3/\widetilde{U}$ in the preimage of the Borromean rings branch set in $E^3/\widehat{U} = S^3$ for which the branching is of type $\{1\}$ is called the upper pseudo branch set.

We can compute the preimage of the branch set in $S^3 = E^3/\widetilde{U}$ from a fundamental domain for $\widetilde{U}$, which consists of 27 $2 \times 2 \times 2$ cubes.

A point in this fundamental domain belongs to the upper branch set if and only if it belongs to an axis of rotation for $\widehat{U}$ but does not belong to an axis of rotation for $\widetilde{U}$. A point in this fundamental domain belongs to the upper pseudo branch set if and only if it belongs to an axis of rotation for $\widetilde{U}$. We do not need to compute the full preimage of the Borromean rings in $S^3 = E^3/\widetilde{U}$, (which turns out to be a 15 component link), but only a certain sublink.

In Figure 23 we give a $6 \times 6 \times 6$ cube fundamental domain for $\widetilde{U}$ and display only those axes of rotation of $\widehat{U}$ that lie in the faces of the cube.

The axes $a'$, $b'$ and $c'$ (and their analogues on the invisible faces) are axes for $\widetilde{U}$ and so give rise to the upper pseudo branch set. The axes $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ (and their analogues on the invisible faces) are axes for $\widehat{U}$, but not $\widetilde{U}$, and so give rise to a sublink of the upper branch set.

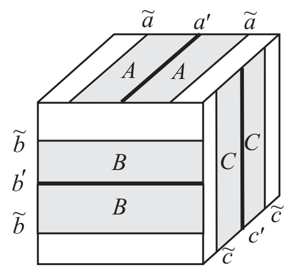


**Figure 23**

Also, axes $a'$ and $\widetilde{a}$ (resp. $b'$ and $\widetilde{b}$; $c'$ and $\widetilde{c}$) lie on the boundary of a rectangle $A$ (resp. $B$; $C$). Something similar occurs on the invisible faces. After identifications are made in the faces of the cubes these rectangles become annuli. In Figure 24 we make the identifications and show that the doubled Borromean rings appear.

We summarize all this in the next proposition.

PROPOSITION (2.7). *The map* $t\colon S^3 = E^3/\widetilde{U} \longrightarrow E^3/\widehat{U} = S^3$ *is a* 27 *to* 1 *irregular branched covering of* $S^3$ *by* $S^3$ *with branch set the Borromean rings.*

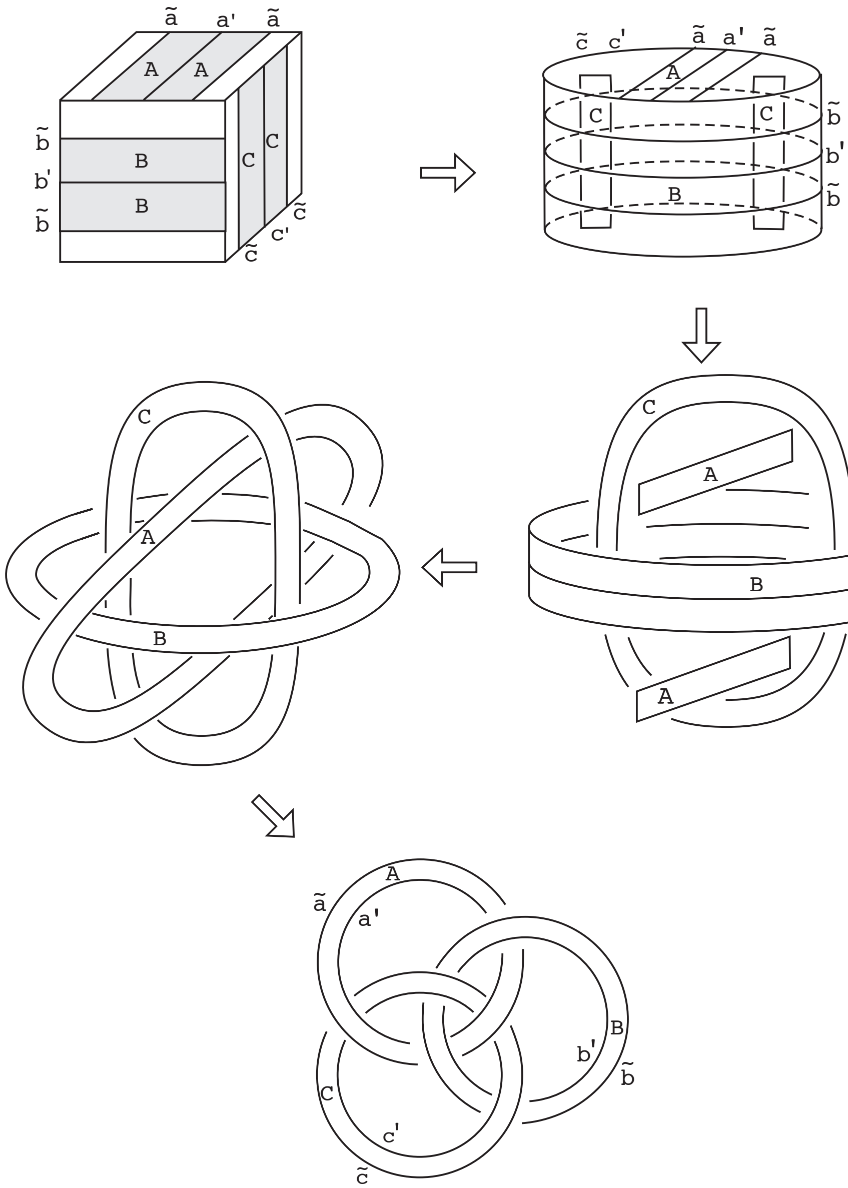


**Figure 24**

*The doubled Borromean rings occur as a sublink of the preimage of the branch set.*

*The doubled Borromean rings consist of three pairs of components. Each pair bounds an annulus disjoint from the other pairs. Each pair is mapped to the same component of the Borromean rings. And each pair contains one component of the upper branch set and one component of the upper pseudo branch set.*

The 27 to 1 map $t$ of Proposition 2.7 can be decomposed into a composition of three 3 to 1 branched coverings of $S^3$ by $S^3$ with branch set a trivial 2-component link. Each of these maps is easier to understand than $t$. We thank the referee for pointing this out to us.

Thus far, starting with an arbitrary closed orientable 3–manifold we have defined a series of branched covering space maps.

$$M^3 \xrightarrow[p]{3-1} S^3 \xrightarrow[f]{m-1} S^3 \xrightarrow[g]{n-1} S^3 \xrightarrow[h]{3-1} S^3 \xrightarrow[t]{27-1} S^3 . \tag{1}$$

The maps $f$ and $g$ only depend on $M^3$ in a superficial way; i.e. they depend on the number of vertical and horizontal components in the link that is the branch set for $p$, but not on the branching itself and the maps $h$ and $t$ don't depend on $M^3$ at all.

In general, when one composes branched covering space maps $a\colon X^3 \longrightarrow Y^3$, $b\colon Y^3 \longrightarrow Z^3$ one obtains a branched covering space map $b \circ a\colon X^3 \longrightarrow Z^3$.

In the sequel we abbreviate branch set by $BS$. Thus $BS(b \circ a) = BS(b) \cup b(BS(a))$. In our case $BS(f) \cap f(BS)(p) = \emptyset$; $BS(g) \cap g(BS(f \circ p)) = \emptyset$; $BS(h) \cap h(BS(g \circ f \circ p)) = \emptyset$. And $h \circ g \circ f \circ p$ is a branched covering space map of $S^3$ by $M^3$, of branching type $\{1, 2\}$ with branch set the doubled Borromean rings. Unlike the previous three compositions $BS(t) = t(BS(h \circ g \circ f \circ p))$ and the map $t \circ h \circ g \circ f \circ p$ is a branched covering space map of $S^3$ by $M^3$ of branching type $\{1, 2, 4\}$.

We summarize this as the next theorem

THEOREM (2.8). *Let $M^3$ be a closed orientable 3-manifold. There is a branched covering space map $q\colon M^3 \to S^3$ with branching type $\{1, 2, 4\}$ and branch set the Borromean rings.*

*There is a 3–fold simple branched covering space map $p\colon M^3 \longrightarrow S^3$ branched over a link, and there is a branched covering space map $\phi\colon S^3 \longrightarrow S^3$ with branch set the Borromean rings such that $q = \phi \circ p$.*

We remark that Theorem (2.8) implies that the Borromean rings are 4-universal. That is, that every closed orientable 3-manifold is a branched covering of $S^3$ with branch set the Borromean rings. This was shown for the first time in [6]. The proof here refines the proof of [6] in the sense that bounds are given on the branching indices. The branching is of type $\{1, 2, 4\}$. The refinement consists in introducing the map $t\colon S^3 \to S^3$ whose definition is based on the Euclidean orbifold structure of $S^3$ with singular set the Borromean rings.

Part of Theorem (2.8), the first two sentences, was first proven in ([5]). But the branched covering space map $q\colon M^3 \longrightarrow S^3$ whose existence was proven there did not factor as in the rest of the statement in Theorem (2.8). This factorization is crucial to the coming proof of Theorem (2.9), the principal result of this paper. Observe that Theorem (2.8) is totally topological in character; hyperbolic geometry appears nowhere in its statement. On the other hand, hyperbolic geometry in the form of the group of hyperbolic isometries $U$ dominates the statement of Theorem (2.9) below.

The sphere $S^3$ has a hyperbolic orbifold structure with singular set the Borromean rings and singular angle $90°$. The maps $t$, $h$, $g$, $f$, and $p$ are used to pull back the hyperbolic orbifold structure on $S^3$ to hyperbolic orbifold structures on $S^3$, $S^3$, $S^3$, $S^3$, and $M^3$ respectively. Thus there is a sequence of groups, orbifold groups, $G \subset G_1 \subset G_2 \subset G_3 \subset G_4 \subset U$ such that $M^3 = H^3/G$, $S^3 = H^3/G_1$, $S^3 = H^3/G_2$, $S^3 = H^3/G_3$, $S^3 = H^3/G_4$, $S^3 = H^3/U$ and the following diagram is commutative. The vertical arrows are homeomorphisms.

$$\begin{array}{ccccccccccc} H^3/G & \longrightarrow & H^3/G_1 & \longrightarrow & H^3/G_2 & \longrightarrow & H^3/G_3 & \longrightarrow & H^3/G_4 & \longrightarrow & H^3/U \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M^3 & \xrightarrow[p]{} & S^3 & \xrightarrow[f]{} & S^3 & \xrightarrow[g]{} & S^3 & \xrightarrow[h]{} & S^3 & \xrightarrow[t]{} & S3 \end{array}$$

In particular we see that $[G_1 : G] = 3$.

If $T$ is any rotation contained in $G_1$ but not contained in $G$ then $G_1 = \langle G, T\rangle$, as $[G_1 : G] = 3$ and 3 is prime. We summarize these remarks in the main theorem of this paper.

THEOREM (2.9). *(Geometric branched covering space theorem.) Let $M^3$ be a closed orientable 3–manifold. Then there are subgroups G and $G_1$ of the universal group U such that $[G_1 : G] = 3$ and $[U : G] < \infty$ and $M^3 = H^3/G$ and $S^3 = H^3/G_1$.*

*The map induced by the inclusion of groups $H^3/G \longrightarrow H^3/G_1$ is a 3–fold simple branched covering of $S^3$ by $M^3$.*

We recall ([5]) that if $M^3$ is as in the statement of Theorem (2.9) then $\pi_1(M^3) \cong G/\operatorname{TOR}(G)$ where $\operatorname{TOR}(G)$ is the subgroup of $G$ generated by rotations. In particular $M^3$ is simply connected if and only if $G = \operatorname{TOR}(G)$. Then $G_1$ is generated by $G$ and any one rotation not in $G$.

An interesting problem is to classify the finite index subgroups of $U$ that are generated by rotations. We begin this classification in [10].

Applying the theory of associated regular coverings to the above situation we obtain an interesting property on the involved groups that restricts their study to a subclass of the class of subgroups of the universal group $U$ defining the same variety. Next we explain this.

In general, given a covering $p\colon M \longrightarrow N$ branched over $L$, with monodromy $\omega\colon \pi_1(N - L) \longrightarrow \Sigma_n$, the *associated regular covering* $q\colon X \longrightarrow N$ is the branched covering determined by the monodromy $\eta \circ \omega\colon \pi_1(N - L) \longrightarrow \Sigma_{\sharp Im(\omega)}$, where $\eta$ is the regular representation of the group $Im(\omega)$. Recall that $q = u \circ p$ where $u\colon X \longrightarrow M$ is a regular (branched or unbranched) covering. Actually $(q|_{X-q^{-1}(L)})_\star(\pi_1(X - q^{-1}(L)) = Ker(\omega)$.

The monodromy of $p\colon M = H^3/G \longrightarrow S^3 = H^3/G_1$ is a homomorphism $\omega\colon \pi_1(S^3 - L) \longrightarrow \Sigma_3$ where the image of every meridian element of $L$ is a transposition $(i, j)$, $1 \le i < j \le 3$. Therefore, $Im(\omega) = \Sigma_6$, and the monodromy $\eta \circ \omega\colon \pi_1(S^3 - L) \longrightarrow \Sigma_6$ sends every meridian element of $L$ to the product of three different transpositions. Thus, $u\colon X \longrightarrow M$ is a 2-fold covering branched over the upper pseudobranch set of $p\colon M = H^3/G \longrightarrow S^3 = H^3/G_1$. The covering $q = u \circ p\colon X \longrightarrow S^3 = H^3/G_1$ is a regular 6-fold covering branched over L with all branching indexes equal to 2. Actually, the map $u$ can be used to pull back the hyperbolic orbifold structure on $M$, so that there exist a normal subgroup $G_0 \lhd G$, such that $[G : G_0] = 2$ and

$$u\colon X = H^3/G_0 \longrightarrow M = H^3/G$$

is an orbifold covering. Observe that $G_0$ is a normal subgroup of $G_1$.

The following diagram of orbifold coverings is commutative, where $G_0 \subset G' \subset G_1$ and $[G_1 : G'] = 2$, $[G' : G_0] = 3$.

$$\begin{tikzcd} H^3/G_0 \arrow[r, "u", "2:1"'] \arrow[d, "3:1"', "p'"] & M = H^3/G \arrow[d, "p"', "3:1"] \\ H^3/G' \arrow[r, "u'", "2:1"'] & S^3 = H^3/G_1 \end{tikzcd}$$

The covering $p'$ is unbranched and $u'$ is the cyclic 2-fold covering branched over $L$. Observe that $G_0$ is a normal subgroup of $G_1$. We summarize these remarks in the following theorem.

THEOREM (2.10). *Let $M^3$ be a closed orientable 3–manifold. Let $G$ and $G_1$ be the subgroups of the universal group $U$ given in Theorem (2.9). Then there exist a subgroup $G_0$ of index 2 of $G$ such that $G_0$ is a normal subgroup of $G_1$.*

Recall that every finite index subgroup $G$ of the universal group $U$ gives rise to a 3-manifold $M = H^3/G$, but infinitely many finite index $G$'s produce the same manifold. The above theorem restricts the class of subgroups of $U$ to consider in order to construct all closed 3-manifolds.

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# ON FINITE INDEX SUBGROUPS OF A UNIVERSAL GROUP

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ABSTRACT. The orbifold group of the Borromean rings with singular angle 90 degrees, $U$, is a universal group, because every closed oriented 3–manifold $M^3$ occurs as a quotient space $M^3 = H^3/G$, where $G$ is a finite index subgroup of $U$. Therefore, an interesting, but quite difficult problem, is to classify the finite index subgroups of the universal group $U$. One of the purposes of this paper is to begin this classification. In particular we analyze the classification of the finite index subgroups of $U$ that are generated by rotations.

## 1. Introduction

A finite covolume discrete group of isometries of hyperbolic 3–space, $H^3$, is said to be *universal* if every closed oriented 3-manifold $M^3$ occurs as a quotient space $M^3 = H^3/G$, where $G$ is a finite index subgroup of the universal group. It was originally shown in [4] that $U$, the orbifold group of the Borromean rings with singular angle 90 degrees is universal. (See [2] for a simpler proof.)

Although there appear to be infinite families of universal groups, the group $U$ is the only one so far known that is associated to a tessellation of $H^3$ by regular hyperbolic polyhedra in that there is a tessellation of $H^3$ by regular dodecahedra with dihedral angles 90 degrees any one of which is a fundamental domain for $U$.

An interesting, important, but quite difficult problem, is to classify the finite index subgroups of $U$. A theorem of Armstrong [1] shows that $\pi_1(M^3) \cong G/TOR(G)$ where $TOR(G)$ is the subgroup of $G$ generated by rotations. In particular $M^3$ is simply connected if and only if $G$ is generated by rotations. One of the purposes of this paper is to begin the classification of the finite index subgroups of $U$ that are generated by rotations. Our main result is Theorem (5.2).

THEOREM (5.2) *For any integer n there is an index n subgroup of U generated by rotations.*

In Theorem (5.3) we illustrate the essential differences between the cases $n$ is odd and $n$ is even.

The organization of the paper is as follows: In Section 2 we define the group $U$, a closely related Euclidean crystallographic group $\widehat{U}$, and a homomorphism $\varphi\colon U \longrightarrow \widehat{U}$. In Section 3 we show there are tessellations of $H^3$ by regular

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dodecahedra and $E^3$ by cubes and we exploit the homomorphism $\varphi\colon U \longrightarrow \widehat{U}$ to define a branched covering space map $p\colon H^3 \longrightarrow E^3$ that respects the two tessellations in the sense that the restriction of $p$ to any one dodecahedron of the tessellation of $H^3$ is a homeomorphism onto a cube of the tessellation of $E^3$. In Section 4 we prove the *rectangle theorem* and we use it to classify the finite index subgroups of $\widehat{U}$ that are generated by rotations. In the final section we use this classification together with the homomorphism defined in Section 2 to prove the main theorem of the paper, Theorem (5.2), and some existence theorems about finite index subgroups of $U$ generated by rotations.

## 2. Definitions of $U$, $\widehat{U}$ and the homomorphism $\varphi\colon U \longrightarrow \widehat{U}$

Let $C_0$ be the cube in $E^3$ with vertices $(\pm 1, \pm 1, \pm 1)$. We obtain a tessellation of $E^3$ by applying compositions of even integer translations in the $x$, $y$, and $z$ directions to $C_0$. In this paper we do not consider any other tessellations of $E^3$ and we refer to this tessellation as "the" tessellation of $E^3$. The intersection of $C_0$ with the positive octant, together with the lines $\widetilde{a} = (t, 0, 1)$, $\widetilde{b} = (1, t, 0)$, and $\widetilde{c} = (0, 1, t)$; $-\infty < t < \infty$, is depicted in Figure 1.

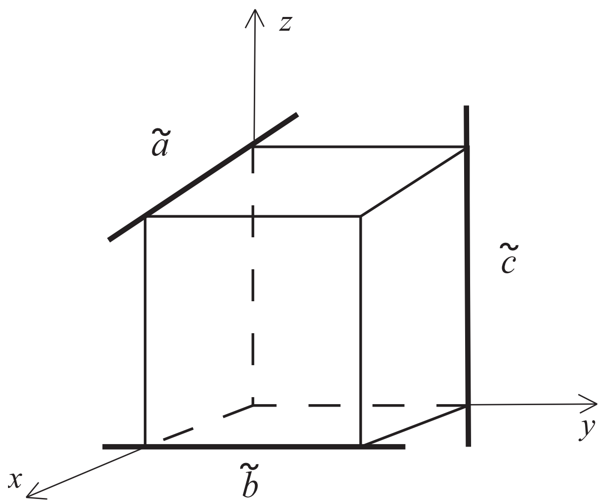


**Figure 1**

The group $\widehat{U}$ is the Euclidean crystallographic group generated by 180 degree rotations $a$, $b$, and $c$ with axes $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$, respectively. We see that $\widehat{U}$ preserves the tessellation and contains the translations $t_x = b(cbc^{-1})$, $t_y = c(aca^{-1})$, $t_z = a(bab^{-1})$, by distances of four, in the $x$, $y$, and $z$ directions, respectively.

The cube $C_0$ is easily seen to be a fundamental domain for $\widehat{U}$, and the axes of rotation in $\widehat{U}$ divide each face of each cube in the tessellation into two rectangles. The quotient space $E^3/\widehat{U}$ is topologically $S^3$ as can be seen by identifying faces of $C_0$ using $a$, $b$, $c$ and other rotations. The group $\widehat{U}$ is the orbifold group of $S^3$ as Euclidean orbifold with singular set the Borromean rings $B$ and singular angle 180 degrees. This construction is due to Thurston. For more details see ([6], [2]). The Borromean rings are depicted in Figure 2.

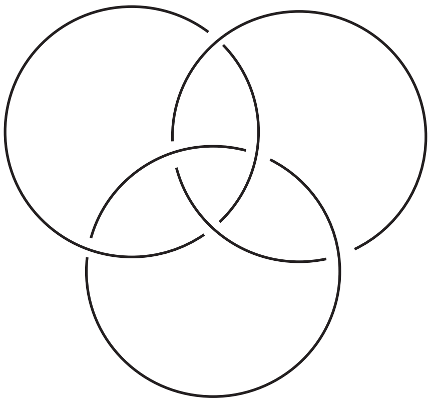

**Figure 2.** Borromean rings

The induced map $p\colon E^3 -$ preimage $B \longrightarrow (E^3 -$ preimage $B)/\widehat{U} \approx S^3 - B$ is a regular covering space map so by the theory of covering spaces

$$\widehat{U} \cong \pi_1(S^3 - B)/p_*\pi_1(E^3 - \text{preimage } B)\,.$$

This gives rise to a presentation for $\widehat{U}$:

(2.1)

$$\widehat{U} = \langle a, b, c \,|\, a\, b\bar{c}\bar{b}c = b\bar{c}\bar{b}c\, a,\ b\, c\bar{a}\bar{c}a = c\bar{a}\bar{c}a\, b,\ c\, a\bar{b}\bar{a}b = a\bar{b}\bar{a}b\, c\,, a^2,\ b^2,\ c^2 \,\rangle\,.$$

The presentation comes from the usual Wirtinger presentation of the group of the Borromean rings with additional relations $a^2$, $b^2$, and $c^2$ arising from $p_*\pi_1(E^3$–preimage $B)$ which is normally generated by squares of meridians about the axes $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}$ of Figure 1.

There is a construction of $S^3$ as hyperbolic orbifold (also due to Thurston) with singular set the Borromean rings analogous to the previous construction. To describe it we shall work in the Klein model for $H^3$.

In the Klein model hyperbolic points are Euclidean points inside a ball of radius $R$ centered at the origin in $E^3$ and hyperbolic lines and planes are the intersections of Euclidean lines and planes with the interior of the ball of radius $R$. Let $D_0$ be a regular Euclidean dodecahedron that is symmetric with respect to reflection in the $xy$, $yz$, and $xz$ planes. The intersection of $D_0$ with the positive octant is depicted in Figure 2.

If $R$ is chosen correctly, (Details are in [5]), then $D_0$ can be considered as a regular hyperbolic dodecahedron with 90 degree dihedral angles. Each pentagonal face contains one edge that lies in either the $xy$, $xz$, or $yz$ plane. Reflection in this plane, restricted to the pentagon, defines an identification in pairs on the pentagonal faces of $D_0$. As in the construction with the cube $C_0$, the resulting topological space is $S^3$. A hyperbolic orbifold structure is thus induced on $S^3$ with singular set the Borromean rings, $B$, and singular angle 90 degrees. The Borromean rings are the image, after identification of the pentagonal edges that lie in the $xy$, $xz$, and $yz$ planes.

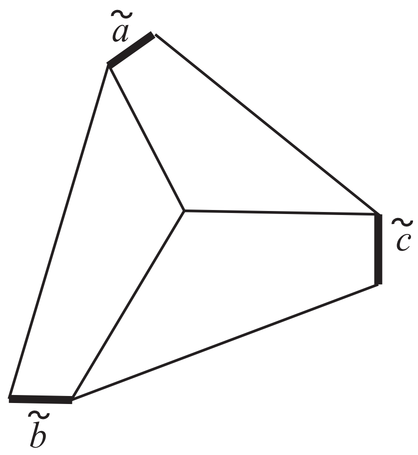


**Figure 3**

There is a 4-fold regular branched cyclic covering $q_1\colon X^3 \longrightarrow S^3$ with branch set the Borromean rings induced by the natural group homomorphisms

$$\pi_1(S^3 - B) \longrightarrow H_1(S^3 - B; Z) \cong Z \oplus Z \oplus Z \longrightarrow Z \text{ mod} 4.$$

The hyperbolic orbifold structure on $S^3$ with singular set the Borromean rings pulls back to a hyperbolic manifold (not orbifold) structure on $X^3$ as meridians are sent to 1 in the above homomorphism.

The hyperbolic manifold $X^3$ has a *tessellation* consisting of four dodecahedra each of which is sent homeomorphically to $D_0$ by the map $p$. The universal covering space map $q_2\colon H^3 \longrightarrow X^3$ is used to pull back the tessellation of $X^3$ by dodecahedra to a tessellation of $H^3$ by dodecahedra. The composition of covering space maps $q_1 \circ q_2\colon H^3 \longrightarrow S^3$ is a regular branched covering space map $H^3 \longrightarrow S^3$ induced by the group of hyperbolic isometries $U$. That is to say there is a quotient branched covering map $H^3 \longrightarrow H^3/U \approx S^3$ and an associated unbranched covering space map $p : H^3 - \text{axes of rotation} = H^3 - \text{preimage } B \longrightarrow (H^3 - \text{preimage } B)/U \approx S^3 - B$. As in the Euclidean case this covering space map gives rise to a presentation for $U$ via covering space theory:

(2.2)

$$U = \langle a, b, c | a\, b\bar{c}\bar{b}c = b\bar{c}\bar{b}c\, a,\ b\, c\bar{a}\bar{c}a = c\bar{a}\bar{c}a\, b,\ c\, a\bar{b}\bar{a}b = a\bar{b}\bar{a}b\, c\,, a^4,\ b^4,\ c^4 \rangle\,.$$

As before the presentation comes from the usual Wirtinger presentation of the group of the Borromean rings with additional relations $a^4$, $b^4$, $c^4$ arising from $p_*\pi_1(H^3 - \text{preimage} B)$ which is normally generated by fourth powers of meridians about the axes $\widetilde{a}$, $\widetilde{b}$ and $\widetilde{c}$.

Examining the presentations for $U$ and $\widehat{U}$ we see that they are the same except for the relations $a^4$, $b^4$, and $c^4$ in $U$ and $a^2$, $b^2$, and $c^2$ in $\widehat{U}$. Nonetheless the map $a \to a$, $b \to b$, and $c \to c$, mapping generators of $U$ to generators of $\widehat{U}$, defines a homomorphism $\varphi\colon U \longrightarrow \widehat{U}$ and an exact sequence.

(2.3) $$1 \longrightarrow K \longrightarrow U \xrightarrow{\varphi} \widehat{U} \longrightarrow 1\,.$$

| Polyhedral Type | Euclidean dihedral angle | Hyperbolic dihedral angle vertices at $\infty$ |
|---|---|---|
| Tetrahedron | $ArcCos[1/3] \approx 70.5288°$ | 60° |
| Cube | ArcCos[0]= 90° | 60° |
| Octahedron | $ArcCos[-1/3] \approx 109.471°$ | 90° |
| Dodecahedron | $ArcCos[-1/\sqrt{5}] \approx 116.565°$ | 60° |
| Icosahedron | $ArcCos[-\sqrt{5}/3] \approx 138.19°$ | 108° |

**Table 1.**

In this exact sequence $K$ is defined to be the kernel of homomorphism $\varphi$.

We say that a group of isometries of $H^3$ or $E^3$ is *associated* to a tessellation of $H^3$ or $E^3$ by regular compact polyhedra if there is a tessellation of $H^3$ or $E^3$ by regular compact polyhedra any one of which is a fundamental domain for the group. Thus the groups $U$ and $\widehat{U}$ are associated to the tessellations of $H^3$ and $E^3$ by regular dodecahedra and cubes, respectively. This is not a common occurrence. For example, of the regular polyhedra only cubes can tessellate $E^3$. In the table below, we have listed the cosines of the dihedral angles of the Euclidean regular polyhedra and also the dihedral angles of the hyperbolic regular polyhedra with vertices on the sphere at infinity. Tetrahedra, octahedra, dodecahedra and icosahedra cannot tessellate $E^3$ because their dihedral angles are not submultiples of 360 degrees so they don't "fit around an edge".

There are five regular Euclidean polyhedra but the corresponding hyperbolic polyhedra occur in one parameter families. One can construct the family of hyperbolic cubes, for example, by starting with $C_0$, the cube with vertices $(\pm 1, \pm 1, \pm 1)$, in the Klein model with the sphere at infinity having Euclidean radius $R = \sqrt{3}$ and let $R$ increase from $\sqrt{3}$ to $\infty$. There is an isometry from the Klein model using the Euclidean ball of radius $R$ to the Poincaré model using the same Euclidean ball (as Thurston has explained), that is the identity on the sphere at infinity. Since the Poincaré model is conformal and Poincaré hyperbolic planes are Euclidean spheres perpendicular to the sphere at infinity, the dihedral angle between two Poincaré planes is the same as the Euclidean angle between the two circles in which the Poincaré planes intersect the sphere at infinity. Thus the dihedral angle between two Klein planes is the same as the angle between the two circles in which they intersect the sphere at infinity. As $R$ increases, in the case of the cube, for example, from $\sqrt{3}$ to infinity, the dihedral angle increases from 60 degrees to 90 degrees. There exists a compact hyperbolic cube with dihedral angle $\theta$ if and only if $0 < \cos\theta < 1/2$. Thus, if it is possible to tessellate $H^3$ with compact hyperbolic cubes they must have dihedral angle 72 degrees as that is the only submultiple of 360 degrees in the range of possible dihedral angles. A glance at the table 1 indicates that it is impossible to tessellate $H^3$ with compact regular octahedra or tetrahedra and if it is possible to tessellate $H^3$ with icosahedra the dihedral angle must be 120 degrees. In the dodecahedral case we have shown that there is a tessellation of $H^3$ by regular compact hyperbolic dodecahedra with dihedral angle 90 degrees. If there were a different tessellation by compact regular dodecahedra the dihedral angle would have to be 72 degrees.

All the above is part of standard 3–dimensional hyperbolic geometry and we explain it mainly so as to highlight the singular nature of the groups $U$ and $\widehat{U}$ and the tessellations with which they are associated and as background for the following conjecture.

*Conjecture* (2.4). The group $U$ is the only universal group associated to a tessellation of $H^3$ by regular hyperbolic polyhedra.

In the next section we study the groups $U$ and $\widehat{U}$ and the tessellations to which they are associated to produce a branched covering of $E^3$ by $H^3$.

## 3. $H^3$ as a branched covering of $E^3$

Let $D_0$ and $C_0$ be the regular dodecahedron and cube in the Klein model for $H^3$ and in $E^3$ respectively, as defined in the previous section. We know that $D_0$ is a fundamental domain for the group $U$ and is also an element of the tessellation of $H^3$ by regular dodecahedra. For any other dodecahedron $D$ in the tessellation there is a unique element $u$ of $U$ such that $u(D_0) = D$. Analogously, $C_0$ is a fundamental domain for the group $\widehat{U}$ and is part of the tessellation of $E^3$ by cubes. For any other cube $C$ in the tessellation there is a unique element $\widehat{u}$ of $\widehat{U}$ such that $\widehat{u}(C_0) = C$.

Let $\alpha_0 \colon D_0 \longrightarrow C_0$ be a homeomorphism that is as nice as possible. Thus $\alpha_0$ should commute with reflections in the $xy$, $xz$, and $yz$ planes and also with the 3–fold rotations about the axes $\{(t, t, t)\}$ in the Klein model for $H^3$ and in $E^3$. The cube $C_0$ becomes a dodecahedron when each of its faces is split in half by an axis of rotation of $\widehat{U}$. Then $\alpha_0$, viewed as a map between dodecahedra takes vertices, edges, and faces to vertices, edges, and faces, respectively.

Now we define a map $p \colon H^3 \longrightarrow E^3$. Let $p = \alpha_0$ on $D_0$. Any other point $A$ in $H^3$ belongs to a dodecahedron $D$ of the tessellation. There is a unique $u \in U$ such that $u(D_0) = D$. Let $\widehat{u} = \varphi(u)$ where $\varphi \colon U \longrightarrow \widehat{U}$ is the homomorphism defined in the previous section. Define the map $p$ by $p(A) = \widehat{u} \circ \alpha_0 \circ u^{-1}(A)$. The map $p$ is well defined for points in the interior of dodecahedra in the tessellation but we must show that $p$ is well defined for the other points. Let $A$ belong to the interior of a pentagonal face $P$ belonging to each of two adjacent dodecahedra $D_1$ and $D_2$.

Then there are unique elements $u_1$ and $u_2$ of $U$ such that $u_1(D_0) = D_1$ and $u_2(D_0) = D_2$. Then $u_1^{-1}(D_2)$ is a dodecahedron, call it $\widehat{D}$, that intersects $D_0$ exactly in a pentagonal face $P_0$. The pentagonal face $P_0$ of $D_0$ intersects exactly one of the six axes of rotation, call it $ax$, that intersect $D_0$ and this axis lies in the $xy$, $xz$, or $yz$ plane of the Klein model. There is a 90 degree rotation about $ax$, call if $rot$, that sends $D_0$ to $\widehat{D}$. Thus $u_1 \circ rot(D_0) = D_2$ which implies $u_1 \circ rot = u_2$, which further implies $\widehat{u}_1 \circ \widehat{rot} = \widehat{u}_2$ in group $\widehat{U}$. Then $\widehat{u}_2 \circ \alpha_0 \circ u_2^{-1} = \widehat{u}_2 \circ \alpha_0 \circ rot^{-1} \circ u_1^{-1} = \widehat{u}_1 \circ \widehat{rot} \circ \alpha_0 \circ rot^{-1} \circ u_1$ so that to show that the map $p$ is well defined on the interior of pentagon $P$ it suffices to show that $\widehat{rot} \circ \alpha_0 \circ rot^{-1} = \alpha_0$ when restricted to pentagonal face $P_0$.

The homomorphism $\varphi\colon U \longrightarrow \widehat{U}$ takes $a$, $b$, and $c$ to $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, respectively where $a$, $b$, and $c$ are 90 degree rotations about axes $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$, respectively of Figure 2 and $\widehat{a}$, $\widehat{b}$, $\widehat{c}$ are 180 degree rotations about axes $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$, respectively of Figure 1. The rotation $rot$ is one of $a$, $b$, $c$, $a^{-1}$, $b^{-1}$, $c^{-1}$, $bab^{-1}$, $cbc^{-1}$, $aca^{-1}$, $ba^{-1}b^{-1}$, $cb^{-1}c^{-1}$, $ac^{-1}a^{-1}$. The rotation $rot$, when restricted to pentagon $P_0$ equals reflection in the $xy$, $yz$, or $xz$ plane depending on which plane axis $rot$ lies in. Similarly, the rotation $\widehat{rot}$ is one of $\widehat{a}$, $\widehat{b}$, $\widehat{c}$, $\widehat{b}\widehat{a}\widehat{b}^{-1}$, $\widehat{c}\widehat{b}\widehat{c}^{-1}$, $\widehat{a}\widehat{c}\widehat{a}^{-1}$ and the rotation $\widehat{rot}$ when restricted to the half square that is the image of $P_0$ under $\alpha$ equals reflection in the $xy$, $xz$, or $yz$ plane depending on which plane axis $\widehat{rot}$ lies in. But $\alpha_0$ commutes with reflections in the $xy$, $xz$, or $yz$ planes so that $\widehat{rot} \circ \alpha_0 \circ rot^{-1} = \alpha_0$ and the map $p$ is well defined on the interiors of dodecahedra in the tessellation and on the interiors of their pentagonal faces. That $p$ is also well defined on edges and vertices of the tessellating dodecahedra now follows by a continuity argument.

We summarize all this in a theorem.

THEOREM (3.1). *There exists a tessellation of $H^3$ by regular hyperbolic dodecahedra with 90 degrees dihedral angle and a tessellation of $E^3$ by cubes and a map $p\colon H^3 \longrightarrow E^3$ such that the following holds.*

1. *Any dodecahedron in the tessellation of $H^3$ is a fundamental domain for the universal group $U$.*
2. *Any cube in the tessellation of $E^3$ is a fundamental domain for the Euclidean crystallographic group $\widehat{U}$.*
3. *The axes of rotation in $\widehat{U}$ divide each face of each cube in the tessellation of $E^3$ into two rectangles so that the cube may be viewed as a dodecahedron.*
4. *The restriction of p to any one dodecahedron is a homeomorphism of that dodecahedron onto a cube in the tessellation of $E^3$. When the cube is viewed as a dodecahedron as in 3 above, the map p sends vertices, edges, and faces to vertices edges and faces respectively. The map p also sends axes of rotation for U homeomorphically, even isometrically, to axes of rotation for $\widehat{U}$.*
5. *The map p is a branched covering space map with all branching of order two.*

In effect, parts 1 through 4 of the theorem have already been proven in the remarks preceding the statement of the theorem. To see that 5 is true, it is only necessary to examine $p$ near an axis of rotation for $U$. The branching is of order two because four dodecahedra *fit* around every axis of rotation in $U$ while only two cubes *fit* around an axis of rotation of $\widehat{U}$.

It is clear from the definition of the map $p$ when restricted to a dodecahedron, $p = \widehat{u} \circ \alpha_0 \circ u^{-1}$, that the group of covering transformations is the kernel of the homomorphism $\varphi\colon U \longrightarrow \widehat{U}$. On the other hand $p$ when restricted to ($H^3$–axes of rotation for $U$) is an unbranched covering of ($E^3$ – axes of rotation for $\widehat{U}$) so that $K = \ker\varphi\colon U \longrightarrow \widehat{U}$ is isomorphic to $\pi_1(E^3$ – axes of rotation for $\widehat{U})$ modulo $p_*\pi_1(H^3$ – axes of rotation for $U)$, by standard covering space theory.

As $\pi_1(E^3$ – axes of rotation for $\widehat{U})$ is a free group generated by meridians, one meridian for each axis of rotation, and $\pi_1(H^3$ – axes of rotation for $U)$ is also generated by meridians it follows that $p_*\pi_1(H^3$ – axes of rotation) is normally generated by squares of meridians, one for each axis of rotation in $\widehat{U}$. We also summarize all this in a theorem.

THEOREM (3.2). *The group of covering transformations for the branched covering $p\colon H^3 \longrightarrow E^3$ is isomorphic to the group $K$ that is the kernel of $\varphi\colon U \longrightarrow \widehat{U}$. The group $K$ is naturally isomorphic to a countable free product of Z mod 2's, one generator for each axis of rotation in $\widehat{U}$. In particular the group $K$ is generated by 180 degree rotations.*

As before, the proof of the theorem is in effect given by the remarks immediately prior to the statement of the theorem.

Theorems (3.1) and (3.2) enable us to "label" each axis of rotation in $U$ with an algebraic integer in the field $Q(\sqrt{-3}\,)$. Note that each axis of rotation for $\widehat{U}$ is a line of parametric equation ($t$, even, odd) or (odd, $t$, even) or (even, odd, $t$), $-\infty < t < \infty$. Any such axis intersects the plane $x + y + z = 0$ in a point (odd, odd, even) or (even, odd, odd) or (odd, even, odd) as zero is even. One can verify that the intersection of the tessellation by cubes of $E^3$ with the plane $x + y + z = 0$ induces a tessellation of the plane $\pi : x + y + z = 0$ by (regular) hexagons and (equilateral) triangles and that cube $C_0$ intersects the plane $x + y + z = 0$ in a hexagon with vertices $\{(\pm 1, \mp 1, 0), (\pm 1, 0, \mp 1), (0, \pm 1, \mp 1)\}$. Using a similarity of the plane $x+y+z = 0$ with center the origin and expansion ratio $1/\sqrt{2}$ we can recoordinatize the plane $x+y+z = 0$ by the complex numbers $\mathbb{C}$ so that the six vertices of this hexagon have coordinates equal to the six roots of unity in $\mathbb{C}$. Then every axis of rotation of $\widehat{U}$ intersects the plane $x+y+z = 0$ in a point whose coordinate is an algebraic integer in the field $Q(\sqrt{-3}\,)$. We label each axis $d$ of rotation of $U$ with the coordinate of $p(d) \cap \pi$. Again we summarize these results in a theorem.

THEOREM (3.3). *In the branched covering $p\colon H^3 \longrightarrow E^3$ each axis of rotation for $U$ is labeled by an algebraic integer of the field $Q(\sqrt{-3}\,)$. The group of covering transformations $K$ preserves labeling. For any two axes of rotation $a$ and $b$ of $U$ with the same label, there is an element $k$ of $K$ such that $k(a) = b$.*

In the next section we classify the subgroups of finite index in $\widehat{U}$ that are generated by rotations.

## 4. Finite index subgroups of $\widehat{U}$ generated by rotations

The group $\widehat{U}$ is the crystallographic group $I2_12_12_1$, number 24 of the International Tables of Crystallography [3] . In this section we describe two families of subgroups of $\widehat{U}$ (defined in Section 2) generated by rotations. And we show that *any* finite index subgroup of $\widehat{U}$ generated by rotations is equivalent (in a sense we make precise) to exactly one member of one of the two families.

The axes of rotation of $\widehat{U}$ have parametric equations of form ($t$, even, odd), (odd, $t$, even) or (even, odd, $t$); $-\infty < t < \infty$ according as to whether they are parallel to the $x$, $y$, or $z$ axes. The distance between axes lying in a plane parallel to the $xy$, $xz$, or $yz$ planes is an even integer.

Let $(m, n, o)$ be a triple of positive integers where $o$ is odd and $m$ and $n$ are arbitrary. We shall define a group $\widehat{G}(m, n, o)$ associated to the triple $(m, n, o)$

and belonging to the first family by defining a rectangular parallelepiped that will turn out to be a fundamental domain for $\widehat{G}(m, n, o)$.

Let $Box(\widehat{G}(m, n, o))$ be the rectangular parallelepiped defined by the following conditions.

a. The front and back faces of $Box(\widehat{G}(m, n, o))$ lie in the planes $x = 2m + 1$ and $x = -2m + 1$, respectively.

b. The right and left faces of $Box(\widehat{G}(m, n, o))$ lie in the planes $y = 2n$ and $y = -2n$, respectively.

c. The top and bottom of $Box(\widehat{G}(m, n, o))$ lie in the planes $z = o$ and $z = 0$, respectively. $Box(\widehat{G}(m, n, o))$ together with certain axes of rotation is pictured in Figure 4.

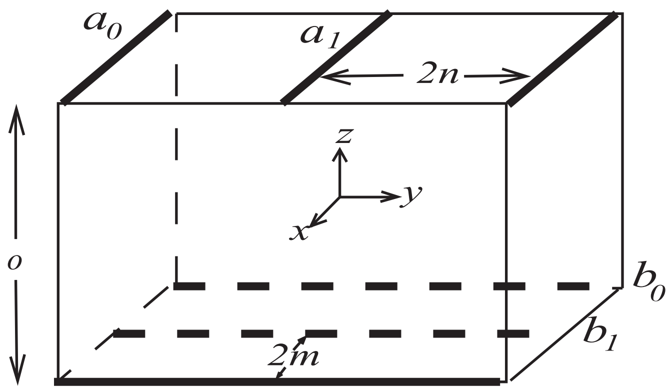


**Figure 4.** $Box(\widehat{H}(m, n, o))$.

Axes $a_0$, $a_1$, $b_0$ and $b_1$ have parametric equations $(t, -2n, o)$, $(t, 0, o)$, $(-2m + 1, t, 0)$ and $(1, t, 0)$, respectively. Then $\widehat{G}(m, n, o)$ is defined to be the subgroup of $\widehat{U}$ generated by $A_0$, $A_1$, $B_0$, and $B_1$, the rotations in the axes $a_0$, $a_1$, $b_0$, and $b_1$, respectively.

Observe that $T_x = B_1B_0$, $T_y = A_1A_0$, and $T_z = (A_0B_1)^2$ are translations by $4m$, $4n$, and $4o$ in the $x$ $y$, and $z$ directions, respectively. Another generating set of $\widehat{G}(m, n, o)$ is $A_1$, $B_1$, $T_x$, and $T_y$. Conjugating a translation $T_x$, $T_y$, or $T_z$ by a rotation $A_1$, $B_1$ either results in the translation itself or its inverse, so there are commutation relations such as $B_1T_x = T_x^{-1}B_1$. Thus any element of $\widehat{G}(m, n, o)$ has form $T$, $A_1T$, $B_1T$ or $A_1B_1T$ where $T$ is a translation that is some product of $T_x$, $T_y$, and $T_z$. With these observations we can see that $Box(\widehat{G}(m, n, o))$ is a fundamental domain for the group $\widehat{G}(m, n, o)$. The volume of $Box(\widehat{G}(m, n, o))$ equals $4m \times 4n \times o$ and the volume of cube $C_0$, which is a fundamental domain for $\widehat{U}$ equals 8. Thus dividing one by the other, the index of $\widehat{G}(m, n, o)$ in $\widehat{U}$ equals $2mno$, an even integer. The group $\widehat{G}(m, n, o)$ is the crystallographic group $P222_1$, number 17 in the International Tables of Crystallography [3].

Let $(p, q, r)$ be a triple of odd positive integers such that $p \leqq q$ and $p \leqq r$ and if $p$, $q$, and $r$ are not all different then $p \leqq q \leqq r$. (The idea here is that any triple of odd positive integers can be cyclicly permuted to a triple satisfying these conditions.) We shall define a group $\widehat{H}(p,q,r)$ in the second family by first defining a rectangular parallelepiped that will turn out to be its fundamental domain. Let $Box(\widehat{H}(p,q,r))$ be the rectangular parallelepiped defined by the following conditions.

The front and back, left and right, top and bottom faces of $Box(\widehat{H}(p,q,r))$ lie in the planes $x=p$, $x=-p$; $y=q$, $y=-q$; $z=r$, $z=-r$, respectively. $Box(\widehat{H}(p,q,r))$ is pictured in Figure 5 along with axes of rotation $a=(t,0,r)$, $b=(p,t,0)$, and $c=(0,q,t)$.

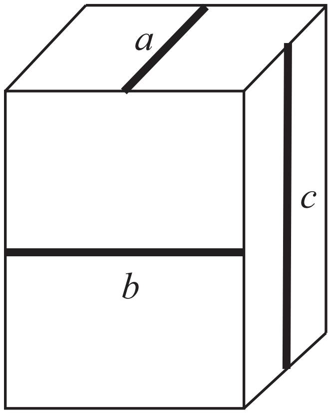


**Figure 5.** $Box(\widehat{H}(p,q,r))$.

The group $\widehat{H}(p,q,r)$ is defined to be the subgroup of $\widehat{U}$ generated by rotations $A$, $B$, and $C$ in axes $a$, $b$, and $c$, respectively. Observe that $T_x = BCBC$, $T_y = CACA$, and $T_z = ABAB$ are translations by $2p$, $2q$, and $2r$ in the $x$, $y$, and $z$ directions, respectively. Also note that conjugating $T_x$, $T_y$, or $T_z$ by ($A$ or $B$ or $C$) results in $T_x$ or $T_x^{-1}$, $T_y$ or $T_y^{-1}$, $T_z$ or $T_z^{-1}$, respectively. These observations imply that any element of group $\widehat{H}(p,q,r)$ equals exactly one of $T$, $AT$, $BT$ or $CT$ where $T$ is a product of $T_x$, $T_y$ and $T_z$. As before, we can see that $Box(H(p,q,r))$ is a fundamental domain for group $\widehat{H}(p,q,r)$. The group $\widehat{H}(p,q,r)$ is again the crystallographic group $I2_12_12_1$, number 24 in [3].

The volume of $Box(\widehat{H}(p,q,r))$ equals $8pqr$ and volume $C_0 = 8$ so, reasoning as before, the index of $\widehat{H}(p,q,r)$ in $\widehat{U}$ is $pqr$ which is an odd integer.

We wish to define an equivalence relation on finite index subgroups of $\widehat{U}$. Let $D$ be the 120 degree rotation about the axis $(t,t,t)$; $-\infty < t < \infty$, which is a main diagonal of cube $C_0$ and let $\widehat{S}$ be the group generated by $D$ and $\widehat{U}$. As $D$ has order three and normalizes $\widehat{U}$ we see that $[\widehat{S} : \widehat{U}] = 3$. We define two subgroups of $\widehat{U}$ to be equivalent if they are conjugate as subgroups of $\widehat{S}$. This equivalence relation when applied to the finite index subgroups of $\widehat{U}$ generated by rotations leads to the least messy classification. We shall show that any finite index subgroup of $\widehat{U}$ generated by rotations is equivalent to exactly one $\widehat{G}(m,n,o)$ or $\widehat{H}(p,q,r)$. We observe that rotation $D$ cyclically permutes the $x$, $y$, and $z$ axes but that there is no element of $\widehat{S}$ that fixes one of these three axes while interchanging the other two.

For each finite index subgroup $\widehat{G}$ of $\widehat{U}$ we define an "integer triple" $(d_1, d_2, d_3)$. The integer $d_1$ is the minimal distance between distinct axes of rotation in $\widehat{G}$ that are parallel to the $x$ axis and $d_2$ and $d_3$ are similarly defined for rotations about axes parallel to the $y$ and $z$ axis. If $\widehat{G}$ contains no rotations the "integer triple" is (none, none, none). Thus the "integer triple" assigned to $\widehat{G}(m, n, o)$ is $(2n, 2m,$ none) and the integer triple assigned to $\widehat{H}(p, q, r)$ is $(2r, 2p, 2q)$. (Recall, the cubes in the tessellation are $2 \times 2 \times 2$.)

Conjugating a $\widehat{G}(m, n, o)$ or an $\widehat{H}(p, q, r)$ by an element of $\widehat{S}$ at most changes a triple by cyclically permuting it. Thus the fact that $\widehat{G}$ contains no axes of rotation parallel to the $z$–axis implies that if $\widehat{G}(m, n, o) \sim \widehat{G}(\widetilde{m}, \widetilde{n}, \widetilde{o})$ then $(m, n, o) = (\widetilde{m}, \widetilde{n}, \widetilde{o})$ and the conditions $p \leqq q$ and $p \leqq r$, etc., imply that if $\widehat{H}(p, q, r) \sim \widehat{H}(\widetilde{p}, \widetilde{q}\, \widetilde{r})$ then $(p, q, r) = (\widetilde{p}, \widetilde{q}, \widetilde{r})$. Also as the index of a $\widehat{G}$ in $\widehat{U}$ is even and the index of an $\widehat{H}$ in $\widehat{U}$ is odd no $\widetilde{G}$ can be equivalent to an $\widehat{H}$. The rest of the classification consists of showing that any finite index subgroup of $\widehat{U}$ generated by rotations is either equivalent to an $\widehat{H}(p, q, r)$ or a $\widehat{G}(m, n, o)$.

Suppose that $\widehat{G}$ is a finite index subgroup of $\widehat{U}$ that is generated by rotations. If $\widehat{G}$ contained only rotations parallel to one of the three axes, it would leave planes perpendicular to this axis invariant and thus have infinite index in $\widehat{U}$. So $\widehat{G}$ either contains rotations about axes parallel to two of the three axes $x$, $y$, and $z$ or it contains rotations about axes parallel to all three. In the former case, we can assume $\widehat{G}$ contains rotations with axes parallel to the $x$ and $y$ axes but doesn't contain rotations with axes parallel to the $z$–axis by conjugating by an element of $\widehat{S}$ if need be. In either case let $\mathcal{P}$ be a plane parallel to the $yz$ plane in which an axis of $\widehat{G}$ parallel to the $y$–axis lies. The set of axes of rotation of $\widehat{G}$ parallel to the $x$–axis intersects $\mathcal{P}$ in a set of points we call axis points.

PROPOSITION (4.1) (The rectangle theorem). *There is a tessellation of $\mathcal{P}$ by congruent rectangles with sides parallel to the y and z axes such that the set of axis points equals the set of vertices of the rectangles. Each rectangle is divided in half by an axis of rotation for $\widehat{G}$ parallel to the y–axis.*

The proof of Proposition (4.1) rests on three facts.

1. If $A$ is a rotation in $\widehat{G}$ with axis $\ell$ and $S \in \widehat{G}$ then $SAS^{-1}$ is a rotation in $\widehat{G}$ with axis $S(\ell)$. In particular if $X$ is an axis point and $S(\mathcal{P}) = \mathcal{P}$, then $S(X)$ is an axis point.

2. If $A$ is a rotation in $\widehat{G}$ with axis $\ell$ and $T$ is a translation in $\widehat{G}$ such that $T(\mathcal{P}) = \mathcal{P}$ and $\ell \cap \mathcal{P} = X$ then $TA$ is also a rotation in $\widehat{G}$ and axis $(TA) \cap \mathcal{P}$ is the midpoint of the line segment $XT(X)$.

3. Group $\widehat{G}$ contains translations in the $x$, $y$, and $z$ directions. (Because $\widehat{U}$ does and $[\widehat{U} : \widehat{G}\,] < \infty$.)

*Proof of Proposition (4.1).* Let $T_y$ and $T_z$ be translations by minimal distance in the $y$ and $z$ directions respectively, belonging to $\widehat{G}$. (Refer to Figure 6.)

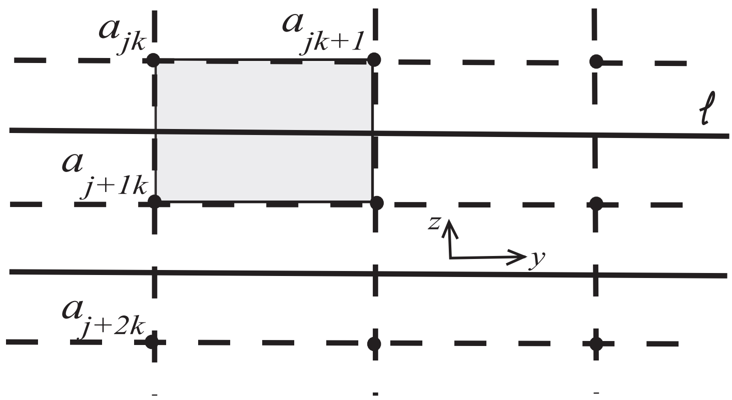


**Figure 6.** The plane $\mathcal{P}$.

Let $a_{00}$ be an axis point. Then by 1 and 2 above, $a_{20} = T_y(a_{00})$ and $a_{10} =$ midpoint $a_{00}a_{20}$ are axis points as are $a_{02} = T_z(a_{00})$, $a_{01} =$ midpoint $a_{00}a_{02}$ and $a_{11} =$ midpoint $a_{01}T_y(a_{01})$. The set of vertices of the tessellation by rectangles referred to in Proposition (4.1) equals $\{T_y^iT_z^j a_{k\ell} \mid i, j \in Z\ k, \ell \in \{0, 1\}\}$.

Suppose $\ell$ is the axis of rotation of $B$ and $\ell$ lies in plane $\mathcal{P}$, is parallel to the $y$–axis and intersects the rectangle $R = \{a_{jk}, a_{j+1k}, a_{jk+1}, a_{j+1k+1}\}$, where axis point $a_{jk}$ corresponds to rotation $A_{jk}$, etc. Then $\ell$ cannot contain the vertices of $R$ as axes of rotation of distinct elements of $\widehat{U}$ don't intersect. And $\ell$ must divide $R$ exactly in half for if $\ell$ lay closer to $a_{jk}$ than to $a_{j+1k}$ the element $A_{jk}(BA_{jk}B^{-1})$ of $\widehat{G}$ would be a translation in the $y$–direction by a distance less than $a_{jk}a_{jk+2}$ contradicting the minimality in the choice of $T_y$. The set of translates of the axes $\ell$ and $A_{jk}(\ell)$ divide every rectangle of the tessellation in half. We must show there are no axis points in $\mathcal{P}$ not of the form $a_{jk}$. Suppose $x$ was such a point corresponding to rotation $X$ and lying in rectangle $R = \{a_{jk}, a_{jk+1}, a_{j+1k}, a_{j+1k+1}\}$. Then $x$ cannot lie on the sides of the rectangle. (For example, if $x$ lay on $a_{jk}a_{jk+1}$, $XA_{yk}X^{-1}A_{jk}$ would be a translation in the $y$ direction by less than length $a_{jk}a_{jk+2}$ contradicting the minimality in the choice of $T_y$.) And $x$ cannot lie on $\ell$. As $x$ belongs to the interior of the rectangle and not on $\ell$, $X(BXB^{-1})$ is a translation in the $y$–direction by a distance less than $a_{jk}a_{jk+2}$ which is impossible. □

The next problem is to construct a fundamental domain for $\widehat{G}$. With this in mind select a plane $\mathcal{P}$ parallel to the $yz$ plane containing an axis $\ell$ in $\widehat{G}$ that is parallel to the $y$–axis. Recall that axes in $\widehat{G}$ parallel to the $x$, $y$, or $z$ axis have parametric equations $(t$, even, odd), (odd, $t$, even) or (even, odd, $t)$ respectively. Thus plane $P$ has equation $x = O$ where $O$ is odd. Define the rectangle $R_1$ in $\mathcal{P}$, as pictured in Figure 7, bounded on one side by $\ell$ with parametric equation $(O, t, e_1)$ with $e_1$ even and having the opposite two vertices be axis points for $\mathcal{P}$ with coordinates $(O, E, o_1)$ and $(O, E + 4n, o_1)$ with $o_1$ odd.

There is a rectangle theorem analogous to Proposition (4.1) but with $x$ substituted for $y$. Let $\mathcal{Q}$ be the plane $y = E$ which contains the $x$ axis from $\widehat{G}$

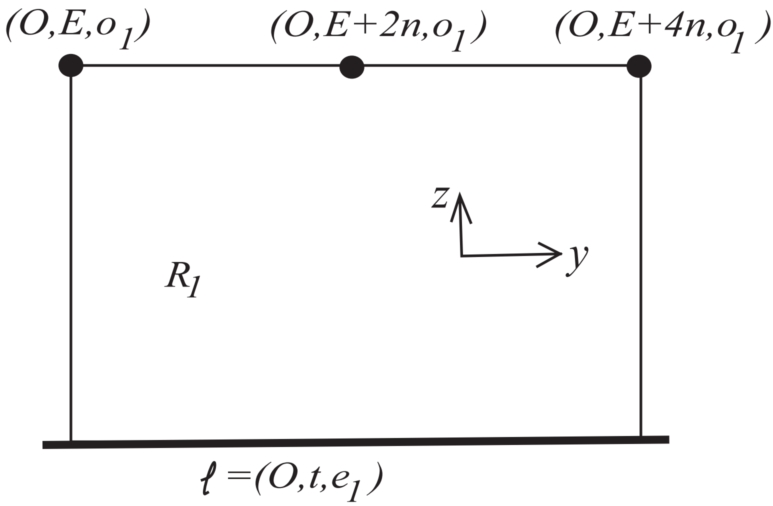


**Figure 7**

with equation $(t, E, o_1)$. Then $\mathcal{Q}$ also is tesselated by rectangles and we define $R_2$ to be the rectangle pictured in Figure 8. Like $R_1$, the rectangle $R_2$ is not part of the tessellation but is formed by gluing two half–rectangles from the tessellation. $R_2$ is bounded on one side by axis $(t, E, o_1)$ and the two vertices of $R_2$ opposite the axis have coordinates $(O, E, e_1)$ and $(O + 4m, E, e_1)$.

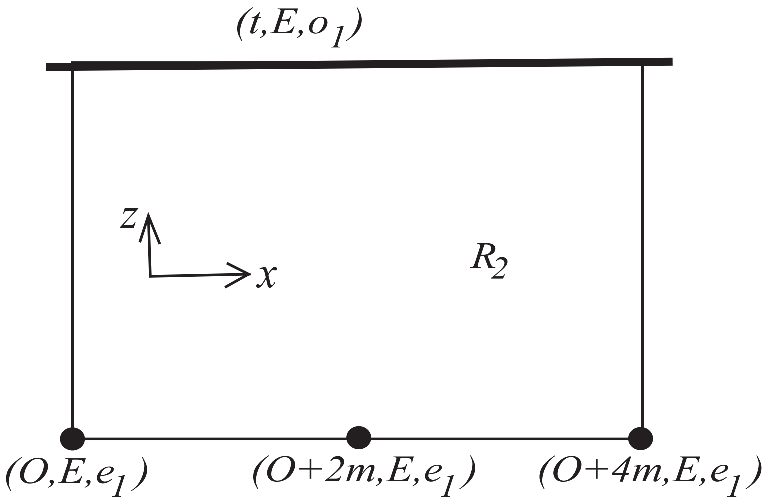


**Figure 8**

Let $BOX$ be that parallelepiped whose projection on planes $\mathcal{P}$ and $\mathcal{Q}$ is rectangles $R_1$ and $R_2$, respectively; i.e.,

$$BOX = \{(x, y, z) \mid O \leqq x \leqq O + 4m, E \leqq y \leqq E + 4n, e_1 \leqq z \leqq o_1\}.$$

So the dimensions of $BOX$ are $4m \times 4n \times o$ where $o = e_1 - o_1$ is odd. We assert $BOX$ is a fundamental domain for $\widehat{G}$.

There is a tessellation of $E^3$ obtained by translating $BOX$ around using translations by $4m$, $4n$, and $o$ in the $x$, $y$, and $z$ directions, respectively. One observes, from the rectangle theorems, that the rotations in $\widehat{G}$, which generate $\widehat{G}$, leave this tessellation invariant. Also $\widehat{G}$ contains translations by $4m$, $4n$, and $4o$ in the $x$, $y$, and $z$ directions, respectively. Using these translations and the rotations which split the faces of $BOX$ we see that any point in $E^3$ is equivalent to a point in BOX. If two points in interior of $BOX$ are equivalent then there is a non–trivial element $\widehat{g}$ of $\widehat{G}$ that leaves $BOX$ invariant. By the Brouwer fixed point theorem, $\widehat{g}$ has a fixed point in $BOX$ and therefore must be a rotation whose axis intersects BOX. Inspecting rectangles $R_1$ and $R_2$ we see that this is impossible. Thus $BOX$ is a fundamental domain for $\widehat{G}$.

We can conjugate $\widehat{G}$ by an element $\widehat{u}$ of $\widehat{U}$ and obtain an equivalent subgroup of $\widehat{U}$. This has the effect of replacing $BOX$ by $\widehat{u}(BOX)$. As $\widehat{U}$ contains translations by 4 in the $x$, $y$, and $z$ directions we may assume without loss of generality that $BOX = \{(x, y, z) \mid \widehat{O} - 2m \leqq x \leqq \widehat{O} + 2m, \widehat{E} - 2n \leqq y \leqq \widehat{E} + 2n, \widehat{e}_1 \leqq z \leqq \widehat{o}_1\}$ where $\widehat{O} = \pm 1$, $\widehat{E} = 0$ or $2$, $\widehat{e}_1 = 0$ or $2$ and $o = \widehat{o}_1 - \widehat{e}_1$. The rotations $\widehat{a}$, $\widehat{b}$, and $\widehat{c}$ of $\widehat{U}$ are given by equations $(x, y, z) \longrightarrow (x, -y, -z + 2)$, $(x, y, z) \longrightarrow (-x + 2, y, -z)$, $(x, y, z) \longrightarrow (-x, -y + 2, z)$ respectively. So applying $\widehat{a}$, $\widehat{b}$, or $\widehat{c}$ if need be we can assume $\widehat{O} = 1$, $\widehat{E} = 0$ and $\widehat{e}_1 = 0$. But then $BOX = Box(G(m, n, o))$ which implies $\widehat{G} = \widehat{G}(m, n, o)$. We have shown that any finite index subgroup of $\widehat{U}$ generated by rotations that contains rotations with axes in only two of the three possible directions is equivalent to a $\widehat{G}(m, n, o)$.

Now suppose $\widehat{G}$ contains rotations with axes parallel to the $x$, $y$, and $z$ directions. For each choice of an ordered pair from the set $\{x$-axis, $y$-axis, $z$-axis$\}$ to play the role of $y$–axis and $z$-axis in Proposition (4.1) we get a rectangle theorem. We don't formally state each of the six propositions but we use the results to get tessellations of planes by rectangles in order to construct a parallelepiped, again called $BOX$, which will turn out to be a fundamental domain for $\widehat{G}$.

Let $\mathcal{P}$ (resp. $\mathcal{Q}$, $\mathcal{R}$) be a plane parallel to the $xy$ (resp. $xz$, $yz$) plane containing an axis $a_x = (t$, even, odd) (resp. $a_z =$(even, odd,$t$), $a_y =$(odd, $t$, even)) parallel to the $x$ (resp. $z$, $y$) axis. Then planes $\mathcal{P}$, $\mathcal{Q}$ , and $\mathcal{R}$ intersect in a point $X = (o_1, o_2, o_3)$ with all odd coordinates. (For example, plane $\mathcal{P}$ contains axis $a_x = (t$,even, odd) and $P$ is parallel to the $xy$ plane and so has equation $z =$ odd.) No point with all odd coordinates belongs to an axis of rotation in $\widehat{U}$.

Consider the tessellation of plane $\mathcal{P}$ by rectangles. Planes $\mathcal{P}$ and $\mathcal{Q}$ intersect in a line $\ell$ (see Figure 9) parallel to the $x$–axis and planes $\mathcal{P}$ and $\mathcal{R}$ intersect in a line $m$ parallel to the $y$–axis. As $\mathcal{Q}$ contains axes from $\widehat{G}$ parallel to the $z$–axis line $\ell$ contains $z$–axis points that are vertices of the tessellation by rectangles. We already know that the axes in $\mathcal{P}$ parallel to the $x$–axis evenly divide the rectangles but the line $m$ which is parallel to the $y$–axis also evenly divides rectangles. To see this translate $\mathcal{P}$ in the $z$–direction to a plane $\widetilde{\mathcal{P}}$ that contains an axis from $\widehat{G}$ that is parallel to the $y$–axis. This translation, in the

$z$–direction, takes vertices of the tessellation of $\mathcal{P}$ by rectangles to vertices of the tessellation of $\widetilde{\mathcal{P}}$ by rectangles, leaves plane $\mathcal{R}$ invariant and sends line $m$ to an axis in $\widehat{G}$ parallel to the $y$–axis that evenly divides a rectangle in $\widetilde{\mathcal{P}}$. Therefore $m$ evenly divides a rectangle of the tessellation of $\mathcal{P}$.

The tessellations of planes $\mathcal{Q}$ and $\mathcal{R}$ by rectangles is also displayed in Figure 9. Planes $\mathcal{Q}$ and $\mathcal{R}$ intersect in line $n$ parallel to the $z$–axis.

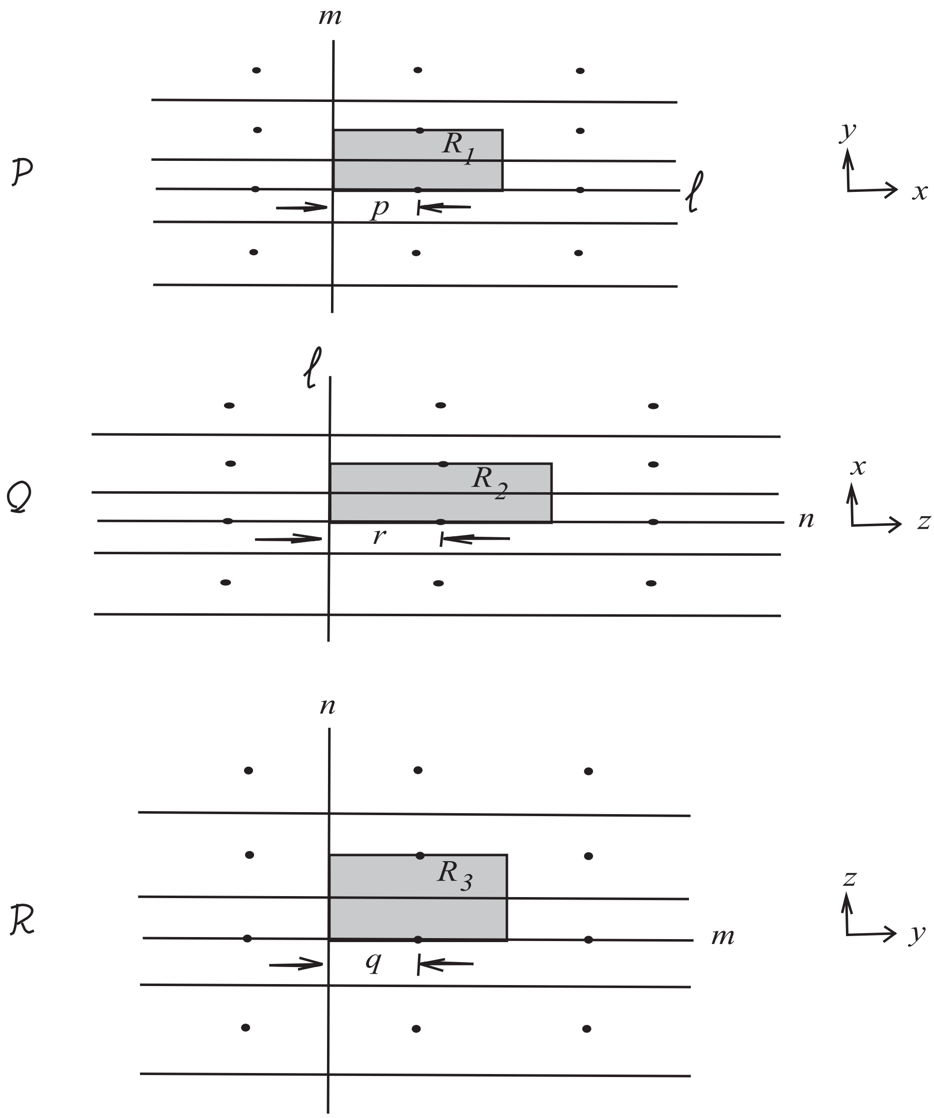


**Figure 9**

The distance from point $X = (o_1, o_2, o_3)$ to the nearest axis in $\widehat{G}$ parallel to the $x$ (resp. $y$, $z$) axis is $q$ (resp. $r$, $p$) as displayed in Figure 9. That $p$, $q$, and $r$ are odd integers can be seen from the parameterizations of the axes in $\widehat{U}$.

Then $BOX$ is defined to be $\{(x, y, z) \mid o_1 \leqq x \leqq o_1 + 2p; o_2 \leqq y \leqq o_2 + 2q; o_3 \leqq z \leqq o_3 + 2r\}$. The projections of $BOX$ on planes $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ are the rectangles $R_1$, $R_2$, $R_3$ displayed in Figure 9. We assert that $BOX$ is a fundamental domain for $\widehat{G}$. Using translations by $4p$, $4q$, $4r$ in the $x$, $y$, and $z$ directions, which are contained in $\widehat{G}$ together with rotations, giving rise to the vertices of the rectangles displayed in Figure 9, we see that any point in $E^3$ can be moved to a point in $BOX$. There is a tessellation of $E^3$ obtained by translating $BOX$ around using translations of $2p$, $2q$, and $2r$ in the $x$, $y$, and $z$ directions. From Figure 9 we see that the rotations in $\widehat{G}$ preserve this tessellation so that $\widehat{G}$ preserves this tessellation. If two points in the interior of $BOX$ are equivalent, say $g(m_1) = m_2$, then $g$ preserves $BOX$, has a fixed point by the Brouwer Theorem and so must be a rotation in $\widehat{G}$ which is impossible.

Therefore $BOX$ is a fundamental domain for $\widehat{G}$. As before we can conjugate by an element of $\widehat{u}$ of $\widehat{U}$ or $\widehat{S}$ which has the effect of replacing $\widehat{G}$ by an equivalent group and $BOX$ by the new fundamental domain $\widehat{u}$(BOX). Since the center of $BOX$ has all even coordinates $(o_1 + p, o_2 + q, o_3 + r)$ we can find an element $\widehat{u}$ which is a product of translations by 4 in the $x$, $y$, and $z$ directions and rotations $\widehat{a}$ and/or $\widehat{b}$ and/or $\widehat{c}$ such that $\widehat{u}$(BOX) is centered at the origin. Finally we can use the 120 degree rotation $D$ in $\widehat{S}$ to cyclically permute $p$, $q$, $r$ so that $p \leqq \max\{q, r\}$ and in the case where $p$, $q$, and $r$ are not all different $p \leqq q \leqq r$. Thus the group to which $\widehat{G}$ is equivalent is $\widehat{H}(p, q, r)$ which has $\widehat{u}$(BOX) as its fundamental domain. We summarize all this as a theorem.

THEOREM (4.2). 1. *Let $\widehat{G}$ be an even index subgroup of $\widehat{U}$ generated by rotations. Then $\widehat{G}$ is equivalent to a unique group in the family $\widehat{G}(m, n, o)$ and $\left[\widehat{U} : \widehat{G}\right] = 2mno$. $\widehat{G}$ contains axes of rotation in two of the three directions x, y, and z. The integers $2m$ (resp. $2n$) represents the distance between adjacent axes of $\widehat{G}(m, n, o)$ that lie in a plane parallel to the xy plane and are parallel to the y–axis (resp. x axis). The odd integer o represents this distance between axes of $\widehat{G}$ that are not parallel but are as close as possible.*

2. *Let $\widehat{G}$ be an odd index subgroup of $\widehat{U}$ generated by rotations. Then $\widehat{G}$ is equivalent to a unique group in the family $\widehat{H}(p, q, r)$ and $\left[\widehat{U} : \widehat{G}\right] = pqr$. Group $\widehat{G}$ contains rotations with axes parallel to each of the three possible directions x, y, and z.*

For each pair of directions $x$ and $y$, $x$ and $z$, $y$ and $z$ there is a distance between a pair of axes in these directions that are not parallel but are as close as possible giving rise to a triple of integers. This triple of integers is $p$, $q$, and $r$, not necessarily in that order.

In the next section, we begin the study of finite index subgroups of $U$ that are generated by rotations.

## 5. Finite index subgroups of $U$ generated by rotations

PROPOSITION (5.1). *Let $\widehat{G}$ be a finite index subgroup of $\widehat{U}$ generated by rotations and let $G = \varphi^{-1}(\widehat{G})$ be the full preimage of G under the homomorphism $\varphi \colon U \to \widehat{U}$ defined in Section 2. Then G is generated by rotations.*

*Proof.* The homomorphism $\varphi\colon G \to \widehat{G}$ defined in Section 2 is surjective, and sends 90 degree rotations in $U$ to 180 degree rotations in $\widehat{U}$. By the classification of the $\widehat{G}$ in Section 4, $\widehat{G}$ is generated by 3 or 4 rotations. Let $S$ be a set of 90 degree rotations in $U$ that is sent to a set of generators for $\widehat{G}$ and let $G_1$ be the subgroup of $U$ generated by $S$. Then $\varphi^{-1}(\widehat{G}) = G_1K$. Since $K$ is generated by rotations (Theorem (3.2)), so is $G_1K$. □

The main theorem now follows easily from Proposition (5.1).

THEOREM (5.2). *Given any positive integer $n$ there is a subgroup $G$ of $U$ of index $n$ that is generated by rotations.*

*Proof.* Let $\widehat{G}$ be a subgroup of $\widehat{U}$ generated by rotations of index $n$ in $\widehat{U}$, which exists by the classification of such subgroups of Section 4. And let $G = \varphi^{-1}(\widehat{G})$. Then $G$ is generated by rotations by Proposition (5.1) and $[U : G] = [\widehat{U} : \widehat{G}] = n$. □

Any axis of rotation $\ell$ in $U$ is the image of the axis of rotation of one of the generators $a$, $b$, $c$ of $U$ under the action of an element $u$ of $U$. This follows from the fact that $D_0$, a dodecahedral fundamental domain of $U$, intersects six axes of rotation in $U$, those of $a$, $b$, $c$, $c^{-1}ac$, $a^{-1}ba$, and $b^{-1}cb$, and if $D$ is any dodecahedron of the tessellation of $H^3$ intersecting $\ell$ there is an element $u_1$ of $U$ such that $u_1(D_0) = D$. Then $u = u_1x^{-1}$ where $x$ is one of $a$, $b$, $c$. Letting $U$ act on the axes of rotation, we get exactly three orbits. (At most three by the argument above and at least three because $\varphi\colon U \longrightarrow \widehat{U}$ preserves orbits and there are three orbits in $\widehat{U}$, those parallel to the $x$, $y$ and $z$ axes.) Thus there are nine conjugacy classes of rotations in $U$ represented by $a$, $a^2$, $a^3$, $b$, $b^2$, $b^3$, $c$, $c^2$, and $c^3$. (This can also be seen by computing $U/[U,U] \cong Z_4 \oplus Z_4 \oplus Z_4$ from the presentation of $U$ in Section 2. For example $a$ is sent to $(1, 0, 0)$, etc.) Similarly there are three conjugacy classes of rotations in $\widehat{U}$ represented by $\widehat{a}$, $\widehat{b}$, and $\widehat{c}$.

THEOREM (5.3). *Let $G$ be a subgroup of $U$ of odd index and generated by rotations. Then $G$ contains a member of each of the nine conjugacy classes of rotations in $U$.*

*Proof.* Let $\widehat{G} = \varphi(G)$ where $\varphi\colon U \longrightarrow \widehat{U}$ is the homomorphism of Section 2 and $K = \ker\varphi$. Then $G \subset GK \subset U$ so that $[U : G] = [U : GK][GK : G]$. But $\varphi : U \longrightarrow \widehat{U}$ induces $\varphi\colon GK \longrightarrow \widehat{G}$ so that $[U : GK] = [\widehat{U} : \widehat{G}]$ and $[U : G] = [\widehat{U} : \widehat{G}] \cdot [GK : G]$. Since $[U : G]$ is odd it follows that $[\widehat{U} : \widehat{G}]$ is odd and thus $\widehat{G}$ contains a member of each of the three conjugacy classes of rotations in $\widehat{U}$ from the classification in Section 4.

We shall show that $G$ contains a member of the conjugacy class of $c$. Let $\widehat{c}_1$ be a rotation in $\widehat{G}$ with axis parallel to the $z$–axis. Suppose $\varphi(g) = \widehat{c}_1$. Then $g$ is a product of rotations, $g = \prod_{i=1}^{n} r_i$, as $G$ is generated by rotations. If $\{r_1, \ldots, r_n\}$ contains a rotation conjugate to $c$ or $c^3$ we are done. Suppose this is not the case. Then $\widehat{c}_1 = \prod_{i=1}^{n} \widehat{r}_i$ where $\widehat{r}_i$ is either the identity or a rotation about an axis parallel to the $x$ or $y$ axes. Each $\widehat{r}_i$ belongs to the group $\widehat{G}(1,1,1)$ defined in Section 4 as that group contains every rotation in $\widehat{U}$ about an axis parallel to the $x$ or $y$ axis. Thus $\widehat{c}_1 \in \widehat{G}(1,1,1)$ but this is impossible as $\widehat{G}(1,1,1)$ contains no rotations with axis parallel to the $z$–axis. Therefore $G$ contains a member of the conjugacy class of $c$.

The two conjugates $D\widehat{G}(1,1,1)D^{-1}$ and $D^2\widehat{G}(1,1,1)D^{-2}$, where $D$ is 120 degree rotation about axis $(t,t,t)$ introduced in Section 4, contain all rotations parallel to the $x$ and $z$ axis and no rotation parallel to the $y$ axis or all rotations parallel to the $y$ and $z$ axes and no rotations parallel to the $x$ axis. We can show that $G$ contains rotations in the conjugacy class of $a$ and $b$ by duplicating the argument for $c$ by replacing $\widehat{G}(1,1,1)$ by $D\widehat{G}(1,1,1)D^{-1}$ or $D^2\widehat{G}(1,1,1)D^{-2}$. $\square$

If $G$ is a finite index subgroup of $U$ that is generated by rotations it is clear that information about the precise placement of $\widehat{G}$ in the classification of Section 4 implies much about group $G$ itself. There are other theorems analogous to Theorems (5.2) and (5.3), but clumsier to state or prove that we could present. We refrain from doing so, so as not to lengthen this paper.

We close by posing a question. If $G$ is a subgroup of $U$ of index $n$, either generated by rotations or not, it is clear that $G$ has a fundamental domain that is a union of $n$ of the dodecahedra in the tessellation associated to $U$. Does $G$ have a fundamental domain that is convex and also the union of $n$ dodecahedra?

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# CLASSIFYING COMBINATORIAL 4–MANIFOLDS UP TO COMPLEXITY

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ABSTRACT. The goal of this paper is to give some theorems which relate to the problem of classifying smooth 4–manifolds up to piecewise -linear (PL) homeomorphism. For this, we use the combinatorial approach to the topology of PL manifolds by means of a special kind of edge–colored graphs, called *crystallizations*. Within this representation theory, Bracho and Montejano introduced in 1987 a nonnegative numerical invariant, called the *reduced complexity*, for any closed $n$–dimensional PL manifold. Here we obtain the complete classification of all closed connected smooth 4–manifolds of reduced complexity less than or equal to 14.

## 1. Definitions and statements

All spaces and maps will be considered in the *PL* category, for which we refer to [24]. The main definitions and results of graph theory can be found in [15]. Survey papers on the representation of PL manifolds by means of edge–colored graphs and crystallizations are, for example, [1], [6], [7], and [26]. Here we briefly recall the necessary definitions to explain the statements of our theorems. An $(n+1)$*–colored graph* $(G, c)$ is a multigraph $G = (V(G), E(G))$, regular of degree $n+1$ (possibly with multiple edges, but without loops), together with a proper edge-coloring $c\colon E(G) \to \Delta_n = \{i \in \mathbb{Z}\colon 0 \leq i \leq n\}$ by $n+1$ colors. This means that any two adjacent edges in $G$ are differently colored. As usual, $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of $G$, respectively; $\Delta_n$ will be called the *color set*, and its elements *the colors*. The cellular complex $K(G)$ *associated* to $G$ is constructed as follows. For each vertex $v$ of $G$, consider a standard $n$–simplex $\sigma^n(v)$, and label its $n+1$ vertices by the colors of $\Delta_n$. If $v$ and $w$ are joined in $G$ by an $i$–colored edge, then identify the $(n-1)$–faces of $\sigma^n(v)$ and $\sigma^n(w)$ opposite to the vertex labelled by $i \in \Delta_n$, so that equally labelled vertices coincide. The complex $K(G)$ is not a classical simplicial complex for two simplexes may meet in more than a single face. On the other hand, it is a *pseudocomplex* in the sense of [16], p.49. This means that any simplex of $K(G)$ is canonically isomorphic to a standard one, and the intersection of two simplexes can be either empty or a union of common faces. By construction, the graph $G$ can be thought as the 1–skeleton of the dual cellular complex of $K(G)$. Let now $M^n$ be a closed connected PL $n$–manifold. We say that $(G, c)$ *represents* $M$ if $M$ is PL homeomorphic to the space underlying $K(G)$. A *crystallization* of $M$ is an $(n+1)$–colored graph $(G, c)$ representing $M$ such that $K(G)$ has exactly $n+1$ vertices (which we shall always assume to be colored by

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$\Delta_n$). In this case, $K(G)$ is called a *contracted triangulation* of $M$. A theorem of Pezzana [22], [23] states that every closed connected PL $n$–manifold admits a crystallization (hence, it can be triangulated by a contracted pseudocomplex). Following [1], we define the *complexity* $\mathbf{c}(M)$ of $M$ as the minimum number of $n$–simplexes which a contracted triangulation of $M$ must have. In other words, $\mathbf{c}(M)$ is the minimum order of a crystallization of $M$, that is,

$$\mathbf{c}(M) = \min\{\text{card } V(G)\colon (G, c) \text{ is a crystallization of } M\}.$$

Since one always has at least two $n$–simplexes (i.e., any crystallization has at least two vertices), it was defined in [1] the *reduced complexity* of $M$ as

$$\widetilde{\mathbf{c}}(M) = \mathbf{c}(M) - 2.$$

The manifold invariant $\widetilde{\mathbf{c}}$ gives a finite-to-one map from the class of closed connected PL $n$-manifolds to the set of nonnegative even integers.

THEOREM (1.1). *The only n-manifold of reduced complexity zero is the standard n-sphere* $\mathbb{S}^n$.

*Proof.* The $n$-sphere $\mathbb{S}^n$ can be represented by the simplest $(n+1)$-colored graph which consists of two vertices joined by $n+1$ differently colored edges. □

Theorem 3.13 of [1] says that $\widetilde{\mathbf{c}}(M) = 4 - 2\chi(M)$, for every closed connected surface $M$. Thus the reduced complexity can be regarded as a generalization of the Euler characteristic, preserving its nice property of classifying manifolds up to a finite ambiguity. Some results on the complexity of closed connected 3-manifolds can be found in [5]. The classification of all closed 3–manifolds with complexity $\leq 28$ was given in [19], Chapter 5, by using a computer algorithm. In the present paper we obtain the complete classification of all closed connected PL 4-manifolds with reduced complexity $\leq 14$. This gives new combinatorial characterizations of $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^3$ (the twisted $\mathbb{S}^3$–bundle over $\mathbb{S}^1$), $\mathbb{C}P^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, and $\mathbb{R}P^4$ among closed 4-manifolds (compare with other characterizations obtained in [3] and [4] by using the concept of regular genus).

MAIN THEOREM. (a) *There are no closed connected 4-manifolds M of reduced complexity* $0 < \widetilde{\mathbf{c}}(M) < 6$. *The unique closed connected 4–manifold of reduced complexity 6 is the complex projective plane* $\mathbb{C}P^2$.

(b) *Let* $M^4$ *be a closed connected 4–manifold. If* $\widetilde{\mathbf{c}}(M) = 8$*, then M is (PL) homeomorphic to either* $\mathbb{S}^1 \times \mathbb{S}^3$ *or* $\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^3$. *There are no closed connected 4-manifolds of reduced complexity 10.*

(c) *The unique closed connected prime 4–manifold of reduced complexity 12 is the topological product* $\mathbb{S}^2 \times \mathbb{S}^2$.

(d) *The unique closed connected prime 4–manifold of reduced complexity 14 is the real projective 4–space* $\mathbb{R}P^4$.

The formulae, used in the proof of the main theorem, imply the following result.

THEOREM (1.2). *Let $M^4$ be a closed connected PL 4–manifold of reduced complexity $\widetilde{\mathbf{c}}(M)$. Then we have*

$$\widetilde{\mathbf{c}}(M) \geq 6\chi(M) + 20\,\mathrm{rk}(M) - 12$$

*where* $\mathrm{rk}(M)$ *denotes the rank of the fundamental group of $M$.*
*If $M$ is simply–connected, then*

$$\widetilde{\mathbf{c}}(M) \geq 6b_2(M)$$

*where $b_2(M)$ denotes the second Betti number of $M$.*

For example, if $T_g$ is the orientable connected surface of genus $g \geq 0$, then $\widetilde{\mathbf{c}}(T_g \times \mathbb{S}^2) \geq 12 + 16g$ and $\widetilde{\mathbf{c}}(T_g \times T_h) \geq 24gh + 16(g+h) + 12$. We conjecture that the equalities hold for such cases (this is true, for example, for $g = h = 0$).

Using Theorem (1.2) and the subadditivity of the reduced complexity (see [1], Theorem 3.9) we get the following consequence.

PROPOSITION (1.3). *For every nonnegative integers $h, k, \ell$, we have*

$$\widetilde{\mathbf{c}}(h(\mathbb{S}^2 \times \mathbb{S}^2)\#k(\pm\mathbb{C}P^2)\#\ell(\mathbb{S}^1 \otimes \mathbb{S}^3)) = 12h + 6k + 8\ell$$

*where $pM$ denotes the connected sum of $p$ copies of the manifold $M$, and the symbol $\mathbb{S}^1 \otimes \mathbb{S}^3$ means either $\mathbb{S}^1 \times \mathbb{S}^3$ or $\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^3$.*

Applying the Freedman classification [9] of simply-connected PL 4-manifolds, up to topological homeomorphism, we obtain another consequence of Theorem (1.2)

PROPOSITION (1.4). *If $M^4$ is a closed simply-connected PL 4-manifold of reduced complexity $\widetilde{\mathbf{c}}(M) \leq 90$, then $M$ is topologically homeomorphic to either $r(\pm\mathbb{C}P^2)$ or $r(\mathbb{S}^2 \times \mathbb{S}^2)$, where $r = b_2(M)$.*

Other applications of these results and some new conjectures, which are related with the 4–dimensional PL Poincaré Conjecture, complete the paper.

## 2. The combinatorics of contracted triangulations

This section is devoted to present some combinatorial relations arising from a contracted triangulation (and a crystallization) of a closed 4-manifold. Some of these formulae were first obtained in [3], [4] and [11], but we prefer to include them (in some cases, with slightly different proofs) to make the reading of the paper self-contained. First we fix some notations which will be used here and in the next sections. Let $M$ be a closed connected 4–manifold, $(G, c)$ a crystallization of $M$ (with color set $\Delta_4 = \{0, 1, 2, 3, 4\}$), and $K = K(G)$ the associated contracted triangulation of $M$. If $\Gamma = \{i, j\} \subset \Delta_4$ (resp. $\Gamma = \{r, s, t\} \subset \Delta_4$), then $g_{ij}$ (resp. $g_{rst}$) represents the number of connected components of the partial subgraph $G_\Gamma = (V(G), c^{-1}(\Gamma))$. Let $p$ denote the *order* of $G$, i.e., the number of vertices in the graph. We always assume that $\{v_i : i \in \Delta_4\}$ is the vertex set of $K$, and that $v_i$ corresponds to $G_{\widehat{i}}$, where $\widehat{i} = \Delta_4 \setminus \{i\}$. Let $q_h(K)$ denote the number of $h$–simplexes in $K$. If $\{i, j\} = \Delta_4 \setminus \{r, s, t\}$, then $K(i, j)$ (resp. $K(r, s, t)$) denotes the subcomplex of $K$ generated by the vertices $v_i$ and $v_j$ (resp. $v_r$, $v_s$ and $v_t$). Furthermore, let $q_h(i, j)$ (resp. $q_h(r, s, t)$) be the number of $h$–simplexes of $K$ containing $v_i$ and $v_j$ (resp. $v_r$, $v_s$ and $v_t$) as their vertices.

Let $N = N(i,j)$ and $N' = N(r,s,t)$ be regular neighborhoods (in $M$) of the polyhedrons underlying $K(i,j)$ and $K(r,s,t)$, respectively, such that $M$ decomposes as $M = N \cup N'$ and $\partial N = \partial N' = N \cap N'$. Of course, $N$ is a handlebody (hence $\partial N$ is a connected sum of copies of $\mathbb{S}^1 \otimes \mathbb{S}^2$), and $N'$ collapses onto a 2–dimensional complex.

LEMMA (2.1). *Let $(G,c)$ be a crystallization of a closed connected 4–manifold $M$, and $K = K(G)$ the associated contracted triangulation. With the above notation, we have:*

(a)
$$q_0(K) = 5 \qquad q_1(K) = \sum_{r<s<t} g_{rst} \qquad q_2(K) = \sum_{i<j} g_{ij}$$
$$q_3(K) = \frac{5}{2}p \qquad q_4(K) = p$$

(b)
$$\begin{array}{ll} q_1(i,j) = g_{rst} & q_2(r,s,t) = g_{ij} \\ q_2(i,j) = g_{rs} + g_{rt} + g_{st} & q_3(r,s,t) = p \\ q_3(i,j) = \frac{3}{2}p & q_4(r,s,t) = p \\ q_4(i,j) = p & \end{array}$$

*where $\{i,j\} = \Delta_4 \setminus \{r,s,t\}$.*

(c)
$$rk(M) \leq \min\{g_{rst} - 1 \colon \{r,s,t\} \subset \Delta_4\}$$
$$\chi(M) = 5 - \sum_{r<s<t} g_{rst} + \sum_{i<j} g_{ij} - \frac{3}{2}p$$

*Here the summations are taken over all pairs $(i,j)$ (resp. triples $(r,s,t)$) of distinct elements in the color set $\Delta_4 = \{0,1,2,3,4\}$.*

*Proof.* For each subset $\Gamma \subset \Delta_4$ with cardinality $h \leq 4$, there is a bijection between the set of connected components of the partial subgraph $G_\Gamma$ and the set of $(4-h)$–simplexes of $K$ whose vertices are labelled by the colors of $\Delta_4 \setminus \Gamma$. This bijection reverses inclusion. So the number of edges (resp. triangles) in $K$ equals the number of 3–colored (resp. 2–colored) connected components in $G$. Furthermore, we have

$$q_2(i,j) = q_2(i,j,r) + q_2(i,j,s) + q_2(i,j,t) = g_{st} + g_{rt} + g_{rs}$$
$$q_3(i,j) = q_3(i,j,r,s) + q_3(i,j,r,t) + q_3(i,j,s,t) = \frac{3}{2}p$$

and

$$q_3(r,s,t) = q_3(r,s,t,i) + q_3(r,s,t,j) = p.$$

The upper bound for the rank of $\pi_1(M)$ follows from [10], where it was described how to deduce a finite presentation of $\pi_1(M)$ from a crystallization of $M$. The procedure can be synthetized as follows. Choose two colors in $\Delta_4$, $i$ and $j$ say, and let $X = \{x_1, \ldots, x_n\}$ be the set of all connected components, but one, of the partial subgraph of $G$ obtained by deleting them (the missing component can be chosen arbitrarily). Then $X$ is a set of generators for $\pi_1(M)$, where $n = g_{rst} - 1$ and $\{r,s,t\} = \Delta_4 \setminus \{i,j\}$. The connected components of the complementary 2–subgraph are simple cycles, whose edges are alternatively colored by $i$ and $j$.

From these components, one can read off the relators (expressed as products of the generators in $X$ and of their inverses) for a finite presentation of $\pi_1(M)$. □

LEMMA (2.2). *Let $(G, c)$ be a crystallization of a closed connected 4–manifold $M$. For each triple $(r, s, t)$ of distinct elements of $\Delta_4$, we have*

$$2g_{rst} = g_{st} + g_{rt} + g_{rs} - \frac{p}{2}$$

*As a consequence, we obtain*

$$2\sum_{r<s<t} g_{rst} = 3\sum_{i<j} g_{ij} - 5p$$

*and*

$$\chi(M) = 5 - \frac{1}{2}\sum_{i<j} g_{ij} + p = 5 - \frac{1}{3}\sum_{r<s<t} g_{rst} + \frac{p}{6}$$

*where the summations are taken over all pairs $(i, j)$ (resp. triples $(r, s, t)$) of distinct elements of $\Delta_4$.*

*Proof.* Let $K$ be the contracted triangulation of $M$ represented by $G$. If $\sigma$ is a simplex of $K$, then the *disjoined star* $\mathrm{std}(\sigma, K)$ is defined as the disjoint union of the 4–simplexes of $K$ containing $\sigma$ with re-identification of the tetrahedra containing $\sigma$ and of their faces. The *disjoined link* $\mathrm{lkd}(\sigma, K)$ is the subcomplex of $\mathrm{std}(\sigma, K)$ formed by all simplexes which do not intersect $\sigma$. Let $e$ be an arbitrary edge of $K(i, j)$. Then the Euler characteristic $\chi(e)$ of $\mathrm{lkd}(e, K)$ is given by $2 = \chi(e) = q_2(e) - q_3(e) + q_4(e)$, where $q_h(e)$ is the number of $h$–simplexes of $K$ containing $e$ as their face. Lemma (2.1) and summation over the edges of $K(i, j)$ give

$$\begin{aligned} 2q_1(i, j) &= q_2(i, j) - q_3(i, j) + q_4(i, j) \\ &= g_{st} + g_{rt} + g_{rs} - \frac{p}{2}. \end{aligned}$$

The proofs of the two consequences in the statement are immediate. □

LEMMA (2.3). *Let $(G, c)$ be a crystallization of a closed connected 4–manifold $M$. For every color $i \in \Delta_4$, we have*

$$\sum_{h,k\neq i} g_{hk} = \sum_{r,s,t\neq i} g_{rst} + p$$

*where the first (second) summation is taken over all pairs (triples) of distinct elements of $\widehat{i} = \Delta_4 \setminus \{i\}$. A further summation over $i \in \Delta_4$ gives again the second formula in Lemma (2.2).*

*Proof.* Let $K$ be the contracted triangulation of $M$ represented by $G$, and $v_0$ a vertex of $K$ (fix, for example, $i = 0 \in \Delta_4$). Then the Euler characteristic $\chi(v_0)$ of $\mathrm{lkd}(v_0, K)$ is given by

$$0 = \chi(v_0) = q_1(v_0) - q_2(v_0) + q_3(v_0) - q_4(v_0)$$

where $q_h(v_0)$ is the number of $h$–simplexes of $K$ containing $v_0$ as their vertex. Now we have (use Lemma (2.1)):

$$q_1(v_0) = q_1(0, 1) + q_1(0, 2) + q_1(0, 3) + q_1(0, 4) = g_{234} + g_{134} + g_{124} + g_{123}$$

$$\begin{aligned} q_2(v_0) &= q_2(0,1,2) + q_2(0,1,3) + q_2(0,1,4) + q_2(0,2,3) + q_2(0,2,4) + q_2(0,3,4) \\ &= g_{34} + g_{24} + g_{23} + g_{14} + g_{13} + g_{12} \end{aligned}$$

and

$$q_3(v_0) = 2p \qquad\qquad q_4(v_0) = p.$$

Substituting these relations in the above expression of $\chi(v_0)$ gives

$$g_{12} + g_{13} + g_{14} + g_{23} + g_{24} + g_{34} = g_{123} + g_{124} + g_{134} + g_{234} + p$$

which is the formula of the statement for $i = 0$. □

*Proof of Theorem (1.2).* Let $(G, c)$ be a crystallization of $M$ with minimum order, i.e., $p = \widetilde{\mathbf{c}}(M) + 2$. For every triple $(r, s, t)$ of distinct elements of $\Delta_4$, we have $g_{rst} \geq \mathrm{rk}(M) + 1$ by Lemma (2.1)c, hence $\sum_{r<s<t} g_{rst} \geq 10\,\mathrm{rk}(M) + 10$. From Lemma (2.2), it follows that

$$6\chi(M) = 30 - 2\sum_{r<s<t} g_{rst} + p$$

hence

$$\widetilde{\mathbf{c}}(M) = p - 2 = 6\chi(M) + 2\sum_{r<s<t} g_{rst} - 32 \geq 6\chi(M) + 20\,\mathrm{rk}(M) - 12$$

as claimed. If $M$ is simply–connected, then $\mathrm{rk}(M) = 0$ and $\chi(M) = 2 + b_2(M)$, so we obtain $\widetilde{\mathbf{c}}(M) \geq 6b_2(M)$. □

Proposition (1.4) follows immediately from the Freedman classification (up to topological homeomorphism) of the simply–connected closed 4–manifolds $M$ with $b_2(M) \leq 15$.

## 3. Four-manifolds of reduced complexity $\leq 6$

The following result, which we are going to prove, implies statement (a) of the main theorem in Section 1.

THEOREM (3.1). *Let $(G, c)$ be a crystallization of a closed connected 4-manifold $M$. If the order of $G$ is $\leq 8$, then $M$ is PL homeomorphic to either $\mathbb{S}^4$ or $\pm\mathbb{C}P^2$. In particular, $\widetilde{\mathbf{c}}(\mathbb{S}^4) = 0$, $\widetilde{\mathbf{c}}(\pm\mathbb{C}P^2) = 6$, and there are no closed connected 4-manifolds $M$ of reduced complexity $0 < \widetilde{\mathbf{c}}(M) < 6$.*

*Proof.* If $p \leq 6$, then $M$ is PL homeomorphic to $\mathbb{S}^4$, as proved in [11], Lemma 1 (only for the orientable case). First we give an alternative proof of that result in the general case. If $p \leq 4$, then the statement is obvious since the unique possible crystallizations, up to colored isomorphism, are those depicted in Figure 1, $a$ and $b$. But they both represent $\mathbb{S}^4$. So let $p = 6$. Since $g_{ij} \leq \frac{p}{2} = 3$, Lemma (2.2) gives $2g_{rst} = g_{st} + g_{rt} + g_{rs} - 3 \leq 6$, hence $g_{rst} \leq 3$ for every three colors $r, s, t \in \Delta_4$. If $g_{rst} = 3$ for some $r \neq s \neq t$, the condition $p = 6$ implies that $G$ must be as in Figure 1c, so it cannot be a crystallization. Thus $g_{rst} \leq 2$ for all $r, s, t \in \Delta_4$. We prove that at least one of $g_{rst}$'s must be equal to 1. Otherwise, if $g_{rst} = 2$ for all three colors $r, s, t \in \Delta_4$, then $\sum_{r<s<t} g_{rst} = 20$, and Lemma (2.2) gives $(p = 6)$ $\chi(M) = 5 - \frac{1}{3}(20) + 1 = -\frac{2}{3}$, which is impossible. So we can always assume $g_{012} = 1$, hence $\pi_1(M) \cong 0$ by Lemma (2.1)c, that

is, $M$ is orientable and simply–connected. Now we prove that $G$ must have two vertices joined by three parallel (differently colored) edges. Suppose that no such a configuration exists. This implies that, for every distinct colors $r, s, t \in \Delta_4$, the partial subgraph $G_{\{r,s,t\}} = (V(G), c^{-1}\{r, s, t\})$ is connected, i.e., $g_{rst} = 1$. As a consequence, $\sum_{r<s<t} g_{rst} = 10$, and Lemma (2.2) gives ($p = 6$) $\chi(M) = 5 - \frac{1}{3}(10) + 1 = \frac{8}{3}$, which is impossible. Thus $G$ must admit either a 3–dipole or a handle, i.e., by [2] and [12] $M$ must be homeomorphic to either $\mathbb{S}^4 \# M'$ or $(\mathbb{S}^1 \otimes \mathbb{S}^3)\# M'$, where $\widetilde{\mathbf{c}}(M') \leq 4$, and hence $M' \cong \mathbb{S}^4$. In the second case, $\pi_1(M) \cong \mathbb{Z}$ against the fact that $M$ is simply–connected. Thus, if $p \leq 6$, then $M$ is the genuine 4–sphere. In particular, $\widetilde{\mathbf{c}}(\mathbb{S}^4) = 0$, and there are no closed connected 4–manifolds $M$ of reduced complexity $0 < \widetilde{\mathbf{c}}(M) \leq 4$.

Let $p = 8$. We divide the proof in two steps.

(I) Suppose that $G$ has no vertices joined by three parallel edges. This implies that for every three colors $r, s, t \in \Delta_4$, the partial subgraph $G_{\{r,s,t\}}$ has at most two components, that is, $g_{rst} \leq 2$. If $g_{rst} = 2$ for every triple $(r, s, t)$, then Lemma (2.2) gives ($p = 8$) $\chi(M) = 5 - \frac{1}{3}(20) + \frac{4}{3} = -\frac{1}{3}$, which is impossible. So there is at least one of $g_{rst}$'s equals to one, say $g_{024} = 1$. Then $\pi_1(M) = 0$ and $M$ is orientable and simply–connected by Lemma (2.1)c. In particular, we have $\chi(M) = 2 + b_2(M) \geq 2$.

(I1) Assume $\chi(M) > 2$. Then we prove that $\sum_{r<s<t} g_{rst} \leq 10$, hence $g_{rst} = 1$ for every distinct colors $r, s, t \in \Delta_4$. Otherwise, if $\sum_{r<s<t} g_{rst} > 10$, Lemma (2.2) gives ($p = 8$)

$$\chi(M) = 5 - \frac{1}{3}\sum_{r<s<t} g_{rst} + \frac{4}{3} < 5 - \frac{1}{3}(10) + \frac{4}{3} = 3$$

which contradicts $\chi(M) > 2$. Suppose that there exists $g_{ij} = 1$ for some distinct colors $i, j \in \Delta_4$, say $g_{13} = 1$. The subcomplex $K(1, 3)$ has exactly one edge since $q_1(1, 3) = g_{024} = 1$. Then $N(1, 3)$ is a 4–ball. The subcomplex $K(0, 2, 4)$ is formed by exactly one triangle since $q_1(0, 2) = g_{134} = 1$, $q_1(2, 4) = g_{013} = 1$, $q_1(0, 4) = g_{123} = 1$, and $q_2(0, 2, 4) = g_{13} = 1$. Then $N(0, 2, 4)$ is a 4–ball, and $M$ is PL homeomorphic to $\mathbb{S}^4$. But this contradicts $\chi(M) > 2$. Hence we can assume $g_{ij} \geq 2$ for every distinct colors $i, j \in \Delta_4$. We prove that $\sum_{i<j} g_{ij} \leq 20$, and hence $g_{ij} = 2$ for every $i, j$. Otherwise, if $\sum_{i<j} g_{ij} > 20$, then Lemma (2.2) gives

$$\chi(M) = 5 - \frac{1}{2}\sum_{i<j} g_{ij} + 8 < 5 - 10 + 8 = 3,$$

which contradicts $\chi(M) > 2$. Reassuming we have $g_{rst} = 1$ and $g_{ij} = 2$ for every triple $(r, s, t)$ and pair $(i, j)$ of distinct colors in $\Delta_4$. This implies that $N(1, 3)$ is a 4–ball, and that $K(0, 2, 4)$ is formed by exactly two triangles $T_1$ and $T_2$ with common boundary. The Mayer-Vietoris exact sequence of the triple $(M, N, N')$, where $N = N(1, 3)$ and $N' = N(0, 2, 4)$, gives $H_0(N') \cong H_2(N') \cong \mathbb{Z}$ and $H_1(N') \cong H_3(N') \cong 0$. By isotopy we can always suppose that $T_1$ is the standard 2–simplex in $M$. Let $\widehat{T}_1$ be the barycenter of $T_1$ and $\mathrm{Sd}^2$ K the second

barycentric subdivision of $K$. Then $N'$ is the orientable bordered 4–manifold obtained by adding a 2–handle (a regular neighborhood of $\widehat{T}_1$ in $\mathrm{Sd}^2\,\mathrm{K}$) onto the boundary of a 4–ball (a small regular neighborhood of $T_2$ in $M$) along a knot $L$. Since the surgery [17] is given by attaching 2–handles in dimension 4, the surgery coefficient associated to $L$ must be an integer and by homological reasons equal to $\pm 1$. Since $\partial N = \partial N' = \mathbb{S}^3$, by [14], Theorem 2, $L$ is the trivial knot (see also [26]). Thus $N'$ is PL homeomorphic to $\pm\mathbb{C}P^2\backslash$(open 4–ball), and $M = N \cup N'$ is the complex projective plane (see, for example, [20], p.47). Now the proof is complete because a crystallization of $\mathbb{C}P^2$ with order 8 was really constructed in [11], Figure 7, p.138 (for convenience, we report it in Figure 1d).
(I2) Assume $\chi(M) = 2$ (and hence $b_2(M) = 0$). Since $M$ is simply-connected, $H_2(M)$ is free, and therefore it is trivial. Since $\partial N = \partial N' = \mathbb{S}^3$, the Mayer-Vietoris exact sequence of the triple $(M, N, N')$ gives $H_2(N') = 0$. This means that $K(0, 2, 4)$ cannot have two triangles with common boundary. So it collapses onto a graph. Since $\partial N' = \mathbb{S}^3$, this graph has no loops. Thus $N$ and $N'$ are 4–balls, and $M$ is the genuine 4–sphere.
(II) Suppose that $G$ has two vertices joined by three parallel edges, i.e., $G$ has either a 3–dipole or a handle. By [2] and [12], $M$ must be homeomorphic to either $\mathbb{S}^4 \# M'$ or $(\mathbb{S}^1 \otimes \mathbb{S}^3)\# M'$, where $M'$ is a closed connected 4-manifold with $\widetilde{\mathbf{c}}(M') \leq 4$, i.e., $M' \cong \mathbb{S}^4$. We prove that $G$ cannot admit a handle. Otherwise, the second case yields $\chi(M) = 0$. Then at least one of $g_{rst}$'s must be equal to 1. On the contrary, if $g_{rst} \geq 2$ for every distinct colors $r, s, t \in \Delta_4$, then $\sum_{r<s<t} g_{rst} \geq 20$, and Lemma (2.2) gives $\chi(M) \leq 5 - \frac{1}{3}(20) + \frac{4}{3} = -\frac{1}{3}$, which contradicts $\chi(M) = 0$. If, for example, $g_{024} = 1$, then $M$ is simply–connected, so it cannot be $\mathbb{S}^1 \otimes \mathbb{S}^3$. This completes the proof. □

## 4. Four-manifolds $M$ of reduced complexity $6 < \widetilde{\mathbf{c}}(M) \leq 10$

We now prove statement (b) of the main theorem in Section 1.

THEOREM (4.1). *Let $(G, c)$ be a crystallization of a closed connected 4-manifold M. If the order of G is $\leq 12$, then M is PL homeomorphic to either $\mathbb{S}^4$, $\mathbb{C}P^2$, $\mathbb{S}^1 \times \mathbb{S}^3$, or $\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^3$. In particular, the unique closed connected 4-manifolds of reduced complexity 8 are $\mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^3$, and there are no closed connected 4-manifolds of reduced complexity 10.*

*Proof.* Let $p = 10$. First suppose that $G$ does not admit two vertices joined by three parallel edges. This implies that $g_{rst} \leq 2$ for all three colors $r, s, t \in \Delta_4$. Suppose that there exists one of $g_{rst}$'s equal to 1, so that $\pi_1(M) = 0$ and $M$ is orientable and simply–connected. In particular, $\chi(M) \geq 2$ and $H_2(M)$ is free. Since $\sum_{r<s<t} g_{rst} \geq 10$, Lemma (2.2) gives ($p = 10$)

$$\chi(M) = 5 - \frac{1}{3}\sum_{r<s<t} g_{rst} + \frac{5}{3} \leq 5 - \frac{10}{3} + \frac{5}{3} = \frac{10}{3}$$

hence $2 \leq \chi(M) \leq 3$. Furthermore, $N = N(1, 3)$ is a 4–ball since, for example, $g_{024} = 1$. If $\chi(M) = 2$, then $H_2(M) \cong 0$, and the Mayer-Vietoris exact sequence of the triple $(M, N, N')$, where $N' = N(0, 2, 4)$, gives $H_2(N') \cong 0$. Thus $N'$

collapses onto a graph. Since $\partial N = \partial N' = \mathbb{S}^3$, we obtain $N' \underset{PL}{\cong} B^4$ and $M \underset{PL}{\cong} \mathbb{S}^4$. If $\chi(M) = 3$, then $H_2(M) \cong \mathbb{Z} \cong H_2(N')$. So $K(0,2,4)$ has two triangles with common boundary. From the Mayer-Vietoris sequence of the triple $(M, N, N')$ we also get $H_1(N') \cong 0$, hence $g_{134} = g_{013} = g_{123} = 1$. By Section 2 we obtain $M \underset{PL}{\cong} \mathbb{C}P^2$. Suppose that $g_{rst} = 2$ for every three colors $r, s, t \in \Delta_4$. Then $\sum_{r<s<t} g_{rst} = 20$ and Lemma (2.2) ($p = 10$) gives

$$\chi(M) = 5 - \frac{1}{3}\sum_{r<s<t} g_{rst} + \frac{5}{3} = 5 - \frac{20}{3} + \frac{5}{3} = 0.$$

Since rank $\pi_1(M) \leq 1$ and $\chi(M) = 0$, we have $\pi_1(M) \cong \mathbb{Z}$ and $b_2(M) = 0$. The manifold $N = N(1,3)$ is PL homeomorphic to either $\mathbb{S}^1 \times B^3$ or $\mathbb{S}^1 \underset{\sim}{\times} B^3$ as $g_{024} = 2$. Since $H_2(M) \cong 0$, the Mayer-Vietoris sequence of the triple $(M, N, N')$, where $N' = N(0,2,4)$, gives $H_2(N') \cong 0$ and $H_1(N') \cong \mathbb{Z}$. Thus $N'$ collapses onto a graph. Since $\partial N = \partial N' = \mathbb{S}^1 \otimes \mathbb{S}^2$, we obtain $N' \underset{PL}{\cong} \mathbb{S}^1 \otimes B^3$. Then $M = N \cup N'$ is PL homeomorphic to $\mathbb{S}^1 \otimes \mathbb{S}^3$ by Theorem 2 of [21] (see also [18]).

Suppose now that $G$ has two vertices joined by three parallel edges. Then $G$ has either a 3-dipole or a handle. If $G$ has a 3-dipole, then $M$ admits a crystallization of order $p \leq 8$, hence $M$ is either $\mathbb{S}^4$ or $\mathbb{C}P^2$. If $G$ has a handle, then $M$ is PL homeomorphic to $M'\#(\mathbb{S}^1 \otimes \mathbb{S}^3)$, where $M'$ is represented by the graph obtained from $G$ by cancelling the handle (see [2] and [12]). Then $M'$ is either $\mathbb{S}^4$ or $\mathbb{C}P^2$ as $\widetilde{\mathbf{c}}(M') \leq 6$. But only the case $M' \cong \mathbb{S}^4$ is allowed. In fact, $\pi_1(M) \neq 0$ implies that $g_{rst} \geq 2$ for every three colors $r, s, t \in \Delta_4$. Then Lemma (2.2) gives $\chi(M) \leq 5 - \frac{20}{3} + \frac{5}{3} = 0$. Now $\chi(M) \leq 0$ and $\chi(M) = \chi(M') + \chi(\mathbb{S}^1 \otimes \mathbb{S}^3) - 2 = \chi(M') - 2$ imply $\chi(M') \leq 2$, hence $M' \cong \mathbb{S}^4$. In Figures 2a and 2b we show two crystallizations of minimum order 10 for $\mathbb{S}^1 \times \mathbb{S}^3$ and $\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^3$, respectively.

Let $p = 12$. First suppose that $G$ does not admit two vertices joined by three parallel edges. This implies that $g_{rst} \leq 3$ for all three colors $r, s, t \in \Delta_4$. If $g_{rst} = 3$ for every $r, s, t \in \Delta_4$, then Lemma (2.2) gives $\chi(M) = 5 - \frac{1}{3}(30) + 2 = -3$. This is impossible since rank $\pi_1(M) \leq g_{rst} - 1 = 2$ and $\chi(M) = 2 - 2b_1 + b_2 \geq -2 + b_2 \geq -2$ (use homology with $\mathbb{Z}_2$–coefficients in the nonorientable case). So there is at least one of $g_{rst}$'s less than or equal to 2, say $g_{024} \leq 2$. If $g_{rst} \geq 2$ for every $r, s, t \in \Delta_4$, then Lemma (2.2) gives $\chi(M) \leq 5 - \frac{1}{3}(20) + 2 = \frac{1}{3}$, hence $\chi(M) \leq 0$, and $M$ cannot be simply–connected (otherwise, $\chi(M) \geq 2$). Suppose that $M$ is orientable; the argument for the nonorientable case is the same by using homology with $\mathbb{Z}_2$–coefficients. Then we have $b_1(M) = 1$ and $b_2(M) = 0$, hence the free part of $H_2(M)$ is trivial. The manifold $N = N(1,3)$ is PL homeomorphic to $\mathbb{S}^1 \times B^3$ as $g_{024} = 2$. The Mayer-Vietoris sequence of the triple $(M, N, N')$, where $N' = N(0,2,4)$, gives $H_2(N') \cong 0$ since $H_2(N')$ is free, $FH_2(M) \cong 0$, and $H_3(M) \cong H_1(N') \cong H_2(\partial N) \cong \mathbb{Z}$. Thus there are no triangles in $K(0,2,4)$ with common boundary. This means that $K(0,2,4)$

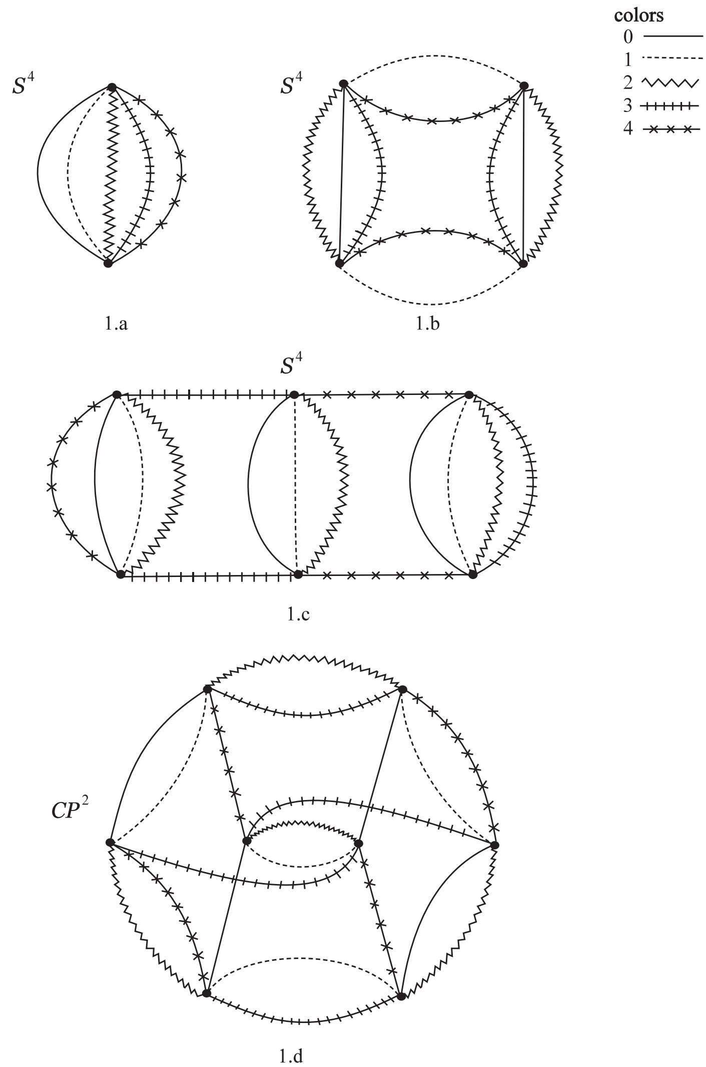


**Figure 1.** Colored graphs and crystallizations of $\mathbb{S}^4$ and $\mathbb{C}P^2$

collapses onto the one–dimensional complex formed by two vertices joined by two edges. Thus $N'$ is PL homeomorphic to $\mathbb{S}^1 \times B^3$, and $M$ is $\mathbb{S}^1 \times \mathbb{S}^3$ by [21]. Suppose now that there exists one of $g_{rst}$'s equal to 1, say $g_{024} = 1$. Then $M$ is simply–connected (hence orientable) and $\chi(M) \geq 2$. By Lemma (2.2) we cannot have $\sum_{r<s<t} g_{rst} = 10$ (resp. 11) because $\chi(M) = \dfrac{11}{3}$ (resp. $\dfrac{10}{3}$), which is impossible. Thus $\sum_{r<s<t} g_{rst} \geq 12$, and Lemma (2.2) gives $\chi(M) \leq 5 - \dfrac{1}{3}(12) + 2 = 3$. Since $\chi(M) = 2 + b_2(M) \leq 3$, we obtain $b_2(M) \leq 1$. If $b_2(M) = 0$, then $H_2(M) \cong 0$. The manifold $N = N(1, 3)$ is a 4–ball as $g_{024} = 1$. The Mayer-Vietoris sequence of the triple $(M, N, N')$, where $N' = N(0, 2, 4)$,

gives $H_2(N') \cong 0$, hence $N'$ collapses onto a graph. Then $\partial N = \partial N' = \mathbb{S}^3$, and $N'$ is a 4–ball. This implies that $M \cong \mathbb{S}^4$. If $b_2(M) = 1$, then $H_2(M) \cong \mathbb{Z}$. From the Mayer-Vietoris sequence, we get $H_2(N') \cong \mathbb{Z}$. Then $K(0, 2, 4)$ has exactly two triangles with common boundary. So we have $M \cong \mathbb{C}P^2$.

Suppose now that $G$ admits two vertices joined by three parallel edges. This means that $G$ has either a dipole or a handle. In the first case, $M$ has a crystallization of order $\leq 10$, hence $M$ is either $\mathbb{S}^4$, $\mathbb{S}^1 \otimes \mathbb{S}^3$, or $\mathbb{C}P^2$. In the second case, $M$ must be homeomorphic to $M'\#(\mathbb{S}^1 \otimes \mathbb{S}^3)$, where $\widetilde{\mathbf{c}}(M') \leq 8$, hence $M'$ is either $\mathbb{S}^4$, $\mathbb{S}^1 \otimes \mathbb{S}^3$, or $\mathbb{C}P^2$. We can immediately exclude the case $M' = \mathbb{S}^1 \otimes \mathbb{S}^3$ as $\mathrm{rk}(M) \leq 1$. Furthermore, $\pi_1(M) \neq 0$ implies that $g_{rst} \geq 2$ for every three colors $r, s, t \in \Delta_4$, hence $\chi(M) \leq 0$, as above. This inequality plus $\chi(M) = \chi(M') - 2$ give $\chi(M') \leq 2$, hence $M' \cong \mathbb{S}^4$. Thus the proof is complete. $\square$

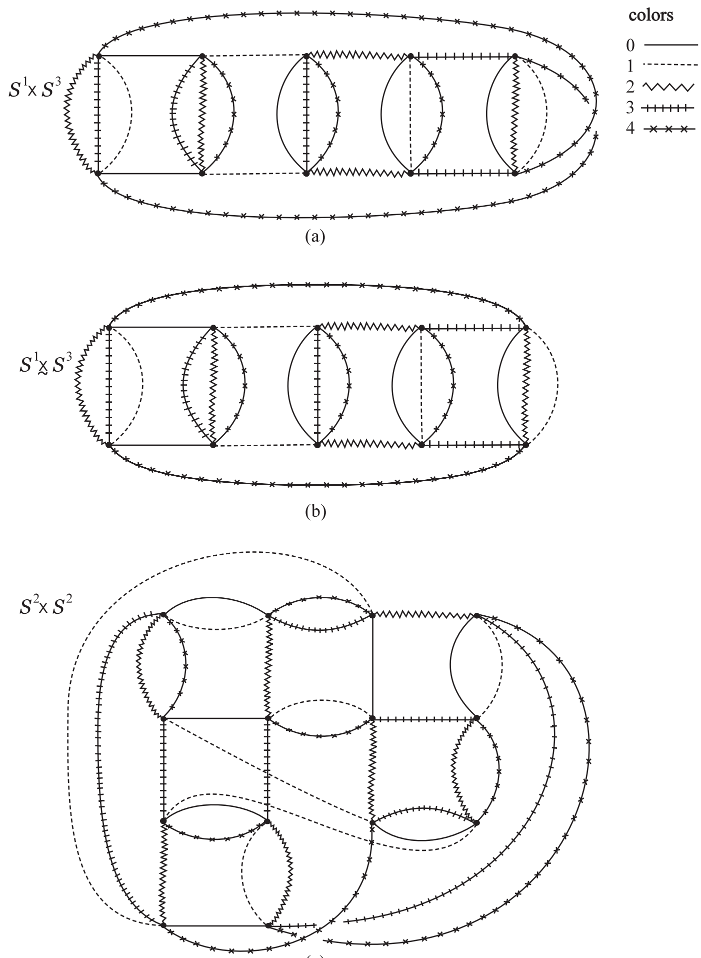


**Figure 2.** Crystallizations of minimum order for $\mathbb{S}^1 \times \mathbb{S}^3$, $\mathbb{S}^1 \underset{\sim}{\times} \mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{S}^2$

## 5. Four-manifolds $M$ of reduced complexity $10 < \tilde{\mathfrak{c}}(M) \leq 14$

Here we prove statements (c) and (d) of the main theorem.
Let $p = 14$. We shall divide the proof in two steps.
(I) Suppose that $G$ has no two vertices joined by three parallel edges. This implies that for every three colors $r, s, t \in \Delta_4$, the partial subgraph $G_{\{r,s,t\}}$ has at most 3 components, that is, $g_{rst} \leq 3$. If $g_{rst} = 3$ for every distinct colors in $\Delta_4$, then Lemma (2.2) gives $\chi(M) = 5 - \frac{1}{3}(30) + \frac{7}{3} = -\frac{8}{3}$, which is impossible. So there is at least one of $g_{rst}$'s less than or equal to 2, hence $\mathrm{rk}(M) \leq 1$. Suppose that all $g_{rst} \geq 2$ for every three colors $r, s, t \in \Delta_4$. Then $\sum_{r<s<t} g_{rst} \geq 20$ and Lemma (2.2) gives

$$\chi(M) = 5 - \frac{1}{3}\sum_{r<s<t} g_{rst} + \frac{7}{3} \leq 5 - \frac{20}{3} + \frac{7}{3} = \frac{2}{3},$$

hence $\chi(M) \leq 0$. This gives $\pi_1(M) \cong \mathbb{Z}$, and we can conclude that $M$ is PL homeomorphic to $\mathbb{S}^1 \otimes \mathbb{S}^3$ as in Section 4. Therefore we can assume that there is at least one of $g_{rst}$'s equal to 1, say $g_{024} = 1$. Hence $\pi_1(M) \cong 0$, i.e., $M$ is simply–connected (hence orientable), and $\chi(M) = 2 + b_2(M)$. If $\sum_{r<s<t} g_{rst} > 10$, then we obtain (use Lemma (2.2) and $p = 14$) $\chi(M) < 5 - \frac{1}{3}(10) + \frac{7}{3} = 4$. Thus $\chi(M) \leq 3$ and $b_2(M) \leq 1$. Reasoning as in Section 3 we obtain that $M$ is PL homeomorphic to either $\mathbb{S}^4$ or $\mathbb{C}P^2$. So we can assume $M$ simply–connected and $\sum_{r<s<t} g_{rst} \leq 10$, hence $g_{rst} = 1$ for every three colors $r, s, t \in \Delta_4$. By Lemma (2.2), we get $\chi(M) = 4$, $b_2(M) = 2$, $H_1(M) \cong H_3(M) \cong 0$, and $H_2(M) \cong \mathbb{Z} \oplus \mathbb{Z}$. The manifold $N = N(1,3)$ is a 4–ball since $g_{024} = 1$. The 1–skeleton of the complex $K(0,2,4)$ is formed by exactly three edges as $g_{013} = g_{123} = g_{134} = 1$, i.e., it is the boundary of a triangle. The Mayer-Vietoris sequence of the triple $(M, N, N')$, where $N' = N(0,2,4)$, gives $H_0(N') \cong \mathbb{Z}$, $H_1(N') \cong H_3(N') \cong 0$ and $H_2(N') \cong \mathbb{Z} \oplus \mathbb{Z}$. Then $K(0,2,4)$ has exactly three triangles $T_1$, $T_2$ and $T_3$ with common boundary. By isotopy we can always suppose the $T_1$ is the standard 2–simplex in $M$. Let $\widehat{T}_i$, $i = 2, 3$, be the barycenter of $T_i$ and $\mathrm{Sd}^2\,\mathrm{K}$ the second barycentric subdivision of $K$. Then $N'$ is the orientable bordered 4–manifold obtained by adding two 2–handles (the regular neighborhoods of $\widehat{T}_i$ in $\mathrm{Sd}^2\,\mathrm{K}$, $i = 2, 3$) onto the boundary of a 4–ball (a small regular neighborhood of $T_1$ in $M$) along knots $L_i$, $i = 2, 3$. Since the surgery [17] is given by attaching 2–handles in dimension 4, the surgery coefficient associated to $L_i$ must be an integer, and by homological reasons equal to either $\pm 1$ or 0. In fact, the integral intersection form $\lambda_M \colon H_2(M) \times H_2(M) \to \mathbb{Z}$ is isomorphic to either $(\pm 1) \oplus (1)$ or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Since $\partial N = \partial N' = \mathbb{S}^3$, by Theorem 2 of [14] (see also [25]) $L_i$ is the trivial knot. Thus $N'$ is PL homeomorphic to one of the surgery manifolds described in Figure 3. In the case on the right side of Figure 3 the linking number is 1 from the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.
From [20], p.47, we get that $N'$ is PL homeomorphic to either $(\pm\mathbb{C}P^2)\#\mathbb{C}P^2 \setminus$(open 4–ball) or $\mathbb{S}^2 \times \mathbb{S}^2 \setminus$ (open 4–ball). Thus $M$ is PL homeomorphic to either $\mathbb{S}^2 \tilde{\times} \mathbb{S}^2 = \mathbb{C}P^2\#(-\mathbb{C}P^2)$, $\mathbb{C}P^2\#\mathbb{C}P^2$ or $\mathbb{S}^2 \times \mathbb{S}^2$. But $M$ is prime, hence $M \cong \mathbb{S}^2 \times \mathbb{S}^2$

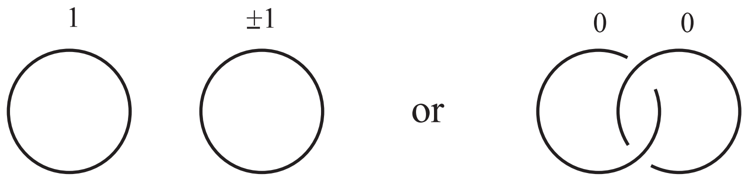


**Figure 3.** The two possible surgery descriptions of the 4-manifold $N'$

as claimed. To complete the proof we have constructed a crystallization of minimum order 14 for $\mathbb{S}^2 \times \mathbb{S}^2$ as shown in Figure 2c. It is obtained by cancelling a 2–dipole from the crystallization of order 16 given in [8], p.343.
(II) If $G$ has two vertices joined by three parallel edges, then $G$ has a 3–dipole or a handle. In the first case, $M$ admits a crystallization of order $\leq 12$, so we reobtain one of the known manifolds. In the second case, $M$ is PL homeomorphic to $(\mathbb{S}^1 \otimes \mathbb{S}^3)\#M'$ by [2] and [12]. Since $M$ is prime, we get $M \cong \mathbb{S}^1 \otimes \mathbb{S}^3$.

Let $p = 16$. Suppose that $G$ has no two vertices joined by three parallel edges. This implies that for every three colors $r, s, t \in \Delta_4$, the subgraph $G_{\{r,s,t\}}$ has at most 4 components, that is, $g_{rst} \leq 4$. If $g_{rst} = 4$ for every distinct colors $r, s, t \in \Delta_4$, then Lemma (2.2) gives $\chi(M) = 5 - \frac{1}{3}(40) + \frac{8}{3} = -\frac{17}{3}$, which is impossible. So there is at least one of $g_{rst}$'s less than or equal to 3, hence $\mathrm{rk}(M) \leq 2$. We cannot have all $g_{rst} \geq 3$ for all $r, s, t \in \Delta_4$. Otherwise, $\sum_{r<s<t} g_{rst} \geq 30$, and Lemma (2.2) gives $\chi(M) \leq 5 - 10 + \frac{8}{3} = -\frac{7}{3}$, hence $\chi(M) \leq -3$. Since $\mathrm{rk}(M) \leq 2$, in the orientable case (otherwise, use homology with $\mathbb{Z}_2$–coefficients) we get $-3 \geq \chi(M) = 2 - 2b_1 + b_2 \geq -2 + b_2$, which is a contradiction. So there is at least one of $g_{rst}$'s less than or equal to 2, hence $\mathrm{rk}(M) \leq 1$. We divide the proof into two steps A and B.

A) Suppose that there is at least one $g_{rst} = 1$. Then $\pi_1(M) \cong 0$, and $M$ is simply–connected (hence orientable) with $\chi(M) \geq 2$. The condition $\sum_{r<s<t} g_{rst} = 10$ and Lemma (2.2) imply $\chi(M) = 5 - \frac{1}{3}(10) + \frac{8}{3} = \frac{13}{3}$, which is impossible. Then we have $\sum_{r<s<t} g_{rst} > 10$, hence $\chi(M) \leq 4$ by Lemma (2.2). If $\chi(M) = 4$, then $\sum_{r<s<t} g_{rst} = 11$, $\pi_1(M) \cong 0$, and $b_2(M) = 2$. Reasoning as in the previous case, we obtain that $M$ is PL homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^2$. If $\chi(M) = 3$, then $\sum_{r<s<t} g_{rst} = 14$, $\pi_1(M) \cong 0$, and $b_2(M) = 1$. From above, $M \cong \mathbb{C}P^2$. If $\chi(M) = 2$, then $\sum_{r<s<t} g_{rst} = 17$, $\pi_1(M) \cong 0$, and $b_2(M) = 0$. Thus $M$ is the genuine 4–sphere.

B) Suppose that all $g_{rst} \geq 2$, and at least one of them is 2, hence $\mathrm{rk}(M) \leq 1$. If $\sum_{r<s<t} g_{rst} = 20$, by Lemma (2.2) $\chi(M) = 1$, hence $M$ is not simply–connected. If $\pi_1(M) \cong \mathbb{Z}$, then $H_2(M) \cong \mathbb{Z}$. Hence $M$ is PL homeomorphic to $(\mathbb{S}^1 \otimes \mathbb{S}^3)\#(\pm\mathbb{C}P^2)$, against the fact that $M$ is prime. If $\pi_1$ is finite cyclic, we must have $\pi_1(M) \cong \mathbb{Z}_{2n}$ as $\chi(M) = 1$. Then $H_1(M; \mathbb{Z}_2) \cong H_2(M; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Since $g_{rst} = 2$ for all three colors $r, s, t \in \Delta_4$, the formula $2g_{123} = g_{12} + g_{23} + g_{13} - \frac{p}{2}$

gives $g_{12} + g_{23} + g_{13} = 12$, hence at least one of $g_{ij}$'s, say $g_{13}$, is $\leq 4$. The manifold $N = N(1,3)$ is PL homeomorphic to $\mathbb{S}^1 \otimes B^3$ as $g_{024} = 2$. The complex $K(0,2,4)$ has at most four triangles as $g_{13} \leq 4$, and its 1–skeleton is homotopy equivalent to the wedge $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^1$ since $g_{013} = g_{134} = g_{123} = 2$. If $g_{13} = 1$, then $N'$ collapses to a graph. Since $\partial N = \partial N' = \mathbb{S}^1 \otimes \mathbb{S}^2$, we get $M \cong \mathbb{S}^1 \otimes \mathbb{S}^3$, which contradicts $\chi(M) = 1$. The Mayer-Vietoris sequence (with $\mathbb{Z}_2$–coefficients) of the triple $(M, N, N')$ gives $H_1(N'; \mathbb{Z}_2) \cong H_2(N'; \mathbb{Z}_2) \cong \mathbb{Z}_2$. If $g_{13} = 2$ and the triangles have no common boundary, we obtain a contadiction as above. If the two triangles have common boundary, then $N'$ is homotopy equivalent to the wedge $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$, which contradicts $H_1(N'; \mathbb{Z}_2) \cong \mathbb{Z}_2$. If $g_{13} = 3$ and the triangles have no common boundary, we are in the previous cases. If the three triangles have common boundary, then $N'$ is homotopy equivalent to a wedge with some 1–spheres and two 2–spheres, which contradicts $H_1(N'; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Thus we can assume $g_{rst} = 2$ for every three colors $r, s, t \in \Delta_4$, and $g_{ij} = 4$ for every two colors $i, j \in \Delta_4$. Furthermore, $N'$ is homotopy equivalent to $e^0 \cup e^1 \cup e^2$, where $\partial e^2 = 2e^1$ (the real projective plane $\mathbb{R}P^2$). In fact, $e^2$ is formed by exactly two triangles of $K(0,2,4)$ with common boundary, hence $\partial e^2 = 2e^1$, and $\pi_1(M) \cong \mathbb{Z}_2$. The universal covering $\widetilde{M}$ of $M$ is a simply–connected 4–manifold with $\chi(\widetilde{M}) = 2\chi(M) = 2$. Then $\widetilde{M}$ is homotopy equivalent to $\mathbb{S}^4$. Thus $\widetilde{M}$ is topologically homeomorphic to $\mathbb{S}^4$ by a celebrated theorem of Freedman [9]. Therefore the only possibilities for $M$ are the finite quotients of $\mathbb{S}^4$. It is well-known that $\mathbb{Z}_2$ is the only nontrivial finite group that can act freely on $\mathbb{S}^4$. So $M$ must be topologically homeomorphic to $\mathbb{R}P^4$ or the unique nonsmoothable homotopy $\mathbb{R}P^4$. But Freedman proved that every smooth fake $\mathbb{R}P^4$ is topologically homeomorphic to the standard $\mathbb{R}P^4$. Really, $M$ is PL homeomorphic to $\mathbb{R}P^4$. In fact, attaching a 3–handle from $N$ onto $N'$ yields a bordered 4–manifold $M \backslash$(open 4–ball)$= N' +$ (3–handle) which collapses onto $\mathbb{R}P^3$. Finally, a crystallization of minimum order 16 is depicted in Figure 4.

Assume now that $\sum_{r<s<t} g_{rst} > 20$. We cannot have $\sum_{r<s<t} g_{rst} = 21$ (resp. 22) because from Lemma (2.2) $\chi(M) = \frac{2}{3}$ (resp. $\frac{1}{3}$), which is impossible. Then $\sum_{r<s<t} g_{rst} \geq 23$, and hence $\chi(M) \leq 0$ by Lemma (2.2). Since $\mathrm{rk}(M) \leq 1$ (and $M$ is not simply–connected), we obtain $\pi_1(M) \cong \mathbb{Z}$. Reasoning as above, we conclude that $M$ is PL homeomorphic to $\mathbb{S}^1 \otimes \mathbb{S}^3$ (recall that $M$ is prime).

*Proof of Proposition (1.3).* Since $\widetilde{\mathbf{c}}(\mathbb{S}^2 \times \mathbb{S}^2) = 12$, the subadditivity of the reduced complexity gives $\widetilde{\mathbf{c}}(h(\mathbb{S}^2 \times \mathbb{S}^2)) \leq 12h$. By Theorem (1.2) we have $\widetilde{\mathbf{c}}(h(\mathbb{S}^2 \times \mathbb{S}^2)) \geq 6(2h+2) - 12 = 12h$ as $\chi(h(\mathbb{S}^2 \times \mathbb{S}^2)) = 2h + 2$ and $\mathrm{rk}(h(\mathbb{S}^2 \times \mathbb{S}^2)) = 0$. Since $\widetilde{\mathbf{c}}(\mathbb{C}P^2) = 6$, we have $\widetilde{\mathbf{c}}(k(\pm\mathbb{C}P^2)) \leq 6k$. Theorem (1.2) implies $\widetilde{\mathbf{c}}(k(\pm\mathbb{C}P^2)) \geq 6(k+2) - 12 = 6k$ as $\chi(k(\pm\mathbb{C}P^2)) = k + 2$ and $\mathrm{rk}(k(\pm\mathbb{C}P^2)) = 0$. Since $\widetilde{\mathbf{c}}(\mathbb{S}^1 \otimes \mathbb{S}^3) = 8$, we obtain $\widetilde{\mathbf{c}}(\ell(\mathbb{S}^1 \otimes \mathbb{S}^3)) \leq 8\ell$. By Theorem (1.2) we have $\widetilde{\mathbf{c}}(\ell(\mathbb{S}^1 \otimes \mathbb{S}^3)) \leq 6(2 - 2\ell) + 20\ell - 12 = 8\ell$ as $\chi(\ell(\mathbb{S}^1 \otimes \mathbb{S}^3)) = 2 - 2\ell$ and $\mathrm{rk}(\ell(\mathbb{S}^1 \otimes \mathbb{S}^3)) = \ell$. Putting these formulae together yields the result of the statement. $\square$

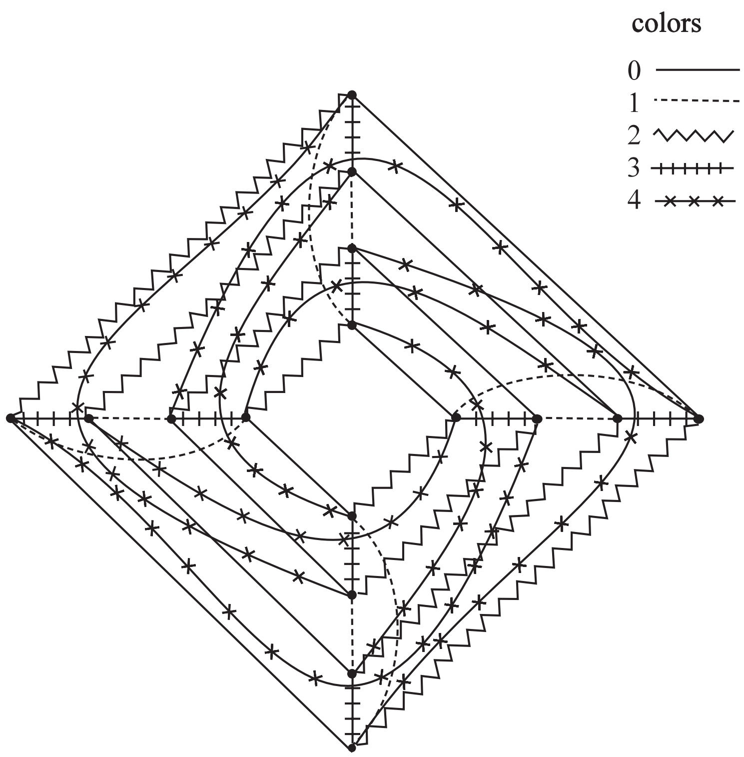


**Figure 4.** Crystallization of minimum order for $\mathbb{R}P^4$

## 6. Relations with the Poincaré Conjecture

Let $P(4)$ be the Poincaré conjecture in the PL 4-dimensional category. Now we state some conjectures which are related with $P(4)$.

Conjecture (6.1) (Additivity of the reduced complexity). *Let $M_1$ and $M_2$ be two closed connected orientable PL n–manifolds. Then*

$$\widetilde{\mathbf{c}}(M_1 \# M_2) = \widetilde{\mathbf{c}}(M_1) + \widetilde{\mathbf{c}}(M_2).$$

By Theorem 3.13 of [1] Conjecture (6.1) is true in dimension 2.

We prove that Conjecture (6.1) implies $P(4)$. For this, let $M$ be a homotopy PL 4–sphere. Then there exists a nonnegative integer $h$ such that $M\#h(\mathbb{S}^2\times\mathbb{S}^2)$ is diffeomorphic to $h(\mathbb{S}^2 \times \mathbb{S}^2)$. Assuming Conjecture (6.1), we get $\widetilde{\mathbf{c}}(M\#h(\mathbb{S}^2 \times \mathbb{S}^2)) = \widetilde{\mathbf{c}}(M) + \widetilde{\mathbf{c}}(h(\mathbb{S}^2 \times \mathbb{S}^2)) = \widetilde{\mathbf{c}}(h(\mathbb{S}^2 \times \mathbb{S}^2))$, hence $\widetilde{\mathbf{c}}(M) = 0$. This implies that $M \underset{PL}{\cong} \mathbb{S}^4$.

Conjecture (6.2). *If $M$ is a closed PL simply–connected 4–manifold, then*

$$\widetilde{\mathbf{c}}(M) = 6\chi(M) - 12 = 6b_2(M).$$

Conjecture (6.2) is equivalent to Conjecture (6.1) for the class of simply–connected closed PL 4–manifolds. Indeed, for sufficiently large $k$, the closed manifold $M\#k\mathbb{C}P^2\#k(-\mathbb{C}P^2)$ is diffeomorphic to $a\mathbb{C}P^2\#b(-\mathbb{C}P^2)$, where $a =$

$b_2^+(M) + k$ and $b = b_2^-(M) + k$ (see, for example, [13], p.161). Assuming Conjecture (6.1), we get

$$\widetilde{\mathbf{c}}(M\#k(\mathbb{C}P^2)\#k(-\mathbb{C}P^2)) = \widetilde{\mathbf{c}}(M) + 12k = 6(a+b) = 12k + 6b_2(M)$$

hence $\widetilde{\mathbf{c}}(M) = 6b_2(M)$. The converse is immediate.

PROPOSITION (6.3). (1) *Let $S_d$ be an algebraic nonsingular hypersurface of degree $d$ in $\mathbb{C}P^3$, then*

$$\widetilde{\mathbf{c}}(S_d) \geq 6(d^3 - 4d^2 + 6d - 2)$$

(2) *Let $V(n)$ denote the set of points $x \in \mathbb{C}P^1 \times \mathbb{C}P^2$ such that $P_n(x) = 0$, for a bihomogeneous polynomial $P_n$ of bidegree $(n, 3)$ in the variables $(y_0, y_1; z_0, z_1, z_2)$. Then we have*

$$\widetilde{\mathbf{c}}(V(n)) \geq 12(6n - 1)$$

(3) *If $\Sigma$ is an Enriques surface, then $\widetilde{\mathbf{c}}(\Sigma) \geq 80$.*

We conjecture that equalities hold in cases (1) and (2) (this follows also from Conjecture (6.2)). For small values of $d$, the statement is true as $S_1 = \mathbb{C}P^2$, $S_2 = \mathbb{S}^2 \times \mathbb{S}^2$, and $S_3 = \mathbb{C}P^2\#6(-\mathbb{C}P^2)$ (see [13], p.23). Furthermore, $S_4$ is the $K3$–surface and we have $\widetilde{\mathbf{c}}(S_4) \geq 132$.

For $n = 1$, $V(n)$ is diffeomorphic to $\mathbb{C}P^2\#9(-\mathbb{C}P^2)$, so the equality holds in case (2). For $n = 2$, $V(n)$ is the $K3$–surface.

*Proof.* By [13], Theorem 1.3.8, p.21, $S_d$ is a smooth simply–connected complex surface. Hence (1) follows from Theorem (1.2) and the formula $b_2(S_d) = d^3 - 4d^2 + 6d - 2$ (see, for example, [13], Lemma 1.3.9, p.21). By [13], Proposition 3.1.11, p.74, $V(n)$ is simply–connected and $b_2(V(n)) = 12n - 2$. So the result in (2) is a consequence of Theorem (1.2). An Enriques surface $\Sigma$ has $\pi_1(\Sigma) \cong \mathbb{Z}_2$ and $\chi(\Sigma) = 12$ (see [13], p.93). So $\widetilde{\mathbf{c}}(\Sigma) \geq 80$ by Theorem (1.2). $\square$

CONJECTURE (6.4). *If $M$ is a PL homotopy 4–sphere, then*

$$\widetilde{\mathbf{c}}(M\#h(\mathbb{S}^2 \times \mathbb{S}^2)) = \widetilde{\mathbf{c}}(M) + 12h.$$

This conjecture is equivalent to $P(4)$. In fact, if $M$ is a smooth homotopy 4–sphere, then there exists a nonnegative integer $h$ such that $M\#h(\mathbb{S}^2 \times \mathbb{S}^2)$ is diffeomorphic to $h(\mathbb{S}^2 \times \mathbb{S}^2)$. Thus we have $\widetilde{\mathbf{c}}(M\#h(\mathbb{S}^2 \times \mathbb{S}^2)) = \widetilde{\mathbf{c}}(M) + 12h = 12h$, and so $\widetilde{\mathbf{c}}(M) = 0$. Then $M \underset{PL}{\cong} \mathbb{S}^4$.

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