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Contenido

ARTÍCULOS DE INVESTIGACIÓN

On the polynomial parametric family of the sets with the property $D(-1; 1)$
A. Filipin 1

Tropical Geometry for fields with a Krull valuation: First definitions and a small result
F. Aroca 9

Characteristic subgroups are not preserved by isomorphisms of tables of marks
V. N. García-Ríos, A. G. Raggi-Cárdenas, and L. Valero-Elizondo 15

An example of a twisted Fusion Algebra
A. N. Duman 19

Geometric differences between the use of Lie algebras and Lie superalgebras
J. R. Cervantes Polanco and O. A. Sánchez-Valenzuela 29

Singular points and automorphisms of unstable foliations of $\mathbb{CP}$
C. R. Alcántara 39

The homotopy groups of L_2-localization of the Ravenel spectra $T(m)/v_1$ at the prime two
I. Ichigi, K. Shimomura, and R. Takeda 53

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ON THE POLYNOMIAL PARAMETRIC FAMILY OF THE SETS WITH THE PROPERTY $D(-1;1)$

ALAN FILIPIN

ABSTRACT. In this paper we prove that if a positive integer d has the property that for an integer $k \geq 1$ each of $(k^{12}+1)d+1$, $(k^{12}+2k^6+2)d+1$ and $(4k^{12}+4k^6+5)d+1$ is a perfect square, then

$$d = 16k^{36}+48k^{30}+100k^{24}+120k^{18}+112k^{12}+60k^6+24.$$

1. Introduction

Let n be an integer. A set of m positive integers is called a Diophantine m-tuple with the property $D(n)$ or simply $D(n)$-m-tuple, if the product of any two of them increased by n is a perfect square.

The first one who studied the problem of finding such sets was Diophantus in the case $n = 1$. He found a set of four positive rational numbers with the above property: $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$. However, Fermat was the first who found a $D(1)$-quadruple, which was the set $\{1, 3, 8, 120\}$. Euler was later able to add the fifth positive rational, $\frac{777480}{8288641}$, to the Fermat's set (see [6], [7] pp. 103–104, 232). Recently, Gibbs [21] found several examples of $D(n)$-sextuples. It is conjectured that there does not exist a $D(1)$-quintuple. This is an immediate consequence of the following stronger version of conjecture (see [1]).

CONJECTURE (1.1). *If $\{a, b, c, d\}$ is a $D(1)$-quadruple such that $a < b < c < d$, then*

$$d = d_+ = a+b+c+2(abc+\sqrt{(ab+1)(ac+1)(bc+1)}).$$

The first result that supports Conjecture (1.1) was proven by Baker and Davenport [2]. Precisely, they proved if $\{1, 3, 8, d\}$ is a $D(1)$-quadruple, then $d = 120$. There are some generalization of this result. First, Dujella [11] proved that if $k \geq 2$ and d are integers and $\{k-1, k+1, 4k, d\}$ is a $D(1)$-quadruple, then $d = 4k(4k^2-1)$. Secondly, Dujella and Pethö [15] proved that if $\{1, 3, c, d\}$ such that $c < d$ is a $D(1)$-quadruple, then $d = c_{\nu+1}$, where

$$c = c_\nu = \frac{1}{6}((2+\sqrt{3})^{2\nu+1}+(2-\sqrt{3})^{2\nu+1}-4), \quad \nu = 1, 2, \ldots,$$

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and finally, Dujella [8] proved that if $\{F_{2k}, F_{2k+2}, F_{2k+4}, d\}$, where $k \geq 1$ is an integer and F_ν denotes the ν-th Fibonacci number, is a $D(1)$-quadruple, then $d = 4F_{2k+1}F_{2k+2}F_{2k+3}$. The first two results have been generalized and it has been proven [5, 19] that if $\{k-1, k+1, c, d\}$ is a $D(1)$-quadruple such that $c < d$, then $d = c_{\nu+1}$, where

$$c = c_\nu = \frac{1}{2(k^2-1)}((k+\sqrt{k^2-1})^{2\nu+1} + (k-\sqrt{k^2-1})^{2\nu+1} - 2k), \ \nu = 1, 2, \ldots .$$

In general, in the case $n = 1$, Dujella [12] proved that there does not exist a $D(1)$-sextuple and that there exist only finitely many $D(1)$-quintuples.

In the case $n = -1$, Dujella [9] proved that the pair $\{1, 2\}$ cannot be extended to a $D(-1)$-quadruple. Moreover, Dujella and Fuchs [14] proved that if $\{a, b, c, d\}$ is a $D(-1)$-quadruple such that $a < b < c < d$, then $a = 1$. There are also results on non-extendibility of $D(-1)$-triples of the form $\{1, b, c\}$ for $b \geq 5$ (see [16, 18, 22]). Recently Dujella *et al.* [13] proved that there exist only finitely many $D(-1)$-quadruples.

Even a $D(-1)$-triple $\{a, b, c\}$ such that $a < b < c$ cannot be conjecturally extended to a $D(-1)$-quadruple, there exist a positive integer d such that each of $ad + 1$, $bd + 1$ and $cd + 1$ is a perfect square. Moreover, $d = d^+$ has such property, where

$$d^+ = 2abc - (a + b + c) + 2\sqrt{(ab-1)(ac-1)(bc-1)}.$$

This leads us to the following definition.

Definition (1.2). A set $\{a, b, c, d\}$ of positive integers is said to have a property $D(-1; 1)$ if $\{a, b, c\}$ is a $D(-1)$-triple and each of $ad + 1$, $bd + 1$ and $cd + 1$ is a perfect square.

Let us mention that a $D(-1)$-triple $\{a, b, c\}$ can be extended to a $D(-1)$-quadruple $\{a, b, c, -d\}$ in the ring $\mathbb{Z}[i]$ of Gaussian integers (see [10], Example 1), which corresponds to our quadruple $\{a, b, c, d\}$ having the property $D(-1; 1)$. In this paper we prove that if $a = k^{12} + 1$, $b = k^{12} + 2k^6 + 2$ and $c = 4k^{12} + 4k^6 + 5$, for an integer $k \geq 1$, then such a d is unique. For $k = 0$ it was already proven by Fujita [17]. There are some similar results on the sets with the property $D(-1; 1)$. Precisely, Fujita [17] recently proved if a set $\{1, 2, c, d\}$ has the property $D(-1; 1)$, then $d = s(3s \pm 2t)$, where $s = \sqrt{c-1}$ and $t = \sqrt{bc-1}$. The same author [20] also proved that if $\{F_{2k+1}, F_{2k+3}, F_{2k+5}, d\}$ has the property $D(-1; 1)$, where $k \geq 0$ is an integer and F_ν denotes the ν-th Fibonacci number, then $d = 4F_{2k+2}F_{2k+3}F_{2k+4}$. Our main result is the following theorem.

THEOREM (1.3). *Let $k \geq 0$ be an integer. If the set*

$$\{k^{12} + 1, k^{12} + 2k^6 + 2, 4k^{12} + 4k^6 + 5, d\}$$

has the property $D(-1; 1)$, then

$$d = 16k^{36} + 48k^{30} + 100k^{24} + 120k^{18} + 112k^{12} + 60k^6 + 24\,.$$

In the proof of theorem we will use already known methods for solving similar problems on extension of $D(n)$-m-tuples. More precisely, we prove our theorem along the same lines as in [13], [17], [20] and we use some useful results which are already proven there.

2. System of Pellian equations

Let us assume that $\{a, b, c, d\}$ has the property $D(-1;1)$. Then there exist positive integers r, s and t such that

$$ab - 1 = r^2, \quad ac - 1 = s^2, \quad bc - 1 = t^2.$$

There also exist positive integers x, y, z such that

$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2.$$

If we eliminate d we obtain the following system of simultaneous Pellian equations:

$$az^2 - cx^2 = a - c, \tag{2.1}$$

$$bz^2 - cy^2 = b - c. \tag{2.2}$$

In the following lemma we describe the sets of solutions of (2.1) and (2.2).

LEMMA (2.3) (cf. [20], Lemma 3). *Let (z, x) and (z, y) be positive solutions of* (2.1) *and* (2.2) *respectively. Then there exist solutions (z_0, x_0) and (z_1, y_1) of* (2.1) *and* (2.2) *respectively such that*

(i) *the following inequalities are satisfied:*

$$0 < x_0 \le \sqrt{a(c-a)},\ |z_0| < \sqrt{c(c-a)},$$
$$0 < y_1 \le \sqrt{b(c-b)},\ |z_1| < \sqrt{c(c-b)}.$$

(ii) *There exist integers $m, n \ge 0$ such that*

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})(2ac - 1 + 2s\sqrt{ac})^m, \tag{2.4}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})(2bc - 1 + 2t\sqrt{bc})^n. \tag{2.5}$$

From (2.4) we conclude that $z = v_m$ for some (z_0, x_0) with the above properties and integer $m \ge 0$, where

$$v_0 = z_0,\ v_1 = (2ac - 1)z_0 + 2scx_0,\ v_{m+2} = (4ac - 2)v_{m+1} - v_m. \tag{2.6}$$

In the same manner from (2.5) we conclude that $z = w_n$ for some (z_1, x_1) with the above properties and integer $n \ge 0$, where

$$w_0 = z_1,\ w_1 = (2ac - 1)z_1 + 2tcy_1,\ w_{n+2} = (4bc - 2)w_{n+1} - w_n. \tag{2.7}$$

So our system of equations (2.1) and (2.2) is thus transformed to finitely many equations of the form $z = v_m = w_n$. Let us mention that here we have the exactly same recurrences as in [13], but with different initial values. So we can use all results proven there that are independent of the fundamental solutions. Also in our case we have

$$a = k^{12} + 1,\ b = k^{12} + 2k^6 + 2,\ c = 4k^{12} + 4k^6 + 5,$$

which yields

$$r = k^{12} + k^6 + 1,\ s = 2k^{12} + k^6 + 2,\ t = 2k^{12} + 3k^6 + 3.$$

In the following lemma we can describe fundamental solutions of (2.1) and (2.2).

LEMMA (2.8). *Let* (x, y, z) *be a positive solution of the system of equations* (2.1) *and* (2.2). *Then we have*

$$(z_0, x_0) = (\pm 1, 1),\ (z_1, y_1) = (\pm 1, 1).$$

Proof. The proof of this is exactly the same as the proof of [20], Lemma 5. The only thing one has to be aware of is that in our case we also have $b < 3a$ and $c < 4b$ which was used there. □

Now we have the following properties.

LEMMA (2.9) (cf. [13], Lemma 2). *If* $v_m = w_n$, $n \neq 0$, *then*

(i) $m \equiv n \pmod 2$,
(ii) $n \leq m \leq 2n$,
(iii) $\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{4c}$.

Using the last lemma, we have in our case that $v_m = w_n$, $n \geq 2$ implies

$$\pm(k^{12}+1)m^2 + (2k^{12}+k^6+2)m \equiv \pm(k^{12}+2k^6+2)n^2 + (2k^{12}+3k^6+3)n \pmod{16k^{12}+16k^6+20}.$$

If we multiply this congruence by 16 we get

$$(16k^6+4)m^2 \mp (16k^6+8)m + (16k^6+12)n^2 \mp (16k^6+8)n \equiv 0 \pmod{16k^{12}+16k^6+20}.$$

Now from $m \geq n \geq 2$ we conclude

$$(16k^6+4)m^2 \mp (16k^6+8)m + (16k^6+12)n^2 \mp (16k^6+8)n > 0,$$

which yields

$$(16k^6+4)m^2 \mp (16k^6+8)m + (16k^6+12)n^2 \mp (16k^6+8)n \geq 16k^{12}+16k^6+20.$$

Now we have

$$\begin{aligned}(64k^6+32)m^2 &> (32k^6+16)m^2 + (32k^6+16)m \\ &> (16k^6+4)m^2 \mp (16k^6+8)m + (16k^6+12)n^2 \mp (16k^6+8)n \\ &\geq 16k^{12}+16k^6+20,\end{aligned}$$

which implies

$$m > \frac{k^3}{2}.$$

So we have just proved the following lemma.

LEMMA (2.10). *If* $v_m = w_n$, $n \geq 2$, *then* $m > \frac{k^3}{2}$.

3. Linear forms in logarithms

In this section we apply Baker's theory to linear forms in three logarithms arising from the sequences (v_m) and (w_n).

LEMMA (3.1). *If $v_m = w_n$, $n \geq 2$, then*

$$0 < m \log \alpha_1 - n \log \alpha_2 + \log \alpha_3 < 2\alpha_1^{-m},$$

where

$$\alpha_1 = (s + \sqrt{ac})^2, \quad \alpha_2 = (t + \sqrt{bc})^2, \quad \alpha_3 = \frac{\sqrt{b}(\sqrt{c} \pm \sqrt{a})}{\sqrt{a}(\sqrt{c} \pm \sqrt{b})}.$$

Proof. Let

$$P = \frac{\sqrt{c} \pm \sqrt{a}}{\sqrt{a}}(s + \sqrt{ac})^{2m}, \qquad Q = \frac{\sqrt{c} \pm \sqrt{b}}{\sqrt{b}}(t + \sqrt{bc})^{2n}.$$

Then $v_m = w_n$ implies

$$P - \frac{c-a}{a}P^{-1} = Q - \frac{c-b}{b}Q^{-1}.$$

We get

$$Q - P = \frac{c-b}{b}Q^{-1} - \frac{c-a}{a}P^{-1} < \frac{c-b}{b}(Q^{-1} - P^{-1}) = \frac{c-b}{b}(P-Q)P^{-1}Q^{-1},$$

and therefore $P > Q$. Furthermore, for $m \geq n \geq 2$, we have

$$Q > P - \frac{c-a}{a}P^{-1} > P - 1.$$

So

$$\frac{P-Q}{P} < P^{-1} < \frac{1}{2}.$$

Hence,

$$0 < \log \frac{P}{Q} = -\log \frac{Q}{P} = -\log\left(1 - \frac{P-Q}{P}\right) < \frac{P-Q}{P} + \left(\frac{P-Q}{P}\right)^2.$$

From this we conclude that

$$0 < \log \frac{P}{Q} < \frac{1}{P} + \frac{1}{P^2} < \frac{2}{P} = \frac{2\sqrt{a}}{\sqrt{c} \pm \sqrt{a}}(s + \sqrt{ac})^{-2m}$$
$$< 2(s + \sqrt{ac})^{-2m} = 2\alpha_1^{-m},$$

which finishes the proof of the lemma. □

Now we give the upper bound for m in the equation $v_m = w_n$. First we need Baker-Wüstholz theorem.

THEOREM (3.2) (cf. [3]). *Let $\Lambda = b_1 \log \alpha_1 + \ldots + b_l \log \alpha_l \neq 0$ be a linear form of l logarithms of algebraic numbers $\alpha_1, \ldots, \alpha_l$ with integer coefficients $b_1, \ldots, b_l$. Then*

(3.3) $$\log \Lambda > -18(l+1)!\,l^{l+1}(32d)^{l+2}h'(\alpha_1)\cdots h'(\alpha_l)\log(2ld)\log B,$$

where $B = \max\{|b_j| : 1 \leq j \leq l\}$, d is a degree of the extension of algebraic number field generated by $\alpha_1, \ldots, \alpha_l$, and $h'(\alpha) = \frac{1}{d}\max\{h(\alpha), \log|\alpha|, 1\}$, where $h(\alpha)$ denotes the standard logarithmic Weil height of α.

We will now apply this theorem to our linear form in logarithms from Lemma (3.1). In the notation of the theorem we have $d = 4$, $l = 3$ and $B = m$, and minimal polynomials of α_1, α_2 and α_3 are given by

$$\alpha_1^2 - (4ac-2)\alpha_1 + 1 = 0,$$
$$\alpha_2^2 - (4bc-2)\alpha_2 + 1 = 0,$$
$$a^2(c-b)^2\alpha_3^4 + 4a^2b(c-b)\alpha_3^3 + 2ab(3ab - c(a+b+c))\alpha_3^2 + 4ab^2(c-a)\alpha_3 + b^2(c-a)^2 = 0.$$

Now we have

$$h'(\alpha_1) = \frac{1}{2}\log\alpha_1 < \frac{1}{2}\log 4ac,$$
$$h'(\alpha_2) = \frac{1}{2}\log\alpha_2 < \frac{1}{2}\log 4bc.$$

We can also easily bound the conjugates of α_3. We have

$$\frac{\sqrt{b}(\sqrt{c}\pm\sqrt{a})}{\sqrt{a}(\sqrt{c}+\sqrt{b})} < \sqrt{\frac{b}{a}}, \quad \frac{\sqrt{b}(\sqrt{c}\pm\sqrt{a})}{\sqrt{a}(\sqrt{c}-\sqrt{b})} < 2\sqrt{\frac{b}{a}}.$$

This implies

$$h'(\alpha_3) < \frac{1}{4}\log\left(a^2(c-b)^2\cdot 4\frac{b^2}{a^2}\right) = \frac{1}{4}\log 4b^2(c-b)^2 < \frac{1}{2}\log 2bc.$$

Finally using Theorem (3.2) together with Lemma (3.1) we get

$$\frac{m}{\log m} < 2.39\cdot 10^{14}\log 4bc\log 2bc\,. \tag{3.4}$$

Now can get upper bounds for b and c in terms of m, using Lemma (2.10), and if we insert that in (3.4), we get $m < 2\cdot 10^{21}$. So we have just proved the following proposition.

PROPOSITION (3.5). *If $v_m = w_n$, $n \geq 2$, then $m < 2\cdot 10^{21}$.*

We can also see from Lemma 2.10 that we have proven our main theorem for $k \geq 1.59\cdot 10^7$.

4. Reduction

In order to deal with the remaining cases $k < 1.59\cdot 10^7$, we will use a Diophantine approximation algorithm, so-called the Baker-Davenport reduction method. The following lemma is a slight modification of the original version of the Baker-Davenport reduction method.

LEMMA (4.1) (cf. [15], Lemma 5a). *Assume that M is a positive integer. Let $\frac{P}{Q}$ be the convergent of the continued fraction expansion of κ such that $Q > 6M$ and let*

$$\eta = \|\mu' Q\| - M\cdot\|\kappa Q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then there is no solution of the inequality

$$0 < m\kappa - n + \mu' < AB^{-m}$$

in integers m and n with

$$\frac{\log\left(AQ/\eta\right)}{\log B} \leq m \leq M\,.$$

We apply Lemma (4.1) with

$$\kappa = \frac{\log\alpha_1}{\log\alpha_2}, \quad \mu' = \frac{\log\alpha_3}{\log\alpha_2}, \quad A = \frac{2}{\log\alpha_2}, \quad B = \alpha_1,$$

and $M = 2\cdot 10^{21}$.

We have done it in Mathematica 5.2. In the first step of reduction for all $k < 1.59\cdot 10^7$ we get $m \leq 11$. Then we can apply Lemma (4.1) with new $M = 11$ and we get $m \leq 2$. So we only have to see what is happening with small indices in equation $v_m = w_n$. But it becomes a polynomial equation in k which are easy to solve. The only solutions we get are $z = v_0 = w_0 = \pm 1$, which implies $d = 0$, which is no real extension of our $D(-1)$-triple to a $D(-1;1)$-quadruple, and $z = v_1 = w_1 = 8k^{24} + 16k^{18} + 26k^{12} + 18k^6 + 11$, which yields

$$d = 16k^{36} + 48k^{30} + 100k^{24} + 120k^{18} + 112k^{12} + 60k^6 + 24\,.$$

This finishes the proof of our main theorem.

5. Concluding remarks

In this paper we have proved the uniqueness of the extension of one parametric polynomial family of $D(-1)$-triples to $D(-1;1)$-quadruples. It is somewhat different from the work done by Fujita [20], where he has considered exponential families which grow much faster for a parameter k. It was the first problem here, because it would be much more interesting to consider a family of $D(-1)$-triples of the form $\{k^2+1, k^2+2k+1, 4k^2+4k+5\}$. But using the methods we have used, we would then get a much larger upper bound for k, and then it would not be possible to do the reduction with a computer program. Also here we cannot use the Bennett's theorem ([4], Theorem 3.2) on simultaneous approximations of algebraic numbers which are close to 1, or any similar result of this type, to get better upper bounds for m and k, because b and c are too close to each other in our case.

Let us mention at the end, that using the same Baker-Davenport reduction and ([13], Theorem 1), we can prove that there does not exist a $D(-1)$-quadruple of the form $\{1, k^{12}+1, k^{12}+2k^6+2, d\}$, for any positive integer k.

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TROPICAL GEOMETRY FOR FIELDS WITH A KRULL VALUATION: FIRST DEFINITIONS AND A SMALL RESULT

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ABSTRACT. Given an algebraically closed field with a Krull valuation, we study the image, via the valuation, of an affine algebraic variety. We define the tropical semi-ring associated with the value group and the tropical variety associated to an ideal of the ring of polynomials with coefficients in the field. We prove the so called "Kapranov's theorem" in the discrete case.

Introduction

For a and b in $\mathbb{R} \cup \{\infty\}$ define the operations $a \oplus b := \min\{a, b\}$ and $a \odot b := a + b$. The semi-ring $\mathbb{T} := (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$ is called *the tropical semi-ring*. A Laurent polynomial $F \in \mathbb{T}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ is called a *tropical polynomial* and the non-linearity locus of the map $F \colon \mathbb{R}^N \to \mathbb{R}$ induced by F is called *the tropical hypersurface* associated to F. Tropical geometry is currently a rapidly growing subject. See for example [3], [2], [8].

Let $(\mathbb{K}, \mathrm{val})$ be a valued field with values in $\mathbb{R}$. A Laurent polynomial $f = \sum_{\alpha \in \Lambda \subset \mathbb{Z}^N} \varphi_\alpha x^\alpha \in \mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ induces a tropical polynomial $\mathcal{T}f := \bigoplus_{\alpha \in \Lambda \subset \mathbb{Z}^N} \mathrm{val}(\varphi_\alpha) \odot x^\alpha$ called the *tropicalization* of f.

Let $\mathcal{I} \subset \mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ be an ideal. Denote by $\mathbf{V}(\mathcal{I})$ the affine algebraic variety of $(\mathbb{K}^*)^N$ defined by $\mathcal{I}$ and consider the map

$$\begin{array}{cccc} \mathrm{val}: & (\mathbb{K}^*)^N & \longrightarrow & \mathbb{R}^N \\ & (\varphi_1, \ldots, \varphi_N) & \mapsto & (\mathrm{val}\,\varphi_1, \ldots, \mathrm{val}\,\varphi_N) \end{array}$$

where $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$.

A well known result in tropical geometry called *Kapranov's theorem* states that the closure in $\mathbb{R}^N$ of $\mathrm{val}(\mathbf{V}(\mathcal{I}))$ equals the intersection of the tropical hypersurfaces associated to every polynomial in $\mathcal{I}$. See [1], [6], [5].

The theory of valuations started at the beginning of the XXth century. Valuations into the reals are just a special type of valuations called *classical* or *archimedean* (see for example [7]). Given a valued field $\mathbb{K}$ with values in a totally ordered group Γ we want to describe the set of values $\mathrm{val}(\mathbf{V}) \subset \Gamma^N$ of an affine algebraic variety $\mathbf{V} \subset \mathbb{K}^N$.

In this note we extend the basic definitions of tropical geometry $(\mathbb{R}, \leq)$ to an arbitrary ordered group $(\Gamma, \leq)$. Then we give, in the discrete case, a very

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simple proof of the extension of Kapranov's theorem for a valued field $\mathbb{K}$ with values in an arbitrary ordered group $(\Gamma, \leq)$.

1. Ordered groups and semi-rings

A *totally ordered group* is an abelian group $(\Gamma, +)$ equipped with a total order such that for all $x, y, z \in \Gamma$ if $x \leq y$ then $x + z \leq y + z$. A totally ordered group is torsion free.

A totally ordered group is isomorphic to a subgroup of $\mathbb{R}$ if and only if given $a > 0$ and b there exists a positive integer k such that $b < ka$ (see for example [4], page 243).

Example (1.1). Through this note we will work with the group $\Gamma = \mathbb{R}^2$ with the lexicographical order. That is $(a, b) \leq_l (a', b')$ if and only if $a < a'$ or $a = a'$ and $b \leq b'$.

A totally ordered group $(\Gamma, +, \leq)$ induces an idempotent semi-ring $\mathbb{G} := (\Gamma \cup \{\infty\}, \oplus, \odot)$. Where

- $a \oplus b := \min\{a, b\}$ and $a \oplus \infty := a$ for $a, b \in \Gamma$
- $a \odot b := a + b$ and $a \odot \infty := \infty$ for $a, b \in \Gamma$.

A semi-ring induced by a totally ordered set will be called a *tropical semi-ring*.

Example (1.2). Consider the group Γ of example (1.1) with tropical multiplication given by standard sum in Γ. We have $(-1, 0) \oplus (1, -8) = (-1, 0)$ and $(-1, 0) \odot (1, -8) = (0, -8)$.

2. Polynomials with coefficients in a tropical semi-ring

Let $\mathbb{G}$ be the tropical semi-ring induced by the group $(\Gamma, \leq)$. A non-zero Laurent polynomial $F \in \mathbb{G}[x_1, x_1^{-1}, \dots, x_N, x_N^{-1}]$ is an expression of the form

$$F = \sum_{\alpha \in \mathcal{E}(F) \subset \mathbb{Z}^N} a_\alpha x^\alpha, \qquad a_\alpha \in \Gamma, \quad \#\mathcal{E}(F) < \infty, \quad x^\alpha := {x_1}^{\alpha_1} \cdots {x_N}^{\alpha_N}.$$

F induces a map $F\colon \Gamma^N \to \Gamma$ given by

$$F\colon \gamma \mapsto \oplus_{\alpha \in \Lambda} a_\alpha \odot \overbrace{\gamma_1 \odot \cdots \odot \gamma_1}^{\alpha_1 \text{ times}} \odot \cdots \odot \overbrace{\gamma_N \odot \cdots \odot \gamma_N}^{\alpha_N \text{ times}}.$$

For $k \in \mathbb{N}$ and $a \in \Gamma$ set $ka := \overbrace{a + \cdots + a}^{k \text{ times}}$ and for $\alpha \in \mathbb{Z}^N$ and $\gamma \in \Gamma^N$ set $\alpha \cdot \gamma := \alpha_1\gamma_1 + \cdots + \alpha_N\gamma_N$.

Example (2.1). $(1, 2) \cdot ((1, 1), (0, -1)) = (1, -1)$.

With this notation

$$F\colon \gamma \mapsto \min_{\alpha \in \mathcal{E}(F)} \{a_\alpha + \alpha \cdot \gamma\}.$$

For each $\alpha \in \mathcal{E}(F)$ set

$$\mathcal{C}_\alpha := \{\gamma \in \Gamma^N \mid \operatorname{val} \varphi_\alpha + \alpha \cdot \gamma \leq \operatorname{val} \varphi_{\alpha'} + \alpha' \cdot \gamma, \ \forall \alpha' \in \mathcal{E}(F)\}.$$

We have $F|_{\mathcal{C}_\alpha}(\gamma) = \operatorname{val} \varphi_\alpha + \alpha \cdot \gamma$ is a linear function on $\mathcal{C}_\alpha$.

Definition (2.2). The *hypersurface associated to F* is the set

$$\mathcal{V}(F) := \bigcup_{\alpha \neq \alpha' \subset \mathcal{E}(F)} \mathcal{C}_\alpha \cap \mathcal{C}_{\alpha'}.$$

Example (2.3). For $F := (0,1)xy^2 + (4,0)x^2 + (0,6)$ we have

$$\begin{aligned}
\mathcal{V}F = \;& \{((a,b),(c,d)) \mid (4,0) + 2(a,b) = (0,6) \le (0,1) + (a,b) + 2(c,d)\} \\
\cup\;& \{((a,b),(c,d)) \mid (0,1) + (a,b) + 2(c,d) = (0,6) \le (4,0) + 2(a,b)\} \\
\cup\;& \{((a,b),(c,d)) \mid (4,0) + 2(a,b) = (0,1) + (a,b) + 2(c,d) \le (0,6)\} \\
=\;& \{((-2,3),(\lambda,\mu)) \mid \lambda,\mu \in \mathbb{R},\ (1,1) \le (\lambda,\mu)\} \\
\cup\;& \{((-2\lambda, 5-2\mu),(\lambda,\mu)) \mid \lambda,\mu \in \mathbb{R},\ (\lambda,\mu) \le (1,1)\} \\
\cup\;& \{((-4+2\lambda, 1+2\mu),(\lambda,\mu)) \mid \lambda,\mu \in \mathbb{R},\ (\lambda,\mu) \le (1,1)\} \\
=\;& \{((-2,3),(1,1)) + \gamma(0,1) \mid \gamma \ge (0,0)\} \\
\cup\;& \{((-2,3),(1,1)) + \gamma(-2,1) \mid \gamma \ge (0,0)\} \\
\cup\;& \{((-2,3),(1,1)) + \gamma(-2,-1) \mid \gamma \ge (0,0)\}.
\end{aligned}$$

3. Valuations

The following definition of valuation was introduced by W. Krull in 1932. It is the one given in most books on commutative algebra (see for example [9]).

A *valued field* with values in Γ is a field $\mathbb{K}$ together with a map $\operatorname{val}\colon \mathbb{K} \to \Gamma \cup \{\infty\}$ called *valuation* such that

1. $\operatorname{val} x = \infty \Leftrightarrow x = 0$,
2. $\operatorname{val}(xy) = \operatorname{val} x + \operatorname{val} y$ for all $x, y \in \mathbb{K}$, and
3. $\operatorname{val}(x+y) \ge \min\{\operatorname{val} x, \operatorname{val} y\}$.

Example (3.1). Consider $\mathbb{K} = \mathbb{C}(t,s)$ the field of two variable complex rational functions and val the only valuation with $\operatorname{val}(t^i s^j) = (i,j)$. That is:

$$\operatorname{val} \frac{\sum_{i,j} a_{i,j} t^i s^j}{\sum_{i,j} b_{i,j} t^i s^j} := \min_{a_{i,j} \neq 0}(i,j) - \min_{b_{i,j} \neq 0}(i,j)$$

where min is taken using the lexicographical order and we set $\min \emptyset = \infty$.

For example, $\operatorname{val}(3t^2s + 7ts^7 + 2ts^3 + t^5) = (1,3)$.

LEMMA (3.2). *Let $E \subset \mathbb{K}$ be a finite set. If $\sum_{\varphi \in E} \varphi = 0$ then the set of elements where the valuation attains its minimum has at least two elements.*

Proof. Let F be the subset of E consisting of elements where the valuation attains its minimum. We have $F = \{\varphi \in E \mid \operatorname{val} \varphi = \min_{\varphi \in E} \operatorname{val} \varphi\}$. Therefore $\sum_{\varphi \in E} \varphi = 0$ implies $\operatorname{val}\left(\sum_{\varphi \in F} \varphi\right) > \min_{\varphi \in E} \operatorname{val} \varphi$. And then $\sum_{\varphi \in E} \varphi = 0 \Rightarrow \#F \ge 2$. □

4. The tropicalization of a polynomial.

Let $(\mathbb{K}, \mathrm{val})$ be a valued field with values in a group Γ and let $\mathbb{G}$ be the tropical semi-ring induced by Γ.

A Laurent polynomial in N variables with coefficients in $\mathbb{K}$,

$$f \in \mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}],$$

is a finite sum of the form:

$$f = \sum_{\alpha \in \Lambda \subset \mathbb{Z}^N} \varphi_\alpha x^\alpha \qquad \varphi_\alpha \in \mathbb{K}, \quad x^\alpha = {x_1}^{\alpha_1} \cdots {x_N}^{\alpha_N}. \tag{4.1}$$

The *set of exponents* of f is the set

$$\mathcal{E}(f) := \{\alpha \in \mathbb{Z}^N \mid \varphi_\alpha \neq 0\}.$$

The set of exponents of $f \in \mathbb{K}[x]$ is a finite set contained in $\mathbb{Z}^N$.

The polynomial f via the valuation val induces an element of $\mathbb{G}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$

$$\mathcal{T}f := \sum_{\alpha \in \Lambda \subset \mathbb{Z}^N} \mathrm{val}(\varphi_\alpha) x^\alpha.$$

This polynomial will be called *the tropicalization of* f.

Example (4.2). For $f = (t+s)xy^2 + t^4x^2 + s^6$ we have $\mathcal{T}f = (0,1)xy^2 + (4,0)x^2 + (0,6)$.

Definition (4.3). The *tropical hypersurface* associated to a polynomial $f \in \mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ is the hypersurface associated to the tropicalization of f:

$$\mathbf{TV}f := \mathcal{V}\mathcal{T}f.$$

Example (4.4). For $f = (t+s)xy^2 + t^4x^2 + s^6$ we have

$$\begin{aligned} \mathbf{TV}P = \; & \{((-2,3),(1,1)) + \gamma(0,1) \mid \gamma \geq (0,0)\} \\ \cup \; & \{((-2,3),(1,1)) + \gamma(-2,1) \mid \gamma \geq (0,0)\} \\ \cup \; & \{((-2,3),(1,1)) + \gamma(-2,-1) \mid \gamma \geq (0,0)\}. \end{aligned}$$

5. Weighted order and initial parts

Let $f \in \mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}] \setminus \{0\}$ be as in (4.1). For $\gamma \in \Gamma^N$ the γ-*order* of f is the element of Γ

$$\mathrm{ord}_\gamma f := \min_{\alpha \in \mathcal{E}(f)} \mathrm{val}(\varphi_\alpha) + \alpha \cdot \gamma$$

and $\mathrm{ord}_\gamma 0 := \infty$.

That is $\mathrm{ord}_\gamma f = \mathcal{T}f(\gamma)$.

Example (5.1). For $\gamma = ((1,1),(0,-1)) \in \left(\mathbb{R}^2\right)^2$ and $P = (t+s)xy^2 + t^4x^2 + s^6 \in \mathbb{C}(t,s)[x,y]$. We have $\mathrm{ord}_\gamma P = \min\{(1,0),(6,2),(0,6)\} = (0,6)$.

Remark (5.2). Consider a binomial $x^\alpha - \varphi$ with $\varphi \in \mathbb{K}$

$$\mathrm{ord}_\gamma (x^\alpha - \varphi) = \begin{cases} \gamma \cdot \alpha & \text{if} \quad \gamma \cdot \alpha \leq \mathrm{val}\, \varphi \\ \mathrm{val}\, \varphi & \text{if} \quad \mathrm{val}\, \varphi \leq \gamma \cdot \alpha. \end{cases}$$

Remark (5.3). $\mathrm{ord}_\gamma \colon \mathbb{K}[x] \longrightarrow \Gamma$ is a valuation.

The γ-*initial part of* f is the polynomial

$$\mathrm{In}_\gamma f := \sum_{\mathrm{val}(\varphi_\alpha)+\gamma\cdot\alpha=\mathrm{ord}_\gamma f} \varphi_\alpha x^\alpha .$$

Example (5.4). For $\gamma = ((1,1),(0,-1)) \in \left(\mathbb{R}^2\right)^2$ and $P = (t+s)xy^2+t^4x^2+s^6 \in \mathbb{C}(t,s)[x,y]$. We have $\mathrm{In}_\gamma P = s^6$.

Remark (5.5).

$$\mathrm{In}_\gamma (x^\alpha - \varphi) = \begin{cases} x^\alpha & \text{if } \ \gamma\cdot\alpha < \mathrm{val}\,\varphi \\ x^\alpha - \varphi & \text{if } \ \gamma\cdot\alpha = \mathrm{val}\,\varphi \\ -\varphi & \text{if } \ \mathrm{val}\,\varphi < \gamma\cdot\alpha. \end{cases}$$

Remark (5.6). $\mathrm{In}_\gamma gh = \mathrm{In}_\gamma \left(\mathrm{In}_\gamma g \, \mathrm{In}_\gamma h\right)$.

LEMMA (5.7). *Let* Γ *be an ordered group and let* $\mathbb{K}$ *be a valued field with values in* Γ. *Given a polynomial* $f \in \mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ *the tropical hypersurface defined by* f *is the subset of* Γ^N

$$\mathbf{TV}f = \{\gamma \in \Gamma^N \mid \mathrm{In}_\gamma f \quad \textit{is not a monomial}\}.$$

6. Tropical Varieties.

Definition (6.1). Let $\mathcal{I} \subset \mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ be an ideal. The set of common zeroes of the elements of $\mathcal{I}$ is called the *algebraic variety* defined by $\mathcal{I}$. That is

$$\mathbf{V}(\mathcal{I}) := \{x \in \mathbb{K}^N \mid f(x) = 0 \quad \forall\, x \in \mathcal{I}\}.$$

Definition (6.2). Let Γ be an ordered group and let $\mathbb{K}$ be a valued field with values in Γ. Given an ideal $\mathcal{I} \subset \mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ the *tropical variety defined by* $\mathcal{I}$ is the subset of Γ^N

$$\mathbf{TV}\mathcal{I} := \bigcap_{f\in\mathcal{I}} \mathbf{TV}f .$$

Let $\mathcal{I}$ be an ideal of $\mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ and $\gamma \in \Gamma^N$. The γ-*initial part of* $\mathcal{I}$ is the set of γ-initial parts of its elements:

$$\mathrm{In}_\gamma \mathcal{I} = \{\mathrm{In}_\gamma f \mid f \in \mathcal{I}\} .$$

By lemma (5.7) we have

$$\mathbf{TV}\mathcal{I} = \{\gamma \in \Gamma^N \mid \mathrm{In}_\gamma \mathcal{I} \quad \text{does not contain a monomial}\}.$$

7. The values of a discrete affine variety

PROPOSITION (7.1). *Let* f *be a polynomial in* $\mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$. *If* $\phi = (\phi_1, \ldots, \phi_N) \in \mathbb{K}^N$ *is a zero of* f, *then* $\gamma = \mathrm{val}\,\phi := (\mathrm{val}\,\phi_1, \ldots, \mathrm{val}\,\phi_N)$ *is in the tropical hypersurface defined by* f.

Proof. For $f = \sum \varphi_\alpha x^\alpha$ we have $\sum \varphi_\alpha \phi^\alpha = 0$. Then, by lemma (3.2), the set of elements α where $\mathrm{val}\,\varphi_\alpha\phi^\alpha$ attains its minimum has at least two elements. Since $\mathrm{ord}_\gamma \varphi_\alpha x^\alpha = \mathrm{val}\,\varphi_\alpha\phi^\alpha$ we have that $\mathrm{In}_\gamma f$ has at least two summands. Hence $\mathrm{In}_\gamma f$ is not a monomial. $\square$

THEOREM (7.2). *Let* $(\mathbb{K}, \mathrm{val})$ *be an algebraically closed valued field with values in* Γ.

Let $\mathcal{I} \subset \mathbb{K}[x_1, x_1^{-1}, \ldots, x_N, x_N^{-1}]$ *be an ideal with a finite number of algebraic zeros. We have that*

$$\mathbf{TV}\mathcal{I} = \mathrm{val}(\mathbf{V}\mathcal{I})\,.$$

Proof. Set

$$\mathbf{V}\mathcal{I} = \{\phi^{(1)}, \ldots, \phi^{(s)}\} \subset \mathbb{K}^N\,.$$

We want to see that

$$\mathbf{TV}\mathcal{I} = \{\mathrm{val}\,\phi^{(1)}, \ldots, \mathrm{val}\,\phi^{(s)}\}\,.$$

The fact that $\mathbf{TV}\mathcal{I} \supset \{\mathrm{val}\,\phi^{(1)}, \ldots, \mathrm{val}\,\phi^{(s)}\}$ is direct consequence of proposition (7.1).

Now suppose that, for $i = 1, \ldots, s$, $\gamma \neq \gamma^{(i)} := \mathrm{val}\,\phi^{(i)}$. Let $j_i \in \{1, \ldots, N\}$ be such that $\gamma_{j_i} \neq {\gamma^{(i)}}_{j_i}$ and set

$$g := \prod_{i=1}^{s} \left(y_{j_i} - {\phi^{(i)}}_{j_i} \right)\,.$$

Since g vanishes on $\mathbf{V}(\mathcal{I})$ there exists $k \in \mathbb{N}$ with $g^k \in \mathcal{I}$.

Now, by remarks (5.5) and (5.6), $\mathrm{In}_\gamma\, g^k$ is a monomial. This implies that γ is not in the tropical variety. □

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CHARACTERISTIC SUBGROUPS ARE NOT PRESERVED BY ISOMORPHISMS OF TABLES OF MARKS

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ABSTRACT. We construct two non-isomorphic groups G and Q of order 96 which have isomorphic tables of marks, but such that the centre of G is mapped to a non-characteristic subgroup of Q under this isomorphism of tables of marks.

1. Introduction

Groups with isomorphic tables of marks may not be isomorphic groups (as proved by Thévenaz in [5]), but one still expects them to have many attributes in common. Indeed, if G and Q are groups with isomorphic tables of marks, then they have isomorphic composition factors (see [2]), and they also have isomorphic Burnside rings (the converse is still an open problem, put forward also in [2]); if two groups have isomorphic Burnside rings and one of them is abelian/Hamiltonian/minimal simple, then the two groups are isomorphic (see [3]), and a similar result is known for several families of simple groups (see [1]).

It is also easy to prove that an isomorphism between tables of marks preserves normal subgroups, maximal subgroups, Sylow p-subgroups, cyclic subgroups, elementary abelian subgroups, the commutator subgroup, and the Frattini subgroup. However, it has been shown that abelian subgroups and the centres of the groups are not always preserved. In this paper we show that characteristic subgroups may not be preserved under an isomorphism between tables of marks.

2. Tables of marks

Let G be a finite group. Let $\mathfrak{C}(G)$ be the family of all conjugacy classes of subgroups of G. We usually assume that the elements of $\mathfrak{C}(G)$ are ordered non-decreasingly. The matrix whose H, K-entry is $\#(G/K)^H$ (that is, the number of fixed points of the set G/K under the action of H) is called the *table of marks* of G (where H, K run through all the elements in $\mathfrak{C}(G)$).

The *Burnside ring* of G, denoted $B(G)$, is the subring of $\mathbb{Z}^{\mathfrak{C}(G)}$ spanned by the columns of the table of marks of G.

Definition (2.1). Let G and Q be finite groups. Let ψ be a function from $\mathfrak{C}(G)$ to $\mathfrak{C}(Q)$. Given a subgroup H of G, we denote by H' any representative of

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$\psi([H])$. We say that ψ is an *isomorphism between the tables of marks of G and Q* if ψ is a bijection and if $\#(Q/K')^{H'} = \#(G/K)^{H}$ for all subgroups H, K of G.

3. Two non-isomorphic groups of order 96 with isomorphic tables of marks

This is a summary of [4].

Let S_3 be the symmetric group or order 6. Let C_8 be the cyclic group of order 8, generated by x, and let C_2 be the cyclic group of order 2, generated by y.

Let δ be the only non-trivial homomorphism from S_3 to C_8. Let W denote the group $S_3 \times C_8$. Let α be the automorphism of W given by $\alpha(\lambda, x^i) = (\lambda, x^i\delta(\lambda))$, and let β be the automorphism of W given by $\beta(\lambda, x^i) = (\lambda, x^{5i}\delta(\lambda))$.

Since α has order two, we can define the group G as the semidirect product of W with C_2 by α, that is, in G we have that $y(\lambda, x^i)y = \alpha(\lambda, x^i)$. Similarly, we define the group Q as the semidirect product of W and C_2 by β; in Q we have that $y(\lambda, x^i)y = \beta(\lambda, x^i)$. We shall denote the elements of both G and Q as $\lambda x^i y^j$.

Note that in G, x and y commute, and the centre of G is therefore the subgroup generated by x, which is a subgroup of order 8; however, x and y do not commute in Q, and the centre of Q is the subgroup generated by x^2, which is a subgroup of order 4. In particular, we also have that G and Q are non-isomorphic groups of order 96.

The following theorem can be found in [4].

THEOREM (3.1). *Let S be a subset of G (and therefore S is also a subset of Q). Then S is a subgroup of G if and only if S is a subgroup of Q. Moreover, two subgroups are conjugate in G if and only if they are conjugate in Q, and the identity map on the family of conjugacy classes of subgroups defines an isomorphism between the tables of marks of G and Q.*

We use this fact to prove our main result.

THEOREM (3.2). *The subgroup of Q generated by x is not a characteristic subgroup. In particular, the isomorphism of tables of marks between G and Q maps the centre of G to a non-characteristic subgroup of Q.*

Proof. We construct an automorphism of Q that does not preserve the subgroup generated by x. Let $\eta : Q \longrightarrow Q$ be given by

$$\eta(\lambda x^i y^j) = \lambda x^{3i+6i^2+\left(1-Sgn(\lambda)\right)(2i+3)} y^{i+j+\frac{1-Sgn(\lambda)}{2}} .$$

We claim that for a generator g of Q and an arbitrary $\lambda x^i y^j$ we have that $\eta(g\lambda x^i y^j) = \eta(g)\eta(\lambda x^i y^j)$, where g can be $(1, 2), (1, 2, 3), x, y$, so η is indeed a homomorphism.

$g = (1, 2)$:

$$\begin{aligned}\eta\big((1,2)(\lambda x^i y^j)\big) &= \eta\big((1,2)\lambda x^i y^j\big)\\ &= (1,2)\lambda x^{5i+6i^2-2iSgn((1,2)\lambda)-3Sgn((1,2)\lambda)+3} y^{i+j+\frac{1-Sgn((1,2)\lambda)}{2}}\\ &= (1,2)\lambda x^{5i+6i^2+2iSgn(\lambda)+3Sgn(\lambda)+3} y^{i+j+\frac{1+Sgn(\lambda)}{2}} .\end{aligned}$$

On the other hand:

$$\begin{aligned}\eta\big((1,2)\big)\eta\big(\lambda x^i y^j\big) &= \big((1,2)x^6y\big)\left(\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}y^{i+j+\frac{1-Sgn(\lambda)}{2}}\right)\\ &= (1,2)\lambda x^{6+5[5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3]+2(1-Sgn(\lambda))}y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}\\ &= (1,2)\lambda x^{i+6i^2-2iSgn(\lambda)-Sgn(\lambda)+7}y^{1+i+j\frac{1-Sgn(\lambda)}{2}}.\end{aligned}$$

These two expressions coincide, because:

$$\begin{aligned}\big(5i+2iSgn(\lambda)+3Sgn(\lambda)+3\big)-\big(i-2iSgn(\lambda)-Sgn(\lambda)+7\big)\\ &= 4i+4iSgn(\lambda)+4Sgn(\lambda)+4\\ &= 4\big((1+Sgn(\lambda))(i+1)\big).\end{aligned}$$

$g=(1,2,3)$:

$$\begin{aligned}\eta\big((1,2,3)(\lambda x^i y^j)\big) &= \eta\big((1,2,3)\lambda x^i y^j\big)\\ &= (1,2,3)\lambda x^{5i+6i^2-2iSgn\big((1,2,3)\lambda\big)-3Sgn\big((1,2,3)\lambda\big)+3}y^{i+j+\frac{1-Sgn\big((1,2,3)\lambda\big)}{2}}\\ &= (1,2,3)\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}y^{i+j+\frac{1-Sgn(\lambda)}{2}}.\end{aligned}$$

On the other hand:

$$\begin{aligned}\eta\big((1,2,3)\big)\eta\big(\lambda x^i y^j\big) &= \big((1,2,3)\big)\left(\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}y^{i+j+\frac{1-Sgn(\lambda)}{2}}\right)\\ &= (1,2,3)\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}.\end{aligned}$$

$g=y$:

$$\begin{aligned}\eta(y\lambda x^i y^j) &= \eta(\lambda x^{5i+2-2Sgn(\lambda)}y^{1+j})\\ &= \lambda x^{5[5i+2-2Sgn(\lambda)]+6[5i+2-2Sgn(\lambda)]^2-2[5i+2-2Sgn(\lambda)]Sgn(\lambda)-3Sgn(\lambda)+3}y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}\\ &= \lambda x^{i+6i^2-2iSgn(\lambda)-Sgn(\lambda)+1}y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}.\end{aligned}$$

On the other hand:

$$\begin{aligned}\eta(y)\eta(\lambda x^i y^j) &= y\left(\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}y^{i+j+\frac{1-Sgn(\lambda)}{2}}\right)\\ &= \lambda x^{5[5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3]+2[1-Sgn(\lambda)]}y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}\\ &= \lambda x^{i+6i^2-2iSgn(\lambda)-Sgn(\lambda)+1}y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}.\end{aligned}$$

$g=x$

$$\begin{aligned}\eta(x\lambda x^i y^j) &= \eta(\lambda x^{1+i}y^j)\\ &= \lambda x^{5(1+i)+6(1+i)^2-2(1+i)Sgn(\lambda)-3Sgn(\lambda)+3}\cdot y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}\\ &= \lambda x^{i+6i^2-2iSgn(\lambda)-5Sgn(\lambda)+6}y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}.\end{aligned}$$

On the other hand:

$$\begin{aligned}\eta(x)\eta(\lambda x^i y^j) &= (xy)\left(\lambda x^{5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3}y^{i+j+\frac{1-Sgn(\lambda)}{2}}\right)\\ &= \lambda x^{1+5[5i+6i^2-2iSgn(\lambda)-3Sgn(\lambda)+3]+2(1-Sgn(\lambda))}y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}\\ &= \lambda x^{2+i+6i^2-2iSgn(\lambda)-Sgn(\lambda)}y^{1+i+j+\frac{1-Sgn(\lambda)}{2}}.\end{aligned}$$

These two expressions coincide because:

$$\big(6 - 5Sgn(\lambda)\big) - \big(2 - Sgn(\lambda)\big) = 4(1 - Sgn(\lambda)).$$

Therefore η is a group homomorphism.

Moreover,

$$\begin{aligned} (1,2) &= \eta((1,2)x^6y), & (1,2,3) &= \eta(1,2,3), \\ x &= \eta(xy), & y &= \eta(y) \end{aligned}$$

so η must be an automorphism. Finally, note that $\eta(x) = xy$, so the subgroup generated by x is not a characteristic subgroup of Q. □

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AN EXAMPLE OF A TWISTED FUSION ALGEBRA

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ABSTRACT. The aim of this paper is to exhibit an explicit non-trivial example of the twisted fusion algebra for a particular finite group. The product is defined for the group $G = (\mathbb{Z}/2)^3$ via the pairing ${}^{\theta_g(\phi)}R(G) \otimes^{\theta_h(\phi)} R(G) \to^{\theta_{gh}(\phi)} R(G)$ where $\theta\colon H^4(G,\mathbb{Z}) \to H^3(G,\mathbb{Z})$ is the inverse transgression map and ϕ is a carefully chosen cocycle class. We find the rank of the fusion algebra $\mathcal{X}(G) = \sum_{g\in G} {}^{\theta_g(\phi)}R(G)$ as well as the relation between its basis elements. We also give some applications to topological gauge theories.

1. Introduction

Inspired by the Chen-Ruan cohomology for orbifolds, it has been shown by Adem, Ruan, Zhang [3] that there is also an internal product in twisted orbifold K-theory ${}^{\alpha}K_{orb}(\mathcal{X})$. The information determining this stringy product lies in $H^4(B\mathcal{X},\mathbb{Z})$ instead of $H^3(B\mathcal{X},\mathbb{Z})$: Given a class $\phi \in H^4(B\mathcal{X},\mathbb{Z})$, it induces a class $\theta(\phi) \in H^3(B\wedge\mathcal{X},\mathbb{Z})$ where $\bigwedge\mathcal{X}$ is the inertia stack. As a result one can define a twisted K-theory on ${}^{\theta(\phi)}K(\bigwedge\mathcal{X})$. The map θ can be thought as the inverse of the classical transgression map.

This construction of this internal product is motivated by the so-called Pontryagin product on $K_G(G)$, for a finite group G. Indeed, if the orbifold is $\mathcal{X} = \bigwedge[*/G]$ one gets the same product for the orbifold K-theory in the untwisted case. There is also an explicit calculation of the inverse transgression map θ for the cohomology of finite groups (see [3]). Using this result we exhibit a non-trivial product structure in the case of $G = (\mathbb{Z}/2)^3$. We use an integral cohomology class $\phi \in H^4(G,\mathbb{Z})$ such that under the inverse transgression it maps non-trivially for every twisted sector, yielding a product structure on the algebra ${}^{\theta(\phi)}K_G(G) = \mathcal{X}(G) = \sum_{g\in G} {}^{\theta_g(\phi)}R(G)$ defined via the pairing ${}^{\theta_g(\phi)}R(G) \otimes^{\theta_h(\phi)} R(G) \to^{\theta_{gh}(\phi)} R(G)$. In this paper we derive the relations between the basis elements of the algebra $\mathcal{X}(G)$ and we prove the uniqueness of this product in this particular case. $G = (\mathbb{Z}/2)^3$ is indeed the abelian group of smallest rank such that it has non-trivial transgressions (see [3], section 5).

These twisted rings have also been worked out in the conformal field theory literature. In [12], the modular invariant (i.e. S and T matrices) of this group $G = (\mathbb{Z}/2)^3$ is calculated. As a result, one can calculate the relations between basis elements of $\mathcal{X}(G)$ using the Verlinde formula. Moreover, the same example is considered in [5] where a decomposition formula for twisted K-theory is given and the product is calculated after tensoring by rational numbers.

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There is also a physical counterpart of this theory. In [8] Dijkgraaf and Witten defined a correspondence between Chern-Simons theories in three dimensions and Wess-Zumino terms in two dimensions via a natural map from $H^4(G,\mathbb{Z})$ to $H^3(G,\mathbb{Z})$. In our case this map is the inverse transgression map, which is actually the map coming from the correspondence between the Chern-Simons action and the Wess-Zumino terms that arise in connecting a specific three dimensional quantum field theory to its related two dimensional quantum field theory. One can see that the Chern-Simons theory associates to each group element $g_i \in G$ a 2-cocycle β_i of the stabilizer group N_{g_i}, which is $G = (\mathbb{Z}/2)^3$ in our abelian case. We use the formulations in [8] to calculate the partition function $Z(S^1 \times S^1 \times S^1)$. It is also worth mentioning that the algebra $\mathcal{X}(G)$ corresponds to a fusion algebra in this physical context.

In this paper we first give some preliminaries and the definition of our fusion algebra. In the second section, we calculate the rank and the uniqueness of this algebra as well as the relation between the basis elements which are the projective representations of $G = (\mathbb{Z}/2)^3$. Finally, we give an application to topological gauge theories by using the formulation in [8].

2. The Twisted Fusion Product for Finite Groups

In this section we review a special case for the product in twisted orbifold K-theory which is defined by Adem, Ruan and Zhang in [3]. We consider the inertia orbifold $\bigwedge[*/G]$ where G is a finite group. In this case the untwisted orbifold K-theory of $\bigwedge[*/G]$ is simply $K_G(G)$, which is additively isomorphic to $\sum_{(g)} R(Z_G(g))$, where $Z_G(g)$ denotes the centraliser of g in G, and the sum is taken over conjugacy classes. The product in $K_G(G)$ is defined as follows. An equivariant vector bundle over G can be thought of as a collection of finite dimensional vector spaces V_g with a G-module structure on $\sum_{g\in G} V_g$ such that $gV_h = V_{ghg^{-1}}$. The product is defined as

$$(V \star W)_g = \bigoplus_{g_1 g_2 = g} V_{g_1} \otimes W_{g_2}.$$

One can give an alternative definition. We first define the maps $e_1\colon G \times G \to G$, $e_2\colon G \times G \to G$ and $e_{12}\colon G \times G \to G$ as $e_1(g,h) = g$, $e_2(g,h) = h$ and $e_{12}(g,h) = gh$ respectively, which are G-equivariant when G acts by conjugation in all coordinates. If α, β are elements in $K_G(G)$ the product is defined as

$$\alpha \star \beta = e_{12*}(e_1^*(\alpha) e_2^*(\beta)).$$

We now need to review the inverse transgression map for finite groups to extend latter definition to twisted K-theory. In order to define the product in twisted K-theory, Adem, Ruan and Zhang [3] define a map to match up the levels which appear in the twistings. This cochain map θ is called inverse transgression map and it induces a homomorphism

$$\theta_*\colon H^{k+1}(B\mathcal{G},\mathbb{Z}) \to H^k(B\wedge \mathcal{G},\mathbb{Z}).$$

If the orbifold $\mathcal{G}$ is $[*/G]$ where G is a finite group, the inverse transgression has a classical interpretation in terms of shuffle product. Recall that $\bigwedge[*/G]$ is equivalent to $\bigsqcup_{(g)}[*/Z_G(g)]$ (see [3]). Hence we would like to focus on the map

$\theta_g : C^k(G, U(1)) \to C^{k-1}(Z_G(g), U(1))$. If G is a finite group then the cochain complex $C^*(G, U(1))$ is in fact equal to $\mathrm{Hom}_G(B_*(G), U(1))$, where $B_*(G)$ is the bar resolution for G (see [6], page 18). If t is the generator of $\mathbb{Z}$ the shuffle product is $B_k(Z_G(g)) \otimes B_1(\mathbb{Z}) \to B_{k+1}(G)$ given by

$$[g_1|g_2|\ldots|g_k] \star [t^i] = \sum_\sigma \sigma[g_1|g_2|\ldots|g_k|g_{k+1}],$$

where $g_{k+1} = g^i$, σ ranges over all $(k, 1)$–shuffles and

$$\sigma[g_1|g_2|\ldots|g_{k+1}] = (-1)^{sign(\sigma)}[g_{\sigma(1)}|g_{\sigma(2)}|\ldots|g_{\sigma(k+1)}].$$

A $(k, 1)$-shuffle is an element in symmetric group $\sigma \in S_{k+1}$ such that $\sigma(i) < \sigma(j)$ for $1 \le i < j \le k$.

We can dualize this using integral coefficients. Given a cocycle $\phi \in C^{k+1}(G, \mathbb{Z})$, one can see that the inverse transgression $\theta_g(\phi) \in C^k(Z_G(G), \mathbb{Z})$ can be defined as

$$\theta_g(\phi)([g_1|g_2|\ldots|g_k]) = \phi([g_1|g_2|\ldots|g_k] \star [g]),$$

where $g_1, g_2, \ldots, g_k \in G$. Hence it induces a map in integral cohomology.

We can now induce the inverse transgression map for $H^*(G, \mathbf{F}_2)$ in the case of $G = (\mathbb{Z}/2)^3$ using the Bockstein homomorphism. We want to find a non-trivial cocycle in the image of inverse transgression map. Notice that $H^*(G, \mathbf{F}_2)$ is a polynomial algebra on three degree one generators x, y and z. In general, for an elementary abelian 2-group, the mod 2 reduction map for $k > 0$ is a monomorphism $H^k(G, \mathbb{Z}) \to H^k(G, \mathbf{F}_2)$ whose image is the kernel of the Bockstein homomorphism $Sq^1 : H^k(G, \mathbf{F}_2) \to H^{k+1}(G, \mathbf{F}_2)$. In order to get a nontrivial cocycle in the image of the inverse transgression map we choose $\alpha = Sq^1(xyz) = x^2yz + xy^2z + xyz^2$ which represents a non-square element in $H^4(G, \mathbb{Z})$. The following lemma is proved in [3] by analyzing the multiplication map in cohomology.

LEMMA (2.1). *Let $g = x^a y^b z^c$ be an element in $G = (\mathbb{Z}/2)^3$, where we are writing it in terms of the standard basis (identified with its dual). Let us consider $\alpha = Sq^1(xyz) = x^2yz + xy^2z + xyz^2$ which represents a non-square element in $H^4(G, \mathbb{Z})$. Then*

$$\theta_g^*(\alpha) = a(y^2z + z^2y) + b(x^2z + xz^2) + c(x^2y + xy^2)$$

and so it is non-zero on every component except the one corresponding to the trivial element in G.

Proof. See [3], lemma 5.2. □

This implies that for all $g, h \in G$, $\theta_g^* + \theta_h^* = \theta_{gh}^*$ in cohomology up to coboundaries. This also implies that the correspondence $g \mapsto \theta_g(\alpha)$ defines a homomorphism. In the case of $G = (\mathbb{Z}/2)^3$ we have the isomorphism $\theta_?(\alpha) : G \to H^3(G, \mathbb{Z}) = G$.

We now define the product as follows:

Definition (2.2). Let τ be a 2-cocycle for the orbifold defined by the conjugation action of a finite group G on itself which is in the image of the inverse transgression. The product on ${}^\tau K_G(G)$ is defined by the following formula: if $\alpha, \beta \in {}^\tau K_G(G)$, then

$$\alpha \star \beta = e_{12*}(e_1^*(\alpha) e_2^*(\beta)).$$

If $\tau = \theta(\phi)$ then we have the following formula proved in [3]:

$$e_1^*\tau + e_2^*\tau = e_{12}^*\tau$$

up to coboundary. Hence the product $e_1(\alpha)e_2(\beta)$ lies in

$$^{e_1^*\tau+e_2^*\tau}K_G(G) =^{e_{12}^*\tau} K_G(G)\,.$$

Applying e_{12*} this is mapped to $^{\tau}K_G(G)$ which gives the product in the twisted K-theory.

Using this identification $\theta_g^* + \theta_h^* = \theta_{gh}^*$ the following product is defined on the algebra

$$^{\theta(\phi)}K_G(G) = \mathcal{X}(G) = \sum_{g\in G} {}^{\theta_g(\phi)}R(G)$$

via the pairing

$$^{\theta_g(\phi)}R(G) \otimes {}^{\theta_h(\phi)}R(G) \to {}^{\theta_{gh}(\phi)}R(G)\,.$$

As a result of the formula $e_{12*}(e_1^*(\alpha)e_2^*(\beta))$ this pairing turns out to be the tensor product of projective representations. In the next section we investigate the properties of this algebra while calculating its rank and the relations between the irreducible projective representations.

3. Calculations

(3.1) 2-cocycles in G with values in $U(1)$. In the rest of the paper we will always assume that $G = (\mathbb{Z}/2)^3$. Recall that for finite dimensional complex vector space a mapping $\rho\colon G \to GL(V)$ is called a *projective representation* of G if there exists a $U(1)$ valued 2-cocycle $\alpha \in Z^2(G; U(1))$ such that $\rho(x)\rho(y) = \alpha(x, y)\rho(x, y)$ for all $x, y \in G$ and $\rho(1) = Id_V$. Hence in order to compute $^{\theta_g(\phi)}R(G)$ we first need to find the 2-cocycles in $H^2(G, U(1))$ corresponding to $\theta_g(\phi)$ in $H^3(G, \mathbb{Z})$ where both cohomology groups are isomorphic to G. For this purpose we consider the following isomorphism

$$H^2(G, U(1)) \to H^3(G, \mathbb{Z})$$

induced by the natural coefficient sequence $0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 1$. As $H^3(G, \mathbb{Z}) \cong G$ we need to find the 8 non-cohomologous 2-cocycles in $H^2(G, U(1))$ corresponding to each $\theta_g(\phi)$ for all $g \in G$.

We now determine the relations in order to obtain the 2-cocycles in $C^2(G, U(1))$. Any 2-cocycle β in $C^2(G, U(1))$ should satisfy:

$$\delta\beta = 1\,.$$

By the boundary formula of the bar resolution of G we derive:

$$\beta(g_2, g_3)\beta(g_1g_2, g_3)^{-1}\beta(g_1, g_2g_3)\beta(g_1, g_2)^{-1} = 1$$

for all $g_i \in G$, $i = 1, 2, 3$. To get some neat relations we plug in $g_1 = g_3 = g$ and $g_2 = 1$ into this formula and we get

$$\beta(g, 1) = \beta(1, g). \tag{3.1.1}$$

Moreover for $g_1 = g_2 = g$ we have

$$\beta(1, g_3)\beta(g, g) = \beta(g, g_3)\beta(g, gg_3). \tag{3.1.2}$$

As β is defining a projective representation, say ρ, it should satisfy $\rho(1)\rho(g) = \beta(1,g)\rho(g)$. This implies $\beta(1,g) = 1$ for all $g \in G$. Now we consider the following tables for 2-cocycles $\beta_i\colon G \times G \to U(1)$ which satisfies the identities (3.1.1) and (3.1.2). We choose β_1 as the trivial co-cycle. We call these cocycles *fundamental cocycles*. Here x_i, y_i and z_i's are in $U(1)$ and they will be determined later.

β_2	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	1	x_1	x_2	x_1	x_2	x_3	x_3
g_3	1	x_1	-1	x_4	$-x_1$	x_6	$-x_4$	$-x_6$
g_4	1	x_2	$-x_4$	1	x_5	x_2	$-x_4$	x_5
g_5	1	x_1	$-x_1$	$-x_5$	-1	x_8	$-x_8$	x_5
g_6	1	x_2	$-x_6$	x_2	$-x_8$	1	$-x_8$	$-x_6$
g_7	1	x_3	x_4	x_4	x_8	x_8	1	x_3
g_8	1	x_3	x_6	$-x_5$	$-x_5$	x_6	x_3	1

β_3	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	-1	y_1	y_2	$-y_1$	$-y_2$	y_3	$-y_3$
g_3	1	y_1	1	y_4	y_1	y_6	y_4	y_6
g_4	1	$-y_2$	y_4	1	y_5	$-y_2$	y_4	y_5
g_5	1	$-y_1$	y_1	$-y_5$	-1	y_8	$-y_8$	y_5
g_6	1	y_2	y_6	y_2	$-y_8$	1	$-y_8$	y_6
g_7	1	$-y_3$	y_4	y_4	y_8	y_8	1	$-y_3$
g_8	1	y_3	y_6	$-y_5$	$-y_5$	y_6	y_3	1

β_4	1	g_2	g_3	g_4	g_5	g_6	g_7	g_8
1	1	1	1	1	1	1	1	1
g_2	1	-1	z_1	z_2	$-z_1$	$-z_2$	z_3	$-z_3$
g_3	1	$-z_1$	1	z_4	z_1	z_6	z_4	z_6
g_4	1	z_2	z_4	-1	z_5	$-z_2$	$-z_4$	$-z_5$
g_5	1	z_1	z_1	z_5	1	z_8	z_8	z_5
g_6	1	$-z_2$	z_6	$-z_2$	$-z_8$	1	$-z_8$	z_6
g_7	1	$-z_3$	z_4	$-z_4$	$-z_8$	z_8	-1	z_3
g_8	1	z_3	z_6	$-z_5$	z_5	$-z_6$	$-z_3$	-1

As the multiplication of two cocycles gives us another cocycle, one can construct 5 more cocycles one of which is the trivial one. Recall that for $\beta \in Z^2(G;U(1))$ an element $g \in G$ is called β-regular if $\beta(g,x) = \beta(x,g)$ for all $x \in C_G(g)$ (see [9], page 107). Thus all of the 2-cocycles have 2 β-regular elements one of which is 1 the other one is different for each cocycles. For example, one can immediately see that the β-regular elements for β_2, β_3 and β_4 are g_2, g_3 and g_4, respectively.

On the other hand from the boundary formula a 2-coboundary should satisfy $\beta(g_1, g_2) = \sigma(g_1)\sigma(g_2)\sigma(g_1g_2)^{-1}$ where σ is in $C^1(G, \mathbb{Z})$. As G is abelian we have

$$\beta(g_i, g_j) = \beta(g_j, g_i)$$

for all g_i and g_j in G. This implies that all of these 8 cocycles represents different cohomology classes as multiplication with a coboundary does not change the β-regular elements. Thus, we have following proposition.

PROPOSITION (3.1.3). *The rank of* ${}^{\theta_g(\phi)}R(G)$ *are* 2 *if* $\theta_g(\phi)$ *is nontrivial.*

Proof. A basic result of projective representations states that ${}^{\alpha}R(G)$ is a free abelian group of rank equal to the number of distinct α-regular conjugacy classes of G (see [9], theorem 6.7). So, the ranks of ${}^{\beta_i}R(G)$ is 2 for non-trivial β_i's.

On the other hand we obtained 8 non-cohomologous cocycles which should correspond to $\theta_g(\phi)$'s because $H^2(G, U(1))$ and $H^3(G, \mathbb{Z})$ are isomorphic to G. The result follows. $\square$

We can therefore conclude:

COROLLARY (3.1.4). *The rank of* $\mathcal{X}(G)$ *is equal to* 22.

(3.2) The projective representations. In order to compute the irreducible projective representations of G it is helpful to determine the x_i, y_i and z_i's. Again from the boundary formula we have the following relations in β_2.

$$\begin{aligned} -1 &= x_1x_3x_4x_5 \\ -1 &= x_2x_3x_5x_8 \\ 1 &= x_6x_8x_3x_1 \,. \end{aligned}$$

By a routine calculation one can check that the other relations depend on these three relations. We can choose $x_1 = x_2 = x_3 = x_4 = -x_5 = x_6 = x_7 = x_8 = 1$ that obviously satisfy these relations. Similarly, we find y_i's and z_i's. The other cocycles are computed by multiplying β_2, β_3 and β_4. We will later show that the choice of x_i, y_j and z_k from the set $\{\pm 1\}$ does not change our representations.

By considering the 2-cocycles that we obtained it is obvious that there is no 1-dimensional projective representation whenever the 2-cocycle is not trivial. For the trivial cocycle we have eight 1 dimensional representations which are just the irreducible linear representations of G. For the other cases we find two 2-dimensional irreducible representations for each of the two cocycles (see [9], Theorem 6.7). Let ρ_1^i and ρ_2^i be the irreducible representations corresponding to the cocycle β_i.

One question that needs to be answered is whether these representations depend on the choice of x_i, y_i and z_i. One can check by calculation that the representations only depend on the values of $\beta_i(g, g)$. More precisely, $\rho(g)\rho(g)$ should be equal to $\beta(g, g)$ for all $g \in G$. For example,

$$\begin{aligned} & & \rho(g_3)\rho(g_8) &= \beta(g_3, g_8)\rho(g_7) \\ &\Leftrightarrow & \rho(g_3)\beta(g_2, g_7)\rho(g_2)\rho(g_7) &= \beta(g_3, g_8)\beta(g_2, g_4)\rho(g_2)\rho(g_4) \\ &\Leftrightarrow & \rho(g_3)\beta(g_2, g_7)\rho(g_2)\beta(g_3, g_4)\rho(g_3)\rho(g_4) &= \beta(g_3, g_8)\beta(g_2, g_4)\rho(g_2)\rho(g_4) \\ &\Leftrightarrow & \rho(g_3)\rho(g_3)\beta(g_2, g_7)\beta(g_3, g_4) &= \beta(g_3, g_8)\beta(g_2, g_4)I \end{aligned}$$

which is true if and only if $\rho(g_3)\rho(g_3) = \beta(g_3, g_3)I$ by the relations we get from the boundary formulas. The other elements can be checked similarly.

Thus we have found the basis of our algebra. We will show that this product is unique up to coboundary. First we prove the following lemma.

PROPOSITION (3.2.1). *If ϕ and ϕ' are cohomologous cocycles in $H^4(G, \mathbb{Z})$ then the fusion algebras corresponding to these cocycles are isomorphic to each other.*

Proof. If ϕ and ϕ' are cohomologous cocycles then $\theta_g(\phi)$ and $\theta_g(\phi')$ represent the same cohomology class in $H^3(G, \mathbb{Z})$. In order to compute the 2-cocycle corresponding to $\theta_g(\phi) \in H^3(G, \mathbb{Z}) \cong G$ we will use the isomorphism induced from the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 1\,.$$

Thus $\theta_g(\phi)$ is mapped to a certain class of 2-cocycles in $H^2(G, U(1)) \cong G$. As we found 8 non-cohomologous 2-cocycles in G it is enough to check how the representations change if we multiply our fundamental 2-cocycles by a 2-coboundary. This is indeed a basic result of projective representation theory (see page 72 in [9]). After multiplying our fundamental 2-cocycles by a 2-coboundary the new projective representation of this cocycle becomes linearly isomorphic to the former one. The result follows from the above argument. $\square$

(3.3) The relations. Now we are able to calculate the relation of this basis using the pairing ${}^{\theta(\phi)_g}R(G) \otimes {}^{\theta(\phi)_h}R(G) \to {}^{\theta(\phi)_{gh}}R(G)$. The calculations are nothing but solving linear equations. Namely one should prove that $\rho_i^k \otimes \rho_i^l$'s are linearly isomorphic to a sum of some basis elements. Here ρ_i^1 denotes the irreducible regular representations of G for $i = 1, 2, \ldots, 8$.

Let us start with $\rho_i^1 \otimes \rho_i^j$ which is linearly isomorphic to ρ_k^j for some $k \in \{1, 2\}$ where $j \neq 1$ because $\rho_i^1 \otimes \rho_i^j$ should be a 2 dimensional β_j representation. Here are the results of these multiplications:

$\otimes$	ρ_1^1	ρ_2^1	ρ_3^1	ρ_4^1	ρ_5^1	ρ_6^1	ρ_7^1	ρ_8^1	ρ_1^2	ρ_2^2	ρ_1^3	ρ_2^3	ρ_1^4	ρ_2^4	ρ_1^5	ρ_2^5	ρ_1^6	ρ_2^6	ρ_1^7	ρ_2^7	ρ_1^8	ρ_2^8
ρ_1^1	ρ_1^1	ρ_2^1	ρ_3^1	ρ_4^1	ρ_5^1	ρ_6^1	ρ_7^1	ρ_8^1	ρ_1^2	ρ_2^2	ρ_1^3	ρ_2^3	ρ_1^4	ρ_2^4	ρ_1^5	ρ_2^5	ρ_1^6	ρ_2^6	ρ_1^7	ρ_2^7	ρ_1^8	ρ_2^8
ρ_2^1	ρ_2^1	ρ_1^1	ρ_5^1	ρ_6^1	ρ_3^1	ρ_4^1	ρ_8^1	ρ_7^1	ρ_2^2	ρ_1^2	ρ_1^3	ρ_2^3	ρ_1^4	ρ_2^4	ρ_2^5	ρ_1^5	ρ_2^6	ρ_1^6	ρ_1^7	ρ_2^7	ρ_2^8	ρ_1^8
ρ_3^1	ρ_3^1	ρ_5^1	ρ_1^1	ρ_7^1	ρ_2^1	ρ_8^1	ρ_4^1	ρ_6^1	ρ_1^2	ρ_2^2	ρ_2^3	ρ_1^3	ρ_1^4	ρ_2^4	ρ_2^5	ρ_1^5	ρ_1^6	ρ_2^6	ρ_2^7	ρ_1^7	ρ_2^8	ρ_1^8
ρ_4^1	ρ_4^1	ρ_6^1	ρ_7^1	ρ_1^1	ρ_8^1	ρ_2^1	ρ_3^1	ρ_5^1	ρ_1^2	ρ_2^2	ρ_1^3	ρ_2^3	ρ_2^4	ρ_1^4	ρ_1^5	ρ_2^5	ρ_2^6	ρ_1^6	ρ_2^7	ρ_1^7	ρ_2^8	ρ_1^8
ρ_5^1	ρ_5^1	ρ_3^1	ρ_2^1	ρ_8^1	ρ_1^1	ρ_7^1	ρ_6^1	ρ_4^1	ρ_2^2	ρ_1^2	ρ_2^3	ρ_1^3	ρ_1^4	ρ_2^4	ρ_1^5	ρ_2^5	ρ_2^6	ρ_1^6	ρ_2^7	ρ_1^7	ρ_1^8	ρ_2^8
ρ_6^1	ρ_6^1	ρ_4^1	ρ_8^1	ρ_2^1	ρ_7^1	ρ_1^1	ρ_5^1	ρ_3^1	ρ_2^2	ρ_1^2	ρ_1^3	ρ_2^3	ρ_2^4	ρ_1^4	ρ_2^5	ρ_1^5	ρ_1^6	ρ_2^6	ρ_2^7	ρ_1^7	ρ_1^8	ρ_2^8
ρ_7^1	ρ_7^1	ρ_8^1	ρ_4^1	ρ_3^1	ρ_6^1	ρ_5^1	ρ_1^1	ρ_2^1	ρ_1^2	ρ_2^2	ρ_2^3	ρ_1^3	ρ_2^4	ρ_1^4	ρ_2^5	ρ_1^5	ρ_2^6	ρ_1^6	ρ_1^7	ρ_2^7	ρ_1^8	ρ_2^8
ρ_8^1	ρ_8^1	ρ_7^1	ρ_6^1	ρ_5^1	ρ_4^1	ρ_3^1	ρ_2^1	ρ_1^1	ρ_2^2	ρ_1^2	ρ_2^3	ρ_1^3	ρ_2^4	ρ_1^4	ρ_1^5	ρ_2^5	ρ_1^6	ρ_2^6	ρ_1^7	ρ_2^7	ρ_2^8	ρ_1^8

Another type of multiplication is $\rho_i^j \otimes \rho_i^j$ which is linearly isomorphic to the sum of four irreducible regular representations ρ_k^1 as $\beta_j(g, h)^2 = 1$ for all j. We calculate all of these by using associativity of our algebra $\mathcal{X}(G)$ and investigating the eigenvalues of the matrices. Of course one can also calculate them by defining the linear isomorphism explicitly. As $\rho_1^i(g_j) = -\rho_2^i(g_j)$ for three g_j's in the definitions of representations we have $\rho_1^i \otimes \rho_1^i = \rho_2^i \otimes \rho_2^i$ as well as $\rho_2^i \otimes \rho_1^i = \rho_1^i \otimes \rho_2^i$.

$\otimes$	ρ_1^2	ρ_2^2
ρ_1^2	$\rho_1^1+\rho_3^1+\rho_4^1+\rho_7^1$	$\rho_2^1+\rho_5^1+\rho_6^1+\rho_8^1$
ρ_2^2	$\rho_2^1+\rho_5^1+\rho_6^1+\rho_8^1$	$\rho_1^1+\rho_3^1+\rho_4^1+\rho_7^1$

$\otimes$	ρ_1^3	ρ_2^3
ρ_1^3	$\rho_1^1+\rho_2^1+\rho_4^1+\rho_6^1$	$\rho_3^1+\rho_5^1+\rho_7^1+\rho_8^1$
ρ_2^3	$\rho_3^1+\rho_5^1+\rho_7^1+\rho_8^1$	$\rho_1^1+\rho_2^1+\rho_4^1+\rho_6^1$

$\otimes$	ρ_1^4	ρ_2^4
ρ_1^4	$\rho_4^1+\rho_6^1+\rho_7^1+\rho_8^1$	$\rho_1^1+\rho_2^1+\rho_3^1+\rho_5^1$
ρ_2^4	$\rho_1^1+\rho_2^1+\rho_3^1+\rho_5^1$	$\rho_4^1+\rho_6^1+\rho_7^1+\rho_8^1$

$\otimes$	ρ_1^5	ρ_2^5
ρ_1^5	$\rho_1^1+\rho_5^1+\rho_6^1+\rho_8^1$	$\rho_2^1+\rho_3^1+\rho_4^1+\rho_7^1$
ρ_2^5	$\rho_2^1+\rho_3^1+\rho_4^1+\rho_7^1$	$\rho_1^1+\rho_5^1+\rho_6^1+\rho_8^1$

$\otimes$	ρ_1^6	ρ_2^6
ρ_1^6	$\rho_1^1+\rho_3^1+\rho_6^1+\rho_8^1$	$\rho_2^1+\rho_4^1+\rho_5^1+\rho_7^1$
ρ_2^6	$\rho_2^1+\rho_4^1+\rho_5^1+\rho_7^1$	$\rho_1^1+\rho_3^1+\rho_6^1+\rho_8^1$

$\otimes$	ρ_1^7	ρ_2^7
ρ_1^7	$\rho_3^1+\rho_4^1+\rho_5^1+\rho_6^1$	$\rho_1^1+\rho_2^1+\rho_7^1+\rho_8^1$
ρ_2^7	$\rho_1^1+\rho_2^1+\rho_7^1+\rho_8^1$	$\rho_3^1+\rho_4^1+\rho_5^1+\rho_6^1$

$\otimes$	ρ_1^8	ρ_2^8
ρ_1^8	$\rho_2^1+\rho_3^1+\rho_4^1+\rho_8^1$	$\rho_1^1+\rho_5^1+\rho_6^1+\rho_7^1$
ρ_2^8	$\rho_1^1+\rho_5^1+\rho_6^1+\rho_7^1$	$\rho_2^1+\rho_3^1+\rho_4^1+\rho_8^1$

The last type of multiplication that we have to consider is $\rho_i^j \otimes \rho_m^n$ where distinct i, m are in $\{1,2\}$ and j and n are in $\{2,3,\dots,8\}$. As $\rho_i^j \otimes \rho_m^n$ is four dimensional it should be linearly isomorphic to $2\rho_1^l$, $2\rho_2^l$ or $\rho_1^l+\rho_2^l$. Neither $-2\rho_1^l$, $-2\rho_2^l$ nor $-\rho_1^l-\rho_2^l$ is possible as they are not β_l representations as one checks from the list of our representations in the previous section. Besides, $2\rho_1^l$ and $2\rho_2^l$ are also impossible by the following associativity argument.

Suppose $\rho_i^j \otimes \rho_m^n = 2\rho_1^l$. By the table above we can always find ρ_k^1 such that $\rho_k^1 \otimes \rho_i^j = \rho_i^j$ and $\rho_k^1 \otimes \rho_1^l = \rho_2^l$. This gives us a contradiction if we multiply each side of $\rho_i^j \otimes \rho_m^n = 2\rho_1^l$ by ρ_k^1.

We can conclude $\rho_i^j \otimes \rho_m^n = \rho_1^l + \rho_2^l$. We have finished calculating all the relations.

4. Topological Gauge Theories

Dijkgraaf and Witten show that three dimensional Chern-Simons gauge theories with a compact gauge group can be classified by the integer cohomology group $H^4(BG,\mathbb{Z})$. Wess-Zumino interactions of such groups G are classified by $H^3(G,\mathbb{Z})$. The relation between three dimensional sigma models involves a certain natural map $H^4(BG,\mathbb{Z})$ to $H^3(G,\mathbb{Z})$ which is the inverse transgression map defined in the second section. Our calculations provide an example of three dimensional topological theories with finite gauge group. In this context our algebra $\mathcal{X}(G)$ is a fusion algebra. In QFT (Quantum field theory) one can associate to a $d+1$ dimensional manifold M a certain number $Z(M)$, the *partition function*. For a detailed discussion one can consult [8].

Consider the partition function of the 3-torus $S^1 \times S^1 \times S^1$. If g, h and k are three commuting gauge fields the partition function is evaluated to give

$$Z(S^1 \times S^1 \times S^1) = \frac{1}{|G|} \sum_{g,h,k \in G} W(g,h,k),$$

where $[g,h] = [h,k] = [k,g] = 1$ which is not important in our abelian case. Define W as

$$W(g,h,k) = \frac{\alpha(g,h,k)\alpha(h,k,g)\alpha(k,g,h)}{\alpha(g,k,h)\alpha(h,g,k)\alpha(k,h,g)}.$$

for $\alpha \in H^3(BG, U(1))$.

The Chern-Simons theory associates to each group element $g_i \in G$ a 2-cocycle β_i, which we calculated above. Again by the result of Witten and Dijkgraaf [8] we can express β_i in terms of 3-cocycle $\alpha \in H^3(G, U(1))$:

$$\beta_i(h_1,h_2) = \frac{\alpha(g_i,h_1,h_2)\alpha(h_1,h_2,g_i)}{\alpha(h_1,g_i,h_2)}.$$

This can also be obtained by the formula for inverse transegression map on page 3:

$$\beta_i(h_1,h_2) = \theta_{g_i}(\alpha)([h_1|h_2]) = \alpha([h_1|h_2] \star [g_i]) = \frac{\alpha([g_i|h_1|h_2])\alpha([h_1|h_2|g_i])}{\alpha([h_1|g_i|h_2])}.$$

Note that the shuffle product on page 3 is defined via additive notation.

Thus the action W can be written in terms of 2-cocycles:

$$W(g_i,h,k) = \beta_i(h,k)\beta_i(k,h)^{-1}.$$

For fixed g, $\epsilon_g(h)$ is defined as $\epsilon_g(h) = \beta(g,h)\beta(h,g)^{-1}$ which is a 1-dimensional representation of G. Thus an element g is β-regular iff $\epsilon_g = 1$. This implies

$$r(G,\beta) = \frac{1}{|G|} \sum_{g,h \in G} \beta(g,h)\beta(h,g)^{-1}$$

where $r(G;\beta)$ denotes the number of irreducible projective representations ${}^{\beta}R(G)$. Our 2-cocycles defined in the second section satisfies this condition.

Comparing all these results we obtain the following result for the partition function of the 3-torus where $G = (\mathbb{Z}/2)^3$:

PROPOSITION (4.1).

$$Z(S^1 \times S^1 \times S^1) = \sum_i r(G;\beta_i) = 22.$$

Using our representations we can find the basis elements ν_α of Hilbert space corresponding to a 3-torus in QFT. These basis elements are given in [8] as

$$\nu_\alpha(g_i,h) = Tr\rho_i(h).$$

In this context our algebra $\mathcal{X}(G)$ can be regarded as the smallest twisted non-trivial fusion algebra for abelian groups.

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GEOMETRIC DIFFERENCES BETWEEN THE USE OF LIE ALGEBRAS AND LIE SUPERALGEBRAS

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ABSTRACT. Let $V = U \oplus W$ be a complex vector space. Then $\mathrm{End}(U \oplus W)$ has natural Lie algebra and Lie superalgebra structures. With given geometries $B_U : U \times U \to \mathbb{C}$, and $B_W : W \times W \to \mathbb{C}$, a geometry B can be defined on V via $B_U \oplus B_W$. We address the question of what determines the choice in using the Lie algebra or the Lie superalgebra structure of $\mathrm{End}(U \oplus W)$ by considering the linear maps that preserve B. It is found that if nontrivial maps $U \to W$ and $W \to U$ are to be included, then the Lie algebra structure of $\mathrm{End}(U \oplus W)$ requires geometries on U and W of the same type —that is, both orthogonal, or both symplectic, or both unitary, or both anti-unitary— whereas the Lie superalgebra requires to combine the geometry-types of U and W in such a way that one is orthogonal and the other symplectic, or one is unitary and the other anti-unitary. The question of defining a geometry B on $U \oplus W$ of *odd degree* is also addressed, and the Lie algebra and Lie superalgebra structures of the subspace of $\mathrm{End}(U \oplus W)$ that preserve such a B are determined.

Introduction

The purpose of this note is to identify exactly to what extent *the geometry* defined on a given vector space formed as the direct sum of two subspaces U and W, *decides* what is the best-suited algebraic structure to use in $\mathrm{End}(U \oplus W)$: either its Lie algebra structure, or its Lie superalgebra structure. It turns out that if a geometry B is defined on $U \oplus W$ via $B = B_U \oplus B_W$, where B_U and B_W are geometries on U, and W, respectively, the Lie algebra structure on $\mathrm{End}(U \oplus W)$ closes nicely with nontrivial maps $U \to W$ and $W \to U$ that are consistent with the given geometries on them, exactly when these geometries are of the same type; namely, both orthogonal, or both symplectic, or both unitary, or both anti-unitary. On the other hand, the Lie superalgebra structure of $\mathrm{End}(U \oplus W)$ provides nontrivial maps $U \to W$ and $W \to U$ consistent with the geometries on the subspaces, exactly when these geometries are mixed in the following way: B_U and B_W are one symplectic and the other orthogonal, or one unitary and the other anti-unitary. From this point of view both, the Lie algebra and the Lie superalgebra structures of $\mathrm{End}(U \oplus W)$ are natural, and somehow complementary to each other, depending only on the way the geometries on U and W are combined. Even though the proof is elementary (see §2 below), this observation makes the result quite illuminating. Besides, the Lie subalgebra and the Lie 'subsuperalgebra' of $\mathrm{End}(U \oplus W)$ that preserve the resulting B,

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both have $\mathfrak{g}_{B_U} \oplus \mathfrak{g}_{B_W} \oplus \mathrm{Hom}(U, W)$ as its underlying space, where $\mathfrak{g}_{B_Z}$ is the Lie algebra $\{T \in \mathrm{End}(Z) \mid B_Z(Tu, v) + B_Z(u, Tv) = 0\}$.

Now, in the Lie superalgebra setting, geometries on $U \oplus W$ of the form $B = B_U \oplus B_W$ are called *even*, or of zero $\mathbb{Z}_2$-degree. There are also the so called *odd* geometries that are defined on $U \oplus W$ when $U \simeq W$, so as to make U and W totally isotropic. To round up this note, we review the conditions to define a geometry B on $U \oplus U$ in terms of a given geometry $B_U : U \times U \to \mathbb{C}$, in such a way that the U direct summands in $U \oplus U$ become totally isotropic with respect to B. It turns out that in order to preserve such a geometry defined by B through elements from $\mathrm{End}(U \oplus U)$, either from its Lie algebra structure or from its Lie superalgebra structure, we deduce the existence of a constant λ satisfying $\lambda^2 = 1$ when B_U is bilinear and $|\lambda|^2 = 1$ when B_U is sesquilinear, and the overall geometry B on $U \oplus U$ is given by $B(u+w, u'+w') = B_U(u, w') + \lambda\, B_U(u', w)$, when B_U is bilinear, and is given by $B(u+w, u'+w') = B_U(u, w') + \bar{\lambda}\, \overline{B_U(u', w)}$, when B_U is sesquilinear. In particular, the overall geometry is orthogonal, or symplectic, depending on whether B_U is bilinear and $\lambda = 1$, or $\lambda = -1$, respectively, and it is unitary, if B_U is sesquilinear. It is quite interesting to note that in any case the geometry defined by B under the imposed conditions becomes isomorphic to the geometry defined on $U \oplus U^*$ by the well-known natural orthogonal, symplectic, or unitary forms. Finally, the underlying space of the Lie subalgebra of $\mathrm{End}(U \oplus U^*)$ that preserves B has the form $\mathfrak{gl}(U) \oplus (A^\lambda(U) \oplus A^\lambda(U))$, with $A^\lambda(U)$ being identified with either the skew-symmetric or the symmetric maps $U \to U$, depending on whether B is orthogonal or symplectic, and being identified with the hermitian (or skew-hermitian) maps $U \to U$, when B is unitary. On the other hand, the underlying space of the Lie subsuperalgebra of $\mathrm{End}(U \oplus U^*)$ that preserves B gets decomposed as $\mathfrak{gl}(U) \oplus (A^\lambda(U) \oplus B^\lambda(U))$, with $A^\lambda(U)$ and $B^\lambda(U)$ being respectively identified with the symmetric and skew-symmetric maps $U \to U$, when B is either orthogonal or symplectic, and being respectively identified with the hermitian and skew-hermitian maps, when B is unitary.

1. Algebraic preliminaries and notation

(1.1) The setting. Let V be a complex vector space with the given direct sum decomposition $U \oplus W$. Then, the associative algebra $\mathrm{End}(U \oplus W)$ gets decomposed into the direct sum $\mathrm{End}(U) \oplus \mathrm{Hom}(U, W) \oplus \mathrm{Hom}(W, U) \oplus \mathrm{End}(W)$ in such a way that a linear transformation $T \in \mathrm{End}(U \oplus W)$ corresponds to the quadruple $(\alpha, \beta, \gamma, \delta)$, as follows: For any $u \in U$ and any $w \in W$, $T(u + w) = (\alpha(u) + \beta(w)) + (\gamma(u) + \delta(w))$, with $\alpha(u) + \beta(w) \in U$, and $\gamma(u) + \delta(w) \in W$. If $T' \in \mathrm{End}(U \oplus W)$ corresponds to the quadruple $(\alpha', \beta', \gamma', \delta')$, then $T \circ T'$ corresponds to $(\alpha \circ \alpha' + \beta \circ \gamma', \alpha \circ \beta' + \beta \circ \delta', \gamma \circ \alpha' + \delta \circ \gamma', \gamma \circ \beta' + \delta \circ \delta')$, which provides the well-known identifications

$$U \oplus W \ni u + w \;\leftrightarrow\; \begin{pmatrix} u \\ w \end{pmatrix} \qquad \text{and} \qquad \mathrm{End}(U \oplus W) \ni T \;\leftrightarrow\; \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

so that

$$T(u+v) \;\leftrightarrow\; \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \qquad \text{and} \qquad T \circ T' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \circ \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

The associative algebra $\mathrm{End}(U \oplus W)$ admits an obvious $\mathbb{Z}_2$-grading decomposition (*cf,* [2]) $\mathrm{End}(U \oplus W) = \mathrm{End}(U \oplus W)_0 \oplus \mathrm{End}(U \oplus W)_1$, where

$$\mathrm{End}(U \oplus W)_0 = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \mid \alpha \in \mathrm{End}(U), \text{ and, } \delta \in \mathrm{End}(W) \right\} \qquad \text{and}$$

$$\mathrm{End}(U \oplus W)_1 = \left\{ \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix} \mid \beta \in \mathrm{Hom}(W, U), \text{ and, } \gamma \in \mathrm{Hom}(U, W) \right\}.$$

Identifying $\mathbb{Z}_2$ with the set $\{0, 1\}$ and equipping it with its usual ring structure, it is immediate to check that for any r and s in $\mathbb{Z}_2$, $(\mathrm{End}(U \oplus W))_r \circ (\mathrm{End}(U \oplus W))_s \subset (\mathrm{End}(U \oplus W))_{r+s}$. We further define a map

$$|\cdot| : (\mathrm{End}(U \oplus W))_0 \cup (\mathrm{End}(U \oplus W))_1 - \left\{ \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right) \right\} \to \mathbb{Z}_2$$

in such a way that $|T| = r$ if and only if $T \in (\mathrm{End}(U \oplus W))_r$. Moreover, there are only two possibilities for defining a map $\varepsilon : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{C}$ in such a way that $\varepsilon(r, s)\varepsilon(r, t) = \varepsilon(r, s + t)$, $\varepsilon(r, s)\varepsilon(t, s) = \varepsilon(r + t, s)$, and $\varepsilon(r, s)\varepsilon(s, r) = 1$ (*cf,* [1]); namely, either $\varepsilon(r, s) = 1$ for all r, s, or else $\varepsilon(r, s) = (-1)^{rs}$ for all r, s. It is well known that defining

$$[T, T']_\varepsilon = T \circ T' - \varepsilon(|T|, |T'|)\, T' \circ T,$$

on any pair of elements T and T' on the domain of $|\cdot|$, and extending this definition bilinearly to all of $\mathrm{End}(U \oplus W)$, one obtains either a Lie algebra structure, or a Lie superalgebra structure on $\mathrm{End}(U \oplus W)$, depending on whether $\varepsilon(r, s) = 1$ for all r, s, or $\varepsilon(r, s) = (-1)^{rs}$ for all r, s (*cf,* [1], or [2]). It is immediate to verify that the difference between these structures comes down to

$$\begin{aligned}
\left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \right]_\varepsilon &= \begin{pmatrix} \alpha \circ \alpha' + \beta \circ \gamma' & \alpha \circ \beta' + \beta \circ \delta' \\ \gamma \circ \alpha' + \delta \circ \gamma' & \gamma \circ \beta' + \delta \circ \delta' \end{pmatrix} \\
&\qquad - \begin{pmatrix} \alpha' \circ \alpha + |\varepsilon|\, \beta' \circ \gamma & \alpha' \circ \beta + \beta' \circ \delta \\ \gamma' \circ \alpha + \delta' \circ \gamma & |\varepsilon|\, \gamma' \circ \beta + \delta' \circ \delta \end{pmatrix} \\
&= \begin{pmatrix} [\alpha, \alpha'] + \beta \circ \gamma' - |\varepsilon|\, \beta' \circ \gamma & \alpha \circ \beta' - \beta' \circ \delta + \beta \circ \delta' - \alpha' \circ \beta \\ \delta \circ \gamma' - \gamma' \circ \alpha + \gamma \circ \alpha' - \delta' \circ \gamma & [\delta, \delta'] + \gamma \circ \beta' - |\varepsilon|\, \gamma' \circ \beta \end{pmatrix},
\end{aligned}$$

where we have written $|\varepsilon|$ in the right hand side to distinguish the case $|\varepsilon| = 1$ obtained when $\varepsilon(r, s) = 1$ for all r, s, from the case $|\varepsilon| = -1$ obtained when $\varepsilon(r, s) = (-1)^{rs}$ for all r, s. Note that $[\,\cdot\,,\,\cdot\,]$ with no further marks stands for the ordinary Lie algebra bracket on $\mathrm{End}(U)$, or $\mathrm{End}(W)$, respectively, and the context makes it clear which one is being used. In summary, the vector space $\mathrm{End}(U \oplus W)$ can be equipped with either a Lie algebra structure (case $|\varepsilon| = 1$), or with a Lie superalgebra structure (case $|\varepsilon| = -1$). When referring to the first one, we shall denote it by $\mathfrak{gl}(U \oplus W)$, and when referring to the second one, by $\mathfrak{gl}(U|W)$, as it is customary.

(1.2) Notation and conventions. Let X be either U or W, and let $B\colon X \times X \to \mathbb{C}$ be a geometry defined on it. Thus, either B is nondegenerate bilinear or nondegenerate sesquilinear. Let X^* be the dual space. We shall follow the convention of letting $B^\flat\colon X \to X^*$ be given by $x \mapsto B(x, \,\cdot\,)$, so that $B^\flat$ becomes a $\mathbb{C}$-antilinear map when B is sesquilinear. The map $B^\flat$ has a left inverse (which is also $\mathbb{C}$-antilinear when B is sesquilinear) given by $B^\sharp\colon X^* \to X$ and characterized by the property $B(B^\sharp(\varphi), x) = \varphi(x)$ for any $\varphi \in X^*$ and any $x \in X$.

If B is bilinear, $B^\flat$ and $B^\sharp$ are both $\mathbb{C}$-linear. If X is finite-dimensional, $B^\flat$ and $B^\sharp$ are inverses of each other. For any $\mathbb{C}$-linear map $\beta\colon X \to Y$ we denote by β^* the $\mathbb{C}$-linear map $Y^* \to X^*$ given by $\varphi \mapsto \beta^*(\varphi) = \varphi \circ \beta$. We shall use the same notation, $\beta^*\colon Y^* \to X^*$, even when $\beta\colon X \to Y$ is $\mathbb{C}$-antilinear, but in this case β^* is given by $\varphi \mapsto \overline{\varphi \circ \beta}$. It is a well-known fact that when B is bilinear (resp., sesquilinear), the subset

$$\mathfrak{g}_B(X) = \{\eta \in \mathrm{End}(X) \mid B^\flat \circ \eta + \eta^* \circ B^\flat = 0\}$$

is a complex (resp., real) subspace of $\mathrm{End}(X)$ which is furthermore, closed under the Lie bracket $[\eta, \eta'] = \eta \circ \eta' - \eta' \circ \eta$. It is thus a complex (resp., real) Lie subalgebra of $\mathfrak{gl}(X)$.

Now, in the *bilinear case* we will assume that there exists a nonzero complex constant, ε_B, such that,

$$B(x_1, x_2) = \varepsilon_B B(x_2, x_1)$$

for any pair of vectors x_1, and x_2 in X. This readily implies that ε_B must be either $+1$ or -1. It also implies that, $\mathfrak{g}_B(X) \simeq \mathfrak{o}(n)$, or $\mathfrak{g}_B(X) \simeq \mathfrak{sp}(n)$, depending on whether ε_B is either $+1$ or -1; in any case, $n = \dim(X)$. On the other hand, in the *sesquilinear case* we will assume that there exists a nonzero complex constant ε_B, such that

$$B(x_1, x_2) = \varepsilon_B \overline{B(x_2, x_1)},$$

for any pair of vectors x_1, and x_2 in X. In particular, this implies that ε_B lies in the unit circle of the complex plane. It also implies that there is an hermitian form $H\colon X \times X \to \mathbb{C}$ and a complex constant ζ depending on ε_B only, such that $H = \zeta\, B$; actually $\bar{\zeta}^2 = \varepsilon_B$ (and similarly, one may obtain an anti-hermitian form $H' = iH = i\zeta\, B$). In any case $\mathfrak{g}_B(X)$ is isomorphic to the unitary Lie algebra $\mathfrak{u}$ of the appropriate hermitian form H. *Convention:* In what follows it will always be assumed that when B is sesquilinear, one can find an appropriate $\zeta = \zeta(\varepsilon_B)$ in the unit circle of the complex plane so as to make $H = \zeta B$ hermitian, or so as to consider the associated anti-hermitian form $H' = iH$, as needed.

2. Statement of the problems

(2.1) The so called 'even' geometries. Let $V = U \oplus W$ as before, and assume we are given nondegenerate bilinear maps $B_U : U \times U \to \mathbb{C}$, and $B_W : W \times W \to \mathbb{C}$, satisfying $B_U(u, u') = \varepsilon_{B_U} B_U(u', u)$, and $B_W(w, w') = \varepsilon_{B_W} B_W(w', w)$, respectively, or nondegenerate sesquilinear maps satisfying $B_U(u, u') = \varepsilon_{B_U} \overline{B_U(u', u)}$, and $B_W(w, w') = \varepsilon_{B_W} \overline{B_W(w', w)}$, respectively. Define $B\colon V \times V \to \mathbb{C}$ by means of

$$B(u + w, u' + w') = B_U(u, u') + B_W(w, w') \tag{2.1.1}$$

for all $u, u' \in U$ and all $w, w' \in W$. Let $T \in \mathrm{End}(U \oplus W)$ be identified with $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{End}(U) \oplus \mathrm{End}(W) \oplus \mathrm{Hom}(U, W) \oplus \mathrm{Hom}(W, U)$ as before. One may then consider, on the one hand, the vector subspace

$$\mathfrak{g}_B(U \oplus W) = \left\{T = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathfrak{gl}(U \oplus W) \mid B^\flat \circ T + T^* \circ B^\flat = 0\right\} \tag{2.1.2}$$

(with $B^\flat = B^\flat_U \oplus B^\flat_W$). On the other hand, given $T \in \mathrm{End}(U \oplus W)$, we may first decompose it in the form $T = T_0 + T_1$ with $T_0 = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right) \in (\mathrm{End}(U \oplus W))_0$, and $T_1 = \left(\begin{smallmatrix} 0 & \beta \\ \gamma & 0 \end{smallmatrix}\right) \in (\mathrm{End}(U \oplus W))_1$, and consider the Lie subsuperalgebra $\mathfrak{g}_B(U|W) = \mathfrak{g}_B(U|W)_0 \oplus \mathfrak{g}_B(U|W)_1$, where

(2.1.3)

$$\mathfrak{g}_B(U|W)_0 = \left\{ T = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \delta \end{smallmatrix}\right) \in \mathfrak{gl}(U|W) \mid B^\flat \circ T\left(\begin{smallmatrix} u \\ w \end{smallmatrix}\right) + T^* \circ B^\flat\left(\begin{smallmatrix} u \\ w \end{smallmatrix}\right) = 0 \right\},$$

$$\mathfrak{g}_B(U|W)_1 = \left\{ T = \left(\begin{smallmatrix} 0 & \beta \\ \gamma & 0 \end{smallmatrix}\right) \in \mathfrak{gl}(U|W) \mid B^\flat \circ T\left(\begin{smallmatrix} u \\ w \end{smallmatrix}\right) + T^* \circ B^\flat\left(\begin{smallmatrix} u \\ -w \end{smallmatrix}\right) = 0 \right\}.$$

We have written the arguments $u \in U$ and $w \in W$ so as to emphasize the fact that the difference with $\mathfrak{g}_B(U \oplus W)$ only occurs in that $B^\flat \circ T$ and $T^* \circ B^\flat$ might be evaluated on different arguments, depending on whether $|T| = 0$ or $|T| = 1$. The customary way to write the defining condition on the elements of $\mathfrak{g}_B(U|W)_\mu$ ($\mu = 0, 1$) is this (*cf,* [2]): $T \in (\mathrm{End}(U \oplus W))_\mu$ such that $B^\flat \circ T(z) + (-1)^{\mu\,|z|} T^* \circ B^\flat(z) = 0$, with the understanding that $|z|$ is equal to either 0 or 1, depending on whether $z \in U$, or $z \in W$, respectively.

We now want to find necessary and sufficient conditions on the geometries B_U and B_W so that $\mathfrak{g}_B(U|W)$ becomes a Lie subalgebra of $\mathfrak{gl}(U \oplus W)$, and $\mathfrak{g}_B(U|W)$ becomes a Lie subsuperalgebra of $\mathfrak{gl}(U|W)$, excluding in both cases the trivial situation in which all maps T satisfy $\beta = 0$, and $\gamma = 0$.

PROPOSITION (2.1.4). *Under the assumption that $B^\flat = B^\flat_U \oplus B^\flat_W$ as above, the linear map $T = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ belongs to:*

(a) $\mathfrak{g}_B(U \oplus W)$ *if and only if its entries α, β, γ, δ satisfy*

$$B^\flat_U \circ \alpha + \alpha^* \circ B^\flat_U = 0, \quad B^\flat_W \circ \delta + \delta^* \circ B^\flat_W = 0, \quad B^\flat_U \circ \beta + \gamma^* \circ B^\flat_W = 0.$$

(b) $\mathfrak{g}_B(U|W)$ *if and only if its entries α, β, γ, δ satisfy*

$$B^\flat_U \circ \alpha + \alpha^* \circ B^\flat_U = 0, \quad B^\flat_W \circ \delta + \delta^* \circ B^\flat_W = 0, \quad B^\flat_U \circ \beta - \gamma^* \circ B^\flat_W = 0.$$

Furthermore, we respectively have:

(a') *If $\varepsilon_{B_U} = \varepsilon_{B_W}$, then* $\mathfrak{g}_B(U \oplus W) \simeq \mathfrak{g}_{B_U}(U) \oplus \mathfrak{g}_{B_W}(W) \oplus \mathrm{Hom}(U, W)$*; otherwise,* $\mathfrak{g}_B(U \oplus W) \simeq \mathfrak{g}_{B_U}(U) \oplus \mathfrak{g}_{B_W}(W)$.

(b') *If $\varepsilon_{B_U} = -\varepsilon_{B_W}$, then* $\mathfrak{g}_B(U|W) \simeq \mathfrak{g}_{B_U}(U) \oplus \mathfrak{g}_{B_W}(W) \oplus \mathrm{Hom}(U, W)$*; otherwise,* $\mathfrak{g}_B(U|W) \simeq \mathfrak{g}_{B_U}(U) \oplus \mathfrak{g}_{B_W}(W)$.

Remark (2.1.5). It follows from Proposition (2.1.4) that the vector space structures of $\mathfrak{g}_B(U \oplus W)$ and $\mathfrak{g}_B(U|W)$ are essentially identical. The actual difference lies in the fact that, for $\mathfrak{g}_B(U \oplus W)$ to be a Lie subalgebra of $\mathfrak{gl}(U \oplus W)$ containing the full subspace $\mathrm{Hom}(U, W)$, the geometries B_U and B_W have to be both orthogonal, or both symplectic, or be both associated to hermitian forms H_U and H_W on U and W, respectively, so as to have $\varepsilon_{B_U} = \varepsilon_{B_W}$. In particular, the direct sum $\mathfrak{g}_{B_U}(U) \oplus \mathfrak{g}_{B_W}(W)$ in this case is isomorphic to the direct sum of two orthogonal, or two symplectic, or two unitary Lie algebras. On the other hand, for $\mathfrak{g}_B(U|W)$ to be a Lie subsuperalgebra of $\mathfrak{gl}(U|W)$ containing the full subspace $\mathrm{Hom}(U, W)$, the geometries B_U and B_W must be one orthogonal and the other symplectic, or be one associated to a hermitian form and the other associated to an anti-hermitian form, so as to satisfy $\varepsilon_{B_U} = -\varepsilon_{B_W}$. The fact that the subspace $\mathrm{Hom}(U, W)$ appears in both $\mathfrak{g}_B(U \oplus W)$ and $\mathfrak{g}_B(U|W)$ reflects the fact that it suffices to know β in order to completely determine γ (or viceversa). The dependence of γ on β is different, depending on whether one is looking at the Lie algebra structure or the Lie superalgebra structue, and of course so does the way of computing the bracket $[\cdot,\cdot]_\varepsilon$, but otherwise they are complementary to each other as far as the geometries defined on U and W are concerned.

Note: All proofs are given in §3 below.

(2.2) The so called 'odd' geometries. When $\dim U = \dim W$, there are other (equally natural and simple) ways to define non-degenerate, bilinear (resp., sesquilinear) maps $B\colon (U \oplus W) \times (U \oplus W) \to \mathbb{C}$; namely, require U and W to become totally isotropic subspaces. Thus, set

$$B(u + w, u' + w') = \Omega(u, w') + \Phi(w, u')$$

with $\Omega\colon U \times W \to \mathbb{C}$, and $\Phi\colon U \times W \to \mathbb{C}$ being both nondegenerate and bilinear (resp., sesquilinear). We may define the $\mathbb{C}$-linear (resp., $\mathbb{C}$-antilinear) maps $\Omega^\flat\colon U \to W^*$ and $\Phi^\flat\colon W \to U^*$, so that $\Omega^\flat(u) = \Omega(u, \cdot\,)$, and $\Phi^\flat(w) = \Phi(w, \cdot\,)$, respectively. In particular, $(\Omega^\flat)^*\colon W \to U^*$, and $(\Phi^\flat)^*\colon U \to W^*$. It is easy to see that $(\Omega^\flat)^*(w)(u) = \Omega(u, w)$ in the bilinear case and $(\Omega^\flat)^*(w)(u) = \overline{\Omega(u, w)}$ in the sesquilinear case. Now set $B^\flat := \Omega^\flat \oplus \Phi^\flat$ (followed by the identification $W^* \oplus U^* \to U^* \oplus W^*$, $\varphi + \psi \mapsto \psi + \varphi$), and look for necessary and sufficient conditions under which the vector subspace (1) yields a Lie subalgebra $\mathfrak{g}_B(U \oplus W)$ of $\mathfrak{gl}(U \oplus W)$ and also look for necessary and sufficient conditions for the subspaces (3) to fit together defining a Lie subsuperalgebra $\mathfrak{g}_B(U|W)$ of $\mathfrak{gl}(U|W)$.

PROPOSITION (2.2.1). *Under the assumption that* $B^\flat := \Omega^\flat \oplus \Phi^\flat$ *as above the linear map* $T = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$ *belongs to:*

(a) $\mathfrak{g}_B(U \oplus W)$ *if and only if its entries* α, β, γ, δ *satisfy*

$$\Phi^\flat \circ \delta + \alpha^* \circ \Phi^\flat = 0, \qquad \Omega^\flat \circ \beta + \beta^* \circ \Phi^\flat = 0, \qquad \Phi^\flat \circ \gamma + \gamma^* \circ \Omega^\flat = 0\,.$$

(b) $\mathfrak{g}_B(U|W)$ *if and only if its entries* α, β, γ, δ *satisfy*

$$\Phi^\flat \circ \delta + \alpha^* \circ \Phi^\flat = 0, \qquad \Omega^\flat \circ \beta - \beta^* \circ \Phi^\flat = 0, \qquad \Phi^\flat \circ \gamma + \gamma^* \circ \Omega^\flat = 0.$$

In any case, there is a constant $\lambda \in \mathbb{C}$, *such that* $(\Omega^\flat)^* = \lambda\,\Phi^\flat$, *and either* $\gamma = \beta = 0$ *or else* $\lambda^2 = 1$ *when* B *is bilinear, and* $|\lambda|^2 = 1$ *when* B *is sesquilinear. Furthermore, we respectively have:*

(a') $\mathfrak{g}_B(U \oplus W) \simeq \mathfrak{gl}(U) \oplus \big(A^\lambda_\Phi(W, U) \oplus A^\lambda_\Phi(U, W)\big)$, *where*

$$A^\lambda_\Phi(W, U) = \{\beta \in \mathrm{Hom}(W, U) \mid \beta^* \circ \Phi^\flat + \bar\lambda(\Phi^\flat)^* \circ \beta = 0\,\},$$
$$A^\lambda_\Phi(U, W) = \{\gamma \in \mathrm{Hom}(U, W) \mid \gamma^* \circ (\Phi^\flat)^* + \lambda\,\Phi^\flat \circ \gamma = 0\,\}.$$

(b') $\mathfrak{g}_B(U|W) \simeq \mathfrak{gl}(U) \oplus \big(A^\lambda_\Phi(U, W) \oplus B^\lambda_\Phi(W, U)\big)$, *where* $A^\lambda_\Phi(U, W)$ *is defined as in* (a'), *and*

$$B^\lambda_\Phi(W, U) = \{\beta \in \mathrm{Hom}(W, U) \mid \beta^* \circ \Phi^\flat - \bar\lambda(\Phi^\flat)^* \circ \beta = 0\,\}.$$

Remark (2.2.2). Since Proposition (2.2.1) applies only in the case $U \simeq W$, we might as well start with a given isomorphism, say $P\colon U \to W$, and define the nondegenerate bilinear form $B_U(u_1, u_2) = \Phi(P(u_1), u_2)$. Since Ω is completely determined by Φ itself, this means that the data to start with can also be a geometry B_U on U, and then define a geometry B on $U \oplus U$ (or in $U \oplus U^*$) in such a way so as to make each direct summand into a totally isotropic subspace. The results of Proposition (2.2.1) remain essentially the same, except for the fact that $\beta^* \circ \Phi^\flat + \bar{\lambda}(\Phi^\flat)^* \circ \beta = 0$, for some $\beta\colon W \to U$, will be replaced by $\beta^* \circ B_U^\flat + \bar{\lambda}\,(B_U^\flat)^* \circ \beta = 0$, this time with $\beta\colon U \to U$, and similarly for γ.

Remark (2.2.3). On the other hand, taking into account that $U \simeq W$, we might as well choose the bases on U and W in such a way that the associated matrix to Φ is simply the unit matrix. We may also assume that λ is ± 1, so that the subspaces $A^\lambda_\Phi(W, U)$ and $A^\lambda_\Phi(U, W)$ in (2.2.1).(a') both become isomorphic to

$$A^\lambda(U) = \{\, \beta \in \mathrm{End}(U) \mid \beta^* + \lambda\beta = 0 \,\},$$

whereas $B^\lambda_\Phi(W, U)$ in (2.2.1).(b') becomes isomorphic to

$$B^\lambda(U) = \{\, \beta \in \mathrm{End}(U) \mid \beta^* - \lambda\beta = 0 \,\}.$$

3. The proofs

(3.1) Proof of Proposition (2.1.4). The linear map $T = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{End}(U \oplus W)$ belongs to $\mathfrak{g}_B(U \oplus W)$ if, and only if, for any $u, \widetilde{u} \in U$ and any $w, \widetilde{w} \in W$,

$$(i)\ (B^\flat \circ T + T^* \circ B^\flat)(u)(\widetilde{u}) = 0, \qquad (ii)\ (B^\flat \circ T + T^* \circ B^\flat)(w)(\widetilde{w}) = 0,$$

$$(iii)\ (B^\flat \circ T + T^* \circ B^\flat)(u)(w) = 0, \qquad (iv)\ (B^\flat \circ T + T^* \circ B^\flat)(w)(u) = 0.$$

It is a straightforward matter to check that

$$\begin{aligned}
(i) &\iff (B_U^\flat \circ \alpha + \alpha^* \circ B_U^\flat)(u)(\widetilde{u}) = 0 \iff B_U^\flat \circ \alpha + \alpha^* \circ B_U^\flat = 0.\\
(ii) &\iff (B_W^\flat \circ \delta + \delta^* \circ B_W^\flat)(w)(\widetilde{w}) = 0 \iff B_W^\flat \circ \delta + \delta^* \circ B_W^\flat = 0.\\
(iii) &\iff (B_W^\flat \circ \gamma + \beta^* \circ B_U^\flat)(u)(w) = 0 \iff B_W^\flat \circ \gamma + \beta^* \circ B_U^\flat = 0.\\
(iv) &\iff (B_U^\flat \circ \beta + \gamma^* \circ B_W^\flat)(w)(u) = 0 \iff B_U^\flat \circ \beta + \gamma^* \circ B_W^\flat = 0.
\end{aligned}$$

Furthermore, (iii) and (iv) must define the same equation for β and γ. From (iii), we have $\beta^* = -B_W^\flat \circ \gamma \circ (B_U^\flat)^{-1}$, whereas from (iv), we obtain, $\beta^* = -(B_W^\flat)^* \circ \gamma \circ ((B_U^\flat)^*)^{-1}$. In order to compare them we must determine the relationship between $(B_U^\flat)^*$ and $B_U^\flat$, and similarly between $(B_W^\flat)^*$ and $B_W^\flat$. We shall work in detail the case when B_U is sesquilinear. The bilinear case is similar and slightly simpler. Now, $B_U^\flat\colon U \to U^*$ is $\mathbb{C}$-antilinear, and so is $(B_W^\flat)^*\colon (U^*)^* \to U^*$. Using the natural identification $i_U\colon U \to (U^*)^*$, it easily follows from the definitions that

$$(B_U^\flat)^*(i_U(\widetilde{u})) = \overline{i_U(\widetilde{u}) \circ B_U^\flat} = \overline{B_U(\,\cdot\,, \widetilde{u})} = \bar{\varepsilon}_{B_U}\, B_U(\widetilde{u}, \,\cdot\,).$$

That is

$$(B_U^\flat)^* \circ i_U = \bar{\varepsilon}_{B_U}\, B_U^\flat, \qquad \text{and similarly} \qquad (B_W^\flat)^* \circ i_W = \bar{\varepsilon}_{B_W}\, B_W^\flat.$$

The only difference with the bilinear case is that the maps involved are all linear and no complex conjugations appear; not in the definition of $(B_U^\flat)^*$, nor in reversing the arguments in B_U. Since $(B_U^\flat)^*$ is $\mathbb{C}$-antilinear, it is easy to see that

$$\overline{\varepsilon}_{B_U}\, i_U \circ (B_U^\flat)^{-1} = ((B_U^\flat)^*)^{-1}$$

and since $B_W^\flat$ is $\mathbb{C}$-antilinear too, we obtain (ommitting the identifications i_U and i_W),

$$\begin{aligned}\beta^* &= -(B_W^\flat)^* \circ \gamma \circ ((B_U^\flat)^*)^{-1}\\ &= -\overline{\varepsilon}_{B_W}\, B_W^\flat \circ \gamma \circ \left(\overline{\varepsilon}_{B_U}\,(B_U^\flat)^{-1}\right)\\ &= -\overline{\varepsilon}_{B_W}\,\varepsilon_{B_U}\; B_W^\flat \circ \gamma \circ (B_U^\flat)^{-1} = \overline{\varepsilon}_{B_W}\,\varepsilon_{B_U}\,\beta^*.\end{aligned}$$

Therefore, $\overline{\varepsilon}_{B_W}\,\varepsilon_{B_U} = 1$, hence $\varepsilon_{B_U} = \varepsilon_{B_W}$, or else $\beta = \gamma = 0$ as claimed. The proof of the second statement is completely analogous. □

Assume we have defined the geometry $B = B_U \oplus B_W$ on $U \oplus W$. The following result states that the conditions $\varepsilon_{B_U} = \varepsilon_{B_W}$ for the Lie algebra $\mathfrak{g}_B(U \oplus W)$, and $\varepsilon_{B_U} = -\varepsilon_{B_W}$ for the Lie superalgebra $\mathfrak{g}_B(U|W)$ are also necessary for consistency in the computation of the Lie brackets in each case.

PROPOSITION (3.1.1). (a) *Let* $\beta_i \in \mathrm{Hom}(W,U)$*,* $\gamma_i \in \mathrm{Hom}(U,W)$*,* $i = 1,2$*, with* $B_U^\flat \circ \beta_i + \gamma_i^* \circ B_W^\flat = 0$*. Then*

$$\left[\begin{pmatrix}0 & \beta_1\\ \gamma_1 & 0\end{pmatrix}, \begin{pmatrix}0 & \beta_2\\ \gamma_2 & 0\end{pmatrix}\right] \in \mathfrak{g}_B(U \oplus W) \quad\Longleftrightarrow\quad \varepsilon_{B_U} = \varepsilon_{B_W}.$$

(b) *On the other hand, if* $B_U^\flat \circ \beta - \gamma^* \circ B_W^\flat = 0$*. Then*

$$\left[\!\left[\begin{pmatrix}0 & \beta_1\\ \gamma_1 & 0\end{pmatrix}, \begin{pmatrix}0 & \beta_2\\ \gamma_2 & 0\end{pmatrix}\right]\!\right] \in \mathfrak{g}_B(U|W) \quad\Longleftrightarrow\quad \varepsilon_{B_U} = -\varepsilon_{B_W}.$$

Proof. We shall only prove statement (a) in the sesquilinear case. The bilinear case can be proved similarly, except that complex conjugations do not arise. Statement (b) is completely analogous. Note first that Proposition (2.1.4) states that

$$\left[\begin{pmatrix}0 & \beta_1\\ \gamma_1 & 0\end{pmatrix}, \begin{pmatrix}0 & \beta_2\\ \gamma_2 & 0\end{pmatrix}\right] \in \mathfrak{g}_B(U \oplus W) \quad\Longleftrightarrow\quad \begin{cases}\beta_1 \circ \gamma_2 - \beta_2 \circ \gamma_1 \in \mathfrak{g}_{B_U}(U).\\ \gamma_1 \circ \beta_2 - \gamma_2 \circ \beta_1 \in \mathfrak{g}_{B_W}(W).\end{cases}$$

Now

$$\begin{aligned}
& & \beta_1 \circ \gamma_2 - \beta_2 \circ \gamma_1 &\in \mathfrak{g}_{B_U}(U)\\
&\Leftrightarrow & B_U^\flat \circ (\beta_1 \circ \gamma_2 - \beta_2 \circ \gamma_1) + (\beta_1 \circ \gamma_2 - \beta_2 \circ \gamma_1)^* \circ B_U^\flat &= 0\\
&\Leftrightarrow & -\gamma_1^* \circ B_W^\flat \circ \gamma_2 + \gamma_2^* \circ B_W^\flat \circ \gamma_1 + (\beta_1 \circ \gamma_2 - \beta_2 \circ \gamma_1)^* \circ B_U^\flat &= 0\\
&\Leftrightarrow & -\varepsilon_{B_W}(\gamma_1^* \circ (B_W^\flat)^* \circ \gamma_2 + \gamma_2^* \circ (B_W^\flat)^* \circ \gamma_1) + (\beta_1 \circ \gamma_2 - \beta_2 \circ \gamma_1)^* \circ B_U^\flat &= 0\\
&\Leftrightarrow & \varepsilon_{B_W}(\gamma_1^* \circ \beta_2^* \circ (B_U^\flat)^* - \gamma_2^* \circ \beta_1^* \circ (B_U^\flat)^*) + (\beta_1 \circ \gamma_2 - \beta_2 \circ \gamma_1)^* \circ B_U^\flat &= 0\\
&\Leftrightarrow & \varepsilon_{B_W}\overline{\varepsilon}_{B_U}(\beta_2 \circ \gamma_1 - \beta_1 \circ \gamma_2)^* \circ B_U^\flat + (\beta_1 \circ \gamma_2 - \beta_2 \circ \gamma_1)^* \circ B_U^\flat &= 0\\
&\Leftrightarrow & (-\varepsilon_{B_W}\overline{\varepsilon}_{B_U} + 1)(\beta_1 \circ \gamma_2 - \beta_2 \circ \gamma_1)^* \circ B_U^\flat &= 0\\
&\Leftrightarrow & \varepsilon_{B_W}\overline{\varepsilon}_{B_U} &= 1\\
&\Leftrightarrow & \varepsilon_{B_U} &= \varepsilon_{B_W}.
\end{aligned}$$

□

We shall now assume that $U \simeq W$ and define a geometry B on $U \oplus W$ as in 2.2 above. We thus proceed to prove Proposition (2.2.1).

(3.2) Proof of Proposition (2.2.1)**.** Just as in the proof of Proposition (2.1.4) in 3.1 above, a direct computation leads to the set of equations shown in the statements (a) and (b) of Proposition (2.2.1). Now the equations $\Phi^\flat \circ \delta + \alpha^* \circ \Phi^\flat = 0$ and $\Omega^\flat \circ \alpha + \delta^* \circ \Omega^\flat = 0$ immediately imply that there is a non-zero complex constant λ such that $(\Omega^\flat)^* = \lambda\,\Phi^\flat$, as $((\Omega^\flat)^*)^{-1} \circ \Phi^\flat$ must commute with any $\delta : W \to W$. Note that the equations $\Omega^\flat \circ \beta \pm \beta^* \circ \Phi^\flat = 0$ and $\Phi^\flat \circ \gamma + \gamma^* \circ \Omega^\flat = 0$ now imply that either $\beta = \gamma = 0$ or $|\lambda|^2 = 1$. In particular, the equations that β and γ satisfy can now be written in terms of Φ and λ as

$$\bar\lambda(\Phi^\flat)^* \circ \beta \pm \beta^* \circ \Phi^\flat = 0 \qquad \text{and} \qquad \Phi^\flat \circ \gamma + \bar\lambda\,\gamma^* \circ (\Phi^\flat)^* = 0,$$

where in the first equation the plus sign corresponds to (a) while the minus sign corresponds to (b). □

The analogue of Proposition (3.1.1) for the geometries described in (2.2), is given by the following

PROPOSITION (3.2.1). (a) *Let* $\beta_i \in \mathrm{Hom}(W, U)$, $\gamma_i \in \mathrm{Hom}(U, W)$, $i = 1, 2$, *with* $\Phi^\flat \circ \gamma_i + \gamma_i^* \circ \Omega^\flat = 0$ *and* $\Omega^\flat \circ \beta_i + \beta_i^* \circ \Phi^\flat = 0$. *Then,*

$$\left[\begin{pmatrix} 0 & \beta_1 \\ \gamma_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta_2 \\ \gamma_2 & 0 \end{pmatrix}\right] \in \mathfrak{g}_B(U \oplus W).$$

(b) *On the other hand, if* $\Phi^\flat \circ \gamma_i + \gamma_i^* \circ \Omega^\flat = 0$ *and* $\Omega^\flat \circ \beta_i - \beta_i^* \circ \Phi^\flat = 0$. *Then*

$$\left[\!\left[\begin{pmatrix} 0 & \beta_1 \\ \gamma_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \beta_2 \\ \gamma_2 & 0 \end{pmatrix}\right]\!\right] \in \mathfrak{g}_B(U|W).$$

Proof. This is also a straightforward computation using the results of Proposition (2.2.1):

$$\begin{aligned} \Omega^\flat \circ (\beta_1 \circ \gamma_2 \mp \beta_2 \circ \gamma_1) &= \mp\beta_1^* \circ \Phi^\flat \circ \gamma_2 + \beta_2^* \circ \Phi^\flat \circ \gamma_1 \\ &= \pm\,\beta_1^* \circ \gamma_2^* \circ \Omega^\flat - \beta_2^* \circ \gamma_1^* \circ \Omega^\flat \\ &= (\pm\,\beta_1^* \circ \gamma_2^* - \beta_2^* \circ \gamma_1^*) \circ \Omega^\flat \\ &= -(\gamma_1 \circ \beta_2 \mp \gamma_2 \circ \beta_1)^* \circ \Omega^\flat. \end{aligned}$$

□

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SINGULAR POINTS AND AUTOMORPHISMS OF UNSTABLE FOLIATIONS OF $\mathbb{CP}$

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ABSTRACT. Let $\mathcal{F}_d$ be the space of holomorphic foliations of $\mathbb{CP}$ of degree d. We study the linear action $\mathrm{PGL}(3,\mathbb{C}) \times \mathcal{F}_d \to \mathcal{F}_d$ given by $gX = DgX \circ (g^{-1})$ in the sense of Mumford in [3]. In this paper we prove that an unstable foliation X of degree $d \geq 2$ satisfies one of the following conditions: it is a Riccati foliation, or its automorphism group $\mathrm{Aut}(X)$ is finite abelian or it is isomorphic to a transitive finite subgroup of $\mathrm{GL}(2,\mathbb{C})$. We also prove the existence of degenerate singularities for unstable foliations; and we give a characterization of foliations on $\mathbb{CP}$ with an infinite automorphism group.

1. Introduction

According to the Geometric Invariant Theory (GIT), it is possible to study the action of a reductive group G on a projective variety V by stratifying the points of the variety in two categories: semistable points and unstable points. By restricting the action of G to the semistable points we obtain what is called a good quotient.

In most of the cases the variety V consists of certain geometric objects such as algebraic curves, hypersurfaces, or pencils of curves. The usual action of G on V is such that objects are in the same orbit if and only if they are isomorphic.

The unstable points in V are in some sense degenerate objects. For example: If we consider the natural action of $\mathrm{PGL}(3,\mathbb{C})$ on $\mathbb{CP}^9$, where $\mathbb{CP}^9$ is the space of plane curves of degree 3, then a cubic plane curve is unstable if and only if it has a triple point, or a cusp, or two components tangent at a point (see [12]).

Another example is the action of $\mathrm{PGL}(2,\mathbb{C})$ in the space of binary forms of degree d. In this case a binary form of degree d is semistable if and only if it has no root of multiplicity greater that $\frac{d}{2}$ (see [12]).

The last example we will mention is the classification of pencils of cubic curves in $\mathbb{CP}$, up to projective automorphism, i.e., the natural action of $\mathrm{PGL}(3,\mathbb{C})$ in the space of pencils of cubic curves. For this case the unstable pencils of cubic are those whose associated elliptic fibration has singularities of types I^*, II^*, III^* in the Kodaira classification (see [11]).

The set which consists of unstable points on V is closed in the Zariski topology. By very well known techniques developed by D. Hilbert and D. Mumford (see [6] and [3]) it is possible to characterize these kind of points. These techniques make use of the 1-parameter subgroups of G, recall that a 1-parameter subgroup of G is a homomorphism, $\lambda\colon \mathbb{C}^* \to G$.

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For a fixed unstable point there exists a set of 1-parameter subgroups, such that these 1-parameter subgroups are, in some sense, special to show the instability of the point (see Theorem (2.8)).

From this set of 1-parameter subgroups, we can get a unique parabolic subgroup of G, which gives us information of the stabilizer of the unstable point (see Corollary (2.10)).

In this work the variety V is the space of holomorphic foliations of $\mathbb{CP}$ of degree d, the group is the automorphism group of $\mathbb{CP}$ and the action is given by the change of coordinates.

We obtain properties of unstable foliations related to the multiplicity and Milnor number of the singular points, the transversality of the foliation respect to a rational fibration (in this case we will say that the foliation is Riccati), and in the existence of algebraic solutions.

We also describe the automorphism group for unstable foliations through the unique parabolic group associated to a special 1-parameter subgroup for the foliation. Finally we give a characterization of foliations on $\mathbb{CP}$ with an infinite automorphism group.

Let $\mathcal{T}\mathbb{CP}(d-1) = \mathcal{T}\mathbb{CP} \otimes \mathcal{O}_{\mathbb{CP}}(d-1)$, the space of holomorphic foliations of $\mathbb{CP}$ of degree d is $\mathcal{F}_d := \mathbb{P}H^0(\mathbb{CP}, \mathcal{T}\mathbb{CP}(d-1))$. The group $\mathrm{PGL}(3,\mathbb{C})$ of automorphisms of $\mathbb{CP}$ acts linearly on $\mathcal{F}_d$:

$$\begin{aligned} \mathrm{PGL}(3,\mathbb{C}) \times \mathcal{F}_d &\to \mathcal{F}_d \\ (g, X) &\mapsto gX = DgX \circ (g^{-1}). \end{aligned}$$

We study this action in the sense of the Geometric Invariant Theory (GIT) and we obtain the following:

THEOREM (1.1). *Let X be an unstable foliation of degree $d \geq 2$ with isolated singularities. Then one of the following holds:*

1. *X is a Riccati foliation;*

2. *the automorphism group* $\mathrm{Aut}(X)$ *of X is finite abelian, X has a singular point p of multiplicity greater than $\frac{d-1}{3}$, and a line solution which contains p, both invariant by* $\mathrm{Aut}(X)$*;*

3. *the automorphism group* $\mathrm{Aut}(X)$ *of X is finite abelian, X has a singular point of multiplicity greater than $\frac{d}{2}$, which is invariant by* $\mathrm{Aut}(X)$*;*

4. *the automorphism group* $\mathrm{Aut}(X)$ *of X, is isomorphic to a transitive finite subgroup of* $\mathrm{GL}(2,\mathbb{C})$ *and X has a singular point of multiplicity greater than $\frac{2d+1}{3}$, which is invariant by* $\mathrm{Aut}(X)$.

THEOREM (1.2). *Let X be a foliation of degree $d \geq 2$ with isolated singularities. Then X has an infinite automorphism group* $\mathrm{Aut}(X)$ *if and only if:*

1. *there exists a 1-PS, λ, such that X is λ-invariant. If this is the case, X is transversal with respect to the rational fibration associated to λ. Or,*

2. *X is in the orbit of the foliation*

$$Y = P(y,z)\frac{\partial}{\partial x} + R(y,z)\frac{\partial}{\partial z} = \begin{pmatrix} P(y,z) \\ 0 \\ R(y,z) \end{pmatrix}.$$

The foliation Y is $\lambda_{(2,-1)}$-unstable, has a singular point with Milnor number greater or equal to $d^2 + d$ and Y is transversal with respect to the rational fibration associated to $\lambda_{(2,-1)}$.

2. Geometric Invariant Theory

The following is a summary of the Geometric Invariant Theory, which will be required for the sequel. All the definitions and results can be found in [12] and [8].

Let V be a projective variety in $\mathbb{CP}^n$, and consider a reductive group G acting linearly on V.

Definition (2.1). Let $x \in V \subset \mathbb{CP}^n$, and consider $\overline{x} \in \mathbb{C}^{n+1}$ such that $\overline{x} \in x$. Denote by $O(\overline{x})$ the orbit of $\overline{x}$ in the affine cone of V. Then

(i) x is *unstable* if $0 \in \overline{O(\overline{x})}$.

(ii) x is *semi-stable* if $0 \notin \overline{O(\overline{x})}$. The set of semi-stable points will be denoted by V^{ss}.

(iii) x is *stable* if it is semistable, the orbit of x, $O(x)$, is closed in V^{ss} and $\dim O(x) = \dim G$. The set of stable points will be denoted by V^s.

The main result in GIT is the following:

THEOREM (2.2) (see page 74 in [12]). (i) *There exists a good quotient (Y, ϕ) of V^{ss} by G, where Y is projective.*

(ii) *There exists an open set $Y^s \subset Y$ such that $\phi^{-1}(Y^s) = V^s$ and (Y^s, ϕ) is a good quotient and an orbit space of V^s by G.*

(iii) *If $x_1, x_2 \in V^{ss}$ then $\phi(x_1) = \phi(x_2)$ if and only if $\overline{O(x_1)} \cap \overline{O(x_2)} \cap V^{ss} \neq \emptyset$.*

Now we describe the Hilbert-Mumford criterion for finding the unstable points for a linear action.

Let $\lambda\colon \mathbb{C}^* \to G$ be a 1-parameter subgroup (1-PS). Then

$$\begin{aligned} \mathbb{C}^* &\to GL(n+1, \mathbb{C}) \\ t &\mapsto \lambda(t) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \\ &\qquad\qquad v \mapsto \lambda(t)v, \end{aligned}$$

is a diagonal representation of $\mathbb{C}^*$. There exists a basis $\{v_0, \ldots, v_n\}$ of $\mathbb{C}^{n+1}$ such that $\lambda(t)v_i = t^{r_i}v_i$, where $r_i \in \mathbb{Z}$. This integer r_i is called the *weight of v_i with respect to the action of $\lambda(t)$ on $\mathbb{C}^{n+1}$.*

Definition (2.3). Let $x \in V$ and let λ be a 1-PS. If $\overline{x} \in x$ and $\overline{x} = \sum_{i=0}^n a_i v_i$, then $\lambda(t)\overline{x} = \sum_{i=0}^n t^{r_i} a_i v_i$. We define the following function:

$$\mu(x, \lambda) := \min\{r_i : a_i \neq 0\}. \tag{2.4}$$

The numerical criterion can now be stated.

THEOREM (2.5) (see Theorem 4.9 of [12]). (i) *x is stable if and only if $\mu(x, \lambda) < 0$ for every 1-PS, λ, of G.*

(ii) *x is unstable if and only if there exists a 1-PS, λ, of G such that $\mu(x, \lambda) > 0$.*

Definition (2.6). If $\mu(x, \lambda) > 0$ we will say that x is *λ-unstable*.

The following is done in order to state a Theorem due to G. Kempf, which will play an important role in the proof of the results of this paper.

Definition (2.7). Let $\lambda\colon \mathbb{C}^* \to G$ be a 1-parameter subgroup. The *parabolic subgroup $P(\lambda)$ of G associated to λ* is the subgroup of points $g \in G$ such that $\lim_{t\to 0} \lambda(t) g \lambda^{-1}(t)$ exists in G.

Let $\Gamma(G)$ be the set of the one-parameter subgroups of G. Let's define a notion of *length* $\|\quad\|$ on $\Gamma(G)$ as a non-negative real-valued function such that:

1. for all $\lambda \in \Gamma(G)$ and $g \in G$, $\|g\lambda g^{-1}\| = \|\lambda\|$ and
2. if T is a maximal torus of G, there exists a positive definite integral-valued bilinear form $(\ ,\ )$ on $\Gamma(T)$ such that $(\lambda, \lambda) = \|\lambda\|^2$ for all $\lambda \in \Gamma(T)$.

Now we are ready to enunciate the Theorem by G. Kempf.

THEOREM (2.8) (see Theorem 3.4 of [8]). *Let G be a reductive group acting linearly on a projective variety V, fix $x \in V$ and a length $\|\quad\|$ on $\Gamma(G)$, then the function*

$$f_x\colon \Gamma(G) \to \mathbb{R}$$

defined by

$$f_x(\lambda) = \frac{\mu(x,\lambda)}{\|\lambda\|} \tag{2.9}$$

has a maximum value B_x on the set $|V, x| = \{\lambda \in \Gamma(G) : \exists \lim_{t\to 0} \lambda(t)x\}$, if this set does not consist only of the trivial subgroup.

B_x exists and is positive if and only if $0 \in \overline{O(\bar{x})}$. If this condition is verified, the set Λ_x of 1-PS, λ, such that $f_x(\lambda) = B_x$, satisfies:

1. *Λ_x is not empty.*
2. *There exists a parabolic subgroup P_x such that $P(\lambda) = P_x$ for all $\lambda \in \Lambda_x$.*
3. *Any maximal torus of P_x contains a unique member of Λ_x.*
4. *Λ_x is a principal homogeneous space under the unipotent radical of P_x.*

COROLLARY (2.10) (see corollary 3.5 of [8]). *In the above situation, suppose that x is unstable. Then*

1. *for all $g \in G$, $gP_xg^{-1} = P_{gx}$,*
2. *P_x contains the stabilizer in G of x.*

The following is a useful tool for the method of 1-PS when $G = \mathrm{SL}(n, \mathbb{C})$. We formulate the result for the case $n = 3$.

LEMMA (2.11) (see [12]). *Every 1-parameter subgroup of* $\mathrm{SL}(3,\mathbb{C})$ *can be written as*

$$g\lambda(t)g^{-1} = g\begin{pmatrix} t^{n_0} & 0 & 0 \\ 0 & t^{n_1} & 0 \\ 0 & 0 & t^{n_2} \end{pmatrix} g^{-1},$$

for some $g \in \mathrm{SL}(3,\mathbb{C})$, where $n_0 \ge n_1 \ge n_2$ and $n_0 + n_1 + n_2 = 0$. We will denote the above diagonal 1-PS, λ, by $\lambda_{(n_0,n_1)}$ and we will assume that the integers are relative primes.

Remark (2.12). If $n_0 \ge n_1 \ge n_2$ and $n_0 + n_1 + n_2 = 0$, then $\frac{1}{2} \le -\frac{n_2}{n_0} \le 2$.

In this paper we use the group $\mathrm{SL}(3,\mathbb{C})$ instead of $\mathrm{PGL}(3,\mathbb{C})$ because they are isogenous, and we will use the length $\|g\lambda_{(n_0,n_1)}g^{-1}\| = \sqrt{n_0^2+n_1^2+n_2^2}$ given by the Killing form.

For purposes of this paper we will also need the following concepts and results related to algebraic groups.

THEOREM (2.13) (see [1]). *Let G be an affine algebraic group acting on an algebraic variety and let $x \in V$. Then the stabilizer in G of x is a closed subgroup of G.*

Definition (2.14). Let $x \in V$, then *the automorphism group of* x is the stabilizer in G of x and we will denote it by $\mathrm{Aut}(x)$.

Definition (2.15). Any finite subgroup G of $\mathrm{GL}(n,\mathbb{C})$ is called a linear group in n variables. If the n variables of the group can be separated into two or more sets, such that the variables of any set are transformed by all the transformation of G into linear functions of the variables of that set only, we say that G is *intransitive*. If such a division is not possible, the group is *transitive*.

3. Foliations of $\mathbb{CP}$

This section provides the definitions and results that we need to know about the holomorphic foliations of $\mathbb{CP}$ for the development of the paper.

Definition (3.1). A holomorphic foliation X of $\mathbb{CP}$ of degree d is a non-trivial morphism of vector bundles:

$$X\colon \mathcal{O}(1-d) \to \mathcal{T}\mathbb{CP},$$

modulo the multiplication by a nonzero scalar. Then the space of foliations of degree d is $\mathcal{F}_d := \mathbb{P}H^0(\mathbb{CP}, \mathcal{T}\mathbb{CP}(d-1))$, where $d \geq 0$.

PROPOSITION (3.2) (see [5]). *Every foliation $X \in \mathcal{F}_d$ can be written as*

$$X = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z} = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix}$$

where $P, Q, R \in \mathbb{C}[x,y,z]$ are homogeneous of degree d, modulo multiplication by a nonzero scalar and if we consider the radial foliation

$$E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z},$$

then X and $X + F(x,y,z)E$ represent the same foliation for all $F \in \mathbb{C}[x,y,z]$ homogeneous of degree $d-1$.

Definition (3.3). A point $p = (a : b : c) \in \mathbb{CP}$ is *singular* for the above foliation X if $(P(a,b,c) : Q(a,b,c) : R(a,b,c)) = (ka : kb : kc)$ for some $k \in \mathbb{C}$. The set of singular points of X will be denoted by $\mathrm{Sing}(X)$.

Definition (3.4). Let $X \in \mathcal{F}_d$ and let p be an isolated singularity of X. Let

$$\begin{pmatrix} Q(y,z) = Q_m(y,z) + Q_{m+1}(y,z) + \cdots \\ R(y,z) = R_n(y,z) + R_{n+1}(y,z) + \cdots \end{pmatrix}$$

be a local generator of X in $p = (1:0:0)$, where Q_i, R_i are forms of degree i, and Q_m, R_n are not identically zero. We define the *Milnor number of p* by $\mu_p(X) := \dim_{\mathbb{C}} \frac{\mathcal{O}_p}{<Q,R>}$ and the *multiplicity of p* by $m_p(X) := \min\{m, n\}$.

Remark (3.5). $\mu_p(X) \geq mn \geq m_P(X)^2$.

PROPOSITION (3.6) (see [2]). *Let X be a foliation of degree d with isolated singularities then*

$$d^2 + d + 1 = \sum_{p \in \mathbb{CP}} \mu_p(X).$$

Definition (3.7). An irreducible plane curve defined by a polynomial $F(x, y, z)$ is an *algebraic solution for X or invariant by X* if and only if there exists a polynomial $H(x, y, z)$ such that:

$$P(x,y,z)\frac{\partial F(x,y,z)}{\partial x} + Q(x,y,z)\frac{\partial F(x,y,z)}{\partial y} + R(x,y,z)\frac{\partial F(x,y,z)}{\partial z} =$$

$$F(x,y,z)H(x,y,z).$$

Definition (3.8). A foliation X is a *Riccati foliation* if there exists a rational fibration on a surface S, obtained from $\mathbb{CP}$ after a finite number of blow-ups, whose generic fiber is transverse to the lifted foliation of X in S.

The following result is about foliations without algebraic solutions, a Theorem by Jouanolou and completed by Lins Neto and Marcio Soares (see [7] and [10]).

THEOREM (3.9). *For $d \geq 2$, the subset $\{X \in \mathcal{F}_d : X$ has no algebraic solutions$\}$ is not empty and dense in $\mathcal{F}_d$ and it contains an open and dense subset.*

The next Theorem give us an open set of stable foliations. This set consists of foliations with $d^2 + d + 1$ different singular points, i.e., every singularity has Milnor number equal to one.

THEOREM (3.10) (see [4]). *If a foliation X of degree d has d^2+d+1 different singular points, then X is stable and every line $L \subset \mathbb{CP}$ has at most $d+1$ singular points.*

Remark (3.11). If a foliation X of degree d has $d^2 + d + 1$ different singular points then X has a line solution L if and only if L has $d + 1$ singular points.

Proof. If the line $L = ax + by + cx$ has $d + 1$ singular points of

$$X = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z},$$

then the polynomial of degree d, $aP(x, y, z)+bQ(x, y, z)+cR(x, y, z)$ and L have $d + 1$ common zeros. Hence by Bézout L is a factor of this polynomial.

Suppose that z is a solution for X, then we can write $X = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y}$, so $\operatorname{Sing} X \cap \{(x:y:z) \in \mathbb{CP} : z \neq 0\} = V(P(x,y,1), Q(x,y,1))$ has at most d^2 different points. Hence $\operatorname{Sing} X \cap V(z) = V(yP(x,y,z) - xQ(x,y,z), z)$ has $d + 1$ points. $\square$

4. Proof of Theorem (1.2)

Suppose that X has an infinite automorphism group $\mathrm{Aut}(X)$. Since $\mathrm{Aut}(X)$ is algebraic and infinite, it contains an algebraic group H of dimension one, then H is either unipotent or a torus.

If H is a torus then it defines a 1-PS, $\lambda\colon \mathbb{C}^* \to \mathrm{Aut}(X)$ such that X is λ-invariant, i.e., $\lambda(t)X = X$ for all $t \in \mathbb{C}^*$.

Using the Lie derivative, it is easy to prove that X is transversal with respect to the rational fibration associated with the flow given by λ.

In case $\lambda = \lambda_{(n_0,n_1)}$ for some $n_0, n_1 \in \mathbb{Z}$ we have that the associated foliation is $X_\lambda = n_0 x\frac{\partial}{\partial x} + n_1 y\frac{\partial}{\partial y} + n_2 z\frac{\partial}{\partial z}$, which has degree 1. This foliation admits a holomorphic first integral $f\colon \mathbb{CP} \to \mathbb{CP}^1$. The irreducible components of the fibers of f are rational curves which are the leaves of X_λ.

Using a Theorem by Seidenberg (see [9]) we can reduce the singularities of X_λ and we obtain a rational fibration $\tilde{f}\colon S \to \mathbb{CP}^1$ (where S is $\mathbb{CP}$ with a finite number of blow-ups) such that the fibers are the separated leaves of X_λ. If $\widetilde{X}$ is the lifting of X on S, then the generic fiber of $\tilde{f}$ is transverse to the leaves of $\widetilde{X}$.

Obviously, if the weight of X with repect to the action of λ is not zero, then X is unstable.

If H is unipotent, then it is isomorphic to $(\mathbb{C}, +)$, and there exists a morphism $\phi\colon \mathbb{C} \to \mathrm{Aut}(X)$, this morphism must be of the form $\phi(t) = \exp(At)$. Since ϕ is an algebraic morphism, then A must be a nilpotent matrix with trace zero, so A is similar to

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \text{or to} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The possible morphisms are

$$\phi_1(t) = g\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}g^{-1}, \quad \text{or} \quad \phi_2(t) = g\begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}g^{-1},$$

where $g \in \mathrm{Aut}(X)$.

Since $\mathbb{C}$ is a unipotent group then its group of characters is trivial, therefore $\phi(t)\overline{X} = \overline{X}$ in the affine cone of $\mathcal{F}_d$ for all $t \in \mathbb{C}$.

We can easily see that the foliation invariant by $\phi_2(t)$ for all $t \in \mathbb{C}$ does not have isolated singularities and the unique foliation invariant by $\phi_1(t)$ for all $t \in \mathbb{C}$ is $Y = P(y, z)\frac{\partial}{\partial x} + R(y, z)\frac{\partial}{\partial z}$, where P, R are homogeneous of degree d in $\mathbb{C}[y, z]$.

In the chart $U_0 = \{(1 : y : z) \in \mathbb{CP}\}$ we have that the local vector field which generates this foliation is

$$\begin{pmatrix} -yP(y, z) \\ R(y, z) - zP(y, z) \end{pmatrix},$$

so $\mu_{(1:0:0)}(Y) \geq d^2 + d$.

The associated foliation to $\lambda_{(2,-1)}$ is $X_{\lambda_{(2,-1)}} = 2x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}$ and $k_1 y - k_2 z = 0$ is a solution for $X_{\lambda_{(2,-1)}}$ for all $k_i \in \mathbb{C}$. After blowing-up the point $(0, 0) \in U_0$

we obtain:

$$\tilde{Y} = \begin{pmatrix} w_1^2 P(1, w_2) \\ -Q(1, w_2) \end{pmatrix},$$

where $w_1 = 0$ is the exceptional divisor and the solutions of $\tilde{X}_{\lambda_{(2,-1)}}$ are $w_2 = k$ for all $k \in \mathbb{C}$, therefore Y is transversal with respect to the flow given by $\lambda_{(2,-1)}$.

The converse of the theorem is obvious.

5. Proof of Theorem (1.1)

The Theorem will be a consequence of the following lemmas and propositions.

To state the first Lemma, we need to define the following types of subgroups of $SL(3, \mathbb{C})$:

(I) An infinite linear algebraic group.

(A) A diagonal finite group: the elements of this group are of the following form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

where $abc = 1$.

(B) Consider the following matrices:

$$\psi_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho_k & 0 \\ 0 & 0 & \rho_k^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad \phi_k = \begin{pmatrix} \rho_k^{-2} & 0 & 0 \\ 0 & \rho_k & 0 \\ 0 & 0 & \rho_k \end{pmatrix},$$

$$\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ρ_k is a $k-$root of the unity, and

$$\omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho_5^3 & 0 \\ 0 & 0 & \rho_5^2 \end{pmatrix}, \quad o = \frac{1}{\sqrt{5}} \begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & \rho_5^4 - \rho_5 & \rho_5^2 - \rho_5^3 \\ 0 & \rho_5^2 - \rho_5^3 & \rho_5 - \rho_5^4 \end{pmatrix},$$

$$\eta = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1+i & -1+i \\ 0 & 1+i & 1-i \end{pmatrix}.$$

(B1a) The group generated by ψ_{2q}, τ, ϕ_{2m}, where $m = n - q \equiv 1 \mod 2$.

(B1b) The group generated by ψ_{2q}, $\tau \circ \phi_{24m}$, where $m \equiv 0 \mod 2$.

(B2a) The the group generated by ψ_4, τ, η, ϕ_{2m}, where $m \equiv 1, 5 \mod 6$.

(B2b) The the group generated by ψ_4, τ, $\eta \circ \phi_{6m}$, where $m \equiv 3 \mod 6$.

(B3) The group generated by ψ_8, τ, η, ϕ_{2m}, where $(6, m) = 1$.

(B4) The group generated by σ, ω, o, ϕ_{2m}, where $(m, 30) = 1$.

LEMMA (5.1). *Let V be a projective variety with a linear action:*

$$SL(3,\mathbb{C}) \times V \to V.$$

Then the automorphism group for an unstable point for this action is, up to linear equivalence, one of the above types (I), (A) or (B).

Proof. Using the classification of finite subgroups of $SL(3,\mathbb{C})$ made in [15] we have a list of 12 groups: $(A)-(L)$. This classification is up to linear equivalence.

Let $x \in V$, since $\mathrm{Aut}(x)$ is a closed subgroup of $\mathrm{SL}(3,\mathbb{C})$, then this group is one of the above list or an infinite linear algebraic group.

On the other hand, for the one parameter subgroup $\lambda_{(n_0,n_1)}$, where $n_0 \geq n_1 \geq n_2$ and $n_0+n_1+n_2=0$, we have that the associated parabolic subgroup $P(\lambda_{n_0,n_1})$ is the group of upper triangular matrices if $n_0 > n_1 > n_2$ and

$$P(\lambda_{(2,-1)}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix} \in SL(3,\mathbb{C}) \right\},$$

$$P(\lambda_{(1,1)}) = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in SL(3,\mathbb{C}) \right\}.$$

If x is unstable and f_x (see Theorem (2.8)) has a maximum positive value in $\lambda_{(n_0,n_1)}$, then by the second part of Corollary (2.10) we obtain that $\mathrm{Aut}(x) \subset P(\lambda_{n_0,n_1})$. Therefore, up to linear equivalence, the automorphism group $\mathrm{Aut}(x)$ could be an infinite linear group or a finite group of the type (A) or (B). The case (B) can occur only if f_x has a maximum value in $\lambda_{(2,-1)}$. □

PROPOSITION (5.2). *Let X be a foliation of degree d such that $\mu(X,\lambda_{(n_0,n_1)}) > 0$, where $n_0 \geq n_1 \geq n_2$. Then the multiplicity of the singular point $p=(1:0:0)$ of X is greater than $\frac{d-1}{3}$.*

Proof. The foliation:

$$X = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z} = \begin{pmatrix} P(x,y,z) \\ Q(x,y,z) \\ R(x,y,z) \end{pmatrix}$$

is represented in affine coordinates $(y,z) \in U_0$ by a vector field of the form:

$$X_0 = \begin{pmatrix} Q_0(y,z)+Q_1(y,z)+Q_2(y,z)+\cdots+Q_d(y,z)-yP_d(y,z) \\ R_0(y,z)+R_1(y,z)+R_2(y,z)+\cdots+R_d(y,z)-zP_d(y,z) \end{pmatrix},$$

where Q_j, R_j and P_d are homogeneous polynomials of degree j and d respectively, in $\mathbb{C}[y,z]$.

Suppose that $Q_j(y,z) = \sum_i a_{ij} y^{j-i} z^i$ is not identically zero. The weight of the monomial field with coefficient a_{ij} respect to $\lambda_{(n_0,n_1)}$ is $n_1 - n_0(d-j) - n_1(j-i) - n_2 i = n_0(2j-i-1) + n_2(j-2i-1) - n_0 d$ and this weight is positive if and only if $\frac{n_2}{n_0}(j-2i-1)+(2j-i-1) > d$.

If there exist i,j such that $a_{ij} \neq 0$ and $j-2i-1 \geq 0$, then

$$d < 2j-i-1+\frac{n_2}{n_0}(j-2i-1) \leq 2j-i-1-\frac{j-2i-1}{2} = \frac{3}{2}j - \frac{1}{2} \quad i.e., \quad \frac{2d+1}{3} < j.$$

If for all i, j such that $j - 2i - 1 \geq 0$, we have $a_{ij} = 0$, then for all i, j with $a_{ij} \neq 0$ we have $j - 2i - 1 < 0$, therefore

$$d < 2j - i - 1 - \frac{n_2}{n_0}(2i - j + 1) \leq 2j - i - 1 + 2(2i - j + 1) = 3i + 1 \quad \textit{i.e.,}\ \frac{d-1}{3} < i.$$

Similarly for $R_j(y, z) = \sum_i b_{ij} y^{j-i} z^i$: If there exist i, j such that $b_{ij} \neq 0$, then $j + 1 - 2i \geq 0$, therefore

$$d < 2j - i + \frac{n_2}{n_0}(j + 1 - 2i) \leq 2j - i - \frac{j + 1 - 2i}{2} = \frac{3}{2}j - \frac{1}{2}, \quad \textit{i.e.,}\ \frac{2d+1}{3} < j,$$

in the other case we have:

$$d < 2j - i - \frac{n_2}{n_0}(2i - j - 1) \leq 2j - i + 2(2i - j - 1) = 3i - 2 \quad \textit{i.e.,}\ \frac{d+2}{3} < i.$$

We have always $j > \frac{d-1}{3}$. Then the multiplicity of the singular point $p = (1 : 0 : 0)$ is greater than $\frac{d-1}{3}$. □

PROPOSITION (5.3). *A foliation X of degree d has a point p of multiplicity greater than $\frac{2d+1}{3}$ if and only if X is $g\lambda_{(2,-1)}g^{-1}-$unstable for some $g \in SL(3, \mathbb{C})$.*

Proof. We will follow the notation of the above proposition, then the weight of the monomial vector field with coefficient a_{ij}, b_{ij} with respect to $\lambda_{(2,-1)}$ is positive if and only if $j > \frac{2d+1}{3}$. □

LEMMA (5.4). *Let X be a foliation of degree d such that $\mu(X, \lambda_{(n_0,n_1)})$ with $n_1 \geq 0$. Then z defines an algebraic solution for X.*

Proof. The weight of the monomial field $x^{d-j}y^j \frac{\partial}{\partial z}$ is $n_2 - n_0(d - j) - n_1 j$. Since $n_1 \geq 0$ and $n_0 > 0$, $n_2 < 0$; this weight is negative or zero. Therefore z divides $R(x, y, z)$, then it is an algebraic solution for X. □

LEMMA (5.5). *Let X be a foliation of degree d. Suppose that $\mu(X, \lambda_{(n_0,n_1)}) > 0$ for some $\lambda_{(n_0,n_1)}$, with $n_1 \leq 0$. Then the multiplicity of the singular point $p = (1 : 0 : 0)$ of X is greater than $\frac{d}{2}$.*

Proof. The proof is similar to that of Proposition (5.2). We must note than $n_1 \leq 0$ if and only if $-\frac{n_2}{n_0} \leq 1$:

If there exists i, j, such that $a_{ij} \neq 0$ and $j - 2i - 1 < 0$ therefore

$$d < 2j - i - 1 - \frac{n_2}{n_0}(2i - j + 1) \leq 2j - i - 1 + 2i - j + 1 = j + i \leq 2j \quad \textit{i.e.,}\ \frac{d}{2} < j.$$

Similarly, for b_{ij} we have:

$$d < 2j - i - \frac{n_2}{n_0}(2i - j - 1) \leq 2j - i + 2i - j - 1 = j + i - 1 \leq 2j - 1 \quad \textit{i.e.,}\ \frac{d+1}{2} < j.$$

□

Now we are ready to prove Theorem (1.1).

Proof. Suppose that X is an unstable foliation. Then, for some $\lambda\in\Gamma(SL(3,\mathbb{C}))$, $f_X(\lambda)$ (see Theorem (2.8)) has a maximum positive value.

We will need the following trivial facts:

Let $g \in SL(3,\mathbb{C})$. Then p is a singular point of X with multiplicity m if and only if $g(p)$ is a singular point of gX with multiplicity m.

$F(x, y, z)$ is an algebraic solutions for X if and only if $F \circ g$ is an algebraic solution for X for all $g \in \mathrm{Aut}(X)$.

Therefore, for our purposes, we can assume that $\lambda = \lambda_{(n_0,n_1)}$ for some integers $n_0 \geq n_1$. We have the following cases:

1. Aut(X) is infinite, in this case we apply Theorem (1.2).

Now suppose that Aut(X) is finite:

2. If $n_1 \geq 0$ we use Proposition (5.2) and Lemmas (5.1), (5.4). Since the singular point is $(1 : 0 : 0)$, the line solution is z and Aut(X) is of type (A), we obtain that the point is in the line and both are invariant by Aut(X).

3. If $n_1 < -1$ we use again Proposition (5.2) and Lemmas (5.1), (5.5).

4. For $n_1 = n_2 = -1$ we have Proposition (5.3). In this case Aut(X) is of type (B). □

With Proposition (5.2) we also obtain the following.

COROLLARY (5.6). *Let X be an unstable foliation of degree d with isolated singularities. Then there exists $p \in \mathbb{CP}$ such that $\mu_p(X) \geq ([\frac{d+2}{3}]+1)([\frac{d-1}{3}]+1)$.*

Proof. This is a consequence of Remark (3.5). In the proof of Proposition (5.2) we have that $m \geq [\frac{d-1}{3}] + 1$ and $n \geq [\frac{d+2}{3}] + 1$. □

The next Corollary is a generalization of Theorem (3.10).

COROLLARY (5.7). *Let $X \in \mathcal{F}_d$, where $d \geq 2$. The foliation X is stable if for all $p \in \mathbb{CP}$ we have:*

$$\mu_p(X) < \frac{(d+2)(d-1)}{9} \quad \textit{when } d \equiv 1 \textit{ modulo } 3, \textit{ and}$$

$$\mu_p(X) < ([\frac{d+2}{3}]+1)([\frac{d-1}{3}]+1) \quad \textit{when } d \equiv 0, 2 \textit{ modulo } 3.$$

Proof. If X is not a stable foliation, there exists $\lambda \in \Gamma(SL(3,\mathbb{C}))$ such that $\mu(X, \lambda) \geq 0$, so we can suppose that $\lambda = \lambda_{(n_0,n_1)}$ for some $n_0 \geq n_1$.

Using again proposition (5.2) we have that $m \geq \frac{d-1}{3}$ and $n \geq \frac{d+2}{3}$ if $d \equiv 1 \pmod 3$, and $m \geq [\frac{d-1}{3}] + 1$ and $n \geq [\frac{d+2}{3}] + 1$ if $d \equiv 0, 2 \pmod 3$. □

Here $[r]$ denotes the interger part of r.

6. Final Remarks

1. The converse of the corollary (5.6) is not necessarily true: The foliation $X = -z^2\frac{\partial}{\partial x} + (y^2 + xz)\frac{\partial}{\partial y} + (y^2 + xz)\frac{\partial}{\partial z}$ is semistable and it has a singularity with Milnor number 5 in $(1 : 0 : 0)$.

2. We are interested in studying the relation between the set of unstable foliations and the set of foliations with algebraic solutions because every known example of foliation without algebraic solution is stable with a finite but rich

automorphism group (see [7] and [16]). In this context Pereira and Sánchez proved the following:

THEOREM (6.1) (see [13]). *Let X be a foliation of $\mathbb{CP}$. If* $\mathrm{Aut}(X)$ *is finite and acts without nontrivial fixed points on the space of cofactors, then either X admits liouvillian first integral or X does not admit an algebraic solution.*

In this paper we obtain that the automorphism group of an unstable foliation is, in some sense, small.

3. In [14] the authors proved the following

THEOREM (6.2) (see [14]). *Let X be a codimension q holomorphic foliation on a projective variety M. Suppose that* $\mathrm{Aut}(X)$ *contains an infinite linear algebraic group. Then X belongs to one of the following classes:*

1. *X has codimension one and it is birationally equivalent to a Riccati foliation.*

2. *There exists a projective variety N and a rational map(possibly with indeterminacy points) $\pi\colon M \to N$ whose fibers are rational curves and such that X is the pull-back of a holomorphic foliation Y on N.*

3. *X has codimension at least* 2 *and is tangent to a holomorphic foliation Y of codimension $q-1$.*

In this paper we obtain a characterization of foliations of $\mathbb{CP}$ with $\mathrm{Aut}(X)$ infinite.

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THE HOMOTOPY GROUPS OF L_2-LOCALIZATION OF THE RAVENEL SPECTRA $T(m)/v_1$ AT THE PRIME TWO

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ABSTRACT. The Ravenel spectra $T(m)$ for non-negative integers m interpolate between the sphere spectrum and the Brown-Peterson spectrum. It admits an essential self-map $\alpha\colon \Sigma^{2p-2}T(m) \to T(m)$, whose cofiber we denote by $T(m)/v_1$. In this note we work in the two-local stable homotopy category and study the homotopy groups of the Bousfield localization of $T(m)/v_1$ with respect to the v_2-inverted Brown-Peterson spectrum.

1. Introduction

In the stable homotopy category of spectra localized at an odd prime number p, the second author, A. Yabe and X. Wang ([11], [9]) determined the structure of the homotopy groups of the sphere spectrum L_2S^0 localized with respect to the v_2-localized Brown-Peterson spectrum $v_2^{-1}BP$ by use of the Adams-Novikov spectral sequence

$$E_2^*(X) = \mathrm{Ext}^*_{BP_*(BP)}(BP_*, BP_*(X)) \Longrightarrow \pi_*(X).$$

Here the E_2-term is the Ext group in the category of $BP_*(BP)$-comodules. At the prime two, the second author and X. Wang ([10]) determined only the E_2-term of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(L_2S^0)$, and we are interested in the stable homotopy category of spectra localized at the prime two. In his book [8], Ravenel constructed the spectrum $T(m)$ for each $m \geq 0$ characterized by

$$(1.1) \qquad BP_*(T(m)) = BP_*[t_1, \dots, t_m] \subset BP_*(BP) = BP_*[t_1, t_2, \dots]$$

as a $BP_*(BP)$-comodule. These spectra admit maps $T(m) \to T(m+1)$ inducing the inclusion on BP_*-homology, and $T(0)$ and $T(\infty)$ are the sphere and the Brown-Peterson spectra, respectively. The homotopy groups of $L_2T(\infty)$ are determined by Ravenel as $BP_* \oplus BP_*/(2^\infty, v_1^\infty, v_2^\infty)$ in [7]. We have partial results [2] and [4] on subgroups of the homotopy groups $\pi_*(L_2T(1))$. We use the 2- and the v_1- Bockstein spectral sequences to determine it for $m \geq 1$ in two different orders:

1) the v_1-Bockstein spectral sequence first and then the 2-Bockstein spectral sequence,

2) the 2-Bockstein spectral sequence first and then the v_1-Bockstein spectral sequence.

As the first step in the order 1), the v_1-Bockstein spectral sequence is computed in [3], and we obtain the homotopy groups of $L_2T(m) \wedge M$ for the modulo

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two Moore spectrum M. In this paper we consider the first step of the order 2).

Let $T(m)/v_1$ denote the cofiber of $\alpha\colon \Sigma^2T(m) \to T(m)$ for $m > 0$ such that $BP_*(\alpha) = v_1 - 2t_1$, whose existence is shown in section two. We then define a spectrum C by the cofiber sequence

$$T(m)/v_1 \xrightarrow{\eta} 2^{-1}T(m)/v_1 \longrightarrow C \longrightarrow \Sigma T(m)/v_1 \tag{1.2}$$

for the localization map $\eta\colon T(m)/v_1 \to 2^{-1}T(m)/v_1$. We first determine the Adams-Novikov E_2-term of L_2C in Proposition (3.8) by use of the 2-Bockstein spectral sequence associated to the cofiber sequence

$$D \xrightarrow{\iota} C \xrightarrow{2} C \xrightarrow{\kappa} \Sigma D, \tag{1.3}$$

where D denotes the spectrum $T(m)/v_1 \wedge M$ for the mod 2 Moore spectrum M. The E_2-term of the Adams-Novikov spectral sequence for $\pi_*(L_2D)$ is determined by Ravenel (*cf.* [8]) as follows:

$$E_2^*(L_2D) = K_m(2)_* \otimes \wedge(g_{10}, g_{11}, g_{20}, g_{21}), \tag{1.4}$$

where

$$K_m(2)_* = v_2^{-1}\mathbb{Z}/2[v_2, \dots, v_{m+2}], \tag{1.5}$$

and g_{ij} denotes the element of bidegree $(1, 2^{j+1}(2^{m+i} - 1))$, which is denoted by $h_{m+i,j}$ in [8]. Next, we show that every element of the Adams-Novikov E_2-term $E_2^*(L_2C)$ is a permanent cycle in Lemma (3.12), and the extension problem of the spectral sequence is trivial in Lemma (3.13). These show the homotopy groups of L_2C are isomorphic to the E_2-term.

In order to state our result, we introduce following notations: the algebra

$$E_m(2)_* = v_2^{-1}\mathbb{Z}_{(2)}[v_1, v_2, \dots, v_{m+2}]$$

such that $K_m(2)_* = E_m(2)_*/(2, v_1)$, the elements

$$u_i = v_{m+i} \in BP_* \quad \text{for } i \geq 1,$$

the algebras

$$\begin{aligned} R &= E_{m-2}(2)_*/(v_1) = v_2^{-1}\mathbb{Z}_{(2)}[v_2, \dots, v_m], \\ R^n &= R\left[u_1^{2^n}, u_2^{2^n}\right] \quad \text{and} \\ R_j^{(n)} &= R\left[u_j^{2^n}\right], \end{aligned}$$

and the submodules of $E_m(2)_*/(2^\infty, v_1) = R[u_1, u_2] \otimes \mathbb{Q}/\mathbb{Z}_{(2)}$:

$$\begin{aligned} \overline{M}(i) &= \bigoplus_{j=1}^{2} R_j^{(i+1)}/(2^{i+1})\left\{u_j^{2^i}/2^{i+1}\right\}, \\ M^0(i) &= R^{i+1}/(2^{i+1})\left\{u_1^{2^i}u_2^{2^{i+1}}/2^{i+1}, u_2^{2^i}u_1^{2^{i+1}}/2^{i+1}, u_1^{2^i}u_2^{2^i}/2^{i+1}\right\} \quad \text{and} \\ M^1(i) &= R^{i+1}/(2^{i+1})\Big\{u_2^{2^i}u_1^{2^{i+1}}\overline{g}_{10}/2^{i+1}, u_1^{2^i}u_2^{2^{i+1}}\overline{g}_{20}/2^{i+1}, \\ &\qquad u_1^{2^i}u_2^{2^i}\overline{g}_{10}/2^{i+1} = u_1^{2^i}u_2^{2^i}\overline{g}_{20}/2^{i+1}\Big\}. \end{aligned}$$

Here $\overline{g}_{j0}$ is an element such that $\overline{g}_{j0}/2 = u_j^{-1}g_{j0}/2$, whose existence is shown in Lemma (3.2).

THEOREM (1.6). *The homotopy groups $\pi_*(L_2C)$ for $m > 1$ are isomorphic, as an R-module, to the tensor product of $\wedge(g_{11}, g_{21})$ and the direct sum of the modules $R/(2^\infty)$, $\overline{M}(i)$, $M^0(i)$ and $M^1(i)$ for $i \geq 0$.*

Since the E_2-term $E_2^*(2^{-1}T(m)/v_1)$ is isomorphic to $H\mathbb{Q}_*(L_2T(m)/v_1) = \mathbb{Q}[v_2, v_3, \dots, v_m]$ by [8], (6.5.7), the homotopy groups of $L_2T(m)/v_1$ are obtained by observing the homotopy exact sequence associated to the cofiber sequence (1.2).

COROLLARY (1.7). *The homotopy groups $\pi_*(L_2T(m)/v_1)$ for $m > 1$ are isomorphic to the direct sum of the modules $\mathbb{Z}_{(2)}[v_2, v_3, \dots, v_m]$, $\Sigma^{-1}R/(2^\infty)$ $\{g_{11}, g_{21}, g_{11}g_{21}\}$ and $\bigoplus_{i\geq 0}\Sigma^{-1}(\overline{M}(i) \oplus M^0(i) \oplus M^1(i)) \otimes \wedge(g_{11}, g_{21})$. Here Σ denotes a shift of dimension.*

We note that the homotopy groups of $L_2T(1)/v_1$ are given in [6]. The structure of $\pi_*(L_2T(m)/v_1)$ for $m > 1$ in Corollary (1.7) is less complicated than that of the case for $m = 1$. So it seems that it is useful to determine the homotopy groups $\pi_*(L_2T(m))$ for $m > 1$ completely. For $m = 1$, we know the structure of subgroups of $\pi_*(L_2T(1))$ (*cf.* [2], [4]).

2. A change of rings theorem and structure maps

We work in the stable homotopy category of spectra localized at the prime two. Let BP denote the Brown-Peterson ring spectrum, and consider the Hopf algebroid (A, Γ) associated with it, where

$$A = \pi_*(BP) = BP_* = \mathbb{Z}_{(2)}[v_1, v_2, \dots],$$
$$\Gamma = BP_*(BP) = BP_*[t_1, t_2, \dots].$$

The Hopf algebroid Γ gives rise to another one

$$(A, \Gamma_m) = (A, \Gamma/(t_1, \dots, t_m)) = (A, BP_*[t_{m+1}, t_{m+2}, \dots]).$$

Recall the Ravenel spectrum $T(m)$ in (1.1) for $m \geq 0$, which is a ring spectrum with multiplication $\mu\colon T(m)\wedge T(m) \to T(m)$. Ravenel showed in [8] the change of rings theorem

$$E_2^*(T(m) \wedge X) = \mathrm{Ext}^*_{\Gamma_m}(A, BP_*(X))$$

for a spectrum X. If X is the sphere spectrum S^0, then we have an element $v_1 \in \mathrm{Ext}^{0,2}_{\Gamma_m}(A, A)$ for $m > 0$. This element is represented by $v_1 - 2t_1$ in the cobar complex $\Omega^0_\Gamma BP_*(T(m))$ for computing $E_2^*(T(m))$. Since $E_2^{s,1+s}(T(m)) = 0$ by observing the reduced cobar complex, the element v_1 survives to a homotopy element $\alpha' \in \pi_2(T(m))$. We now let $T(m)/v_1$ denote the cofiber of the composite

$$\alpha\colon \Sigma^2T(m) = T(m) \wedge S^2 \xrightarrow{1\wedge\alpha'} T(m) \wedge T(m) \xrightarrow{\mu} T(m).$$

Let M and M_∞ be the modulo two Moore spectrum and the cofiber of the localization map $S^0 \to S\mathbb{Q}$, respectively. In this paper we consider the spectra

$$D = T(m)/v_1 \wedge M \quad \text{and} \quad C = T(m)/v_1 \wedge M_\infty.$$

These fit in the cofiber sequence (1.3). The BP_*-homologies of the L_2-localizations of these spectra are

$$BP_*(L_2D) = v_2^{-1}BP_*/(2, v_1)[t_1, \dots, t_m] \quad \text{and}$$
$$BP_*(L_2C) = v_2^{-1}BP_*/(2^\infty, v_1)[t_1, \dots, t_m].$$

Consider a spectrum

$$E_m(2) = v_2^{-1}BP\langle m+2\rangle$$

for the Johnson-Wilson spectrum $BP\langle m+2\rangle$ such that $\pi_*(BP\langle m+2\rangle) = \mathbb{Z}_{(2)}[v_1, v_2, \dots, v_{m+2}]$. Since

$$v_2^{-1}BP_*/J \xrightarrow{1\otimes\eta_R} E_m(2)_*/J \otimes_A \Gamma_m$$

for an invariant regular ideal J of length two is a faithfully flat extension, we have an isomorphism

$$\text{(2.1)} \qquad \operatorname{Ext}^*_{\Gamma_m}(A, v_2^{-1}BP_*/J) \cong \operatorname{Ext}^*_{\Sigma_m(2)}(E_m(2)_*, E_m(2)_*/J)$$

shown by the same way as the proofs of the change of rings theorem in [1]. Here

$$\Sigma_m(2) = E_m(2)_* \otimes_A \Gamma_m \otimes_A E_m(2)_*$$

is the induced Hopf algebroid, and

$$\text{(2.2)} \qquad \Sigma_m(2) = E_m(2)_*[t_1, t_2, \dots]/(\eta_R(v_{m+k}) : k > 2).$$

Note that $m+2$ is the smallest number n such that

$$v_2^{-1}BP_*/J \xrightarrow{1\otimes\eta_R} v_2^{-1}BP\langle n\rangle_*/J \otimes_A \Gamma_m$$

is a faithfully flat extension.

PROPOSITION (2.3). *The Adams-Novikov E_2-terms for computing $\pi_*(L_2C)$ and $\pi_*(L_2D)$ are isomorphic to*

$$E_2^*(L_2C) = \operatorname{Ext}^*_{\Sigma_m(2)}(E_m(2)_*, E_m(2)_*/(2^\infty, v_1)) \quad \textit{and}$$
$$E_2^*(L_2D) = \operatorname{Ext}^*_{\Sigma_m(2)}(E_m(2)_*, E_m(2)_*/(2, v_1)).$$

Proof. The isomorphism on $E_2^*(L_2D)$ follows from (2.1). Since $L_2C = \text{hoco-}\lim_k L_2T(m) \wedge M_k$ for the mod 2^k Moore spectrum M_k, the change of rings theorem (2.1) also shows the isomorphism on $E_2^*(L_2C)$. □

Consider the Hopf algebroid $(E_m(2)_*, \Sigma_m(2))$ (see (2.2)). We read off the behavior of the right unit $\eta_R\colon E_m(2)_* \to \Sigma_m(2)$ and the diagonal $\Delta\colon \Sigma_m(2) \to \Sigma_m(2) \otimes_{E_m(2)_*} \Sigma_m(2)$ from that of Γ_m. Hereafter we set $v_2 = 1$ and use the notation

$$u_i = v_{m+i} \quad \text{and} \quad s_i = t_{m+i}$$

for $i = 1, 2$. Recall the Hazewinkel and the Quillen formulas:

$$v_n = 2\ell_n - \sum_{k=1}^{n-1} \ell_k v_{n-k}^{2^k} \in \mathbb{Q} \otimes A = \mathbb{Q}[\ell_1, \ell_2, \dots],$$

$$\eta_R(\ell_n) = \sum_{k=0}^{n} \ell_k t_{n-k}^{2^k} \in \mathbb{Q} \otimes \Gamma = \mathbb{Q} \otimes A[t_1, t_2, \dots], \quad \text{and}$$

$$\sum_{i+j=n} \ell_i \Delta(t_j^{2^i}) = \sum_{i+j+k=n} \ell_i t_j^{2^i} \otimes t_k^{2^{i+j}} \in \mathbb{Q} \otimes \Gamma \otimes_A \Gamma.$$

Then a routine computation shows

LEMMA (2.4). *The right unit $\eta_R \colon A \to \Gamma_m$ and the diagonal $\Delta \colon \Gamma_m \to \Gamma_m \otimes_A \Gamma_m$ act on generators as follows:*

$$\begin{aligned} \eta_R(v_n) &= v_n \quad \text{for } n \le m, \\ \eta_R(u_1) &= u_1 + 2s_1, \\ \eta_R(u_2) &\equiv u_2 + 2s_2 \mod (v_1), \\ \Delta(s_1) &= s_1 \otimes 1 + 1 \otimes s_1, \\ \Delta(s_2) &\equiv s_2 \otimes 1 + 1 \otimes s_2 \mod (v_1). \end{aligned}$$

3. The Adams-Novikov E_2-term for $\pi_*(L_2C)$

We begin introducing the cocycles of cobar complexes that represent generators g_{j1} and $\overline{g}_{j0}$.

LEMMA (3.1). *The elements $s_j^2 + u_j s_j$ for $j = 1, 2$ are cocycles of the cobar complex $\Omega^1_{\Gamma_m} E_m(2)_*/(v_1)$.*

Proof. Since $d(u_j) \equiv 2s_j$ and $d(s_j) \equiv 0$,

$$d(s_j^2 + u_j s_j) \equiv -2s_j \otimes s_j + 2s_j \otimes s_j \equiv 0 \mod (v_1). \qquad \square$$

LEMMA (3.2). *There are elements*

$$2w_j = \sum_{n>0} (-1)^{n-1} \frac{1}{n} (2u_j^{-1} s_j)^n \in \Omega^1_{\Gamma_m} u_j^{-1} E_m(2)/(2^k, v_1)$$

for $j = 1, 2$ such that $d(w_j) = 0$.

Proof. Note that $\sum_{n>0} (-1)^{n-1} \frac{1}{n} (2u_j^{-1} s_j)^n = \log(1 + 2u_j^{-1} s_j) = \log(\eta_R(u_j)) - \log(u_j)$. Since

$$\begin{aligned} \log(\eta_R(u_j)) &\equiv \log(1 - (1 - \eta_R(u_j))) = -\sum_{n>0} \frac{1}{n} (1 - \eta_R(u_j))^n \\ &\equiv -\sum_{n>0} \eta_R(\frac{1}{n}(1 - u_j)^n) = -\eta_R(\sum_{n>0} \frac{1}{n}(1 - u_j)^n) \\ &\equiv \eta_R(-\sum_{n>0} \frac{1}{n}(1 - u_j)^n) = \eta_R(\log(1 - (1 - u_j))) \\ &\equiv \eta_R(\log(u_j)), \end{aligned}$$

we see that $d(2w_j) = d(\log(1+2u_j^{-1}s_j)) = dd(\log(u_j)) = 0$. Therefore, $d(w_j) = 0$ as desired. □

Note that $u_j^{2^{k-1}} w_j$ in $\Omega^1_{\Gamma_m} E_m(2)/(2^k, v_1)$ for each $k > 0$. Let

$$g_{j1} \quad \text{and} \quad u_j^{2^{k-1}} \overline{g}_{j0} \in E_2^1(L_2T(m)/v_1 \wedge M_k)$$

denote the homology classes of the cocycles of Lemma (3.1) and $u_j^{2^{k-1}} w_j$, respectively, for each $k > 0$, where M_k denotes the mod 2^k Moore spectrum.

Consider the subalgebras

$$\begin{aligned} F &\equiv K_{m-2}(2)_* = R/(2) = v_2^{-1}\mathbb{Z}/2[v_2, \ldots, v_m], \\ F^{(n)} &\equiv F\left[u_1^{2^n}, u_2^{2^n}\right], \quad \text{and} \\ F_j^{(n)} &\equiv F\left[u_j^{2^n}\right], \end{aligned} \tag{3.3}$$

and the submodules

$$\begin{aligned} \overline{N}(i) &\equiv \bigoplus_{j=1}^{2} u_j^{2^i} F_j^{(i+1)} \quad \text{and} \\ N^0(i) &\equiv F^{(i+1)}\left\{u_1^{2^i}u_2^{2^{i+1}}, u_2^{2^i}u_1^{2^{i+1}}, u_1^{2^i}u_2^{2^i}\right\} \end{aligned}$$

of the polynomial algebra $K_m(2)_* = F[u_1, u_2]$. Then, as an F-module,

$$\begin{aligned} K_m(2)_* &\equiv (F[u_1] + F[u_2]) \oplus \bigoplus_{i\ge 0} N^0(i) \\ &\equiv F \oplus \bigoplus_{i\ge 0}\left(\overline{N}(i) \oplus N^0(i)\right), \\ u_jK_m(2)_* &\equiv u_jF[u_j] \oplus \bigoplus_{i\ge 0} N^0(i) \quad \text{and} \\ u_1u_2K_m(2)_* &\equiv \bigoplus_{i\ge 0} N^0(i) \end{aligned} \tag{3.4}$$

for $j = 1, 2$. Under these notations, we rewrite (1.4) as follows:

$$E_2^*(L_2D) = \wedge(g_{11}, g_{21}) \otimes \left(K_m(2)_* \otimes \wedge(u_1\overline{g}_{10}, u_2\overline{g}_{20})\right). \tag{3.5}$$

The factor $K_m(2)_* \otimes \wedge(u_1\overline{g}_{10}, u_2\overline{g}_{20})$ is decomposed into the direct sum

$$K_m(2)_* \oplus \overline{g}_{10}(u_1K_m(2)_*) \oplus \overline{g}_{20}(u_2K_m(2)_*) \oplus \overline{g}_{10}\overline{g}_{20}(u_1u_2K_m(2)_*). \tag{3.6}$$

We consider the connecting homomorphism $\delta\colon E_2^s(L_2C) \to E_2^{s+1}(L_2D)$ on the factor $K_m(2)_* \otimes \wedge(u_1\overline{g}_{10}, u_2\overline{g}_{20})$. The behavior of δ is read off from the following lemma:

LEMMA (3.7). *The connecting homomorphism δ acts as an R-module map on the elements of $E_2^0(L_2C)$ as follows:*

$$\begin{aligned} \delta(1/2^i) &\equiv 0 \quad \textit{and} \\ \delta(u_1^{2^is}u_2^{2^it}/2^{i+1}) &\equiv su_1^{2^is}u_2^{2^it}\overline{g}_{10} + tu_1^{2^is}u_2^{2^it}\overline{g}_{20}, \end{aligned}$$

where s, t and i are non-negative integers.

Proof. Note that $u_j^{a-1}s_j$ represents $u_j^a\overline{g}_{j0}$. The lemma follows then immediately from the relations $d(u_j) \equiv 2s_j$ and the binomial coefficient theorem. $\square$

PROPOSITION (3.8). *The Adams-Novikov E_2-term $E_2^*(L_2C)$ is isomorphic to the module given in Theorem* (1.6).

Proof. Put $E^* = K_m(2)_* \otimes \wedge(u_1\overline{g}_{10}, u_2\overline{g}_{20})$, $B^0 = R/(2^\infty) \oplus \bigoplus_{i\geq 0}\left(\overline{M}(i) \oplus M^0(i)\right)$ and $B^1 = \bigoplus_{i\geq 0} M^1(i)$. By [5], Remark 3.11, it suffices to show that the sequence

$$0 \longrightarrow E^0 \longrightarrow B^0 \xrightarrow{2} B^0 \xrightarrow{\delta} E^1 \longrightarrow B^1 \xrightarrow{2} B^1 \xrightarrow{\delta} E^2 \longrightarrow 0 \tag{3.9}$$

is exact. In fact, $B^* \otimes \wedge(g_{11}, g_{21}) \subset E_2^*(L_2C)$ by Lemma (3.7), and the exact sequence (3.9) induces a commutative diagram

$$\begin{tikzcd}
(E^* \otimes \Lambda)^s \arrow[r] \arrow[d, equal] & (B^* \otimes \Lambda)^s \arrow[r] \arrow[d] & (B^* \otimes \Lambda)^s \arrow[r, "\delta"] \arrow[d] & (E^* \otimes \Lambda)^{s+1} \arrow[d, equal] \\
E_2^s(L_2D) \arrow[r, "\iota_*"] & E_2^s(L_2C) \arrow[r, "2"] & E_2^s(L_2C) \arrow[r, "\delta"] & E_2^{s+1}(L_2D)
\end{tikzcd}$$

of exact sequences, where $\Lambda = \wedge(g_{11}, g_{21})$. Then, the middle maps are isomorphisms by [5], Remark 3.11.

By (3.6) and (3.4),

$$\begin{aligned}
E^0 &\equiv F \oplus \bigoplus_{i\geq 0}\left(\overline{N}(i) \oplus N^0(i)\right), \\
E^1 &\equiv \left(\bigoplus_{j=1}^{2} \overline{g}_{j0}u_jF[u_j]\right) \oplus \bigoplus_{i\geq 0}\left(E^{1,I}(i) \oplus E^{1,C}(i)\right) \\
&\equiv \bigoplus_{i\geq 0}\left(\overline{N}^1(i) \oplus E^{1,I}(i) \oplus E^{1,C}(i)\right), \\
E^2 &\equiv \bigoplus_{i\geq 0} \overline{g}_{10}\overline{g}_{20}N^0(i),
\end{aligned}$$

where

$$\begin{aligned}
E^{1,I}(i) &\equiv F^{(i+1)}\left\{u_1^{2^i}u_2^{2^{i+1}}\overline{g}_{10}, u_2^{2^i}u_1^{2^{i+1}}\overline{g}_{20}, u_1^{2^i}u_2^{2^i}\overline{g}_{10}\right\}, \\
E^{1,C}(i) &\equiv F^{(i+1)}\left\{u_1^{2^i}u_2^{2^{i+1}}\overline{g}_{20}, u_2^{2^i}u_1^{2^{i+1}}\overline{g}_{10}, u_1^{2^i}u_2^{2^i}\overline{g}_{20}\right\}, \quad \text{and} \\
\overline{N}^1(i) &\equiv \bigoplus_{j=1}^{2} \overline{g}_{j0}u_j^{2^i}F_j^{(i+1)}.
\end{aligned}$$

Note that $u_jF[u_j] = \bigoplus_{i\geq 0} u_j^{2^i}F_j^{(i+1)}$. Each summand of E^0 fits in one of the exact sequences

$$0 \longrightarrow F \longrightarrow F/(2^\infty) \xrightarrow{2} F/(2^\infty) \longrightarrow 0,$$

$$0 \longrightarrow \overline{N}(i) \longrightarrow \overline{M}(i) \xrightarrow{2} \overline{M}(i) \xrightarrow{\delta} \overline{N}^1(i) \longrightarrow 0, \quad \text{and}$$

$$0 \longrightarrow N^0(i) \longrightarrow M^0(i) \xrightarrow{2} M^0(i) \xrightarrow{\delta} E^{1,I}(i) \longrightarrow 0$$

by Lemma (3.7), and the direct sum of these shows the exact sequence

$$0 \longrightarrow E^0 \longrightarrow B^0 \xrightarrow{2} B^0 \xrightarrow{\delta} E^1 \longrightarrow \bigoplus_{i\geq 0} E^{1,C}(i) \longrightarrow 0. \tag{3.10}$$

Lemma (3.7) also shows the exact sequence

$$0 \longrightarrow E^{1,C}(i) \longrightarrow M^1(i) \xrightarrow{2} M^1(i) \xrightarrow{\delta} \overline{g}_{10}\overline{g}_{20}N^0(i) \longrightarrow 0$$

and the direct sum yields the exact sequence

$$0 \longrightarrow \bigoplus_{i\geq 0} E^{1,C}(i) \longrightarrow B^1 \xrightarrow{2} B^1 \xrightarrow{\delta} E^2 \longrightarrow 0. \tag{3.11}$$

Splice the exact sequences (3.10) and (3.11) and we obtain the desired exact sequence (3.9). □

Since the Adams-Novikov E_2-term $E_2^s(L_2C)$ for $s > 3$ is trivial by Proposition (3.8), every element of $E_2^s(L_2C)$ for $0 < s \leq 3$ is a permanent cycle in the Adams-Novikov spectral sequence. For $s = 0$, we have

LEMMA (3.12). *Every element of $E_2^0(L_2C)$ is a permanent cycle in the Adams-Novikov spectral sequence.*

Proof. Let $x/2^i \in E_2^0(L_2C)$. Suppose that $d_3(x/2^i) = y/2^j \neq 0$. If $x/2^{i+1} \in R/(2^\infty)$, then there exist elements $y_k = d_3(x/2^k)$ for $k > i$ such that $2y_k = y_{k-1}$ and $2y_{i+1} = y/2^j \neq 0$, and so the y_k's generate a module isomorphic to $R/(2^\infty)$ in $E_2^3(L_2C)$. This contradicts Proposition (3.8). So we may assume that $x/2^{i+1}$ belongs to $\overline{M}(i)$ or $M^0(i)$. Then, $d_3(x/2^l) = y/2 \neq 0$ for $l = i - j + 1$. Since $x \in E_2^0(L_2D)$ is a permanent cycle by Ravenel [8], the integer l is greater than one. Then, the element $x/2^{l-1}$ is a permanent cycle and survives to a homotopy element $[x/2^{l-1}]$ such that $\kappa_*([x/2^{l-1}]) = [y] \in \pi_*(L_2D)$, where κ is the map in (1.3), and $[z]$ denotes the homotopy element detected by an element z in the E_2-term. Since $y \in E_2^3(L_2D)$, there is an element $h \in \{g_{ji} : j = 1, 2, i = 0, 1\}$ such that $yh \neq 0 \in E_2^4(L_2D)$. Note that it detects $[yh] \neq 0 \in \pi_*(L_2D)$. By Proposition (3.8), we see that $xh/2^l \in E_2^1(L_2C)$, which is a permanent cycle since $E_2^s(L_2C) = 0$ for $s > 3$. This implies a contradiction: $0 \neq [yh] = \kappa_*([xh/2^{l-1}]) = \kappa_* 2_*([xh/2^l]) = 0$. We notice here that $xg_{j0}/2^{i+1} \in E_2^1(L_2C)$ since the cochain $xs_j/2^{i+1}$ is a cocycle. □

LEMMA (3.13). *In the Adams-Novikov spectral sequence, the extension problem as an R-module is trivial.*

Proof. Let $\xi \in \pi_*(L_2C)$ be elements detected by $x/2^j \in E_\infty^0(L_2C) = E_2^0(L_2C)$. It suffices to show that $2^j\xi = 0$. Indeed, the relation $2^j(x/2^j) = 0$ in the E_2-term gives that of the homotopy. Since $x \in E_2^0(L_2D)$ is a permanent cycle (*cf.* [8]) and $2^{j-1}\xi$ is detected by $x/2$, $2^{j-1}\xi$ is in the image of the induced map $\iota_* \colon \pi_*(L_2D) \to \pi_*(L_2C)$ from the map in (1.3). It follows that $2^j(\iota_*([x])) = \iota_*([2x]) = 0$ as desired. □

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