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ENCUENTROS CON MISCHA COTLAR

En memoria de Mischa Cotlar, quien falleció el 16 de enero de 2007 en Buenos Aires.

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Antes de comenzar mi relato, quiero llamar la atención del lector sobre el volumen *Analysis and Partial Differential Equations* [20], editado por la profesora Cora Sadosky en honor del septuagésimo quinto cumpleaños de Mischa Cotlar. Allí encontrará la biografía de Mischa, tres ensayos sobre su personalidad, un análisis de su obra matemática y la lista de sus publicaciones matemáticas hasta 1989.

1. Primeros encuentros

Entre 1948 y 1951 yo viví en el Colegio de España, situado en la Ciudad Universitaria de París. Desde la guerra civil española el colegio había sido administrado por el gobierno francés y en la época de mi residencia, también albergaba a varios refugiados españoles. Entre ellos se encontraban algunos matemáticos radicados en la Argentina, por ejemplo Manuel Balanzat, profesor de la Universidad de Cuyo en San Luis y alumno de Luis A. Santaló. El nombre de Santaló ya yo lo había oído en Budapest, porque László Fejes-Tóth había explicado su demostración de la desigualdad isoperimétrica en un curso de geometría que dictó en 1946. Vivían también en el Colegio de España algunos portugueses opuestos al régimen de Salazar y establecidos en Brasil. Es de todos ellos que oí por primera vez los nombres de Antonio Monteiro, Leopoldo Nachbin y Mischa Cotlar.

En mayo de 1951, viajando de París a Bogotá, hice una escala de algunas semanas en los Estados Unidos. Con mi compañero de estudios Steve Gaal fuimos a New Haven para visitar a Shizuo Kakutani en la Universidad Yale. Kakutani nos dijo que Mischa Cotlar estaba en New Haven con una beca Guggenheim para hacer estudios de doctorado en Yale. Además Kakutani nos contó detalles de la vida de Mischa: que llegó al Uruguay como refugiado ruso, que se ganó la vida tocando el piano en los bares del puerto de Montevideo, que nunca había tenido la posibilidad de seguir cursos en una escuela secundaria y en particular que no tenía ningún diploma de estudios. Kakutani añadió que Mischa era un autodidacta en matemáticas y que había publicado artículos matemáticos desde 1936, cuando tenía 23 años.

Al tiempo de su estadía en New Haven, Mischa estaba interesado en la teoría ergódica, motivo por el cual había ido a trabajar con Kakutani. En efecto, Kakutani era considerado en aquel entonces el mejor conocedor de la teoría, tanto es así que en el Congreso Internacional de Matemáticos de 1950 en

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Cambridge, Massachussetts, fue él quien se encargó de presentar un informe sobre el tema [17]. Recuerdo un detalle divertido sobre aquellos tiempos antediluvianos sin fotocopiadoras, sin calculadoras y sin correo electrónico: Kakutani tenía un único ejemplar del texto de su conferencia, aún no publicada. Le pedí que me lo prestara, lo que él muy gentilmente hizo, después de yo jurarle que se lo devolvería.

Volviendo a ese mayo de 1951, Cotlar vino a cenar con nosotros y pasamos un rato muy agradable. Al fin de la cena tuve la primera muestra de la increíble humildad de Mischa. Tenía que tomar el tren para viajar a Nueva York y los colegas me acompañaron a la estación del ferrocarril. Cotlar, quien tenía once años más que yo, insistió en cargar mi maleta.

Continué con mi viaje y al llegar a Bogotá, supe que Yale había decidido que no podía otorgarle el título de doctor a quien no tenía ni siquiera un diploma de estudios secundarios. Cuando Marshall Stone, quien conoció a Cotlar en Buenos Aires, se enteró de la situación, sugirió a Mischa que fuera a la Universidad de Chicago, donde los trámites se arreglaron con más liberalidad.

Así Cotlar llegó al territorio de Antoni Zygmund y Alberto P. Calderón, donde el foco de interés de los analistas era las integrales singulares. Allí, en un momento de inspiración genial, Cotlar combinó el tema de las integrales singulares con su propio interés en la teoría ergódica. Su tesis doctoral, de la cual hablaré más abajo, lleva el título *Una teoría unificada de la transformación de Hilbert y de los teoremas ergódicos*.

Mischa Cotlar en Chicago, 1952.

2. Encuentros en Latino América

Mi próximo encuentro personal con Mischa fue en julio de 1954 en ocasión del *Segundo simposio sobre algunos problemas matemáticos que se están* *estudiando en Latino América*. En la ciudad de Mendoza, al pie de la Cordillera de los Andes, la Universidad Nacional de Cuyo había establecido un instituto de matemática, en el Departamento de Investigaciones Científicas. Cotlar era el director de este instituto desde su regreso de los Estados Unidos y el instituto fue el organizador del simposio. La conferencia inaugural estuvo a cargo de don Julio Rey Pastor, gran figura de las matemáticas iberoamericanas y tuvo lugar en el Colegio San Juan de Mendoza. Al día siguiente todos los participantes fuímos llevados en autobús a un lujoso hotel en Villavicencio, en la sierra. Allí nos alojaron y es allí que tuvieron lugar las ponencias del simposio. Recuerdo que Cotlar dió una excelente charla panorámica con el título *El problema de los momentos y la teoría de operadores hermitianos* [6]. Quiero mencionar que en el primer simposio, que se hizo en Punta del Este, Uruguay, en diciembre de 1951, Cotlar habló *Sobre los fundamentos de la teoría ergódica* [5].

Cuando el instituto comenzó a publicar la Revista Matemática Cuyana, el famoso segundo fascículo del primer volumen contenía cuatro contribuciones de Cotlar. Las tres primeras presentan resultados auxiliares, que luego se aplican en la última parte. Una nota al pie de página nos informa que las partes esenciales de los tres últimos artículos son tomados de la tesis doctoral del autor defendida en la Universidad de Chicago en 1953.

El primer artículo [8] contiene uno de los más conocidos y más citados resultados de Cotlar, el que dió lugar al concepto de casi-ortogonalidad. Elías M. Stein dedica una gran parte del capítulo séptimo de su libro [21] a discutir este concepto. El teorema principal se puede enunciar así:

Sea $\mathcal{A}$ un anillo conmutativo normado, sean T_k, $1 \leq k \leq n$, elementos de $\mathcal{A}$ y pongamos $T = \sum_{k=1}^{n} T_k$. Si para $1 \leq i, j \leq n$ se cumple la condición de casi-ortogonalidad

$$\left\|T_i T_j\right\| \leq 2^{-|i-j|} \tag{1}$$

y si además $\|T_i\| \leq 1$, entonces $\left\|T^k\right\| \leq 2^{3k} k k^{k^{3/4}} n$. Como consecuencia inmediata se sigue que si $\{T_i\}_{i\in\mathbb{N}}$ es una sucesión de operadores hermitianos sobre un espacio prehilbertiano que cumplen (1) y que satisfacen además la condición $\|T_i\| \leq C$, entonces $\sum_{i=1}^{n} T_i$ converge en norma cuando $n \to \infty$ hacia un operador T tal que $\|T\| \leq 5C$.

La demostración original de este resultado, basada en un argumento combinatorio, es complicada. Un poco más tarde Béla Szőkefalvi-Nagy encontró una demostración simple [23]. Tanto Cotlar como Stein generalizaron el resultado al caso de operadores que no conmutan, y Stein usó esta generalización en un trabajo escrito en colaboración con A. W. Knapp [22] sobre la teoría de los grupos de Lie semi-simples. La desigualdad desempeña un papel importante en trabajos de Calderón yVaillancourt [3] y de Coifman y Meyer [4].

Un teorema clásico de Marcel Riesz, llamado Teorema de Interpolación de M. Riesz, tiene como caso particular lo siguiente:

Sea $\mathcal{D}$ un espacio de funciones denso en todo espacio L^p, por ejemplo las funciones en escalera, para $1 \leq p \leq \infty$. Si $T : \mathcal{D} \to \mathcal{D}$ es un operador lineal que cumple la condición (C_r),

$$\|Tf\|_r \leq M_r \|f\|_r$$

para $r = 1$ y $r = p$, entonces (C_r) es cierta para todo r entre 1 y p. Además la cota M_r óptima es una función logarítmicamente convexa de r, es decir

$$M_r \leq M_1^{\frac{r-p}{1-p}} M_p^{\frac{r-1}{p-1}}.$$

Es por esto que el teorema se solía llamar Teorema de Convexidad de Marcel Riesz.

Ahora bien, hay casos en los cuales T no cumple (C_1) pero sin embargo cumple (C_r) para $1 < r \leq p$; el ejemplo más importante es la transformada de Hilbert,

$$Hf(x) = \frac{1}{\pi} vp \int_{\mathbb{R}} \frac{f(t)}{x-t} dt.$$

Se puede uno preguntar si al reemplazar $\|Tf\|_1$ en la condición (C_1) por una cantidad más pequeña, la condición así obtenida no implicará aún que (C_r) se cumple para $1 < r \leq p$. Una respuesta se obtiene de la desigualdad de Chebišov,

$$\|f\|_1 \geq \int_{\{|f| \geq \lambda\}} |f| \geq \lambda |\{|f| \geq \lambda\}|,$$

donde $\{|f| \geq \lambda\} = \{x \in \mathbb{R} : |f(x)| \geq \lambda\}$ y $|E|$ es la medida de Lebesgue del conjunto E. Esta desigualdad sugiere el reemplazar la condición (C_1) por la condición $\left(C_1^*\right)$,

$$|\{|Tf| \geq \lambda\}| \leq \frac{M_1}{\lambda} \|f\|_1 .$$

De hecho, junto a muchas otras condiciones semejantes, Cotlar encontró que (C_1^*) y (C_p) implican que (C_r) vale para $1 < r < p$. Cuando Zygmund supo de los resultados de Cotlar, le dijo que esta condición (C_1^*) ya había sido descubierta por su discípulo polaco Jozef Marcinkiewicz quien la publicó sin demostración en los Comptes Rendus de París [19] en 1939, poco antes que el Ejército Rojo lo asesinara en el bosque de Katin. Sobre este tema, hay una carta de Cotlar a Peetre que se puede leer en [15], páginas 46-47. Debo agregar que la demostración del resultado de Marcinkiewicz fue publicada por Zygmund sólo en 1956 [25]. Creo que es el primer lugar donde aparece la frase "interpolación de operadores".

Similarmente a lo que precede, si $T : \mathcal{D} \to \mathcal{D}$ cumple las condiciones

$$\|Tf\|_1 \leq K_1 \|f\|_1 \tag{2}$$

$$|\{|Tf| > \lambda\}| \leq \frac{K_2}{\lambda^2} \|f\|_2^2 \tag{3}$$

para todo $\lambda > 0$, entonces $\|Tf\|_p \leq K_p \|f\|_p$ para $1 < p < 2$. Cotlar procede a debilitar la condición (2), reemplazando $\|Tf\|_1$ por una "norma modificada" que es considerablemente menor que $|| \cdot ||_1$. Para definirla escoge $\mathcal{D}$ de una manera especial e introduce el concepto de soporte generalizado $S_L(f)$ definido con la ayuda de un operador $L : \mathcal{D} \to \mathcal{D}$ que posee algunas de las propiedades del operador identidad. Todo esto conduce al teorema principal de [9].

En su tercer artículo en la Revista Matemática Cuyana [10], Cotlar presenta generalizaciones de desigualdades relativas al operador maximal de Hardy-Littlewood

$$\Lambda f(x) = \sup_{Q(x)} \frac{1}{|Q(x)|} \int_{Q(x)} |f(t)| \, dt,$$

donde $Q(x)$ representa los cubos de $\mathbb{R}^n$ con centro x y con aristas paralelas a los ejes de coordenadas.

Si $M\colon \mathcal{D} \to \mathcal{D}$ es un operador que satisface

$$\left|M\left(f+g\right)(x)\right| \leq |M(f)(x)| + |Mg(x)|$$

y $T\colon \mathcal{D} \to \mathcal{D}$ es otro operador que cumple la misma condición, Cotlar define dos condiciones de subordinación local entre M y T:

Se escribe $|M| \preceq O_1 |T|$ si para $f \in \mathcal{D}$ y todo $x \in \mathbb{R}^n$ existe un cubo $Q(x)$ tal que

$$|Mf(x)| \leq O_1 \frac{1}{|Q(x)|} \int_{Q(x)} |Tf(t)| dt.$$

Por otra parte, $|M| \ll O_1 |T|$ significa que

$$|Mf(x)| \leq O_1 \frac{1}{|Q(x)|} \int_{Q(x)} T(\varphi_{Q(x)} f)(t)| dt,$$

donde φ_E es la función característica del conjunto E.

Usando la notación $|M_\alpha f| = |Mf|^\alpha$ y $|T_\alpha f| = |Tf|^\alpha$, Cotlar demuestra que si $|M_\alpha| \ll |T_\alpha|$ y si $\|Tf\|_p \leq K_p \|f\|_p$, $p > \alpha$, entonces

$$\|Mf\|_q \leq K_q \|f\|_q$$

para todo $q > p$.

Además si

$$|\{|Tf| > \lambda\}| \leq O_p \lambda^{-p} \int |f|^p \, dx$$

para $p > \alpha$, entonces el operador M también satisface esta "condición débil". Resultados semejantes valen para la otra condición de subordinación.

Refiriéndose a estos resultados, Cora Sadosky ([20], p. 722) escribe: "El artículo se ocupa sobre todo de operadores maximales que son para un operador T lo que el operador maximal de Hardy-Littlewood es para el operador identidad. Se presentan también teoremas maximales en espacios producto, anticipando muchos trabajos ulteriores sobre el tema." En particular, Cotlar menciona la desigualdad

$$T_* f(x) \leq C(Tf)^*(x) + \|T\| \, f^*(x)$$

que Coifman y Meyer ([4], p. 95) llaman "la desigualdad de Cotlar" y a la que se refieren literalmente como "el corazón de la demostración" de su Teorema 21. Este teorema trata la convergencia en casi todos los puntos de un operador con núcleo K de Calderón-Zygmund. La notación g^* indica la función maximal de Hardy-Littlewood, mientras que

$$T_* f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y| \geq \varepsilon} K(x,y) f(y) dy \right|.$$

Tanto Coifman y Meyer ([4], p. 102) como Sadosky mencionan una característica particular de esta desigualdad, a saber que T figura en ambos lados de la desigualdad.

La cuarta y última parte [11] de este grupo de trabajos de Cotlar, utiliza las herramientas forjadas en las tres primeras partes para deducir las contribuciones más importantes del fascículo. Cotlar considera una función integrable

K sobre $\mathbb{R}^n$ y escribe para $j \in \mathbb{Z}$,

$$K_j(x) = \frac{1}{2^{nj}} K\left(\frac{x}{2^j}\right).$$

Cotlar considera también $\Omega = \{P\}$, un "espacio" provisto de una medida μ, y $\{\sigma_x : x \in \mathbb{R}^n\}$ un conjunto de aplicaciones $\sigma_x : \Omega \to \Omega$ que conservan la medida, es decir $\mu(\sigma_x(E)) = \mu(E)$ para todo $x \in \mathbb{R}^n$ y $E \subset \Omega$ medible. Además, estas aplicaciones satisfacen la condición $\sigma_x \circ \sigma_y = \sigma_{x+y}$. Si f es una función μ-medible definida en Ω, Cotlar define

$$H_m f(P) = \sum_{j=-m}^{m} \int_{\mathbb{R}^n} f(\sigma_x P) K_j(x) dx,$$

y se pregunta si $H_m f$ converge hacia una función Hf cuando $m \to \infty$, sea en casi todos los puntos $P \in \Omega$, sea en promedio.

Cuando $\Omega = \mathbb{R}^n$, $\sigma_x(t) = x + t$, $K(x) = \omega(x)|x|^{-n}$ y $\int_{|x|=1} \omega(x) dx = 0$, el operador H_m es una generalización n-dimensional de la transformada de Hilbert. Cuando Ω es un espacio de medida general, $K(x) = -1$ para $|x| < 1$ y $K(x) = 1$ para $1 \leq |x| \leq 2$, entonces H_m es el operador ergódico.

La función $H_m f$ converge a Hf cuando $m \to \infty$ en promedio si $f \in L^p(\Omega, \mu)$, $p > 1$: este es el teorema de von Neumann; además, la función $H_m f$ converge a Hf en casi todos los puntos: este es el teorema de Birkhoff. Una sección del artículo estudia el caso en que en vez de $\mathbb{R}^n$ se considera un grupo abeliano localmente compacto.

El Instituto de Matemática de Cuyo se disolvió apenas dos años después de su apertura, mientras que de la Revista Matemática Cuyana salieron sólo tres números. En 1957 Mischa fué nombrado profesor de matemática en la Escuela de Ciencias de la Universidad de Buenos Aires. Con la colaboración de Cora Ratto de Sadosky, Cotlar editó una serie de *Cursos y Seminarios de Matemática* para la cual obtuvo las contribuciones de un grupo de destacados matemáticos, entre ellos Laurent Schwartz, Jean-Pierre Kahane, Alberto P. Calderón, Guido Weiss y Esteban Vági.

El mismo Cotlar es el autor de tres fascículos. El primero y más grueso, con 353 páginas, es el número 2 que lleva por título: *Condiciones de Continuidad de Operadores Potenciales y de Hilbert* [12]. Yo tuve el placer de escribir la reseña para Zentralblatt für Mathematik (Zbl. **99**, 377) y la comencé así: "Este es un relato altamente legible de la teoría reciente de operadores potenciales y de integrales singulares, debida principalmente a Sobolev, Thorin, Calderón y Zygmund y al mismo Cotlar." Le presté el fascículo a Jacques-Louis Lions, quien estaba de visita en la Universidad de Maryland y quien precisamente en aquellos tiempos hacía investigaciones en la teoría de interpolación de aplicaciones lineales. Me lo devolvió diciendo: "El que ha escrito esto, sabe mucho". Cora Sadosky menciona que Béla Szőkefalvi-Nagy sugirió publicar una traducción del fascículo en la colección *Ergebnisse der Mathematik und Ihrer Grenzgebiete* de la cual él era uno de los editores, de la Editorial Springer. Es una lástima que esto no se haya realizado.

Los otros dos fascículos de *Cursos y Seminarios* escritos por Cotlar son el número 11, *Introducción a la Teoría de Representación de Grupos* [13] y el número 15, *Equipación con Espacios de Hilbert* [14].

3. Encuentros estadounidenses

Como Cora Sadosky dice, el período de oro de las matemáticas en Buenos Aires se acabó en 1966 cuando los militares irrumpieron en la Universidad, maltratando a profesores y alumnos. Como resultado de estas acciones, unos cuatrocientos miembros del cuerpo docente renunciaron a sus puestos. Mischa se fué primero a Montevideo y en 1967 fué nombrado profesor en la Universidad Rutgers, que es la universidad del estado de New Jersey en los Estados Unidos. Nosotros, sus amigos, pensamos que este nombramiento le agradaría ya que su discípulo y coautor Rodolfo Ricabarra era en ese entonces profesor en la cercana Universidad de Delaware. Por cierto, Rutgers queda cerca también de la Universidad de Maryland y mis colegas y yo nos alegramos de ver a Ricabarra asistir frecuentemente a nuestro seminario de análisis funcional que se reunía los martes a las ocho de la noche.

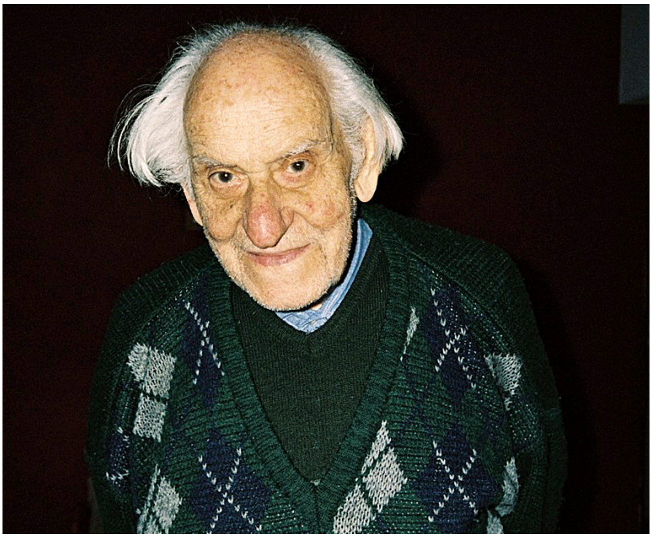
Durante el tiempo en que Cotlar fue profesor en Rutgers, Dieudonné lo invitó a pasar un período en Niza. La nostalgia que Cotlar sentía por el ambiente hispano americano, lo llevó en 1971 a aceptar una posición de profesor en Caracas, Venezuela. Luego pasó dos años en Argentina, entre Buenos Aires y La Plata, estableciéndose definitivamente en Caracas en 1974.

En 1972 el Centro de Investigaciones en Matemáticas Aplicadas, Sistemas y Servicios de la Universidad Nacional Autónoma de México me invitó a dar un curso de verano sobre espacios localmente convexos. Al mismo tiempo, Lucien Waelbroeck me invitó a enseñar un curso sobre ese tema en la Escuela de Verano sobre Espacios Vectoriales Topológicos en Bruselas en septiembre. Los cursos de esta escuela se han publicado ([24], [18]). El primer problema del cual decidí hablar fue la generalización del teorema de Hahn-Banach a semigrupos. Consulté el artículo de Georg Aumann [1] que comienza con las siguientes palabras: “Los teoremas de M. Cotlar [7] sobre la extensión de funciones aditivas monótonas sobre grupos parcialmente ordenados pueden generalizarse sin cambio esencial en el método de demostración y toman así una forma notablemente redondeada.” En mi curso, mencioné esta referencia al trabajo de Cotlar y al final de la clase vino a hablarme Carlos Alberto Berenstein quien acababa de recibir el doctorado con Leon Ehrenpreis. Carlos me dijo que era argentino, que entre mis oyentes había otros argentinos y que a todos ellos les había dado placer el oír el nombre de Mischa Cotlar. Después de la escuela de verano, Benno Fuchssteiner produjo nuevos resultados en la teoría. Sus resultados se pueden ver en [16], en los apuntes [24], pp. 45-46, y en la nueva edición de Espacios Vectoriales Topológicos de Bourbaki ([2], II, p. 78, ejercicio 7).

En 1975 empezó la colaboración entre Mischa Cotlar y Cora Sadosky que produjo unos cincuenta artículos, ocasionalmente con otros coautores. El trabajo en común hizo que Cotlar viniera a la ciudad de Washington casi cada año, ya que Cora era y es profesora en la Universidad Howard. En la Universidad de Maryland aprovechamos de las estancias de Cotlar, invitándolo a dar conferencias en el coloquio departamental y en el seminario de Israel Gojberg. Este último solía pasar cada año una larga temporada en College Park y tenía intereses comunes con Cotlar, en particular los operadores de

Toeplitz. Recuerdo que cuando invitamos a Cotlar a que diera una conferencia, su respuesta fué: "¿pojque invitan ustedes a un ignojante?".

Mischa Cotlar y su esposa Yanny Frenkel en Caracas, 2001.



Mischa Cotlar en Buenos Aires, 2006.

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SOME REMARKS ON SIERPIŃSKI NUMBERS AND RELATED PROBLEMS

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ABSTRACT. Here, we prove a few results about Sierpiński numbers of various shapes. We also show that there exist infinitely many even positive integers not of the forms $2^k+\phi(m)$, or $2^k-\phi(m)$, or $\phi(m)-2^k$ for some positive integers k and m, respectively, where $\phi(m)$ is the Euler function of m.

1. Introduction

In 1960, W. Sierpiński (see [16]) showed that there are infinitely many odd positive integers k such that $2^n k+1$ is composite for all n. Since then, these positive integers are referred to as *Sierpiński numbers*. In 1962, J. Selfridge showed that $k=78557$ is a Sierpiński number. This is now believed to be the smallest Sierpiński number. As of the writing of this paper, there are only 6 odd candidates k smaller than 78557 whose status with respect to being Sierpiński or not remains to be decided (see [15]).

Typically, the way to find Sierpiński numbers is the following. Assume that $(a_i, b_i, p_i)_{i=1}^{i=t}$ are triples of positive integers with the following properties

(i) for each integer n there exists $i\in\{1,\ldots,t\}$ such that $n\equiv a_i \pmod{b_i}$;

(ii) $p_1,\ldots,p_t$ are distinct prime numbers such that $p_i \mid 2^{b_i}-1$ for all $i=1,\ldots,t$.

Then one creates Sierpiński numbers k by imposing that

$$2^{a_i}k\equiv -1 \pmod{p_i} \qquad \text{for } i=1,\ldots,t. \tag{1.1}$$

Since the primes p_i are all odd for $i=1,\ldots,t$, it follows that for each i, the above congruence (1.1) is solvable and puts k into a certain arithmetic progression modulo p_i. The fact that the congruences (1.1) are simultaneously solvable for all $i=1,\ldots,t$ follows from the fact that the primes $p_1,\ldots,p_t$ are distinct via the Chinese Remainder Theorem. Every odd positive integer k in the resulting arithmetic progression has the property that $2^n k+1$ is always a multiple of one of the numbers p_i for $i=1,\ldots,t$, and if $k>\max\{p_i : i=1,\ldots,t\}$, it follows that $2^n k+1$ can never be a prime. The number $k=78557$ was proved to be Sierpiński by the above procedure for the triples

$$(0,2,3),\ (1,4,5),\ (1,3,7),\ (11,12,13),\ (15,36,19),\ (27,36,37),\ (3,36,73).$$

Solving the system of congruences (1.1) for $i=1,\ldots,t$ creates an arithmetic progression $k_0 \pmod{p_1\cdots p_t}$, where k_0 is coprime to $p_1\cdots p_t$. By Dirichlet's

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Theorem on Primes in Arithmetic Progressions, this progression contains infinitely many primes. Thus, there are infinitely many primes k which are also Sierpiński.

2. Sierpiński numbers of various shapes

If k is a Sierpiński number, it follows that $2^n k \neq q-1$ for any prime q. Let $\phi(m)$ be the Euler function of the positive integer m. Since $q-1=\phi(q)$, it makes sense to ask about Sierpiński numbers k such that $2^n k$ is in the image of the Euler function. Here are a couple of results in this direction.

THEOREM (2.1). (i) *If k is a Sierpiński prime and $2^n k=\phi(m)$ holds for some positive integers n and m, then k is a Fermat number.*

(ii) *There exist infinitely many Sierpiński numbers k such that for each one of them, $2^n k=\phi(m)$ holds for some positive integers n and m.*

Proof. For (i), assuming that $2^n k=\phi(m)$, it follows from the fact that k is prime, that either $k \mid q-1$ for some prime factor q of m, or $k^2 \mid m$. The first case leads to $q-1=2^a k$ for some positive integer $a \leq n$, which is impossible since k is Sierpiński. The second case leads to $k(k-1) \mid \phi(m) \mid 2^n k$, implying that $k-1=2^a$ for some positive integer a. Thus, $k=2^a+1$ is a prime, therefore it is a Fermat prime.

For (ii), consider the system of triples $(a_i, b_i, p_i)_{i=1}^{i=7}$ given by

$$
\begin{aligned}
&(1,2,3),\ (2,4,5),\ (4,8,17),\ (8,16,257),\\
&(16,32,65537),\ (32,64,641),\ (0,64,6700417).
\end{aligned} \tag{2.2}
$$

Let $k_0>6700417$ be an odd positive integer such that $2^{a_i} k_0 \equiv -1 \pmod{p_i}$ for $i=1,\ldots,7$ for the above system. For $i=1$, the above congruence is $2k_0 \equiv -1 \pmod 3$, therefore $k_0 \equiv 1 \pmod 3$. In particular, $(k_0/3)=1$, where here and from now on for an odd prime p and an integer a we write (a/p) for the Legendre symbol of a with respect to p. For $i \geq 2$, the congruence is $2^{a_i} k_0 \equiv -1 \pmod{p_i}$, where a_i is even and $p_i \equiv 1 \pmod 4$. Hence, $(k_0/p_i)=(-1/p_i)=1$ for all $i=2,\ldots,7$. In particular, for each $i=1,\ldots,7$, there exists λ such that $\lambda^2 \equiv k_0 \pmod{p_i}$. Let λ_i and $-\lambda_i$ be the two solutions λ modulo p_i of the above congruence. It is clear that $2\lambda_i+1$ and $-2\lambda_i+1$ cannot both be congruent to 0 modulo p_i, since otherwise we would get $2 \equiv (2\lambda_i+1)+(-2\lambda_i+1) \equiv 0 \pmod{p_i}$, which is impossible. For each $i \in \{1,\ldots,7\}$, we let $\mu_i \in \{-2\lambda_i+1, 2\lambda_i+1\}$ be such that $\mu_i \not\equiv 0 \pmod{p_i}$. The system

$$
q \equiv \mu_i \pmod{p_i} \qquad \text{for } i=1,\ldots,7
$$

puts q in an arithmetic progression $q_0 \pmod{p_1 \cdots p_7}$ with q_0 coprime to the modulus $p_1 \cdots p_7$. Thus, again by Dirichlet's Theorem on Primes in Arithmetic Progressions, there exist infinitely many primes $q \equiv q_0 \pmod{p_1 \cdots p_7}$. Let q_1 and q_2 be two such primes which additionally are congruent to 3 modulo 4. Then

$$
(q_1-1)(q_2-1) \equiv (q_0-1)^2 \equiv 4\lambda_i^2 \equiv 4k_0 \pmod{p_i} \qquad \text{for all } i=1,\ldots,7.
$$

Thus, the positive integer $(q_1-1)(q_2-1)/4$ is an odd number congruent to $k_0 \pmod{p_1 \cdots p_7}$, which implies that $k=(q_1-1)(q_2-1)/4$ is a Sierpiński number. Clearly, $4k=\phi(q_1q_2)$, therefore the desired equation $2^n k=\phi(m)$

holds with $n = 2$ and $m = q_1q_2$. Since there are infinitely many possibilities for the primes q_1 and q_2, we get infinitely many possibilities for k as well. $\square$

3. Even positive integers not of the form $2^k + \phi(m)$

In 1849, A. de Polignac asked if all odd positive integers can be expressed as a sum between a power of 2 and a prime. P. Erdős [6], showed, by means of a system of triples $(a_i, b_i, p_i)_{i=1}^{i=t}$ satisfying the conditions (i) and (ii), that there is an arithmetic progression of odd integers $k \equiv k_0 \pmod{2p_1 \cdots p_t}$ such that no member of this progression is of the form $2^n + p$ for some prime p. In particular, there are infinitely many even integers not of the form $2^k + p - 1$. Since $p-1 = \phi(p)$, it makes sense to ask if, more generally, there are infinitely many positive integers not of the form $2^k + \phi(m)$ for some positive integers k and m. The answer is given by the following result.

THEOREM (3.1). (i) *There are infinitely many even positive integers n not of the form $2^k + \phi(m)$ for some positive integers k and m.*
(ii) *There are infinitely many even positive integers n not of the form $2^k - \phi(m)$ for some positive integers k and m.*
(iii) *There are infinitely many even positive integers not of the form $\phi(m) - 2^k$ for some positive integers k and m.*

Proof. For (i), let $(a_i, b_i, p_i)_{i=1}^{i=t}$ be a system of triples as in the Introduction and let n_0 be such that $n_0 \equiv 2 \pmod 4$ and $n_0 \equiv 2^{a_i} - 1 \pmod{p_i}$ for all $i = 1, \ldots, t$. Let n be such that $n \equiv n_0 \pmod{4p_1 \cdots p_t}$. Let x be a large positive real number. The number of positive integers $n \le x$ in the arithmetic progression $n_0 \pmod{4p_1 \cdots p_t}$ is $\gg x$. Assume now that $n = 2^k + \phi(m)$. If $k = 1$, then $n = 2 + \phi(m)$. Thus, $\phi(m) \le x - 2$. The number of values of the Euler function which are at most x is $x/(\log x)^{1+o(1)}$ as $x \to \infty$ (see, for example, [7] or [13]). Thus, there can be at most $x/(\log x)^{1+o(1)}$ as $x \to \infty$ values of the positive integer $n \le x$ such that $k = 1$.

We discard these and therefore assume from now on that $k \ge 2$. Since $2\|n$, we get that $2\|\phi(m)$, so $m \in \{4, p^\gamma, 2p^\gamma\}$ for some odd prime p and positive integer γ. If $m = 4$, we get $n = 2^k + 2$. The number of such positive integers $n \le x$ is $O(\log x)$. We discard these too. Thus, we may assume that $m \in \{p^\gamma, 2p^\gamma\}$ and since $\phi(p^\gamma) = \phi(2p^\gamma)$, we shall further assume that $m = p^\gamma$. Assume first that $\gamma = 1$. Then $n = 2^k + p - 1$. Since $k \equiv a_i \pmod{b_i}$ for some $i = 1, \ldots, t$, we get that $p = n - 2^k + 1 \equiv n_0 - 2^{a_i} + 1 \equiv 0 \pmod{p_i}$, leading to $p = p_i$. Thus, $n = 2^k + p_i - 1 \le x$ for some $i = 1, \ldots, t$, and the number of such positive integers n is $O(\log x)$. We discard these values of n too. Thus, $\gamma \ge 2$. Then $n = 2^k + p^{\gamma-1}(p-1) \le x$, giving $2^k \le x$ and $p^\gamma \ll p^{\gamma-1}(p-1) \le x$. Hence, $k \ll \log x$, $\gamma \ll \log x$ and $p \ll x^{1/2}$, so the number of such possibilities for n is $O(x^{1/2}(\log x)^2)$. Discarding these values too, we arrive at the conclusion that there exist a number $\gg x$ of even positive integers $n \le x$ such that $n \ne 2^k + \phi(m)$ for any positive integers k and m.

For (ii), we take the following system of triples $(a_i, b_i, p_i)_{i=1}^{i=7}$ given by

$$(0, 2, 3),\ (1, 4, 5),\ (3, 8, 17),\ (7, 16, 257),$$
$$(15, 32, 65537),\ (31, 64, 641),\ (63, 64, 6700417),$$

and let $n_0 \equiv 2 \pmod 4$ and $n_0 \equiv 2^{a_i} + 1 \pmod{p_i}$ for all $i = 1, \ldots, 7$. We show that there exists n such that $2n^6 \equiv n_0 \pmod{4p_1 \cdots p_7}$. Note first that $(2n_0/p_i) = 1$ for all $i = 1, \ldots, 7$. Indeed, for $i = 1, 2, \ldots, 6$, one checks easily that $2^{a_i} \equiv -2^{-1} \pmod{p_i}$, therefore $2^{a_i} + 1 \equiv -2^{-1} + 1 \equiv 2^{-1} \pmod{p_i}$, which in turn implies that $(2n_0/p_i) = (1/p_i) = 1$. For $i = 7$, we have that $2^{a_7} \equiv 2^{-1} \pmod{p_7}$, therefore $2^{a_7} + 1 \equiv 2^{-1} + 1 = 3 \cdot 2^{-1} \pmod{p_7}$. Thus, $(2n_0/p_7) = (3/p_7) = (p_7/3) = 1$, by the Quadratic Reciprocity Law and the fact that $p_7 \equiv 1 \pmod{12}$. Since $(2n_0/p_i) = 1$ for all $i = 1, \ldots, 7$, the arithmetic progression $n_0 \pmod{4p_1 \cdots p_7}$ contains infinitely many numbers of the form $2n_1^2$ for some positive integer n_1 which must necessarily be odd. Furthermore, of all our primes $p_1, \ldots, p_7$, only p_7 is congruent to 1 modulo 3. In particular, the congruence $n_1 \equiv n_2^3 \pmod{p_i}$ has a solution $n_2 \pmod{p_i}$ for all $i = 1, \ldots, 6$. Finally, for p_7 we simply checked that the congruence $n_0 \equiv 2n_2^6 \pmod{p_7}$ also has a solution $n_2 \pmod{p_7}$ and, in fact, six of them, the smallest one being $n_2 \equiv 1137449 \pmod{p_7}$. Hence, the arithmetic progression $n_0 \pmod{4p_1 \cdots p_7}$ contains infinitely many positive integers of the form $2n^6$.

Let x be a large positive real number. The number of positive integers of the form $2n^6 \le x$ in the arithmetic progression $n_0 \pmod{4p_1 \cdots p_7}$ is $\gg x^{1/6}$.

Assume now that n is such a positive integer with the additional property that $2n^6 = 2^k - \phi(m)$ for some positive integers k and m. If $k = 1$, we then get that $\phi(m) < 2$ must an even positive integer, which is a contradiction. Thus, $k \ge 2$. Since $2 \| 2n^6$, we get that $2 \| \phi(m)$, therefore again $m \in \{4, p^\gamma, 2p^\gamma\}$ for some odd prime p and positive integer γ. The case $m = 4$ leads to $2n^6 = 2^k - 2$, or $n^6 = 2^{k-1} - 1$. This Diophantine equation has no positive integer solutions with $n > 1$, since the only positive integer solutions of the Catalan equation $y^l - z^m = 1$ in integers y, z, l, m all exceeding 1 is $3^2 - 2^3 = 1$ (see [1], for example). From now on, we assume that $m \in \{p^\gamma, 2p^\gamma\}$, and since $\phi(p^\gamma) = \phi(2p^\gamma)$, it follows that we may further assume that $m = p^\gamma$. Since $k \ge 2$ and n is odd, we get that $p \equiv 3 \pmod 4$. Suppose that $\gamma = 1$. Then there must exist $i \in \{1, \ldots, 7\}$ such that $k \equiv a_i \pmod{p_i}$. Thus,

$$p = 2^k + 1 - 2n^6 \equiv 2^{a_i} + 1 - n_0 \equiv 0 \pmod{p_i},$$

therefore $p = p_i$. Hence, $2n^6 = 2^k - p_i + 1$ for some $i = 1, \ldots, 7$. Since $2n^6 \le x$, it follows that each one of these seven Diophantine equations has at most $O(\log x)$ positive integer solutions (n, k), which we may discard for large x. Now suppose that $\gamma > 1$.

Assume that k is odd and let us write it as $k = 2k_0 + 1$. Then

$$p^{\gamma-1}(p-1) = 2^k - 2n^6 = 2^{2k_0+1} - 2n^6 = 2(2^{k_0} - n^3)(2^{k_0} + n^3).$$

Since n is odd, it follows that $2^{k_0} - n^3$ and $2^{k_0} + n^3$ are coprime. Thus, $p^{\gamma-1}$ divides one of $2^{k_0} - n^3$ or $2^{k_0} + n^3$. Assume first that $\gamma \ge 3$. Then $p^{\gamma-1} \ge p^2 > p - 1$ and $2^{k_0} + n^3 > 2^{k_0} - n^3 > 0$, so we must have

$$p^2 \mid 2^{k_0} + n^3 \qquad \text{and} \qquad 2^{k_0} - n^3 \mid p - 1.$$

In particular, $2^{k_0} + n^3 \ge p^2$ and $0 < 2^{k_0} - n^3 < p$. This yields

$$|2^{k_0} - n^3| = O(2^{k_0/2}),$$

which in turn leads to

$$|2^{k_0/3} - n| = O(2^{-k_0/6}). \tag{3.2}$$

If $3 \mid k_0$, then the above relation leads to $n = 2^{k_0/3}$ for sufficiently large values of n, which is impossible because n is odd. Thus, for large n, we have $k_0 \equiv r \pmod 3$, where $r \in \{1, 2\}$. The above relation (3.2) leads to

$$\left|2^{r/3} - \frac{n}{2^{\lfloor k_0/3 \rfloor}}\right| \ll \frac{1}{(2^{\lfloor k_0/3 \rfloor})^{3/2}}. \tag{3.3}$$

A well-known theorem of Ridout [14], tells us that the above Diophantine approximation (3.3) has only finitely many positive integer solutions (n, k_0), which we may discard for large x.

Now assume that $\gamma = 2$. We then get

$$2n^6 = 2^{2k_0+1} - p(p-1), \tag{3.4}$$

or

$$8n^6 - 1 = 2^{2k_0+3} - (2p-1)^2.$$

If $2^{2k_0+1} > x^3$, then $|2^{2k_0+3} - (2p-1)^2| = 8n^6 - 1 \ll x \ll 2^{2k_0/3}$. This leads to

$$|2^{(2k_0+3)/2} - (2p-1)| \ll 2^{-k_0/3},$$

or

$$\left|2^{1/2} - \frac{2p-1}{2^{k_0+1}}\right| \ll \frac{1}{(2^{k_0})^{4/3}}. \tag{3.5}$$

Again Ridout's theorem tells us that the above Diophantine approximation (3.5) has only finitely many positive integer solutions (n, k_0), which we may discard for large x.

Thus, we may assume that $2^{2k_0+1} < x^3$. We now rewrite our equation (3.4) as

$$(2p-1)^2 + 2(2n^3)^2 = 2^{k+2} + 1. \tag{3.6}$$

Put $\Delta = 2^{k+2} + 1 \ll x^3$. The above relation (3.6) implies that $(X, Y) = (2p - 1, 2n^3)$ is a positive integer solution of the norm form equation $X^2 + 2Y^2 = \Delta$. In turn, this implies that the algebraic integer $X + i\sqrt{2}Y$ is a divisor of Δ in $\mathbb{Z}[i\sqrt{2}]$. Let $\tau_1(\Delta)$ be the number of divisors of Δ in $\mathbb{Z}[i\sqrt{2}]$. Since the estimate $\tau_1(\Delta) = \Delta^{o(1)}$ holds as $\Delta \to \infty$, it follows that for k fixed, the number of possibilities (n, p) in equation (3.6) is $\Delta^{o(1)} = x^{o(1)}$ as $x \to \infty$. Since $2^{2k_0+1} \le x^3$, it follows that k (hence, Δ) can take only $O(\log x)$ values. Thus, the triple (k, n, p) can take only $x^{o(1)}$ values as $x \to \infty$. Eliminating these $x^{o(1)}$ possibilities out of the $\gg x^{1/6}$ available for large x, we arrive at the conclusion that we may assume that k is even.

Put $k = 2k_0$. Then

$$p^{\gamma-1}(p-1) = 2^{2k_0} - 2n^6. \tag{3.7}$$

Since $\gamma \ge 2$, we get that $2^{2k_0} \equiv 2(n^3)^2 \pmod p$. Thus, $(2/p) = 1$, and since $p \equiv 3 \pmod 4$, we get that $p \equiv 7 \pmod 8$. Hence,

$$p^{\gamma-1}\left(\frac{p-1}{2}\right) = 2^{2k_0-1} - n^6. \tag{3.8}$$

We may assume that $k_0 \geq 2$, otherwise the right hand side above is ≤ 0, which is impossible. In relation (3.8), the factor $(p-1)/2$ is congruent to -1 modulo 4, while the right hand side of this relation is also congruent to -1 modulo 4. Thus, reducing relation (3.8) modulo 4, we learn that $p^{\gamma-1} \equiv 1 \pmod 4$, implying that γ is odd. We now reduce the equation

$$2n^6 = 2^{2k_0} - p^\gamma + p^{\gamma-1} \tag{3.9}$$

modulo 3 under the assumption that $p \neq 3$ to get that

$$2 \equiv 1 - p^\gamma + 1 \pmod 3,$$

therefore $3 \mid p^\gamma$, which is a contradiction. Thus, it must be the case that $p = 3$, but in this case the right hand side of equation (3.9) is 1 modulo 3 while the left hand side of it is 2 modulo 3.

This shows that the case k even is also impossible. Hence, we showed that there are $\gg x^{1/6}$ even positive integers $\leq x$ which are not of the form $2^k - \phi(m)$ for any positive integers k and m.

We now look at (iii). We choose the system of triples given by (2.2) and let $n_0 \equiv 2 \pmod 4$ and $n_0 \equiv -(2^{a_i}+1) \pmod{p_i}$ for $i = 1, \ldots, 7$. It turns out that the progression $n_0 \pmod{4p_1 \cdots p_7}$ consists of the set of positive integers n representable as

$$2p_1 \cdots p_6(a + 2p_7\ell)$$

for some nonnegative integer ℓ, where a is an odd positive integer whose class modulo p_7 is the class of $-(p_1 \cdots p_6)^{-1}$. One can take $a = 9517519$. We choose positive integers of the above form which are also of the form $6n^2$ for some positive integer n. Thus, $n = p_2 \cdots p_6 n_1$ for some positive integer n_1 with the property that

$$a + 2p_7\ell = p_2 \cdots p_6 n_1^2.$$

This is possible if and only if $(ap_2 \cdots p_6/p_7) = 1$. Since $a \equiv -(p_1 \cdots p_6)^{-1} \pmod{p_7}$, we get that $(ap_2 \cdots p_6/p_7) = (-p_1^{-1}/p_7) = (-3/p_7) = 1$, where the last equality holds because $p_7 \equiv 1 \pmod{12}$. Hence, it is possible to choose such n_1. Computationally, we can choose $n_1 \equiv 6495551 \pmod{2p_7}$. Writing

$$n_1 = 6495551 + 2p_7 n_2$$

for some nonnegative integer n_2, one checks that

$$2 + 6n^2 = 8p_7(A + Bn_2 + Cn_2^2),$$

where

$$\begin{aligned} A &= 397726654787030777543306093850 7, \\ B &= 164108291295882514131557823959 25, \\ C &= 169284173866063591478202585144 75. \end{aligned}$$

The polynomial $f(X) = A + BX + CX^2 \in \mathbb{Z}[X]$ is irreducible, has $C > 0$, A odd, and negative discriminant equal to

$$-3(p_2 \cdots p_6)^2.$$

By a well-known theorem of H. Iwaniec (see [9]) on almost-primes represented by quadratic polynomials, for a large positive real number x there are $\gg x^{1/2}/\log x$ such values of n having the property that $6n^2 \leq x$ and such that furthermore $f(n_2)$ is either prime, or it has exactly two prime factors each of which exceeds $c_{10}x^{1/10}$, where $c_{10} > 0$ is some positive constant.

From now on, we work with such numbers n. Assume that $6n^2 = \phi(m) - 2^k$ holds for some positive integers m and k. Assume first that $k \geq 2$. Then $2 \| \phi(m)$, therefore $m \in \{4, p^\gamma, 2p^\gamma\}$ for some odd prime p and some positive integer γ. The case $m = 4$ gives a negative value for $\phi(m) - 2^k$, which is not possible. Thus, we may assume that $m = p^\gamma$. Since $k \geq 2$ and n is odd, it follows that $p \equiv 3 \pmod 4$. In particular, $p \neq p_7$. Assume that $\gamma = 1$. Let $i \in \{1, \dots, 7\}$ be such that $k \equiv a_i \pmod{p_i}$. Then

$$p = 6n^2 + 1 + 2^k \equiv n_0 + 1 + 2^{a_i} \equiv 0 \pmod{p_i},$$

leading to $p = p_i$. This shows that $6n^2 < p_7$, which is impossible.

Assume now that $\gamma > 1$. Since $3 \mid 6n^2$, it follows easily that p cannot be 3, and it cannot be congruent to 1 modulo 3 either. Hence, $p \equiv 2 \pmod 3$. Let us now suppose that $k \geq 3$. Then $p^{\gamma-1}(p-1) = 6n^2 + 2^k$. Assume first that k is odd and write it as $k = 2k_0 + 1$. Then $p \mid p^{\gamma-1} \mid 3n^2 + (2^{k_0})^2$, which shows that $(-3/p) = 1$. Since $p \equiv 3 \pmod 4$, we get $1 = (-3/p) = -(3/p) = (p/3) = -1$ because $p \equiv 2 \pmod 3$, which is a contradiction. Thus, k cannot be odd. Assume now that $k = 2k_0$ is even. Then $p \mid p^{\gamma-1} \mid 3n^2 + 2(2^{k_0-1})^2$, leading to $(-6/p) = 1$. Since $p \equiv 3 \pmod 4$ and $p \equiv 2 \pmod 3$, we get $(-3/p) = -(3/p) = (p/3) = -1$. Now the relation $(-6/p) = 1$ leads to $(2/p) = -1$, so $p \equiv 3 \pmod 8$. Thus, $(p-1)/2 \equiv 1 \pmod 4$. Reducing the relation

$$3n^2 = p^{\gamma-1}\left(\frac{p-1}{2}\right) - 2^{k-1}$$

modulo 4 and using the fact that $k \geq 3$, shows now that $p^{\gamma-1} \equiv 3 \pmod 4$. Hence, γ is even. Now the relation

$$6n^2 = p^\gamma - p^{\gamma-1} - 2^{2k_0}$$

reduced modulo 3 gives $0 \equiv 1 - p^{\gamma-1} - 1 \pmod 3$, therefore $3 \mid p$, which is again a contradiction.

This shows that the case $k \geq 3$ is impossible. If $k = 2$, we then get $p^{\gamma-1}(p-1) = 6n^2 + 4$. If $\gamma \geq 3$, then $p^\gamma \ll x$, which in turn leads to $\gamma \ll \log x$ and $p \ll x^{1/3}$. Thus, there are at most $O(x^{1/3}\log x)$ possibilities for the pair (n, k) in this case, and we can discard these possibilities for large x from our totality of $\gg x^{1/2}/\log x$ possibilities for n. When $\gamma = 2$, we get the equation $p(p-1) = 6n^2 + 4$, which can be rewritten as $(2p-1)^2 - 6(2n)^2 = 17$. Reducing this last relation modulo 17, we learn that $(6/17) = 1$. However, this is impossible because $(6/17) = (2/17)(3/17) = (3/17) = (17/3) = (2/3) = -1$.

Thus, it remains to deal with the case $k = 1$. This is equivalent to $2 + 6n^2 = \phi(m)$. Then $2 + 6n^2 = 2(1 + 3n^2) \equiv 8 \pmod{16}$, therefore m is divisible by at most three odd primes and none of them is congruent to 1 modulo 3. Furthermore, assume that $p^2 \mid m$. Then $p \mid 2(1 + 3n^2)$. In particular, $(-3/p) = 1$. Since $(-3/p) = (p/3)$, it follows that $p \equiv 1 \pmod 3$, which is false. Thus, we have showed that the odd part of m (i.e., the largest odd divisor of m) is squarefree. Assume now that m has three odd prime factors $p_1 < p_2 < p_3$. Since $8 \| \phi(m)$, it follows that $m \in \{p_1p_2p_3, 2p_1p_2p_3\}$ and $\phi(m) = (p_1-1)(p_2-1)(p_3-1)$. Thus, $2 + 6n^2 = (p_1 - 1)(p_2 - 1)(p_3 - 1)$. If $p_1 > 3$, then $p_1 \equiv p_2 \equiv p_3 \equiv 2$

(mod 3), and reducing the last equation above modulo 3 we would get $2 \equiv 1 \pmod 3$, which is impossible. Thus, $p_1 = 3$ and we get

$$1 + 3n^2 = (p-1)(q-1), \tag{3.10}$$

where we use $p = p_2 < q = p_3$. Assume now that m has only two prime factors p_1 and p_2. Then $m \in \{p_1p_2, 2p_1p_2, 4p_1p_2\}$, leading to

$$2 + 6n^2 = (p_1 - 1)(p_2 - 1), \qquad \text{or} \qquad 1 + 3n^2 = (p_1 - 1)(p_2 - 1).$$

The second equation above is the same as (3.10). The first equation above leads, as before, to the contradiction $2 \equiv 1 \pmod 3$ assuming that $p_1 > 3$. Thus, $p_1 = 3$ and $p_2 = 2 + 3n^2$. Hence,

$$2 + 3n^2 = p, \tag{3.11}$$

where we write $p = p_2$. If m has only one odd prime factor p, then $m \in \{p, 2p, 4p, 8p\}$, therefore

$$2 + 6n^2 = p - 1, \qquad \text{or} \qquad 2 + 6n^2 = 2p - 2, \qquad \text{or} \qquad 2 + 6n^2 = 4p - 4.$$

The first and last equation lead to $3 \mid p$, therefore $p = 3$, so $n = 1$. The second equation is the same as (3.11). Finally, if m has no odd prime factors, then $m \le 2^4$, leading to $2 + 6n^2 \le 8$, therefore $n = 1$, which is impossible.

Thus, it remains to deal with the odd positive integers n such that $6n^2 \le x$ and such that either (3.10) holds for some odd primes $p < q$, or (3.11) holds for some prime p. However, let us recall that for us

$$2 + 6n^2 = 8p_7u,$$

where u is either prime or a product of two primes each exceeding $c_0x^{1/10}$ for some positive constant c_0. Thus, in the case of equation (3.11), we have

$$3n^2 + 1 = 4p_7u, \qquad \text{and} \qquad 3n^2 + 2 = p,$$

where p is prime and u is either prime or a product of two primes each exceeding $c_0x^{1/10}$. By the Brun method, it follows that if we put

$$\rho(q) = \#\{0 \le n \le q - 1 : (3n^2 + 1)(3n^2 + 2) \equiv 0 \pmod q\},$$

then the number of our n with $6n^2 \le x$ and such that (3.11) holds is

$$\ll x^{1/2} \prod_{p_7 < q < c_0x^{1/10}} \left(1 - \frac{\rho(q)}{q}\right) \ll x^{1/2} \exp\left(-\sum_{p_7 < q < c_0x^{1/10}} \frac{\rho(q)}{q}\right).$$

It is easy to see that $\rho(q) = 4$ for $q \equiv 1, 7 \pmod{24}$, $\rho(q) = 2$ for $q \equiv 5, 11, 13, 19 \pmod{24}$, and $\rho(q) = 0$ for $q \equiv 17, 23 \pmod{24}$. A little calculation together with Dirichlet's Theorem on Primes in Arithmetic Progressions shows that

$$\sum_{p_7 < q < c_0x^{1/10}} \frac{\rho(q)}{q} = \left(\frac{4 \cdot 2}{8} + \frac{2 \cdot 4}{8}\right) \log\log x + O(1) = 2\log\log x + O(1)$$

as $x \to \infty$. This shows that the number of such n for which (3.11) holds is $\ll x^{1/2}/(\log x)^2$, and we may discard these values of n out of our $\gg x^{1/2}/\log x$ possibilities.

Finally, assume that we are in the case of equation (3.10). Note that if $p = 3$, we then get $1 + 3n^2 = 2q - 2$, leading to $3 \mid q$, which is impossible.

Furthermore, observe that $2p_7+1$ is not a prime since it is a multiple of 3. Thus, $4p_7u = (p-1)(q-1)$, and it now follows that $u = q_1q_2$ is a product of two primes each exceeding $c_0x^{1/10}$ and $\{p,q\} = \{2p_7q_1+1, 2q_2+1\}$. We let $\mathcal{P}$ be the set of all primes p such that $(p-1)/2$ is either a prime, or p_7 times a prime. Clearly, p and q are in $\mathcal{P}$. For a positive real number t we put $\mathcal{P}(t) = \#(\mathcal{P}\cap[2,t])$. It follows easily from Brun's sieve that the estimate $\mathcal{P}(t) \ll t/(\log t)^2$ holds for all $t\ge 2$.

Let $y = \exp((\log x)^{1/2})$. We distinguish the following two cases:

Case 1. $p < x^{1/2}/y$.

Let $\mathcal{A}$ be the set of such n. Let us fix p and assume that $p = 2p_7q_1+1$, since the case when $p = 2q_1+1$ can be dealt with similarly. We need to bound from above the number of odd positive integers n with $6n^2 \le x$ such that $1+3n^2$ is a multiple of p_7q_1 and such that $q_2 = (1+3n^2)/(4p_7q_1)$ is a prime. By Brun's sieve, the number of such n is

$$\ll \frac{x^{1/2}}{p_7q_1} \prod_{\substack{p_7<r\le x^{1/2}/(p-1)\\ r\equiv 1 \pmod 3}} \left(1-\frac{2}{r}\right) \ll \frac{x^{1/2}}{p\log y} \ll \frac{x^{1/2}}{p(\log x)^{1/2}}.$$

Summing up over p, we get the number of n in our situation is

$$\#\mathcal{A} \ll \frac{x^{1/2}}{(\log x)^{1/2}} \sum_{\substack{p\in\mathcal{P}\\ p\ge c_0x^{1/10}}} \frac{1}{p}. \tag{3.12}$$

By Abel's summation formula and the fact that $\mathcal{P}(t) \ll t/(\log t)^2$, we get that

$$\sum_{\substack{p\in\mathcal{P}\\ p\ge c_0x^{1/10}}} \frac{1}{p} \ll \frac{\mathcal{P}(t)}{t}\Big|_{t=c_0x^{1/10}}^{t=\infty} + \int_{c_0x^{1/10}}^{\infty} \frac{\mathcal{P}(t)}{t^2}dt \ll \frac{1}{\log x},$$

which together with the estimate (3.12) gives

$$\#\mathcal{A} \ll \frac{x^{1/2}}{(\log x)^{3/2}} = o\left(\frac{x^{1/2}}{\log x}\right),$$

so for large x we can discard the numbers n in $\mathcal{A}$.

Case 2. $p \ge x^{1/2}/y$.

Let $\mathcal{B}$ be the set of such numbers. Again, we may assume that $p = 2p_7q_1+1$, since the case when $p = 2q_1+1$ can be dealt with analogously. In this case, the number of n with $6n^2 \le x$ and $1+3n^2$ divisible by q_1 is $\ll x^{1/2}/q_1 \ll x^{1/2}/p$. Summing up over all primes $p \in \mathcal{P}\cap[x^{1/2}/y, x^{1/2}]$, we get that

$$
\begin{aligned}
\#\mathcal{B} &\ll x^{1/2} \sum_{\substack{p\in\mathcal{P}\\ x^{1/2}/y\le p\le x^{1/2}}} \frac{1}{p} \ll x^{1/2}\left(\frac{\mathcal{P}(t)}{t}\Big|_{t=x^{1/2}/y}^{t=x^{1/2}} + \int_{x^{1/2}/y}^{x^{1/2}} \frac{\mathcal{P}(t)}{t^2}dt\right)\\
&\ll x^{1/2}\left(\frac{1}{(\log x)^2} + \int_{x^{1/2}/y}^{x^{1/2}} \frac{dt}{t(\log t)^2}\right)\\
&\ll x^{1/2}\left(\frac{1}{(\log x)^2} - \frac{1}{\log t}\Big|_{t=x^{1/2}/y}^{t=x^{1/2}}\right)\\
&\ll x^{1/2}\left(\frac{1}{(\log x)^2} + \frac{\log y}{(\log x)^2}\right) \ll \frac{x^{1/2}}{(\log x)^{3/2}},
\end{aligned}
$$

so for large x we can discard the numbers from $\mathcal{B}$ too from our totality of $\gg x^{1/2}/\log x$ positive integers n.

In conclusion, we have shown that there are $\gg x^{1/2}/\log x$ even positive integers $\le x$ not of the form $\phi(m) - 2^k$ for any positive integers m and k. □

4. Comments and Remarks

It is known that there are infinitely many odd positive integers n not of the form $\pm p^a \pm q^b$ for any prime powers p^a and q^b (see, for example, [2], [4], [5] and [17]). Yong-Gao Chen [3], has found 5 consecutive odd positive integers none of which is expressible as a sum of two prime powers. It makes sense to ask if there are infinitely many even positive integers not of the form $\pm\phi(m) \pm 2^k$ for some positive integers m and k. We believe this to be so, and, in fact, we believe that the number of such even positive integers $n \le x$ is $\gg x$ for large x. We leave this question to the reader. Regarding Theorem 2.1 (i), it is not known if there are infinitely many Fermat primes. In fact, it is believed that $F_4 = 65537$ is the largest Fermat prime (see [10] for more information on Fermat numbers). Nevertheless, one may ask if there are infinitely many Fermat numbers which are Sierpiński. How about infinitely many Mersenne numbers which are also Sierpiński, where we recall that a Mersenne number is a number of the form $2^p - 1$ for some prime p? On a related note, we point out that Luca and Stănică showed in [12] that there are infinitely many Fibonacci numbers which are not a difference or a sum of two prime powers, while in the recent paper [11], we showed that there are infinitely many Fibonacci numbers which are Sierpiński. Finally, we mention that similar questions to the ones dealt with in this paper can be asked for Riesel numbers, and some results in this direction have been recently obtained by Marcos González [8].

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FACTORIZATIONS AND REPRESENTATIONS OF SECOND ORDER LINEAR RECURRENCES WITH INDICES IN ARITHMETIC PROGRESSIONS

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ABSTRACT. In this paper we consider second order recurrences $\{V_k\}$ and $\{U_n\}$. We give second order linear recurrences for the sequences $\left\{V_{\pm kn}\right\}$ and $\left\{U_{\pm kn}\right\}$. Using these recurrence relations, we derive relationships between the determinants of certain matrices and these sequences. Further, as generalizations of the earlier results, we give representations and trigonometric factorizations of these sequences by matrix methods and methods relying on Chebyshev polynomials of the first and second kinds. We give the generating functions and some combinatorial representations of these sequences.

1. Introduction

Let A and B be nonnegative integers such that $A^2 + 4B \neq 0$. The generalized Lucas sequence $\left\{V_n\left(A,B\right)\right\}$ and the generalized Fibonacci sequence $\left\{U_n\left(A,B\right)\right\}$ are defined by: for $n > 0$

$$\begin{aligned} V_{n+1}\left(A,B\right) &= AV_n\left(A,B\right) + BV_{n-1}\left(A,B\right) \\ U_{n+1}\left(A,B\right) &= AU_n\left(A,B\right) + BU_{n-1}\left(A,B\right) \end{aligned}$$

where $V_0\left(A,B\right) = 2, V_1\left(A,B\right) = A$ and $U_0\left(A,B\right) = 0, U_1\left(A,B\right) = 1$, respectively. We will frequently use the notations V_n and U_n instead of $V_n\left(A,B\right)$ and $U_n\left(A,B\right)$. The Binet formulas of the sequences $\{U_n\}$ and $\{V_n\}$ are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad V_n = \alpha^n + \beta^n$$

where α and β are the roots of the equation $t^2 - At - B = 0$.

When $A = B = 1$, then $V_n(1,1) = L_n$ (nth Lucas number) and $U_n(1,1) = F_n$ (nth Fibonacci number).

Lind (cf. [8], p. 478) first gave the following trigonometric factorization of Fibonacci numbers and then, two years later Zeitlin derived a factorization of

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the Lucas numbers using trigonometric factorizations of the Chebyshev polynomials of the first kind [19]:

$$F_n = \prod_{k=1}^{n} (1 - 2i \cos \left(k\pi/n\right),$$

$$L_n = \prod_{k=0}^{n-1} \left(1 - 2i \cos \left(2k+1\right) \pi/2n\right).$$

In [17] and [4], the authors gave complex factorization of the Fibonacci numbers by considering the roots of Fibonacci polynomials. In [10], the author established the following representations:

$$F_n = i^{n-1} \frac{\sin\left(n \cos^{-1}\left(-\frac{i}{2}\right)\right)}{\sin(\cos^{-1}(-\frac{i}{2}))}, \; L_n = 2i^n \cos\left(n \cos^{-1}\left(-\frac{i}{2}\right)\right), \; n \geq 1$$

Also in [3], the authors obtained the same results on the trigonometric factorizations of the Fibonacci and Lucas numbers by matrix methods. The matrix method was first used by them.

Recently, in [7], the authors consider the backward second order linear recurrences and they gave the trigonometric factorizations and representations of these sequences. Note that this case will be special case with $k = 1$ of the present paper.

The second order linear recurrences have been studied by many authors. For example, in [1], the author gave the following combinatorial representation:

$$V_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} A^{n-2k} B^k \tag{1.1}$$

$$U_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} A^{n-2k} B^k \tag{1.2}$$

There are many relationships between linear recurrence relations and determinants of certain matrices. For example, the generalized Lucas sequence can be obtained by the following determinant (see [15], [16], [5], [6]):

$$\begin{vmatrix} A & -2B & & & \\ 1 & A & -B & & \\ & 1 & A & \ddots & \\ & & \ddots & \ddots & -B \\ & & & 1 & A \end{vmatrix} = V_n.$$

Especially the case $A = B = 1$ can be found in [2]. Furthermore one can find similar special relationships in [16], [15], [9], [5], [6].

In this paper, we consider the positively and negatively kn subscripted terms of the sequences $\{V_n\}$ and $\{U_n\}$, and we derive relationships between these and the determinants of certain tridiagonal matrices. Then we give the general trigonometric factorizations and representations of terms of these sequences $\{V_{\mp kn}\}$ and $\{U_{\mp kn}\}$. We also present generating functions and combinatorial representations of these sequences.

2. Forward and Backward Generalized Lucas Sequences $\{V_{kn}\}, \{V_{-kn}\}$

In this section, for an arbitrary positive integer k, we consider the terms V_{kn} and give a second order linear recurrence relation for the sequence $\{V_{kn}\}$. We start with the following useful lemma.

LEMMA (2.1). *For $k > 0$, and $n > 1$, the terms V_{kn} satisfy the following recurrence relation*

$$V_{kn} = V_k V_{k(n-1)} + (-1)^{k+1} B^k V_{k(n-2)}.$$

Proof. From the Binet formula and since $\alpha\beta = -B$, we can write

$$\begin{aligned}V_k V_{k(n-1)} + (-1)^{k+1} B^k V_{k(n-2)} &= V_k V_{k(n-1)} - (-B)^k V_{k(n-2)} \\ &= \left((\alpha^k + \beta^k)\left(\alpha^{k(n-1)} + \beta^{k(n-1)}\right) - (\alpha\beta)^k\left(\alpha^{k(n-2)} + \beta^{k(n-2)}\right)\right) \\ &= \alpha^{kn} + \beta^{kn} = V_{kn},\end{aligned}$$

which is as desired. □

Now we describe a relationship between the terms of sequence $\{V_{kn}\}$ and the determinant of a certain tridiagonal matrix.

Define the $n \times n$ tridiagonal matrix $T_n = \left(t_{ij}\right)$ by

$$T_n = \begin{bmatrix} V_k & 2(-B)^{k/2} & & & \\ (-B)^{k/2} & V_k & (-B)^{k/2} & & \\ & (-B)^{k/2} & V_k & \ddots & \\ & \ddots & & \ddots & (-B)^{k/2} \\ & & & (-B)^{k/2} & V_k \end{bmatrix}.$$

THEOREM (2.2). *For $n > 1$*

$$\det T_n = V_{kn},$$

where $\det T_1 = V_k$.

Proof. We will use the principle of mathematical induction to show that $\det T_n = V_{kn}$. If $n = 2$, then, by Lemma (2.1), we obtain

$$\det T_2 = \begin{vmatrix} V_k & 2(-B)^{k/2} \\ (-B)^{k/2} & V_k \end{vmatrix} = V_{2k}.$$

Suppose that the equation holds for $n-1$. Then we show that the equation holds for n. Expanding $\det T_n$ by the Laplace expansion of a determinant according to the last row, we obtain

$$\det T_n = V_k \det T_{n-1} - (-B)^k \det T_{n-2}.$$

By our assumption and the result of Lemma (2.1), we have the required conclusion:

$$\det T_n = V_k V_{k(n-1)} - (-B)^k V_{k(n-2)} = V_{kn}.$$

□

In the remaining of the section we consider the terms of the backward Lucas sequence $\{V_{-kn}\}$, and we give a second order linear recurrence relation for these, similar to the positively subscripted terms. Then we determine a certain matrix whose successive determinants equal the terms V_{-kn}.

LEMMA (2.3). *For $k \geq 1$ and $n > 1$,*

$$V_{-kn} = (-B)^{-k} \left(V_k V_{-k(n-1)} - V_{-k(n-2)} \right).$$

Proof. From the Binet formulas of sequence $\{V_{-n}\}$, we have that $\alpha\beta = -B$ and so $V_{-n} = V_n (-B)^{-n}$. The proof follows from Lemma (2.1). □

Now we give a relationship between the determinant of a certain tridiagonal matrix and the terms of the backward general Lucas sequence.

Define the $n \times n$ tridiagonal matrix H_n by

$$H_n = \begin{bmatrix} V_{-k} & 2(-B)^{-k/2} & & & \\ (-B)^{-k/2} & V_{-k} & (-B)^{-k/2} & & \\ 0 & (-B)^{-k/2} & V_{-k} & \ddots & \\ & & \ddots & \ddots & (-B)^{-k/2} \\ & & & (-B)^{-k/2} & V_{-k} \end{bmatrix}.$$

As a consequence of Theorem (2.2), we have the following result.

COROLLARY (2.4). *For $n > 1$,*

$$\det H_n = V_{-kn}$$

where $\det H_1 = V_{-k}$.

Proof. From the definitions of the matrices H_n and T_n, using the identity $V_{-n} = (-B)^{-n} V_n$, it is seen that $H_n = (-B)^{-k} T_n$, and so

$$\det H_n = (-B)^{-kn} \det T_n. \tag{2.5}$$

By Theorem (2.2) and equation (2.5), we obtain

$$\det H_n = (-B)^{-kn} V_{kn} = V_{-kn}.$$

The proof is complete. □

3. Trigonometric factorizations of the General Lucas sequences $\{V_{kn}\}$ and $\{V_{-kn}\}$

In this section we give the trigonometric factorizations and representations of the generalized Lucas sequences $\{V_{kn}\}$ and $\{V_{-kn}\}$ by matrix methods.

Define the $n \times n$ tridiagonal matrix Q as below:

$$Q = \begin{bmatrix} 0 & 2 & & \\ 1 & 0 & \ddots & \\ & \ddots & & 1 \\ & & 1 & 0 \end{bmatrix}.$$

The characteristic equation of the matrix Q satisfies the following equation

$$t_{n+1}(\lambda) = -\lambda t_n(\lambda) - t_{n-1}(\lambda), \; n > 0,$$

where $t_0(\lambda) = -\lambda$ and $t_1(\lambda) = \lambda^2 - 2$.

The Chebyshev polynomials of the first kind are defined by the following equation

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),\ n > 1,$$

where $T_0(x) = 1,\ T_1(x) = x$.

The zeros of the Chebyshev polynomials of the first kind are given by (for more details see [11], [12], [14])

$$x_k = \cos\frac{(2k-1)\pi}{2n},\ k = 1, 2, \ldots, n.$$

If we take $\lambda \equiv -2x$, then the sequence $\{t_n(\lambda)\}$ is reduced to the sequence of Chebyshev polynomials of the first kind, $\{2T_n(x)\}$. Therefore the zeros of the characteristic equation of matrix Q are given by

$$\lambda_k = -2\cos\frac{(2k-1)\pi}{2n},\ \text{for } k = 1, 2, \ldots, n \tag{3.1}$$

From the definitions of Q and T_n, we can write $T_n = V_k I_n + (-B)^{k/2} Q$ where I_n is the $n \times n$ unit matrix.

THEOREM (3.2). *For $n > 1$,*

$$V_{kn} = \prod_{r=1}^{n}\left[V_k - 2(-B)^{k/2}\cos\left(\tfrac{(2r-1)\pi}{2n}\right)\right].$$

Proof. Let λ_r, $r = 1, 2, \ldots, n$, be the eigenvalues of Q with respect to eigenvectors x_r. Then, for all r

$$T_n x_k = \left[V_k I_n + (-B)^{k/2} Q\right] x_r = V_k I_n x_r + (-B)^{k/2} Q x_r = \left[V_k + (-B)^{k/2}\lambda_r\right] x_r.$$

Thus $\mu_r = V_k + (-B)^{k/2}\lambda_r$, $r = 1, 2, \ldots, n$, are the eigenvalues of T_n. Thus by (3.1)

$$\det T_n = \prod_{r=1}^{n}\left[V_k + (-B)^{k/2}\lambda_r\right] = \prod_{r=1}^{n}\left[V_k - 2(-B)^{k/2}\cos\left(\tfrac{(2r-1)\pi}{2n}\right)\right],$$

and the proof is complete. □

As a corollary, we obtain Lind's result [8], p. 478.

COROLLARY (3.3). *When $A = B = k = 1$, then by the above theorem, we obtain*

$$L_n = \prod_{r=1}^{n}\left[1 - 2i\cos\left(\tfrac{(2r-1)\pi}{2n}\right)\right].$$

As a consequence of Theorem (3.2), we give the following corollary.

COROLLARY (3.4). *For $n > 1$,*

$$V_{-kn} = \prod_{r=1}^{n}\left[V_{-k} - 2(-B)^{-k/2}\cos\left(\tfrac{(2r-1)\pi}{2n}\right)\right].$$

Proof. From Theorem (3.2), we have

$$V_{kn} = \prod_{r=1}^{n} \left[V_k - 2\left(-B\right)^{k/2} \cos\left(\tfrac{(2r-1)\pi}{2n}\right) \right].$$

Multiplying the above equation by $(-B)^{kn}$, we have the conclusion since $V_{-n} = (-B)^{-n} V_n$. □

Alternatively, one can consider the equation $H_n = V_{-k} I_n + (-B)^{-k/2} Q$, and the next result follows.

THEOREM (3.5). *For $k \geq 1$ and $n > 1$,*

$$V_{\mp kn} = (-B)^{\mp kn/2} \cos\left(n \cos^{-1}\left(\tfrac{V_{\mp k}}{2(-B)^{\mp k/2}}\right)\right).$$

Proof. First, we consider the case V_{kn}. If the $n \times n$ matrix G_n has the following form

$$G_n(x) = \begin{bmatrix} 2x & 2 & & 0 \\ 1 & 2x & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 2x \end{bmatrix}, \tag{3.6}$$

then it is seen that $\det G_n(x) = 2T_n(x)$ where $\{T_n(x)\}$ is the sequence of the Chebyshev polynomials of the first kind. Thus

$$\det T_n = (-B)^{kn/2} \det G_n\left(\tfrac{V_k}{2(-B)^{k/2}}\right) = (-B)^{kn/2}\, T_n\left(\tfrac{V_k}{2(-B)^{k/2}}\right)$$

Defining $x = \cos\theta$ allows the Chebyshev polynomials of the second kind to be written as (see [12])

$$T_n(x) = \cos n\,\theta. \tag{3.7}$$

Then by (3.7) and the value of determinant of matrix T_n, we obtain

$$V_{kn} = (-B)^{kn/2} \cos\left(n \cos^{-1}\left(\tfrac{V_k}{2(-B)^{k/2}}\right)\right).$$

For the case of V_{-kn}, we consider

$$\det H_n = (-B)^{-kn/2} \det G\left(\tfrac{V_{-k}}{2(-B)^{-k/2}}\right) = (-B)^{-kn/2}\, T_n\left(\tfrac{V_{-k}}{2(-B)^{-k/2}}\right),$$

from which the proof follows. □

COROLLARY (3.8). *For $k \geq 1$ and $n > 1$ even,*

$$V_{\mp kn} = \prod_{r=1}^{\lfloor n/2 \rfloor} \left[V_{\mp k}^2 - 4\left(-B\right)^{\mp k} \cos^2\left(\tfrac{(2r-1)\pi}{2n}\right) \right]$$

and for $n > 1$ odd

$$V_{\mp kn} = V_{\mp k} \prod_{k=1}^{(n-1)/2} \left[V_{\mp k}^2 - 4\left(-B\right)^{\mp k} \cos^2\left(\tfrac{(2r-1)\pi}{2n}\right) \right].$$

Proof. These are immediate consequences of Theorem (3.2) and Corollary (3.4), since, for $1 \leq k < n/2$, $\cos\left(k\pi/n\right) = -\cos\left(\left(n-k\right)\pi/n\right)$. □

4. The Generalized Fibonacci Sequence $\{U_n(A,B)\}$

In this section, we consider the recurrence $\{U_n\}$ and then obtain two recurrence relations for the sequences $\{U_{kn}\}$ and $\{U_{-kn}\}$. Also we determine certain tridiagonal matrices and then we obtain relationships between the determinants of these matrices and the sequences $\{U_{kn}\}$ and $\{U_{-kn}\}$. Therefore, we obtain trigonometric factorizations and representations of these sequences. We start with the following useful lemma.

LEMMA (4.1). *For $k \geq 1$ and $n > 1$,*

$$U_{kn} = V_k U_{k(n-1)} + (-1)^{k+1} B^k U_{k(n-2)}.$$

Proof. From the Binet formula of the sequences $\{U_n\}$, $\{V_n\}$ and since $\alpha\beta = -B$, we can write

$$\begin{aligned} V_k U_{k(n-1)} + (-1)^{k+1} B^k U_{k(n-2)} &= V_k U_{k(n-1)} - (-B)^k U_{k(n-2)} \\ &= \left((\alpha^k + \beta^k) \left(\tfrac{\alpha^{k(n-1)} - \beta^{k(n-1)}}{\alpha - \beta} \right) - (\alpha\beta)^k \left(\tfrac{\alpha^{k(n-2)} - \beta^{k(n-2)}}{\alpha - \beta} \right) \right) \\ &= \tfrac{\alpha^{kn} - \beta^{kn}}{\alpha - \beta} = U_{kn}. \end{aligned}$$

The proof is complete. □

Define the $n \times n$ tridiagonal Toeplitz matrix M_n by

$$M_n = \begin{bmatrix} V_k & (-B)^{k/2} & & \\ (-B)^{k/2} & V_k & \ddots & \\ & \ddots & \ddots & (-B)^{k/2} \\ & & (-B)^{k/2} & V_k \end{bmatrix}.$$

Since $U_{2k} = U_k V_k$, we have immediately the following result.

THEOREM (4.2). *For $n > 1$,*

$$\det M_n = \frac{U_{k(n+1)}}{U_k},$$

where $\det M_1 = U_{2k}/U_k$.

Proof. It is known that the tridiagonal Toeplitz matrices satisfy

$$\det M_n = V_k \det M_{n-1} - (-B)^k \det M_{n-2}.$$

From the principle of mathematical induction and Lemma (4.1), the result follows. □

LEMMA (4.3). *For $k \geq 1$ and $n > 1$,*

$$U_{-k(n+1)} = (-B)^{-k} \left(V_k U_{-kn} - U_{-k(n-1)} \right) = V_{-k} U_{-kn} - (-B)^{-k} U_{-k(n-1)}$$

Proof. By the Binet formulas of $\{U_{-n}\}$ and $\{V_{-n}\}$, we can write

$$\begin{aligned}(-B)^{-k}V_kU_{-kn} - (-B)^{-k}U_{-k(n-1)} &= (-B)^{-k}\left(V_kU_{-kn} - U_{-k(n-1)}\right)\\ &= (-B)^{-k}\left(\left(\alpha^k+\beta^k\right)\left(\tfrac{\alpha^{-kn}-\beta^{-kn}}{\alpha-\beta}\right) - \left(\tfrac{\alpha^{-k(n-1)}-\beta^{-k(n-1)}}{\alpha-\beta}\right)\right)\\ &= (-B)^{-k}\left(\tfrac{\alpha^{-kn+k}-\beta^{-kn+k}-\alpha^k\beta^{-kn}+\beta^k\alpha^{-kn}-\alpha^{-kn+k}+\beta^{-kn+k}}{\alpha-\beta}\right)\\ &= \left(\alpha^{-k}\beta^{-k}\right)\left(\tfrac{-\alpha^k\beta^{-kn}+\beta^k\alpha^{-kn}}{\alpha-\beta}\right)\\ &= \tfrac{\alpha^{-kn-k}-\beta^{-kn-k}}{\alpha-\beta} = U_{-k(n+1)},\end{aligned}$$

and the conclusion follows. □

Define the $n\times n$ tridiagonal Toeplitz matrix E_n as shown below:

$$E_n = \begin{bmatrix} V_{-k} & (-B)^{-k/2} & & \\ (-B)^{-k/2} & V_{-k} & \ddots & \\ & \ddots & \ddots & (-B)^{-k/2} \\ & & (-B)^{-k/2} & V_{-k} \end{bmatrix}.$$

COROLLARY (4.4). *For $n > 1$,*

$$\det E_n = \frac{U_{-k(n+1)}}{U_{-k}}$$

where $\det E_1 = U_{2k}/U_k$.

Proof. Since $E_n = (-B)^{-k} M_n$, $\det E_n = (-B)^{kn} \det M_n$. By Theorem (4.2), the proof follows easily. □

5. Trigonometric factorization of the General Fibonacci sequences $\{U_{kn}\}$ and $\{U_{-kn}\}$

In this section, we give the trigonometric factorizations and representations of sequences $\{U_{kn}\}$ and $\{U_{-kn}\}$ by matrix methods and the Chebyshev polynomials of the second kind.

Define the $n\times n$ tridiagonal matrix W as shown:

$$W = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & \ddots & \\ & \ddots & & 1 \\ & & 1 & 0 \end{bmatrix}.$$

The characteristic equation of the matrix W satisfies the following recurrence

$$f_{n+1}(y) = -yf_n(y) - f_{n-1}(y),\ n > 0,$$

where $f_0(y) = -y$ and $f_1(y) = y^2 - 1$.

The Chebyshev polynomials of the second kind are defined by the recurrence relation for $n > 1$

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x)$$

where $U_0(x) = 1,\ U_1(x) = 2x$.

The zeros of the Chebyshev polynomials of the second kind is given by (see [11], [12], [14]):

$$x_k = \cos \tfrac{k\pi}{n+1}, \quad k = 1, 2, \dots, n,$$

Taking $\lambda \equiv -2x$, the sequence $\{f_n(y)\}$ is reduced to the sequence $\{U_n(x)\}$. Then the zeros of the characteristic equation of matrix W are given by

$$y_k = -2\cos \tfrac{k\pi}{n}, \quad \text{for } k = 1, 2, \dots, n. \tag{5.1}$$

By the definitions of W, M_n and E_n, we write $M_n = V_k I_n + (-B)^{k/2} W$ and $E_n = V_{-k} I_n + (-B)^{-k/2} W$ where I_n is the $n \times n$ unit matrix.

THEOREM (5.2). *Then for $n > 1$,*

$$U_{\mp k(n+1)} = U_{\mp k} \prod_{r=1}^{n} \left[V_{\mp k} - 2(-B)^{\mp k/2} \cos\left(\tfrac{r\pi}{n+1}\right) \right].$$

Proof. Let y_r, $r = 1, 2, \dots, n$, be the eigenvalues of matrix W with respect to the eigenvectors x_r. Then, for all $r = 12, \dots, n$

$$M_n x_k = \left[V_k I_n + (-B)^{k/2} W \right] x_r = V_k I_n x_r + (-B)^{k/2} W x_r = \left[V_k + (-B)^{k/2} y_r \right] x_r$$

Thus $\omega_r = V_k + (-B)^{k/2} y_r$, $r = 1, 2, \dots, n$, are the eigenvalues of M_n. Thus by (5.1) and Theorem (4.2),

$$\det M_n = U_k \prod_{r=1}^{n} \left[V_k + (-B)^{k/2} y_r \right] = U_k \prod_{r=1}^{n} \left[V_k - 2(-B)^{k/2} \cos\left(\tfrac{r\pi}{n}\right) \right].$$

Similarly, one can obtain that $c_r = V_{-k} + (-B)^{-k/2} y_r$, $r = 1, 2, \dots, n$, are the eigenvalues of the matrix E_n. Thus we obtain

$$\det E_n = U_{-k} \prod_{r=1}^{n} \left[V_{-k} + (-B)^{-k/2} y_r \right] = U_{-k} \prod_{r=1}^{n} \left[V_{-k} - 2(-B)^{-k/2} \cos\left(\tfrac{r\pi}{n+1}\right) \right].$$

Considering the value of $\det E_n$, the proof is complete. □

For example, when $k = 5$, $A = 1$, $B = 1$ in sequence $\{U_n(A, B)\}$, then

$$F_{5(n+1)} = 5 \prod_{r=1}^{n} \left[11 - 2i \cos\left(\tfrac{r\pi}{n+1}\right) \right].$$

COROLLARY (5.3). *For an arbitrary positive integer k and $n > 1$ even,*

$$U_{\mp k(n+1)} = U_{\mp k} \prod_{r=1}^{\lfloor n/2 \rfloor} \left[V_{\mp k}^2 - 4(-B)^{\mp k} \cos^2\left(\tfrac{r\pi}{n+1}\right) \right]$$

and for $n > 1$ odd

$$U_{\mp k(n+1)} = U_{\mp 2k} \prod_{k=1}^{\lfloor n/2 \rfloor} \left[V_{\mp k}^2 - 4(-B)^{\mp k} \cos^2\left(\tfrac{r\pi}{n+1}\right) \right].$$

Proof. The proof follows from Theorem (5.2), since, for $1 \le k < n/2$, $\cos(k\pi/n) = -\cos((n-k)\pi/n)$ and $U_n V_n = U_{2n}$. □

THEOREM (5.4). *For $k \geq 1$ and $n > 1$,*

$$U_{\mp k(n+1)} = U_{\mp k} \frac{(-B)^{\mp kn/2} \sin\left[(n+1)\cos^{-1}\left(\frac{V_{\mp k}}{2(-B)^{\mp k/2}}\right)\right]}{\sin\left(\cos^{-1}\left(\frac{V_{\mp k}}{2(-B)^{\mp k/2}}\right)\right)}.$$

Proof. Let the matrix K_n be defined by

$$K_n(x) = \begin{bmatrix} 2x & 1 & & 0 \\ 1 & 2x & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 2x \end{bmatrix}_{n \times n}. \tag{5.5}$$

It is known that $\det K_n(x) = U_n(x)$, where $\{U_n(x)\}$ is the sequence of the Chebyshev polynomials of the second kind. Thus we obtain

$$\det M_n = U_k(A, B)(-B)^{kn/2} \det K_n\left(\frac{V_k(A,B)}{2(-B)^{k/2}}\right) = (-B)^{kn/2} U_n\left(\frac{V_k(A,B)}{2(-B)^{k/2}}\right)$$

If $x = \cos\theta$, the Chebyshev polynomials of the second kind can be written as (see [12])

$$U_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta}. \tag{5.6}$$

Then by (5.6) and the value of the determinant of the matrix M_n, we obtain

$$U_{k(n+1)}(A, B) = \frac{U_k(A,B)(-B)^{kn/2} \sin\left[(n+1)\cos^{-1}\left(\frac{V_k(A,B)}{2(-B)^{k/2}}\right)\right]}{\sin\left(\cos^{-1}\left(\frac{V_k(A,B)}{2(-B)^{k/2}}\right)\right)}.$$

From

$$\det E_n = (-B)^{-kn/2} \det K_n\left(\frac{V_{-k}}{2(-B)^{-k/2}}\right) = (-B)^{-kn/2} U_n\left(\frac{V_{-k}}{2(-B)^{-k/2}}\right)$$

and the values of the determinants of matrices E_n and $K_n(x)$, we obtain the conclusion. □

6. Generating Functions

In this section, we give combinatorial representations and generating functions for the terms of sequences $\{V_{kn}\}$ and $\{V_{-kn}\}$, thus generalizing in one direction some results of [13].

THEOREM (6.1). *For an arbitrary positive integer k and $n > 0$,*

$$V_{\mp kn} = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{n}{n-r}\binom{n-r}{r} V_{\mp k}^{n-2r} B^{\mp kr}$$

$$U_{\mp k(n+1)} = U_{\mp k} \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n-r}{r} V_{\mp k}^{n-2r} B^{\mp kr}$$

Proof. We leave the proof of this theorem to the reader. □

Generating functions are useful tools for solving linear homogeneous recurrence relations with constant coefficients (for more details about generating functions of recurrence relations see [18]). Now we give the generating functions for any power of the sequences $\{V_{kn}\}$ and $\{V_{-kn}\}$, generalizing known identities.

THEOREM (6.2). *Let $\{V_n\}$ be the generalized Lucas sequence.*
(a) *If r is odd, then*

$$\sum_{n=0}^{\infty} V_{\pm kn}^{r} x^{n} = \sum_{i=0}^{(r-1)/2} \binom{r}{i} \frac{2-(-B)^{ki} V_{\pm k(r-2i)} x}{1-(-B)^{ki} V_{\pm k(r-2i)} x + (-B)^{\pm kr} x^{2}}.$$

(b) *If r is even, then*

$$\sum_{n=0}^{\infty} V_{\pm kn}^{r} x^{n} = \sum_{i=0}^{r/2-1} \binom{r}{i} \frac{2-(-B)^{ki} V_{\pm k(r-2i)} x}{1-(-B)^{ki} V_{\pm k(r-2i)} x + (-B)^{\pm kr} x^{2}} + \binom{r}{r/2} \frac{1}{1-(-B)^{\pm kr/2} x}.$$

Proof. For easy writing, let

$$G_{k,r}(x) = \sum_{n=0}^{\infty} V_{kn}^{r} x^{n}.$$

We deal with the case $r = 1$ separately, as it can be handled easily by a slightly different method from the general case. First we consider the positively subscripted case:

$$\begin{aligned}\left(1 - V_k x + (-B)^k x^2\right) G_{k,1}(x) = V_0 &+ (V_k - V_k V_0)\, x \\ &+ \left(V_{2k} - V_k V_k + (-B)^k V_0\right) x^2 + \cdots \\ &+ \left(V_{kn} - V_k V_{k(n-1)} + (-B)^k V_{k(n-2)}\right) x^n + \cdots .\end{aligned}$$

Since $V_0 = 2$ and by the recurrence relation of $\{V_{kn}\}$ of Lemma (2.1) the coefficients of x^n for $n > 1$ are all 0. Thus

$$\left(1 - V_k x + (-B)^k x^2\right) G_{k,1}(x) = V_0 + V_k (1 - V_0)\, x$$

and so

$$\sum_{n=0}^{\infty} V_{kn} x^n = \frac{V_0 + V_k (1 - V_0)\, x}{1 - V_k x + (-B)^k x^2}.$$

For the negatively subscripted case, the proof for $r = 1$ case follows from Lemma (2.3) and an argument similar to the one above.

We shall give the proof for arbitrary power r, only for positively subscripted case, since the negatively subscripted one is similar. We write

$$V_{kn}^{r} = \left(\alpha^{kn} + \beta^{kn}\right)^{r} = \sum_{i=0}^{r} \binom{r}{i} \alpha^{kni} \beta^{kn(r-i)},$$

and so,

$$\begin{aligned}
G_{k,r}(x) &= \sum_{n=0}^{\infty}\sum_{i=0}^{r}\binom{r}{i}\left(\alpha^{ki}\beta^{k(r-i)}x\right)^n \\
&= \sum_{i=0}^{r}\binom{r}{i}\sum_{n=0}^{\infty}\left(\alpha^{ki}\beta^{k(r-i)}x\right)^n \\
&= \sum_{i=0}^{r}\binom{r}{i}\frac{1}{1-\alpha^{ki}\beta^{k(r-i)}x}.
\end{aligned}$$

We will deal with the case of r odd, only, since the case of r even is similar. Thus, under r odd, using $\binom{r}{i}=\binom{r}{r-i}$ and $\alpha\beta=-B$, we get

$$\begin{aligned}
G_{k,r}(x) &= \sum_{i=0}^{(r-1)/2}\binom{r}{i}\left(\frac{1}{1-\alpha^{ki}\beta^{k(r-i)}x}+\frac{1}{1-\alpha^{k(r-i)}\beta^{ki}x}\right) \\
&= \sum_{i=0}^{(r-1)/2}\binom{r}{i}\frac{2-\alpha^{k(r-i)}\beta^{ki}x-\alpha^{ki}\beta^{k(r-i)}x}{1-\alpha^{k(r-i)}\beta^{ki}x-\alpha^{ki}\beta^{k(r-i)}x+(\alpha\beta)^{kr}x^2} \\
&= \sum_{i=0}^{(r-1)/2}\binom{r}{i}\frac{2-(-B)^{ki}V_{k(r-2i)}x}{1-(-B)^{ki}V_{k(r-2i)}x+(-B)^{kr}x^2},
\end{aligned}$$

since $\alpha^{k(r-i)}\beta^{ki}=(-B)^{ki}\alpha^{k(r-2i)}$, $\alpha^{ki}\beta^{k(r-i)}=(-B)^{ki}\beta^{k(r-2i)}$, and $\alpha^{k(r-2i)}+\beta^{k(r-2i)}=V_{k(r-2i)}$. □

Regarding the generalized Fibonacci sequence, we can show the following theorem that generalizes Theorem 1 of [13], and [8], Formulas 1 and 17 on p. 230. Since the proof is somewhat similar to the proof of Theorem (6.2), we will leave it to the interested reader.

THEOREM (6.3). *Let $\{U_n\}$ be the generalized Fibonacci sequence.*

(*a*) *If r is odd, then*

$$\sum_{n=0}^{\infty}U_{\pm kn}^{r}x^n=\delta^{r-1}\sum_{i=0}^{(r-1)/2}(-1)^i\binom{r}{i}\frac{(-B)^{\pm ki}U_{k(r-2i)}x}{1-(-B)^{\pm ki}V_{\pm k(r-2i)}x+(-B)^{\pm kr}x^2}.$$

(*b*) *If r is even, then*

$$\begin{aligned}
\sum_{n=0}^{\infty}U_{\pm kn}^{r}x^n=\delta^{r}\sum_{i=0}^{r/2-1}(-1)^i\binom{r}{i}&\frac{2-(-B)^{\pm ki}V_{\pm k(r-2i)}x}{1-(-B)^{\pm ki}V_{\pm k(r-2i)}x+(-B)^{\pm kr}x^2} \\
&+\delta^r(-1)^{r/2}\binom{r}{r/2}\frac{1}{1-B^{\pm kr/2}x}.
\end{aligned}$$

There is nothing special about the arithmetic progression kn, so one can obtain similar formulas for indices in any other arithmetic progression modulo k. We chose this particular one, namely kn, since it is consistent with the first part of the paper, and the results are easier to state.

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SOME SPECIAL CHARACTER SUMS OVER ELLIPTIC CURVES

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ABSTRACT. Let $\mathbf{E}(\mathbb{F}_q)$ be the set of $\mathbb{F}_q$-rational points on an elliptic curve $\mathbf{E}$ over a finite field $\mathbb{F}_q$ of q elements given by an affine Weierstraß equation. We use $x(P)$ to denote the x-component of a point $P = (x(P), y(P)) \in \mathbf{E}$ and for an integer n consider character sums

$$S_n(a,b) = \sum_{P \in \mathbf{E}(\mathbb{F}_q)} \psi\left(ax(P) + bx(nP)\right), \qquad a, b \in \mathbb{F}_q,$$

with an additive character ψ of $\mathbb{F}_q$. In the case when $\gcd(n, \#\mathbf{E}(\mathbb{F}_q))$ is sufficiently large, we obtain a new bound for such sums. In particular, we show that for any positive integer $n \mid \#\mathbf{E}(\mathbb{F}_q)$, we have $S_n(a,b) = O(q^{9/10})$ uniformly over n, $a \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$.

1. Introduction

Let $\mathbf{E}$ be an elliptic curve defined over a finite field $\mathbb{F}_q$ of q elements given by an affine Weierstraß equation

$$\mathbf{E}: \; y^2 + (a_1x + a_3)y = x^3 + a_2x^2 + a_4x + a_6,$$

with some $a_1, \ldots, a_6 \in \mathbb{F}_q$, see [2], [4], [11].

We recall that the set of all points on $\mathbf{E}$ forms an Abelian group, with the "point at infinity" $\mathcal{O}$ as the neutral element, and we use $\oplus$ to denote the group operation. As usual, we write every point $P \neq \mathcal{O}$ on $\mathbf{E}$ as $P = (x(P), y(P))$.

Let $\mathbf{E}(\mathbb{F}_q)$ denote the set of $\mathbb{F}_q$-rational points on $\mathbf{E}$. Various character sums over points of elliptic curves have been considered in a number of papers, see [3], [5], [6], [8], [9], [7], [10] and references therein.

For a fixed additive character ψ of $\mathbb{F}_q$ and an integer n, we consider the character sums

$$S_n(a,b) = \sum_{P \in \mathbf{E}(\mathbb{F}_q)} \psi\left(ax(P) + bx(nP)\right), \qquad a, b \in \mathbb{F}_q.$$

It follows immediately from a much more general result of [8], Corollary 5 (which is turn is based on a bound of [7] that if at least one of a and b is a non-zero element of $\mathbb{F}_q$ and $n > 0$ then

$$S_n(a,b) = O(n^2 q^{1/2}). \tag{1.1}$$

Here we show that if $\gcd\left(n, \#\mathbf{E}(\mathbb{F}_q)\right)$ is not very small then the ideas of Akulinichev [1] can be combined with bounds of character sums from [7], [8], [10] and lead to improvements of (1.1).

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Throughout the paper, the implied constants in symbols 'O' and '$\ll$' may depend on an integer parameter $\nu \geq 1$ and are absolute otherwise (we recall that $U \ll V$ and $U = O(V)$ are both equivalent to the inequality $|U| \leq cV$ with some constant $c > 0$).

2. Main Result

We start with the case when $d = \gcd\left(n, \#\mathbf{E}(\mathbb{F}_q)\right)$ is very large.

THEOREM (2.1). *Let $n > 0$ be an arbitrary integer and let $d = \gcd\left(n, \#\mathbf{E}(\mathbb{F}_q)\right)$. Then for any $a \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$, we have*

$$S_n(a, b) \ll q^{3/2}/d.$$

Proof. Let $\mathcal{H}_d \subseteq \mathbf{E}(\mathbb{F}_q)$ be the subgroup of $\mathbf{E}(\mathbb{F}_q)$ consisting of the d-torsion points $Q \in \mathbf{E}(\mathbb{F}_q)$, that is, of points Q with $dQ = \mathcal{O}$.

It is well-known, see [2], [4], [11], that the group $\mathbf{E}(\mathbb{F}_q)$ is isomorphic to

$$\mathbf{E}(\mathbb{F}_q) \cong \mathbb{Z}_M \times \mathbb{Z}_L \tag{2.2}$$

for some unique integers M and L with

$$L \mid M, \qquad LM = \mathbf{E}(\mathbb{F}_q), \qquad L \mid q - 1. \tag{2.3}$$

Since $d \mid \#\mathbf{E}(\mathbb{F}_q)$ we see from (2.2) and (2.3) that we can write $d = d_1 d_2$ where $d_1 = \gcd(d, M)$ and $d_2 | d_1$. It is now easy to see that

$$\#\mathcal{H}_d \geq d, \tag{2.4}$$

(clearly $\mathcal{H}_d$ is a subgroup of the group $\mathbf{E}[d]$ of d-torsion points on $\mathbf{E}$, thus we also have $\#\mathcal{H}_d \leq d^2$, see [2], [4], [11]).

We remark that for any point $Q \in \mathbf{E}(\mathbb{F}_q)$ we have

$$S_n(a, b) = \sum_{P \in \mathbf{E}(\mathbb{F}_q)} \psi\left(ax(P \oplus Q) + bx(n(P \oplus Q))\right).$$

Therefore, we obtain

$$\begin{aligned} S_n(a, b) &= \frac{1}{\#\mathcal{H}_d} \sum_{Q \in \mathcal{H}_d} \sum_{P \in \mathbf{E}(\mathbb{F}_q)} \psi\left(ax(P \oplus Q) + bx(n(P \oplus Q))\right) \\ &= \frac{1}{\#\mathcal{H}_d} \sum_{Q \in \mathcal{H}_d} \sum_{P \in \mathbf{E}(\mathbb{F}_q)} \psi\left(ax(P \oplus Q) + bx(nP)\right) \\ &= \frac{1}{\#\mathcal{H}_d} \sum_{P \in \mathbf{E}(\mathbb{F}_q)} \psi\left(bx(nP)\right) \sum_{Q \in \mathcal{H}_d} \psi\left(ax(P \oplus Q)\right). \end{aligned}$$

Hence,

$$|S_n(a, b)| \leq \frac{1}{\#\mathcal{H}_d} \sum_{P \in \mathbf{E}(\mathbb{F}_q)} \left| \sum_{Q \in \mathcal{H}_d} \psi\left(ax(P + Q)\right) \right|.$$

The inner sum has been estimated in [7], [8] as $O(q^{1/2})$, which together with (2.4) implies the result. $\square$

Clearly the bound of Theorem (2.1) is nontrivial whenever $d/q^{1/2} \to \infty$ as $q \to \infty$.

For smaller values of $d = \gcd\left(n, \#\mathbf{E}(\mathbb{F}_q)\right)$ we use a different approach.

THEOREM (2.5). *Let $n > 0$ be an arbitrary integer and let $d = \gcd\left(n, \#\mathbf{E}(\mathbb{F}_q)\right)$. Then for any $a \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$, we have*

$$S_n(a, b) \ll qd^{-1/2} + q^{3/4}.$$

Proof. As in the proof of Theorem (2.1), we have

$$S_n(a, b) = \frac{1}{\#\mathcal{H}_d} \sum_{Q \in \mathcal{H}_d} \sum_{P \in \mathbf{E}(\mathbb{F}_q)} \psi\left(ax(P \oplus Q) + bx(nP)\right). \tag{2.6}$$

We now recall the main result of [10]. Let

$$T_{\rho,\vartheta}(a, \mathcal{P}, \mathcal{Q}) = \sum_{P \in \mathcal{P}} \sum_{Q \in \mathcal{Q}} \rho(P)\vartheta(Q)\psi(ax(P \oplus Q)), \tag{2.7}$$

where $\mathcal{P}, \mathcal{Q} \subseteq \mathbf{E}(\mathbb{F}_q)$, $\rho(P)$ and $\vartheta(Q)$ are arbitrary complex functions supported on $\mathcal{P}$ and $\mathcal{Q}$ with

$$|\rho(P)| \le 1,\ P \in \mathcal{P}, \qquad \text{and} \qquad |\vartheta(Q)| \le 1,\ Q \in \mathcal{Q}.$$

For the sum $T_{\rho,\vartheta}(a, \mathcal{P}, \mathcal{Q})$ given by (2.7) it is shown in [10] that for any fixed integer $\nu \ge 1$ and $a \in \mathbb{F}_q^*$, we have

$$T_{\rho,\vartheta}(a, \mathcal{P}, \mathcal{Q}) \ll (\#\mathcal{P})^{1-1/(2\nu)}(\#\mathcal{Q})^{1/2}q^{1/(2\nu)} + (\#\mathcal{P})^{1-1/(2\nu)}\#\mathcal{Q}q^{1/(4\nu)}. \tag{2.8}$$

We now recall (2.6) and use (2.8) with $\nu = 1$, $\mathcal{P} = \mathbf{E}(\mathbb{F}_q)$ and $\mathcal{Q} = \mathcal{H}_d$. This leads us to the bound

$$|S_n(a, b)| \ll \frac{1}{\#\mathcal{H}_d}\left(q(\#\mathcal{H}_d)^{1/2} + q^{3/4}\#\mathcal{H}_d\right).$$

Using (2.4), we conclude the proof. □

Clearly the bound of Theorem (2.5) is nontrivial whenever $d \to \infty$ as $q \to \infty$, however for $d > q^{3/4}$ it is weaker than that of Theorem (2.1).

If $n \mid \gcd\left(n, \#\mathbf{E}(\mathbb{F}_q)\right)$ then the bound (1.1) (used for $n < q^{1/5}$) and Theorem (2.5) (used for $n \ge q^{1/5}$) can be combined to derive a bound which does not depend on n.

COROLLARY (2.9). *Let $n > 0$ be an arbitrary integer with $n \mid \#\mathbf{E}(\mathbb{F}_q)$. Then for any $a \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$, we have*

$$S_n(a, b) \ll q^{9/10}.$$

Clearly, our method applies to more general sums of the shape

$$\widetilde{S}_n(\Psi, a) = \sum_{P \in \mathbf{E}(\mathbb{F}_q)} \Psi(nP)\psi\left(ax(P)\right), \qquad a \in \mathbb{F}_q,$$

for any function Ψ defined on the points of $\mathbf{E}(\mathbb{F}_q)$. In particular, the bounds of Theorems (2.1) and (2.5) remain unchanged (but Corollary (2.9) does not hold anymore as (1.1) does not apply to $\widetilde{S}_n(\Psi, a)$).

Finally, we remark that several other character sums can be estimated by the same method. In particular, analogues of several other results of [1] can also be obtained to character sums over elliptic curves.

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STRANGE CURVES IN CHARACTERISTIC TWO

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ABSTRACT. We study the singularities of a general member of the family of all strange curves of degree d over an algebraically closed field of characteristic two. As a consequence we get a genus formula for such a member.

1. Introduction

A strange curve is a non-linear irreducible projective curve with the property that all its tangent lines at simple points pass through a fixed point, called the center of the curve. These curves were first studied by E. Lluis in [L], A. Holme [Hl] and V. Bayer and A. Hefez in [BH]. E. Lluis showed that, except for the conics in characteristic two, strange curves are always singular. (This result is usually credited to P. Samuel [S]). In [C], A. Campillo introduced an algebraic notion of strange branches and showed that such branches appear as branches of strange curves. He also introduced a special sequence of multiplicities, associated to the blowing up sequence, that characterizes strange branches. Campillo's paper is a report of the conclusions of the seminar "*Equisingularity and saturation of singularities*" he delivered at UNAM in Mexico City in the summer of 1984. Our approach is different from Campillo's, once we study singularities of a general member of a family of strange curves. Strange curves were introduced in connection with the problem of projecting space curves. They are non-reflexive curves and, due to this, they can only occur in positive characteristic. In characteristic two strange curves behave differently from the general case. Our goal in this paper is to fulfill some gaps left by Bayer and Hefez's work. In that paper there were a few unsolved questions in this situation. We made some attempts to complete the study but did not succeed. A question (described at the end of this work) is still open. Apart from the results in and of themselves, our resuming the subject after several years has the intention of phrasing the question more explicitly.

No new tools are used in the paper, rather a careful analysis of the situation.

Strange curves are in the context of algebraic curves in positive characteristic. Lately, positive characteristic has earned a new impulse due its applications in coding theory.

In Section 2, we review the fundamental facts on the general theory. The basic reference is [BH]. In Section 3 we study the characteristic two case. The main results are Theorems 3.9 and 3.10. Theorem 3.4 is fundamental, although so much technical.

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2. Review of the general theory

Let Z be a projective strange curve with center S, defined over an algebraically closed field k of characteristic p. For the notation and the basic facts about duality and reflexivity we refer to [HK], [H] or [K].

Let $C(Z)$ and Z' be respectively the conormal variety and the dual variety of Z. Let $\pi\colon C(Z) \to Z$ and $\pi'\colon C(Z) \to Z'$, be the natural projections. We denote by π'_i and π'_s respectively the inseparable and separable degrees of π'. Note that the dual Z' of a non-linear curve Z is a hypersurface. In our case Z is a strange curve and its dual hypersurface is contained in the hyperplane corresponding, by duality, to the center S, so it must coincide with it. This will imply that a strange curve is non-reflexive. In fact, reflexive curves must satisfy biduality, while in our case, $Z'' = \{S\} \neq Z$. This tells us that strange curves only exist in positive characteristic. So, from now on we assume $p > 0$.

It is well known that (see [HK], 3.5) for a general point H' of Z', and $P \in \pi((\pi')^{-1}(H'))$, one has $\pi'_i = I(P, Z \cdot H)$, where H is the hyperplane corresponding to H' and $I(P, Z \cdot H)$ is the intersection multiplicity of Z and H at P. Since $\pi'_s = \#\pi((\pi')^{-1}(H'))$ and $\deg(\pi') = \pi'_i\pi'_s$, we have by Bezout's Theorem that $\deg(\pi') \leq \deg(Z)$. What is missing on the left hand side of this inequality for it to become an equality is the multiplicity of the center S in the curve Z. This was shown by Holme-Lluis in [Hl]: $\deg(Z) = \deg(\pi') + m_S(Z)$.

A general projection of a strange space curve is also strange. Conversely, if a general projection of a space curve is strange, then the curve is strange. Since $\deg(Z)$, π', π'_i, π'_s, $m_S(Z)$ and the genus of Z are invariant under general projections, it follows that some questions, such as the determination of the genus, may be reduced to analogues for plane curves.

Let d and e be positive integers and set $q = p^e$ and $N = \frac{d(d+3)}{2}$. If $q < d$, we denote by W^q_d the subset of the N-dimensional projective space which parametrizes all strange projective plane curves of degree d satisfying the polynomial identities for some $S = (a_0 : a_1 : a_2)$ in the projective plane:

$$\sum_{n_0+n_1+n_2=n} a_0^{n_0} a_1^{n_1} a_2^{n_2} D^{n_0}_{X_0} D^{n_1}_{X_1} D^{n_2}_{X_2} G(X_0, X_1, X_2) = 0, \quad n = 1, \ldots, p^e - 1.$$

The set W^q_d is an irreducible subvariety of $\mathbb{P}^N$ and its dimension is given by $\frac{1}{2}(2m + sq + 2)(s + 1) + 1$, where s and m are respectively the quotient and remainder of the euclidean division of d by q (see [BH]). Furthermore, a strange projective plane curve Z is in W^q_d if, and only if, $\pi'_i \geq p^e$. It follows that the general member of W^q_d is an irreducible polynomial and the corresponding curve Z is such that its inseparability degree is q, its separability degree is s and its multiplicity at S is m. If $m_S(Z) > 1$ then S is an ordinary multiple point of Z. If we choose $S = (0 : 0 : 1)$, we see that any strange projective plane curve is projectively equivalent to a curve with an equation of the form

$$a_m(X_0, X_1){X_2}^{sq} + a_{m+q}(X_0, X_1){X_2}^{(s-1)q} + \cdots + a_{m+sq}(X_0, X_1) = 0,$$

where X_0, X_1, X_2 are homogeneous coordinates in $\mathbb{P}^2$, each a_j is an homogeneous polynomial of degree j and q is a power of the characteristic p satisfying $q \leq \pi'_i$.

Most of the following facts can be found in [BH].

PROPOSITION (2.1) ([BH], Lemma 4.1). *Let $Z : g(x, y^q) = 0$, with $q = p^n$, be an irreducible curve through the origin in the affine plane. If $(x(t), y(t))$ is a parametrization of a branch of Z at the origin, then $x(t) \in k[[t^q]]$.*

PROPOSITION (2.2) ([BH], Proposition 4.2). *Let Z be an irreducible plane strange curve with center S and with $\pi_i' \geq q$. Let Γ be a branch of Z centered at a point $P \neq S$ with tangent line T.*

1. *If $m_P(\Gamma) < q$, then T is the line $\overline{PS}$. In particular, if Γ is smooth, then T passes through S.*
2. *For any Γ, we have $I(P, \Gamma \cdot T) \geq q$.*

Let $Z : G = 0$ be a projective plane curve and $Q = (b_0 : b_1 : b_2) \in \mathbb{P}^2$. The *polar* of Z centered at Q is the curve defined by the polynomial

$$\varphi_Q(G) = b_0 G_{X_0} + b_1 G_{X_1} + b_2 G_{X_2}.$$

If T is a projective transformation of $\mathbb{P}^2$ and $F(X_0, X_1, X_2)$ is a polynomial, we denote by

$$F^T(X_0, X_1, X_2) = F(T(X_0, X_1, X_2)).$$

One can easily, using the chain rule, deduce the formula:

$$(\varphi_Q(G))^T = \varphi_{T^{-1}(Q)}(G^T)$$

PROPOSITION (2.3) ([BH], Proposition 4.6). *Let $Z : G = 0$ be a plane strange curve with center S. Let P and Q be arbitrary points such that P, Q and S are not collinear. If $m_P(Z) = m$, then*

$$I(P, G \cdot \varphi_Q(G)) \geq m^2 - 1.$$

PROPOSITION (2.4) ([BH], Theorem 4.7). *Let $Z : G = 0$ be a plane strange curve with center S and with $\pi_i' \geq q$. Let P and Q be points such that P, Q and S are not collinear. Then*

$$I(P, G \cdot \varphi_Q(G)) \geq q(m_P(Z) - 1).$$

Let $F, G \in k[X_0, X_1, X_2]$ be polynomials. We denote by $R_{X_2}(F, G)$ the resultant of F and G with respect to the indeterminate X_2. If F and G are homogeneous, so is $R_{X_2}(F, G)$. The following is a well known result. (See for example [F, Example 1.2.2].)

PROPOSITION (2.5). *Let F and G be homogeneous polynomials in $k[X_0, X_1, X_2]$ and let $S = (0 : 0 : 1)$ in $\mathbb{P}^2$. Then $R_{X_2}(F, G)$ is divisible by the polynomial $L_1^{r_1} \cdots L_s^{r_s}$, where the L_i's are distinct linear polynomials defining the lines joining S to the points of intersection of F and G other than S, and*

$$r_i = \sum_{P \in L_i - \{S\}} I(P, F \cdot G).$$

The following two Propositions are also well known.

PROPOSITION (2.6). *Let $F = a_m X_2^\lambda + a_{m+r} X_2^{\lambda-1} + \cdots + a_{m+\lambda r}$ and $G = b_n X_2^\mu + b_{n+r} X_2^{\mu-1} + \cdots + b_{n+\mu r}$, where m, n, λ, μ and r are positive integers, a_i and b_j are homogeneous polynomials in $k[X_0, X_1]$ of degrees respectively i and j, such* *that $a_m b_n \neq 0$. Then R_{X_2}, as a polynomial in $k[X_0, X_1]$ is either zero or is homogenenous of degree*

$$d = \frac{1}{r}(\deg(F) \cdot \deg(G) - m \cdot n)$$

where $\deg(F)$ *and* $\deg(G)$ *mean the total degree of F and G, respectively, in X_0, X_1, X_2.*

PROPOSITION (2.7). *Let F, G be homogeneous polynomials in $k[X_0, X_1, X_2]$ and let T be a projective transformation of $\mathbb{P}^2$ which fixes the point* $(0:0:1)$. *Then*

$$R_{X_2}(F^T, G^T) = (R_{X_2}(F, G))^T.$$

PROPOSITION (2.8) ([BH], Lemma 5.4). *Let $F(X_0, X_1, X_2) = H_1(X_0, X_1, X_2^q)$ and $G(X_0, X_1, X_2) = H_2(X_0, X_1, X_2^q)$ be homogeneous polynomials in the ring $k[X_0, X_1, X_2]$, where q is a power of p. Then $R_{X_2}(F, G)$ is a homogeneous polynomial such that*

$$R_{X_2}(F, G) = (R_{X_2}(H_1(X_0, X_1, X_2), H_2(X_0, X_1, X_2)))^q$$

PROPOSITION (2.9) ([BH], Lemma 5.5). *Let $F = a_m X_2^{\lambda q} + a_{m+q} X_2^{(\lambda-1)q} + \cdots + a_{m+\lambda q}$ and $G = a_n X_2^{\mu q} + b_{n+q} X_2^{(\mu-1)q} + \cdots + b_{n+\mu q}$ be homogeneous polynomials in $k[X_0, X_1, X_2]$ with a_i and b_j homogeneous polynomials in $k[X_0, X_1]$ of degrees respectively i and j. Then either $R_{X_2}(F, G)$ is zero or it is a homogeneous polynomial of degree* $(\deg(F) \cdot \deg(G) - m \cdot n)q$.

3. The characteristic two case

PROPOSITION (3.1). *Let Z_1 and Z_2 be two plane strange curves with the same center S and both with $\pi_i' \geq q$. Let $P \in Z_1 \cap Z_2 - \{S\}$. We have*

1. *If Z_1 or Z_2 is irreducible, then $I(P, Z_1 \cdot Z_2)$ is a multiple of q.*
2. *If* $\min\{m_P(Z_1), m_P(Z_2)\} < q$, *then*

$$I(P, Z_1 \cdot Z_2) \geq q \cdot \min\{m_P(Z_1), m_P(Z_2)\}.$$

Proof. We change coordinates in $\mathbb{P}^2$ in order that S and P are respectively transformed into $(0:0:1)$ and $(1:0:0)$. The affine equations of Z_1 and Z_2 will then take the form $g(x, y^q) = 0$ and $h(x, y^q) = 0$.

For statement 1, we may assume that Z_1 is irreducible. Let $\Gamma = (x(t), y(t))$ be a parametrization of a branch of Z_1 at P. By Proposition 2.1, we have $x(t) = (u(t))^q$, hence

$$h(x(t), (y(t))^q) = h((u(t))^q, (y(t))^q).$$

It follows that $I(P, \Gamma \cdot Z_2)$ is a multiple of q and therefore $I(P, Z_1 \cdot Z_2)$ is also a multiple of q.

To prove 2, we may assume $m_P(Z_2) \leq m_P(Z_1)$. Then we have

$$\min\{m_P(Z_1), m_P(Z_2)\} = m_P(Z_2) \leq m_P(Z_1) \text{ and } m_P(Z_2) < q.$$

So, we may write

$$h(x, y^q) = x^m a(x) + y^q b(x, y^q) \quad \text{and} \quad g(x, y^q) = x^s c(x) + y^q d(x, y^q),$$

with $a(0) \neq 0$ and $c(0) \neq 0$, if $c(x)$ is a non-zero polynomial. In this case, we have $s \geq m_P(Z_1) \geq m = m_P(Z_2)$, and it follows that

$$\begin{aligned} I(P, Z_1 \cdot Z_2) &= I(P, h(x, y^q) \cdot [a(x)g(x, y^q) - c(x)x^{s-m}h(x, y^q)]) \geq mq \\ &= q \cdot \min\{m_P(Z_1), m_P(Z_2)\}. \end{aligned}$$

If $c(x)$ is the zero polynomial the inequality is obviously true. $\square$

LEMMA (3.2). *Let Z be an irreducible plane strange curve with center S and with $\pi_i' \geq q$, where q is a power of 2. Let $P \neq S$ be a singular point of Z. Assume that $S = (0:0:1)$ and let $F(X_0, X_1, X_2) = 0$ be an equation for Z. Then*

$$I(P, Z \cdot F_{X_1}) \geq 2q.$$

Proof. Firstly suppose that we have $\min\{m_P(F_{X_1}), m_P(Z)\} \geq q$. Then $I(P, Z \cdot F_{X_1}) \geq q^2$. Since $q \geq 2$ we are done. On the other hand, suppose that $\min\{m_P(F_{X_1}), m_P(Z)\} < q$. Then, from 2. of Proposition 3.1, we have $I(P, Z \cdot F_{X_1}) \geq q \cdot \min\{m_P(F_{X_1}), m_P(Z)\}$. Since $m_p(Z) \geq 2$ and $p = 2$, by a direct computation we see that $m_P(F_{X_1}) \geq 2$ and again we get the result. $\square$

LEMMA (3.3). *Let Z be an irreducible plane strange curve of characteristic $p = 2$ with center S and with $\pi_i' \geq q$. Let $F(X_0, X_1, X_2)$ be an equation for Z. If $P \in Z - \{S\}$ and $I(P, Z \cdot F_{X_1}) = 2q$, then $m_P(Z) \leq 3$.*

Proof. We may assume that P is a singular point. After changing coordinates, we can apply Proposition 2.4 and we get $I(P, Z \cdot F_{X_1}) \geq q(m_P(Z) - 1)$. Hence $2q \geq q(m_P(Z) - 1)$, and so, $m_P(Z) \leq 3$. $\square$

Next, we keep the notation W_d^q for the space of strange curves introduced in Section 2.

THEOREM (3.4). *Let Z be a general member of $W_d^q(S)$, where $S = (0:0:1)$ is its center and q is power of 2. Suppose that $P_1, P_2, \ldots, P_\mu$ are the singular points of Z distinct from S. Then the lines $\overline{SP_1}, \overline{SP_2}, \ldots, \overline{SP_\mu}$ are distinct and none of the P_i is on the line $X_0 = 0$. If L_i is a linear polynomial defining the line $\overline{SP_i}$, for $i = 1, \ldots, \mu$, then $R_{X_2}(F, F_{X_1})$ is associated to the polynomial*

$${X_0}^{sq}(L_1 \cdots L_\mu)^{2q},$$

where s is the quotient of the division of d by q.

Proof. We need the following proposition.

PROPOSITION (3.5). *With the same notation as in Theorem 3.4, there is an open set of curves in $W_d^q(S)$ satisfying the conditions there.*

Proof. By abuse of notation, let us use the same symbol for a polynomial and the curve defined by it. Let $F \in W_d^q(S)$. Note that Lemma 3.2 yields, for any singular point $P \neq S$ of F, that $I(P, F \cdot F_{X_1}) \geq 2q$. On the other hand, Euler's relation yields $X_0 F_{X_0} \equiv -X_1 F_{X_1} (\mathrm{mod} F)$. So for any P,

$$I(P, F_{X_1} \cdot F) + I(P, X_1 \cdot F) = I(P, F_{X_0} \cdot F) + I(P, X_0 \cdot F)$$

But, recalling that s and m are respectively the quotient and the remainder of the euclidean division of d by q, since

$$\sum_{P\in(X_0)-\{S\}} I(P, X_1\cdot F)=0,$$

and for a general F, $I(S, X_0\cdot F)=m$, it follows that,

$$\begin{aligned}\sum_{P\in(X_0)-\{S\}} I(P, F_{X_1}\cdot F) &\geq \sum_{P\in(X_0)-\{S\}} I(P, X_0\cdot F)\\ &= \deg(F)-I(S, X_0\cdot F)=\deg(F)-m=sq.\end{aligned}$$

Now, the open set we are looking for is the intersection of the open subset of $W_d^q(S)$ of curves with ordinary singular point at S with the open subset of curves for which no singular point other than S is on the line $X_0=0$ and no pair of them are collinear with S and the open set for which $I(P_i, F\cdot F_{X_1})\leq 2q$, for $i=1,\ldots,\mu$ and $\sum_{P\in(X_0)-\{S\}} I(P, F_{X_1}\cdot F)\leq sq$. $\square$

To conclude the proof of Theorem 3.4, we have to show that the open set of Proposition 3.5 is not empty. To show this, we are going to exhibit for each d and q a strange curve with center $S=(0:0:1)$ verifying the conditions of Theorem 3.4. We will need the following two lemmas.

LEMMA (3.6) ([BH], Lemma 6.3). *Given polynomials $p_1, p_2, q_1, q_2\in k[X]$ without common roots in k and such that $p_1q_2-p_2q_1\neq 0$, then there exists $\rho\in k$ such that $f_1=\rho p_1+q_1$ and $f_2=\rho p_2+q_2$ have no common roots in k.*

LEMMA (3.7) ([BH], Lemma 6.4). *Let $G(X_0, X_1, X_2)=AX_2^{\lambda q}+BX_2^q+C$, where A, B and C are homogeneous polynomials in $k[X_0, X_1]$ of degrees m, $(\lambda-1)q+m$ and $\lambda q+m$, respectively. Then we have*

$$R_{X_2}(G, G_{X_1})=[(A'B-AB')^{\lambda-1}(BC'-B'C)+(AC'-A'C)^{\lambda}]^q,$$

where the prime sign indicates the partial derivation with respect to X_1.

LEMMA (3.8). *Let λ, q and m be integers such that $\lambda\geq 1$, q is a power of 2 and $0\leq m<q$. Then the general member of the family of plane curves*

$$F=AX_2^{\lambda q}+BX_2^q+C$$

with

$$A=\alpha_0X_1^m+\alpha_1X_1^{m-1}X_0+\alpha_2X_0^q,\quad B=\beta X_1X_0^{(\lambda-1)q+m-1},$$

and

$$C=\gamma_0X_1^{\lambda q+m}+\gamma_1X_1^{\lambda q+m-1}X_0+\gamma_2X_1^2X_0^{\lambda q+m-2}+\gamma_3X_1X_0^{\lambda q+m-1}+\gamma_4X_0^{\lambda q+m}$$

satisfies the conditions of Theorem 3.4.

Proof. Put

$$\hat{h}=R_{X_2}(\hat{F}, \hat{F_{X_1}})(1,x),\quad \text{where}\quad \hat{F}(X_0, X_1, X_2)=F(X_0, X_1, X_2^{\frac{1}{q}}).$$

Using Lemma 3.7 to compute $R_{X_2}(F, F_{X_1})$ and in view of Proposition 2.8, we can prove easily that the highest power of X_0 which divides $R_{X_2}(\hat{F}, \hat{F_{X_1}})$ is X_0^{λ}. $\square$

Now, put
$$a = A(1, x), \quad b = B(1, x) \quad \text{and} \quad c = C(1, x).$$
We will split our analysis in two cases:

a) $m \equiv 1 \bmod 2$.

In this case we specialize the family to
$$a = x^m + x^{m-1} + \tilde{\alpha}, \quad b = \tilde{\beta}x, \quad c = x^{\lambda q+m} + \tilde{\gamma}x^2 + 1.$$
Using Lemma 3.7 and Proposition 2.8 we see that
$$\hat{h} = (ab' - a'b)^{\lambda-1} \cdot (bc' - b'c) + (ac' - a'c)^{\lambda}$$
and since char $k = 2$, we have
$$\hat{h} = \tilde{\beta}^{\lambda}(x^{m-1} + \tilde{\alpha})^{\lambda-1}(\tilde{\gamma}x^2 + 1) + (x^{\lambda q+2m-2} + \tilde{\alpha}x^{\lambda q+m-1} + \tilde{\gamma}x^{m+1} + 1)^{\lambda}.$$
Now observe that $\hat{h}$ is a perfect square, that is, $\hat{h} = h^2$ where
$$h = \beta^{\lambda}(x^{\frac{m-1}{2}} + \alpha)^{\lambda-1}(\gamma x + 1) + (x^{\frac{\lambda q+2m-2}{2}} + \alpha x^{\frac{\lambda q+m-1}{2}} + \gamma x^{\frac{m+1}{2}} + x^{\frac{m-1}{2}})^{\lambda},$$
and $\tilde{\alpha} = \alpha^2$, $\tilde{\beta} = \beta^2$ and $\tilde{\gamma} = \gamma^2$. We can write $h = \beta^{\lambda}p_1 + q_1$, where
$$p_1 = (x^{\frac{m-1}{2}} + \alpha)^{\lambda-1}(\gamma x + 1) \quad \text{and} \quad q_1 = (x^{\frac{\lambda q+2m-2}{2}} + \alpha x^{\frac{\lambda q+m-1}{2}} + \gamma x^{\frac{m+1}{2}} + x^{\frac{m-1}{2}})^{\lambda}.$$
Now, an easy computation shows that p_1 and q_1 have no common roots. By differentiation, we get $h' = \beta^{\lambda}p_2 + q_2$, where $p_2 = p_1'$ and $q_2 = q_1'$.

Again, by a computation, we can prove that for a good choice of values of α and γ one has
$$p_1q_2 - p_2q_1 \neq 0.$$
So, from Lemma 3.6, choosing β in such way that $\rho = \beta^{\lambda}$ we have that h and h' have no common roots.

b) $m \equiv 0 \bmod 2$.

In this case we specialize the family to
$$a = x^m + \tilde{\alpha}_1x^{m-1} + \tilde{\alpha}_2, \quad b = \tilde{\beta}x, \quad c = x^{\lambda q+m} + \tilde{\gamma}_1x^{\lambda q+m-1} + \tilde{\gamma}_2x^2 + \tilde{\gamma}_1\tilde{\gamma}_2x + \tilde{\gamma}_3.$$
Using Lemma 3.7, we see that
$$\hat{h} = (ab' - a'b)^{\lambda-1} \cdot (bc' - b'c) + (ac' - a'c)^{\lambda}$$
and since char $k = 2$, we have
$$\hat{h} = \tilde{\beta}^{\lambda}(x^m + \tilde{\alpha}_2)^{\lambda-1}(x^{\lambda q+m} + \tilde{\gamma}_2x^2 + \tilde{\gamma}_3)$$
$$+ [(\tilde{\gamma}_1 + \tilde{\alpha}_1)x^{\lambda q+2m-2} + \tilde{\alpha}_2\tilde{\gamma}_1x^{\lambda q+m-2} + (\tilde{\gamma}_1 + \tilde{\alpha}_1)\tilde{\gamma}_2x^m + \tilde{\alpha}_1\tilde{\gamma}_3x^{m-2} + \tilde{\alpha}_2\tilde{\gamma}_1\tilde{\gamma}_2]^{\lambda}.$$
Now we observe again that $\hat{h}$ is a perfect square, that is, $\hat{h} = h^2$, where
$$h = \beta^{\lambda}(x^{\frac{m}{2}} + \alpha_2)^{\lambda-1}(x^{\frac{\lambda q+m}{2}} + \gamma_2x + \gamma_3)$$
$$+ [(\gamma_1 + \alpha_1)x^{\frac{\lambda q+2m-2}{2}} + \alpha_2\gamma_1x^{\frac{\lambda q+m-2}{2}} + (\gamma_1 + \alpha_1)\gamma_2x^{\frac{m}{2}} + \alpha_1\gamma_3x^{\frac{m-2}{2}} + \alpha_2\gamma_1\gamma_2]^{\lambda},$$
and $\tilde{\alpha_1} = \alpha_1^2$, $\tilde{\alpha_2} = \alpha_2^2$, $\tilde{\beta} = \beta^2$ and $\tilde{\gamma_1} = \gamma_1^2$, $\tilde{\gamma_2} = \gamma_2^2$ and $\tilde{\gamma_3} = \gamma_3^2$.

Write $h = \beta^{\lambda}p_1 + q_1$, where $p_1 = \beta^{\lambda}(x^{\frac{m}{2}} + \alpha_2)^{\lambda-1}(x^{\frac{\lambda q+m}{2}} + \gamma_2x + \gamma_3)$ and
$$p_2 = [(\gamma_1 + \alpha_1)x^{\frac{\lambda q+2m-2}{2}} + \alpha_2\gamma_1x^{\frac{\lambda q+m-2}{2}} + (\gamma_1 + \alpha_1)\gamma_2x^{\frac{m}{2}} + \alpha_1\gamma_3x^{\frac{m-2}{2}} + \alpha_2\gamma_1\gamma_2]^{\lambda}.$$
Now, a straightforward computation can show that p_1 and q_1 have no common roots for some choice of the parameters .

By differentiation, we get $h' = \beta^\lambda p_2 + q_2$, where, $p_2 = p_1'$ and $q_2 = q_1'$.

Again, by a computation, we can prove that for a good choice of values of α and γ one has

$$p_1 q_2 - p_2 q_1 \neq 0.$$

So, from Lemma 3.6, choosing β in such way that $\rho = \beta^\lambda$ we have that h and h' have no common roots. $\square$

THEOREM (3.9). *Let Z be a general member of $W_d^q(S)$, where $S = (0:0:1)$ is its center and q is a power of 2. Let P be a singular point of Z distinct from S. Then Z has a unique branch Γ at P. The multiplicity of Z at P is 2 or 3. The semigroup of values of Γ is generated by*

2 and $2q-1$ if $m_P(Z) = 2$,
3 and q if $m_P(Z) = 3$.

In both cases, the contribution of the singularity P to the genus of Z is

$$\delta_P = q - 1.$$

Proof. Let $F(X_0, X_1, X_2) = 0$ be a homogeneous equation for Z. After a projective change of coordinates we may assume that $S = (0:0:1)$ and $P = (1:0:0)$. From Lemma 3.3, we have that $m_P(Z) \leq 3$. An affine equation of Z can be written as

$$f(X_1, X_2) = a(X_1) + u(X_1, X_2)X_2^q.$$

In the case $m_P(Z) = 3$, we have $\mathrm{ord}_{X_1}(a) = 3$ and $u(0,0) \neq 0$, so u is a unit in $k[[X_1, X_2]]$. It is easy to verify that there is just one branch of Z at P and the semigroup of values of the branch is $V = [3, q]$, and its conductor is $c = 2 \cdot (q-1)$.

In the case $m_P(Z) = 2$, we have $\mathrm{ord}_{X_1}(a) = 2$ and $u(0,0) \neq 0$. The situation now is quite different from the first case because now q and $m_P(Z)$ are not coprime. Write $a(X_1) = X_1^2 + a_3X_1^3 + R(X_1)X_1^4$, where $a_3 \in k$ and $R(X_1) \in k[X_1]$. For $q = 2$ we write down the equations and we see that there is just one tangent line at P to Z. Although we do not know at the beginning whether there is only one branch at P, we blow up P and we find out that the new curve is non singular at P. In particular Z has just one branch at P. Furthermore, the semigroup associated to the branch is $V = [2, 3]$. For $q \geq 4$, again we have just one tangent line at P. We consider the blowing up sequence $f^{(j)}(X_1, X_2)$. We observe that $f^{(\frac{q}{2})}(X_1, X_2)$ is again a strange curve with inseparability degree equal to $\frac{q}{2}$. This fact allows us to use induction on the exponent e of $q = 2^e$. From this we can conclude that the semigroup is $V = [2, 2q-1]$. The conductor of V is again $c = 1 \cdot (2q-2) = 2 \cdot (q-1)$. The contribution of P for the genus of Z is $\delta_P = \frac{c}{2} = q - 1$. $\square$

THEOREM (3.10). *Let Z be a general member of $W_d^q(S)$, where S is its center and q is power of a prime 2. Then we have*

1. *$\pi_i' = q$ and $\pi_s' = s$ where $s = \left[\frac{d}{q}\right]$, where [ ] is the integer part function.*
2. *The multiplicity of Z at its center S is $m = d - sq$. If $m > 1$, S is an ordinary multiple point of Z.*

3. *Z has* $\frac{1}{2}s(d+m-2)$ *multiple points other than S. At each such a point there is just one branch with semigroup* $[2, 2q-1]$ *or* $[3, q]$. *The tangent line at each singular point contains the center S.*
4. *The genus of Z is given by*

$$g = \frac{1}{2}(s-1)(d+m-2).$$

Proof. The Statements (1) and (2) are proven in [BH, thm 7.2]. Statement (3) follows from a count using Proposition 2.9 and Theorem 3.4. Statement (4) follows from the above parts of the theorem, from Theorem 3.9 and from the genus formula. □

Remarks.

1. It is interesting to compare Theorem 3.10 with Theorem 7.2 in [BH]. The number of singular points distinct from the center of a generic strange curve in characteristic two is half of the number in other characteristics. On the other hand, the contribution of each singular point in characteristic two is twice the contribution in the other characteristic. So, at the end of the computation, the final formula for the genus is the same in any case.

2. An open question: We don't know whether the multiplicity of a singular point, distinct from the center, of a generic strange curve in characteristic two assumes or doesn't assume the value three. Our conjecture is that this multiplicity is always two.

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INTEGRAL TRANSFORMATIONS ASSOCIATED WITH THE MUCKENHOUPT MATRIX WEIGHTS

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ABSTRACT. For any $(n \times n)$-matrix weight $w^2(x) \geqslant 0$ which satisfies the condition (A_2) on $\mathbb{R}$ there exists an outer matrix-valued function $w_-(z)$ in the domain $\operatorname{Im} z < 0$, such that $w^2(x) \overset{a.e.}{=} w_-(x - i0)w_-^*(x - i0)$, $x \in \mathbb{R}$, where non-tangent limit values w_- are denoted by $w_-(x - i0)$. The matrix kernel $Y_w(z, t)$ is defined by formula

$$\int_0^\infty Y_w(z, t)e^{-i\lambda t}dt = w_-(\lambda)(\lambda - z)^{-1}, \quad \operatorname{Im} \lambda < 0, \quad \operatorname{Im} z > 0.$$

We study properties of the integral transformation

$$(\mathcal{D}_w f)(z) := \frac{1}{\sqrt{2\pi}} \int_0^\infty Y_w(z, t)f(t)dt,$$

defined on the set of continuous finite vector-valued functions. In the case $w^2(x) \equiv E_n$ (the identity matrix) the transformation $\mathcal{D}_w$ is the same as the classic Fourier transformation.

1. Preliminary Results

In this section we will state some facts touching upon the matrix (A_2)-Muckenhoupt weight on $\mathbb{R}$, i.e. $w^2(x) > 0$ for almost all $x \in \mathbb{R}$ and the following condition is fulfilled:

$$(A_2) \qquad \sup_\Delta \left\{ \left\| \left(|\Delta|^{-1} \int_\Delta w^2(x)dx \right)^{1/2} \left(|\Delta|^{-1} \int_\Delta w^{-2}(x)dx \right)^{1/2} \right\| \right\} < \infty.$$

Here Δ is an interval of the real axis and $|\Delta|$ is its length.

Every A_2-weight w^2 admits factorizations, whose properties result in the next theorem. Let the domains $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$ be denoted by $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. Let the $(n \times n)$-matrix-valued functions with the elements from the scalar Hardy classes $H^2_\pm$ be denoted by $H^2_\pm(\mathbb{C}^{n\times n})$. Further, a matrix-valued function is called an outer function in the $H^2_\pm(\mathbb{C}^{n\times n})$ if its determinant is an outer function [4] in the domain $\mathbb{C}_\pm$.

THEOREM (1.1). *Let w^2 be a matrix A_2-weight on the real axis. Then the following statements hold:*

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1) *In the domains $\mathbb{C}_+$ and $\mathbb{C}_-$ there exist analytic matrix-valued functions $w_+(z)$ and $w_-(z)$, respectively, such that*

$$w^2(x) \overset{a.e.}{=} w_+^*(x+i0)w_+(x+i0) \overset{a.e.}{=} w_-^*(x-i0)w_-(x-i0), \quad x \in \mathbb{R} \tag{1.2}$$

Here we denote by $w_+(x+i0)$ and $w_-(x-i0)$ the non-tangent limit values of matrix-valued functions $w_+(z)$ and $w_-(z)$, respectively, which exist for almost all $x \in \mathbb{R}$.

2) *The matrix-valued functions $w_+(z)(z+i)^{-1}$ and $w_-(z)(z-i)^{-1}$ are outer functions in $H_+^2(\mathbb{C}^{n\times n})$ and $H_-^2(\mathbb{C}^{n\times n})$, respectively.*
3) *Every factorization (1.2) is unique up to multiplication from the left by a constant unitary factor.*

We note that in the above theorem the last statement means the following. If we assume $w^2(x) \overset{a.e.}{=} \tilde{w}_+^*(x+i0)\tilde{w}_+(x+i0)$ where $\tilde{w}_+(z)$ is analytic in the $\mathbb{C}_+$ matrix-valued function with the same properties as $w_+(z)$, then

$$\tilde{w}_+(z) = u\, w_+(z), \quad z \in \mathbb{C}_+$$

where u is some constant unitary matrix. It is clear that the analogous statement is correct for the second factorization (1.2).

Theorem (1.1) can be deduced from the results of papers [6, 1]. Indeed, let $w^2(x)$ be an A_2-weight on the real axis. We will denote by $\lambda_k(x)$, $1 \leqslant k \leqslant n$ the eigenvalues of matrix $w(x) := \|w_{kj}(x)\|$. Then

$$\begin{aligned}\left(\det w^2(x)\right)^{\frac{1}{n}} &= \left(\prod_{k=1}^{n} \lambda_k^2(x)\right)^{\frac{1}{n}} \leqslant \frac{1}{n}\sum_{k=1}^{n} \lambda_k^2(x) = \frac{1}{n}\sum_{k,j=1}^{n} \left|w_{kj}(x)\right|^2 \\ &= \frac{1}{n}\sum_{j=1}^{n} \left\|w(x)e_j\right\|^2 = \frac{1}{n}\sum_{j=1}^{n} \left(w^2(x)e_j, e_j\right),\end{aligned}$$

where $\{e_j\}_1^n$ is the standard orthonormal basis in the space $\mathbb{C}^n$. As the weight $w^{-2}(x)$ also satisfies the conditions (A_2), it follows from previous considerations that

$$\left(\det w^2(x)\right)^{-\frac{1}{n}} \leqslant \frac{1}{n}\sum_{j=1}^{n} \left(w^{-2}(x)e_j, e_j\right).$$

The weights $\left(w^{\pm 2}(x)e_j, e_j\right)$, $1 \leqslant j \leqslant n$, satisfy the scalar condition (A_2) [6] and therefore from [4] we have

$$\int_{\mathbb{R}} \left(w^{\pm 2}(x)e_j, e_j\right)(1+x^2)^{-1}dx < \infty, \quad 1 \leqslant j \leqslant n.$$

Thus

$$\int_{\mathbb{R}} \left(\det w^2(x)\right)^{\pm\frac{1}{n}} (1+x^2)^{-1}dx < \infty,$$

and so

$$\int_{\mathbb{R}} \left|\log\det w^2(x)\right| (1+x^2)^{-1}dx < \infty.$$

Moreover, the integral [6] converges:

$$\int_{\mathbb{R}} \|w^2(x)\| \, (1+x^2)^{-1} dx.$$

Now all three statements of theorem (1.1) follow from the paper [1] (lemma 10.4).

Let v^2 be an almost everywhere nonnegative matrix weight on $\mathbb{R}$. In what follows we denote by $L_2^{(n)}(\mathbb{R}, v^2)$ the Hilbert space of $\mathbb{C}^n$-valued vector-valued functions (columns) with the norm

$$\|f\|^2 := \int_{\mathbb{R}} \big(v^2(x)f(x), f(x)\big)_{\mathbb{C}^n} \, dx < \infty.$$

Here the brackets mean the Euclidean inner product in $\mathbb{C}^n$. The spaces $L_2^{(n)}(\mathbb{R}_+, v^2)$ and $L_2^{(n)}([a,b], v^2)$ are defined analogously. In the case $v^2(x) \equiv E_n$ the spaces introduced will be denoted by $L_2^{(n)}(\mathbb{R})$, $L_2^{(n)}(\mathbb{R}_+)$ and $L_2^{(n)}(a,b)$, respectively.

The Treil-Volberg deep result [6] states that Hilbert operator

$$(\mathcal{H}f)(x) := \frac{1}{\pi i} \fint_{\mathbb{R}} \frac{f(t)}{t-x} dx$$

is bounded on the space $L_2^{(n)}(\mathbb{R}, w^2)$ if and only if the weight w^2 satisfies condition (A_2).

The Treil-Volberg theorem permits us to introduce the Hardy weight classes in the half-planes $\mathbb{C}_\pm$. In what follows we will denote by $H^2_\pm(\mathbb{C}^n)$ the Hardy class of $\mathbb{C}^n$-valued vector-valued functions (columns) in the upper (lower) half-plane. We recall that if $F \in H^2_+(\mathbb{C}^n)$, then the norm is defined by the formula

$$\|F\|_+^2 := \sup_{y>0} \int_{\mathbb{R}} \|F(x+iy)\|^2_{\mathbb{C}^n} dx,$$

where the norm in $\mathbb{C}^n$ is Euclidean.

Let w^2 be a matrix (A_2)-weight on $\mathbb{R}$. By definition, the Hardy weight class $H^2_+(\mathbb{C}^n, w^2)$ consists of all $\mathbb{C}^n$-valued functions F such that $w_+(z)F(z) \in H^2_+(\mathbb{C}^n)$, and the norm is defined by

$${}^w\|F\|_+^2 := \|w_+F\|_+^2 = \sup_{y>0} \int_{\mathbb{R}} \|w_+(x+iy)F(x+iy)\|^2_{\mathbb{C}^n} \, dx.$$

Since $\det w_+(z) \neq 0$ for $z \in \mathbb{C}_+$, every function F from the Hardy weight class is analytic in $\mathbb{C}_+$ and has non-tangent limits $F(x+i0)$, $x \in \mathbb{R}$, almost everywhere; moreover, $F(x+i0) \in L_2^{(n)}(\mathbb{R}, w^2)$.

Analogously, the Hardy class $H^2_-(\mathbb{C}^n, w^2)$ consists of all vector-valued functions G such that $w_-(z)G(z) \in H^2_-(\mathbb{C}^n)$; the norm is given by

$${}^w\|G\|_-^2 := \|w_-(z)G(z)\|_-^2 = \sup_{y<0} \int_{\mathbb{R}} \|w_-(x+iy)G(x+iy)\|^2_{\mathbb{C}^n} \, dx.$$

In the next theorem we will list the Hardy weight classes' properties.

THEOREM (1.2). *For a matrix (A_2)-weight w^2 on the real axis the following statements hold:*

1) *Let $F \in H^2_+(\mathbb{C}^n, w^2)$. Then there exist non-tangent limit values $F(x + i0)$ belonging to the space $L_2^{(n)}(\mathbb{R}, w^2)$ for almost every $x \in \mathbb{R}$;*
2) *For every function $f \in L_2^{(n)}(\mathbb{R}, w^2)$, the Cauchy integral*

$$F(z) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(x)dx}{x - z}, \qquad z \in \mathbb{C}_+$$

belongs to $H^2_+(\mathbb{C}^n, w^2)$ and the following estimate is valid:

$${}^{w}\|F\|_+^2 \leqslant C \int_{\mathbb{R}} \big(w^2(x)f(x), f(x)\big)_{\mathbb{C}^n} \, dx$$

3) *Every function $F \in H^2_+(\mathbb{C}^n, w^2)$ is related to its limit values $F(x + i0)$ by the formula*

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{F(x + i0)dx}{x - z}, \qquad z \in \mathbb{C}_+;$$

4) *The direct decomposition*

$$L_2^{(n)}(\mathbb{R}, w^2) = H^2_+(\mathbb{C}^n, w^2) \dotplus H^2_-(\mathbb{C}^n, w^2)$$

is valid, where elements from $H^2_\pm(\mathbb{C}^n, w^2)$ are identified with their limit values. The following formulas hold for the projections $\mathbb{P}_\pm$ from $L_2^{(n)}(\mathbb{R}, w^2)$ onto $\mathbb{H}^2_\pm(\mathbb{C}^n, w^2)$:

$$\mathbb{P}_\pm f = \frac{1}{2}(I \pm \mathcal{H})f, \quad (\mathcal{H}f)(x) = \frac{1}{\pi i} \fint_{\mathbb{R}} \frac{f(t)dt}{t - x}.$$

The theorems formulated in this section are either contained in the papers [6, 1] or are easily derived from these papers' results.

2. Integral transformations

Let

$$\begin{aligned} \mathring{w}_+(z) &:= w_-^*(\bar{z}), \qquad & z \in \mathbb{C}_+ \\ \mathring{w}_-(z) &:= w_+^*(\bar{z}), \qquad & z \in \mathbb{C}_- \end{aligned}$$

From (1.2) it follows that

$$w^2(x) \overset{a.e.}{=} \mathring{w}_+(x + i0)\mathring{w}_+^*(x + i0) \overset{a.e.}{=} \mathring{w}_-(x - i0)\mathring{w}_-^*(x - i0). \tag{2.1}$$

We note that from theorem (1.1) the elements of the matrix $\mathring{w}_-(z)(z - i)^{-1}$ belong to the class H^2_-. Thus from the Wiener-Paley theorem [4] it follows that

$$\mathring{w}_-(z) = (z - i) \int_0^\infty e^{-izt} h(t)dt, \quad z \in \mathbb{C}_-,$$

where matrix $h(t)$ elements belong to $L_2(\mathbb{R}_+)$. The last equality can be rewritten as

$$\mathring{w}_-(z) = z\int_0^\infty e^{-izt}Y_w(t)dt, \quad z \in \mathbb{C}_-, \tag{2.2}$$

where the matrix $Y_w(t)$ is defined by the formula

$$Y_w(t) = h(t) + \int_0^t h(s)ds, \quad t \geqslant 0. \tag{2.3}$$

We denote by $Y_w(z,t)$, $t \geqslant 0$, $z \in \mathbb{C}$, the solution of the matrix integral equation

$$Y_w(z,t) - iz\int_0^t Y_w(z,s)ds = Y_w(t). \tag{2.4}$$

From (2.3) it follows that elements $Y_w(t)$ belong to $L_2^{\mathrm{loc}}(\mathbb{R}_+)$. Thus $Y_w(z,t)$ also obeys property (2.4) and the integral operator

$$(\mathcal{D}_w f)(z) := \frac{1}{\sqrt{2\pi}}\int_0^\infty Y_w(z,t)f(t)dt \tag{2.5}$$

is defined on the set of all the finite continuous $\mathbb{C}^n$-valued functions $f(t)$. The next theorem states that the operator $\mathcal{D}_w$ can be extended by continuity onto the space $L_2^{(n)}(\mathbb{R}_+)$.

THEOREM (2.6). *Let w^2 be a matrix (A_2)-weight on $\mathbb{R}$. Then there exists a constant M such that for each function $f \in L_2^{(n)}(\mathbb{R}_+)$*

$$\int_{\mathbb{R}_+} \|f(x)\|^2_{\mathbb{C}^n}\, dx \leqslant \int_{\mathbb{R}} \big(w^{-2}(x)(\mathcal{D}_w f)(x), (\mathcal{D}_w f)(x)\big)_{\mathbb{C}^n}\, dx \leqslant M\int_{\mathbb{R}_+} \|f(x)\|^2_{\mathbb{C}^n}\, dx.$$

Moreover, the following inversion formula holds:

$$f(t) \overset{a.e.}{=} \frac{1}{\sqrt{2\pi} i}\int_{\mathbb{R}} e^{-ixt}\mathring{w}_-^{-1}(x-i0)(\mathcal{D}_w f)(x)dx, \quad t \geqslant 0 \tag{2.7}$$

Proof. Let us apply the Fourier-Laplace transform ($\lambda \in \mathbb{C}_-$) to the both parts of equation (2.4):

$$\int_0^\infty Y_w(z,t)e^{-i\lambda t}dt - iz\int_0^\infty\int_0^t Y_w(z,s)ds\, e^{-i\lambda t}dt = \int_0^\infty Y_w(t)e^{-i\lambda t}dt.$$

Taking into account (2.2) we get the formula

$$\int_0^\infty Y_w(z,t)e^{-i\lambda t}dt = \mathring{w}_-(\lambda)(\lambda - z)^{-1}, \quad z \in \mathbb{C}_+$$

and thus

$$\int_0^\infty Y_w(z,t)e^{-ixt}dt \overset{a.e.}{=} \frac{\mathring{w}_-(x-i0)}{x-z}, \quad z \in \mathbb{C}_+, \quad x \in \mathbb{R}$$

Applying the Fourier transform to the k-component of the vector $(\mathcal{D}_w f)(z)$, $z \in \mathbb{C}_+$, we have

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \sum_{j=1}^n Y_w^{kj}(z,t) f_j(t) dt = \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{j=1}^n \frac{\mathring{w}_-^{kj}(x-i0)}{x-z} \hat{f}_j(x) dx,$$

where

$$\hat{f}_j(x) := \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ixt} f_j(t) dt, \quad \hat{f}(x) := \mathrm{col}\, \left(\hat{f}_j(x)\right)_1^n.$$

Thus we arrive at

$$(\mathcal{D}_w f)(z) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\mathring{w}_-(x-i0)}{x-z} \hat{f}(x) dx, \quad z \in \mathbb{C}_+ \tag{2.8}$$

We note that the equality

$$w^{-2}(x) \overset{a.e.}{=} \left(\mathring{w}_-^{-1}(x-i0)\right)^* \left(\mathring{w}_-^{-1}(x-i0)\right), \quad x \in \mathbb{R}$$

follows from the factorization (2.1). Moreover, the function $F(x) := \mathring{w}_-(x - i0)\hat{f}(x)$ belongs to the space $L_2^{(n)}(\mathbb{R}, w^{-2})$. Since the weight w_-^{-2} also satisfies the Muckenhoupt condition, and equality (2.8) can be represented in the form $(\mathcal{D}_w f)(x) = i(\mathbb{P}_+ F)(x)$, the estimate

$$\begin{aligned}\int_{\mathbb{R}} \left(w^{-2}(x)(\mathcal{D}_w f)(x), (\mathcal{D}_w f)(x)\right)_{\mathbb{C}^n} dx &\leqslant \|P_+\| \int_{\mathbb{R}} \left(w^{-2}(x)F(x), F(x)\right)_{\mathbb{C}^n} dx \\ &= \|P_+\| \int_{\mathbb{R}} \left\|\hat{f}(x)\right\|_{\mathbb{C}^n}^2 dx = \|P_+\| \cdot \|f\|^2_{L_2^{(n)}(\mathbb{R}_+)}\end{aligned}$$

follows from the theorem (1.3).

Next, from the formula for $\mathbb{P}_+$ we deduce

$$\mathring{w}_-^{-1}(x-i0)(\mathcal{D}_w f)(x) = \frac{i}{2}\hat{f}(x) + \frac{\mathring{w}_-^{-1}(x-i0)}{2\pi} \cdot \fint_{\mathbb{R}} \frac{\mathring{w}_-(u-i0)}{u-x} \hat{f}(u) du.$$

Thus taking into account the formula $\mathbb{P}_- F = 1/2F - 1/2\mathcal{H}F$, we have

$$\mathring{w}_-^{-1}(x-i0)(\mathcal{D}_w f)(x) = i\hat{f}(x) - i\mathring{w}_-^{-1}(x-i0)\mathbb{P}_-(\mathring{w}_-(u-i0)\hat{f}(u)). \tag{2.9}$$

By theorem (1.3) with the Muckenhoupt weight w^{-2}, the second summand in (2.9) belongs to the class $H_-^2(\mathbb{C}^n)$. Thus the summands in (2.9) are orthogonal, and therefore

$$\int_{\mathbb{R}_+} \|f(x)\|^2_{\mathbb{C}^n} dx = \int_{\mathbb{R}} \|\hat{f}(x)\|^2_{\mathbb{C}^n} dx \leqslant \int_{\mathbb{R}} \|\mathring{w}_-^{-1}(x-i0)(\mathcal{D}_w f)(x)\|^2_{\mathbb{C}^n} dx.$$

Taking into account the Wiener-Paley theorem, the Fourier transform (with kernel $\exp\{-ixt\}$) of the second summand in (2.9) is equal to 0 almost everywhere for $t \geqslant 0$. Thus the inverse of formula (2.7) easily follows from equality (2.9). The theorem is proved. □

The next result shows that the range of the transformation $\mathcal{D}_w$ coinsides with the Hardy class $H_+^2(\mathbb{C}^n, w^{-2})$.

THEOREM (2.10). *Let w^2 be a matrix (A_2)-weight on $\mathbb{R}$. The vector-valued function F belongs to the class $H^2_+(\mathbb{C}^n, w^{-2})$ if and only it the representation*

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty Y_w(z,t) f(t) dt, \quad z \in \mathbb{C}_+ \tag{2.11}$$

holds for a certain function $F \in L_2^{(n)}(\mathbb{R}_+)$.

Proof. It follows from theorem (1.3) and formula (2.8) that every function F of type (2.11) belongs to $H^2_+(\mathbb{C}^n, w^{-2})$.

Conversely, let $F \in H^2_+(\mathbb{C}^n, w^{-2})$. Taking into account (2.1), we deduce that

$$\int_{\mathbb{R}} \left\| \mathring{w}_-^{-1}(x - i0) F(x + i0) \right\|^2_{\mathbb{C}^n} dx < \infty.$$

The unitarity of the Fourier transform $\mathbb{C}^n$ on $L_2^{(n)}(\mathbb{R})$ implies

$$\mathring{w}_-^{-1}(x - i0) F(x + i0) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixt} f(t) dt, \quad f \in L_2^{(n)}(\mathbb{R}). \tag{2.12}$$

For a function f, we define the function

$$\tilde{F}(z) := \frac{i}{\sqrt{2\pi}} \int_0^\infty Y_w(z,t) f(t) dt, \quad z \in \mathbb{C}_+$$

which belongs to the class $H^2_+(\mathbb{C}^n, w^{-2})$. Observe that formula (2.9) holds for its limit values $\tilde{F}(x + i0)$, i.e.

$$\mathring{w}_-^{-1}(x - i0) \tilde{F}(x + i0) = i\hat{f}(x) + G(x), \quad G \in H^2_-(\mathbb{C}^n).$$

Together with (2.12) this means that $\mathring{w}_-^{-1}(x - i0) \left(\tilde{F}(x + i0) - F(x + i0) \right)$ belongs to $H^2_-(\mathbb{C}^n)$, or $(F - \tilde{F}) \in H^2_-(\mathbb{C}^n, w^{-2})$. On the other hand $(F - \tilde{F}) \in H^2_+(\mathbb{C}^n, w^{-2})$ and then $F = \tilde{F}$. The theorem is proved. □

The following result generalizes the classical Paley-Wiener theorem [4]. For an (A_2)-weight w^2 we denote by $A^2_\sigma(\mathbb{R}, w^{-2})$ the class of entire vector-valued function $F(z) = \operatorname{col}\left(F_k(z)\right)_1^n$ of exponential type which satisfy the following conditions:

1)

$$\int_{\mathbb{R}} \left(w^{-2}(x) F(x), F(x) \right)_{\mathbb{C}^n} dx < \infty,$$

2) For every k

$$h(F_k, \pi/2) \leqslant 0, \quad h(F_k, -\pi/2) \leqslant \sigma, \quad 1 \leqslant k \leqslant n,$$

where h is the growth indicator

$$h(F_k, \alpha) = \limsup_{r \to \infty} r^{-1} \log \left| F_k(re^{i\alpha}) \right|, \quad \pi < \alpha \leqslant -\pi.$$

THEOREM (2.13). *The entire vector-valued function F belongs to the class $A^2_\sigma(\mathbb{R}, w^{-2})$ if and only if it admites the integral representation:*

$$F(z) = \int_0^\sigma Y_w(z,t)f(t)dt, \quad z \in \mathbb{C} \tag{2.14}$$

for a certain function $f \in L_2^{(n)}(0,\sigma)$.

Proof. From theorem (2.6) it follows that every function of type (2.14) satisfies the condition 1). The estimates in 2) follow from the fact that the solution of equality (2.4) can be written as

$$Y_w(z,t) = \frac{d}{dt}\int_0^t Y_w(t-s)e^{izs}ds, \quad z \in \mathbb{C}.$$

Conversely, let a function F satisfy the conditions 1)–2) of the above statement. For every $\lambda \in \mathbb{C}_-$ we estimate the Euclidean norm

$$\begin{aligned}\left\|F(x)(x-\lambda)^{-1}\right\| &= \left\|w(x)w^{-1}(x)F(x)(x-\lambda)^{-1}\right\| \\ &\leqslant |x-\lambda|^{-1}\|w(x)\|\cdot\left\|w^{-1}(x)F(x)\right\| \\ &\leqslant C|x-\lambda|^{-1}\|w(x)\|_2\left\|w^{-1}(x)F(x)\right\|,\end{aligned}$$

where C is a constant, $\|w(x)\|_2$ is the Hilbert-Schmidt norm. It is easy to see that all scalar weights $\left(w^2(x)e_k, e_k\right)$ (where e_k belongs to the standard orthonormal base in $\mathbb{C}^n$) satisfy the Muckenhoupt scalar condition. Therefore the weight $\|w(x)\|_2^2 = \sum_{k=1}^n \left(w^2(x)e_k, e_k\right)$ satisfies scalar condition (A_2), and thus the function $\|w(x)\|_2^2/|x-\lambda|^2$ is summable on $\mathbb{R}$ [4]. Returning to the previous estimate we have

$$\int_{\mathbb{R}}\left\|\frac{F(x)}{x-\lambda}\right\|dx \leqslant C\left(\int_{\mathbb{R}}\frac{\|w(x)\|_2^2}{|x-\lambda|^2}dx\right)^{1/2}\cdot\left(\int_{\mathbb{R}}\left\|w^{-1}(x)F(x)\right\|_{\mathbb{C}^n}^2dx\right)^{1/2} < \infty. \tag{2.15}$$

Thus, for every $\varepsilon > 0$, the vector-valued function $Q(z) := e^{i\varepsilon z}F(z)(z-\lambda)^{-1}$ ($\lambda \in \mathbb{C}_-$) is summable on $\mathbb{R}$. Furthermore the condition $h(F_k, \pi/2) \leqslant 0$, $1 \leqslant k \leqslant n$ implies that Q is also summable on the straight line $\{iy, y \in \mathbb{R}_+\}$. As Q has the exponential growth in $\mathbb{C}_+$, all components of Q belong to the Hardy class H^1_+ [5] and therefore [4]

$$\int_{\mathbb{R}}\frac{e^{ix\varepsilon}F(x)}{x-\lambda}dx = 0,$$

for every $\lambda \in C_-$. From theorem (1.3) it follows that the entire function $e^{i\varepsilon z}F(z)$ belongs to the class $H^2_+\left(\mathbb{C}^n, w^{-2}\right)$ for every $\varepsilon > 0$. Choosing a sequence $\varepsilon_n \to 0$ such that the sequence of functions $e^{i\varepsilon_n z}F(z)$ converges weakly, we conclude that $F \in H^2_+\left(\mathbb{C}^n, w^{-2}\right)$.

Now we consider the analytic vector-valued function

$$Q(z) := e^{-i(\sigma+\varepsilon)z}F(z)(z-\lambda)^{-1}, \quad \text{where} \quad \varepsilon > 0, \lambda \in \mathbb{C}_+.$$

From estimate (2.15) and the theorem's condition 2), the summability of Q on $\mathbb{R}$ and on the straight line $\{iy, y<0\}$ follows. Repeating the previous arguments we conclude that $e^{-i\sigma z}F(z)$ belongs to the class $H^2_-(\mathbb{C}^n, w^{-2})$. Therefore, $\mathring{w}_-^{-1}(x-i0)F(x)\cdot e^{-i\sigma x} \in H^2_-(\mathbb{C}^n)$, and thus we have the following representation,

$$\mathring{w}_-^{-1}(x-i0)F(x) = \frac{i}{\sqrt{2\pi}} \int\limits_{-\infty}^{\sigma} e^{ixt} f(t)dt, \quad f \in L_2^{(n)}(-\infty;\sigma) \tag{2.16}$$

We consider now the function

$$F_1(z) := \frac{1}{\sqrt{2\pi}} \int\limits_0^{\sigma} Y_w(z,t) f(t)dt,$$

where f is contained in the previous representation. Comparing equality (2.9) written for F_1 and equality (2.16), we deduce that $\mathring{w}_-^{-1}(x-i0)\big(F(x)-F_1(x)\big) \in H^2_-(\mathbb{C}^n)$ i.e. $(F-F_1) \in H^2_-(\mathbb{C}^n, w^{-2})$. On the other hand, $(F-F_1) \in H^2_+(\mathbb{C}^n, w^{-2})$ and thus $F = F_1$. The theorem is proved. □

From the above theorem the corollary of independent interest follows.

COROLLARY (2.17). *Let v^2 be a matrix (A_2)-weight on the real axis. Then the class of entire vector-valued functions $A^2_\sigma(\mathbb{R}, v^2)$ is a Hilbert space with the inner product:*

$$(F,G) := \int\limits_{\mathbb{R}} \big(v^2(x)F(x), G(x)\big)_{\mathbb{C}^n}\, dx.$$

Now we consider a number of simple examples. Let $w^2(x) = \operatorname{diag}\,\{w_k^2(x)\}$, where w_k^2 are scalar (A_2)-weights on $\mathbb{R}$ $(1 \leqslant k \leqslant n)$. Then

$$w_k^2(x) \overset{a.e.}{=} \left|w_-^k(x-i0)\right|^2 \overset{a.e.}{=} \left|w_+^k(x+i0)\right|^2, \quad x \in \mathbb{R},$$

where $w_+^k(w_-^k)$ is an outer function in the domain $\mathbb{C}_+(\mathbb{C}_-)$.

The representation (2.2) is equal to union of equalities

$$w_-^k(z) = z\int\limits_0^{\infty} e^{-izt} Y_w^k(t)dt, \quad 1 \leqslant k \leqslant n. \tag{2.18}$$

In this case the kernel $Y_w(z,t)$ is diagonal:

$$Y_w(z,t) = \operatorname{diag}\left\{\frac{d}{dt}\int\limits_0^t Y_w^k(t-s)e^{izs}ds\right\} \tag{2.19}$$

Introduce now the special Muckenhoupt power weights

$$w_k^2(x) := |x|^{2(1-\alpha_k)}, \quad \frac{1}{2} < \alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_n < \frac{3}{2}, \quad x \in \mathbb{R}.$$

From (2.18) it follows [3] that

$$Y_w^k(t) = \Gamma^{-1}(\alpha_k)t^{\alpha_k - 1}, \quad 1 \leqslant k \leqslant n.$$

Therefore, from (2.19) and [2] there follows

$$Y_w(z,t) = \operatorname{diag}\,\left\{t^{\alpha_k-1}E_1(izt, \alpha_k)\right\},$$

where the Mittag-Leffler entire function $E_1(z,\alpha)$ is defined by the following power series:

$$E_1(z,\alpha)=\sum_{k=0}^{\infty}\Gamma^{-1}(\alpha+k)z^k,\quad z\in\mathbb{C}.$$

The integral transformations with kernels $t^{\alpha-1}E_1(izt,\alpha)$ $(1/2<\alpha<3/2)$ were introduced and studied in M.M. Dzhrbashyan's papers [2]. If all $\alpha_k=0$ then from the last formula it follows that $Y_w(z,t)=\operatorname{diag}\left\{e^{izt}\right\}$, and, therefore, the integral transformation $\mathcal{D}_w$ is the same as the classical Fourier transform.

Now we assume that Π is analytic in the $\mathbb{C}_-(n\times n)$-matrix-valued function with bounded elements. As the elements $(z-i)^{-1}\Pi(z)$ belong to the class H^2_-, the following representation of type (2.2) holds:

$$\Pi(z)=z\int_0^{\infty}e^{-izt}Y_\Pi(t)dt.$$

Let the constant invertible matrix A be such that

$$\sup_{z\in\mathbb{C}_-}\left\|A^{-1}\Pi(z)\right\|<1,$$

where the matrix norm satisfies the condition $\|E_n\|=1$, $\|AB\|\leqslant\|A\|\cdot\|B\|$ for any matrices A, B.

Then the weight $w^2(x):=\big(A+\Pi(x-i0)\big)\cdot\big(A^*+\Pi^*(x-i0)\big)$ satisfies the (A_2)-condition and

$$A+\Pi(z)=z\int_0^{\infty}e^{-izt}\left(iA+Y_\Pi(t)\right)dt,\quad z\in\mathbb{C}_-,$$

i.e. $Y_w(t)=iA+Y_\Pi(t)$, $t\in\mathbb{R}_+$. Therefore the kernel of the integral transformation $\mathcal{D}_w$ has the form

$$Y_w(z,t)=iAe^{izt}+\frac{d}{dt}\int_0^{t}Y_\Pi(t-s)e^{izs}ds. \tag{2.20}$$

In the above case there exist numbers $M,m>0$ such that

$$m(f,f)\overset{a.e.}{\leqslant}\left(w^2(x)f,f\right)\overset{a.e.}{\leqslant}M(f,f),\quad f\in\mathbb{C}^n,\quad x\in\mathbb{R}.$$

Therefore, the integral transformation $\mathcal{D}_w$ with kernel of type (2.20) acts on the spaces without weight as the classical Fourier transform.

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A CATEGORICAL VIEW OF DI-UNIFORM TEXTURE SPACES

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ABSTRACT. In this paper the authors introduce the category **dfDiU** of di-uniformities and uniformly bicontinuous difunctions. This category is shown to be topological over the category **dfTex** of textures and difunctions and its products are characterized. Relations with some other categories, particularly **dfDitop** and the constructs **Unif** and **QUnif** are discussed. Finally the construct **fDiU** of di-uniformities and uniformly continuous **fTex**-morphisms is introduced and briefly discussed.

1. Introduction

The theory of texture spaces was introduced by L. M. Brown in 1992 under the name "fuzzy structure", and results on this topic appear in several papers including [3, 4, 5, 6, 7, 8], to which the reader may refer for background material and motivation.

On the other hand di-uniformities on a texture were introduced in [14], which will be our main reference. The reader is also referred to [15, 16, 17] for further material on di-uniformities. In this section we recall some basic notions regarding textures and di-uniform texture spaces.

Texture space: [4] Let S be a set. We work within a subset $\mathcal{S}$ of the power set $\mathcal{P}(S)$ called a *texturing*. A texturing is a point-separating, complete, completely distributive lattice with respect to inclusion, which contains S and $\emptyset$, and for which arbitrary meets coincide with intersections, and finite joins with unions. If $\mathcal{S}$ is a texturing of S the pair $(S, \mathcal{S})$ is called a *texture*.

For $s \in S$ the sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\} \text{ and } Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\} = \bigvee\{P_u \mid u \in S,\ s \notin P_u\}$$

are called respectively, the p-sets and q-sets of $(S, \mathcal{S})$. These sets are used in the definition of many textural concepts.

Clearly $(S, \mathcal{S})$ is a texture if and only if $(S, \mathcal{S}^c)$ is a C-space [11]. It is shown in [3] that if ω is the interior relation on $(S, \mathcal{S})$ then $s_1\, \omega\, s_2 \iff P_{s_2} \not\subseteq Q_{s_1}$.

In a texture, arbitrary joins need not coincide with unions, and clearly this will be so if and only if $\mathcal{S}$ is closed under arbitrary unions, or equivalently if $P_s \not\subseteq Q_s$ for all $s \in S$, that is if ω is reflexive. In this case $(S, \mathcal{S})$ is said to be *plain*.

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On the other hand, $M \in \mathcal{S}$ is called a *molecule* if $M \neq \emptyset$ and $M \subseteq A \cup B$, $A, B \in \mathcal{S}$ implies $M \subseteq A$ or $M \subseteq B$. The texture $(S, \mathcal{S})$ is called *simple* if the sets P_s, $s \in S$ are the only molecules in $\mathcal{S}$.

Complementation: [4] A mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A,\ \forall A \in \mathcal{S}$ and $A \subseteq B \implies \sigma(B) \subseteq \sigma(A),\ \forall A, B \in \mathcal{S}$ is called a *complementation* on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is then said to be a *complemented texture*.

Example (1.1).

1. For any set X, $(X, \mathcal{P}(X), \pi_X)$ is the complemented *discrete texture* representing the usual set structure of X. Here the complementation $\pi_X(Y) = X \setminus Y$, $Y \subseteq X$, is the usual set complement. Clearly, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$ for all $x \in X$, so $(X, \mathcal{P}(X))$ is both plain and simple.
2. For $\mathbb{I} = [0, 1]$ define $\mathcal{I} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$, $\iota([0, t]) = [0, 1 - t)$ and $\iota([0, t)) = [0, 1 - t]$, $t \in [0, 1]$. Again $(\mathbb{I}, \mathcal{I}, \iota)$ is a complemented texture, which we will refer to as the *unit interval texture*. Here $P_t = [0, t]$ and $Q_t = [0, t)$ for all $t \in I$. This time $(I, \mathcal{I})$ is plain but not simple since the sets Q_s, $0 < s \leq 1$, are also molecules.
3. The *product texture* $(S \times T, \mathcal{S} \otimes \mathcal{T})$ of textures $(S, \mathcal{S})$ and $(T, \mathcal{T})$. Here the product texturing $\mathcal{S} \otimes \mathcal{T}$ of $S \times T$ consists of arbitrary intersections of sets of the form $(A \times T) \cup (S \times B)$, $A \in \mathcal{S}$ and $B \in \mathcal{T}$.

   For $(s, t) \in S \times T$, $P_{(s,t)} = P_s \times P_t$ and $Q_{(s,t)} = (Q_s \times T) \cup (S \times Q_t)$.
4. Let $B = \{a, b, c\}$, $\mathcal{B} = \{\emptyset, \{a\}, \{a, b\}, B\}$, $\beta(\emptyset) = B$, $\beta(\{a\}) = \{a, b\}$, $\beta(\{a, b\} = \{a\}$ and $\beta(B) = \emptyset$. Then $(B, \mathcal{B}, \beta)$ is a complemented texture. Clearly $P_a = \{a\}$, $P_b = \{a, b\}$, $P_c = B$ and $Q_a = \emptyset$, $Q_b = \{a\}$, $Q_c = \{a, b\}$, so $(B, \mathcal{B})$ is both plain and simple.

Ditopology: A *dichotomous topology* on $(S, \mathcal{S})$ or *ditopology* for short, is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

1. $S,\ \emptyset \in \tau$,
2. $G_1,\ G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
3. $G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

1. $S,\ \emptyset \in \kappa$,
2. $K_1,\ K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and
3. $K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets.

Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures. In the following definition we consider $\mathcal{P}(S) \otimes \mathcal{T}$. To avoid confusion $\overline{P}_{(s,t)}$, $\overline{Q}_{(s,t)}$ are used to denote the p–sets and q–sets for $(S \times T, \mathcal{P}(S) \otimes \mathcal{T})$.

Direlations: [7] Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures.

(1) $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies

R1 $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$.

R2 $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists\, s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.

(2) $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies

CR1 $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$.

CR2 $\overline{P}_{(s,t)} \not\subseteq R \implies \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.

A pair (r, R) consisting of a relation r and corelation R is now called a *direlation*. We will denote by $\mathcal{DR}$ the family of all direlations on a given texture.

Example (1.2). For any texture $(S, \mathcal{S})$ the *identity direlation* (i, I) on $(S, \mathcal{S})$ is given by

$$i = \bigvee\{\overline{P}_{(s,s)} \mid s \in S\} \text{ and } I = \bigcap\{\overline{Q}_{(s,s)} \mid s \in S\}.$$

Direlations are ordered by $(r_1, R_1) \sqsubseteq (r_2, R_2) \iff r_1 \subseteq r_2$ and $R_2 \subseteq R_1$, and the direlation (r, R) on $(S, \mathcal{S})$ is called *reflexive* if $(i, I) \sqsubseteq (r, R)$. We will denote by $\mathcal{RDR}$ the family of all reflexive direlations on a given texture.

Greatest lower bound of direlations: [14] Let (p, P), (q, Q) be direlations on $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then the greatest lower bound $(p, P) \sqcap (q, Q)$ of (p, P), (q, Q) is the direlation $(p, P) \sqcap (q, Q) = (p \sqcap q, P \sqcup Q)$, where

1. $p \sqcap q = \bigvee\{\overline{P}_{(s,t)} \mid \exists v \in S \ P_s \not\subseteq Q_v \text{ and } p, q \not\subseteq \overline{Q}_{(v,t)}\}$, and
2. $P \sqcup Q = \bigcap\{\overline{Q}_{(s,t)} \mid \exists v \in S \ P_v \not\subseteq Q_s \text{ and } \overline{P}_{(v,t)} \not\subseteq P, Q\}$.

Inverse of a direlation: [7] The *inverse* of (r, R) from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is the direlation $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ from $(T, \mathcal{T})$ to $(S, \mathcal{S})$ given by

$$r^{\leftarrow} = \bigcap\{\overline{Q}_{(t,s)} \mid r \not\subseteq \overline{Q}_{(s,t)}\}, \quad R^{\leftarrow} = \bigvee\{\overline{P}_{(t,s)} \mid \overline{P}_{(s,t)} \not\subseteq R\}.$$

A direlation (r, R) on $(S, \mathcal{S})$ is called *symmetric* if $(r, R) = (r, R)^{\leftarrow}$.

Complement of a direlation: [7] Let (r, R) be a direlation between the complemented textures $(S, \mathcal{S}, \sigma)$ and $(T, \mathcal{T}, \theta)$.

1. The *complement r' of the relation r* is the co-relation
   $r' = \bigcap\{\overline{Q}_{(s,t)} \mid \exists u, v \text{ with } r \not\subseteq \overline{Q}_{(u,v)}, \sigma(Q_s) \not\subseteq Q_u \text{ and } P_v \not\subseteq \theta(P_t)\}$.
2. The *complement R' of the co-relation R* is the relation
   $R' = \bigvee\{\overline{P}_{(s,t)} \mid \exists u, v \text{ with } \overline{P}_{(u,v)} \not\subseteq R, P_u \not\subseteq \sigma(P_s) \text{ and } \theta(Q_t) \not\subseteq Q_v\}$.
3. The *complement $(r, R)'$ of the direlation (r, R)* is the direlation
   $(r, R)' = (R', r')$.

A direlation (r, R) on $(S, \mathcal{S})$ is said to be *complemented* if $(r, R)' = (r, R)$.

Another important concept for direlations is that of composition. We recall the following:

Compositions of direlations [7] Let $(S, \mathcal{S})$, $(T, \mathcal{T})$, $(U, \mathcal{U})$ be textures.

1. If p is a relation on $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and q a relation on $(T, \mathcal{T})$ to $(U, \mathcal{U})$ then their *composition* is the relation $q \circ p$ on $(S, \mathcal{S})$ to $(U, \mathcal{U})$ defined by
   $$q \circ p = \bigvee\{\overline{P}_{(s,u)} \mid \exists t \in T \text{ with } p \not\subseteq \overline{Q}_{(s,t)} \text{ and } q \not\subseteq \overline{Q}_{(t,u)}\}.$$
2. If P is a co-relation on $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and Q a co-relation on $(T, \mathcal{T})$ to $(U, \mathcal{U})$ then their *composition* is the co-relation $Q \circ P$ on $(S, \mathcal{S})$ to $(U, \mathcal{U})$ defined by
   $$Q \circ P = \bigcap\{\overline{Q}_{(s,u)} \mid \exists t \in T \text{ with } \overline{P}_{(s,t)} \not\subseteq P \text{ and } \overline{P}_{(t,u)} \not\subseteq Q\}.$$

3. With $p, q; P, Q$ as above, the *composition* of the direlations (p, P), (q, Q) is the direlation

$$(q, Q) \circ (p, P) = (q \circ p, Q \circ P).$$

If (r, R) is a direlation on $(S, \mathcal{S})$ then $(r, R) \circ (r, R) = (r \circ r, R \circ R)$ is also a direlation on $(S, \mathcal{S})$, which we denote by $(r, R)^2$.

Direlational Uniformity: [14] Let $(S, \mathcal{S})$ be a texture and $\mathcal{U}$ a family of direlations on $(S, \mathcal{S})$. If $\mathcal{U}$ satisfies the conditions

1. $(i, I) \sqsubseteq (d, D)$ for all $(d, D) \in \mathcal{U}$. That is, $\mathcal{U} \subseteq \mathcal{RDR}$.
2. $(d, D) \in \mathcal{U}$, $(e, E) \in \mathcal{DR}$ and $(d, D) \sqsubseteq (e, E)$ implies $(e, E) \in \mathcal{U}$.
3. $(d, D), (e, E) \in \mathcal{U}$ implies $(d, D) \sqcap (e, E) \in \mathcal{U}$.
4. Given $(d, D) \in \mathcal{U}$ there exists $(e, E) \in \mathcal{U}$ satisfying $(e, E) \circ (e, E) \sqsubseteq (d, D)$.
5. Given $(d, D) \in \mathcal{U}$ there exists $(c, C) \in \mathcal{U}$ satisfying $(c, C)^{\leftarrow} \sqsubseteq (d, D)$.

then $\mathcal{U}$ is called a *direlational uniformity* on $(S, \mathcal{S})$, and $(S, \mathcal{S}, \mathcal{U})$ is known as a *direlational uniform texture space*.

For a given direlational uniformity $\mathcal{U}$ on $(S, \mathcal{S}, \sigma)$ the direlational uniformity $\mathcal{U}' = \{(d, D)' \mid (d, D) \in \mathcal{U}\}$ is called the *complement* of $\mathcal{U}$. The di-uniformity $\mathcal{U}$ is said to be *complemented* if $\mathcal{U}' = \mathcal{U}$.

Example (1.3). [14] Let $(\mathbb{I}, \mathcal{I})$ be the unit interval texture and for $\epsilon > 0$ define $d_\epsilon = \{(r, s) \mid r, s \in \mathbb{I},\ s < r + \epsilon\}$, $D_\epsilon = \{(r, s) \mid r, s \in \mathbb{I},\ s \leq r - \epsilon\}$. Clearly (d_ϵ, D_ϵ) is a reflexive, symmetric direlation on $(\mathbb{I}, \mathcal{I})$ and

$$\mathcal{U}_{\mathbb{I}} = \{(d, D) \mid (d, D) \in \mathcal{DR} \text{ and } \exists\, \epsilon > 0 \text{ with } (d_\epsilon, D_\epsilon) \sqsubseteq (d, D)\}$$

is a direlational uniformity on $(\mathbb{I}, \mathcal{I})$. We will call $\mathcal{U}_{\mathbb{I}}$ *the usual direlational uniformity on* $(\mathbb{I}, \mathcal{I})$.

Difunction: [7] A *difunction* from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is a direlation (f, F) from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ satisfying the conditions

(DF1) For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \implies \exists\, t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.

(DF2) For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \implies P_{t'} \not\subseteq Q_t$.

If (f, F) is a difunction on $(S, \mathcal{S})$ to $(T, \mathcal{T})$ then (f, F) is called *surjective* if it satisfies the condition

SUR. For $t, t' \in T$, $P_t \not\subseteq Q_{t'} \implies \exists\, s \in S$ with $f \not\subseteq \overline{Q}_{(s,t')}$ and $\overline{P}_{(s,t)} \not\subseteq F$.

Likewise, (f, F) is called *injective* if it satisfies the condition

INJ. For $s, s' \in S$ and $t \in T$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F \implies P_s \not\subseteq Q_{s'}$.

Example (1.4). Let $(X, \mathcal{P}(X), \pi_X)$, $(Y, \mathcal{P}(Y), \pi_Y)$ be complemented discrete textures. The pair (φ, ψ) of point relations from X to Y is a difunction if and only if φ is a point function $\varphi : X \to Y$ and $\psi = \varphi^c$, where ^{c} denotes set complementation.

Inverse of a direlation under a difunction: [14] Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures, (r, R) a direlation on $(T, \mathcal{T})$ and (f, F) a difunction on $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Then

$$(f, F)^{-1}(r) = \bigvee\{\overline{P}_{(s_1,s_2)} \mid \exists P_{s_1} \not\subseteq Q_{s_1'} \text{ so that } \overline{P}_{(s_1',t_1)} \not\subseteq F, f \not\subseteq \overline{Q}_{(s_2,t_2)} \implies \overline{P}_{(t_1,t_2)} \subseteq r\}$$

$$(f, F)^{-1}(R) = \bigcap\{\overline{Q}_{(s_1,s_2)} \mid \exists P_{s_1'} \not\subseteq Q_{s_1} \text{ so that} f \not\subseteq \overline{Q}_{(s_1',t_1)}, \overline{P}_{(s_2,t_2)} \not\subseteq F, \implies R \subseteq \overline{Q}_{(t_1,t_2)}\}$$

$$(f, F)^{-1}(r, R) = ((f, F)^{-1}(r), (f, F)^{-1}(R)).$$

Uniformly bicontinuous difunction: [14] Let $\mathcal{U}$ be a direlational uniformity on $(S, \mathcal{S})$, $\mathcal{V}$ a direlational uniformity on $(T, \mathcal{T})$ and (f, F) a difunction from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. If $(d, D) \in \mathcal{V} \implies (f, F)^{-1}(d, D) \in \mathcal{U}$ then the difunction (f, F) is said to be $\mathcal{U}$–$\mathcal{V}$ *uniformly bicontinuous*. In the case of complemented texture spaces, (f, F) is $\mathcal{U}$–$\mathcal{V}$ uniformly bicontinuous if and only if $(f, F)'$ is $\mathcal{U}'$–$\mathcal{V}'$ uniformly bicontinuous.

Dicovers: A difamily $\mathcal{C} = \{(A_j, B_j) \mid j \in J\}$ of elements of $\mathcal{S} \times \mathcal{S}$ which satisfies $\bigcap_{j\in J_1} B_j \subseteq \bigvee_{j\in J_2} A_j$ for all partitions (J_1, J_2) of J, including the trivial partitions, is called a *dicover* of $(S, \mathcal{S})$. An important example is the family $\mathcal{P} = \{(P_s, Q_s) \mid s \in S^\flat\}$ [3, Page 338] which is a dicover for any texture $(S, \mathcal{S})$. If $\mathcal{D}$ is a dicover we often write $L\mathcal{D}M$ in place of $(L, M) \in \mathcal{D}$. We recall the following notions for dicovers.

1. $\mathcal{C}$ is a *refinement* of $\mathcal{D}$ if given $j \in J$ we have $L\,\mathcal{D}\,M$ so that $A_j \subseteq L$ and $M \subseteq B_j$. In this case we write $\mathcal{C} \prec \mathcal{D}$.
2. If $\mathcal{C}$, $\mathcal{D}$ are dicovers then $\mathcal{C} \wedge \mathcal{D} = \{(A \cap C, B \cup D) \mid A\,\mathcal{C}\,B,\ C\,\mathcal{D}\,D\}$ is the greatest lower bound (meet) of $\mathcal{C}$, $\mathcal{D}$ with respect to refinement.
3. The *star* and *co-star* of $C \in \mathcal{S}$ with respect to $\mathcal{C}$ are respectively the sets
   $\mathrm{St}(\mathcal{C}, C) = \bigvee\{A_j \mid J \in j,\ C \not\subseteq B_j\} \in \mathcal{S}$, and
   $\mathrm{CSt}(\mathcal{C}, C) = \bigcap\{B_j \mid j \in j,\ A_j \not\subseteq C\} \in \mathcal{S}$.
   We say that $\mathcal{C}$ is a *delta refinement* of $\mathcal{D}$, and write $\mathcal{C} \prec_{(\Delta)} \mathcal{D}$, if
   $\mathcal{C}^\Delta = \{(\mathrm{St}(\mathcal{C}, P_s), \mathrm{CSt}(\mathcal{C}, Q_s)) \mid s \in S^\flat\} \prec \mathcal{D}$.
   We say that $\mathcal{C}$ is a *star refinement* of $\mathcal{D}$, and write $\mathcal{C} \prec_{(\star)} \mathcal{D}$, if
   $\mathcal{C}^\star = \{(\mathrm{St}(\mathcal{C}, A_i), \mathrm{CSt}(\mathcal{C}, B_i)) \mid i \in I\} \prec \mathcal{D}$.

A family $\mathcal{C} \subseteq \mathcal{S} \times \mathcal{S}$ is called an *anchored dicover* if it satisfies:

1. $\mathcal{P} \prec \mathcal{C}$, and
2. Given $A\,\mathcal{C}\,B$ there exists $s \in S$ satisfying
   (i) $A \not\subseteq Q_u \implies \exists\, A'\,\mathcal{C}\,B'$ with $A' \not\subseteq Q_u$ and $P_s \not\subseteq B'$, and
   (ii) $P_v \not\subseteq B \implies \exists\, A''\,\mathcal{C}\,B''$ with $P_v \not\subseteq B''$ and $A'' \not\subseteq Q_s$.

The family of all dicovers (anchored dicovers) on $(S, \mathcal{S})$ is denoted by $\mathcal{DC}$ (resp, $\mathcal{ADC}$). Just as in the classical case uniformities can be described using covers, so diuniformities can be described in terms of dicovers, the resulting structure being called a dicovering uniformity.

Dicovering uniformity: [14] Let $(S, \mathcal{S})$ be a texture. If υ is a family of dicovers of S satisfying the following conditions:

1. Given $\mathcal{C} \in \boldsymbol{v}$ there exists $\mathcal{D} \in \boldsymbol{v} \cap \mathcal{ADC}$ with $\mathcal{D} \prec \mathcal{C}$,
2. $\mathcal{C} \in \boldsymbol{v}$, $\mathcal{D} \in \mathcal{DC}$ and $\mathcal{C} \prec \mathcal{D}$ implies $\mathcal{D} \in \boldsymbol{v}$,
3. $\mathcal{C}, \mathcal{D} \in \boldsymbol{v}$ implies $\mathcal{C} \wedge \mathcal{D} \in \boldsymbol{v}$.
4. Given $\mathcal{C} \in \boldsymbol{v}$ there exists $\mathcal{D} \in \boldsymbol{v}$ with $\mathcal{D} \prec_{(\star)} \mathcal{C}$,

we say $\boldsymbol{v}$ is a *dicovering uniformity* on $(S, \mathcal{S})$, and call $(S, \mathcal{S}, \boldsymbol{v})$ a *dicovering uniform texture space*.

Since we will use the relationship between direlational and dicovering uniformities in a categorical result we recall the following theorem.

THEOREM (1.5). [14] *Let $(S, \mathcal{S})$ be a texture.*

1. *To each direlational uniformity $\mathcal{U}$ on $(S, \mathcal{S})$ we may associate a dicovering uniformity $\boldsymbol{v} = \Gamma(\mathcal{U}) = \{\mathcal{C} \in \mathcal{DC} \mid \exists\,(c, C) \in \mathcal{U}$ and $\gamma(c, C) \prec \mathcal{C}\}$.*
2. *To each dicovering uniformity $\boldsymbol{v}$ on $(S, \mathcal{S})$ we may associate a direlational uniformity $\mathcal{U} = \Delta(\boldsymbol{v}) = \{(d, D) \in \mathcal{RDR} \mid \exists\, \mathcal{C} \in \boldsymbol{v}$ with $\delta(\mathcal{C}) \sqsubseteq (d, D)\}$.*
3. *$\Delta(\Gamma(\mathcal{U})) = \mathcal{U}$ for every direlational uniformity $\mathcal{U}$ on $(S, \mathcal{S})$.*
4. *$\Gamma(\Delta(\boldsymbol{v})) = \boldsymbol{v}$ for every dicovering uniformity $\boldsymbol{v}$ on $(S, \mathcal{S})$.*

Inverse of a dicover under a difunction: [14] Let $(S, \mathcal{S})$,$(T, \mathcal{T})$ be textures, (f, F) a difunction on $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and $\mathcal{C}$ a dicover of $(T, \mathcal{T})$. Then

$$(f, F)^{-1}(\mathcal{C}) = \{(F^{\leftarrow}(A), f^{\leftarrow}(B)) \mid A\, \mathcal{C}\, B\}.$$

Uniform Ditopology: [14] Just as a uniformity in the classical sense determines a topology called the uniform topology, so a di-uniformity determines a ditopology called the *uniform ditopology*. If $(S, \mathcal{S}, \boldsymbol{v})$ is a dicovering uniform texture space we recall from [14, Definition 4 and 5] that the uniform ditopology $(\tau_{\boldsymbol{v}}, \kappa_{\boldsymbol{v}})$ on $(S, \mathcal{S})$ is given by

$$\tau_{\boldsymbol{v}} = \{G \in \mathcal{S} \mid G \not\subseteq Q_s \implies \exists\, \mathcal{C} \in \boldsymbol{v},\ \mathrm{St}(\mathcal{C}, P_s) \subseteq G\},$$
$$\kappa_{\boldsymbol{v}} = \{K \in \mathcal{S} \mid P_s \not\subseteq K \implies \exists\, \mathcal{C} \in \boldsymbol{v},\ K \subseteq \mathrm{CSt}(\mathcal{C}, Q_s)\}.$$

For general textures it is impossible to associate a difunction with a general point function, but under some conditions on the point function this is possible. The next lemma gives a way of obtaining a difunction from a suitable point function, and the susequent propositions consider the converse situation in two special cases.

LEMMA (1.6). [7] *Suppose that the point function φ on S to T satisfies the condition*

(a) $P_s \not\subseteq Q_{s'} \implies P_{\varphi(s)} \not\subseteq Q_{\varphi(s')}$ *for all* $s, s' \in S$.

Then the equalities

$$f = f_\varphi = \bigvee\{\overline{P}_{(s,t)} \mid \exists u \in S \text{ satisfying } P_s \not\subseteq Q_u \text{ and } P_{\varphi(u)} \not\subseteq Q_t\},$$
$$F = F_\varphi = \bigcap\{\overline{Q}_{(s,t)} \mid \exists u \in S \text{ satisfying } P_u \not\subseteq Q_s \text{ and } P_t \not\subseteq Q_{\varphi(u)}\},$$

define a difunction (f, F) on $(S, \mathcal{S})$ to $(T, \mathcal{T})$. Moreover, for $B \in \mathcal{T}$, $F^{\leftarrow}B = \varphi^{\leftarrow}B = f^{\leftarrow}B$, where $\varphi^{\leftarrow}B = \bigvee\{P_s \mid P_{\varphi(u)} \subseteq B\ \forall u \in S$ with $P_s \not\subseteq Q_u\}$.

Clearly condition (a) just says that φ is ω-preserving. Now let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be textures with $(T, \mathcal{T})$ simple, and let (f, F) be a difunction from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. For $s \in S$ the set P_s is a molecule and it follows that $f^{\to}P_s$ is also a molecule, whence there exists $t \in T$, necessarily unique since $\mathcal{T}$ separates the points of T, such that $f^{\to}P_s = P_t$. In this way we obtain a function $\varphi = \varphi_{(f,F)} : S \to T$ characterized by the equality $P_{\varphi(s)} = f^{\to}P_s$ for all $s \in S$ [7].

PROPOSITION (1.7). [7] *The function $\varphi = \varphi_{(f,F)} : S \to T$ corresponding as above to the difunction $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$, with $(T, \mathcal{T})$ simple, is ω-preserving and in addition satisfies the property:*

(b) $P_{\varphi(s)} \not\subseteq B$, $B \in \mathcal{T} \implies \exists\, s' \in S$ *with* $P_s \not\subseteq Q_{s'}$ *for which* $P_{\varphi(s')} \not\subseteq B$.

Conversely, if $\varphi : S \to T$ is any ω-preserving function satisfying condition (b) above then there exists a unique difunction $(f_\alpha, F_\alpha) : (S, \mathcal{S}) \to (T, \mathcal{T})$ satisfying $\varphi = \varphi_{(f_\alpha, F_\alpha)}$.

If we consider plain textures we obtain the same class of point functions. Indeed, assume that $(S, \mathcal{S})$ is plain and consider a difunction $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$. For $s \in S$ we now have $P_s \not\subseteq Q_s$, so by *DF1* there exists $t \in T$ satisfying $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t)} \not\subseteq F$. Moreover, t is easily seen to be uniquely determined by s, so we obtain a point function $\psi = \psi_{(f,F)} : S \to T$ characterized by $f \not\subseteq \overline{Q}_{(s,\psi(s))}$ and $\overline{P}_{(s,\psi(s))} \not\subseteq F$ [7].

PROPOSITION (1.8). [7] *The function $\psi : S \to T$ corresponding as above to the difunction $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$, with $(S, \mathcal{S})$ plain, is ω-preserving and satisfies condition (b) of Proposition 1.7.*

Conversely, if $\psi : S \to T$ is any ω-preserving function satisfying condition (b) of Proposition 1.7 then there exists a unique difunction $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$ satisfying $\psi = \psi_{(f,F)}$.

2. The Category dfDiU

In this section we introduce the category **dfDiU** of di-uniform spaces and uniformly bicontinuous difunctions, and consider several related categories.

The term di-uniformity is used to denote both direlational and dicovering uniformities on a texture, so to justify its use in categorical terms we must show that direlational uniformities and dicovering uniformities, together with their appropriate class of uniformly continuous difunctions, produce categories which are concretely isomorphic [1, Remark 5.12].

Direlational uniformities and uniformly bicontinuous difunctions form a category since uniform bicontinuity is preserved under composition of difunctions [14, Proposition 5.12], the identity difunction (i, I) on $(S, \mathcal{S}, \mathcal{U})$ is $\mathcal{U}$–$\mathcal{U}$ uniformly bicontinuous [14, Example 5.10], and finally the identity difunctions are identities for composition and composition is associative [7, Proposition 2.17(3)]. We will denote this category by **A**.

Similarly, dicovering uniformities and uniformly bicontinuous difunctions form a category which we will denote by **B**.

Denoting by $U : \mathbf{A} \to \mathbf{dfTex}$, $V : \mathbf{B} \to \mathbf{dfTex}$ the obvious forgetful functors we see that $(\mathbf{A}, U), (\mathbf{B}, V)$ are concrete categories over the category **dfTex** of textures and difunctions.

THEOREM (2.1). *The categories* $(\mathbf{A}, U), (\mathbf{B}, V)$ *are concretely isomorphic.*

Proof. By [14, Proposition 5.20], $F : \mathbf{A} \to \mathbf{B}$ given by

$$F((S_1, \mathcal{S}_1, \mathcal{U}_1) \xrightarrow{(f,F)} (S_2, \mathcal{S}_2, \mathcal{U}_2)) = ((S_1, \mathcal{S}_1, \Gamma(\mathcal{U}_1)) \xrightarrow{(f,F)} (S_2, \mathcal{S}_2, \Gamma(\mathcal{U}_2)))$$

is a functor. By Theorem 1.5 it is easy to verify that F is an isomorphism, and clearly $V \circ F = U$, so it is a concrete isomorphism. □

This justifies the following definition:

Definition (2.2). The category whose objects are di-uniformities and whose morphisms are uniformly bicontinuous difunctions will be denoted by **dfDiU**.

If we take as objects di-uniformities on a simple texture we obtain the full subcategory **dfSDiU** and inclusion functor $\mathfrak{S} : \mathbf{dfSDiU} \hookrightarrow \mathbf{dfDiU}$.

Also we obtain the full subcategory **dfPDiU** and inclusion functor $\mathfrak{P} : \mathbf{dfPDiU} \hookrightarrow \mathbf{dfDiU}$ by taking as objects di-uniformities on a plain texture.

In the same way we can use **dfPSDiU** to denote the category whose objects are di-uniformities on a plain simple texture, and whose morphisms are uniformly bicontinuous difunctions. Between these new categories we may write the inclusion functors with explanatory subscripts. For example $\mathfrak{P}_s : \mathbf{dfPSDiU} \hookrightarrow \mathbf{dfSDiU}$.

Again we will use the term complemented di-uniformity to refer to complemented direlational and complemented dicovering uniformities in general. We denote the corresponding subcategory of **dfDiU** by **dfCdiU** and the inclusion functor by $\mathfrak{C} : \mathbf{dfCdiU} \hookrightarrow \mathbf{dfDiU}$. Note that the morphisms in this category are not restricted to be complemented difunctions, should we wish to make this additional restriction the resulting (non-full) subcategory of **dfDiU** will be denoted by **cdfDiU** in accordance with the convention established in [7]. Example 2.13 in [16] and the discussion following [16, Theorem 3.5] suggests that in certain circumstances **dfCdiU** may be more useful than **cdfDiU**.

We will return to these subcategories later. In the meantime we give some basic results about the category **dfDiU**. We begin with some results about the morphisms of **dfDiU**. The reader is referred to [1, 13] for the relevant terms from category theory.

PROPOSITION (2.3). *In the category* **dfDiU***:*

1. *Every section is injective as a difunction.*
2. *Every injective morphism is a monomorphism.*
3. *Every retraction is surjective as a difunction.*
4. *Every surjective morphism is an epimorphism.*
5. *A morphism is an isomorphism if and only if it is bijective as a difunction and its inverse is bicontinuous.*

Proof. Clear from [7, Proposition 3.14]. □

As mentioned earlier **dfDiU** is concrete over **dfTex** with respect to the forgetful functor which we will denote by $\mathfrak{G}$. We wish to show that $\mathfrak{G}$ is topological. We recall from [1] that $\mathfrak{G}$ is topological if every $\mathfrak{G}$-structured source has a unique $\mathfrak{G}$-initial lift. In many cases the existence of $\mathfrak{G}$-initial lifts is related to the existence of initial structures, and the following lemma shows this to be the case for di-uniformities.

LEMMA (2.4). *The source* $((S,\mathcal{S},\mathcal{U}),((S,\mathcal{S},\mathcal{U}) \xrightarrow{(f_j,F_j)} (S_j,\mathcal{S}_j,\mathcal{U}_j))_{j\in J})$ *in* **dfDiU** *is $\mathfrak{G}$-initial if and only if* $\beta = \{(f_j,F_j)^{-1}(d,D) \mid (d,D)\in\mathcal{U}_j, j\in J\}$ *is a subbase for* $\mathcal{U}$. *That is, if and only if* $\mathcal{U}$ *is the coarsest direlational uniformity on* $(S,\mathcal{S})$ *for which the difunctions* (f_j,F_j), $j\in J$, *are uniformly bicontinuous.*

Proof. Necessity is clear. For the sufficiency suppose that β is a base of $\mathcal{U}$ and consider the following diagrams in **dfDiU** and **dfTex**, respectively.

$$\begin{tikzcd} (S,\mathcal{S},\mathcal{U}) \arrow[dr, "(f_j,F_j)"] & \\ (S',\mathcal{S}',\mathcal{U}') \arrow[u, "(k,K)"] \arrow[r, "(h_j,H_j)"'] & (S_j,\mathcal{S}_j,\mathcal{U}_j) \end{tikzcd} \qquad \begin{tikzcd} (S,\mathcal{S}) \arrow[dr, "(f_j,F_j)"] & \\ (S',\mathcal{S}') \arrow[u, "(k,K)"] \arrow[r, "(h_j,H_j)"'] & (S_j,\mathcal{S}_j) \end{tikzcd}$$

Let (k,K) be a morphism in **dfTex** making the right-hand diagram commutative and suppose in the left-hand diagram that (h_j,H_j) is uniformly bicontinuous. Because $\mathfrak{G}$ is faithful it will be sufficient to show that (k,K) is a morphism in **dfDiU**. So let $(k,K)^{-1}(d,D)\in\mathcal{U}'\,\forall(d,D)\in\beta$. But $(d,D)\in\beta$ has the form $(f_j,F_j)^{-1}(r,R)$, $(r,R)\in\mathcal{U}_j$ and $(k,K)^{-1}((f_j,F_j)^{-1}(d,D)) = ((f_j,F_j)\circ(k,K))^{-1}(d,D) = (h_j,H_j)^{-1}\,(d,D)\in\mathcal{U}'$ since (h_j,H_j) is uniformly bicontinuous. Hence (k,K) is uniformly bicontinuous, as required. □

THEOREM (2.5). *The functor* $\mathfrak{G}$: **dfDiU** $\to$ **dfTex** *is topological. In other words,* **dfDiU** *is topological over* **dfTex** *with respect to the functor* $\mathfrak{G}$.

Proof. Let us take $(S_j,\mathcal{S}_j,\mathcal{U}_j)$, $j\in J$ as an object in **dfDiU** and $(S,\mathcal{S}) \xrightarrow{(f_j,F_j)} (S_j,\mathcal{S}_j)$ in $\mathfrak{G}$(**dfDiU**) $=$ **dfTex**. Let $\mathcal{U}$ be the di-uniformity on $(S,\mathcal{S})$ with subbase $\beta = \{(f_j,F_j)^{-1}(r,R)\mid(r,R)\in\mathcal{U}_j, j\in J\}$. Then by Lemma 2.4,

$$((S,\mathcal{S},\mathcal{U}),((S,\mathcal{S},\mathcal{U}) \xrightarrow{(f_j,F_j)} (S_j,\mathcal{S}_j,\mathcal{U}_j))_{j\in J})$$

is the unique $\mathfrak{G}$-initial source, which maps to $((S,\mathcal{S}),((S,\mathcal{S}) \xrightarrow{(f_j,F_j)} (S_j,\mathcal{S}_j))_{j\in J})$ under $\mathfrak{G}$. □

Since a topological functor preserves limits, and from [8, Theorem 3.10] we know **dfTex** has products, we have at once by [1, Proposition 10.53] that:

COROLLARY (2.6). **dfDiU** *has products.* □

Thus, to construct the product in **dfDiU** of $(S_j,\mathcal{S}_j,\mathcal{U}_j)$, $j\in J$, we construct the product of $(S_j,\mathcal{S}_j)$, $j\in J$, in **dfTex** and then take the $\mathfrak{G}$-initial lift to the objects $(S_j,\mathcal{S}_j,\mathcal{U}_j)$, which by Lemma 2.4 produces the initial di-uniformity with respect to the spaces $(S_j,\mathcal{S}_j,\mathcal{U}_j)$ and the projection difunctions (π_j,Π_j) [8].

In order to discuss products in **dfCdiU** let $(S, \mathcal{S}, \sigma)$ be the complemented product of the complemented textures $(S_j, \mathcal{S}_j, \sigma_j)$, $j \in J$, where J is a set and $(S_j, \mathcal{S}_j, \sigma_j)$ are complemented textures.

Now let $(S_j, \mathcal{S}_j, \sigma_j, \mathcal{U}_j)$ be complemented diuniformities and $\mathcal{U}$ the product di-uniformity on $(S, \mathcal{S}, \sigma)$. For all $j \in J$ the j^{th} projection difunction (π_j, Π_j) is complemented [8], and since $(d_j, D_j)' = (d_j, D_j)$ for all $(d_j, D_j) \in \mathcal{U}_j$ we have

$$((\pi_j, \Pi_j)^{-1}(d_j, D_j))' = (((\pi_j, \Pi_j)')^{-1}(d_j, D_j)') = (\pi_j, \Pi_j)^{-1}(d_j, D_j).$$

Thus, the product di-uniformity $\mathcal{U}$ is complemented and we have:

COROLLARY (2.7). *The category* **dfCdiU** *has products.* □

Similarly the categories **dfSDiU**, **dfPDiU**, **dfSCdiU**, and **dfPCdiU** also have products.

3. Relations with other categories

In this section we consider the relation of **dfDiU** and some of its subcategories with other known categories. We begin by noting that just as the category **Unif** of uniformities and uniformly continuous functions is related to the construct **Top** of topological spaces and continuous functions via the uniform topology, so **dfDiU** is related to the category **dfDitop** of ditopological textures and bicontinuous difunctions via the uniform ditopology. Specifically, working in terms of dicovering uniformities we define $\mathfrak{F}$: **dfDiU** $\to$ **dfDiTop** by

$$\mathfrak{F}((S_1, \mathcal{S}_1, \upsilon_1) \xrightarrow{(f,F)} (S_2, \mathcal{S}_2, \upsilon_2)) = ((S_1, \mathcal{S}_1, \tau_{\upsilon_1}, \kappa_{\upsilon_1}) \xrightarrow{(f,F)} (S_2, \mathcal{S}_2, \tau_{\upsilon_2}, \kappa_{\upsilon_2})).$$

This is a functor since if (f, F) is υ_1–υ_2 uniformly bicontinuous it is $(\tau_{\upsilon_1}, \kappa_{\upsilon_1})$–$(\tau_{\upsilon_2}, \kappa_{\upsilon_2})$ bicontinuous by [14, Proposition 5.13] and Theorem 1.5. It is also concrete with respect to the forgetful functors to **dfTex**. Moreover, we know by [14, Theorem 4.14] that the uniform ditopology is always completely biregular. Hence, denoting by **dfCbiReg** the full subcategory of **dfDitop** whose objects are completely biregular ditopological texture spaces, we may regard $\mathfrak{F}$ as a concrete functor $\mathfrak{F}$: **dfDiU** $\to$ **dfCbiReg** over **dfTex**. Since by [14, Theorem 5.15] every completely biregular ditopological texture space has a compatible di-uniformity, $\mathcal{F}$ is surjective on objects.

LEMMA (3.1). *The functor* $\mathfrak{F}$: **dfDiU** $\to$ **dfCbiReg** *preserves initial sources.*

Proof. We must show that if $((S, \mathcal{S}, \upsilon), ((S, \mathcal{S}, \upsilon) \xrightarrow{(f_j, F_j)} (S_j, \mathcal{S}_j, \upsilon_j))_{j \in J})$ is a $\mathfrak{G}$-initial source in **dfDiU** then

$$((S, \mathcal{S}, \tau_\upsilon, \kappa_\upsilon), ((S, \mathcal{S}, \tau_\upsilon, \kappa_\upsilon) \xrightarrow{(f_j, F_j)} (S_j, \mathcal{S}_j, \tau_{\upsilon_j}, \kappa_{\upsilon_j}))_{j \in J})$$

is a $\mathfrak{U}$-initial source in **dfDitop**, where $\mathfrak{U}$ is the forgetful functor $\mathfrak{U}$: **dfCbiReg** $\to$ **dfTex** (cf. [8]). In view of the characterizations of initial lifts given in Lemma 2.4 and [8, Theorem 3.5] respectively, this is equivalent to showing that the uniform ditopology of the initial di-uniformity υ is the initial ditopology (τ, κ) of the corresponding uniform ditopologies.

Since a uniformly bicontinuous difunction is bicontinuous for the uniform ditopology [14, Proposition 5.13], it is clear that $\tau \subseteq \tau_\upsilon$ and $\kappa \subseteq \kappa_\upsilon$. To obtain the opposite inclusions take $H \in \tau_v$ and $s \in S$ with $H \not\subseteq Q_s$. Then there exists $\mathcal{C} \in v$ with $St(\mathcal{C}, P_s) \subseteq H$ by [14, Definition 4.5]. Now we have $j_1, j_2, \ldots, j_n \in J$ with

$$\mathcal{D} = \left(\bigwedge_{k=1}^{n} (f_{j_k}, F_{j_k})^{-1} (\mathcal{C}_{j_k})^{\Delta} \right)^{\Delta} \prec \mathcal{C},$$

for some $\mathcal{C}_{j_k} \in v_{j_k}$, $k = 1, 2, \ldots, n$. By [14, Proposition 4.8] each v_{j_k} has a base of open, coclosed dicovers, so without loss of generality we may assume that $\mathcal{C}_{j_k}$ is open, coclosed for each $k = 1, \ldots, n$. By [14, Definition 5.17] we see that $(f_{j_k}, F_{j_k})^{-1}(\mathcal{C}_{j_k})$ is open, coclosed since (f_{j_k}, F_{j_k}) is bicontinuous. We now easily deduce that $\mathcal{D}$ is open, coclosed, and hence $H \in \tau$. Thus we have $\tau_v \subseteq \tau$, and likewise $\kappa_v \subseteq \kappa$. □

THEOREM (3.2). *The concrete functor $\mathfrak{F}$: **dfDiU** $\to$ **dfCbiReg** has a left adjoint which is also the finest section of $\mathfrak{F}$.*

Proof. By Lemma 3.1 we may define a functor $\mathfrak{L} = \langle \mathbf{dfDiU} \rangle_{\mathfrak{F}} : \mathbf{dfCbiReg} \to \mathbf{dfDiU}$ using an adaptation of the *spanning construction* described in [9, 10]. Specifically for $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}\,\mathbf{dfCbiReg}$ we let $v(\tau, \kappa)$ be the $\mathfrak{G}$-initial di-uniformity generated by all $(T, \mathcal{T}, v) \in \mathrm{Ob}\,\mathbf{dfDiU}$ and bicontinuous $(f, F) : (S, \mathcal{S}, \tau, \kappa) \to \mathfrak{F}(T, \mathcal{T}, v)$, and then

$$\mathfrak{L}((S_1, \mathcal{S}_1, \tau_1, \kappa_1) \xrightarrow{(g,G)} (S_2, \mathcal{S}_2, \tau_2, \kappa_2)) = ((S_1, \mathcal{S}_1, v(\tau_1, \kappa_1)) \xrightarrow{(g,G)} (S_2, \mathcal{S}_2, v(\tau_2, \kappa_2))).$$

By Lemma 3.1 and the proof of [14, Theorem 5.16] we see that $v(\tau, \kappa)$ is compatible with the completely biregular space $(S, \mathcal{S}, \tau, \kappa)$ and so $\mathfrak{L}$ is a section of $\mathfrak{F}$ (i.e. satisfies $\mathfrak{F}\mathfrak{L} = \mathbf{1}_{\mathbf{dfCbiReg}}$).

Furthermore, $\mathfrak{L}$ is the finest section of $\mathfrak{F}$, in the sense that we (clearly) have a natural transformation $j : \mathfrak{L}\mathfrak{F} \to \mathbf{1}_{\mathbf{dfDiU}}$ such that $\mathfrak{F}j$ is the identity transformation on $\mathbf{1}_{\mathbf{dfCbiReg}}$. Clearly j is the co-unit of an adjunction in which the unit is the identity transformation $\mathbf{1}_{\mathbf{dfCbiReg}} \to \mathfrak{F}\mathfrak{L}$. □

Now let us look at the link between di-uniformities and quasi-uniformities. This was investigated in [16]. Our aim will be to put the relations between di-uniformities on a discrete texture and (quasi-) uniformities on a set discussed in [16] into a categorical setting.

The construct **Unif** of uniform structures and uniformly continuous functions is well known. The objects in **Unif** may be regarded as either diagonal or covering uniformities, since it is well known that these lead to concretely isomorphic constructs. The diagonal representation of quasi-uniformities is also well known, that by dual covers less so. We wish to be sure that in the construct **QUnif** of quasi-uniformities and quasi-uniformly continuous functions we may again use either the diagonal or dual-covering representation.

In [16] a one-to-one mapping Γ^* is defined between the diagonal quasi-uniformities on a set and the dual-covering quasi uniformities on that set. If we can show that the quasi-uniformly continuous functions are preserved under this mapping it will follow immediately that they form concretely isomorphic constructs. We recall from [16] that if ρ is a binary relation on X then

$$\gamma^*(\rho) = \{ \big(\rho[x], \rho^{-1}[x]\big) \mid x \in X \}$$

is a dual cover of X, and that for a diagonal quasi-uniformity $\mathcal{Q}$, that is a family of binary relations on X satisfying all but the symmetry condition of a uniformity, then

$$\Gamma^*(\mathcal{Q}) = \{U \mid U \text{ is a dual cover}, \exists\, d \in \mathcal{Q} \text{ with } \gamma^*(d) \prec U\}$$

is the corresponding dual-covering quasi-uniformity. On the other hand a function $f : (X_1, \mathcal{Q}_1) \to (X_2, \mathcal{Q}_2)$ is quasi-uniformly continuous if $d \in \mathcal{Q}_2 \implies (f^2)^{-1}[d] \in \mathcal{Q}_1$, while for dual-covering quasi-uniformities, $f : (X_1, \mathcal{U}_1) \to (X_2, \mathcal{U}_2)$ is quasi-uniformly continuous if $U \in \mathcal{U}_2 \implies f^{-1}(U) \in \mathcal{U}_2$. Here, for a dual cover U, $f^{-1}(U) = \{\big(f^{-1}[A], f^{-1}[B]\big) \mid A\, U\, B\}$, and it is trivial to show that for $x \in X_1$ and a binary relation ρ on X_2,

$$f^{-1}\big(\rho[f(x)]\big) = \big((f^2)^{-1}[\rho]\big)[x] \text{ and } f^{-1}\big(\rho^{-1}[f(x)]\big) = \big((f^2)^{-1}[\rho]\big)^{-1}[x].$$

LEMMA (3.3). *$f : (X_1, \mathcal{Q}_1) \to (X_2, \mathcal{Q}_2)$ is quasi-uniformly continuous if and only if $f : (X_1, \Gamma^*(\mathcal{Q}_1)) \to (X_2, \Gamma^*(\mathcal{Q}_2))$ is quasi-uniformly continuous.*

Proof. For $d \in \mathcal{Q}_2$ we have $f^{-1}(\gamma^*(d)) = \gamma^*((f^2)^{-1}[d])$ by the above equalities, and the result now follows immediately from the definitions of quasi-uniform continuity for diagonal and dual-covering quasi-uniformities. □

This confirms that the objects of **QUnif** may be taken to be diagonal or dual-covering quasi-uniformities, as the need arises.

In [16] a bijection was established between dual-covering quasi-uniformities on a set X and dicovering uniformities on the discrete texture $(X, \mathcal{P}(X))$. Specifically, for a dual cover U on X, $u^*(U) = \{(A, X \setminus B) \mid A\, U\, B\}$ was shown to be a dicover of $(X, \mathcal{P}(X))$, anchored if $A\, U\, B \implies A \cap B \neq \emptyset$, and then setting

$$u^*(\mathcal{U}) = \{\mathcal{C} \mid \mathcal{C} \text{ a dicover of } (X, \mathcal{P}(X)) \text{ with } u^*(U) \prec \mathcal{C} \text{ for some } U \in \mathcal{U}\}$$

gives the required bijection $\mathcal{U} \to u^*(\mathcal{U})$ [16, Corollary 3.12]. Now let $f : (X_1, \mathcal{U}_1) \to (X_2, \mathcal{U}_2)$ be a point function between the dual-covering quasi-uniformities $(X_1, \mathcal{U}_1)$ and $(X_2, \mathcal{U}_2)$. By [16, Lemma 3.1 (13)], $(f, f') : (X_1, \mathcal{P}(X_1)) \to (X_2, \mathcal{P}(X_2))$ is a difunction, and indeed every difunction between these discrete textures has this form.

LEMMA (3.4). *$f : (X_1, \mathcal{U}_1) \to (X_2, \mathcal{U}_2)$ is quasi-uniformly continuous if and only if $(f, f') : (X_1, \mathcal{P}(X_1), u^*(\mathcal{U}_1)) \to (X_2, \mathcal{P}(X_2), u^*(\mathcal{U}_2))$ is uniformly bicontinuous.*

Proof. $(\Rightarrow)$ Take $\mathcal{C} \in u^*(\mathcal{U}_2)$. Since $u^*(\mathcal{U}_2)$ has a base of anchored dual covers there will be no loss of generality in taking $\mathcal{C}$ to be anchored, whence $\mathcal{C} \prec \mathcal{C}^\Delta$. By definition we have $U \in \mathcal{U}_2$ with $u^*(U) \prec \mathcal{C}$, and so $f^{-1}(U) \in \mathcal{U}_1$ as f is quasi-uniformly continuous. By [16, Lemma 3.1 (11)] and the fact that f is a function, for $B \in \mathcal{P}(X_2)$ and the difunction (f, f') we have $f^{\leftarrow}B = f^{-1}[B] = (f')^{\leftarrow}B$. It is now straightforward to verify that

$$u^*(f^{-1}(U)) \prec (f, f')^{-1}(\mathcal{C}) \prec (f, f')^{-1}(\mathcal{C}^\Delta) = (f, f')^{-1}(\mathcal{C})^\Delta,$$

so $(f, f')^{-1}(\mathcal{C})^\Delta \in u^*(\mathcal{U}_1)$, which proves (f, f') is uniformly bicontinuous.

$(\Leftarrow)$ The proof is similar, and we omit the details. □

We denote by **dfDdiU** the full subcategory of **dfDiU** whose objects are di-uniformities on a discrete texture. Then:

THEOREM (3.5). *The functor* $\mathfrak{T}$: **QUnif** $\to$ **dfDdiU** *defined by*

$$\mathfrak{T}((X_1, \mathcal{U}_1) \xrightarrow{f} (X_2, \mathcal{U}_2)) = (X_1, \mathcal{P}(X_1), u^*(\mathcal{U}_1)) \xrightarrow{(f,f')} (X_2, \mathcal{P}(X_2), u^*(\mathcal{U}_2))$$

is a concrete isomorphism.

Proof. Clearly $\mathfrak{T}$ preserves identities and composition [16, Lemma 3.1], while by Lemma 3.4 if f is a morphism in **QUnif** then $\mathfrak{T}(f) = (f, f')$ is a morphism in **dfDdiU**. Hence, $\mathfrak{T}$ is a functor. The functor $\mathfrak{T}$ is clearly faithful, and it is full, again by Lemma 3.4. Finally, is bijective on objects by [16, Corollary 3.12]. Hence, $\mathfrak{T}$ is a concrete isomorphism. □

We know from [16] that a quasi uniformity is a uniformity if and only if the corresponding di-uniformity on $(X, \mathcal{P}(X), \pi_X)$ is complemented [16]. In general there will be many different complementations on $(X, \mathcal{P}(X))$, and complemented di-uniformities with respect to these complementations, so we must distinguish between the full subcategory **dfCDdiU** of **dfDiU** consisting of complemented di-uniformities on a discrete texture, and its full subcategory **df$\boldsymbol{\pi}$DdiU**, where the complementation is restricted to be π. Denoting by $\mathfrak{T}_\pi$: **Unif** $\to$ **df$\boldsymbol{\pi}$DdiU** the restriction of $\mathfrak{T}$ to the construct **Unif** of uniform structures and uniformly continuous functions we have at once:

COROLLARY (3.6). *The functor* $\mathfrak{T}_\pi$: **Unif** $\to$ **df$\boldsymbol{\pi}$DdiU** *is a concrete isomorphism.* □

Note that, since each difunction (f, f') is complemented, we may also write **cdfDdiU** in place of **dfCDdiU**.

Since a discrete texture is both plain and simple we may regard **dfDdiU** as a full subcategory of **dfPSDiU**. However, we will not be interested in simplicity in this paper, and so will consider **dfDdiU** as a full subcategory of the larger category **dfPDiU**. With the appropriate inclusion functors this gives the following commutative diagram:

$$\begin{tikzcd}
\mathbf{Unif} \arrow[r, "\mathfrak{T}_\pi"] \arrow[d] & \mathbf{df\pi DdiU} \arrow[r] & \mathbf{dfCDdiU} \arrow[r] \arrow[d] & \mathbf{dfCPDiU} \arrow[d] \\
\mathbf{QUnif} \arrow[rr, "\mathfrak{T}"] & & \mathbf{dfDdiU} \arrow[r] & \mathbf{dfPDiU}
\end{tikzcd}$$

The family of quasi-uniformities corresponding to di-uniform texture spaces in **dfCDdiU** is potentially larger than the family of uniformities, and its members can still be expected to exhibit some form of symmetry. An internal characterization is at the moment an open question, and the authors hope to study this question in a future paper.

We are interested in the nature of the embedding of **dfDdiU** in **dfPDiU**.

THEOREM (3.7). **dfDdiU** *is a concretely coreflective subcategory of* **dfPDiU**.

Proof. Let $(S, \mathcal{S}, \upsilon)$ be a dicovering uniform texture space with $(S, \mathcal{S})$ plain, and set

$$\upsilon_D = \{\mathcal{C} \mid \mathcal{C} \text{ a dicover of } (S, \mathcal{P}(S)) \text{ with } \mathcal{D} \prec \mathcal{C} \text{ for some } \mathcal{D} \in \upsilon\}.$$

Since join coincides with union in $(S, \mathcal{S})$, a dicover of $(S, \mathcal{S})$ is also a dicover of $(S, \mathcal{P}(S))$. Moreover, for $A \in \mathcal{S}$ and $s \in S$, $P_s \not\subseteq A \iff \{s\} \not\subseteq A$ and by the plainess of $(S, \mathcal{S})$, $A \not\subseteq Q_s \iff A \not\subseteq S \setminus \{s\}$. Since $\{s\}$ is the p-set and $S \setminus \{s\}$ the q-set of s in $(S, \mathcal{P}(S))$, we see that an anchored dicover of $(S, \mathcal{S})$ is also anchored in $(S, \mathcal{P}(S))$, while for any two dicovers of $(S, \mathcal{S})$ the least upper bound and the property of being a star refinement carry over to $(S, \mathcal{P}(S))$. We conclude that υ_D is a dicovering uniformity on $(S, \mathcal{P}(S))$.

Now consider the identity point function $\iota : (S, \mathcal{P}(S), \upsilon_D) \to (S, \mathcal{S}, \upsilon)$. Clearly ι is ω-preserving so by Lemma 1.6 we may consider the difunction $(f_\iota, F_\iota) : (S, \mathcal{P}(S), \upsilon_D) \to (S, \mathcal{S}, \upsilon)$. This difunction is uniformly bicontinuous since $\upsilon \subseteq \upsilon_D$, and hence a **dfPDiU**-morphism. We claim that $((S, \mathcal{P}(S), \upsilon_D), (f_\iota, F_\iota))$ is a coreflective arrow. To prove this let $(X, \mathcal{P}(X), \nu)$ be an object in **dfDdiU**, and $(f, F) : (X, \mathcal{P}(X), \nu) \to (S, \mathcal{S}, \upsilon)$ a **dfPDiU**-morphism. Bearing in mind the form of the morphisms in **dfDdiU** we must prove the existence of a unique function $\varphi : X \to S$ for which the diagram below is commutative.

$$\begin{tikzcd}
(S, \mathcal{P}(S), \upsilon_D) \arrow[r, "(f_\iota, F_\iota)"] & (S, \mathcal{S}, \upsilon) \\
(X, \mathcal{P}(X), \nu) \arrow[u, dashed, "(\varphi, \varphi')"] \arrow[ur, "(f, F)"'] &
\end{tikzcd}$$

Since $(X, \mathcal{P}(X))$ is plain we have a unique point function $\phi : X \to S$ for which $(f, F) = (f_\phi, F_\phi)$. Also the difunction (φ, φ') corresponds to φ in the same way, so $(f_\iota, F_\iota) \circ (\varphi, \varphi') = (f_\phi, F_\phi) \iff \iota \circ \varphi = \phi$, which means the only possible choice for φ is ϕ. Hence, it remains to show $(\phi, \phi') : (X, \mathcal{P}(X), \nu) \to (S, \mathcal{P}(S), \upsilon_D)$ is uniformly bicontinuous. Take $\mathcal{C} \in \upsilon_D$. Then we have $\mathcal{D} \in \upsilon$ with $\mathcal{D} \prec \mathcal{C}$, and $(f, F)^{-1}(\mathcal{D}) \in \nu$. But for $A \in \mathcal{P}(S)$ we have $\phi^{\leftarrow} A = \phi^{-1}[A]$, and for $B \in \mathcal{S}$ we have $f^{-1}B = \phi^{-1}[B]$, so $(f, F)^{-1}(\mathcal{D}) \prec (\phi, \phi')^{-1}(\mathcal{C})$ and we obtain $(\phi, \phi')^{-1}(\mathcal{C}) \in \nu$, as required.

Finally, regarding these categories as concrete categories over **dfTex** it is easy to see that **dfDdiU** is a concretely coreflective subcategory of **dfPDiU**. $\square$

The authors do not know if **dfCDdiU** is coreflective in **dfPCDiU**, but would conjecture that it is.

4. The construct fDiU

In the category **dfDiU** we have used uniformly bicontinuous difunctions as our morphisms. This parallels the choice of bicontinuous difunctions as the morphisms in **dfDitop** [8]. As we have noted earlier, however, it is possible to represent difunctions by ordinary point functions in certain situations and this brings to our attention ω-preserving functions satisfying the condition (b) of Proposition 1.7. The construct **fDitop**, where the objects are ditopological texture spaces and the morphisms such bicontinuous point functions, was therefore introduced in [8], and we wish to define a similar construct of di-uniform texture spaces. First, however, we will need to say what we mean by the inverse of a direlation under a point function.

Definition (4.1). Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be textures, (r, R) a direlation on $(T, \mathcal{T})$ and $\varphi : S \to T$ an ω-preserving point function. Then we set

(i) $$\varphi^{-1}(r) = \bigvee\{\overline{P}_{(s_1,s_2)} \mid \exists\, P_{s_1} \not\subseteq Q_{s_1'} \text{ so that } P_{t_1} \not\subseteq Q_{\varphi(s_1')}, P_{\varphi(s_2)} \not\subseteq Q_{t_2}, \implies \overline{P}_{(t_1,t_2)} \subseteq r\}$$

(ii) $$\varphi^{-1}(R) = \bigcap\{\overline{Q}_{(s_1,s_2)} \mid \exists\, P_{s_1'} \not\subseteq Q_{s_1} \text{ so that } P_{\varphi(s_1')} \not\subseteq Q_{t_1}, P_{t_2} \not\subseteq Q_{\varphi(s_2)}, \implies R \subseteq \overline{Q}_{(t_1,t_2)}\}$$

(iii) $\varphi^{-1}(r, R) = (\varphi^{-1}(r), \varphi^{-1}(R))$.

Since we assume φ is ω-preserving in Definition 4.1, we may associate a difunction (f_φ, F_φ) with φ by Lemma 1.6. We must therefore show that Definition 4.1 is consistent with [14, Definition 5.1].

PROPOSITION (4.2). *Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be textures, $\varphi : S \to T$ an ω-preserving point function and $(f_\varphi, F_\varphi) : (S, \mathcal{S}) \to (T, \mathcal{T})$ the corresponding difunction. Then for a direlation (r, R) on $(T, \mathcal{T})$ we have $\varphi^{-1}(r, R) = (f_\varphi, F_\varphi)^{-1}(r, R)$.*

Proof. Suppose that $\varphi^{-1}(r) \not\subseteq (f_\varphi, F_\varphi)^{-1}(r)$. Then from the definition of $\varphi^{-1}(r)$ $\exists\, s_1, s_2 \in S$ with $\overline{P}_{(s_1,s_2)} \not\subseteq (f_\varphi, F_\varphi)^{-1}(r)$ so that for some $s_1{}' \in S$ with $P_{s_1} \not\subseteq Q_{s_1'}$ we have

$$P_{w_1} \not\subseteq Q_{\varphi(s_1')}, P_{\varphi(s)} \not\subseteq Q_{w_2} \implies \overline{P}_{(w_1,w_2)} \subseteq r. \tag{4.3}$$

On the other hand, from the definition of $(f_\varphi, F_\varphi)^{-1}(r)$ $\exists\, t_1, t_2 \in S$ with $\overline{P}_{(s_1',t_1)} \not\subseteq F_\varphi$ and $f_\varphi \not\subseteq \overline{Q}_{(s_2,t_2)}$, $\overline{P}_{(t_1,t_2)} \not\subseteq r$. From the definition of F_φ $\exists\, t_1', n \in S$ with $P_{t_1} \not\subseteq Q_{t_1'}$, $P_n \not\subseteq Q_{s_1'}$ and $P_{t_1'} \not\subseteq Q_{\varphi(n)}$. Using that φ is ω-preserving easily leads to $P_{t_1} \not\subseteq Q_{\varphi(s_1')}$, and likewise the definition of f_φ and the ω-preserving property gives $P_{\varphi(s_2)} \not\subseteq Q_{t_2}$. Taking $w_1 = t_1$, $w_2 = t_2$ in (4.3) now gives the contradiction $\overline{P}_{(t_1,t_2)} \subseteq r$. This establishes $\varphi^{-1}(r) \subseteq (f_\varphi, F_\varphi)^{-1}(r)$ and the proof of the inverse inclusion is similar.

The proof of $\varphi^{-1}(R) = (f_\varphi, F_\varphi)^{-1}(R)$ is dual and is omitted. □

The above result justifies the following definition:

Definition (4.4). Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be texture spaces, $\mathcal{U}$ a direlational uniformity on $(S, \mathcal{S})$, $\mathcal{V}$ a direlational uniformity on $(T, \mathcal{T})$ and $\varphi : S \to T$ an ω-preserving point function. If $(r, R) \in \mathcal{V} \implies \varphi^{-1}(r, R) \in \mathcal{U}$ then φ is said to be $\mathcal{U}$-$\mathcal{V}$ (texturally) *uniformly bicontinuous*.

Specifically:

COROLLARY (4.5). *Suppose that $\varphi : (S_1, \mathcal{S}_1) \to (S_2, \mathcal{S}_2)$ is ω-preserving and that $\mathcal{U}_k$ is a direlational uniformity on $(S_k, \mathcal{S}_k)$, $k = 1, 2$. Then $\varphi : (S_1, \mathcal{S}_1, \mathcal{U}_1) \to (S_2, \mathcal{S}_2, \mathcal{U}_2)$ is uniformly bicontinuous if and only if $(f_\varphi, F_\varphi) : (S_1, \mathcal{S}_1, \mathcal{U}_1) \to (S_2, \mathcal{S}_2, \mathcal{U}_2)$ is uniformly bicontinuous.*

If $\varphi : S \to T$ is an ω-preserving point function satisfying the condition (b) of Proposition 1.7, then the corresponding difunction $(f_\varphi, F_\varphi) : (S, \mathcal{S}) \to (T, \mathcal{T})$ satisfies [14, Proposition 5.3] and [14, Lemma 5.11]. The identity function is $\mathcal{U}$-$\mathcal{U}$ (texturally) uniformly bicontinuous and this property is preserved by functional composition of ω-preserving functions satisfying (b) of Proposition 1.7.

Also we have $\varphi^{-1}(r,R) = (f_\varphi, F_\varphi)^{-1}(r,R)$, which means that all the properties satisfied by $(f_\varphi, F_\varphi)^{-1}(r,R)$ are also satisfied by the point function $\varphi^{-1}(r,R)$. Hence,

COROLLARY (4.6). *Di-uniformities and $\mathcal{U}$-$\mathcal{V}$ (texturally) uniformly bicontinuous ω-preserving point functions satisfying the condition (b) of Proposition 1.7 form a concrete category over* **fTex**.

The concrete category of dicovering uniform textures and (texturally) uniformly bicontinuous ω-preserving point functions satisfying (b) of Proposition 1.7 may also be defined and shown to be concretely isomorphic to the above category. Both may be denoted by **fDiU**. Likewise, restricting the objects to be complemented di-uniformities we obtain the full subcategory **fCDiU**.

PROPOSITION (4.7). *In the category* **fDiU***:*

1. *Every section is an* **fDiU***-embedding.*
2. *Every injective morphism is a monomorphism.*
3. *Every retraction is an* **fDiU***-quotient.*
4. *Every surjective morphism is an epimorphism.*
5. *A morphism is an isomorphism if and only if it is a textural isomorphism and its inverse is uniformly bicontinuous.*

Proof. The first four results hold because **fDiU** is a construct. For (5) it is known from [7, Proposition 3.15] that a morphism in **fTex** is an isomorphism if and only if it is a textural isomorphism in the sense of [4], whence (5) follows easily. □

We define $\mathfrak{D}$: **fDiU** → **dfDiU** by

$$\mathfrak{D}((S,\mathcal{S},\mathcal{U}) \xrightarrow{\varphi} (T,\mathcal{T},\mathcal{V})) = (S,\mathcal{S},\mathcal{U}) \xrightarrow{(f_\varphi,F_\varphi)} (T,\mathcal{T},\mathcal{V}).$$

THEOREM (4.8). $\mathfrak{D}$: **fDiU** → **dfDiU** *defined above is a functor. The restriction* $\mathfrak{D}_p$: **fPDiU** → **dfPDiU** *is an isomorphism with inverse* $\mathfrak{V}_p$: **dfPDiU** → **fPDiU** *given by*

$$\mathfrak{V}_p((S,\mathcal{S},\mathcal{U}) \xrightarrow{(f,F)} (T,\mathcal{T},\mathcal{V})) = (S,\mathcal{S},\mathcal{U}) \xrightarrow{\varphi_{(f,F)}} (T,\mathcal{T},\mathcal{V}).$$

Likewise we have isomorphisms between **fSDiU** *and* **dfSDiU***, and between the various corresponding subconstructs of these contructs.*

Proof. It is easy to show that $\mathfrak{D}(\iota_S) = (i_S, I_S)$. Now let $(S,\mathcal{S})$, $(T,\mathcal{T})$, $(U,\mathcal{U})$ be textures, $\varphi : S \to T$, $\psi : T \to U$ ω-preserving point functions satisfying (b) of Proposition 1.7. We have $(f_{\psi\circ\varphi}, F_{\psi\circ\varphi}) = (f_\psi, F_\psi) \circ (f_\varphi, F_\varphi)$ by [7, Theorem 3.10]. We can also say a that a point function is (textural) uniformly bicontinuous if and only if the corresponding difunction is uniformly bicontinuous. Thus $\mathfrak{D}$: **fDiU** → **dfDiU** is a functor. If we restrict to $\mathfrak{D}_p$: **fPUnif** → **dfPDiU** we again obtain a functor. Now let us define $\mathfrak{V}_p$: **dfPDiU** → **fPDiU** by $\mathfrak{V}_p(S,\mathcal{S},\mathcal{U}) = (S,\mathcal{S},\mathcal{U})$ and $\mathfrak{V}_p(f,F) = \varphi_{(f,F)}$ which is also a functor and the inverse of $\mathfrak{D}_p$. This means that $\mathfrak{D}_p$ is an isomorphism. The other isomorphisms can be proved similarly. □

As a corollary we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathbf{Unif} \arrow[r, "\mathfrak{V}_\pi \circ \mathfrak{T}_\pi"] \arrow[d] & \mathbf{f\pi DdiU} \arrow[r] & \mathbf{fCDdiU} \arrow[r] \arrow[d] & \mathbf{fCPDiU} \arrow[d] \\
\mathbf{QUnif} \arrow[rr, "\mathfrak{V}_d \circ \mathfrak{T}"] & & \mathbf{fDdiU} \arrow[r] & \mathbf{fPDiU}
\end{tikzcd}$$

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SEMICONTINUOUS ORDER-REPRESENTABILITY OF TOPOLOGICAL SPACES

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ABSTRACT. We study the semicontinuous order-representability property on a topological space and its relationship with the topological structure of the given space.

1. Introduction

A topology τ defined on a nonempty set X is said to have the semicontinuous order-representability property (SRP) if every lower semicontinuous total preorder $\precsim$ defined on X admits a numerical representation by means of a lower semicontinuous order-monomorphism $u\colon X \to \mathbb{R}$ such that $x \precsim y \iff u(x) \leq u(y)$ $(x, y \in X)$.

There are different and complementary motivations to study this concept. For instance, an important one arises in Optimization Theory in the search for minimal elements of orderings defined on a set. Also, the semicontinuous order-representability property (SRP) on topological spaces is a tool to analyze order-extension properties of topological spaces (see [4]), as well as to characterize various classical topological properties of a Banach space (see [3]), in Functional Analysis.

Having these motivations in mind, given a topological space it is important to know which total preorders defined on it admit a representation through a semicontinuous real-valued order-monomorphism. In the present paper, we pay attention to the relationship between countability properties of the topological space and the semicontinuous order-representability property.

2. Preliminaries

A *preorder* $\precsim$ on an arbitrary nonempty set X is a binary relation on X which is reflexive and transitive.

An antisymmetric preorder is said to be an *order*. A *total preorder* $\precsim$ on a set X is a preorder such that if $x, y \in X$ then $x \precsim y$ or $y \precsim x$.

If $\precsim$ is a preorder on X, its *asymmetric* relation $\prec$ is defined by $x \prec y \iff (x \precsim y) \wedge \neg(y \precsim x)$, and its *equivalence* relation $\sim$ is given by $x \sim y \iff (x \precsim y) \wedge (y \precsim x)$.

Let $(X, \precsim)$ be a totally preordered set and let $X/\sim$ be the set of equivalence classes. If $x \in X$ we denote the equivalence class of x by $[x]$. The preorder $\precsim$ on X induces a total order $\preceq$ on $X/\sim$ defined by $[x] \preceq [y] \iff x \precsim y$. Let $[x], [y]$ be two equivalence classes in $X/\sim$. Then we say that the ordered

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pair $([x],[y]) \in (X/\sim) \times (X/\sim)$ is a *jump* if there is no $[z] \in X/\sim$ such that $[x] < [z] < [y]$, where $<$ denotes the asymmetric part of $\preceq$.

A totally preordered set $(X, \precsim)$ is said to be *dense* if it has no jumps. A subset Z of X is said to be *order-dense in X with respect to* $\precsim$, if $x, y \in X$ and $x \prec y$ imply that there exists $z \in Z$ such that $x \precsim z \precsim y$. $(X, \precsim)$ is said to be *order-separable* if it has a countable order-dense subset.

If $(X, \precsim)$ is a preordered set then a real-valued function $u\colon X \to \mathbb{R}$ is said to be an *order-monomorphism* if for every $x, y \in X$, $x \precsim y \implies u(x) \leq u(y)$ and $x \prec y \implies u(x) < u(y)$. If such a function u does exist then $\precsim$ is said to be *representable*.

If a nonempty set X is endowed with a topology τ then a *total* preorder $\precsim$ on X is said to be *continuously* (*semicontinuously*) *representable* if there exists an order-monomorphism that is continuous (lower semicontinuous) with respect to the topology τ and the usual topology on the real line $\mathbb{R}$.

Let $(X, \precsim)$ be a totally preordered set. The family of all sets of the form $L(x) = \{a \in X : a \prec x\}$ and $G(x) = \{a \in X : x \prec a\}$, where $x \in X$ is a subbasis for a topology $\tau_{\precsim}$ on X, called the *order topology*, that is the intersection of the *lower topology* $\tau^l_{\precsim}$ and the *upper topology* $\tau^u_{\precsim}$, respectively generated by all the sets $G(x) = \{a \in X : x \prec a\}$ $(x \in X)$ and all the sets $L(x) = \{a \in X : a \prec x\}$ $(x \in X)$.

If $(X, \precsim)$ is a preordered set and τ is a topology on X, then the preorder $\precsim$ is said to be τ-continuous on X if for each $x \in X$ the sets $\{a \in X : x \precsim a\}$ and $\{b \in X : b \precsim x\}$ are τ-closed in X. In addition, the preorder $\precsim$ is said to be τ-lower semicontinuous on X if for each $x \in X$ the set $\{a \in X : a \precsim x\}$ is τ-closed in X.

Given a topological space (X, τ), the topology τ on X is said to have the *continuous order-representability property* (CRP) if every continuous total preorder $\precsim$ defined on X admits a representation by means of a continuous order-monomorphism. These topologies were studied in [8], [3], [4], [5]. Similarly, the topology τ on X is said to have the *semicontinuous order-representability property* (SRP) if every lower semicontinuous total preorder $\precsim$ defined on X admits a representation by means of a lower semicontinuous order-monomorphism. Topologies having SRP were studied in [1], [3], [4].

On a topological space (X, τ), the topology τ is said to be *lower preorderable* if it is the lower topology $\tau^l_{\precsim}$ of a *total preorder* $\precsim$ defined on X.

A family $\mathcal{N}$ of subsets of a topological space (X, τ) constitutes a *network* for the topology τ if for every open subset $O \in \tau$ and every $x \in O$, there exists an element $N \in \mathcal{N}$ such that $x \in N \subseteq O$.

3. Properties of order topologies

We present here some key results about the representability of totally preordered sets and some countability properties of (lower) order topologies. These results will be very useful when considering SRP on general topological spaces in section 4.

LEMMA (3.1). *Let X be a nonempty set endowed with a total preorder $\precsim$. Let $\tau_{\precsim}$ be the order topology on X. Then the following conditions are equivalent:*

i) *There exists an order-monomorphism from* $(X, \precsim)$ *into the real line* $\mathbb{R}$ *endowed with the usual order* $\leq$.

ii) *The total preorder* $\precsim$ *is continuously representable (considering the order topology* $\tau_{\precsim}$ *on* X *and the usual topology on* $\mathbb{R}$*) through an order-monomorphism.*

iii) *The order topology* $\tau_{\precsim}$ *is second countable.*

iv) *The totally preordered set* $(X, \precsim)$ *is order-separable.*

Proof. See Th. 1.6.11 and Th. 3.2.9 in [2]. □

LEMMA (3.2). *Let* X *be a nonempty set endowed with a total preorder* $\precsim$ *and a topology* τ *such that* $\tau^l_{\precsim} \subseteq \tau$. *If* τ *has a countable network* $\mathcal{N}$, *then* $\tau^l_{\precsim}$ *is second countable.*

Proof. Let $\mathcal{N} = \{N_i : i \in \mathbb{N}\}$ be a countable network for the topology τ. Suppose that there exist $a, b \in X$ such that $a \prec b$ and there is no $c \in X$ such that $a \prec c \prec b$ (in other words, $([a], [b])$ is a jump). Since $G(a)$ is τ-open, there exists at least one element $N_k \in \mathcal{N}$ such that $b \in N_k$ and $b \prec d$ for every $d \in N_k \setminus \{x \in X : x \sim b\}$. Since $\mathcal{N}$ is countable, we conclude that the family of possible jumps that $\precsim$ defines on X is, at most, countable. Let $\mathcal{J} = \{([a_i], [b_i])\}_{i\in\mathbb{N}}$ be the family of all the jumps. For any jump $([a_k], [b_k]) \in \mathcal{J}$, set $C_k = G(a_k)$ and $D_k = G(b_k)$. (Observe that D_k could eventually be empty if b_k is maximal in X with respect to $\precsim$). Given $N_k \in \mathcal{N}$, let $A_k = \bigcup_{x\in N_k} G(x)$. It is obvious that each A_k is a τ-open set for the lower topology $\tau^l_{\precsim}$. (By the way, A_k might eventually be empty if X has a maximal element $s \in X$ and there is a last jump $([r], [s]); r \prec s$ in X with respect to $\precsim$). Consider the family $\mathcal{G}$ of $\tau^l_{\precsim}$-open subsets given by: $\mathcal{G} = \{A_i : A_i \neq \emptyset,\ i \in \mathbb{N}\} \bigcup \{C_i : i \in \mathbb{N}\} \bigcup \{D_i : D_i \neq \emptyset,\ i \in \mathbb{N}\}$. Let $x \in X$ such that $G(x) \neq \emptyset$ (so that $G(x)$ is a basic open set of $\tau^l_{\precsim}$). Let $y \in G(x)$. If $([x], [y])$ is a jump in X then $G(x) \in \mathcal{G}$ by construction. If otherwise there exists $z \in X$ such that $x \prec z \prec y$, then since $G(x)$ is also τ-open and $\mathcal{N}$ is a network of $\tau_{\precsim}$, there exists an element $N_k \in \mathcal{N}$ such that $z \in N_k$ and consequently $y \in A_k \subset G(x)$ and $A_k \in \mathcal{G}$. Thus we conclude that $\mathcal{G}$ is a countable basis for the lower topology $\tau^l_{\precsim}$. Therefore the lower topology $\tau^l_{\precsim}$ is second countable. □

THEOREM (3.3). *Let* X *be a nonempty set endowed with a total preorder* $\precsim$. *Then the following conditions are equivalent:*

i) *The order topology* $\tau_{\precsim}$ *is second countable.*

ii) *The order topology* $\tau_{\precsim}$ *has a countable network.*

iii) *The lower topology* $\tau^l_{\precsim}$ *is second countable.*

Proof. The fact i) $\implies$ ii) is immediate, and the implication ii) $\implies$ iii) follows from Lemma (3.2). We only prove that iii) $\implies$ i).

Suppose that the lower topology $\tau^l_{\precsim}$ is second countable. Let $\mathcal{G} = \{G_n : n \in \mathbb{N}\}$ be a countable base of the lower topology $\tau^l_{\precsim}$. Observe that each element $G_n \in \mathcal{G}$ satisfies that $z \in G_n$ for every $y \in G_n$ and $z \in X$ with $x \prec z$. Consequently, if $n, k \in \mathbb{N}$ and $n \neq k$ it follows that $G_n \subsetneq G_k$ or $G_k \subsetneq G_n$. Thus, for every $n, k \in \mathbb{N}$ with $n \neq k$ we can select an element $x_{nk} \in (G_n \setminus G_k) \cup (G_k \setminus G_n)$. It is plain that the set $\mathcal{X} = \{x_{nk} : n, k \in \mathbb{N}, n \neq k\}$ is countable. Suppose that there exist $a, b \in X$ such that $a \prec b$ and $([a], [b])$ is a jump in X. Since $G(a)$ is $\tau^l_{\precsim}$-open, there exists at least one element $G_k \in \mathcal{B}$ such that $b \in G_k$ and $b \prec d$ for every $d \in G_k \setminus \{x \in X : x \sim b\}$ (i.e.: b is a minimal element of G_k as regards $\precsim$). Since $\mathcal{G}$ is a countable basis for the lower topology $\tau^l_{\precsim}$, the family of possible jumps that $\precsim$ defines on X is at most countable. Let $\mathcal{J} = \{([a_i], [b_i])\}_{i \in \mathbb{N}}$ be the family of all those possible jumps. For a given jump $([a_k], [b_k]) \in X$ we select two elements u_k and v_k such that $u_k \in [a_k]$ and $v_k \in [b_k]$. Let $\mathcal{Y} = \{u_i, v_i : ([a_i], [b_i])$ is a jump in X with respect to $\precsim$ $(i \in \mathbb{N})\}$. Let $\mathcal{Z} = \mathcal{X} \cup \mathcal{Y}$. Observe that $\mathcal{Z}$ is countable, by definition. Let $a, b \in X$ such that $a \prec b$ and $([a], [b])$ is not a jump. Take an element $c \in X$ such that $a \prec c \prec b$. If either $([a], [c])$ or $([c], [b])$ is a jump, then there exists $c' \in [c]$ such that $c' \in \mathcal{Y}$. If there exist $p, q \in X$ such that $a \prec p \prec c \prec q \prec b$, then since $G(a)$ and $G(p)$ are both $\tau^l_{\precsim}$-open and $\mathcal{G}$ is a basis for $\tau^l_{\precsim}$, set $G(a) = \bigcup_{n \in \mathbb{N}} G_{r_n}$ and $G(p) = \bigcup_{n \in \mathbb{N}} G_{s_n}$. Since $p \in G(a) \setminus G(p)$, there exists $k \in \mathbb{N}$ such that $p \in G_{r_k}$. Hence $G(p) \subseteq G_{r_k}$. Because $c \in G(p) \setminus G(c)$, there exists $n \in \mathbb{N}$ such that $c \in G_{s_n}$. Hence $G(c) \subseteq G_{s_n}$. By construction, the element $x_{r_k, s_n} \in \mathcal{X}$ lies also in $G(r_k) \setminus G(s_n) \subset G(a) \setminus G(c)$. Thus $a \prec x_{r_k, s_n} \precsim c \prec b$. Therefore, for every $a, b \in X$ such that $a \prec b$ we may always find an element $z \in \mathcal{Z}$ such that $a \precsim z \precsim b$. This implies that the totally ordered set $(X, \precsim)$ is order-separable and consequently the order topology $\tau_{\precsim}$ is second countable by Lemma (3.1). $\square$

4. The semicontinuous order-representability property (SRP) on topological spaces

We search for topological conditions on a topological space (X, τ) in order for τ to have the semicontinuous order-representability property (SRP). Our first result is a characterization theorem.

THEOREM (4.1). *Let (X, τ) be a topological space. The topology τ has SRP if and only if all its lower preorderable subtopologies are second countable.*

Proof. If τ' is a lower preorderable subtopology of τ, the total preorder $\precsim$ that defines $\tau' = \tau_{\precsim}$ is obviously τ-lower semicontinuous. Conversely, if $\precsim$ is a τ-lower semicontinuous preorder on X, then its order topology $\tau_{\precsim}$ is a lower preorderable subtopology of τ. Thus we see that there is a bijection between τ-lower semicontinuous preorders defined on X and lower preorderable subtopologies of τ.

Suppose that the topology τ has SRP. In particular, every semicontinuous total preorder $\precsim$ is representable through an order-monomorphism, so that the topology $\tau_{\precsim}$ is second countable by Lemma (3.1). Hence the topology $\tau^l_{\precsim}$ is also second countable by Theorem (3.3). Due to the aforementioned bijection, this implies that all the lower preorderable subtopologies of τ are second countable.

Conversely, let us assume that all the lower preorderable subtopologies of τ are second countable. Suppose that $\precsim$ is a lower semicontinuous preorder defined on X. This means that $\tau^l_{\precsim} \subseteq \tau$. By hypothesis, $\tau^l_{\precsim}$ is second countable. By Theorem (3.3) this implies that $\tau_{\precsim}$ is also second countable. By Lemma (3.1) $\precsim$ is representable by an order-monomorphism $f : X \to \mathbb{R}$ that is continuous if we consider on X the *order* topology $\tau_{\precsim}$ and on $\mathbb{R}$ the usual topology. Given

$a \in \mathbb{R}$ we have that $f^{-1}(a, +\infty) = \bigcup_{\{x \in X : f(x) \geq a\}} G(x)$ is an open set in $\tau^l_{\precsim}$, hence in τ. Therefore f is lower semicontinuous (now considering the topology τ on X and the usual topology on $\mathbb{R}$). □

COROLLARY (4.2). *The semicontinuous order-representability property (SRP) implies the continuous order-representability property (CRP). The converse is not true in general.*

Proof. Let (X, τ) be a topological space such that τ has SRP. Let $\precsim$ be a τ-continuous total preorder on X. Since the lower topology $\tau^l_{\precsim} \subseteq \tau$ is obviously lower preorderable, it is second countable by Theorem (4.1). Thus, by Theorem (3.3), the topology $\tau_{\precsim}$ is also second countable, which implies that $\precsim$ is continuously representable (with respect to the topology τ on X and the Euclidean topology on $\mathbb{R}$) by Lemma (3.1). Therefore, τ has CRP. To see that the converse is not true, consider the first uncountable ordinal ω_1. Endowed with the lower topology ω_1 trivially has CRP but is not second countable, so that by Theorem (4.1) it does not have SRP. □

THEOREM (4.3). *SRP is invariant under continuous surjections.*

Proof. Let (X, τ_X) be a topological space such that τ_X has SRP. Let (Y, τ_Y) be a topological space such that there exists a continuous surjection $\pi : X \to Y$. We observe that a total preorder $\precsim_Y$ on Y induces a total preorder $\precsim_X$ on X by declaring that $x_1 \precsim_X x_2 \iff \pi(x_1) \precsim_Y \pi(x_2)$. It is clear that, if the preorder $\precsim_Y$ is τ_Y- lower semicontinuous, then the corresponding preorder $\precsim_X$ defined on X is τ_X-lower semicontinuous, as a consequence of the continuity of π: Given an element $x \in X$, $\{z \in X : z \prec_X x\} = \{z \in X : \pi(z) \prec_Y \pi(x)\} = \pi^{-1}(\{s \in Y : s \prec_Y \pi(x)\})$. Similarly $\{t \in X : x \prec_X t\} = \{t \in X : \pi(x) \prec_Y \pi(t)\} = \pi^{-1}(\{r \in Y : \pi(x) \prec_Y r\})$. In addition, since π is a surjection we have that given an element $y \in Y$ there exists at least one element $x_y \in X$ such that $\pi(x_y) = y$. Moreover $\pi(\{z \in X : z \prec_X x_y\}) = \{s \in Y : s \prec_Y y\}$ and $\pi(\{t \in X : x_y \prec_X t\}) = \{r \in Y : y \prec_Y r\}$. Therefore, if we consider on X the order topology induced by $\precsim_X$ and on Y the order topology induced by $\precsim_Y$, then with respect to these topologies the map π is not only a continuous surjection, but also an *open* map. Since τ_X has SRP and the preorder $\precsim_X$ on X is τ_X-lower semicontinuous, the lower topology $\tau^l_{\precsim_X}$ is second countable by Theorem (4.1). Thus the order topology $\tau_{\precsim_X}$ is second countable by Theorem (3.3). Hence the order topology $\tau_{\precsim_Y}$ that $\precsim_Y$ induces on Y is also second countable, because second countability is indeed invariant under continuous *open* surjections (see [7], Th. 6.2 on p. 174). Again by Theorem (4.1), we conclude that τ_Y has SRP. □

COROLLARY (4.4). *SRP is invariant under topological quotients.*

THEOREM (4.5). *Let (X, τ) be a topological space such that the topology τ has a countable network. Then τ has SRP.*

Proof. This is a direct consequence of Lemma (3.2) and Theorem (4.1). □

Definition. A T_3 topological space (X, τ) is said to be *cosmic* (see [11]) if it is the continuous image of a separable metric space.

COROLLARY (4.6). i) *Continuous images of second countable spaces have SRP.*

ii) *Cosmic spaces have SRP.*

iii) *Second countable spaces have SRP. The converse is not true in general.*

Proof. All these facts follow from Theorem (4.3) and Theorem (4.5). Observe also that any countable space has SRP, but not all such spaces are first countable. □

For the particular case of totally preordered sets we have the following result which is a consequence of the previous ones already proved.

COROLLARY (4.7). *Let $(X, \precsim)$ be a totally preordered set. Then the following conditions are equivalent:*

i) $\tau_{\precsim}$ *has SRP.*

ii) $\tau_{\precsim}$ *has CRP.*

iii) $\tau_{\precsim}$ *is second countable.*

iv) $\tau^l_{\precsim}$ *is second countable.*

v) $\tau^l_{\precsim}$ *has SRP.*

Proof. i) $\implies$ ii) follows from Corollary (4.2). If $\tau_{\precsim}$ has CRP $\precsim$ is in particular continuously representable, so that $\tau_{\precsim}$ is second countable by Lemma (3.1). This proves ii) $\implies$ iii). The implications iii) $\implies$ iv) $\implies$ v) follow from Theorem (3.3) and Corollary (4.6) iii. Finally, if $\tau^l_{\precsim}$ has SRP it is plain that $\precsim$ is (semicontinuously) representable. Indeed, by Lemma (3.1) $\precsim$ is continuously representable and $\tau_{\precsim}$ is second countable. Therefore $\tau_{\precsim}$ has SRP by Corollary (4.6) iii. □

Now we show a *necessary* condition for SRP.

THEOREM (4.8). *Let (X, τ) be a topological space. If τ has SRP, then it is hereditarily separable and hereditarily Lindelöf. The converse is not true in general.*

Proof. See Lemma 4.1 and Proposition 4.2 in [1], and Lemma 2.3 in [3]. For the converse, observe that the Alexandroff's double arrow $\mathbb{R} \times \{0, 1\}$ endowed with the order topology associated to the lexicographic ordering is hereditarily separable and hereditarily Lindelöf (see [10], Th. 2.2 and Th. 3.3) but not second countable; hence by Theorem (4.1), it does not have SRP. □

The semicontinuous order-representability property SRP of a given topology has been characterized through the second countability of all its lower preorderable subtopologies. To conclude this section, we characterize these subtopologies.

THEOREM (4.9). *Let (X, τ) be a topological space. The topology τ is lower preorderable if and only if it has a basis $\mathcal{B} = \{X\} \cup \{O_\alpha \subseteq X : \alpha \in A\}$, satisfying the following two conditions:*

a) *For every $\alpha, \beta \in A$, $O_\alpha \subseteq O_\beta$ or $O_\beta \subseteq O_\alpha$. (In other words, $\mathcal{B}$ consists of a nested family of τ-open subsets).*

b) *For every* $\alpha \in A$, $\bigcap_{\gamma\in A, O_\alpha \subsetneq O\gamma} (O\gamma \setminus O_\alpha) \neq \emptyset$.

(A denotes a nonempty set of indexes).

Proof. Assume first the existence of a basis of τ satisfying conditions a) and b) of the statement. Let us define a binary relation $\precsim$ on X by declaring $x \precsim y \iff (x \in O_\alpha \implies y \in O_\alpha)$, for every $\alpha \in A$ $(x, y \in X)$. It is straightforward to see that $\precsim$ is a total preorder. For any $x \in X$, we may observe that $G(x) = \bigcup_{\alpha \in A, x \notin O_\alpha} O_\alpha$. Hence $G(x) \in \tau$ $(x \in X)$. Thus the preorder $\precsim$ is τ-lower semicontinuous, or, in other words, the lower topology $\tau^l_{\precsim}$ satisfies that $\tau^l_{\precsim} \subseteq \tau$. Also, for every $\alpha \in A$, by condition b) there exists an element $x \in \bigcap_{\gamma\in A, O_\alpha \subsetneq O\gamma} (O\gamma \setminus O_\alpha)$, and the upper contour set $G(x)$ is indeed O_α. Hence $\tau \subseteq \tau^l_{\precsim}$. Therefore $\tau^l_{\precsim} = \tau$, so that the topology τ is lower preorderable.

For the converse, suppose that τ is lower preorderable. Let $\precsim$ be a total preorder on X such that τ coincides with $\tau^l_{\precsim}$. For every $x \in X$ we observe that the upper contour set $G(x)$ is τ-open by hypothesis. Moreover, $x \in \bigcup_{z\in X, G(x) \subsetneq G(z)} (G(z) \setminus G(x))$. In particular, $\bigcup_{z\in X, G(x)\subsetneq G(z)} (G(z) \setminus G(x)) \neq \emptyset$. Henceforth, the family $\{X\} \cup \{G(x) : x \in X\}$ satisfies the conditions a) and b) of the statement, and it is obviously a basis of the topology $\tau^l_{\precsim} = \tau$. □

5. The semicontinuous order-representability property (SRP) on Banach spaces

On Banach spaces, SRP has particular characteristics.

THEOREM (5.1). *Let* $(X, ||\cdot||)$ *be a Banach space endowed with the norm topology. The following statements are equivalent:*

i) $(X, ||\cdot||)$ *has SRP,*

ii) $(X, ||\cdot||)$ *is hereditarily separable and hereditarily Lindelöf,*

iii) $(X, ||\cdot||)$ *is separable,*

iv) (X, ω) *is separable (where* ω *stands for the weak topology on* X*),*

v) (X, ω) *has SRP,*

vi) (X, ω) *is hereditarily separable and hereditarily Lindelöf,*

vii) (X, ω) *is hereditarily separable.*

Proof. i) $\implies$ ii) $\implies$ iii) $\implies$ i).

This follows directly from Theorem (4.8) and Corollary (4.6) because separability and second countability are equivalent properties on metric spaces.

iii) $\iff$ iv).

This is a well known fact in the theory of Banach spaces (see [6] or p. 196 in [9]).

i) $\implies$ v).

This is a direct consequence of Theorem (4.3), since the weak topology is a subtopology of the norm topology.

v) $\implies$ vi) $\implies$ vii) $\implies$ iv).

This follows from Theorem (4.8). □

We now characterize when the weak-star topology ω^* of the dual Banach space X^* has the SRP. To do so, we need the following lemma

LEMMA (5.2). *Let X be a non-separable Banach space. Then (X^*, ω^*) is not hereditarily Lindelöf.*

Proof. See Lemma 4.1 in [3]. □

THEOREM (5.3). *Let X be a Banach space. The following statements are equivalent:*

i) $(X, ||\cdot||)$ *is separable,*

ii) (X^*, ω^*) *has SRP,*

iii) (X^*, ω^*) *is hereditarily separable and hereditarily Lindelöf,*

iv) (X^*, ω^*) *is hereditarily Lindelöf.*

Proof. i) $\Longrightarrow$ ii).

Since $(X, ||\cdot||)$ is separable, the closed unit ball U of $(X^*, ||\cdot||)$ is compact and *metrizable*, with respect to the weak-star topology ω^*. This also happens for $nU \ (n \in \mathbb{N})$.

If $\precsim$ is a total preorder on X^*, and it is lower semicontinuous with respect to ω^*, the restriction of $\precsim$ to $nU \ (n \in \mathbb{N})$ is also lower semicontinuous with respect to the restriction of ω^* to nU, which is a second countable topology. By Corollary (4.6), the restriction of $\precsim$ to nU is, in particular, representable by means of a real-valued order-monomorphism, hence, it is *order-separable* by Lemma (3.1). Since $X^* = \bigcup_{n\in\mathbb{N}} (nU)$, it is straightforward to prove that $\precsim$ is actually order-separable on the whole X^*. Thus, there exists a real-valued order-monomorphism $f : X^* \longrightarrow \mathbb{R}$ that represents $\precsim$ and is continuous if we consider on X^* the order topology $\tau_{\precsim}$ and on $\mathbb{R}$ the usual topology. In particular, since $\precsim$ is lower semicontinuous with respect to ω^*, the map f is lower semicontinuous with respect to the weak-star topology on X^* and the usual Euclidean topology on the real line $\mathbb{R}$. Thus we conclude that (X^*, ω^*) also has SRP.

ii) $\Longrightarrow$ iii) $\Longrightarrow$ iv) $\Longrightarrow$ i).

This has been proved in Theorem (4.8) and Lemma (5.2). □

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ON DECOMPOSING SUSPENSIONS OF SIMPLICIAL SPACES

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Abstract. Let $X_\bullet$ denote a simplicial space. The purpose of this note is to record a decomposition of the suspension of the individual spaces X_n occurring in $X_\bullet$ in case the spaces X_n satisfy certain mild topological hypotheses and where these decompositions are natural for morphisms of simplicial spaces. In addition, the summands of X_n which occur after one suspension are stably equivalent to choices of filtration quotients of the geometric realization $|X_\bullet|$. The purpose of recording these decompositions is that they imply decompositions of the single suspension of certain spaces of representations [1, 2] as well as other varieties and are similar to decompositions of suspensions of moment-angle complexes [4] which appear in a different context.

1. Introduction and Statement of Results

Let $X_\bullet$ denote a simplicial space. The purpose of this note is to give a decomposition of the suspension of the individual spaces X_n occurring in $X_\bullet$ in case the spaces X_n satisfy certain mild topological hypotheses. These decompositions are natural for morphisms of simplicial spaces. In addition, the summands of X_n which occur after one suspension are stably equivalent to choices of filtration quotients of the geometric realization $|X_\bullet|$.

These structures occur in several contexts in useful ways and the following spaces admit decompositions of the type discussed above.

1. The suspension of the the loop space for a (path-connected) suspension of a CW-complex Y is homotopy equivalent to a bouquet of the suspension of smash products of Y [7, 11].
2. Spaces of ordered commuting n-tuples in a Lie group G, $\mathrm{Hom}(\oplus_n \mathbb{Z}, G)$, assemble to give a simplicial space denoted $\mathrm{Hom}(\mathbb{Z}_\bullet, G)$. If G is a closed subgroup of $GL_r(\mathbb{C})$, there are natural homotopy equivalences

$$\Sigma\,\mathrm{Hom}(\oplus_n \mathbb{Z}, G) \to \bigvee_{1\leq k\leq n} \Sigma \bigvee^{\binom{n}{k}} \mathrm{Hom}(\oplus_k \mathbb{Z}, G)/S_k(G)$$

   where $S_k(G)$ denotes the *singular subspace* defined as those commuting k-tuples where at least one entry is equal to 1 (see [1]). The associated spaces of representations

$$Rep(\oplus_n \mathbb{Z}, G) = \mathrm{Hom}(\oplus_n \mathbb{Z}, G)/G^{ad}$$

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where G acts by conjugation also assemble into a simplicial space, with similar decompositions (see [3]). For G a finite group, the simplicial spaces $\mathrm{Hom}(\mathbb{Z}_\bullet, G)$ and $Rep(\mathbb{Z}_\bullet, G)$ have natural connections with the cohomology of finite groups [2].

3. The suspension of moment-angle complexes as well as their generalizations are homotopy equivalent to a bouquet of *smash product moment-angle complexes* (see [4]).
4. A compact real algebraic variety as given in §5 is homeomorphic to the geometric realization of a simplicial space $V_\bullet$ for which each space V_n decomposes after a single suspension.

It is the purpose of this note to show that many of these decompositions carry over into the context of simplicial spaces which satisfy a mild cofibration condition and to put these in a coherent picture. Recall that a simplicial space $X_\bullet$ is a set of topological spaces X_n, $n \geq 0$, together with continuous maps $d_i : X_n \to X_{n-1}$ and $s_j : X_n \to X_{n+1}$ which satisfy the simplicial identities. A natural filtration of each space X_n is defined next.

Definition (1.1)**.** Define subspaces $S^t(X_n) = \cup s_{i_1}s_{i_2}\cdots s_{i_t}(X_{n-t}) \subset X_n$ with $S^0(X_n) = X_n$ and $S^1(X_n) = S(X)$ (for notational convenience). This defines a natural *decreasing filtration* for the spaces X_n in a simplicial space $X_\bullet$, where

$$s_0^n(X_0) = S^n(X_n) \subset \cdots \subset S^t(X_n) \subset \cdots \subset S(X_n) \subset S^0(X_n) = X_n$$

and $S^{n+1}(X_n)$ is empty by convention.

The following concepts appear in [9], Definition 11.2.

Definition (1.2)**.** A pair of spaces (X, A) is said to be a *strong NDR pair* provided that there are maps $u : X \to [0, 1]$ and a homotopy $h : X \times [0, 1] \to X$ such that (X, A) is an NDR pair, namely

1. $A = u^{-1}(0)$,
2. $h(0, x) = x$ for all $x \in X$,
3. $h(a, t) \in A$ for all $(a, t) \in A \times [0, 1]$,
4. if $u(x) < 1$ then $h(x, 1) \in A$.

and if $u(x) < 1$ then $u(h(x, t)) < 1$.

Definition (1.3)**.** A simplicial space $X_\bullet$ is said to be *proper* if each pair $(X_n, S(X_n))$ is a strong NDR-pair for all n.

Definition (1.4)**.** A simplicial space $X_\bullet$ is said to be *simplicially NDR* if each

$$(S^{t-1}(X_n), S^t(X_n))$$

is an NDR pair for all $t - 1 \geq 0$ and all n.

Note that every degenerate element x in X_n has a unique decomposition as

$$x = s_{j_r}s_{j_{r-1}}\cdots s_{j_1}(y)$$

where y is in X_{n-r} with y non-degenerate and $j_r > j_{r-1} > \cdots > j_1$. Given any sequence $I = (i_r, i_{r-1}\cdots, i_1)$, write $s_I(X_{n-r}) = s_{i_r}s_{i_{r-1}}\cdots s_{i_1}(X_{n-r})$ with $|I| = r$.

Definition (1.5)**.** The sequence $I = (i_r, i_{r-1}\cdots, i_1)$ is said to be *admissible* provided $i_r > i_{r-1} > \cdots > i_1$. In case I is admissible, define $s_I\widehat{(X_{n-r})} = s_I(X_{n-r})/s_I S(X_{n-r})$.

The point-set topological properties of $X_\bullet$ are basic in these results. One instance is illustrated by the natural inclusion $\iota : S(X_n) \to X_n$ with mapping cone denoted $K(\iota)$. The proof of the main Theorem 1.6 implies the suspension of $K(\iota)$ is a retract of the suspension $\Sigma(X_n)$. On the other-hand, the quotient space $X_n/S(X_n)$ sometimes has independent useful features such as the case in [1] where these spaces are sometimes identified as natural Spanier-Whitehead duals of certain choices of Lie groups. To ensure that the properties of $X_n/S(X_n)$ are reflected in the structure of Theorem 1.6, it is useful to know that the inclusion $S(X_n) \to X_n$ is a cofibration.

The precise point-set topology for $|X_\bullet|$ admits several natural choices. Milnor originally topologized $|X_\bullet|$ by the natural quotient topology [12]. Milgram topologized BG, the geometric realization of a simplicial space similarly [10]. Subsequently, Steenrod topologized BG by the natural compactly generated topology [17]. Finally, May topologized $|X_\bullet|$ with the natural compactly generated, weak Hausdorff topology which is both elegant and convenient. This topology is used throughout the current article.

THEOREM (1.6). *Assume that the simplicial space $X_\bullet$ is simplicially NDR. Then the spaces X_n in the simplicial space $X_\bullet$ are naturally filtered where*

$$s_0^n(X_0) = S^n(X_n) \subset \cdots S^r(X_n) \subset \cdots \subset S(X_n) \subset S^0(X_n) = X_n.$$

Furthermore, these filtrations are split up to homotopy after suspending once. Thus there are homotopy equivalences which are natural for morphisms of simplicial spaces

1. $$\Theta(n) : \Sigma(X_n) \longrightarrow \bigvee_{0 \le r \le n} \Sigma(S^r(X_n)/S^{r+1}(X_n)),$$

2. $$H(n) : \Sigma(X_n) \longrightarrow \bigvee_{0 \le r \le n} \bigvee_{J} \Sigma(s_J\widehat{(X_{n-r})})$$

   where

   $$J = (j_r, j_{r-1} \cdots, j_1)$$

   is admissible with $|J| = r$ and $0 \le r \le n$ and
3. *the map $H(n)$ restricts to a homotopy equivalence*

   $$H(n)\big|_t : \Sigma(S^t(X_n)) \longrightarrow \bigvee_{t \le r \le n} \bigvee_{J} \Sigma(s_J\widehat{(X_{n-r})}).$$

Remarks:
(1) The splitting maps above in Theorem 1.6 are induced by the natural transformation from the identity to the decomposition maps $\Theta(n)$ regarded as functors from simplicial spaces to spaces.
(2) The finer decompositions obtained using the maps $H(n)$ arise from spaces $s_{J_t}\widehat{(X_{n-t})}$ with fixed $t = |J_t|$. In case t is fixed, the spaces $s_{J_t}\widehat{(X_{n-t})}$ are homeomorphic, but not equal.
(3) The notation in the proof and statement of Theorem 1.6 simplifies considerably if for fixed t, these differences of the $s_{J_t}\widehat{(X_{n-t})}$ are not addressed. Since the proof that moment-angle complexes admit stable decompositions in [4] uses an

analogous proof which keeps track of these differences, the more technically complicated statement as well as proof are retained here.

The following was proved by J. P. May as Lemma 11.3 [9].

PROPOSITION (1.7). *Assume that the simplicial space $X_\bullet$ is proper. Then the geometric realization $|X_\bullet|$ is naturally filtered by $F_j|X_\bullet|$ with induced homeomorphisms*

$$\Sigma^j(X_j/S(X_j)) \to F_j|X_\bullet|/F_{j-1}|X_\bullet|.$$

The next corollary follows from Theorem 1.6 and Proposition 1.7. Notice that Corollary 1.8 implies that the stable summands in Theorem 1.6 are all given in terms of the filtration quotients $F_j|X_\bullet|/F_{j-1}|X_\bullet|$. In addition, the natural d^1-differential in homology arising from the natural spectral sequence first investigated by G. Segal [15] then admits a geometric interpretation in terms of this decomposition, a point not developed here.

COROLLARY (1.8). *Assume that the simplicial space $X_\bullet$ is proper and simplicially NDR. Then there are natural homotopy equivalences*

$$K(n,t) : \Sigma^{n+1}(S^t(X_n)/S^{t+1}(X_n)) \longrightarrow \bigvee_{J_t} \Sigma^{t+1}(F_{n-t}|X_\bullet|/F_{n-t-1}|X_\bullet|)$$

where J_t runs over all admissible sequences with $t = |J_t|$ and t is a fixed integer such that $0 \le t \le n$. Thus by Theorem 1.6, there are natural homotopy equivalences

$$\Theta(n) : \Sigma^{n+1}(X_n) \longrightarrow \bigvee_{0\le t\le n} \bigvee_{J_t} \Sigma^{t+1}(F_{n-t}|X_\bullet|/F_{n-t-1}|X_\bullet|)$$

where J_t runs over all admissible sequences with $t = |J_t|$.

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2. Simplicial spaces

The purpose of this section is to recall standard properties of simplicial spaces to be used in the proof of the main theorem. Throughout this article $X_\bullet$ is assumed to be a simplicial space which is proper. Standard properties are stated in the next lemma.

LEMMA (2.1). *If $X_\bullet$ is a simplicial space which is simplicially NDR then it satisfies the following properties.*

1. *The pair*

$$(s_{i_r}s_{i_{r-1}}\cdots s_{i_1}S^{t-1}(X_n), s_{i_r}s_{i_{r-1}}\cdots s_{i_1}S^t(X_n))$$

is an NDR pair for all sequences $(i_r, i_{r-1}\cdots, i_1)$ and all n and t with $r \le n-t$. Thus the maps

$$s_{j_r}s_{j_{r-1}}\cdots s_{j_1}S^t(X_n) \to s_{j_r}s_{j_{r-1}}\cdots s_{j_1}S^{t-1}(X_n)$$

are cofibrations for all sequences $(i_r, i_{r-1}\cdots, i_1)$ and all n and t with $r \le n-t$.

2. *If* $0 \leq r \leq n-1$, *there are homeomorphisms which are natural for morphisms of simplicial spaces*

$$\gamma(n, r) : \vee_J \widehat{s_J(X_{n-r})} \longrightarrow S^r(X_n)/S^{r+1}(X_n)$$

where
(a) $J = (j_r, j_{r-1} \cdots, j_1)$ *is admissible with* $|J| = r$,
(b) $\widehat{s_J(X_{n-r})} = s_J(X_{n-r})/s_J S(X_{n-r})$ *and*
(c) $S^n(X_n)$ *is equal to* $s_0^n(X_0)$.

Proof. The pair $(S^{t-1}(X_n), S^t(X_n))$ is an NDR pair for all n and t with $t \leq n$ by hypothesis. Let $I = (i_r, i_{r-1} \cdots, i_1)$. Since the map $s_I = s_{i_r} s_{i_{r-1}} \cdots, s_{i_1}$ is a homeomorphism onto its image with one choice of inverse $d_{i_1} \cdots d_{i_{r-1}} d_{i_r}$, the pair

$$(s_{i_r} s_{i_{r-1}} \cdots s_{i_1} S^{t-1}(X_n), s_{i_r} s_{i_{r-1}} \cdots s_{i_1} S^t(X_n))$$

is an NDR pair for all sequences. The first part of Lemma 2.1 follows.

To prove the second part of Lemma 2.1, observe that

$$S^r(X_n) = \cup_I s_{i_r} s_{i_{r-1}} \cdots s_{i_1}(X_{n-r})$$

for $I = (i_r, i_{r-1} \cdots, i_1)$ admissible with $|I| = r$. Now consider $J = (j_r, j_{r-1} \cdots, j_1)$ admissible for $I \neq J$. Since $I \neq J$, let t denote the largest integer for which the entries i_t and j_t are not equal; without loss of generality we can assume that $i_t > j_t$. Applying $d_{i_t} d_{i_{t+1}} \cdots d_{i_{r-1}} d_{i_r}$ and using the simplicial identities gives that $s_I(X_{n-r}) \cap s_J(X_{n-r}) \subset S^{r+1}(X_n)$.

Now if $J = (j_r, j_{r-1} \cdots, j_1)$ is admissible with $|J| = r$, then the inclusions give rise to a relative homeomorphism

$$F(n, r) : (\sqcup_I s_J(X_{n-r}), \sqcup_J s_J(S(X_{n-r}))) \to (S^r(X_n), S^{r+1}(X_n))$$

which thus induces a natural map

$$\gamma(n, r) : \vee_J \widehat{s_J(X_{n-r})} \to S^r(X_n)/S^{r+1}(X_n)$$

that is a continuous bijection. Hence, it suffices to check that the map $\gamma(n, r)$ is open. Note that there is a commutative diagram

$$\begin{array}{ccc}
\sqcup_J s_J(X_{n-r}) & \xrightarrow{F(n,r)} & S^r(X_n) \\
\pi_1 \Big\downarrow & & \Big\downarrow \pi_2 \\
\vee_J \widehat{s_J(X_{n-r})} & \xrightarrow{\gamma(n,r)} & S^r(X_n)/S^{r+1}(X_n).
\end{array}$$

where π_1 and π_2 are the natural projection maps. $F(n, r)$ is an open map as it is a local homeomorphism, and so it follows that the induced map $\gamma(n, r)$ is also open. The second part of Lemma 2.1 follows. $\square$

3. The Proof of Theorem 1.6

Theorem 1.6 gives two different decompositions:

1. One decomposition arises from the equivalence

$$\Theta(n) : \Sigma(X_n) \longrightarrow \bigvee_{0 \leq r \leq n} \Sigma(S^r(X_n)/S^{r+1}(X_n)).$$

2. The second decomposition arises by using the equivalences
$$\widehat{H(n)|_t} : \Sigma(S^t(X_n)/S^{t+1}(X_n)) \longrightarrow \Sigma(\bigvee_{J_t} s_{J_t}\widehat{(X_{n-t})})$$
induced by the $H(n)|_t$.

One direct proof arises from giving both splittings at once as given below. This proof is the precise setting of the analogue of classical James-Hopf invariant maps and how they fit into a simplicial setting as well as a splitting of simplicial spaces.

The details of proof come from a construction of the maps $\widehat{H(n)|_t}$ together with some tedious verifications using simplicial identities. The main work requires definitions of the analogue of James-Hopf invariants.

Let
$$D(n,r) = \vee_{|J|=r} s_J\widehat{(X_{n-r})}$$
where

1. $J = (j_r, j_{r-1} \cdots, j_1)$ is admissible,
2. $s_J\widehat{(X_{n-r})} = s_J(X_{n-r})/s_J S(X_{n-r})$ and
3. $S^n(X_n)$ is equal to $s_0^n(X_0)$.

The method of proof is to exhibit a map
$$H(n) : \Sigma(X_n) \to \bigvee_{0\le r\le n} \Sigma(D(n,r))$$
as described next with the following properties.

LEMMA (3.1). *Assume that the simplicial space $X_\bullet$ is simplicially NDR. Then there is a map*
$$H(n) : \Sigma(X_n) \to \bigvee_{0\le r\le n} \Sigma(D(n,r))$$
with the following properties.

1. *The map $H(n)$ restricts to a map*
$$H(n)|_t : \Sigma(S^t(X_n)) \to \bigvee_{t\le r\le n} \Sigma(D(n,r)).$$
2. *There is a morphisms of cofibrations*
$$\begin{array}{ccc}
\Sigma(S^{t+1}(X_n)) & \xrightarrow{H(n)|_{t+1}} & \bigvee_{t+1\le r\le n}\Sigma(D(n,r)) \\
\Big\downarrow{\scriptstyle i(n,t+1)} & & \Big\downarrow{\scriptstyle \bar{i}(n,t+1)} \\
\Sigma(S^t(X_n)) & \xrightarrow{H(n)|_t} & \bigvee_{t\le r\le n}\Sigma(D(n,r)) \\
\Big\downarrow{\scriptstyle q(t)} & & \Big\downarrow{\scriptstyle \bar{q}(t)} \\
\Sigma(S^t(X_n)/S^{t+1}(X_n)) & \xrightarrow{\widehat{H(n)|_t}} & \Sigma(D(n,t))
\end{array}$$
where $i(n,t+1)$, $\bar{i}(n,t+1)$, $q(t)$ and $\bar{q}(t)$ are the natural inclusions and projections.
3. *The map $\widehat{H(n)|_t} : \Sigma(S^t(X_n)/S^{t+1}(X_n)) \to \Sigma(D(n,t))$ is induced by $H(n)$.*
4. *The map $\widehat{H(n)|_t}$ is a homotopy equivalence, and*

5. *the map* $H(n)|_{t+1} : \Sigma(S^{t+1}(X_n)) \to \bigvee_{t+1\le r\le n} \Sigma(D(n,r))$ *is a homotopy equivalence by downward induction on* t *starting with* $t = n$ *which is the equivalence* $\widehat{H(n)|_n} : \Sigma(S^n(X_n) = s_0^n(X_0) \to \Sigma(D(n,n))$.

Notice that Theorem 1.6 is an immediate consequence of Lemma 3.1. The key step is to define the map $H(n)$; the verification of its properties will be left to the reader.

Consider admissible sequences $I = (i_r, i_{r-1} \cdots, i_1)$ with $n \ge i_r > i_{r-1} > \cdots > i_1 \ge 0$. Thus $s_I(X_{n-r})$ is a subspace of X_n. Define $\chi(I) = (i_1, i_2, \cdots i_{r-1}, i_r)$. Thus

$$d_{\chi(I)} = d_{i_1} d_{i_2} \cdots d_{i_{r-1}} d_{i_r} \text{ and } \mathrm{d}_{\chi(\mathrm{I})} \circ \mathrm{s_I(x) = x}.$$

The natural lexicographical total ordering on such admissible sequences is obtained next from a partial ordering on all such sequences (not necessarily admissible).

Definition (3.2)**.** If $I = (i_r, i_{r-1} \cdots, i_1)$ and $J = (j_r, j_{r-1} \cdots, j_1)$ are sequences with $|I| = |J| = r$ and $I \ne J$, define $I < J$ provided there exists a $p \le r$ such that $i_k = j_k$ if $p < k \le r$ with $i_p < j_p$.

Since an admissible sequence $I = (i_r, i_{r-1} \cdots, i_1)$ satisfies $n \ge i_r > i_{r-1} > \cdots > i_1 \ge 0$, there are exactly $\binom{n+1}{r}$ choices of admissible sequences with $|I| = r$. Furthermore, the partial ordering in Definition 3.2 restricts to a total ordering on admissible sequences which satisfy $|I| = r$. The next lemma is a direct verification with details omitted.

LEMMA (3.3). *Assume that* $X_\bullet$ *is a simplicial space, both* I *and* J *are admissible with* $|I| = |J| = r$ *and that* $x \in X_{n-r}$. *Then*

$$d_{\chi(I)} s_J(x) = \begin{cases} x & \text{if } I = J \text{ and} \\ s_m(y) & \text{for some } s_m \text{ and some } y \text{ if } I < J. \end{cases}$$

Definition (3.4)**.** Restrict to admissible sequences I_s with $|I_s| = r$. Define

$$\delta(r) : X_n \to (X_{n-r})^{\binom{n+1}{r}}$$

by

$$\delta(r) = d_{\chi(I_1)} \times d_{\chi(I_2)} \times \cdots \times d_{\chi(I_{\alpha(n,r)})}$$

where $I_s < I_{s+1}$ for all $1 \le s$ and $\alpha(n,r) = \binom{n+1}{r}$.

If $r = 0$, then

$$\delta(0) : X_n \to X_n$$

is the identity map by convention.

The next lemma follows at once from the definitions.

LEMMA (3.5). *Assume that* $X_\bullet$ *is a simplicial space. Then the map*

$$\delta(r) : X_n \to (X_{n-r})^{\binom{n+1}{r}}$$

restricts to a map

$$\delta(r) : S^{r+1}(X_n) \to (S(X_{n-r}))^{\binom{n+1}{r}}$$

for all $0 \leq r \leq n$*. Thus there is a commutative diagram*

$$\begin{array}{ccc}
S^{r+1}(X_n) & \xrightarrow{\delta(r)} & (S(X_{n-r}))^{\binom{n+1}{r}} \\
{\scriptstyle i(n,r+1)}\big\downarrow & & \big\downarrow{\scriptstyle i(n-r,1)^{\binom{n+1}{r}}} \\
S^r(X_n) & \xrightarrow{\delta(r)} & (X_{n-r})^{\binom{n+1}{r}} \\
{\scriptstyle i(n,r)}\big\downarrow & & \big\downarrow{\scriptstyle 1} \\
X_n & \xrightarrow{\delta(r)} & (X_{n-r})^{\binom{n+1}{r}}.
\end{array}$$

The definition of the map $H(n)$ is given next. Recall the maps

$$\delta(r) : X_n \to (X_{n-r})^{\binom{n+1}{r}}$$

given by

$$\delta(r) = d_{\chi(I_1)} \times d_{\chi(I_2)} \times \cdots \times d_{\chi(I_{\alpha(n,r)})}$$

where $I_s < I_{s+1}$ for all $s \geq 1$. The coordinates in $(X_{n-r})^{\binom{n+1}{r}}$ are indexed by $\chi(I)$ for I admissible with $|I| = r$. Let

$$P_{\chi(I)} : (X_{n-r})^{\binom{n+1}{r}} \to X_{n-r}$$

denote the projection map to the $\chi(I)$-th coordinate.

Recall that

$$D(n, r) = \vee_{|J|=r} s_J\widehat{(X_{n-r}})$$

where

1. $J = (j_r, j_{r-1} \cdots, j_1)$ is admissible,
2. $s_J\widehat{(X_{n-r}}) = s_J(X_{n-r})/s_J S(X_{n-r})$ and
3. $S^n(X_n)$ is equal to $s_0^n(X_0)$.

Definition (3.6). 1. Let

$$\nu(n, J) : \Sigma((X_{n-r})^{\binom{n+1}{r}}) \longrightarrow s_J\widehat{(X_{n-r}})$$

denote the composite $\nu(n, J) = \bar{\sigma}_J \circ P_{\chi(J)}$.

2. Let

$$\lambda(n, J) : \Sigma(X_n) \longrightarrow s_J\widehat{(X_{n-r}})$$

denote the composite $\lambda(n, J) = \nu(n, J) \circ \Sigma(\delta(r))$.

3. Let

$$\Phi(n, r) : \Sigma(X_n) \longrightarrow \bigvee_{|J|=r} s_J\widehat{(X_{n-r}})$$

denote the sum

$$\Phi(n, r) = \sum_{|J|=r} \lambda(n, J)$$

where the index is over all admissible sequence J with a fixed order of summation.

4. Define

$$H(n) : \Sigma(X_n) \to \bigvee_{0 \leq r \leq n} \Sigma(D(n, r))$$

as the sum

$$H(n) = \sum_{0 \le r \le n} \Phi(n, r)$$

with a fixed order of summation.

Therefore this defines the desired map

$$H(n) : \Sigma(X_n) \to \bigvee_{0 \le r \le n} \Sigma(D(n, r))$$

which when restricted to $S^t(X_n)$ makes the following diagram commute (up to homotopy)

$$\begin{tikzcd} \Sigma(S^t(X_n)) \arrow[r, "H(n)|_t"] \arrow[d, "i(n,t)"'] & \bigvee_{t \le r \le n} \Sigma(D(n, r)) \arrow[d, "\bar{i}(n,t)"] \\ \Sigma(X_n) \arrow[r, "H(n)"] & \bigvee_{0 \le r \le n} \Sigma(D(n, r)). \end{tikzcd}$$

The desired properties of $H(n)$ can be readily verified and are left to the reader.

4. Simplicial sets and real algebraic sets

The purpose of this section is to (i) recall standard properties of simplicial complexes as well as (ii) the way in which Theorem 1.6 can be applied to real algebraic varieties.

Let K denote an abstract simplicial complex on m vertices labeled by the set

$$[m] = \{1, 2, \dots, m\}.$$

Thus a simplex σ of K is given by an ordered sequence

$$\sigma = (i_1, \cdots, i_k)$$

with $1 \le i_1 < \cdots < i_k \le m$ such that if $\tau \subset \sigma$, then τ is a simplex of K. Recall the simplicial set $\Delta(K)$ obtained from an abstract simplicial complex as defined in [6] page 234.

Definition (4.1)**.** The simplicial set $\Delta(K)$ is defined as follows.

- $\Delta(K)$ has n-simplices given by the $(n+1)$-tuples of vertices $(v_0, v_1, \cdots, v_n)$ for which $v_0 \le v_1 \le \cdots \le v_n$
- the face and degeneracy operators are given by

  $$d_i(v_0, v_1, \cdots, v_n) = (v_0, \cdots v_{i-1}, v_{i+1}, \cdots, v_n)$$

  and

  $$s_i(v_0, v_1, \cdots, v_n) = (v_0, \cdots v_i, v_i, \cdots, v_n).$$

As pointed out in [6], the following result follows from the two paragraphs before Theorem 15 on page 111 in [16].

THEOREM (4.2). *The geometric realization $|\Delta(K)|$ is homeomorphic to $|K|$.*

The benefits of this construction are expressed in the following result, which is left to the reader for verification.

PROPOSITION (4.3). *The simplicial space $\Delta(K)$ is proper and simplicially NDR. Thus Theorem 1.6 applies to $\Delta(K)$.*

Definitions and basic properties of real algebraic and semi-algebraic varieties are listed next with main reference [5].

Definition (4.4)**.** An affine real algebraic set is the the common zero set of a finite number of real polynomials in $\mathbb{R}[x_1, ..., x_k]$ for some $k \geq 1$. A real algebraic set has a topology induced from $\mathbb{R}^k$ equipped with the Euclidean topology which is called the classical topology.[1] A real semi-algebraic set is a subset of $\mathbb{R}^k$ for some k, which is a finite union of sets each determined by a finite number of polynomial inequalities.

The following theorem was proven in [5] section 9.4.1.

THEOREM (4.5). *If X be a compact semi-algebraic set then it is triangulable, i.e. it is homeomorphic to the geometric realization of a finite simplicial complex.*

Combining the fact that a compact semi-algebraic set is triangulable with Theorem 1.6 above, the next corollary follows at once.

COROLLARY (4.6). *Let X be a compact semi-algebraic set which is the geometric realization of a finite simplicial complex K with order complex $\Delta(K)$. Then $\Delta(K)$ is a simplicial space for which the n-space $\Delta(K)_n$ admits a decomposition after suspension given in Theorem 1.6.*

5. Examples and Problems

This section gives a list of examples of decompositions which arise from the above method.

Example (5.1)**.** Let G denote a closed subgroup of $GL_r(\mathbb{C})$ and consider the space of ordered commuting n–tuples in G, denoted $\mathrm{Hom}(\oplus_n \mathbb{Z}, G)$ and topologized as a subspace of the product G^n. These spaces can be assembled to form a simplicial space which is proper and simplicially NDR, and thus Theorem 1.6 can be applied, yielding in particular the decompositions described in §1 (see [1] and [2] for details).

Example (5.2)**.** The group G acts on $\mathrm{Hom}(\oplus_n \mathbb{Z}, G)$ by conjugation with orbit space denoted $Rep(\oplus_n \mathbb{Z}, G)$. These spaces also form a simplicial space satisfying the hypotheses of 1.6 and thus admit analogous stable decompositions in case G is a compact Lie group (see [3]).

Example (5.3)**.** A product of spaces $X_1 \times X_2 \times \cdots X_n$ decomposes as a wedge of suspensions after suspending once. These decompositions induce related decompositions of moment-angle complexes, in special cases, homotopy equivalent to various varieties obtained from complements of coordinate planes in Euclidean space (see [4]). The decompositions of simplicial spaces given here are more general than the decompositions of generalized moment-angle complexes given in [4]; however they are also coarser.

[1] The Zariski topology, for which closed sets are given by algebraic subsets, will not be used here.

It seems useful to conclude by formulating some problems associated to the topics in this paper:

Problems:

1. Give interesting examples for which Corollary 4.6 can be used to provide useful information concerning semi-algebraic sets.
2. Identify useful conditions for which the splittings of Theorem 1.6 imply decompositions for the suspension of $|X_\bullet|$ for more general simplicial spaces $X_\bullet$.
3. Identify useful conditions when $Tot_\bullet(X)$ can be stably split.

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BAND SUM OPERATIONS YIELDING TRIVIAL KNOTS

KAI ISHIHARA AND KIMIHIKO MOTEGI

ABSTRACT. Let L be a 2-component link in the 3-sphere S^3 consisting of a knot K and its meridian c. Let $b\colon [0,1] \times [0,1] \to S^3$ be an embedding such that $b([0,1]\times[0,1]) \cap K = b([0,1]\times\{0\})$ and $b([0,1]\times[0,1]) \cap c = b([0,1]\times\{1\})$. Then we obtain a knot L_b by replacing $b([0,1]\times\{0,1\})$ in L with $b(\{0,1\}\times[0,1])$. We call L_b a *band sum* of L with the band b. If K is a trivial knot, i.e. L is a Hopf link, then Thompson has proved that only obvious band can create a trivial knot L_b. In the present paper we will show that K is a nontrivial knot and L_b is a trivial knot for some band b if and only if K has unknotting number one. As a particular case, if K is a torus knot, we determine the band b with L_b a trivial knot.

1. Introduction

Let $L = K_1 \cup K_2$ be a 2-component link in the 3-sphere S^3. Let $b\colon [0,1] \times [0,1] \to S^3$ be an embedding such that $b([0,1]\times[0,1]) \cap K_1 = b([0,1]\times\{0\})$ and $b([0,1]\times[0,1]) \cap K_2 = b([0,1]\times\{1\})$. Then we obtain a knot L_b by replacing $b([0,1]\times\{0,1\})$ in L with $b(\{0,1\}\times[0,1])$, see Figure 1. We call L_b a *band sum* of L with the band b. In the following, for simplicity, we use the same symbol b to denote the image $b([0,1]\times[0,1])$.

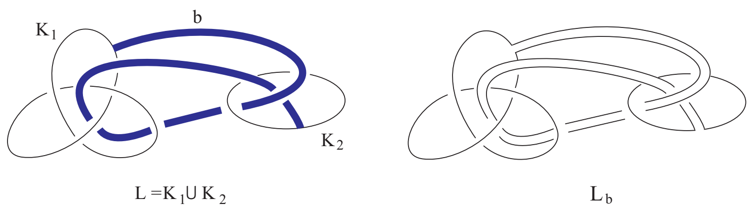

Figure 1. Band sum operation

It follows from [6] and [8] that we completely understand when we can obtain a trivial knot from a split link, i.e. there is a 2-sphere (called *splitting sphere*) in S^3 which separates K_1 and K_2, by band sum operation.

THEOREM (1.1) (Scharlemann [6], Thompson [8]). *Let L be a 2-component split link. If a band sum L_b is a trivial knot in S^3, then both K_1 and K_2 are unknotted and the band b is trivial, i.e. there is a splitting 2-sphere S so that $b \cap S$ consists of a single arc.*

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Now we start with a 2-component link obtained from a split link by a single crossing change in different components. To make precise we say that a 2-component link $L = K_1 \cup K_2$ is a *Hopf sum* of K_1 and K_2 and denote by $K_1 \cup_H K_2$ if L is obtained from K_1, K_2 and the Hopf link by conneced sum operation in a suitable way, see Figure 2. This operation depends on the orientations of K_1, K_2 and the Hopf link, but in the following such an ambiguity is irrelevant.

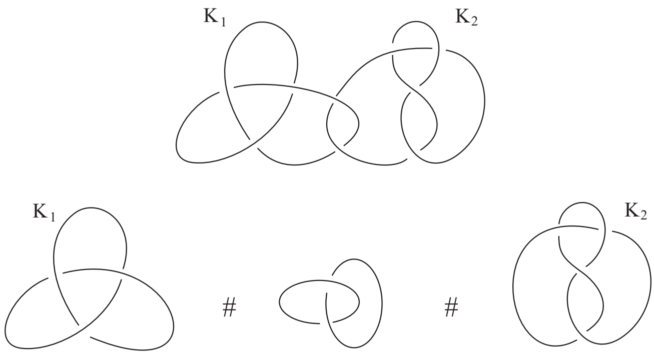

Figure 2. Hopf sum $K_1 \cup_H K_2$

In the case where both K_1 and K_2 are unknotted, i.e. L is a Hopf link H, Thompson [9], Corollary 3, has proved the following. See [3] for an alternate proof and a generalization where L is a $(2, 2p)$-torus link.

THEOREM (1.2) (Thompson). *Let L be a Hopf link. Then L_b is a trivial knot if and only if $L \cup b(\{\frac{1}{2}\} \times [0, 1])$ has a planar projection with exactly two crossings and b is untwisted or half-twisted.*

Furthermore, following Eudave-Muñoz [1], Corollary 2, we have:

THEOREM (1.3) (Eudave-Muñoz). *Let L be a Hopf sum $K_1 \cup_H K_2$ of K_1 and K_2.*

(1) *If both K_1 and K_2 are knotted, then L_b is knotted for any band b.*

(2) *If K_1 is a composite knot and K_2 is a trivial knot, then L_b is knotted for any band b.*

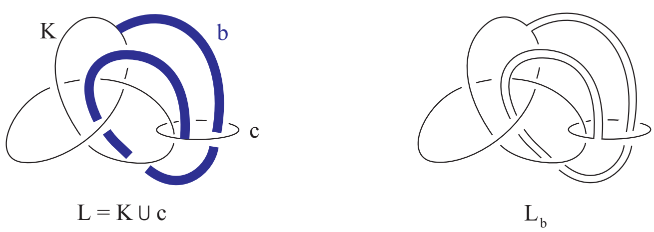

Figure 3. Band sum of a Hopf sum

In the present note we will consider the remaining case: K_1 is a (nontrivial) prime knot and K_2 is a trivial knot. Then L consists of the prime knot $K_1 = K$ and its meridian $K_2 = c$, see Figure 3.

THEOREM (1.4). *Let L be a 2-component link $K \cup c$ consisting of a prime knot K and its meridian c. Then a band sum L_b is a trivial knot for some band b if and only if K has unknotting number one.*

Theorem (1.4), together with a result of Kobayashi [4] ([7]), we have the following result which is motivated by a study of Seifert surgery on knots [2].

THEOREM (1.5). *Let L be a 2-component link $T_{p,q} \cup c$ consisting of a (p, q)-torus knot $T_{p,q}$ ($|p| > q \geq 2$) and its meridian c. If a band sum L_b is a trivial knot, then $(p, q) = (\pm 3, 2)$ and the band b is given by Figure 4 up to isotopy in S^3; the isotopy is not necessarily leaving the link L invariant.*

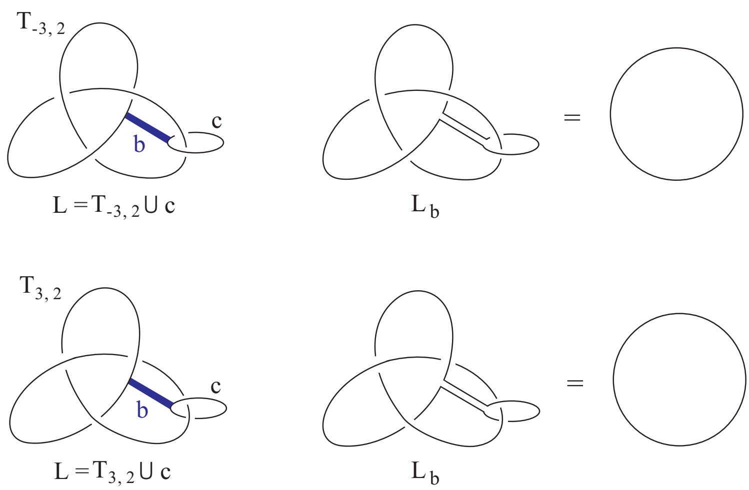

Figure 4. Trivializing band for $T_{p,q} \cup c$

2. Proof of Theorem (1.4)

Proof of if part. Let us assume that K is an unknotting number one knot. Then as indicated in Figure 4, we can choose a band b so that the band sum with b corresponds to a crossing change which converts K into a trivial knot. Thus L_b is a trivial knot.

Proof of only if part. The idea of a proof of the only if part of Theorem (1.4) is showing that the band sum producing a trivial knot is actually a crossing change converting K into a trivial knot.

Let b be a band such that L_b is unknotted. Then since L is a composite link $K\#H$, Eudave-Muñoz [1], Theorem 3, has shown:

LEMMA (2.1) (Eudave-Muñoz [1]). *There exists a decomposing 2-sphere S intersecting L transversely in two points with the following properties.*

(1) *Neither of the 3-balls bounded by S intersects L in a single unknotted spanning arc.*

(2) *S crosses the band b in a single arc parallel to $b([0, 1] \times \{\frac{1}{2}\})$.*

CLAIM (2.2). *The 2-sphere S gives a decomposition of L as $K\#H$.*

Proof. It is easy to see that the two points in $S \cap L$ belong to K and the other component c, a meridian of K, is entirely contained in a 3-ball, say B, bounded by S. Then $K \cap B'$, where B' is the opposite side of S in S^3, is a knotted spanning arc. Since K is prime, $K \cap B$ is an unknotted spanning arc. Hence S gives the required decomposition. □

Using a result of Hirasawa and Shimokawa [3], we put further restriction on a position of the band b. To apply [3], Theorem 1.6, choose an orientation of K arbitrarily and then choose an orientation of c so that L and L_b have coherent orientations except for the band b.

Then [3], Theorem 1.6, asserts:

LEMMA (2.3) (Hirasawa-Shimokawa [3]). *There exists a minimal genus Seifert surface F for the link L which contains the band b.*

Let D be a disk bounded by c which intersects K exactly once.

LEMMA (2.4). *We may assume, if necessary after sliding the band b along L, that b does not intersect the interior of D, i.e. $b \cap D = b([0, 1] \times \{1\})$.*

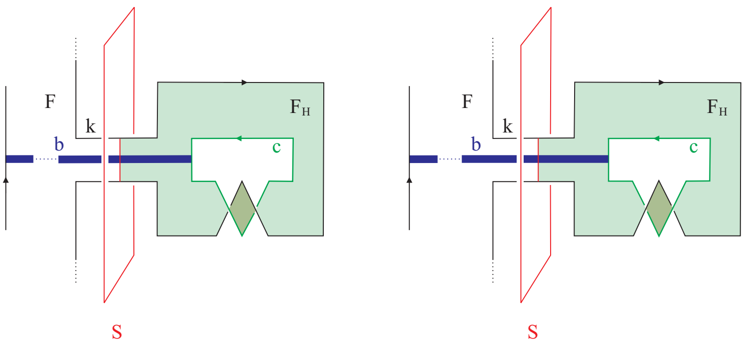

Figure 5. Position of the band b

Proof. Recall first that L intersects the 2-sphere S transversely in two points and that the band b intersects S transversely in a single arc $b([0, 1] \times \{\frac{1}{2}\})$. As usual we can isotope F keeping S, b, L invariant so that F intersects S transversely. Then $F \cap S$ consists of a single arc componet and circle components. By an innermost disk argument we can eliminate the circle components (keeping S, b, L invariant) to obtain a minimal genus Seifert surface F intersecting S transversely in a single arc which contains the arc $b \cap S$. Since S crosses the band b in a single arc parallel to $b([0, 1] \times \{\frac{1}{2}\})$ (Lemma (1.3)), b crosses the arc $F \cap S$ just once.

Cutting the minimal genus Seifert surface F along the arc $F \cap S$, we obtain a minimal genus Seifert surface F_H of the Hopf link H, which is an annulus. Since F_H is an annulus, we can slide the band b along c so that b does not intersects the interior of D as desired, see Figure 5. □

It follows from Lemma (2.4) that the band sum operation can be regarded as a crossing change. Thus the knot K becomes a trivial knot after the single crossing change, and hence K has unknotting number one as desired.

This completes a proof of the only if part of Theorem (1.4).

3. Proof of Theorem (1.5).

Let L be a 2-component link $T_{p,q} \cup c$ consisting of a torus knot $T_{p,q}$ ($|p| > q \geq 2$) and its meridian c. It is known that the unknotting number of a torus knot $T_{p,q}$ ($|p| > q \geq 2$) is $\frac{(|p|-1)(q-1)}{2}$ ([5]), hence Theorem (1.4) shows that L_b is a trivial knot for some band b if and only if $(p, q) = (\pm 3, 2)$.

Let us prove the uniqueness of such a trivializing band. Suppose that $K = T_{\pm 3,2}$ and b is a band connecting K and c such that L_b is a trivial knot. Let D be a disk bounded by c intersecting K exactly once. From Lemma (2.4), we may assume that b does not intersect the interior of D. Following Kobayashi [4], we call a disk Δ intersecting K in two points of opposite orientations a *crossing disk* for K. Now we associate a crossing disk Δ_b to the band b. Denote the core $b(\{\frac{1}{2}\} \times [0, 1]) \subset b$ by τ. Extending the 1-complex $\tau \cup D$ to obtain a disk Δ_b so that (i) the linking number between K and $\partial\Delta_b$ is zero, and (ii) $\Delta_b \cap \mathrm{int} b = \mathrm{int}\tau$, see Figure 6. Thus a band b determines a crossing disk Δ_b with a surgery slope $\varepsilon(= \pm 1)$ on $\partial\Delta_b$ so that the band sum is realized by the surgery on $\partial\Delta_b$.

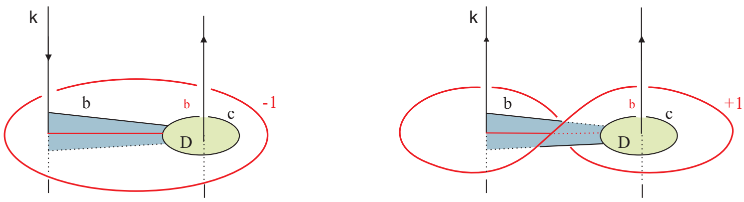

Figure 6. In the left, L_b is the result of (-1)-surgery on $\partial\Delta_b$; in the right, L_b is the result of $(+1)$-surgery on $\partial\Delta_b$

Conversely a crossing disk Δ with a meridian c (or D) and a surgery slope ε on $\partial\Delta$ determines a band b uniquely.

Since $K = T_{\pm 3,2}$ has unknotting number one, we can choose a band b so that L_b is a trivial knot. Let Δ_b be a crossing disk associated to the band b. Then using [4] we have a minimal genus Seifert surface F for K such that (1) F is obtained from two Hopf bands by a plumbing along a disk, and (2) $K \cup \Delta_b$ has a position given by Figure 7 up to isotopy; a presentation of a trefoil knot $T_{\pm 3,2}$ as the plumbing of two Hopf bands is unique up to isotopy. (In [4] Kobayashi describes a position of $\partial\Delta_b$, but his proof shows that we have the same conclusion for the crossing disk Δ_b.)

For the crossing disk Δ_b we have two possibilities for the position of the disk D as in Figure 7. In Figure 7 we indicate cores τ of two possible bands b_1 (in the left) and b_2 (in the right) depending on the positions of D. It is easy to see that there is an isotopy of S^3 deforming $L \cup b_1$ to $L \cup b_2$.

Furthermore, we can easily isotope $L \cup b_1$ to $L \cup b$ given in Figure 4 as required.

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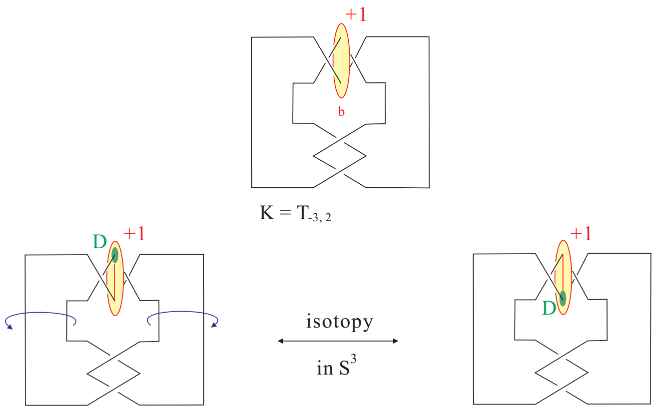

Figure 7

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