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HARMONIC SUMS AND RATIONAL MULTIPLES OF ZETA FUNCTIONS

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ABSTRACT. In this paper we develop some interesting new identities associated with harmonic numbers and give their integral representations. In particular cases we prove the infinite sums may be expressed as rational multiples of zeta functions.

1. Introduction

In this paper we are interested in expressing sums of the type

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^3 \binom{n+k}{k}^2}$$

in closed form. Firstly we develop some general theorems involving the integral representation of the more general sums

$$\sum_{n=1}^{\infty} \frac{t^n \, Q^{(m)}(a,j)}{\binom{bn+k}{k}\binom{cn+l}{l}},$$

where $Q^{(m)}(a,j)$ is the m^{th} derivative of the reciprocal binomial coefficient

$$\binom{an+j}{j}^{-1} = \frac{\Gamma(j+1)\Gamma(an+1)}{\Gamma(an+j+1)}.$$

There are many striking formulas of the type

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{(n+1)^2} = \zeta(3) \text{ and } \sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^3} = \frac{5}{4}\zeta(4),$$

where the Riemann zeta function

$$\zeta(z) = \sum_{r=1}^{\infty} \frac{1}{r^z}, \; \mathbb{R}(z) > 1$$

and the generalized harmonic number in power α is defined as

$$H_n^{(\alpha)} = \sum_{r=1}^{n} \frac{1}{r^\alpha}.$$

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The n^{th} harmonic number

$$H_n^{(1)} = H_n = \int_0^1 \frac{1-t^n}{1-t} dt = \sum_{r=1}^{n} \frac{1}{r} = \gamma + \psi(n+1),$$

where γ denotes the Euler-Mascheroni constant, defined by

$$\gamma = \lim_{n\to\infty} \left(\sum_{r=1}^{n} \frac{1}{r} - \log(n) \right) = -\psi(1) \approx 0.57721566490153286060651 2.........$$

Other striking representations are

$$2\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^q} = (q+2)\zeta(q+1) - \sum_{r=1}^{q-2} \zeta(r+1)\zeta(q-r), \tag{1.1}$$

due to Euler [6], see also [5] , and

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{n^{2q+1}} = \frac{1}{2}\sum_{r=2}^{2q} (-1)^r \zeta(r)\zeta(2q-r+2) \tag{1.2}$$

due to Georghiou and Philippou [8]. Further work in the summation of harmonic numbers and binomial coefficients has also been done by Flajolet and Salvy [7] and Basu [4]. The works of [1], [2], [3], [9], [10], [11], [12], [13], [14], [15] and [17] also investigate various representations of binomial sums and zeta functions in simpler form by the use of the Beta function and other techniques. Zhao and Zhao, [19] have also recently given some asymptotic expansions of certain sums involving powers of binomial coefficients.

2. Harmonic Numbers

LEMMA (2.1). *Let a be a positive real number with $j \geq 0$, $n > 0$ and $Q(a,j) = \binom{an+j}{j}^{-1}$ is an analytic function in j then,*

$$Q^{(1)}(a,j) = \frac{dQ}{dj} = \begin{cases} -Q(a,j)P(a,j), \ \text{where} \\ P(a,j) = \sum_{r=1}^{an} \frac{1}{r+j} \ \text{for } j > 0 \\ = -Q(a,j)\left[\psi(j+1+an) - \psi(j+1)\right] \\ -H_n^{(1)}, \ \text{for } j = 0 \text{ and } a = 1 \end{cases}$$

and

$$Q^{(\lambda)}(a,j) = \frac{d^\lambda Q}{dj^\lambda} = -\sum_{\rho=0}^{\lambda-1} \binom{\lambda-1}{\rho} Q^{(\rho)}(a,j) P^{(\lambda-1-\rho)}(a,j), \ \text{for } \lambda \geq 2$$

where $P^{(0)}(a,j) = \sum_{r=1}^{an} \frac{1}{r+j}$, for $n = 1,2,3,...$, and $Q^{(0)}(a,j) = Q(a,j)$. For $i = 1,2,3...$

$$P^{(i)}(a,j) = \frac{d^i P}{dj^i} = \frac{d^i}{dj^i}\left(\sum_{r=1}^{an} \frac{1}{r+j}\right).$$

The polygamma functions $\psi^{(k)}(z)$, $k \in \mathbb{N}$ *are defined by*

$$\psi^{(k)}(z) \;:\; = \frac{d^{k+1}}{dz^{k+1}} \log \Gamma(z) = \frac{d^k}{dz^k}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)$$

$$= -\int_0^1 \frac{[\log(t)]^k \; t^{z-1}}{1-t} dt, \; k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{N} = \{1,2,3,...\}$$

and $\psi^{(0)}(z) = \psi(z)$, *denotes the Psi, or digamma function, defined by*

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

Now we list some particular case of Lemma 2.1, when a *is a positive integer*

$$Q^{(3)}(a,j) = -\binom{an+j}{j}^{-1} \left[\begin{array}{c} \left(\sum\limits_{r=1}^{an} \frac{1}{r+j}\right)^3 + 2\sum\limits_{r=1}^{an} \frac{1}{(r+j)^3} \\ +3\sum\limits_{r=1}^{an} \frac{1}{(r+j)^2} \sum\limits_{r=1}^{an} \frac{1}{r+j} \end{array} \right]$$

and

$$Q^{(4)}(a,j) = \binom{an+j}{j}^{-1} \left[\begin{array}{c} 6\sum\limits_{r=1}^{an} \frac{1}{(r+j)^2} \left(\sum\limits_{r=1}^{an} \frac{1}{r+j}\right)^2 \\ +8\sum\limits_{r=1}^{an} \frac{1}{(r+j)^3} \sum\limits_{r=1}^{an} \frac{1}{r+j} + 3\left(\sum\limits_{r=1}^{an} \frac{1}{(r+j)^2}\right)^2 \\ +\left(\sum\limits_{r=1}^{an} \frac{1}{r+j}\right)^4 + 6\sum\limits_{r=1}^{an} \frac{1}{(r+j)^4} \end{array} \right],$$

when a *is a real number the sums maybe written in terms of polygamma functions. In the special case when* $a = 1$ *and* $j = 0$ *we may write*

$$Q^{(3)}(1,0) = \left(H_n^{(1)}\right)^3 + 3H_n^{(1)}H_n^{(2)} + 2H_n^{(3)},$$

and

$$Q^{(4)}(1,0) = 6H_n^{(2)}\left(H_n^{(1)}\right)^2 + 8H_n^{(3)}H_n^{(1)} + 3\left(H_n^{(2)}\right)^2 + \left(H_n^{(1)}\right)^4 + 6H_n^{(4)}.$$

Proof. A proof of this Lemma may be seen in [10]. □

Now consider

THEOREM (2.3). *Let* $a \geq 0$, $b \geq 0$, $c \geq 0$ *be real positive numbers* $j,k,l \geq 0$, *and* $|t| \leq 1$. *Then*

$$\sum_{n=1}^{\infty} \frac{t^n \sum\limits_{r=1}^{an} \frac{1}{r+j}}{n^3 \binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}} = \sum_{n=1}^{\infty} \frac{t^n \left[\psi(j+1+an) - \psi(j+1)\right]}{n^3 \binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}} \tag{2.4}$$

$$= -abct \int_0^1 \int_0^1 \int_0^1 \frac{(1-x)^j (1-y)^k (1-z)^l x^{a-1} y^{b-1} z^{c-1} \log(1-x)}{1 - tx^a y^b z^c} dxdydz. \tag{2.5}$$

Proof. The proof of Theorem 2.3 follows similar arguments as that used for Theorem 2.17 and will not be detailed here. □

Some related results are the following:

REMARK (2.6). *For $a=b=c=1$, $t=1$ and $j=l=0$ we have that*

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{\binom{n+k}{k}} = -k\int_0^1\int_0^1\frac{(1-y)^{k-1}y\log(1-x)}{(1-xy)^2}dxdy$$

$$= \frac{k}{(k-1)^2} \text{ for } k>1, \tag{2.7}$$

this is a result that is communicated by Cloitre and cited in [16], *and later proved by Sofo* [13]. *Also for $a=b=c=1$, $t=\frac{1}{2}$ and $j=l=0$, Sofo* [13] *obtained the new result*

$$\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)^n H_n^{(1)}}{\binom{n+k}{k}} = (-1)^{k+1}k\ln^2(2)+2(-1)^{k+1}k\sum_{r=1}^{k-1}\binom{k-1}{r}\frac{(-1)^r}{r}\ln(2)$$
$$+2(-1)^{k+1}k\sum_{r=1}^{k-1}\binom{k-1}{r}\frac{(-1)^r(1-2^r)}{r^2}.$$

The following new results are consequences of Theorem 2.3.

COROLLARY (2.8). *Let $a=b=c=1$, $j=0$, $k=0$, $l\geq 1$, and $t=1$. Then*

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3\binom{n+l}{l}} = -\int_0^1\int_0^1\int_0^1\frac{(1-z)^{l-1}\ln(1-x)}{1-xyz}dxdydz \tag{2.9}$$

$$= \frac{5}{4}\zeta(4)-2H_l^{(1)}\zeta(3)+\frac{1}{2}\left[\left(H_l^{(1)}\right)^2+H_l^{(2)}\right]\zeta(2) \tag{2.10}$$
$$+\frac{1}{2}\sum_{r=1}^{l}\binom{l}{r}\frac{(-1)^{r+1}}{r^2}\left(\left(H_{r-1}^{(1)}\right)^2+H_{r-1}^{(2)}\right).$$

Proof. To prove (2.10), consider (2.9) and expand as follows:

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3\binom{n+l}{l}} = \sum_{n=1}^{\infty}\frac{l!H_n^{(1)}}{n^3(n+1)_{l+1}}$$
$$= \sum_{n=1}^{\infty}\frac{l!H_n^{(1)}}{n^3}\sum_{r=1}^{l}\frac{A_r}{(n+r)}$$

where

$$A_r = \lim_{n\to(-r)}\left\{\frac{n+r}{\prod_{r=1}^{l}(n+r)}\right\} = \frac{(-1)^{r+1}}{(l-r)!(r-1)!}$$
$$= \frac{(-1)^{r+1}}{l!}\binom{l}{r}\binom{r}{1},$$

here, $(\alpha)_r$ is Pochhammer's symbol defined by $(\alpha)_r = \alpha(\alpha+1)(\alpha+2)...(\alpha+r-1)$, $r>0$, $(\alpha)_0 = 1$. Now, by rearrangement

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{l!H_n^{(1)}}{n^3}\left[\sum_{r=1}^{l}\frac{(-1)^{r+1}}{(n+r)\,l!}\binom{l}{r}\binom{r}{1}\right]\\ =\;&\sum_{r=1}^{l}\frac{(-1)^{r+1}}{l!}\binom{l}{r}\binom{r}{1}\sum_{n=1}^{\infty}\frac{l!H_n^{(1)}}{n^3(n+r)}.\end{aligned}$$

Since we have that

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3(n+r)} = \frac{5}{4r}\zeta(4)-\frac{2}{r^2}\zeta(3)+\frac{1}{r^3}\zeta(2)+\frac{\left(H_{r-1}^{(1)}\right)^2+H_{r-1}^{(2)}}{2r^3},$$

then we may write

$$\begin{aligned}&\sum_{r=1}^{l}\frac{(-1)^{r+1}}{l!}\binom{l}{r}\binom{r}{1}\sum_{n=1}^{\infty}\frac{l!H_n^{(1)}}{n^3(n+r)}\\ =&\sum_{r=1}^{l}(-1)^{r+1}\binom{l}{r}\binom{r}{1}\left[\begin{array}{c}\frac{5}{4r}\zeta(4)-\frac{2}{r^2}\zeta(3)+\frac{1}{r^3}\zeta(2)\\ +\frac{\left(H_{r-1}^{(1)}\right)^2+H_{r-1}^{(2)}}{2r^3}\end{array}\right].\end{aligned}$$

With the aid of Mathematica, and after some simplification we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3\binom{n+l}{l}} &= \frac{5}{4}\zeta(4)-2H_l^{(1)}\zeta(3)+\frac{1}{2}\left[\left(H_l^{(1)}\right)^2+H_l^{(2)}\right]\zeta(2)\\ &\quad+\frac{1}{2}\sum_{r=1}^{l}\binom{l}{r}\frac{(-1)^{r+1}}{r^2}\left(\left(H_{r-1}^{(1)}\right)^2+H_{r-1}^{(2)}\right),\end{aligned}$$

which is the result (2.10). □

REMARK (2.11). *The degenerate case, when $l=0$, furnishes the well known result* [8], *or from (1.1) or (1.2),*

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3} &= -\int_0^1\int_0^1\int_0^1\frac{\ln(1-x)}{1-xyz}dxdydz\\ &= -\int_0^1\frac{\ln(1-x)\psi^{(2)}(x)}{x}dx\\ &= \frac{1}{2}(\zeta(2))^2=\frac{5}{4}\zeta(4),\end{aligned}$$

and

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3\binom{n+8}{8}} = \frac{5}{4}\zeta(4)-\frac{761}{140}\zeta(3)+\frac{3144919}{705600}\zeta(2)-\frac{8269549}{4064256}.$$

The next result is another consequence of Theorem 2.3.

THEOREM (2.12). *Let $a=b=c=1$, $j=0$, $l=k$, $k\geq 1$, and $t=1$. Then*

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3\binom{n+k}{k}^2} = -\int_0^1\int_0^1\int_0^1\frac{((1-y)(1-z))^k\ln(1-x)}{1-xyz}dxdydz$$

$$= \frac{5}{4}\zeta(4) + \sum_{r=1}^{k}\left(-\lambda_r\,\zeta(3) + \mu_r\,\zeta(2) + \nu_r\right) \tag{2.13}$$

where

$$\begin{aligned}\lambda_r &= \left(r\binom{k}{r}\right)^2\left(\frac{5}{r^3} + \frac{4}{r^2}\left(H_{r-1}^{(1)} - H_{k-r}^{(1)}\right)\right),\\ \mu_r &= \left(r\binom{k}{r}\right)^2\left(\frac{3}{r^4} - \frac{H_{r-1}^{(1)}}{r^3} + \frac{2}{r^3}\left(H_{r-1}^{(1)} - H_{k-r}^{(1)}\right)\right)\end{aligned}$$

and

$$\nu_r = \left(r\binom{k}{r}\right)^2\left(\begin{array}{c}\frac{3}{2r^4}\left[\left(H_{r-1}^{(1)}\right)^2 + H_{r-1}^{(2)}\right] + \frac{1}{r^3}\left[H_{r-1}^{(1)}H_{r-1}^{(2)} + H_{r-1}^{(3)}\right]\\ +\frac{1}{r^3}\left(\left(H_{r-1}^{(1)}\right)^2 + H_{r-1}^{(2)}\right)\left(H_{r-1}^{(1)} - H_{k-r}^{(1)}\right)\end{array}\right).$$

Proof. Consider the following expansion:

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3\binom{n+k}{k}^2} = \sum_{n=1}^{\infty}\frac{(k!)^2\ H_n^{(1)}}{n^3\left((n+1)_{k+1}\right)^2}$$

$$= \sum_{n=1}^{\infty}\frac{(k!)^2 H_n^{(1)}}{n^3}\sum_{r=1}^{k}\left[\frac{A_r}{(n+r)} + \frac{B_r}{(n+r)^2}\right]. \tag{2.14}$$

Now

$$\begin{aligned}B_r &= \lim_{n\to(-r)}\left\{\frac{(n+r)^2}{\prod_{r=1}^{k}(n+r)^2}\right\} = \frac{1}{[(k-r)!\,(r-1)!]^2};\ k\geq 1\\ &= \left(\frac{1}{k!}\binom{k}{r}\binom{r}{1}\right)^2\end{aligned}$$

and

$$A_r = \lim_{n\to(-r)}\frac{d}{dn}\left\{\frac{(n+r)^2}{\prod_{r=1}^{k}(n+r)^2}\right\} = -2\left(\frac{1}{k!}\binom{k}{r}\binom{r}{1}\right)^2\left(H_{k-r}^{(1)} - H_{r-1}^{(1)}\right).$$

From (2.14)

$$\sum_{n=1}^{\infty}\frac{(k!)^2 H_n^{(1)}}{n^3}\sum_{r=1}^{k}\left(\frac{1}{k!}\binom{k}{r}\binom{r}{1}\right)^2\left[\frac{1}{(n+r)^2} - \frac{2\left(H_{k-r}^{(1)} - H_{r-1}^{(1)}\right)}{n+r}\right]$$

$$= \sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3(n+r)^2}$$

$$+2\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left(H_{r-1}^{(1)} - H_{k-r}^{(1)}\right)\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3(n+r)}. \tag{2.15}$$

First, after some lengthy algebra and with the aid of Mathematica [18], we note that

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3(n+r)}=\frac{5}{4r}\zeta(4)-\frac{2}{r^2}\zeta(3)+\frac{1}{r^3}\zeta(2)+\frac{\left(H_{r-1}^{(1)}\right)^2+H_{r-1}^{(2)}}{2r^3},$$

and

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3(n+r)^2} &= \frac{5}{4r^2}\zeta(4)-\frac{5}{r^3}\zeta(3)+\left(\frac{3}{r^4}-\frac{H_{r-1}^{(1)}}{r^3}\right)\zeta(2)\\ &\quad+\frac{3}{2r^4}\left(\left(H_{r-1}^{(1)}\right)^2+H_{r-1}^{(2)}\right)+\frac{1}{r^3}\left(H_{r-1}^{(1)}H_{r-1}^{(2)}+H_{r-1}^{(3)}\right).\end{aligned}$$

Now we can write from (2.15)

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3\binom{n+k}{k}^2}\\ &=\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left[\begin{array}{c}\frac{5}{4r^2}\zeta(4)-\frac{5}{r^3}\zeta(3)+\left(\frac{3}{r^4}-\frac{H_{r-1}^{(1)}}{r^3}\right)\zeta(2)\\ +\frac{3}{2r^4}\left(\left(H_{r-1}^{(1)}\right)^2+H_{r-1}^{(2)}\right)+\frac{1}{r^3}\left(H_{r-1}^{(1)}H_{r-1}^{(2)}+H_{r-1}^{(3)}\right)\end{array}\right]\\ &\quad+2\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left(H_{r-1}^{(1)}-H_{k-r}^{(1)}\right)\left[\begin{array}{c}\frac{5}{4r}\zeta(4)-\frac{2}{r^2}\zeta(3)+\frac{1}{r^3}\zeta(2)\\ +\frac{\left(H_{r-1}^{(1)}\right)^2+H_{r-1}^{(2)}}{2r^3}\end{array}\right]\end{aligned}$$

so that collecting $\zeta(4)$, $\zeta(3)$, $\zeta(2)$ and constant terms we have

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3\binom{n+k}{k}^2}\\ &=\frac{5}{4}\zeta(4)\\ &\quad-\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left(\frac{5}{r^3}+\frac{4}{r^2}\left(H_{r-1}^{(1)}-H_{k-r}^{(1)}\right)\right)\zeta(3)\\ &\quad+\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left(\frac{3}{r^4}-\frac{H_{r-1}^{(1)}}{r^3}+\frac{2}{r^3}\left(H_{r-1}^{(1)}-H_{k-r}^{(1)}\right)\right)\zeta(2)\\ &\quad+\sum_{r=1}^{k}\left(r\binom{k}{r}\right)^2\left(\begin{array}{c}\frac{3}{2r^4}\left[\left(H_{r-1}^{(1)}\right)^2+H_{r-1}^{(2)}\right]+\frac{1}{r^3}\left[H_{r-1}^{(1)}H_{r-1}^{(2)}+H_{r-1}^{(3)}\right]\\ +\frac{1}{r^3}\left(\left(H_{r-1}^{(1)}\right)^2+H_{r-1}^{(2)}\right)\left(H_{r-1}^{(1)}-H_{k-r}^{(1)}\right)\end{array}\right)\end{aligned}$$

which is the result (2.13). Here we have used the identity

$$\sum_{r=1}^{k}\binom{k}{r}^2\left(1-2r\left(H_{k-r}^{(1)}-H_{r-1}^{(1)}\right)\right)=1.$$

□

REMARK (2.16). *For $k=6$ we have*

$$\sum_{n=1}^{\infty}\frac{H_n^{(1)}}{n^3\binom{n+6}{6}^2}=\frac{5}{4}\zeta(4)-\frac{1421}{4}\zeta(3)-\frac{501011}{1200}\zeta(2)+\frac{720882841}{648000}.$$

The following is an extension of Theorem 2.3.

THEOREM (2.17). *Let $a\ge 0,\ b\ge 0,\ c\ge 0$ be real positive numbers $j,k,l\ge 0,\ p\ge 1,\ k\ge p$, and $|t|\le 1$. $Q^{(m)}(a,j)$ is an analytic function in j, as given in Lemma 2.1, then:*

(2.18)
$$\sum_{n=1}^{\infty}\frac{n^p\ t^n\ Q^{(m)}(a,j)}{\binom{bn+k}{k}\binom{cn+l}{l}}$$
(2.19)
$$=akl\int_0^1\int_0^1\int_0^1\frac{(1-x)^j}{x}(1-y)^{k-1}(1-z)^{l-1}(\log(1-x))^m\operatorname{Li}_{(-p-1)}\left(tx^ay^bz^c\right)dxdydz,$$

where the PolyLog function $\operatorname{Li}_q\left(\beta\right)=\sum_{r=1}^{\infty}\frac{\beta^r}{r^q}$.

Proof. To prove (2.19), consider the following expansion:

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{t^n}{\binom{an+j}{j}\binom{bn+k}{k}\binom{cn+l}{l}}\\&=\sum_{n=1}^{\infty}\frac{t^n ankl\ \Gamma(an)\ \Gamma(j+1)\ \Gamma(bn+1)\ \Gamma(k)\Gamma(cn+1)\ \Gamma(l)}{\Gamma(an+j+1)\ \Gamma(bn+k+1)\Gamma(cn+l+1)}\\&=akl\sum_{n=1}^{\infty}t^n nB(an,j+1)B(bn+1,k)B(cn+1,l)\\&=akl\sum_{n=1}^{\infty}t^n nB(bn+1,k)B(cn+1,l)\int_0^1 x^{an-1}(1-x)^j\,dx,\end{aligned}$$

where the Beta function

$$B(s,t)=\int_0^1 z^{s-1}(1-z)^{t-1}dz=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)},\ \text{for } \mathbb{R}(s)>0 \text{ and } \mathbb{R}(t)>0,$$

and the Gamma function

$$\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}dt,\ \text{for } \mathbb{R}(z)>0.$$

Now we can differentiate, m times, both sides with respect to the parameter j, and by the use of Lemma 2.1, we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{t^n\ Q^{(m)}(a,j)}{\binom{bn+k}{k}\binom{cn+l}{l}} \\
&= akl \sum_{n=1}^{\infty} t^n nB(bn+1,k)B(cn+1,l)\int_0^1 x^{an-1}(1-x)^j(\ln(1-x))^m\,dx \\
&= akl \sum_{n=1}^{\infty} t^n n\int_0^1 x^{an-1}(1-x)^j\ (\ln(1-x))^m\,dx\int_0^1 y^{bn}(1-y)^{k-1}\,dy \\
&\quad\times\int_0^1 z^{cn}(1-z)^{l-1}\,dz.
\end{aligned}$$

Now we apply the consecutive derivative operator $t\frac{d}{dt}(\cdot)$, p- times so that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n^p\ t^n\ Q^{(m)}(a,j)}{\binom{bn+k}{k}\binom{cn+l}{l}} &= akl\sum_{n=1}^{\infty} t^n n^{p+1}\int_0^1 x^{an-1}(1-x)^j\ (\ln(1-x))^m\,dx \\
&\quad\times\int_0^1 y^{bn}(1-y)^{k-1}\,dy\int_0^1 z^{cn}(1-z)^{l-1}\,dz.
\end{aligned}$$

By an allowable change of order of sum and integral, we may write

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n^p\ t^n\ Q^{(m)}(a,j)}{\binom{bn+k}{k}\binom{cn+l}{l}} = akl\int_0^1\int_0^1\int_0^1 \frac{(1-x)^j(1-y)^{k-1}(1-z)^{l-1}(\ln(1-x))^m}{x} \\
&\times\sum_{n=1}^{\infty} n^{p+1}\left(tx^a y^b z^c\right)^n dxdydz
\end{aligned}$$

and with the requirement $\left|tx^a y^b z^c\right| < 1$ we may write

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n^p\ t^n\ Q^{(m)}(a,j)}{\binom{bn+k}{k}\binom{cn+l}{l}} &= akl\int_0^1\int_0^1\int_0^1 \frac{(1-x)^j(1-y)^{k-1}(1-z)^{l-1}(\log(1-x))^m}{x} \\
&\quad\times \mathrm{Li}_{(-p-1)}\left(tx^a y^b z^c\right)dxdydz,
\end{aligned}$$

which is the result (2.19). □

The following Corollary is a particular case of the above Theorem.

COROLLARY (2.20). *Let $a = b = c = 1$, $j = 0$, $t = 1$, $p \geq 1$, $k \geq p$, $l = k$ and put $m = 1$, then*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{n^p\ H_n^{(1)}}{\binom{n+k}{k}^2} \\
&= k^2\int_0^1\int_0^1\int_0^1 \frac{[(1-y)(1-z)]^{k-1}\log(1-x)\mathrm{Li}_{(-p-1)}(xyz)}{x}dxdydz \\
&= \sum_{r=2}^{k}\left(\alpha_r\,\zeta(3)+\beta_r\,\zeta(2)+\vartheta_r\right) \qquad (2.21)
\end{aligned}$$

where

$$\alpha_r = (-1)^{p-1}r^{p+1}\binom{k}{r}^2\left[(r-1)\left(p+2r\left(H_{k-r}^{(1)}-H_{r-2}^{(1)}\right)\right)-2r\right],$$

$$\beta_r = (-1)^p\, r^{p+2} \binom{k}{r}^2 H^{(1)}_{r-1} \text{ and}$$

$$\vartheta_r = (-1)^p\, r^{p+1} \binom{k}{r}^2 \left[\begin{array}{c} \frac{1}{2}\left(p+2r\left(H^{(1)}_{k-r} - H^{(1)}_{r-2}\right)\right)\left(H^{(2)}_{r-1} + \left(H^{(1)}_{r-1}\right)^2\right) \\ -r\left(H^{(1)}_{r-1}H^{(2)}_{r-1} + H^{(3)}_{r-1} + \frac{\left(H^{(2)}_{r-1}+\left(H^{(1)}_{r-1}\right)^2\right)}{(r-1)}\right) \end{array} \right].$$

Proof. Consider the expansion

$$\sum_{n=1}^{\infty} \frac{n^p\, H^{(1)}_n}{\binom{n+k}{k}^2} = \sum_{n=1}^{\infty} \frac{n^p\, H^{(1)}_n\, (k!)^2}{(n+1)^2 \prod_{r=2}^{k} (n+r)^2} = \sum_{n=1}^{\infty} \frac{H^{(1)}_n\, (k!)^2}{(n+1)^2} \left[\sum_{r=2}^{k} \frac{A_r}{(n+r)} + \frac{B_r}{(n+r)^2} \right]$$

where

$$B_r = \lim_{n\to(-r)} \left[\frac{n^p\,(n+r)^2}{\prod_{r=2}^{k}(n+r)^2} \right] = \frac{(-r)^p}{((k-r)!\,(r-2)!)^2} = (-1)^p\, r^p \left(\frac{2}{k!} \binom{k}{r} \binom{r}{2} \right)^2$$

and

$$A_r = \lim_{n\to(-r)} \frac{d}{dn} \left\{ \frac{n^p\,(n+r)^2}{\prod_{r=2}^{k}(n+r)^2} \right\} = (-r)^{p-1} \left(\frac{2}{k!} \binom{k}{r} \binom{r}{2} \right)^2 \left[p + 2r\left(H^{(1)}_{k-r} - H^{(1)}_{r-2}\right) \right].$$

Now

$$\sum_{n=1}^{\infty} \frac{H^{(1)}_n\, (k!)^2}{(n+1)^2} \left[\sum_{r=2}^{k} \frac{A_r}{(n+r)} + \frac{B_r}{(n+r)^2} \right]$$

$$(2.22) \qquad = \sum_{n=1}^{\infty} \frac{H^{(1)}_n}{(n+1)^2} \sum_{r=2}^{k} \left[\begin{array}{c} \frac{(-r)^{p-1}\left(2\binom{k}{r}\binom{r}{2}\right)^2 \left[p+2r\left(H^{(1)}_{k-r}-H^{(1)}_{r-2}\right)\right]}{(n+r)} \\ + \frac{(-1)^p r^p \left(2\binom{k}{r}\binom{r}{2}\right)^2}{(n+r)^2} \end{array} \right].$$

Now by an allowable change the order of summation we have, from (2.22)

$$\sum_{r=2}^{k} (-r)^{p-1} \left(2\binom{k}{r}\binom{r}{2}\right)^2 \left[p + 2r\left(H^{(1)}_{k-r} - H^{(1)}_{r-2}\right)\right] \sum_{n=1}^{\infty} \frac{H^{(1)}_n}{(n+1)^2\,(n+r)}$$

$$(2.23) \qquad + \sum_{r=2}^{k} (-1)^p\, r^p \left(2\binom{k}{r}\binom{r}{2}\right)^2 \sum_{n=1}^{\infty} \frac{H^{(1)}_n}{(n+1)^2\,(n+r)^2},$$

and after some lengthy algebra and with the aid of Mathematica [18], we note that

$$\sum_{n=1}^{\infty} \frac{H^{(1)}_n}{(n+1)^2\,(n+r)} = \frac{1}{2(r-1)^2} \left[2(r-1)\zeta(3) - H^{(2)}_{r-1} - \left(H^{(1)}_{r-1}\right)^2 \right],$$

and

$$\sum_{n=1}^{\infty} \frac{H_n^{(1)}}{(n+1)^2 (n+r)^2} = \sum_{n=1}^{\infty} H_n^{(1)} \left[\begin{array}{c} \frac{1}{(r-1)^2(n+1)^2} + \frac{1}{(r-1)^2(n+r)^2} \\ -\frac{2}{(r-1)^2(n+1)(n+r)} \end{array} \right]$$

$$= \frac{2\zeta(3)}{(r-1)^2} + \frac{\zeta(2) H_{r-1}^{(1)}}{(r-1)^2} - \frac{H_{r-1}^{(1)} H_{r-1}^{(2)} + H_{r-1}^{(3)}}{(r-1)^2} - \frac{\left(H_{r-1}^{(1)}\right)^2 + H_{r-1}^{(2)}}{(r-1)^3}.$$

Using these results we can now put, from (2.23)

$$\begin{aligned} &\sum_{r=2}^{k} (-r)^{p-1} \left(2 \binom{k}{r} \binom{r}{2}\right)^2 \left[p + 2r\left(H_{k-r}^{(1)} - H_{r-2}^{(1)}\right)\right] \\ &\times \frac{1}{2(r-1)^2} \left[2(r-1)\zeta(3) - H_{r-1}^{(2)} - \left(H_{r-1}^{(1)}\right)^2\right] \\ &+ \sum_{r=2}^{k} (-1)^p r^p \left(2 \binom{k}{r} \binom{r}{2}\right)^2 \\ &\times \left[\frac{2\zeta(3)}{(r-1)^2} + \frac{\zeta(2) H_{r-1}^{(1)}}{(r-1)^2} - \frac{H_{r-1}^{(1)} H_{r-1}^{(2)} + H_{r-1}^{(3)}}{(r-1)^2} - \frac{\left(H_{r-1}^{(1)}\right)^2 + H_{r-1}^{(2)}}{(r-1)^3}\right] \end{aligned}$$

and collecting $\zeta(3)$, $\zeta(2)$ and constant terms we have

$$\sum_{r=2}^{k} \left(\alpha_r \, \zeta(3) + \beta_r \, \zeta(2) + \vartheta_r\right)$$

where α_r, β_r and ϑ_r are given in (2.21) which is the required result. □

Some particular values for $\sum_{n=1}^{\infty} \frac{n^p \, H_n^{(1)}}{\binom{n+k}{k}^2}$ are below:

k	p	$\sum_{n=1}^{\infty} \frac{n^p \, H_n^{(1)}}{\binom{n+k}{k}^2}$
4	1	$\frac{72748}{27} - \frac{3160}{3}\zeta(2) - 800\zeta(3)$
5	3	$\frac{63575675}{288} - \frac{372375}{4}\zeta(2) - 56250\zeta(3)$
4	4	$-\frac{1799668}{27} + \frac{81928}{4}\zeta(2) + 18080\zeta(3)$
8	11	$-\frac{18482164770224189961 2351}{400075200} + \frac{273862031177771641}{1260}\zeta(2)$ $+86883993177916\zeta(3)$
	......	....

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ON COMPOSITE INTEGERS n FOR WHICH $\varphi(n) \mid n-1$

FLORIAN LUCA AND CARL POMERANCE

ABSTRACT. Let φ denote Euler's function. Clearly $\varphi(n) \mid n-1$ if $n=1$ or if n is a prime. In 1932, Lehmer asked if any composite numbers n have this property. Improving on some earlier results, we show that the number of composite integers $n \le x$ with $\varphi(n) \mid n-1$ is at most $x^{1/2}/(\log x)^{1/2+o(1)}$ as $x \to \infty$. Key to the proof are some uniform estimates of the distribution of integers n where the largest divisor of $\varphi(n)$ supported on primes from a fixed set is abnormally small.

1. Introduction

Let $\varphi(n)$ be the Euler function of n. Lehmer [6] asked if there exist composite positive integers n such that $\varphi(n) \mid n-1$. In 1977, the second author [8] proved that if one sets

$$\mathscr{L}(x) = \{n \le x : \varphi(n) \mid n-1 \text{ and } n \text{ is composite}\},$$

then

$$\#\mathscr{L}(x) \ll x^{1/2}(\log x)^{3/4}.$$

This was followed by subsequent improvements in the exponent of the logarithm, by first replacing the above bound by $x^{1/2}(\log x)^{1/2}(\log\log x)^{-1/2}$ in [9], next by $x^{1/2}(\log\log x)^{1/2}$ in [2], and recently by $x^{1/2}(\log x)^{-\Theta+o(1)}$ as $x \to \infty$ in [1], where $\Theta = 0.129398\ldots$ is the least positive solution of the transcendental equation

$$2\Theta(\log\Theta - 1 - \log\log 2) = -\log 2.$$

Here, we continue this trend and present the following result.

THEOREM (1.1). *As $x \to \infty$, we have*

$$\#\mathscr{L}(x) \le \frac{x^{1/2}}{(\log x)^{1/2+o(1)}}. \tag{1.2}$$

The function $o(1)$ appearing in the above exponent is of order of magnitude $O((\log\log\log\log x)^{1/2}/(\log\log\log x)^{1/3})$. As in the previous works on the subject, the above bound is also an upper bound for the cardinality of the set

$$\mathscr{L}_a(x) = \{n \le x : \varphi(n) \mid n-a \text{ and } n \ne ap \text{ where } p \nmid a \text{ is a prime}\},$$

where $a \ne 0$ is any fixed integer. In that case, the function $o(1)$ in (1.2) depends on a.

We point out that in spite of all these improvements, there is still no known composite number n with $\varphi(n) \mid n-1$. It is reasonable to conjecture that $\#\mathscr{L}(x) \le x^{o(1)}$ as $x \to \infty$, but we seem to be a long way from improving the exponent 1/2 on x in the upper bound to anything smaller.

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While the proof follows the general approach from [1], we add a detailed study of the distribution of those integers n where the contribution to $\varphi(n)$ from primes in a given set $\mathcal{Q}$ is below normal. Such results (see Proposition 2.3 in the case when $\mathcal{Q}$ is a small set and Proposition 2.12 in the case when $\mathcal{Q}$ is large) can be viewed as a generalization of the Hardy–Ramanujan estimates for the distribution of integers with fewer than the normal number of prime divisors, which integers usually have the 2-part of $\varphi(n)$ smaller than normal. Hopefully these propositions will have some independent interest.

We use the symbols O, o and $\ll$, $\gg$ with their usual meaning. We also use p and q for prime numbers. For a positive integer n, we use $\omega(n)$ for the number of primes that divide n. For a prime q and a positive integer n we write $v_q(n)$ for the exponent of q in the factorization of n; that is, $q^{v_q(n)}\| n$.

2. Some auxiliary results

It follows from the Hardy–Ramanujan inequality that

$$\begin{aligned} \#\{n\le t:\omega(n)\ge \lambda\log\log t\} &\ll \frac{e^{\lambda}t}{(\log t)^{1+\lambda\log(\lambda/e)}},\\ \#\{n\le t:\omega(n)\le \lambda\log\log t\} &\ll \frac{t}{(\log t)^{1+\lambda\log(\lambda/e)}} \end{aligned} \tag{2.1}$$

hold uniformly for all $\lambda\ge 1$, and $0<\lambda\le 1$, respectively. (For λ fixed, a somewhat stronger estimate is known, see Erdős and Nicolas [5, Prop. 3].) These estimates played key roles in the proof in [1].

Since all prime divisors of a positive integer n with at most one possible exception are odd, the bound (2.1) gives us that the inequality

$$\#\{n\le t: v_2(\varphi(n))\le \lambda\log\log t\}\ll \frac{t}{(\log t)^{1+\lambda\log(\lambda/e)}} \tag{2.2}$$

holds for all t uniformly in $\lambda\in(0,1]$. While the above inequality is correct, it does not capture the full contribution to $v_2(\varphi(n))$ arising from primes p with $p-1$ a multiple of 4, 8, or a larger power of 2.

In this section, we prove a stronger and more general inequality than (2.2). Let $\mathcal{Q}\subset[1,M]$ be a set of primes. Put

$$F_{\mathcal{Q}}(n):=\prod_{q\in\mathcal{Q}} q^{v_q(\varphi(n))}$$

for the *$\mathcal{Q}$-part* of $\varphi(n)$. In analogy with (2.1) and (2.2), for $\lambda>0$ put

$$\mathcal{B}_{\mathcal{Q},\lambda}(t):=\{n\le t: F_{\mathcal{Q}}(n)\le(\log t)^{\lambda}\}.$$

Our first result addresses the cardinality of $\mathcal{B}_{\mathcal{Q},\lambda}(t)$. Letting

$$c_{\mathcal{Q}}(s):=\prod_{q\in\mathcal{Q}}\left(\frac{q-2}{q-1}+\frac{1}{q^{s+1}-1}\right),$$

we have the following inequality.

PROPOSITION (2.3). *For $\mathcal{Q}\subset[1,M]$ a set of primes, the estimate*

$$\#\mathcal{B}_{\mathcal{Q},\lambda}(t)\le \frac{t}{(\log t)^{1-\lambda s-c_{\mathcal{Q}}(s)}}\exp\big(O((\log M)^3)\big) \tag{2.4}$$

holds uniformly in $\mathcal{Q}$, $M\ge 2$, $\lambda>0$, $s\ge 0$, and $t\ge 2$.

Note that we are free to choose the number $s \ge 0$ above. Obviously, when $\mathcal{Q}$ and λ are given we would like to choose s in such a way that $\lambda s + c_{\mathcal{Q}}(s)$ is minimal. Before proving Proposition 2.3, let us give an application.

Take $\mathcal{Q} = \{2\}$. We have $F_{\{2\}}(n) = 2^{v_2(\varphi(n))}$ and $c_{\{2\}}(s) = 1/(2^{s+1}-1)$. To find the minimum of $\lambda s + c_{\{2\}}(s)$ as a function of s, we take its derivative with respect to s and set it to equal zero getting

$$\lambda = \frac{2^{s+1}\log 2}{(2^{s+1}-1)^2}.$$

Putting $x = 2^{s+1}$, we get the quadratic equation

$$(x-1)^2 = \frac{\log 2}{\lambda}x,$$

whose solutions are

$$x_\lambda = 1 + \frac{\log 2}{2\lambda} \pm \sqrt{\frac{\log 2}{\lambda} + \frac{(\log 2)^2}{4\lambda^2}}.$$

The one with the negative sign leads to a solution $x_\lambda < 1$, which is impossible because $x = 2^{s+1} \ge 2$. Thus, we must pick the solution x_λ with the positive sign whose corresponding s equals

$$s = \frac{1}{\log 2}\log\left(1 + \frac{\log 2}{2\lambda} + \sqrt{\frac{\log 2}{\lambda} + \frac{(\log 2)^2}{4\lambda^2}}\right) - 1.$$

This number is non-negative only when $\lambda \in (0, 2\log 2]$. The above calculation applied to $\lambda \log 2$ implies the following improvement of (2.2).

COROLLARY (2.5). *Given any $\lambda \in (0,2]$, we have the estimate*

$$\begin{aligned} \#\{n \le t : v_2(\varphi(n)) \le \lambda \log\log t\} &= \#\mathcal{B}_{\{2\},\lambda\log 2}(t) \\ &\ll \frac{t}{(\log t)^{1+\lambda\log 2 - \lambda\log\left(1+\frac{1+\sqrt{4\lambda+1}}{2\lambda}\right) - \frac{2\lambda}{1+\sqrt{4\lambda+1}}}}. \end{aligned} \tag{2.6}$$

When $\mathcal{Q}$ contains more than one element, finding the optimal value of s leads to solving a polynomial-like equation but with transcendental exponents. In this case one may solve for s via numerical methods.

Taking say $\lambda = 1/2$ in (2.2), we get the value $0.1534264097\cdots$ for the exponent of the logarithm, while taking $\lambda = 1/2$ in (2.6), we get the value $0.3220692380\cdots$ for the exponent of the logarithm.

If one goes through the arguments from [1] and replaces inequality (2.2) by the inequality (2.6), then one gets that with λ the solution of the equation

$$1 + \lambda\log 2 - \lambda\log\left(1 + \frac{1+\sqrt{4\lambda+1}}{2\lambda}\right) - \frac{2\lambda}{1+\sqrt{4\lambda+1}} = \lambda\log 2,$$

the inequality $\#\mathcal{L}(x) \le x/(\log x)^{\Theta+o(1)}$ holds as $x \to \infty$, where $\Theta = \lambda(\log 2)/2$. Calculation reveals that $\lambda = 0.4815450284\cdots$, so that $\Theta = 0.1668907893\cdots$, which is already better than the main result from [1]. The improvement to $\Theta = 1/2$ in our Theorem 1.1 arises by allowing more primes into the set $\mathcal{Q}$.

Now that we have hopefully convinced the reader of the usefulness of Proposition 2.3, let's get to its proof.

Proof. We need the following theorem which appears in [10, III, sec. 3.5].

LEMMA (2.7). *Let f be a multiplicative function such that $f(n) \ge 0$ for all n, and such that there exist numbers A and B such that for all $x > 1$ both inequalities*

$$\sum_{p \le x} f(p) \log p \le Ax \tag{2.8}$$

and

$$\sum_{p} \sum_{\alpha \ge 2} \frac{f(p^\alpha)}{p^\alpha} \log(p^\alpha) \le B \tag{2.9}$$

hold. Then, for $x > 1$, we have

$$\sum_{n \le x} f(n) \le (A + B + 1) \frac{x}{\log x} \sum_{n \le x} \frac{f(n)}{n}.$$

We apply Lemma 2.7 to the multiplicative function $F_{\mathcal{Q}}(n)^{-s}$ whose range is in the set $(0, 1]$. Clearly, the estimates (2.8) and (2.9) hold with some absolute constants A and B independent of $\mathcal{Q}$ or s. Since $F_{\mathcal{Q}}(n)^{-s} \le 1$,

$$\begin{aligned}
\sum_{n \le t} \frac{1}{F_{\mathcal{Q}}(n)^s} &\ll \frac{t}{\log t} \prod_{p \le t} \left(1 + \frac{1}{F_{\mathcal{Q}}(p)^s p} + \frac{1}{F_{\mathcal{Q}}(p^2)^s p^2} + \cdots \right) \\
&\le \frac{t}{\log t} \prod_{p \le t} \left(1 + \frac{1}{F_{\mathcal{Q}}(p)^s p} + O\left(\frac{1}{p^2}\right)\right) \\
&\ll \frac{t}{\log t} \exp\left(\sum_{p \le t} \frac{1}{F_{\mathcal{Q}}(p)^s p}\right).
\end{aligned}$$

We now compute the sum within the above exponential. Let $\mathcal{M}_{\mathcal{Q}}$ be the set of all positive integers m whose prime factors are contained in $\mathcal{Q}$. Then

$$\sum_{p \le t} \frac{1}{F_{\mathcal{Q}}(p)^s p} = \sum_{m \in \mathcal{M}_{\mathcal{Q}}} \frac{1}{m^s} \sum_{\substack{p \le t \\ F_{\mathcal{Q}}(p) = m}} \frac{1}{p}.$$

Given $m \in \mathcal{M}_{\mathcal{Q}}$, then p is a prime such that $F_{\mathcal{Q}}(p) = m$ precisely when $m \mid p - 1$ and $(p-1)/m$ is coprime to $Q := \prod_{q \in \mathcal{Q}} q$. We use the following estimate:

$$\sum_{\substack{p \le t \\ p \equiv 1 \pmod{\ell}}} \frac{1}{p} = \frac{\log\log t}{\varphi(\ell)} + O\left(\frac{\log \ell}{\ell}\right), \tag{2.10}$$

(see [7] for example). For each $m \in \mathcal{M}_{\mathcal{Q}}$, we have, by the Principle of Inclusion and Exclusion, that

$$\sum_{\substack{p \le t \\ F_{\mathcal{Q}}(p) = m}} \frac{1}{p} = \sum_{d \mid Q} \mu(d) \sum_{\substack{p \le t \\ p \equiv 1 \pmod{md}}} \frac{1}{p}.$$

Using estimate (2.10) we get that

$$\sum_{\substack{p \le t \\ F_{\mathcal{Q}}(p) = m}} \frac{1}{p} = (\log\log t) \sum_{d \mid Q} \frac{\mu(d)}{\varphi(md)} + O\left(\sum_{d \mid Q} \frac{\log(md)}{md}\right).$$

Certainly,

$$\sum_{d \mid Q} \frac{\log(dm)}{dm} \le \frac{1}{m} \sum_{d \mid Q} \frac{\log d}{d} + \frac{\log m}{m} \sum_{d \mid Q} \frac{1}{d} \ll \frac{(\log M)^2 + (\log m) \log M}{m}.$$

We thus get that

$$\begin{aligned}\sum_{p\le t}\frac{1}{F_{\mathcal{Q}}(p)^s p} &= (\log\log t)\sum_{\substack{m\in\mathcal{M}_{\mathcal{Q}}\\ d\mid Q}}\frac{\mu(d)}{m^s\varphi(md)}\\ &\quad + O\left(\sum_{m\in\mathcal{M}_{\mathcal{Q}}}\frac{(\log M)^2+(\log m)\log M}{m}\right).\end{aligned}$$

Observe that the error term is $O((\log M)^3)$. Thus,

$$\sum_{p\le t}\frac{1}{F_{\mathcal{Q}}(p)^s p} = (\log\log t)\sum_{\substack{m\in\mathcal{M}_{\mathcal{Q}}\\ d\mid Q}}\frac{\mu(d)}{m^s\varphi(md)} + O((\log M)^3). \tag{2.11}$$

The double sum above is a multiplicative function of the parameter Q (where $\mathcal{Q}$ is the set of prime factors of Q). Its value when $Q=q$ is a prime is

$$1-\frac{1}{q}+\sum_{\alpha\ge 1}\left(\frac{1}{q^{\alpha s}\varphi(q^\alpha)}-\frac{1}{q^{\alpha s}\varphi(q^{\alpha+1})}\right)=\frac{q-2}{q-1}+\frac{1}{q^{s+1}-1},$$

so that the main term in (2.11) above is our familiar $c_{\mathcal{Q}}(s)$ multiplied by $\log\log t$. We have shown that

$$\sum_{n\le t}\frac{1}{F_{\mathcal{Q}}(n)^s} \ll \frac{t}{\log t}\exp\left(c_{\mathcal{Q}}(s)\log\log t + O((\log M)^3)\right).$$

Since $s\ge 0$, we deduce immediately that

$$\begin{aligned}\#\mathcal{B}_{\mathcal{Q},\lambda}(t) &\le \frac{t}{\log t}\exp\left((\lambda s + c_{\mathcal{Q}}(s))\log\log t + O((\log M)^3)\right)\\ &= \frac{t}{(\log t)^{1-\lambda s - c_{\mathcal{Q}}(s)}}\exp\left(O((\log M)^3)\right),\end{aligned}$$

which is what we wanted to prove. □

For a specific set $\mathcal{Q}$ of primes that one has in mind, one can use Proposition 2.3 with a choice of s that minimizes the estimate for $\#\mathcal{B}_{\mathcal{Q},\lambda}(t)$ as we did above in the case $\mathcal{Q}=\{2\}$. It turns out that to prove Theorem 1.1, we will want to take choices for $\mathcal{Q}$ as large sets of primes and λ far below its "normal" value, in which case we will push up against a best-possible estimate $\#\mathcal{B}_{\mathcal{Q},\lambda}(t)\le t/(\log t)^{1+o(1)}$. In this case it is not necessary to choose the absolute optimal s, merely a "pretty good" value.

For a finite set of primes $\mathcal{Q}$, let

$$T_{\mathcal{Q}} = \exp\left(\sum_{q\in\mathcal{Q}}\frac{1}{q}\right).$$

We now prove the following consequence of Proposition 2.3.

PROPOSITION (2.12). *Suppose that $\mathcal{Q}\subset[1,M]$ is a set of primes with $0<R\le 1$, where $R:=\lambda(\log\log M)/T_{\mathcal{Q}}$. We have, uniformly for $t\ge 2$,*

$$\#\mathcal{B}_{\mathcal{Q},\lambda}(t)\le \frac{t}{(\log t)^{1+O(R^{1/2})}}\exp\left(O((\log M)^3)\right). \tag{2.13}$$

Proof. We shall apply Proposition 2.3 with s chosen as the number

$$s = R^{1/2}/\lambda.$$

Thus, the term $-\lambda s$ in the exponent on $\log t$ in (2.4) is absorbed into the O-estimate in (2.13). It remains to show that $c_{\mathcal{Q}}(s)$ is likewise majorized.

We have

$$c_{\mathcal{Q}}(s) \le \prod_{\substack{q\in\mathcal{Q}\\ q>2}} \left(\frac{q-2}{q-1}\right) \exp\left(\sum_{\substack{q\in\mathcal{Q}\\ q>2}} \frac{q-1}{(q-2)(q^{1+s}-1)}\right). \tag{2.14}$$

The product satisfies

$$\prod_{\substack{q\in\mathcal{Q}\\ q>2}} \left(\frac{q-2}{q-1}\right) = \exp\left(-\sum_{\substack{q\in\mathcal{Q}\\ q>2}} \frac{1}{q} + O(1)\right) \ll T_{\mathcal{Q}}^{-1}. \tag{2.15}$$

We have

$$\begin{aligned}\sum_{\substack{q\in\mathcal{Q}\\ 2<q\le\exp(R^{1/2}T_{\mathcal{Q}})}} \frac{q-1}{(q-2)(q^{1+s}-1)} &\le \sum_{\substack{q\in\mathcal{Q}\\ 2<q\le\exp(R^{1/2}T_{\mathcal{Q}})}} \frac{1}{q-2}\\ &\le \log(R^{1/2}T_{\mathcal{Q}}) + O(1).\end{aligned}$$

Also,

$$\begin{aligned}\sum_{\substack{q\in\mathcal{Q}\\ q>\exp(R^{1/2}T_{\mathcal{Q}})}} \frac{q-1}{(q-2)(q^{1+s}-1)} &\le \exp(-sR^{1/2}T_{\mathcal{Q}}) \sum_{\substack{q\in\mathcal{Q}\\ q>2}} \frac{q-1}{(q-2)(q-q^{-s})}\\ &\ll \exp(-sR^{1/2}T_{\mathcal{Q}})\log\log M.\end{aligned}$$

Since $sR^{1/2}T_{\mathcal{Q}} = \log\log M$, we have from these calculations that

$$\sum_{\substack{q\in\mathcal{Q}\\ q>2}} \frac{q-1}{(q-2)(q^{1+s}-1)} \le \log(R^{1/2}T_{\mathcal{Q}}) + O(1),$$

so that with (2.14) and (2.15), we get

$$c_{\mathcal{Q}}(s) \ll T_{\mathcal{Q}}^{-1} \exp\left(\log(R^{1/2}T_{\mathcal{Q}})\right) = R^{1/2}.$$

Thus, we may also absorb $c_{\mathcal{Q}}(s)$ into the O-estimate in the exponent on $\log t$ in (2.13), completing the proof of the proposition. □

Finally, we shall need an upper bound on the number of $n \le t$ whose Euler function is coprime to the primes $q \in \mathcal{Q}$ for $\mathcal{Q}$ a set of *odd* primes with $\mathcal{Q} \subset [1, M]$. For such a set of primes, put again $Q := \prod_{q\in\mathcal{Q}} q$, let

$$\mathcal{S}_{\mathcal{Q}}(t) = \{n \le t : \gcd(\varphi(n), Q) = 1\},$$

and let

$$g_{\mathcal{Q}} = \prod_{q\in\mathcal{Q}} \frac{q-2}{q-1}.$$

LEMMA (2.16). *Let $t, M \ge 2$ and let $\mathcal{Q} \subset [1, M]$ be a set of odd primes. We have the uniform estimate*

$$\#\mathcal{S}_{\mathcal{Q}}(t) \le \frac{t}{(\log t)^{1-g_{\mathcal{Q}}}} \exp\left(O((\log M)^2)\right).$$

Proof. Writing $f(n)$ for the characteristic function of the numbers n having $\varphi(n)$ coprime to Q, Lemma 2.7 applied to $f(n)$ shows that

$$\begin{aligned}\#\mathscr{S}_{\mathscr{Q}}(t) &\ll \frac{t}{\log t} \prod_{\substack{p\le t\\ (p(p-1),Q)=1}} \left(1+\frac{1}{p-1}\right) \prod_{\substack{p\le t\\ (p(p-1),Q)=p}} \left(1+\frac{1}{p}\right)\\ &\ll \frac{t}{\log t}\exp\left(\sum_{\substack{p\le t\\ (p-1,Q)=1}} \frac{1}{p}\right).\end{aligned}$$

The Principle of Inclusion and Exclusion together with estimate (2.10) shows that

$$\begin{aligned}\sum_{\substack{p\le t\\ (p-1,Q)=1}} \frac{1}{p} &= \sum_{d\mid Q}\mu(d) \sum_{\substack{p\le t\\ p\equiv 1 \pmod d}} \frac{1}{p}\\ &= (\log\log t)\sum_{d\mid Q}\frac{\mu(d)}{\varphi(d)} + O\left(\sum_{d\mid Q}\frac{\log d}{d}\right)\\ &= (\log\log t)\prod_{q\in\mathscr{Q}}\left(1-\frac{1}{q-1}\right) + O((\log M)^2)\\ &= g_{\mathscr{Q}}\log\log t + O((\log M)^2).\end{aligned}$$

The desired conclusion about $\#\mathscr{S}_{\mathscr{Q}}(t)$ now follows. □

3. The Proof of Theorem 1.1

Let x be large and let $\mathscr{D}(x) = \mathscr{L}(x) \cap (x/2, x]$. It suffices to show that inequality (1.2) holds with the left hand side replaced by $\#\mathscr{D}(x)$, since afterwards the resulting inequality will follow from the obvious fact that

$$\#\mathscr{L}(x) \le \sum_{0\le k\le \lfloor(\log x)/(\log 2)\rfloor} \#\mathscr{D}(x/2^k).$$

If $n \in \mathscr{D}(x)$, we have that n is squarefree. Let $K = \omega(n)$ be the number of prime divisors of n. In [1], it was shown that the inequality $K < 20\log\log x$ holds with at most $O(x^{1/2}/\log x)$ exceptional numbers n, which is acceptable for us. So, we shall assume that $K < 20\log\log x$.

A result of the second author from [8] shows that n has a divisor d such that $d \in [y/(2K), y]$, where we take $y := x^{1/2}/(\log x)^{1/2}$. We let $m = n/d$ be the corresponding cofactor. Clearly,

$$d \in \left[\frac{y}{2K}, y\right], \quad m \in \left[\frac{y\log x}{2}, 2Ky\log x\right].$$

In the remainder of the proof we take

$$M = \log\log x$$

and assume that x is large enough that $M \ge 3$. We let D be any odd divisor of $\prod_{q\le M} q$ and study the contribution to $\mathscr{D}(x)$ of those n having

$$D = \gcd(n, \prod_{q\le M} q).$$

Let $\mathscr{Q}_D$ be the set of prime factors of D and let $\bar{\mathscr{Q}}_D$ be the set of primes $q \le M$ not dividing D. Observe that $(n,\varphi(n)) = 1$, so that $(m,\varphi(m)) = 1$. In particular, $(\varphi(m), \prod_{q\in\mathscr{Q}_D} q) = 1$. We distinguish 3 possibly overlapping cases:

1. $\sum_{q\in\mathcal{Q}_D} 1/q \ge (1/3)\log\log M$;
2. $\sum_{q\in\bar{\mathcal{Q}}_D} 1/q \ge (2/3)\log\log M$ and $F_{\bar{\mathcal{Q}}_D}(m) \le (1/2)\log x$;
3. $F_{\bar{\mathcal{Q}}_D}(m) > (1/2)\log x$.

Since $\sum_{q\le M} 1/q > \log\log M$ for all sufficiently large values of x, these 3 cases cover all possibilities.

In case 1, we have $g_{\mathcal{Q}_D} \ll (\log M)^{-1/3}$, so that by Lemma 2.16 the number of possibilities for $m \le 2Ky\log x$ is

$$\le \frac{Ky(\log x)\exp\left(O((\log M)^2)\right)}{(\log x)^{1+O((\log M)^{-1/3})}} = y\exp\left(O(M/(\log M)^{1/3})\right).$$

Since $dm \equiv 1 \pmod{\varphi(d)\varphi(m)}$, it follows that $d \le x/m$ is uniquely determined modulo $\varphi(m)$, and since $m\varphi(m) > x$ for large values of x, we get that m determines n uniquely.

In case 2, we use Proposition 2.12 with $t = 2Ky\log x$ and $\lambda = 1$. Note that $T_{\bar{\mathcal{Q}}_D} \ge (\log M)^{2/3}$. We get that the number of possibilities for m, and hence for n, is at most

$$\frac{Ky(\log x)\exp\left(O((\log M)^3)\right)}{(\log x)^{1+O((\log\log M)^{1/2}/(\log M)^{1/3})}} = y\exp\left(O\left(\frac{M(\log\log M)^{1/2}}{(\log M)^{1/3}}\right)\right).$$

Assume next that $F_{\bar{\mathcal{Q}}_D}(m) > (1/2)\log x$. In particular, there exists a divisor ℓ of $\varphi(m)$ in the interval $[(\log x)/(2M), (\log x)/2]$ with each prime factor of ℓ in $[1, M]$. Let us fix this number ℓ. The number of choices for ℓ is at most $\psi(\log x, M)$, where $\psi(X,Y)$ denotes the number of integers in $[1,X]$ composed of primes in $[1,Y]$. Using a result of Erdős [4] (see also [3]) that $\psi(X, \log X) \le 4^{(1+o(1))(\log X)/\log\log X}$ as $X\to\infty$, we have

$$\psi(\log x, M) \le \exp(O(M/\log M)).$$

Let us fix also d. Then the congruence $dm \equiv 1 \pmod{\varphi(d)\varphi(m)}$ puts $m \le x/d$ in a congruence class modulo $\varphi(d)\ell$. Thus, the number of choices for m is at most $1 + x/(d\varphi(d)\ell)$. Summing over $d \in [y/(2K), y]$, we have for this ℓ that the number of possibilities for m, hence for n, is

$$\le \sum_{d\in[y/(2K),y]} \left(1 + \frac{x}{d\varphi(d)\ell}\right) \ll y + \frac{Kx}{y\ell} \le (2KM+1)y \ll M^2 y.$$

Multiplying this expression by the number of choices for ℓ we get a contribution of at most $y\exp(\,O(M/\log M))$ choices for m, hence for n, in this case.

Thus, we have at most $y\exp\left(O\left(M(\log\log M)^{1/2}/(\log M)^{1/3}\right)\right)$ choices for $n \in \mathcal{D}(x)$ in each case. This bound is to be multiplied by the number of odd D with $D \mid Q$, which is $2^{\pi(M)-1} \ll \exp(M/\log M)$. We therefore have

$$\#\mathcal{D}(x) \le \frac{x^{1/2}}{(\log x)^{1/2+o(1)}},$$

where $o(1)$ here has the order $O((\log\log\log\log x)^{1/2}/(\log\log\log x)^{1/3})$. This concludes our proof.

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PRODUCTOS VECTORIALES DE POLINOMIOS

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RESUMEN. Se introduce un producto vectorial entre polinomios homogéneos de un grado fijo en tres variables con coeficientes en un campo. Aunado a un producto interior adecuado, se pueden detectar propiedades geométricas de curvas algebraicas. En este artículo nos concretamos a propiedades de tangencia o incidencia de curvas algebraicas en el plano proyectivo sobre un campo F de característica cero.

ABSTRACT. A vector product is introduced between homogeneous polynomials of fixed degree in three variables with coefficients in a field. Combined with an appropriate inner product, geometric properties of algebraic curves may be detected. In this article we focus on tangency or incidence properties of algebraic curves in the projective plane over a field F of characteristic zero.

1. Introducción

Introduciremos un producto vectorial y un producto interior en el espacio vectorial $V_F(n)$ que consiste de todos los polinomios homogéneos de grado n en tres indeterminadas x_1, x_2, x_3 y con coeficientes en un campo F de característica 0. Obviamente tenemos que incluir el polinomio 0 en $V_F(n)$. Como aplicación podremos expresar diversas condiciones de incidencia y tangencia entre curvas algebraicas.

2. Definiciones y ejemplos preliminares

DEFINICIÓN (2.1). *Sea V un espacio vectorial sobre un campo F. Un producto interior en V es una función* $: V \times V \to F$, $(v,w) \mapsto v \cdot w \in F$, *la cual cumple las siguientes propiedades:*

(i) $v \cdot (w + w') = v \cdot w + v \cdot w', \quad v, w, w' \in V$

(ii) $v \cdot w = w \cdot v, \quad v, w \in V$

(iii) $v \cdot (\lambda w) = (\lambda v) \cdot w = \lambda(v \cdot w), \quad v, w \in V, \lambda \in F.$

Tenemos tres ejemplos fundamentales:

EJEMPLO (2.2). $V = \mathbb{R}^n \quad (n \in \mathbb{N})$

$$(v_1, v_2, \ldots, v_n) \cdot (w_1, w_2, \ldots, w_n) = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n\,.$$

EJEMPLO (2.3). *V = matrices $n \times n$ con entradas en F. Si $A, B \in V$,*

$$A \cdot B = \mathrm{Tr}(AB^t),\ B^t = \text{ matriz transpuesta de } B\,.$$

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EJEMPLO (2.4). $V = \left\{ \sum_{\substack{i+j+k=n \\ 0\le i,j,k}} a_{ijk} x_1^i x_2^j x_3^k \,|\, a_{ijk} \in F \right\}$

$$\left(\sum_{i,j,k} a_{ijk} x_1^i x_2^j x_3^k\right) \cdot \left(\sum_{i,j,k} b_{ijk} x_1^i x_2^j x_3^k\right) = \frac{1}{n!} \sum_{i,j,k} a_{ijk} b_{ijk} i! j! k!,$$

DEFINICIÓN (2.5). *Sea V un espacio vectorial sobre un campo F y sea $\cdot$ un producto interior en V.*

(a) Dos vectores $v, w \in V$ son ortogonales si $v \cdot w = 0$.

(b) Una colección $\mathscr{S}$ de vectores de V es ortogonal si cada vez que $v, w \in \mathscr{S}$, con $v \neq w$, se tiene $v \cdot w = 0$.

(c) Un vector $v \in V$, $v \neq \overline{0}$ es isotrópico si $v \cdot v = 0$ y anisotrópico si $v \cdot v \neq 0$.

(d) V es anisotrópico si no contiene vectores isotrópicos.

(e) Para cada subespacio L de V, se define

$$L^{\perp} = \{v \in V | v \cdot w = 0 \ \forall\, w \in L\}$$

(f) El producto interior $\cdot$ es regular si $V^{\perp} = \{\overline{0}\}$.

Se dice también en este caso que el espacio $(V, \cdot)$ es regular.

Cuando $F = \mathbb{R}$, los espacios descritos en los ejemplos 2.2, 2.3 y 2.4 son anisotrópicos. Obviamente un espacio anisotrópico es regular. El siguiente es un ejemplo de un espacio regular con vectores isotrópicos.

EJEMPLO (2.6). *Si F tiene característica $\neq 2$, sea*

$$V = \{a(x^2 + y^2) + 2gxz + 2fyz + cz^2 | a, g, f, c \in F\}$$

Defínase:

$$[a(x^2 + y^2) + 2gxz + 2fyz + cz^2] \cdot [a'(x^2 + y^2) + 2g'xz + 2f'yz + c'z^2]$$
$$= 2gg' + 2ff' - ac' - a'c.$$

Los vectores $x^2 + y^2$ y z^2 son obviamente isotrópicos. Sin embargo, V es regular.

La demostración de los siguientes teoremas se puede encontrar en [2], Capítulo 12.

TEOREMA (2.7). *Para cada subespacio L de V, $L^{\perp}$ es también un subespacio de V. Si* $\dim V < \infty$, *se tienen la fórmula:*

$$\dim L + \dim L^{\perp} = \dim V + \dim(L \cap V^{\perp}).$$

Por tanto, si V es regular:

$$\dim L + \dim L^{\perp} = \dim V.$$

TEOREMA (2.8). *Todo espacio vectorial de dimension finita con producto interior admite una base ortogonal.*

COROLARIO (2.9). *Todo espacio vectorial de dimension finita con producto interior regular admite una base ortogonal con vectores anisotrópicos*

TEOREMA (2.10). *Sea V un espacio vectorial de dimension finita n con producto interior y sea L un subespacio de V. Entonces:*

(a) $\dim L^{\perp\perp} = \dim L + \dim V^{\perp}/L \cap V^{\perp}$. *Por tanto,* $L^{\perp\perp} = L$ *si y sólo si* $V^{\perp} \subseteq L$.

(b) *Si L es un subespacio regular de V (es decir, si el producto interior restringido a $L \times L$ es regular), entonces:*

$$V = L + L^{\perp}, \quad L \cap L^{\perp} = \{\overline{0}\} \text{ y } L^{\perp\perp} = L$$

(c) *L y $L^{\perp}$ son ambos regulares si y sólo si L y V son ambos regulares.*

DEFINICIÓN (2.11). *Sea V un espacio vectorial sobre un campo F. Un producto vectorial en V es una función $: V \times V \to V$, $(v,w) \mapsto v \times w \in V$, la cual satisface:*

(i) $v \times (w + w') = v \times w + v \times w'$, $v, w, w' \in V$

(ii) $v \times (\lambda w) = (\lambda v) \times w = \lambda(v \times w)$, $v, w \in V, \lambda \in F$

(iii) $v \times w = w \times v \quad \forall v, w \in V$ *o*

(iii)' $v \times w = -(w \times v) \quad \forall v, w \in V$.

Si $\times$ satisface i), ii) y iii) diremos que $\times$ es *simétrico*. Si $\times$ satisface i), ii) y iii)', diremos que $\times$ es *asimétrico*.

EJEMPLO (2.12). *El producto vectorial usual de $\mathbb{R}^3$:*

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

es asimétrico.

EJEMPLO (2.13). *El producto vectorial en $\mathbb{R}^2$:*

$$(a, b) \times (a', b') = (bb', ab' + a'b)$$

es simétrico.

EJEMPLO (2.14). *Sea V el conjunto de matrices simétricas 3×3 con entradas en un campo F. Defínase:*

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \times \begin{pmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{pmatrix}$$

$$= \begin{pmatrix} bc' + b'c - 2ff' & fg' + f'g - hc' - h'c & hf' + h'f - bg' - b'g \\ fg' + f'g - hc' - h'c & ac' + a'c - 2gg' & hg' + h'g - af' - a'f \\ hf' + h'f - bg' - b'g & hg' + h'g - af' - a'f & ab' + a'b - 2hh' \end{pmatrix}$$

Este es un producto vectorial simétrico.

En la sección siguiente definiremos un producto vectorial en $V_F(n)$ el cual será simétrico si n es par y asimétrico si n es impar. Además, con el producto interior en $V_F(n)$ definido en (2.4), siempre tendremos:

$$F \cdot (G \times H) = (F \times G) \cdot H, \quad F, G, H \in V_F(n).$$

3. Producto vectorial de polinomios

Convertiremos al espacio vectorial $V_F(n)$ definido en la introducción de este trabajo en un triple espacio. El producto interior en $V_F(n)$ se define como en el Ejemplo (2.4). Todos los elementos de $V_F(n)$ pueden expresarse en la forma:

$$E(x_1,x_2,x_3) = \sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} a_{ijk}x_1^i x_2^j x_3^k, \qquad a_{ijk}\in F.$$

La dimensión de $V_F(n)$ coincide con el número de soluciones enteras no negativas de la ecuación:

$$i+j+k=n,$$

esto es, $\dim V_F(n) = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$. Por ejemplo, $\dim V_F(1)=3$, $\dim V_F(2)=6$, $\dim V_F(3)=10$.

Las derivadas parciales $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ son transformaciones lineales de $V_F(n)$ en $V_F(n-1)$, en donde:

$$\begin{aligned}\frac{\partial x_1^i x_2^j x_3^k}{\partial x_1} &= i x_1^{i-1}x_2^j x_3^k,\\ \frac{\partial x_1^i x_2^j x_3^k}{\partial x_2} &= j x_1^{i}x_2^{j-1} x_3^k\\ \frac{\partial x_1^i x_2^j x_3^k}{\partial x_3} &= k x_1^{i}x_2^j x_3^{k-1}.\end{aligned}$$

Tenemos entonces las fórmulas usuales para sumas y productos:

$$\begin{aligned}\frac{\partial(E+G)}{\partial x_i} &= \frac{\partial E}{\partial x_i}+\frac{\partial G}{\partial x_i}, && i=1,2,3;\\ \frac{\partial(\lambda E)}{\partial x_i} &= \lambda\frac{\partial E}{\partial x_i}, && \lambda\in F,\ i=1,2,3;\\ \frac{\partial EG}{\partial x_i} &= E\frac{\partial G}{\partial x_i}+G\frac{\partial E}{\partial x_i}, && i=1,2,3;\\ \frac{\partial E^s}{\partial x_i} &= sE^{s-1}\frac{\partial E}{\partial x_i}, && s\in\mathbb{N},\ i=1,2,3;\end{aligned}$$

$$\frac{\partial x_t}{\partial x_i} = \begin{cases}1 & \text{si } t=i\\ 0 & \text{si } t\neq i\end{cases}.$$

Dados $E,G\in V_F(n)$, definiremos el producto vectorial $E\times G$ y el producto interior $E\cdot G$ de una manera recursiva. La definición recursiva de $E\cdot G$ coincidirá con la definición dada en el Ejemplo (2.4).

Si $n=0$, $V_F(0)$ coincide con el campo F y los productos $\times,\cdot$ en $V_F(0)$ coinciden ambos con la multiplicación ordinaria en F. Suponiendo que ambos productos ya han sido definidos en $V_F(n-1)$, en donde $n \geq 1$, y si $E, G \in V_F(n)$, ponemos:

$$n^2(E \times G) = \left(\frac{\partial E}{\partial x_2} \times \frac{\partial G}{\partial x_3} - \frac{\partial E}{\partial x_3} \times \frac{\partial G}{\partial x_2}\right) x_1 + \left(\frac{\partial E}{\partial x_3} \times \frac{\partial G}{\partial x_1} - \frac{\partial E}{\partial x_1} \times \frac{\partial G}{\partial x_3}\right) x_2 + \left(\frac{\partial E}{\partial x_1} \times \frac{\partial G}{\partial x_2} - \frac{\partial E}{\partial x_2} \times \frac{\partial G}{\partial x_1}\right) x_3 \,;$$

$$n^2(E \cdot G) = \frac{\partial E}{\partial x_1} \cdot \frac{\partial G}{\partial x_1} + \frac{\partial E}{\partial x_2} \cdot \frac{\partial G}{\partial x_2} + \frac{\partial E}{\partial x_3} \cdot \frac{\partial G}{\partial x_3}.$$

Por ejemplo, si $n = 1$ o $n = 2$,

$$(a_1x_1 + a_2x_2 + a_3x_3) \times (b_1x_1 + b_2x_2 + b_3x_3) = (a_2b_3 - a_3b_2)x_1 + (a_3b_1 - a_1b_3)x_2 + (a_1b_2 - a_2b_1)x_3 \,;$$

$$(a_1x_1 + a_2x_2 + a_3x_3) \cdot (b_1x_1 + b_2x_2 + b_3x_3) = a_1b_1 + a_2b_2 + a_3b_3 \,;$$

$$(ax_1^2 + 2hx_1x_2 + bx_2^2 + 2gx_1x_3 + 2fx_2x_3 + cx_3^2) \times (a'x_1^2 + 2h'x_1x_2 + b'x_2^2 + 2g'x_1x_3 + 2f'x_2x_3 + c'x_3^2)$$
$$= (bc' + b'c - 2ff')x_1^2 + 2(fg' + f'g - hc' - h'c)x_1x_2 + (ac' + a'c - 2gg')x_2^2 + 2(hf' + h'f - bg' - b'g)x_1x_3 + 2(hg' + h'g - af' - a'f)x_2x_3 + (ab' + a'b - 2hh')x_3^2 \,;$$

$$(ax_1^2 + 2hx_1x_2 + bx_2^2 + 2gx_1x_3 + 2fx_2x_3 + cx_3^2) \cdot (a'x_1^2 + 2h'x_1x_2 + b'x_2^2 + 2g'x_1x_3 + 2f'x_2x_3 + c'x_3^2) = aa' + bb' + cc' + 2hh' + 2gg' + 2ff'.$$

Reunimos en un sólo teorema las propiedades esenciales de estos productos.

TEOREMA (3.1). *Sean $E, G, H \in V_F(n)$ y sea $\lambda \in F$. Para cada $L \in V_F(n)$, denotemos $\frac{\partial L}{\partial x_i}$ como L_i. Entonces:*

(i) $E \cdot (G + H) = E \cdot G + E \cdot H$,

(ii) $E \cdot G = G \cdot E$; $E \times G = (-1)^n G \times E$,

(iii) $E \cdot (\lambda G) = \lambda(E \cdot G)$; $E \times (\lambda G) = (\lambda E) \times G = \lambda(E \times G)$,

(iv) $E \times (G + H) = E \times G + E \times H$,

(v) $n(E \times G)_1 = E_2 \times G_3 - E_3 \times G_2$; $n(E \times G)_2 = E_3 \times G_1 - E_1 \times G_3$ *y* $n(E \times G)_3 = E_1 \times G_2 - E_2 \times G_1$,

(vi) $E \cdot (G \times H) = (E \times G) \cdot H$.

Demostración. (i) Procederemos por inducción respecto a n. El caso $n = 0$ es simplemente la ley distributiva en F. Suponiendo el resultado cierto en $V_F(n-1)$, tenemos:

$$\begin{aligned} n^2[E \cdot (G + H)] &= E_1 \cdot (G + H)_1 + E_2 \cdot (G + H)_2 + E_3 \cdot (G + H)_3 \\ &= E_1 \cdot (G_1 + H_1) + E_2 \cdot (G_2 + H_2) + E_3 \cdot (G_3 + H_3) \\ &= E_1 \cdot G_1 + E_1 \cdot H_1 + E_2 \cdot G_2 + E_2 \cdot H_2 + E_3 \cdot G_3 + E_3 \cdot H_3 \\ &= (E_1 \cdot G_1 + E_2 \cdot G_2 + E_3 \cdot G_3) + (E_1 \cdot H_1 + E_2 \cdot H_2 + E_3 \cdot H_3) \\ &= n^2(E \cdot G) + n^2(E \cdot H) \\ &= n^2(E \cdot G + E \cdot H). \end{aligned}$$

Dividiendo por n^2, obtenemos la fórmula deseada.

(ii) Procederemos nuevamente por inducción respecto a n:

$$\begin{aligned} n^2(E\cdot G) &= E_1\cdot G_1 + E_2\cdot G_2 + E_3\cdot G_3 \\ &= G_1\cdot E_1 + G_2\cdot E_2 + G_3\cdot E_3 \\ &= n^2(G\cdot E); \end{aligned}$$

$$\begin{aligned} n^2(E\times G) &= (E_2\times G_3 - E_3\times G_2)x_1 + (E_3\times G_1 - E_1\times G_3)x_2 \\ &\quad + (E_1\times G_2 - E_2\times G_1)x_3 \\ &= (-1)^{n-1}[(G_3\times E_2 - G_2\times E_3)x_1 + \cdots] \\ &= (-1)^{n}[(G_2\times E_3 - G_3\times E_2)x_1 + \cdots] \\ &= (-1)^{n} n^2(G\times E). \end{aligned}$$

(iii) (Obvio).

(iv) Esta parte se obtiene razonando como en i).

(v) Procederemos por inducción. Tomando derivadas parciales respecto a x_1 en la ecuación

$$n^2(E\times G) = [(E_2\times G_3) - (E_3\times G_2)]x_1 + \cdots,$$

obtenemos

$$\begin{aligned} n^2(E\times G)_1 &= (E_2\times G_3) - (E_3\times G_2) + [(E_2\times G_3)_1 - (E_3\times G_2)_1]x_1 \\ &\quad + [(E_3\times G_1)_1 - (E_1\times G_3)_1]x_2 + [(E_1\times G_2)_1 - (E_2\times G_1)_1]x_3. \end{aligned}$$

Por la hipótesis inductiva:

$$\begin{aligned} (n-1)[(E_3\times G_1)_1 - (E_1\times G_3)_1] &= (E_{23}\times G_{13}) - (E_{33}\times G_{12}) - (E_{12}\times G_{33}) + (E_{13}\times G_{23}) \\ &= (n-1)[(E_2\times G_3)_2 - (E_3\times G_2)_2]. \end{aligned}$$

Análogamente:

$$(n-1)[(E_1\times G_2)_1 - (E_2\times G_1)_1] = (n-1)[(E_2\times G_3)_3 - (E_3\times G_2)_3].$$

Por tanto:

$$\begin{aligned} n^2(n-1)(E\times G)_1 &= (n-1)[(E_2\times G_3) - (E_3\times G_2)] \\ &\quad + (n-1)[(E_2\times G_3)_1 - (E_3\times G_2)_1]x_1 \\ &\quad + (n-1)[(E_2\times G_3)_2 - (E_3\times G_2)_2]x_2 \\ &\quad + (n-1)[(E_2\times G_3)_3 - (E_3\times G_2)_3]x_3. \end{aligned}$$

Por homogeneidad:

$$n^2(n-1)(E\times G)_1 = (n-1)[(E_2\times G_3) - (E_3\times G_2)] + (n-1)^2[(E_2\times G_3) - (E_3\times G_2)].$$

Por tanto:

$$n(E\times G)_1 = (E_2\times G_3) - (E_3\times G_2).$$

En forma similar se prueban las fórmulas para $n(E\times G)_2$ y $n(E\times G)_3$.

(vi) Usamos (v) e inducción:

$$\begin{aligned} n^2[E\cdot(G\times H)] &= E_1\cdot(G\times H)_1 + E_2\cdot(G\times H)_2 + E_3\cdot(G\times H)_3 \\ &= \frac{1}{n}[E_1\cdot((G_2\times H_3) - (G_3\times H_2)) + E_2\cdot((G_3\times H_1) - (G_1\times H_3)) \\ &\quad + E_3\cdot((G_1\times H_2) - (G_2\times H_1))] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n}[(E_1 \times G_2)\cdot H_3 - (E_1 \times G_3)\cdot H_2 + (E_2 \times G_3)\cdot H_1 \\
&\qquad - (E_2 \times G_1)\cdot H_3 + (E_3 \times G_1)\cdot H_2 - (E_3 \times G_2)\cdot H_1] \\
&= (E \times G)_1 \cdot H_1 + (E \times G)_2 \cdot H_2 + (E \times G)_3 \cdot H_3 \\
&= n^2[(E \times G)\cdot H].
\end{aligned}$$

□

Obtendremos a continuación fórmulas para los productos vectoriales e interiores $E \times G$ y $E \cdot G$, en donde E y G son productos de n elementos de $V_F(1)$. Denotamos como S_n al grupo de permutaciones del conjunto $\{1,2,\ldots,n\}$.

TEOREMA (3.2). *Sean $L_1,\ldots,L_n$; $M_1,\ldots,M_n \in V_F(1)$. Entonces:*

(a) $n!L_1L_2\cdots L_n \times M_1\cdots M_n = \sum_{\tau\in S_n}(L_1 \times M_{\tau(1)})\cdots(L_n \times M_{\tau(n)})$

(b) $n!L_1L_2\cdots L_n \cdot M_1M_2\cdots M_n = \sum_{\tau\in S_n}(L_1 \cdot M_{\tau(1)})\cdots(L_n \cdot M_{\tau(n)})$.

Demostración. (a) Usamos inducción y 3.1 (v). Sean

$$L_i = \lambda_{i1}x_1 + \lambda_{i2}x_2 + \lambda_{i3}x_3\,; \quad M_j = \mu_{j1}x_1 + \mu_{j2}x_2 + \mu_{j3}x_3\,, \quad i,j = 1,2,\ldots,n.$$

Tenemos

$$n\frac{\partial}{\partial x_1}[L_1L_2\cdots L_n \times M_1M_2\cdots M_n] =$$

$$\begin{aligned}
&= \frac{\partial}{\partial x_2}(L_1L_2\cdots L_n) \times \frac{\partial}{\partial x_3}(M_1M_2\cdots M_n) \\
&\qquad - \frac{\partial}{\partial x_3}(L_1L_2\cdots L_n) \times \frac{\partial}{\partial x_2}(M_1M_2\cdots M_n) \\
&= \left(\sum_{i=1}^{n}\lambda_{i_2}L_1L_2\cdots L_{i-1}L_{i+1}\cdots L_n\right) \times \left(\sum_{j=1}^{n}\mu_{j_3}M_1M_2\cdots M_{j-1}M_{j+1}\cdots M_n\right) \\
&- \left(\sum_{i=1}^{n}\lambda_{i_3}L_1L_2\cdots L_{i-1}L_{i+1}\cdots L_n\right) \times \left(\sum_{j=1}^{n}\mu_{j_2}M_1M_2\cdots M_{j-1}M_{j+1}\cdots M_n\right) \\
&= \sum_{i,j}\left(\lambda_{i2}\mu_{j3} - \lambda_{i3}\mu_{j2}\right)(L_1L_2\cdots L_{i-1}L_{i+1}\cdots L_n) \times M_1M_2\cdots M_{j-1}M_{j+1}\cdots M_n).
\end{aligned}$$

Llamando T_{ij} al conjunto de biyecciones de $\{1,\ldots,i-1,i+1,\ldots,n\}$ sobre $\{1,\ldots,j-1,j+1,\ldots,n\}$ y aplicando la inducción, concluimos:

$$\begin{aligned}
&\sum_{i,j}(\lambda_{i2}\mu_{j3} - \lambda_{i3}\mu_{j2})\left(L_1L_2\cdots L_{i-1}L_{i+1}\cdots L_n\right) \times \left(M_1M_2\cdots M_{j-1}M_{j+1}\cdots M_n\right) \\
&= \frac{1}{(n-1)!}\sum_{\substack{i,j\\ \sigma\in T_{ij}}}(\lambda_{i2}\mu_{j3} - \lambda_{i3}\mu_{j2})(L_1 \times M_{\sigma(1)})\cdots(L_{i-1} \times M_{\sigma(i-1)})(L_{i+1} \times M_{\sigma(i+1)})\cdots \\
&= \frac{1}{(n-1)!}\sum_{\substack{i,j\\ \sigma\in T_{ij}}}\frac{\partial(L_i \times M_j)}{\partial x_1}(L_1 \times M_{\sigma(1)})\cdots(L_{i-1} \times M_{\sigma(i-1)})(L_{i+1} \times M_{\sigma(i+1)})\cdots.
\end{aligned}$$

Obtenemos fórmulas similares para

$$n\frac{\partial}{\partial x_2}(L_1L_2\cdots L_n \times M_1M_2\cdots M_n) \text{ y } n\frac{\partial}{\partial x_3}(L_1L_2\cdots L_n \times M_1M_2\cdots M_n).$$

Por tanto

$$
\begin{aligned}
&n^2(L_1L_2\cdots L_n \times M_1M_2\cdots M_n)=\\
&= n\sum_{k=1}^{3}\frac{\partial}{\partial x_k}(L_1L_2\cdots L_n \times M_1M_2\cdots M_n)x_k\\
&= \frac{1}{(n-1)!}\sum_{\substack{i,j\\ \sigma\in T_{ij}}}(L_i\times M_j)(L_1\times M_{\sigma(1)})\cdots(L_{i-1}\times M_{\sigma(i-1)})(L_{i+1}\times M_{\sigma(i+1)})\cdots\\
&= \frac{n}{(n-1)!}\sum_{\sigma\in S_n}(L_1\times M_{\tau(1)})\cdots(L_n\times M_{\tau(n)}).
\end{aligned}
$$

(De hecho, cada $\sigma\in T_{ij}$ puede extenderse a una permutación $\tau\in S_n$ si definimos $\tau(i)=j$).

Por tanto

$$
n!(L_1L_2\cdots L_n \times M_1M_2\cdots M_n)=\sum_{\tau\in S_n}(L_1\times M_{\tau(1)})\cdots(L_n\times M_{\tau(n)}).
$$

(b) Usando inducción y la definición de producto interior:

$$
\begin{aligned}
&n^2(L_1L_2\cdots L_n \cdot M_1M_2\cdots M_n)=\\
&=\sum_{k=1}^{3}\left(\frac{\partial L_1L_2\cdots L_n}{\partial x_k}\cdot\frac{\partial M_1M_2\cdots M_n}{\partial x_k}\right)\\
&=\sum_{k=1}^{3}\left[\left(\sum_{i=1}^{n}\lambda_{ik}L_1\cdots L_{i-1}L_{i+1}\cdots L_n\right)\cdot\left(\sum_{j=1}^{n}\mu_{jk}M_1M_2\cdots M_{j-1}M_{j+1}\cdots M_n\right)\right]\\
&= \frac{1}{(n-1)!}\sum_{\substack{i,j\\ \sigma\in T_{ij}}}(L_i\cdot M_j)(L_1\cdot M_{\sigma(1)})\cdots(L_{i-1}\cdot M_{\sigma(i-1)})(L_{i+1}\cdot M_{\sigma(i+1)}\cdots\\
&= \frac{n}{(n-1)!}\sum_{\tau\in S_n}(L_1\cdot M_{\tau(1)})\cdots(L_n\cdot M_{\tau(n)})
\end{aligned}
$$

Por tanto

$$
n!(L_1L_2\cdots L_n \cdot M_1M_2\cdots M_n)=\sum_{\tau\in S_n}(L_1\cdot M_{\tau(1)})\cdots(L_n\cdot M_{\tau(n)}).
$$

□

COROLARIO (3.3). *Sean $L_1,L_2,\dots,L_n,M\in V_F(1)$, $E\in V_F(s)$, $G\in V_F(t)$ con $n\in\mathbb{N}$ y $s+t=n$. Entonces:*

(a) $L_1L_2\cdots L_n\times M^n=(L_1\times M)(L_2\times M)\cdots(L_n\times M)$,

(b) $L_1L_2\cdots L_n\cdot M^n=(L_1\cdot M)(L_2\cdot M)\cdots(L_n\cdot M)$,

(c) $EG\times M^n=(E\times M^s)(G\times M^t)$,

(d) $EG\cdot M^n=(E\cdot M^s)(G\cdot M^t)$.

En el siguiente teorema calculamos los productos vectoriales e interiores de monomios.

TEOREMA (3.4). *Sea $n\in\mathbb{N}$ y sean i,j,k,i',j',k' enteros no negativos tales que $n=i+j+k=i'+j'+k'$.*

(a) *Si* $(i-i')^2+(j-j')^2+(k-k')^2>0$, *entonces*

$$x_1^i x_2^j x_3^k \cdot x_1^{i'} x_2^{j'} x_3^{k'} = 0\,.$$

(b) $x_1^i x_2^j x_3^k \cdot x_1^i x_2^j x_3^k = \frac{i!j!k!}{n!}$.

(c) *Si* $\max\{i+i', j+j', k+k'\} > n$, *entonces*

$$x_1^i x_2^j x_3^k \times x_1^{i'} x_2^{j'} x_3^{k'} = 0.$$

(d) *Si* $\max\{i+i', j+j', k+k'\} \le n$ *y si* $i''=n-(i+i')$, $j''=n-(j+j')$, $k''=n-(k+k')$, *entonces*

$$x_1^i x_2^j x_3^k \times x_1^{i'} x_2^{j'} x_3^{k'} = \lambda x_1^{i''} x_2^{j''} x_3^{k''},$$

en donde

$$\lambda = \frac{i!j!k!}{n!}\sum_{p=\ell}^{t}(-1)^{i'+p-k}\binom{i'}{p+j-k'}\binom{j'}{i-p}\binom{k'}{p},$$

$$\ell = \max\{k'-j, 0, i-j'\},$$

$$t = \min\{k', i, i'+k'-j\}\,.$$

De hecho, $t-\ell = \min\{i,j,k,i',j',k',i'',j'',k''\}$*. Por tanto, si* $\ell = t$*, se tiene*

$$\lambda = (-1)^{i'+\ell-k}\,\frac{i!j!k!}{n!}\binom{i'}{\ell+j-k'}\binom{j'}{i-\ell}\binom{k'}{\ell} \neq 0.$$

(e) $x_1^{n-s}x_2^s \times x_1^i x_2^j x_3^k = (-1)^{s-k}\dfrac{\binom{k}{s-i}}{\binom{n}{s}} x_1^{s-i} x_2^{n-s-j} x_3^{n-k}$.

(f) $x_1^i x_2^j x_3^k \times x_1^{i'} x_2^{k-i'} x_3^{n-k} = \dfrac{(-1)^{i+i'-k}}{\binom{n}{k}} x_1^{n-(i+i')} x_2^{i+i'}$.

Demostración. (a) Supongamos, por ejemplo, que $i < i'$. Entonces, para cada $\tau \in S_n$, existe un entero s, $1 \le s \le i'$ tal que $\tau(s) > i$. Aplicando (3.2), (b) con el producto interior $x_1^{i'} x_2^{j'} x_3^{k'} \cdot x_1^i x_2^j x_3^k$, deducimos que cada sumando tiene un factor de la forma $x_1 \cdot x_2$ o de la forma $x_1 \cdot x_3$. Por tanto, cada sumando es igual a cero y

$$x_1^i x_2^j x_3^k \cdot x_1^{i'} x_2^{j'} x_3^{k'} = x_1^{i'} x_2^{j'} x_3^{k'} \cdot x_1^i x_2^j x_3^k = 0\,.$$

(b) Se sigue directamente de (3.2), (b).

(c) Supongamos, por ejemplo, que $i+i' > n$. Entonces $i' > n-i = j+k$ y cada sumando no nulo en el desarrollo de $n!x_1^{i'} x_2^{j'} x_3^{k'} \times x_1^i x_2^j x_3^k$ sería de la forma:

$$\pm(x_1\times x_2)^{\lambda}(x_1\times x_3)^{i'-\lambda}(x_2\times x_3)^{\mu}(x_2\times x_1)^{j'-\mu}(x_3\times x_1)^{\nu}(x_3\times x_2)^{k'-\nu}\,.$$

Pero tales términos no pueden existir, pues la desigualdad $i' > j+k$ implica que $\lambda > j$ o $i'-\lambda > k$.

(d) Cada sumando no nulo en el desarrollo de

$$n!x_1^i x_2^j x_3^k \times x_1^{i'} x_2^{j'} x_3^{k'}$$

es de la forma

$$(*) \quad (x_1\times x_3)^p (x_1\times x_2)^{i-p}(x_2\times x_3)^{k'-p}(x_2\times x_1)^{j-k'+p}(x_3\times x_1)^{i'+k'-j-p}(x_3\times x_2)^{j'-i+p}$$

$$= (-1)^{i'-k+p} x_1^{n-i-i'} x_2^{n-j-j'} x_3^{n-k-k'},$$

en donde p es un entero tal que los seis exponentes p, $i-p$, $k'-p$, $j-k'+p$, $i'+k'-j-p$, $j'-i+p$ son todos ≥ 0.

Por tanto, $p \geq \max\{k'-j, 0, i-j'\} = \ell$ y $p \leq \min\{k', i, i'+k'-j\} = t$. Para una p fija, $\ell \leq p \leq t$, tenemos

$$\binom{k'}{p}\binom{i}{p} p! \binom{j}{k'-p}(k'-p)! \binom{j'}{i-p}\binom{k}{j'-i+p}(j'-i+p)!(i-p)!$$

$$\cdot \binom{i'}{j-k'+p}(j-k'+p)!(i'+k'-j-p)! = i!j!k! \binom{i'}{p+j-k'}\binom{j'}{i-p}\binom{k'}{p}$$

sumandos del tipo $(*)$. Esto completa el cálculo del coeficiente λ. La diferencia $t-\ell$ es igual al mínimo de todas las diferencias posibles $a-b$, en donde $a \in \{k', i, i'+k'-j\}$ y $b \in \{k'-j, 0, i-j'\}$. Este conjunto de nueve diferencias coincide con el conjunto $\{i, j, k, i', j', k', i'', j'', k''\}$. En el caso $t-\ell = 0$, el coeficiente λ tiene un único término:

$$\lambda = (-1)^{i'+\ell-k} \frac{i!j!k!}{n!} \binom{i'}{\ell+j-k'}\binom{j'}{i-\ell}\binom{k'}{\ell} \neq 0.$$

(e) y (f) son casos especiales de esta última fórmula. □

TEOREMA (3.5). *Sea $n \in \mathbb{N}$, sean i, j, k, i', j', k' enteros no negativos tales que $n = i+j+k = i'+j'+k'$ y sean $L_1, L_2, L_3 \in V_F(1)$. Entonces*

(a) *Si $(i-i')^2 + (j-j')^2 + (k-k')^2 > 0$, tenemos*

$$L_1^i L_2^j L_3^k \cdot (L_2 \times L_3)^{i'} (L_3 \times L_1)^{j'} (L_1 \times L_2)^{k'} = 0.$$

(b) $L_1^i L_2^j L_3^k \cdot (L_2 \times L_3)^i (L_3 \times L_1)^j (L_1 \times L_2)^k = \dfrac{i!j!k!}{n!} (L_1 \times L_2 \cdot L_3)^n$.

(c) *Si* $\max\{i+i', j+j', k+k'\} > n$, *entonces*

$$L_1^i L_2^j L_3^k \times L_1^{i'} L_2^{j'} L_3^{k'} = 0.$$

(d) *Si* $\max\{i+i', j+j', k+k'\} \leq n$, *entonces*

$$L_1^i L_2^j L_3^k \times L_1^{i'} L_2^{j'} L_3^{k'} = \lambda (L_2 \times L_3)^{n-i-i'} (L_3 \times L_1)^{n-j-j'} (L_1 \times L_2)^{n-k-k'},$$

en donde λ se define exactamente como en (3.4), (d).

COROLARIO (3.6). *Sean $n \in \mathbb{N}$, $E, G \in V_F(n)$ y sean $L_1, L_2, L_3 \in V_F(1)$. Entonces:*

(a) $E(L_1, L_2, L_3) \times G(L_1, L_2, L_3) = (E \times G)(L_2 \times L_3, L_3 \times L_1, L_1 \times L_2)$.

(b) $E(L_1, L_2, L_3) \cdot G(L_2 \times L_3, L_3 \times L_1, L_1 \times L_2) = (L_1 \times L_2 \cdot L_3)^n (E \cdot G)$.

TEOREMA (3.7). *Sean $n \in \mathbb{N}$, $E(x_1, x_2, x_3) \in V_F(n)$ y sea $P = a_1x_1 + a_2x_2 + a_3x_3 \in V_F(1)$. Entonces:*

(a) $E \cdot P^n = E(a_1, a_2, a_3)$.

(b) $E \times P^n = 0$ *si y sólo si P es un factor de E.*

(c) *Si P no es un factor de E, $E \times P^n$ se factoriza en la forma*

$$E \times P^n = P_1^{\alpha_1} \cdots P_s^{\alpha_s},$$

en donde $P_1, P_2, \ldots, P_s$ son elementos de $V_F(1)$ los cuales cumplen las propiedades

$$E \cdot P_i^n = 0 = P \cdot P_i \quad \text{para cada} \quad i = 1, \ldots, s$$

y con $\alpha_1 + \cdots + \alpha_s = n$. *Cualquier otro elemento* $Q \in V_F(1)$ *que cumpla* $E \cdot Q^n = 0 = P \cdot Q$ *es múltiplo escalar de alguna* P_i.

(d) *Tenemos la identidad*

$$\left[(E \times P^n) \times P^n\right] \times P^n = (-1)^n (P \cdot P)^n (E \times P^n).$$

(e) *Si* $P \cdot P \neq 0$, *se tiene* $E = T + T'$, *en donde:*

$$T = E - (-1)^n \frac{(E \times P^n) \times P^n}{(P \cdot P)^n}$$

y

$$T' = (-1)^n \frac{(E \times P^n) \times P^n}{(P \cdot P)^n}.$$

Además, $T \times P^n = 0$.

Demostración. (a) Por el Teorema de Desarrollo Multinomial, tenemos:

$$P^n = (a_1x_1 + a_2x_2 + a_3x_3)^n = \sum_{\substack{i+j+k=n \\ i,j,k \geq 0}} \binom{n}{i\ j\ k} a_1^i a_2^j a_3^k x_1^i x_2^j x_3^k.$$

Usando ((3.3)), (b) obtenemos

$$E(a_1, a_2, a_3) = E \cdot P^n.$$

(b) Obviamente, si P es factor de E, se tiene $E \times P^n = 0$ (úsese el Corolario (3.3), (a)). Supongamos ahora que $E \times P^n = 0$. Por el Corolario (3.6), podemos suponer, sin pérdida de generalidad, que $P = x_3$.

Podemos escribir E en la forma

$$E = aL_1^{\alpha_1} \cdots L_s^{\alpha_s} + x_3 G,$$

en donde $\alpha_1, \ldots, \alpha_s$ son enteros positivos con suma n, $a \in F$, $L_i \cdot x_3 = 0$ para cada $i = 1, \ldots, s$ y con $G \in V_F(n-1)$. Por tanto

$$0 = E \times x_3^n = a(L_1 \times x_3)^{\alpha_1} \cdots (L_s \times x_3)^{\alpha_s} = aP_1^{\alpha_1} \cdots P_s^{\alpha_s},$$

en donde $P_i = L_i \times x_3$ $(i = 1, \ldots, s)$. Por tanto, $a = 0$ y x_3 es un factor de E.

(c) Conservemos la notación de (b). En este caso, $a \neq 0$ y $P_i \cdot x_3 = 0 = E \cdot P_i^n$ para cada $i = 1, 2, \ldots, s$.

Si $Q \in V_F(1)$ y $Q \cdot x_3 = 0 = E \cdot Q^n$, entonces

$$0 = E \cdot Q^n = a(L_1 \cdot Q)^{\alpha_1} \cdots (L_s \cdot Q)^{\alpha_s},$$

Por tanto, para alguna $i \in \{1, 2, \ldots, s\}$, se tiene $L_i \cdot Q = 0$. Como también $L_i \cdot P_i = 0$ y $x_3 \cdot P_i = x_3 \cdot Q = 0$, debemos tener que P_i y Q son linealmente dependientes.

(d) Si $E \in V_F(1)$, podemos probar directamente que $(E \times P) \times P = -(P \cdot P)E + (E \cdot P)P$. Por tanto

$$[(E \times P) \times P] \times P = -(P \cdot P)(E \times P).$$

Sin pérdida de generalidad, podemos suponer que $P \cdot x_3 \neq 0$. Entonces x_1, x_2 y P constituyen una base de $V_F(1)$ y existen escalares $\lambda_1, \lambda_2, \lambda_3$ tales que:

$$x_3 = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 P.$$

Por tanto,

$$\begin{aligned} E(x_1,x_2,x_3) &= E(x_1,x_2,\lambda_1 x_1+\lambda_2 x_2+\lambda_3 P) \\ &= G(x_1,x_2,P) \\ &= H(x_1,x_2)+PK(x_1,x_2,P). \end{aligned}$$

Usando el corolario (3.3), tenemos:

$$E\times P^n = H(x_1,x_2)\times P^n = H(x_1\times P, x_2\times P),$$

$$(E\times P^n)\times P^n = H((x_1\times P)\times P,(x_2\times P)\times P),$$

$$\begin{aligned} \left[(E\times P^n)\times P^n\right]\times P^n &= H([(x_1\times P)\times P]\times P,[(x_2\times P)\times P]\times P) \\ &= H(-(P\cdot P)(x_1\times P),-(P\cdot P)(x_2\times P)) \\ &= (-1)^n(P\cdot P)^n H(x_1,x_2)\times P^n \\ &= (-1)^n(P\cdot P)^n(E\times P^n) \end{aligned}$$

y la demostración está completa.

(e) Claramente $E=T+T'$. Falta probar que $T\times P^n=0$. Como $T\times P^n = E\times P^n - (-1)^n\frac{[(E\times P^n)\times P^n]\times P^n}{(P\cdot P)^n}$, d) implica que $T\times P^n=0$. □

DEFINICIÓN (3.8). *a) Sean $S\in V_F(n)$ y $P\in V_F(1)$. Decimos que P es punto singular de S si:*

$$S_1\cdot P^{n-1} = S_2\cdot P^{n-1} = S_3\cdot P^{n-1} = 0.$$

b) $L\in V_F(1)$ es tangente a $S\in V_F(n)$ si se presenta una de las dos situaciones siguientes:

(i) *L no contiene ningún punto singular de S y $S\times L^n$ tiene un factor múltiple.*

o

(ii) *L contiene un punto singular de S.*

OBSERVACIÓN (3.9). *Si $S\in V_F(n)$ no tiene factores lineales y si L pasa por un punto singular P de S, entonces $P^2\,|\,S\times L^n$.*

Demostración. Sin pérdida de generalidad, podemos suponer que $L=x_1$ y $P=x_3$. Las condiciones:

$$S_1\cdot x_3^{n-1} = S_2\cdot x_3^{n-1} = S_3\cdot x_3^{n-1} = 0$$

implican que todos los términos distintos de cero de S son de la forma $ax_1^i x_2^j x_3^k$ con $k\le n-2$. Por otro lado, $ax_1^i x_2^j x_3^k\times x_1^n = 0$ si $i>0$ y $ax_2^{n-k}x_3^k\times x_1^n = (-1)^{n-k}ax_2^k x_3^{n-k}$. Por tanto, como $k\le n-2$, tenemos $x_3^2\,|\,S$. □

COROLARIO. *Si $S\in V_F(n)$ no tiene factores lineales y si $L\in V_F(1)$, entonces L es tangente a S si y sólo si $S\times L^n$ tiene un factor múltiple.*

Podemos aplicar los resultados anteriores para hallar condiciones de tangencia entre una recta L y una curva $S\in V_F(n)$.

TEOREMA (3.10). *Sean $L\in V_F(1)$ y $S\in V_F(n)$ una curva irreducible.*

(i) *Si $n=2$, entonces L es tangente a S si y sólo si $S\times S\cdot L^2=0$.*

(ii) *Si $n=3$, entonces L es tangente a S si y sólo si $S^2\times S^2\cdot L^6=0$.*

(iii) *Si $n=4$, entonces L es tangente a S si y sólo si*

$$\left[11(S^3\times S^3)-5(S^2\times S^2)(S\times S)\right]\cdot L^{12}=0.$$

Demostración. Por el Teorema (3.7), incisos (c) y (e), podemos escribir S de la forma $S = T + T'$, en donde $L \mid T$ y T' se factoriza como:

$$T' = \prod_{i=1}^{n} (P + \lambda_i Q)$$

en donde $P, Q \in V_F(1)$ y $P \cdot L = Q \cdot L = 0$. Para cada $k \in \mathbb{N}$, tenemos

$$S^k \times S^k \cdot L^{nk} = S^k \cdot (S^k \times L^{nk})$$
$$= S^k \cdot \left(\sum_{j=0}^{k} \binom{k}{j} T^{k-j} T'^j \right) \times L^{nk}$$
$$= S^k \cdot (T'^k \times L^{nk}) = (-1)^{nk} \left(\sum_{j=0}^{k} \binom{k}{j} T^{k-j} T'^j \right) \times L^{nk} \cdot T'^k$$
$$= (-1)^{nk} (T'^k \times L^{nk} \cdot T'^k) = T'^k \times T'^k \cdot L^{nk}.$$

Si $n = 2$ y $k = 1$, tenemos:

$$S \times S \cdot L^2 = (T' \times T') \cdot L^2 = \left[-\frac{1}{2} (\lambda_1 - \lambda_2)^2 (P \times Q)^2 \right] \cdot L^2.$$

Como $P \times Q$ y L son linealmente dependientes, existe una constante $\lambda \in F$ tal que

$$S \times S \cdot L^2 = \lambda (\lambda_1 - \lambda_2)^2$$

L es tangente a S si y sólo si $\lambda_1 = \lambda_2$. Por tanto, L es tangente a S si y sólo si $S \times S \cdot L^2 = 0$.

Si $n = 3$ y $k = 2$, tenemos

$$S^2 \times S^2 \cdot L^6 = T'^2 \times T'^2 \cdot L^6 =$$
$$(P + \lambda_1 Q)^2 (P + \lambda_2 Q)^2 (P + \lambda_3 Q)^2 \times (P + \lambda_1 Q)^2 (P + \lambda_2 Q)^2 (P + \lambda_3 Q)^2 \cdot L^6.$$

Aplicando (3.5), (d), tenemos:

$$S^2 \times S^2 \cdot L^6 = \lambda (\lambda_1 - \lambda_2)^2 (\lambda_2 - \lambda_3)^2 (\lambda_3 - \lambda_1)^2,$$

en donde $\lambda \in F$, $\lambda \neq 0$. Por tanto, L es tangente a S si y sólo si $S^2 \times S^2 \cdot L^6 = 0$.

Si $n = 4$, tenemos

$$\left[11 (S^3 \times S^3) - 5 (S^2 \times S^2)(S \times S) \right] \cdot L^{12}$$
$$= \left[11 (T'^3 \times T'^3) - 5 (T'^2 \times T'^2)(T' \times T') \right] \cdot L^{12}$$
$$= \varphi(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$$

en donde $\varphi(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ es un polinomio simétrico en $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Sin embargo, si igualamos dos de estas variables, por ejemplo, si $\lambda_3 = \lambda_4$,

$$\varphi(\lambda_1, \lambda_2, \lambda_3, \lambda_3)$$
$$= \{ 11 \left[(P + \lambda_1 Q)^3 (P + \lambda_2 Q)^3 (P + \lambda_3 Q)^6 \times (P + \lambda_1 Q)^3 (P + \lambda_2 Q)^3 (P + \lambda_3 Q)^6 \right]$$
$$- 5 \left[(P + \lambda_1 Q)^2 (P + \lambda_2 Q)^2 (P + \lambda_3 Q)^4 \times (P + \lambda_1 Q)^2 (P + \lambda_2 q)^2 (P + \lambda_3 Q)^4 \right]$$
$$\cdot \left[(P + \lambda_1 Q)(P + \lambda_2 Q)(P + \lambda_3 Q)^2 \times (P + \lambda_1 Q)(P + \lambda_2 Q)(P + \lambda_3 Q)^2 \right] \} \cdot L^{12}.$$

Usando la fórmula en (3.5) d), tenemos:

$$(P + \lambda_1 Q)^3 (P + \lambda_2 Q)^3 (P + \lambda_3 Q)^6 \times (P + \lambda_1 Q)^3 (P + \lambda_2 Q)^3 (P + \lambda_3 Q)^6$$

$$= (-1)^{3+3-6}\frac{3!3!6!}{12!}\binom{3}{3+3-6}\binom{3}{3-3}\binom{6}{3}$$
$$= \frac{1}{84\cdot 11}(P\times Q)^{12},$$

$$(P+\lambda_1 Q)^2(P+\lambda_2 Q)^2(P+\lambda_3 Q)^4 \times (P+\lambda_1 Q)^2(P+\lambda_2 Q)^2(P+\lambda_3 Q)^4$$
$$= (-1)^{2+2-4}\frac{2!2!4!}{8!}\binom{2}{2+2-4}\binom{2}{2-2}\binom{4}{2}(P\times Q)^8$$
$$= \frac{1}{70}(P\times Q)^8$$

y

$$(P+\lambda_1 Q)(P+\lambda_2 Q)(P+\lambda_3 Q)^2 \times (P+\lambda_1 Q)(P+\lambda_2 Q)\times(P+\lambda_3 Q)^2$$
$$= (-1)^{1+1-2}\frac{1!1!2!}{4}\binom{2}{1}(P\times Q)^4$$
$$= \frac{1}{6}(P\times Q)^4.$$

Por tanto,

$$\varphi(\lambda_1,\lambda_2,\lambda_3,\lambda_3) = \left[11\left[\frac{1}{84\cdot 11}\right] - 5\left[\frac{1}{70\cdot 6}\right]\right][(P\times Q)\cdot(P\times Q)]^{12} = 0.$$

Esto implica que $\lambda_3-\lambda_4$ debe ser factor de $\varphi(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. Por tanto, existe un polinomio homogéneo de grado 11, digamos $\psi(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, tal que

$$\varphi(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (\lambda_3-\lambda_4)\psi(\lambda_1,\lambda_2,\lambda_3,\lambda_4).$$

Como φ es simétrico, tenemos

$$\varphi(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \varphi(\lambda_1,\lambda_2,\lambda_4,\lambda_3) = (\lambda_4-\lambda_3)\psi(\lambda_1,\lambda_2,\lambda_4,\lambda_3).$$

Por tanto, $\psi(\lambda_1,\lambda_2,\lambda_4,\lambda_3) = -\psi(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. Esto implica que $\psi(\lambda_1,\lambda_2,\lambda_3,\lambda_3) = 0$. Por tanto, $\psi(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ puede escribirse en la forma

$$\psi(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = (\lambda_3-\lambda_4)\nu(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$$

en donde $\eta(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ es un polinomio homogéneo de grado 10 en $\lambda_1,\lambda_2,\lambda_3,\lambda_4$. Concluimos entonces que $(\lambda_3-\lambda_4)^2 \mid \varphi(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$. En forma similar se prueba que $(\lambda_i-\lambda_j)^2 \mid \varphi(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$, en donde $i,j\in\{1,2,3,4\}\ \ i\neq j$. Por tanto, existe una constante $a\neq 0$ tal que

$$\varphi(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = a(\lambda_1-\lambda_2)^2(\lambda_1-\lambda_3)^2(\lambda_1-\lambda_4)^2(\lambda_2-\lambda_3)^2(\lambda_2-\lambda_4)^2(\lambda_3-\lambda_4)^2.$$

La demostración se concluye entonces como en los casos anteriores. □

Para concluir este trabajo, podemos afirmar que $V_F(n)$ no es otra cosa que un álgebra no asociativa con un producto interior regular. Para el lector interesado en familiarizarse con este tipo de álgebras, recomendamos el libro [1].

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ON SOME NICE CLASS OF NON-GALOIS COVERINGS FOR ALGEBRAS

PIOTR DOWBOR AND ADAM HAJDUK

ABSTRACT. Almost Galois G-covering functors $F : R \to R'$ of type L are studied in context of preserving representation type, nice properties of the pair $(F_\lambda, F_\bullet)$ of functors associated to F and connections to geometric degenerations of algebras. In particular, we prove that in certain situations R' is representation-finite (resp. tame) if R is so (Theorem 2.4.1), and that $(F_\lambda, F_\bullet)$ have some nice properties, similar to these for Galois coverings (Theorems 3.1.1 and 4.3.1).

Dedicated to Professor Jose Antonio de la Peña

Introduction

One of the most important and efficient tools of modern representation theory of finite dimensional algebras over a field is covering technique. It was invented thirty years ago by Bongartz, Gabriel and Riedtmann for studying and classifying representation-finite algebras ([17, 25, 4, 16], see also [21, 29]).

An especially important role have been always played by Galois coverings. They are usually used to reduce a problem for modules over an algebra Λ to an analogous one, often much simpler, for its cover category $\tilde{\Lambda}$. In particular, they yield a method of determining representation type and constructing indecomposable modules for an algebra Λ by applying of the so-called "push-down" functor $F_\lambda : \operatorname{mod} \tilde{\Lambda} \to \operatorname{mod} \Lambda$, the left adjoint to the "pull-back" functor $F_\bullet : \operatorname{mod} \Lambda \to \operatorname{Mod} \tilde{\Lambda}$ associated to the Galois covering $F : \tilde{\Lambda} \to \Lambda$ (see for example [16, Theorem 3.6], [11, Theorem], [13, Theorem 3.6]). This kind of treatment allows to answer many interesting theoretical questions and obtain classifications for various classes of algebras (respectively, matrix problems) in representation-finite or tame case.

General (non-Galois) covering functors have been applied in a little bit different way. In contrast to Galois ones, they appeared first "on a module category level". The key result for theory of representation-finite algebras, [4], 3.1; says that there exists always a covering functor $F : k(\tilde{\Gamma}_\Lambda) \to (\operatorname{ind} \Lambda)_0$ with nice properties, where $k(\tilde{\Gamma}_\Lambda)$ is the mesh category of universal covering $\tilde{\Gamma}_\Lambda$ of the translation Auslander-Reiten quiver Γ_Λ of the algebra Λ and $(\operatorname{ind} \Lambda)_0$ a full subcategory of mod Λ formed by a fixed complete selection of nonisomorphic finite dimensional indecomposable Λ-modules. This fact allows to treat all algebras, standard and nonstandard ones, in a coherent way and to prove that each (nonstandard) representation-finite algebra Λ degenerates in the sense of algebraic geometry to its standard form $\bar{\Lambda} = \tilde{\Lambda}/G$,

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where $G = \Pi(\Gamma_\Lambda)$ is a (free) fundamental group of the translation quiver Γ_Λ (see [4], 5.2). Moreover, the restriction $F' = F_{|\tilde{\Lambda}} : \tilde{\Lambda} \to \Lambda$ of F to $\tilde{\Lambda}$ is clearly a covering functor, which is stable on G-orbits in ob $\tilde{\Lambda}$, and the "push-down" functor $F'_\lambda : \text{mod}\, \tilde{\Lambda} \to \text{mod}\, \Lambda$ associated to F' is again a covering functor between the respective module categories, in particular, it allows to reconstruct all indecomposable Λ-modules from those over $\tilde{\Lambda}$.

In this paper we consider the situation in some sense similar to that described above, but for algebras which are not necessarily representation-finite. Our main aim is to study properties of certain class of (non-Galois) covering functors $F : R \to R'$ between locally bounded k-categories, associated with a free action of a fixed group G of k-linear automorphisms of a cover R (R' is arbitrary), in a context of preserving representation type, nice properties of the pair $(F_\lambda, F_\bullet)$ of functors associated to F and connections to geometric degenerations of algebras. In particular, we try to discuss similarity and difference of their behaviour in comparison to Galois coverings situation, and also eventual analogy to the representation-finite case. A direct inspiration for this studies is an idea that non-Galois coverings, especially the considered class, are supposed to be a proper tool for better understanding of the structure and the way of constructing for nonstandard algebras of infinite-representation type.

In the paper we introduce the notion of an almost Galois G-covering functor of type L, where L is a totally ordered factor group of G. We discuss its practical meaning in situation of bounded quiver categories (Theorem 2.3.3). It occurs that the proposed approach explains many interesting and classical examples of nonstandard algebras (i.e. these from [26]). We prove that for any almost Galois G-covering $F : R \to R'$ of integral type, the representation type of R' is not more complicated as that of R, eventually R/G (Theorem 2.4.1). We also show (Theorem 3.1.1, Proposition 3.8.3) that for some almost Galois coverings F, under certain Hom-Ext conditions depending on N, the functor F_λ preserves indecomposability and Auslander-Reiten sequences, and we have $(*) : F_\bullet F_\lambda N \cong \bigoplus_{g \in G} {}^g N$, for N in mod R. (The presence of R-isomorphism $(*)$ has some special meaning, if we study dimension of module varieties for certain degeneration of algebras, cf. [8]). It occurs that for certain important family examples associated with nonstandard algebras, the pairs $(F_\lambda, F_\bullet)$ of functors behave even in a more regular way; in particular, the isomorphism $(*)$ depends not so strongly on N as in a general situation (Theorems 4.1.1 and 4.3.1).

The paper is organized as follows. In Section 1 we recall basic definitions concerning covering functors and introduce the notion of G-covering, discussing some of its aspects. In Section 2 we give the definition of almost Galois G-covering of type L and prove two results, Theorems 2.3.3 and 2.4.1, described briefly above. Section 3 is devoted to the analysis of properties of the pair $(F_\lambda, F_\bullet)$ of functors for almost Galois coverings F. There we also prove Theorem 3.1.1 and Proposition 3.8.3. In Section 4 we study certain interesting family of examples of almost Galois coverings between bounded quiver categories; in particular, we prove Theorem 4.1.1, and also formulate Theorem 4.3.1.

In this article we use mainly the well known notations and definitions [3, 4, 13, 16, 18]. Nevertheless, for a benefit of the reader, we recall the general situation and the most important notions we deal with. For basic information concerning representation theory of algebras (respectively, rings, category theory, algebraic geometry) we refer to [3] (respectively, [2], [23], [27]).

Throughout the paper we denote by $\mathbb{Z}$ (resp. $\mathbb{N}$) the set of all integers (resp. nonnegative integers). For a positive $n \in \mathbb{N}$ we usually use the abbreviated notation $[n] = \{1,\dots,n\}$.

1. Covering functors

(1.1) Let k be a field (not necessarily algebraically closed). Following [4] a k-category R (each set $R(x,y)$ of morphisms from x to y in R, $x,y \in \operatorname{ob} R$, is a k-linear space and composition of morphisms in R is k-bilinear) is called *locally bounded k-category*, if all objects of R have local endomorphism rings, the different objects are nonisomorphic, and the sums $\sum_{y\in R} \dim_k R(x,y)$ and $\sum_{y\in R} \dim_k R(y,x)$ are finite for each $x \in R$. The Jacobson radical of R is always denoted by $J(R)$. By $\underline{\dim} R$ we mean the family $(\dim_{\mathrm{k}} J(x,y))_{x,y\in \operatorname{ob} R}$, where $J = J(R)$. Note that for locally bounded categories R and S, any full, faithful and dense functor $F : R \to S$ is always invertible. Consequently, R and S are equivalent if and only if they are isomorphic (we write then $R \simeq S$).

For a fixed locally bounded k-category R we denote by $\mathscr{F}(R)$ the category whose objects are k-functors $F_1 : R \to R_1$, where R_1 is a k-category, and the morphism sets $\mathscr{F}(F_1,F_2)$ consists of k-functors $E : R_1 \to R_2$ such that $EF_1 = F_2$. By an R-module we mean a contravariant k-linear functor from R to the category of all k-vector spaces. An R-module M is *locally finite dimensional* (resp. *finite dimensional*) if $\dim_k M(x)$ is finite for each $x \in R$ (resp. the *dimension* $\dim_k M = \sum_{x\in R} \dim_k M(x)$ of M is finite). We denote by $\operatorname{MOD} R$ the category of all R-modules, by $\operatorname{Mod} R$ (resp. $\operatorname{mod} R$) the full subcategory of $\operatorname{MOD} R$ formed by all locally finite dimensional (resp. finite dimensional) R-modules and by $\mathscr{J}_R$ the Jacobson radical of the category $\operatorname{MOD} R$. By the *support* of any object M in $\operatorname{MOD} R$ we shall mean the full subcategory $\operatorname{supp} M$ of R formed by the set $\{x \in R : M(x) \neq 0\}$.

For any finite k-algebra A we denote analogously by $\operatorname{MOD} A$ (respectively, $\operatorname{mod} A$) the category of all (respectively, all finite-dimensional) right A-modules and by $J(A)$ the Jacobson radical of A.

To any finite locally bounded k-category R we can attach the finite-dimensional algebra $A(R) = \bigoplus_{x,y\in \operatorname{ob} R} R(x,y)$ endowed with the multiplication given by the composition in R. It is well known that the mapping $M \mapsto \bigoplus_{x\in \operatorname{ob} R} M(x)$ yields an equivalence

$$\operatorname{mod} C \simeq \operatorname{mod} A(R).$$

Conversely, if A is finite dimensional algebra, $\operatorname{MOD} A$ can be interpreted as MOD $R(A)$, where $R(A)$ is a full subcategory of $\operatorname{mod} A$ formed by fixed set of representatives of isomorphism classes of indecomposable projective A-modules.

A specially important role in representation theory of algebras is played by algebras and categories of quivers with relations.

Let $Q = (Q_0,Q_1)$ be a quiver, where Q_0 is the set vertices and Q_1 the set of arrows, together with the function $s,t : Q_1 \to Q_0$ attaching to each arrow its source and sink, respectively. Recall that by walk of length $n \geq 0$ in Q starting at x and ending at y, for $x,y \in Q_0$, we mean a sequence $w = \delta_1 \dots \delta_n$ consisting of arrows and their formal inverses such that $s(\delta_i) = x_{i-1}$ and $t(\delta_i) = x_i$, if $\delta_i \in Q_1$, respectively, $s(\delta_i^{-1}) = x_i$ and $t(\delta_i^{-1}) = x_{i-1}$, if $\delta_i^{-1} \in Q_1$, for some $x_0,x_1,\dots,x_n \in Q_0$, $x_0 = x$,

$x_n = y$. A trivial walk from x to x is always denoted by ε_x. We say that a walk w is an (oriented) path, if $\delta_i \in Q_1$ for all i. For a path w as above we fix the abbreviate notation: $|w| = n$, $Q_0(w) = \{x_0, \ldots, x_n\}$ and $Q_1(w) = \{\delta_1, \ldots, \delta_n\}$. The sets of all walks (resp. all paths, arrows) from x to y we denote by $\mathscr{W}(x,y) = \mathscr{W}_Q(x,y)$ (resp. $\mathscr{P}(x,y) = \mathscr{P}_Q(x,y)$, $Q_1(x,y)$). For walks $w \in \delta_1 \ldots \delta_n \in \mathscr{W}(x,y)$ and $w' = \delta'_1 \ldots \delta'_m \in \mathscr{W}(y,z)$, the walk $ww' = \delta_1 \ldots \delta_n \delta'_1 \ldots \delta'_m \in \mathscr{W}(x,y)$ is called a composition of w and w'.

The path algebra of a finite quiver Q is the k-algebra $A(Q) = (\bigoplus_{w \in \mathscr{P}} kw, \cdot)$ with unit $1 = \sum_{x \in Q_0} \varepsilon_x$, where $\mathscr{P} = \bigcup_{x,y \in Q_0} \mathscr{P}(x,y)$ and $\cdot$ is induced by composing of paths. If I is an admissible ideal in $A(Q)$ then we set $A(Q,I) = A(Q)/I$. The algebra $A(Q,I)$ is finite dimensional an is called the algebra of the bounded quiver (Q,I). It is well known that we have

$$\operatorname{mod} A(Q) \simeq \operatorname{rep}_k(Q,I)$$

where $\operatorname{rep}_k(Q,I)$ denotes the category of all finite dimensional representation of the quiver Q over k, satisfying the relations from I (see [3]).

Similarly, given a quiver Q we construct the path k-category $R(Q)$ of Q. We set $\operatorname{ob} R(Q) = Q_0$ and $R(Q)(x,y) = \bigoplus_{w \in \mathscr{P}(y,x)} kw$, for $x, y \in Q_0$. The composition $\circ$ of morphisms in $R(Q)$ is again defined as the k-bilinear map given on basis by composition of paths in Q. If I is an admissible ideal in the category $R(Q)$ (in the sense of [4]) then the factor category $R(Q,I) = R(Q)/I$ is locally bounded and it is called *a category of the bounded quiver* (Q,I). It easy to check that analogously as above we have

$$\operatorname{mod} R(Q) \simeq \operatorname{rep}_k(Q,I)$$

It is easily seen that if Q is finite then $A(R(Q,I)) \cong A(Q,I')$ and $R(A(Q,I')) \cong R(Q,I)$, where $I' = \bigoplus_{x,y \in Q_0} I(x,y)$.

We will often not distinguish between a category and an algebra of a finite bounded quiver (Q,I).

(1.2) Let R and R' be a pair of locally bounded k-categories. Recall [4] that a functor $F : R \to R'$ is called a covering functor if F is dense and for any pair of objects $x \in \operatorname{ob} R$ and $a \in \operatorname{ob} R'$, F induces k-isomorphisms

$$(*) \qquad {}_a f^x = (F_{x,y})_y : \bigoplus_{y \in F^{-1}(a)} R(x,y) \to R'(F(x),a)$$

and

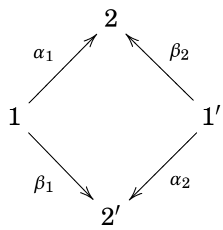
$$(**) \qquad {}^x f_a = (F_{y,x})_y : \bigoplus_{y \in F^{-1}(a)} R(y,x) \to R'(a,F(x))$$

where F_{z_1,z_2} is given by $\alpha \mapsto F(\alpha)$, for $z_1, z_2 \in \operatorname{ob} R$. Note that if R is connected then we can omit the assumption on density of F. If F is dense and all homomorphisms ${}_a f^x$ (resp. ${}^x f_a$) are isomorphisms then we say that F is a left (resp. right) covering functor.

It is easy to see that there exist left covering functors which are not the right ones, and conversely.

EXAMPLE (1.2.1). *Let $R = R(Q)$ and $R' = R(Q')$, where*

Q :



and

$$Q' : \qquad 1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} 2$$

Then the functor $F : R \to R'$ given by $F(\alpha_1) = \alpha + \beta = F(\beta_1)$, $F(\alpha_2) = \alpha$ and $F(\beta_2) = \beta$ is a left covering functor but it is not a right one.

Given a covering functor $F : R \to R_1$ one can study interrelations between the module categories $\mathrm{MOD}R$ and $\mathrm{MOD}R'$ applying the pair of functors

$$\mathrm{MOD}R \underset{F_\bullet}{\overset{F_\lambda}{\rightleftarrows}} \mathrm{MOD}R'$$

where $F_\bullet : \mathrm{MOD}R' \to \mathrm{MOD}R$ is the "pull-up" functor associated with functor F, assigning to each X in $\mathrm{MOD}R'$ the R-module $X \circ F$, and the "push-down" functor $F_\lambda : \mathrm{MOD}R \to \mathrm{MOD}R'$ is the left adjoint to $F_\bullet$. The R-module $F_\lambda(N)$, for N in $\mathrm{MOD}R$, is defined as follows:

$$F_\lambda(N)(a) = \bigoplus_{x \in F^{-1}(a)} N(x)$$

for $a \in \mathrm{ob}R'$, and

$$F_\lambda(N)(\alpha) = [N({}_x\alpha_y)] : \bigoplus_{x \in F^{-1}(a)} N(x) \to \bigoplus_{y \in F^{-1}(b)} N(y)$$

for $\alpha \in R'(b,a)$, where $({}_x\alpha_y) = ({}^x f_b)^{-1}(\alpha) \in \bigoplus_{y \in F^{-1}(b)} R(y,x)$, for any $x \in F^{-1}(a)$. We have also at our disposal the right adjoint

$$F_\rho : \mathrm{MOD}R \to \mathrm{MOD}R'$$

to the functor $F_\bullet$, where the R_1-module $F_\rho(N)$ is given by the formulas

$$F_\rho(N)(b) = \prod_{y \in F^{-1}(b)} N(y)$$

for $b \in \mathrm{ob}R'$, and

$$F_\rho(N)(\alpha) = [N({}_x\alpha_y)] : \prod_{x \in F^{-1}(a)} N(x) \to \prod_{y \in F^{-1}(b)} N(y)$$

for $\alpha \in R'(b,a)$, where $({}_x\alpha_y) = ({}_a f^y)^{-1}(\alpha) \in \bigoplus_{x \in F^{-1}(a)} R(y,x)$, for any $y \in F^{-1}(b)$.

It is easily seen that for any $x \in \mathrm{ob}R$, the family $({}^x f_a)_{a \in \mathrm{ob}R'}$ yields the isomorphism $\sigma_x : F_\lambda(R(-,x)) \to R'(-,Fx)$ of R-modules.

LEMMA (1.2.2). *For any $x \in \mathrm{ob}R$, σ_x induces R-isomorphisms $F_\lambda(\mathcal{J}_R(-,x)) \cong \mathcal{J}_{R'}(-,Fx)$ and $F_\lambda(R(-,x)/\mathcal{J}_R(-,x)) \cong R'(-,Fx)/\mathcal{J}_{R'}(-,Fx)$. In consequence, $F_\lambda(w_x)$ is a right minimal almost split map, where $w_x : \mathcal{J}_R(-,x) \hookrightarrow R(-,x)$ is a canonical embedding.*

Proof. Fix $x \in \mathrm{ob}R$. We set $E = R(x,x)$, $E' = R'(a,a)$, $I = (\bigoplus_{x \neq y \in F^{-1}(a)} R(y,x)) \oplus J(E)$ and $I' = \sigma_x(a)(I)$, where $a = F(x)$. To prove the assertions it suffices to show that $I' = J(E')$ and that the canonical embedding of factor fields $E/J(E) \to E'/J(E')$, induced by the local algebra embedding $F_{x,x} : E \to E'$, is an isomorphism. Note that I' is a right ideal in E' since for any $\tilde{\alpha} \in R(y,x)$ such that $\alpha = F(\tilde{\alpha})$ and $\beta = \sum_{z \in F^{-1}(a)} F(\tilde{\beta}_z) \in E'$, $(\tilde{\beta}_z) \in \bigoplus_{y \in F^{-1}(a)} R(y,x)$, we have $\alpha\beta = \sum_{z \in F^{-1}(a)} F(\tilde{\alpha}\tilde{\beta}_z)$, but $\tilde{\alpha}$ is form $\mathscr{J}_R(y,x)$, so $\tilde{\alpha}\tilde{\beta}_x$ belongs to $\mathscr{J}_R(x,x) = J(E)$. Consequently, we have $I' \subseteq J(E')$, so $I' = J(E')$ and $E/J(E) \cong E'/J(E')$, since $\dim_{\mathrm{k}} E'/I' = \dim_{\mathrm{k}} E/J(E) \leq \dim_{\mathrm{k}} E'/J(E') \leq \dim_{\mathrm{k}} E'/I'$. In this way the proof is complete. □

COROLLARY (1.2.3). *(a) For any $x \in \mathrm{ob}R$, $F_\rho(D_R(R(x,-))) \cong D_{R'}(R'(Fx,-))$.*
(b) For any $x \in \mathrm{ob}R$, $F_\lambda(p_x)$ is a minimal left almost spilt, where $p_x : D(R(-,x)) \to D(R(-,x))/\mathrm{Soc}(D(R(-,x)))$ is a canonical projection.

Proof. (a) We have a canonical isomorphism $F_\rho \cong D_{R'} \circ F_\lambda^{\mathrm{op}} \circ D_R$ of functors from $\mathrm{mod}\, R$ to $\mathrm{mod}\, R'$, where $F^{\mathrm{op}} : R^{\mathrm{op}} \to R'^{\mathrm{op}}$ is the functor between opposite categories induced by F. Consequently, $F_\rho(D_R(R(x,-))) \cong D_{R'}(R'(Fx,-))$ for any $x \in \mathrm{ob}R$, since the family $({}_a f^x)_{a \in \mathrm{ob}R'}$ yields an R-isomorphism $\sigma'_x : F_\lambda^{\mathrm{op}}(R(x,-)) \to R'(Fx,-)$.

The proof of (b) follows immediately form (a) and the lemma. □

(1.3) The most important role in representation theory is played by the so-called Galois coverings.

Let G be a group acting by k-linear automorphisms on a locally bounded k-category R. We assume that G acts freely on the objects of R (i.e. that is the stabilizer G_x is trivial for every $x \in \mathrm{ob}R$) so it can be regarded as a subgroup of $\mathrm{Aut}_{k-\mathrm{cat}}(R)$. Note that then G acts also on the category $\mathrm{MOD}R$ by translations ${}^g(-)$, which assign to each M in $\mathrm{MOD}R$ the R-module ${}^gM = M \circ g^{-1}$ and to each $f : M \to N$ in $\mathrm{MOD}R$ the R-homomorphism ${}^gf : {}^gM \to {}^gN$ given by the family of k-linear maps $\left(f(g^{-1}(x))\right)_{x \in R}$. Given M in $\mathrm{MOD}R$, the subgroup

$$G_M = \{g \in G : {}^gM \simeq M\}$$

of G is called the *stabilizer* of M.

We denote by $\mathscr{F}^G$ be the full subcategory of $\mathscr{F} = \mathscr{F}(R)$ formed by all functors $F : R \to R'$ such that $Fg = F$, for all $g \in G$. The subcategory category $\mathscr{F}^G$ is closed under isomorphisms and it admits zero objects $F_0 : R \to R_0$. (F_0 is an object in $\mathscr{F}^G$ determined uniquely up to isomorphism by the property, that for any $F : R \to R'$ in $\mathscr{F}^G$, there exists exactly one k-functor $E : R_0 \to R'$ such that $EF_0 = F$). The distinguished one among them is the canonical projection functor $\bar{F} : R \to \bar{R} = R/G$, where R/G is an orbit category of the action of G on R. Recall [16] that $\mathrm{ob}R/G = (\mathrm{ob}R/G)$, and the morphism space $(R/G)(G\bar{x}, G\bar{y})$, for $\bar{x}, \bar{y} \in \mathrm{ob}R$, are formed by sums of G-orbits of morphisms in R, which can be presented as fix points of the induced G-action on the product, by the following formula:

$$(R/G)(G\bar{x}, G\bar{y}) = \left(\prod_{x \in G\bar{x}} \prod_{y \in G\bar{y}} R(x,y) \right)^G .$$

It is easily seen that the orbit category R/G constructed above is again a locally bounded k-category. Moreover, the canonical projection functor $\bar{F}$ given by $\bar{F}(x) = Gx$, for $x \in \mathrm{ob}R$, and $\bar{F}(\alpha) = G\alpha$, for $\alpha \in R(x,y)$, belongs to $\mathscr{F}^G$ and it is a covering functor. Note that if we fix the set $(\mathrm{ob}R)_0$ of representative of G-orbits in $\mathrm{ob}R$ then for any $\bar{x},\bar{y}\in(\mathrm{ob}R)_0$ we have a k-space decomposition

$$(R/G)(G\bar{x},G\bar{y})=\left(\prod_{x\in G\bar{x}}\prod_{y\in G\bar{y}}R(x,y)\right)^G\cong\bigoplus_{g\in G}\bar{R}(\bar{x},\bar{y})_g$$

where

$$\bar{R}(\bar{x},\bar{y})_g=\left(\prod_{(g_1,g_2):g_2^{-1}g_1=g}R(g_1\bar{x},g_2\bar{y})\right)^G$$

for $g\in G$. One can show that the decomposition above define G-grading on the category R/G. Observe that a different choice of the set $(\mathrm{ob}R)_0$ leads to different grading of summand spaces appearing in the decompositions above, but the summands themselves remain the same. More precisely, $\bar{R}(g_{\bar{x}}\bar{x},g_{\bar{y}}\bar{y})_g=\bar{R}(\bar{x},\bar{y})_{g_{\bar{y}}^{-1}gg_{\bar{x}}}$ for any $g\in G$, where $\{g_{\bar{x}}\}_{\bar{x}\in(\mathrm{ob}R)_0}\subseteq G$ is arbitrarily fixed. Finally, we have the standard isomorphisms $R(g_1\bar{x},g_2\bar{y})\cong\bar{R}(\bar{x},\bar{y})_{g_2^{-1}g_1}$ induced by $\bar{F}$, and they commute with the action of G. From now on we will use the isomorphism above as a standard identification(!) without any extra comment.

Denote by $\mathscr{F}_0^G$ the isomorphism class of $\bar{F}$ in $\mathscr{F}$. Observe that $\mathscr{F}_0^G$ consists precisely of all covering functors $F:R\to R'$ belonging to $\mathscr{F}^G$ and having the extra property that G acts transitively on $F^{-1}(a)$, for every $a\in\mathrm{ob}R'$ (in fact, its suffices only to claim that F is a left or right covering functor). The elements of $\mathscr{F}_0^G$ are called Galois covering functors with group G.

REMARK (1.3.1). *If F is a Galois covering functor then restrictions of the functors F_λ and F_ρ to mod R are canonically isomorphic.*

(1.4) The most significant example of Galois covering functors is related to the combinatorial construction of universal covering $(\tilde{Q},\tilde{I})$ of quiver with relations (Q,I). We briefly recall this construction, following [24], and next we give a description, we use further in the paper.

Let (Q,I) be a bounded quiver, where $Q=(Q_0,Q_1)$ is a quiver and I an admissible ideal in the path k-category $R(Q)$ of Q. Fix $a_\bullet\in Q_0$. We define first the "topological" fundamental group $\Pi_1(Q)$ and universal covering $\tilde{\tilde{Q}}$ of Q (at $a_\bullet$) setting $\Pi_1(Q)=\mathscr{W}_Q(a_\bullet,a_\bullet)/\sim_H$ and $\tilde{\tilde{Q}}_0=(\bigcup_{a\in Q_0}\mathscr{W}_Q(a_\bullet,a))/\sim_H$, where $\sim_H$ is the classical homotopy in graph version.

The construction of $(\tilde{Q},\tilde{I})$ is based on the notion of a minimal relation. An element $\rho=\sum_{i=1}^n t_i\omega_i\in I(b,a)$, where $2\leq n$ and $t_i\in k$, is called a minimal (linear) relation, if $\sum_{i\in I}t_i\omega_i\notin I(a,b)$, for any nonempty $I\subsetneq\{1,\ldots,n\}$. Let $N=N(Q,I)$ be a subgroup in $\Pi_1(Q)$ generated by all elements $c_{u,\omega,\omega'}=[C(u,\omega,\omega')]_{\sim_H}$, where $C(u,\omega,\omega')=u\omega\omega'^{-1}u^{-1}$ for $u\in\mathscr{W}_Q(a_\bullet,a)$ and $\omega,\omega'\in\mathscr{P}_Q(a,b)$ such that $\omega=\omega_1$ and $\omega'=\omega_2$ for some minimal linear relation $\rho\in I(b,a)$ as above. It is easily seen that $c_{u,\omega,\omega'}^{-1}=c_{u,\omega',\omega}$ and that N is a normal subgroup of $\Pi_1(Q)$. We set $\Pi_1(Q,I)=\Pi_1(Q)/N$ and $\tilde{Q}=\tilde{\tilde{Q}}/N$. Clearly, the mapping $N[v_a]_{\sim_H}\mapsto a$, $v_a\in\mathscr{W}_Q(a_\bullet,a)$, yields a quiver morphism $p:\tilde{Q}\to Q$. The map p is a Galois covering of quivers with group $\Pi_1(Q,I)$. In fact $p:(\tilde{Q},\tilde{I})\to(Q,I)$ is a morphism of bounded quivers, where $\tilde{I}$ by definition is the ideal generated by liftings of all minimal and zero relations in I

to $\tilde{Q}$. Moreover, the functor $\bar{F}: R(\tilde{Q},\tilde{I}) \to R(Q,I)$ induced by p is a Galois covering functor with group $\Pi_1(Q,I)$.

Following the idea from [4] one can define $\Pi_1(Q,I) = \Pi_1((Q,I),a_\bullet)$ and $\tilde{Q} = \tilde{Q}(a_\bullet)$ using one step procedure.

Let $\mathscr{W} = \bigcup_{a,b\in Q_0} \mathscr{W}(a,b)$ and $\mathscr{P} = \bigcup_{a,b\in Q_0} \mathscr{P}(a,b)$. Consider the equivalence relation $\sim \subseteq \mathscr{W} \times \mathscr{W}$ generated by the relations of the following two types:

(i) $u\delta\delta^{-1}v \sim uv$

(ii) $u\omega v \sim u\omega' v$

where $u,v \in \mathscr{W}$, $\delta \in Q_1$ or $\delta^{-1} \in Q_1$, and $\omega,\omega' \in \mathscr{P}$ are as above. The equivalence $\sim$ depends only on the set of all minimal linear relations in I and it is a congruence w.r.t composition of walks. Then the set

$$\dot{\Pi}_1(Q,I) = \mathscr{W}(a_\bullet,a_\bullet)/\sim$$

carries the structure of a group. Similarly we define a quiver $\dot{\tilde{Q}}$. We set

$$\dot{\tilde{Q}}_0 = (\bigcup_{a\in Q_0} \mathscr{W}_Q(a_\bullet,a))/\sim$$

and

$$\dot{\tilde{Q}}_1([v_a],[v_b]) = \{([v_a],\alpha) : \alpha \in Q_1(a,b), v_a\alpha \sim v_b\}$$

(resp. $\dot{\tilde{Q}}_1([v_a],[v_b]) = \{(\alpha,[v_b]) : \alpha \in Q_1(a,b), v_a\alpha \sim v_b\}$), for any $[v_a] \in \mathscr{W}_Q(a_\bullet,a)/\sim$, $[v_b] \in \mathscr{W}_Q(a_\bullet,b)/\sim$, where $[v] = [v]_\sim$ for any $v \in \mathscr{W}$. Analogously, $\mathscr{P}_{\dot{\tilde{Q}}}([v_a],[v_b]) = \{([v_a],\omega) : \omega \in \mathscr{P}_Q(a,b), v_a\omega \sim v_b\} = \{(\omega,[v_b]) : \omega \in \mathscr{P}_Q(a,b), v_a\omega \sim v_b\}$. The pairs $([v_a],\omega)$ and $(\omega,[v_b])$ represent the same path $\tilde{\omega}$, which dependent on presentation is called a lifting of ω to $\dot{\tilde{Q}}$ starting at $[v_a]$, respectively, ending at $[v_b]$.

It is clear $\dot{\Pi}_1(Q,I)$ operates by quiver automorphisms on $\dot{\tilde{Q}}$, defined by composition of the appropriate walks. One proves, applying definition of equivalence generated by some relations and properties of the group $N = N(Q,I)$, that the natural mapping $[v] \mapsto N[v]_{\sim_H}$, $v \in \mathscr{W}$, yields the expected isomorphims $\dot{\Pi}_1(Q,I) \cong \Pi_1(Q,I)$ and $\dot{\tilde{Q}} \cong \tilde{Q}$, which are compatible with the respective group actions.

We finish this section with some practical remark, which is useful, if one wants to compute universal covering $(\tilde{Q},\tilde{I})$ for a concrete example. The admissible ideal I in $R(Q)$ is usually given in the form $I = \langle \rho_l : l \in \Lambda \rangle$, where all ρ_l are elements of the spaces $I(b,a)$, $a,b \in Q_0$, and Λ is a finite set, in case Q is finite. Without loss of generality one can always assume that each ρ_l is either zero or minimal linear relation. Then one proves that the equivalence relation $\sim$ is generated by all relations of the type (i) and only these relation of type (ii) for those ω and ω' appear in some minimal linear relation ρ_l, $l \in \Lambda$.

(1.5) Now we extend the notion of Galois covering. Let G be as above and $F : R \to R'$ be a covering functor. We say that F is a *covering functor with group* G (shortly, G-*covering*), if $F(gx) = F(x)$ for all $g \in G$, $x \in \mathrm{ob}R$, and the action of G on $F^{-1}(a)$ is transitive for every $a \in \mathrm{ob}R'$. Analogously, we define the notion of right G-covering (resp. left G-covering) functor. Note that in this situation as above we can identify $\mathrm{ob}R'$ with $(\mathrm{ob}R)_0$ and $\mathrm{ob}R$ with $G \times (\mathrm{ob}R)_0$, where $(\mathrm{ob}R)_0$ is as above. Moreover, the conditions $(*)$ and $(**)$ from the definition of covering functor can be rephrased as follows:

for any pair $\bar{x}, \bar{y} \in (\text{ob} R)_0$ and $g_1 \in G$, the k-homomorphisms

$$(*)_{(\bar{x},\bar{y},g_1)} \qquad {}_{\bar{y}}f_{\bar{x}}^{g_1} = ({}_{\bar{y}}^{g_2}f_{\bar{x}}^{g_1})_{g_2} : \bigoplus_{g_2 \in G} R(g_1\bar{x}, g_2\bar{y}) \to R'(\bar{x},\bar{y})$$

and

$$(**)_{(\bar{x},\bar{y},g_1)} \qquad {}_{\bar{y}}^{g_1}f_{\bar{x}} = ({}_{\bar{y}}^{g_1}f_{\bar{x}}^{g_2})_{g_2} : \bigoplus_{g_2 \in G} R(g_2\bar{x}, g_1\bar{y}) \to R'(\bar{x},\bar{y})$$

are isomorphisms, where ${}_{\bar{y}}^{g'}f_{\bar{x}}^{g''} = F_{g''\bar{x},g'\bar{y}}$ for any $g', g'' \in G$.

Denote by $\mathscr{F}'$ (resp. $\mathscr{F}'_l$ and $\mathscr{F}'_r$) the full subcategory of $\mathscr{F}$ formed by all G-coverings (resp. left and right G-coverings). It is clear that $\mathscr{F}'$, $\mathscr{F}'_l$, $\mathscr{F}'_r$ are closed under isomorphisms, $\mathscr{F}' = \mathscr{F}'_l \cap \mathscr{F}'_r$, and $\mathscr{F}_0^G = \mathscr{F}^G \cap \mathscr{F}' = \mathscr{F}^G \cap \mathscr{F}'_l = \mathscr{F}^G \cap \mathscr{F}'_r$. (Note that the full subcategory of $\mathscr{F}$ formed by all F such that $F(gx) = F(x)$ for all $g \in G$, $x \in \text{ob} R$, does not admit zero elements).

There exist two simple invariants describing isomorphism classes in $\mathscr{F}'_l$ and $\mathscr{F}'_r$, respectively, we use in the paper. With a functor $F : R \to R'$ in $\mathscr{F}'_l$ we associate a family $\varphi(F) = (\varphi_{g_1})_{g_1 \in G}$ of k-linear maps of the following form:

$$\varphi_{g_1} = \bigoplus_{\bar{x},\bar{y} \in (\text{ob}R)_0} \varphi_{g_1}(\bar{x},\bar{y}) : \bigoplus_{\bar{x},\bar{y} \in (\text{ob}R)_0} \bigoplus_{g \in G} \bar{R}(\bar{x},\bar{y})_g \longrightarrow \bigoplus_{\bar{x},\bar{y} \in (\text{ob}R)_0} \bigoplus_{g' \in G} \bar{R}(\bar{x},\bar{y})_{g'}$$

where

$$\varphi_{g_1}(\bar{x},\bar{y}) = [\varphi_{g_1}(\bar{x},\bar{y})^{(g',g)}] : \bigoplus_{g \in G} \bar{R}(\bar{x},\bar{y})_g = \bigoplus_{g_2 \in G} R(g_1\bar{x}, g_2\bar{y}) \xrightarrow{{}_{\bar{y}}f_{\bar{x}}^{g_1}} R'(\bar{x},\bar{y}) \xrightarrow{({}_{\bar{y}}f_{\bar{x}}^{e})^{-1}} \bigoplus_{g'_2 \in G} R(\bar{x}, g'_2\bar{y}) = \bigoplus_{g' \in G} \bar{R}(\bar{x},\bar{y})_{g'}$$

$(g = g_2^{-1}g_1,\ g' = (g'_2)^{-1})$. Analogously, with $F : R \to R'$ in $\mathscr{F}'_r$ we associate the family $\varphi'(F) = (\varphi'_{g_2})_{g_2 \in G}$,

$$\varphi'_{g_2} = \bigoplus_{\bar{x},\bar{y} \in (\text{ob}R)_0} \varphi'_{g_2}(\bar{x},\bar{y}) : \bigoplus_{\bar{x},\bar{y} \in (\text{ob}R)_0} \bigoplus_{g \in G} \bar{R}(\bar{x},\bar{y})_g \longrightarrow \bigoplus_{\bar{x},\bar{y} \in (\text{ob}R)_0} \bigoplus_{g' \in G} \bar{R}(\bar{x},\bar{y})_{g'}$$

where

$$\varphi'_{g_2}(\bar{x},\bar{y}) = [\varphi'_{g_2}(\bar{x},\bar{y})^{(g',g)}] : \bigoplus_{g \in G} \bar{R}(\bar{x},\bar{y})_g = \bigoplus_{g_1 \in G} R(g_1\bar{x}, g_2\bar{y}) \xrightarrow{{}_{\bar{y}}^{g_2}f_{\bar{x}}} R'(\bar{x},\bar{y}) \xrightarrow{({}_{\bar{y}}^{e}f_{\bar{x}})^{-1}} \bigoplus_{g'_1 \in G} R(g'_1\bar{x}, \bar{y}) = \bigoplus_{g' \in G} \bar{R}(\bar{x},\bar{y})_{g'}$$

$(g = g_2^{-1}g_1,\ g' = g'_1)$. Clearly all maps φ_{g_1}, $g_1 \in G$ (resp. $(\varphi_{g_2}$, $g_2 \in G)$ are k-isomorphisms and φ_e (resp. φ'_e) is an identity map.

LEMMA (1.5.1). *Let $F_1 : R \to R_1$, $F_2 : R \to R_2$ and $F : R \to R'$ be left (resp. right) G-coverings.*

(a) $F_1 \cong F_2$ in $\mathscr{F}'_l$ (resp. in $\mathscr{F}'_r$) if and only if $\varphi(F_1) = \varphi(F_2)$ (resp. $\varphi'(F_1) = \varphi'(F_2)$).

(b) F is a Galois covering if and only if all components φ_{g_1}, $g_1 \in G$, (resp. φ'_{g_2}, $g_2 \in G$) of $\varphi(F)$ (resp. $\varphi'(F)$) are equal to identity maps.

Proof. We prove our assertions only for left coverings, the proof of the dual versions are analogous.

(a) Assume that $F_1 \cong F_2$. Then there exists an isomorphism $E : R_1 \to R_2$ of k-categories such that $F_2 = EF_1$. Denote by ${}_{\bar{y}}e_{\bar{x}}$, $\bar{x}, \bar{y} \in (\text{ob}R)_0$, the k-space isomorphisms induced by E, by ${}_{\bar{y}}\dot{f}_{\bar{x}}^{g_1}$ and ${}_{\bar{y}}\ddot{f}_{\bar{x}}^{g_1}$, $g_1 \in G$, the isomorphims induced by F_1

and F_2 (see $(*)_{(\bar{x},\bar{y},g_1)}$), and by $\dot{\varphi}_{g_1}$ and $\ddot{\varphi}_{g_1}$, $g_1 \in G$, the components of $\varphi(F_1)$ and $\varphi(F_2)$, respectively. Then we have

$$_{\bar{y}}\ddot{f}_{\bar{x}}^{g_1} = {}_{\bar{y}}\epsilon_{\bar{x}}\, {}_{\bar{y}}\dot{f}_{\bar{x}}^{g_1}$$

Consequently,

$$\ddot{\varphi}_{g_1} = ({}_{\bar{y}}\ddot{f}_{\bar{x}}^{g_1})^{-1}\, {}_{\bar{y}}\ddot{f}_{\bar{x}}^{e} = ({}_{\bar{y}}\epsilon_{\bar{x}}\, {}_{\bar{y}}\dot{f}_{\bar{x}}^{e})^{-1}\, {}_{\bar{y}}\epsilon_{\bar{x}}\, {}_{\bar{y}}\dot{f}_{\bar{x}}^{g_1} = ({}_{\bar{y}}\dot{f}_{\bar{x}}^{e})^{-1}\, {}_{\bar{y}}\dot{f}_{\bar{x}}^{g_1} = \dot{\varphi}_{g_1}$$

for all $\bar{x}, \bar{y} \in (\mathrm{ob}R)_0$ and $g_1 \in G$, hence $\varphi(F_1) = \varphi(F_2)$.

To prove the opposite implication we start by some general observation. Let $F : R \to R'$ be an arbitrary left G-covering functor. Then the associated k-space isomorphisms ${}_{\bar{y}}f_{\bar{x}}^{e} : \bar{R}(\bar{x},\bar{y}) = \bigoplus_{g\in G} \bar{R}(\bar{x},\bar{y})_g \to R'(\bar{x},\bar{y})$, $\bar{x},\bar{y} \in (\mathrm{ob}R)_0$, given by F, induces by transportation the structure $\bar{R}_F$ of locally bounded k-category on the family $\bar{R}(\bar{x},\bar{y})$, $\bar{x},\bar{y} \in (\mathrm{ob}R)_0$, of k-spaces, such that $\mathrm{ob}\bar{R}_F = \mathrm{ob}\bar{R} = (\mathrm{ob}R)_0$, $\bar{R}_F(\bar{x},\bar{y}) = \bar{R}(\bar{x},\bar{y})$ for all $\bar{x},\bar{y} \in (\mathrm{ob}R)_0$, and the family $(({}_{\bar{y}}f_{\bar{x}}^{e})^{-1})_{\bar{x},\bar{y}}$ yields a functor from $E = E_F : R' \to \bar{R}_F$ being an isomorphism of k-categories. Consequently, $\check{F} = E \circ F : R \to \bar{R}_F$ is a G-covering functor and $\check{F} \cong F$ in $\mathscr{F}'$. Note that for any $g_1 \in G$ we have ${}_{\bar{y}}\check{f}_{\bar{x}}^{g_1} = \varphi_{g_1}$, where ${}_{\bar{y}}\check{f}_{\bar{x}}^{g_1} : \bigoplus_{g_2\in G} R(g_1\bar{x}, g_2\bar{y}) \to \bar{R}_F(\bar{x},\bar{y})$, $\bar{x},\bar{y} \in (\mathrm{ob}R)_0$, are k-homomorphisms defined by the functor $\check{F}$. In particular, the map $\check{F}_{g_1\bar{x},g_2\bar{y}} : R(g_1\bar{x},g_2\bar{y}) \to \bar{R}_F(\bar{x},\bar{y})$ coincides with gth component $(\varphi_{g_1}(\bar{x},\bar{y})^{(g',g)})_{g'}$ of $\varphi_{g_1}(\bar{x},\bar{y})$, where $g = g_2^{-1}g_1$, for any $g_1, g_2 \in G$ and $\bar{x},\bar{y} \in (\mathrm{ob}R)_0$. Consequently, the composition $\bullet$ of elements $\alpha \in \bar{R}(\bar{x},\bar{y})_g = R(\bar{x}, g^{-1}\bar{y})$ and $\beta \in \bar{R}(\bar{y},\bar{z})_h$ in $\bar{R}_F$ is given by the formula

$$(***) \qquad \beta \bullet \alpha = \sum_{g_2\in G} \tilde{\beta}_{g_2} \cdot \alpha$$

where $(\tilde{\beta}_{g_2})_{g_2} = (\varphi_{g^{-1}}(\bar{y},\bar{z}))^{-1}(\beta) \in \bigoplus_{g_2\in G} R(g^{-1}\bar{y}, g_2\bar{z}) = \bigoplus_{g_2\in G} \bar{R}(\bar{y},\bar{z})_{(gg_2)^{-1}}$ and $\cdot$ denotes the composition in $\bar{R}$. Note that we have

$$\beta \bullet \alpha = \Big(\sum_{g_2\in G} \check{F}(\tilde{\beta}_{g_2})\Big) \bullet \check{F}(\alpha) = \sum_{g_2\in G} \check{F}(\tilde{\beta}_{g_2}\alpha) = \sum_{g_2\in G} \tilde{\beta}_{g_2} \cdot \alpha$$

since by the definition $\sum_{g_2\in G} \check{F}(\tilde{\beta}_{g_2}) = \beta$, and $\varphi_e(\bar{x},\bar{y})$ (resp. $\varphi_e(\bar{y},\bar{z})$) is an identity map, so $\check{F}(\alpha) = \alpha$ (resp. $\check{F}(\tilde{\beta}_{g_2}\alpha) = \tilde{\beta}_{g_2} \cdot \alpha$, for every $g_2 \in G$).

Assume now that $\varphi(F_1) = \varphi(F_2)$. Then by the considerations above $\bar{R}_{F_1} = \bar{R}_{F_2}$ and $\check{F}_1 = \check{F}_2$. Consequently, $F_1 \cong F_2$, since $F_1 \cong \check{F}_1$ and $F_2 \cong \check{F}_2$ in $\mathscr{F}'$. In this way the proof is complete.

To prove (b) observe that φ_{g_1} is an identity map, for $g_1 \in G$, if and only if $F(g_1^{-1}\alpha) = F(\alpha)$ for all $\alpha \in R(g_1\bar{x}, g_2\bar{y})$, $g_2 \in G$. Consequently, the property: φ_{g_1} is an identity map for every $g_1 \in G$, means exactly, that F belongs to $\mathscr{F}^G$. Now the assertion follows immediately from the equality $\mathscr{F}_0^G = \mathscr{F}^G \cap \mathscr{F}_l'$. □

REMARK (1.5.2). *(a) If F is a Galois covering functor then $\check{F} = \bar{F}$.*

(b) The formula $(***)$ seems to be useful in the eventual searching of theoretical description of all up to isomorphism G-covering functors with (resp. algebras admitting a G-covering functor by) a fixed cover category R.

2. Almost Galois coverings and representation type

(2.1) Let H be a group. We say that H is *L-totally ordered* by the relation $\preceq$, where $L = (L, \leq)$ is a totally ordered (so torsionfree) group, if there exists a surjective group homomorphism $\pi : H \to L$ such that $h_1 \prec h_2$ if and only if $\pi(h_1) < \pi(h_2)$ (resp. $h_1 \preceq h_2$ if and only if $h_1 \prec h_2$ or $h_1 = h_2$), for any $h_1, h_2 \in H$.

In the group L we always use the additive notation; in particular, the neutral element is denoted by 0. In case $L = (\mathbb{Z}, \leq)$, we mean by $\leq$ the standard ordering relation in $\mathbb{Z}$.

Note that each group H is always $\{0\}$-totally ordered, and that if H is a free (resp. an abelian free) group then H is $\mathbb{Z}$-totally ordered in a canonical way (free generators are mapped to 1).

LEMMA (2.1.1). *$(H \preceq)$ is an ordered group.*

Proof. An easy check on definition. □

An ordering $\preceq$ as above we called later an L-total order in H.

Let $\psi = [\psi_{h',h}] : \bigoplus_{h \in H} W_h \to \bigoplus_{h' \in H} W_{h'}$ be a k-linear endomorphism of the space $W = \bigoplus_{h \in H} W_h$ such that $W_h = 0$ for almost all $h \in H$. We say that ψ is *lower $\preceq$-triangular* (resp. *$\prec$-triangular*), if $\psi_{h',h} = 0$ for all $h, h' \in H$ such that $h' \prec h$ (resp. $\pi(h') \leq \pi(h)$). We say that ψ is lower $\preceq$*-unitriangular* if ψ is lower $\preceq$- triangular, $\psi_{h',h} = \mathrm{id}_{V_h}$ for all $h \in H$, and $\psi_{h',h} = 0$ for all $h, h' \in H$ such that or h, h' are incomparable. Note that then ψ is clearly an isomorphism and ψ^{-1} is also lower $\preceq$-unitriangular. Moreover, the composition $\psi'\psi$ is again lower $\preceq$-unitriangular if so is ψ'. Finally, for any $h \in H$ and $l \in L$ the subspaces $W_h \oplus (\bigoplus_{h' : h \prec h'} W_{h'})$ and $\bigoplus_{h' : l \leq \pi(h')} W_{h'}$ of W are ψ-invariant. Later on we will use the abbreviated names *$\preceq$-triangular*, *$\prec$-triangular* and *$\preceq$-unitriangular*, respectively.

We also fix some handy notation. For any $h \in H$ and $i \in L$ we set $H^{\prec h} = \{h' \in H : h' \prec h\}$ and $H^{<i} = \{h' \in H : \pi(h') < i\}$. In similar way we define the sets $H^{\preceq h}$, $H^{\leq i}$; $H^{\succ h}$, $H^{>i}$; and $H^{\succeq h}$, $H^{\geq i}$. Note that $H^{\prec h} = H^{<\pi(h)}$, but $H^{\preceq h} \subsetneq H^{\leq \pi(h)}$, if $\mathrm{Ker}\, \pi \neq \{e\}$.

(2.2) Now we introduce the most important notion of this paper.

DEFINITION (2.2.1). *Let $G \subseteq \mathrm{Aut}(R)$ and $(\mathrm{ob}R)_0$ be as above. A left G-covering functor $F : R \to R'$ is called* an almost Galois covering (of type L) with group G*, if G admits an L-total order $\preceq$ such that all automorphisms*

$$\varphi_{g_1}(\bar{x}, \bar{y}) : \bigoplus_{g \in G} \bar{R}(\bar{x}, \bar{y})_g \to \bigoplus_{g' \in G} \bar{R}(\bar{x}, \bar{y})_{g'}$$

for $g_1 \in G$ and $\bar{x}, \bar{y} \in (\mathrm{ob}R)_0$, are $\preceq$-unitriangular. In case $L = \mathbb{Z}$ (resp. $L = \{0\}$), then F is called an almost Galois G-covering functor of integral (resp. trivial) type.

Observe that the definition above does not depend on the choice of the set $(\mathrm{ob}R)_0$ since the maps $g_{\bar{y}}^{-1}(-)g_{\bar{x}} : G \to G$ preserve the ordering $\preceq$, for any fixed collection $\{g_{\bar{x}}\}_{\bar{x} \in (\mathrm{ob}R)_0} \subseteq G$ (cf. 1.3).

PROPOSITION (2.2.2). *(a) An almost Galois covering functor $F : R \to R'$ with group G is a right covering functor, so a covering functor, such that all automorphisms*

$$\varphi'_{g_1}(\bar{x}, \bar{y}) : \bigoplus_{g \in G} \bar{R}(\bar{x}, \bar{y})_g \to \bigoplus_{g' \in G} \bar{R}(\bar{x}, \bar{y})_{g'}$$

for $g_1 \in G$ and $\bar{x}, \bar{y} \in (\mathrm{ob} R)_0$, are $\preceq$-unitriangular.

(b) The functor $F : R \to R'$ is an almost Galois covering of trivial type if and only if F is a Galois covering.

Proof. (b) By Definition 2.2.1, a left G-covering functor F is an almost Galois covering of trivial type if and only if φ_{g_1} is an identity map, for every $g_1 \in G$. Now the assertion (b) is an immediate consequence of Lemma 1.5.1(b).

To prove (a) it suffices to show that the assertion holds for the functor $\check{F}$, where $\check{F}$ is as in 1.5. Since $\check{F}_{g_1\bar{x}, g_2\bar{y}} = (\varphi_{g_1}(\bar{x},\bar{y})^{(g',g)})_{g'}$, where $g = g_2^{-1}g_1$, the k-linear map ${}^{g_2}_{\bar{y}}\check{f}_{\bar{x}} : \bigoplus_{g_1 \in G} R(g_1\bar{x}, g_2\bar{y}) \to \bar{R}(\bar{x},\bar{y}) = \bigoplus_{g' \in G} \bar{R}(\bar{x},\bar{y})_{g'}$, induced for any fixed g_2 and $\bar{x},\bar{y} \in (\mathrm{ob} R)_0$ by $\check{F}$, has the form ${}^{g_2}_{\bar{y}}\check{f}_{\bar{x}} = [\varphi_{g_2 g}(\bar{x},\bar{y})^{(g',g)}]$ under the standard identification $\bigoplus_{g_2 \in G} R(g_2\bar{x}, g_1\bar{y}) = \bigoplus_{g \in G} \bar{R}(\bar{x},\bar{y})_g$. Observe that

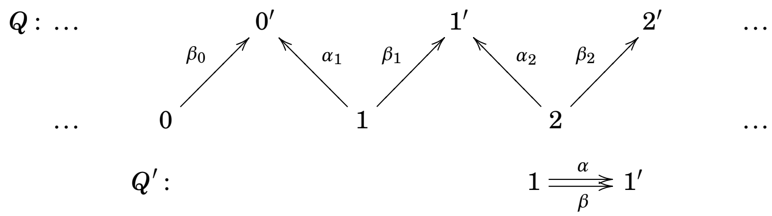
$$[\varphi_{g_2 g}(\bar{x},\bar{y})^{(g',g)}] : \bigoplus_{g \in G} \bar{R}(\bar{x},\bar{y})_g \to \bigoplus_{g' \in G} \bar{R}(\bar{x},\bar{y})_{g'}$$

is $\preceq$-unitriangular, since all maps $\varphi_{g_2 g}(\bar{x},\bar{y})$, $g \in G$, are $\preceq$-unitriangular, so for any $g \in G$ we have $\varphi_{g_2 g}(\bar{x},\bar{y})^{(g,g)} = \mathrm{id}_{\bar{R}(\bar{x},\bar{y})_g}$ and $\varphi_{g_2 g}(\bar{x},\bar{y})^{(g',g)} = 0$, if $g \prec g'$ or g' is incomparable to g. Consequently, the map ${}^{g_2}_{\bar{y}}\check{f}_{\bar{x}}$ is a k-isomorphism, and $({}^{e}_{\bar{y}}\check{f}_{\bar{x}})^{-1}\,{}^{g_2}_{\bar{y}}\check{f}_{\bar{x}}$ regarded as an endomorphism of $\bar{R}(\bar{x},\bar{y}) = \bigoplus_{g' \in G} \bar{R}(\bar{x},\bar{y})_{g'}$ is also $\preceq$-unitriagular. □

REMARK (2.2.3). *(a) The statement of Remark 1.5.2(b) remain valid, if we restrict our attention only to almost Galois G-covering functors of a fixed type.*

(b) If for a left G-covering functor $F : R \to R'$, where G admits an L-total order, not all the maps $\varphi_{g_1}(\bar{x},\bar{y})$ are $\preceq$-unitriangular, then it can happen that F is not a right covering functor (see example below).

EXAMPLE (2.2.4). *Let $R = R(Q)$ $R' = R(Q')$ and $G \subseteq \mathrm{Aut}_{k-cat}(R)$ be the infinite cyclic group generated by g, where*



and $g : R \to R$ is given by $g(i) = i+1$, $g(i') = (i+1)'$, for $i \in \mathbb{Z}$. Then the functor $F : R \to R'$ defined by $F(\alpha_2) = \alpha + \beta = F(\beta_2)$, and $F(\alpha_i) = \alpha$ (resp. $F(\beta_i) = \beta$), if $i \neq 2$, is a left G-covering functor such that $\varphi_2(0',0)$ is not $\preceq$-unitriangular, where $(\mathrm{ob}R)_0 = \{0, 0'\}$, $L = \mathbb{Z}$ and $p : G \to L$ is equal to $\mathrm{id}_{\mathbb{Z}}$ under the identification $G = \mathbb{Z}$, $g^n \mapsto n$. (Note that $\varphi_1(0',0)$ is $\preceq$-unitriangular and $\varphi_i(0',0)$ is an identity map for all $i \neq 1,2$). Clearly, F is not a right covering (cf. Example 1.2.1).

(2.3) Now we present a class of almost Galois coverings, containing many natural and important examples.

Let $Q = (Q_0, Q_1)$ be a finite connected quiver, I an admissible ideal in the path category $R(Q)$ and $(\tilde{Q}, \tilde{I})$ a universal covering of (Q, I), where $\tilde{Q} = \tilde{Q}(a_\bullet)$ for a fixed vertex $a_\bullet \in Q_0$ (see 1.4). Assume that the fundamental group $G = \Pi_1((Q,I), a_\bullet)$ is an L-totally ordered group. Then the ordering relation $<$ in L induces the relations $\prec' = \prec'_{(b,a)} \subseteq \mathscr{W}(b,a)\times\mathscr{W}(b,a)$, for $a,b\in Q_0$. For $\omega_1,\omega_2\in\mathscr{W}(b,a)$, we set $\omega_1 \prec' \omega_2$ if and only if $[\varepsilon_{a_\bullet}] \prec [w_a\omega_1^{-1}\omega_2 w_a^{-1}]$ in G, where $w_a\in\mathscr{W}(a_\bullet,a)$ is a fixed walk (respectively, $\omega_1 \preceq' \omega_2$ if and only if $\omega_1=\omega_2$ or $\omega_1\prec'\omega_2$). Note that the definition above does not depend on the choice of the walk w_a, and that in case $a=b$ the relation $\prec'$ is a lifting of the relation being the transportation of $\prec\subseteq G\times G$ to $\Pi_1((Q,I),a)$ via the isomorphism given by the mapping $[w]\mapsto[w_a^{-1}w\,w_a]$, for $[w]\in G$.

LEMMA (2.3.1). *(a) For any $\omega_1,\omega_2\in\mathscr{W}(b,a)$ the following condition are equivalent:*

- $\omega_1\prec'\omega_2$,
- $[\varepsilon_{a_\bullet}]\prec[w_b\,\omega_2\,\omega_1^{-1}w_b^{-1}]$,
- $[w_b\,\omega_1 w_a^{-1}]\prec[w_b\,\omega_2 w_a^{-1}]$,

Moreover, for any $\omega_1',\omega_2'\in\mathscr{W}(b,a)$ such that $\omega_1'\sim\omega_1$ and $\omega_2'\sim\omega_2$, $\omega_1'\prec'\omega_2'$ if and only if $\omega_1\prec'\omega_2$.

(b) $\prec'=\prec'_{(b,a)}$ is an ordering relation in $\mathscr{W}(b,a)$.

(c) For any $\omega_1,\omega_2\in\mathscr{W}(b,a)$ and $v_1,v_2\in\mathscr{W}(c,b)$, $\omega_1\prec'_{(b,a)}\omega_2$ and $v_1\prec'_{(c,b)}v_2$ implies $v_1\omega_1\prec'_{(c,a)}v_2\omega_2$ in $\mathscr{W}(c,a)$.

Proof. (a), (b). An easy check on definitions.

(c) Note that for any $\omega\in\mathscr{W}(b,a)$ and $v\in\mathscr{W}(c,a)$, we have $v\omega_1\prec'_{(c,a)}v\omega_2$ and $v_1\omega\prec'_{(c,a)}v_2\omega$ in $\mathscr{W}(a,c)$, since $[\varepsilon_{a_\bullet}]\prec[w_a{\omega_1}^{-1}\omega_2 w_a^{-1}]=[w_a\omega_1^{-1}v^{-1}v\omega_2 w_a^{-1}]$ and $[\varepsilon_{a_\bullet}]\prec[w_a'\omega^{-1}{v_1}^{-1}v_2\omega {w'}_a^{-1}]$, where $w_a'=w_b\omega$. Now the inequalities from the assertion follow immediately. □

For any $\omega\in\mathscr{W}(b,a)$ we set $\deg(\omega)=[w_b\omega w_a^{-1}]$.

COROLLARY (2.3.2). *The function*

$$\deg:\mathscr{W}\to G$$

has the following properties (we keep the notation from the lemma):

(a) $\deg(\omega_1)=\deg(\omega_2)$ *if and only if* $\omega_1\sim\omega_2$,

(b) $\omega_1\prec'\omega_2$ *if and only if* $\deg(\omega_1)\prec\deg(\omega_2)$,

(c) $\deg(v_1\omega_1)=\deg(v_1)\deg(\omega_1)$.

Proof. Follows immediately from the definition and the proof above. □

Note that due to the properties above, the function deg allows for better understanding the situation we deal with, in terms similar to grading.

THEOREM (2.3.3). *Let $(Q,I),a_\bullet,G$ be as above and I' be an admissible ideal in the path category $R(Q)$ such that $\underline{\dim}R(Q,I)=\underline{\dim}R(Q,I')$. Assume that $F: R(\tilde{Q})\to R(Q,I')$ is a k-functor satisfying the following conditions:*

(a) $F_{\mathrm{ob}}:\mathrm{ob}R(\tilde{Q})\to\mathrm{ob}R(Q,I')$ is given by $p_0:\tilde{Q}_0\to Q_0$,

(b) for any lifting $\tilde{\alpha}\in\tilde{Q}_1$ of $\alpha\in Q_1$

$$F(\tilde{\alpha})=(\alpha+\sum_{\alpha\prec'\omega}a_{\omega,\tilde{\alpha}}\,\omega)+I'$$

where ω are oriented paths in Q (not belonging to I') and $a_{\omega,\tilde{\alpha}}\in k$,

(c) $F(\tilde{I})=0$.

Then the functor $F' : R(\tilde{Q},\tilde{I}) \to R(Q,I')$ induced by F is an almost Galois G-covering functor of type L.

Proof. We start by fixing an abbreviate notation. We set $\bar{R} = R(Q,I)$, $R' = R(Q,I')$, $\tilde{R} = R(\tilde{Q},\tilde{I})$ and $\mathscr{P} = \mathscr{P}_Q$, $\tilde{\mathscr{P}} = \mathscr{P}_{\tilde{Q}}$. Next we prove that F' is a left G-covering functor.

Due to the assumption $\underline{\dim}\bar{R} = \underline{\dim}R'$, it suffices to show that for any pair $(b,x) \in Q_0 \times \tilde{Q}_0$, the map ${}_bf^x = (F_{x,y})_y : \bigoplus_{y\in F^{-1}(b)} R(\tilde{Q})(x,y) \to R'(F(x),b)$ is surjective. Note that this is the case, then so are all the maps ${}_b(f')^x = (F'_{x,y})_y : \bigoplus_{y\in F^{-1}(b)} \tilde{R}(x,y) \to R'(F(x),b)$. In consequence, they are k-isomorphisms, since so are the maps ${}_b\bar{f}^x = (\bar{F}_{x,y})_y : \bigoplus_{y\in F^{-1}(b)} \tilde{R}(x,y) \to \bar{R}(F(x),a)$, where $\bar{F} : \tilde{R} \to \bar{R}$ is a canonical Galois covering functor.

In fact we have to show that for any nontrivial path $\omega \in \mathscr{P}(b,a)$ such that $\omega \notin I'$, the coset $\omega + I'$ belongs to $\mathrm{Im}_b f^x$, for any x such that $F(x) = a$. Note that I' is admissible, so there exists $m \in \mathbb{N}$ such that all ω as above belong to $\mathscr{P}^{<m}$, where $\mathscr{P}^{<l} = \mathscr{P}^{<l}(b,a)$ consists of all paths in $\mathscr{P}(b,a)$ of length smaller l, for $l \in \mathbb{N}$.

We start by some general observation. Let $\omega \in \mathscr{P}(b,a)$ be a nontrivial path, $x \in \tilde{Q}_0$ a fixed vertex such that $F(x) = a$ and $\tilde{\omega} \in \tilde{\mathscr{P}}(y,x)$ a lifting of ω to $\tilde{Q}$, where $y \in F^{-1}(b)$. Then by the assumption (b) and Lemma 2.3.1(b) we have

$$(*) \qquad F(\tilde{\omega}) = \omega + \sum_{\omega' \in \mathscr{P}^{<m} : \omega \prec' \omega'} a_{\omega',\tilde{\omega}}\, \omega' + I'$$

for some $a_{\omega',\tilde{\omega}} \in k$, where $\prec' = \prec'_{(b,c)}$.

To prove our claim fix $a, b \in Q_0$ and x such that $F(x) = a$. We show by induction with respect to $\prec'$ that for any $\omega \in \mathscr{P}^{<m}$ the coset $\omega + I'$ belongs to $\mathrm{Im}_b f^x$.

Assume first that ω is a maximal element in $\mathscr{P}^{<m}$. From $(*)$ it follows that $F(\tilde{\omega}) = \omega + I'$, and the assertion is trivially satisfied. Next assume that $\omega \in \mathscr{P}^{<m}$ is not maximal and that the assertion holds for all $\omega' \in \mathscr{P}^{<m}$ such that $\omega \prec' \omega'$. Then, by the inductive assumption, $\omega = F(\tilde{\omega}) - \sum_{\omega' \in \mathscr{P}^{<m} : \omega \prec' \omega'} a_{\omega',\tilde{\omega}}\, \omega' + I'$ belongs to $\mathrm{Im}_b f^x$, since $F(\tilde{\omega}) \in \mathrm{Im}_b f^x$. Observe that in fact we have showed something more; namely, that coset of any $\omega \in \mathscr{P}$ has a presentation

$$(**) \qquad \omega + I' = F(\tilde{\omega}) + \sum_{\omega' \in \mathscr{P}^{<m} : \omega \prec' \omega'} a'_{\omega',\omega} F(\tilde{\omega'}) + I'$$

where $\tilde{\omega'} \in \tilde{\mathscr{P}}(y',x)$, for $y' \in F^{-1}(b)$, are liftings of ω' to $\tilde{Q}$, and $a'_{\omega',\omega} \in k$.

In this way our claim is proved and F' is a left G-covering functor. Similarly one can show that F' is a right G-covering functor.

Now we show that F' is an almost Galois G-covering functor. We start by some preparatory remarks.

Let $w_a \in \mathscr{W}(a_\bullet, a)$, $a \in Q_0$, be a collection of fixed walks as in definition of $\prec'$. Then as a set $(\mathrm{ob}\tilde{R})_0$ of representatives of G-orbits in $\mathrm{ob}\tilde{R} = \tilde{Q}_0$ we take just the set $\{[w_a] : a \in Q_0\}$. In particular, in the canonical presentation $x = g_x\bar{x}$ of an object $x = [v_a] \in \tilde{Q}_0$, for $v_a \in \mathscr{W}(a_\bullet, a)$, we have $\bar{x} = [w_a]$ and $g_x = [v_a w_a^{-1}]$. Moreover, the identification isomorphism $\tilde{R}(x,y) \cong \bar{R}(\bar{x},\bar{y})_{g_y^{-1}g_x}$ defining $\bar{R}(\bar{x},\bar{y})_{g_y^{-1}g_x}$, given by $\bar{F}$, in our situation has the form

$$\tilde{R}([v_a],[v_b]) \cong \tilde{R}([w_a], g_x^{-1}g_y[w_b]) \cong \bar{R}(a,b)_{[w_b v_b^{-1} v_a w_a^{-1}]}$$

where $y = [v_b]$, $v_b \in \mathscr{W}(a_\bullet, b)$. Recall that $\tilde{R}([v_a],[v_b]) = \sum_{\tilde{\omega} \in \tilde{\mathscr{P}}([v_b],[v_a])} k(\tilde{\omega} + \tilde{I})$, $\bar{R}(a,b) = \sum_{\omega \in \mathscr{P}(b,a)} k(\omega + I)$, and that $\bar{F}$ is given by the mapping $\tilde{\omega} + \tilde{I} \mapsto \omega + I$, for

$\tilde{\omega} = ([v_b],\omega) = (\omega,[v_a]) \in \tilde{\mathscr{P}}([v_b],[v_a])$, where $\omega \in \mathscr{P}(b,a)$ satisfies $v_b\omega \sim v_a$. Consequently,

$$\bar{R}(a,b)_{[w_b v_b^{-1} v_a w_a^{-1}]} = \sum_{\omega\in\mathscr{P}(b,a):\,\omega\sim v_b^{-1}v_a} k(\omega+I)$$

and

$$\bar{R}(a,b)_{[u]} = \sum_{\omega\in\mathscr{P}(b,a):\,\omega\sim w_b^{-1}uw_a} k(\omega+I)$$

for any $[u]\in G$, $u\in\mathscr{W}(a,a)$. Note that in particular, we have $\omega+I\in\bar{R}(a,b)_{[w_b\omega w_a^{-1}]}$, for any $\omega\in\mathscr{P}(b,a)$.

To prove our main assertion we have to show that all automorphisms

$$\varphi_{g_1}(a,b): \bigoplus_{g\in G}\bar{R}(a,b)_g \to \bigoplus_{g'\in G}\bar{R}(a,b)_{g'}$$

for $g_1\in G$ and $a,b\in Q_0$, are $\preceq$-unitriangular, where $(\varphi_{g_1})_{g_1\in G}=\varphi(F')$ (we replaced $[w_a]$ by a in the definition of φ_{g_1}, cf. 1.5).

Fix g_1, or equivalently $[v_a]$ such that $v_a w_a^{-1}=g_1$, where $v_a\in\mathscr{W}(a_\bullet,a)$. Then φ_{g_1} has the form

$$\oplus_{g\in G}\bar{R}(a,b)_g=\oplus_{[v_b]\in F^{-1}(b)}\tilde{R}([v_a],[v_b]) \to R'(a,b) \to \oplus_{[v_b']\in F^{-1}(b)}\tilde{R}([w_a],[v_b'])=\oplus_{g'\in G}\bar{R}(a,b)_{g'}$$

($g=[w_b v_b^{-1} v_a w_a^{-1}]$ and $g'=[w_b v_b'^{\,-1}]$). Fix g and $\omega+I\in\bar{R}(a,b)_g$, where $\omega\in\mathscr{P}(b,a)$ is such that $[\omega]=[w_b^{-1}]g[w_a]=[v_b^{-1}v_a]$, or equivalently, $[v_b]\in F^{-1}(b)$ and $\tilde{\omega}=([v_b],\omega)\in\tilde{\mathscr{P}}([v_b],[v_a])$. By previous considerations we have only to show that

$$\varphi_{g_1}(\omega+I)=(\omega+I)+\sum_{\omega'\in\mathscr{P}^{<m}_{[\omega]}} c_{\omega'}(\omega'+I)$$

for some $c_{\omega'}\in k$, where $\mathscr{P}^{<m}_{[\omega]}=\{\omega'\in\mathscr{P}^{<m} : [w_b\omega w_a^{-1}]\prec[w_b\omega' w_a^{-1}]\}$. F' is a covering functor, therefore by Lemma 2.3.1(a), the precise description of component isomorphisms forming φ_{g_1} and the definition of F', it is equivalent to fact that

$$(***)\qquad F(\tilde{\omega})+I'=\sum_{\varsigma'\in\mathscr{P}^{<m}:\,\omega\preceq'\varsigma'} c'_{\varsigma'}F(\tilde{\varsigma}')+I'$$

for some $c'_{\varsigma'}\in k$ such that $c'_\omega=1$, where $\tilde{\varsigma}'=([v_b'],\varsigma')\in\tilde{\mathscr{P}}([v_b'],[w_a])$, for $[v_b']\in F^{-1}(b)$.

To show the last claim we apply the formulas $(*)$ and $(**)$. We have

$$F(\tilde{\omega})+I'=\omega+\sum_{\varsigma\in\mathscr{P}^{<m}:\,\omega\prec'\varsigma} a_{\varsigma,\tilde{\omega}}\,\varsigma+I',$$

and

$$\varsigma+I'=F(\tilde{\varsigma})+\sum_{\varsigma'\in\mathscr{P}^{<m}:\,\varsigma\prec'\varsigma'} a'_{\varsigma',\varsigma}F(\tilde{\varsigma}')+I'$$

for any $\varsigma\in\mathscr{P}^{<m}$ such that $\omega\prec'\varsigma$ or $\omega=\varsigma$, where $a_{\varsigma,\tilde{\omega}}, a'_{\varsigma',\varsigma}\in k$ and $\tilde{\varsigma},\tilde{\varsigma}'\in\tilde{\mathscr{P}}([v_b'],[w_a])$ for $[v_b']\in F^{-1}(b)$. Now the required formula $(***)$ follows easily and the proof is complete. □

REMARK (2.3.4). *(a) For any $\omega\in\mathscr{P}(b,a)$, $\omega\in I'$ implies $\omega\in I$. Note that by the formula $(**')$ being a variant of $(**)$ for F', $\omega_1-\omega_2\in I'$ we have $F'(\tilde{\omega}+\tilde{I})\in {}_b(f')^{[w_a]}(\tilde{R}([w_a],[v_b]))$, where v_b is such that $[w_b v_b^{-1}]=[w_b\omega w_a^{-1}](=\deg\omega)$, and $\sum_{\omega'\in\mathscr{P}^{<m}:\,\omega\prec'\omega'} a'_{\omega',\omega}F'(\tilde{\omega}'+\tilde{I})\in {}_b(f')^{[w_a]}(\bigoplus_{[v_b']\in\tilde{Q}^{(\omega)}}\tilde{R}([w_a],[v_b']))$, where $\tilde{Q}^{(\omega)}=\{[v_b']:$*

$\deg\omega \prec [w_b v_b'^{-1}]\}$. *Consequently, if* $\omega \in I'$ *then* $F'(\tilde{\omega}+\tilde{I}) = I'$, *so* $\tilde{\omega} \in \tilde{I}$, *since* F' *is a faithful functor, and hence* $\omega \in I$.

(b) More generally, $\rho = \sum_{i=1}^{m} t_i\omega_i \in I(b,a)$, *if* $\rho = \sum_{i=1}^{n} t_i\omega_i \in I'(b,a)$, *where* $\deg\omega_1 = \cdots = \deg\omega_m \prec \deg\omega_{m+1} \preceq \cdots \preceq \deg\omega_n$, *for* $1 \le m \le n$, *and* $t_1,\dots,t_n \in k$ *(apply similar arguments).*

(c) For $\omega_1,\omega_2 \in \mathscr{P}(b,a)$, *if* $\omega_1-\omega_2 \in I'$ *and* $\omega_1 \notin I$ *then* $\omega_1,\omega_2 \notin I'$ *and* $\deg\omega_2 \preceq' \deg\omega_1$, *moreover,* $\deg\omega_1 = \deg\omega_2$ *if and only if* $\omega_1-\omega_2 \in I$ *and* $\omega_2 \notin I$.

The remark can be also explained by means of the natural filtration on the category R' defined as follows:

Given $a,b \in Q_0$, for any $g \in G$ we set

$$R'(a,b)_{(g)} = \sum_{\omega\in\mathscr{P}(b,a):\, g\preceq\deg\omega} k(\omega+I')$$

Note that from Corollary 2.3.2, for any $a,b,c \in Q_0$ and $g,h \in G$ we have

$$R'(b,c)_{(h)}R'(a,b)_{(g)} \subseteq R'(a,c)_{(hg)}.$$

Now we give an alternative description of the situation we deal with (we keep the notation from the proof).

PROPOSITION (2.3.5). *For fixed* $a,b \in Q_0$ *and* $[v_a] \in F^{-1}(a)$, *respectively* $[v_b] \in F^{-1}(b)$, *the functor* F' *induces a* k*-isomorphism*

$$(****) \qquad \bigoplus_{g\preceq h} \bar{R}(a,b)_h = \bigoplus_{[v_b]\in F^{-1}(b):\, g\preceq\deg[v_b^{-1}v_a]} \tilde{R}([v_a],[v_b]) \xrightarrow{\cong} R'(a,b)_{(g)}$$

respectively,

$$(****)' \qquad \bigoplus_{g\preceq h} \bar{R}(a,b)_h = \bigoplus_{[v_a]\in F^{-1}(a):\, g\preceq\deg[v_b^{-1}v_a]} \tilde{R}([v_a],[v_b]) \xrightarrow{\cong} R'(a,b)_{(g)}.$$

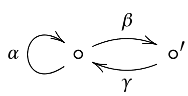
Moreover, for any $\omega \in \mathscr{P}(b,a)$, $(F(\tilde{\omega})-\omega)+I'$ *belongs to the image of* $\bigoplus_{\deg\omega\prec h} \bar{R}(a,b)_h$ *via* $(****)$, *(respectively,* $(****)'$*).*

Proof. Follows immediately from the previous considerations. □

Finally we present two examples of coverings, which fit to the scheme described above.

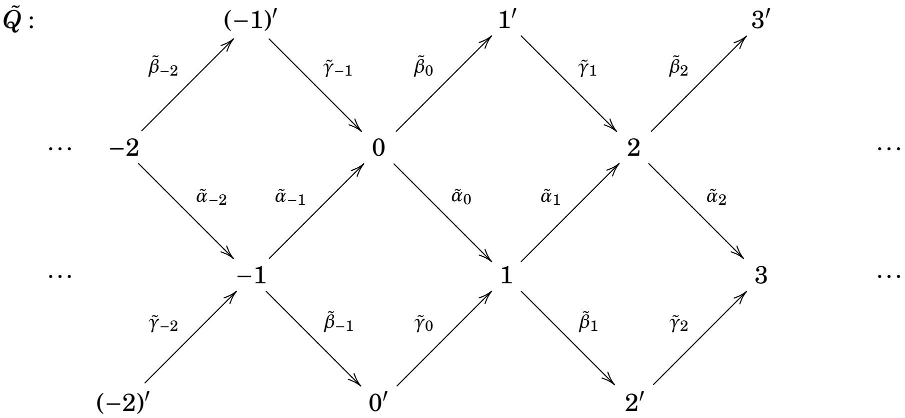
EXAMPLE (2.3.6). *Let* (Q,I) *and* (Q,I') *be bounded quivers, where*

Q :



$I = \langle\alpha^2-\beta\gamma, \alpha^2\beta, \gamma\alpha^2, \gamma\beta\rangle$ *and* $I' = \langle\alpha^2-\beta\gamma, \alpha^2\beta, \gamma\alpha^2, \gamma\alpha\beta-\gamma\beta\rangle$. *It is easily seen that* $\underline{\dim}R = \underline{\dim}R'$, *where* $R = R(Q,I)$ *and* $R' = R(Q,I')$, *and that* $J(R)^4 = 0 = J(R')^4$. *More precisely, the basis of the corresponding morphism spaces in* R *and* R' *are given by (pairwise different) cosets of paths from the same sets* $\{\varepsilon_\circ, \alpha, \alpha^2, \alpha^3\}$, $\{\varepsilon_{\circ'}, \gamma\alpha\beta\}$, $\{\beta, \alpha\beta\}$ *and* $\{\gamma, \gamma\alpha\}$, *respectively. Recall, that* R *and* R' *are isomorphic if and only if* $\operatorname{char}k \neq 2$, *and that* (R',R) *forms the first historical Riedmann's example of the pair of (nonisomorphic) representation finite algebras having the same Auslander-Reiten quiver. In particular, if* $\operatorname{char}k = 2$, *the algebra* R' *is a non-standard one, whereas* R *is its standard form (see* [25, 26], *also* [4]*).*

Set $a_\bullet = \circ$. Then $G = \Pi_1(Q,I)$ is an infinite cyclic group generated by $[\alpha]$, $(\tilde{Q},\tilde{I})$ is given by the quiver



and the ideal $\tilde{I} = \langle\{\tilde{\alpha}_n\tilde{\alpha}_{n+1} - \tilde{\tilde{\beta}}_n\tilde{\gamma}_{n+1}, \tilde{\alpha}_n\tilde{\alpha}_{n+1}\tilde{\beta}_{n+2}, \tilde{\gamma}_{n-1}\tilde{\alpha}_n\tilde{\alpha}_{n+1}, \tilde{\gamma}_n\tilde{\beta}_{n+1} : n \in \mathbb{Z}\}\rangle$, where $w_\circ = \varepsilon_\circ$, $w_{\circ'} = \beta$, $n = [\alpha^n]$, $n' = [\alpha^{n-1}\beta]$, $\tilde{\alpha}_n = (n,\alpha) = (\alpha,(n+1)')$, $\tilde{\beta}_n = (n,\beta) = (\beta,(n+1)')$ and $\tilde{\gamma}_n = (n',\gamma) = (\gamma, n+1)$, for $n \in \mathbb{Z}$. Clearly, G is $\mathbb{Z}$-totally ordered, where $\pi : G \to \mathbb{Z}$ is given by $\pi([\alpha^n]) = n$, for $n \in \mathbb{Z}$. In particular, we obtain a family of order relations $\prec'_{(a_1,a_2)}$ in $\mathscr{W}(a_1,a_2)$, for $a_1,a_2 \in Q_0 = \{\circ,\circ'\}$.

For any $(b,c) \in k^{\mathbb{Z}} \times k^{\mathbb{Z}}$, $b = (b_n)$, $c = (c_n)$, we define a functor

$$F_{b,c} : R(\tilde{Q}) \to R(Q,I')$$

setting $F_{b,c}(\tilde{\alpha}_n) = \alpha + I'$, $F_{b,c}(\tilde{\beta}_n) = \beta + b_n\alpha\beta + I'$ and $F_{b,c}(\tilde{\gamma}_n) = \gamma + c_n\gamma\alpha + I'$, for $n \in \mathbb{Z}$. Note that

$$\beta \prec' \alpha\beta \quad \text{and} \quad \gamma \prec' \gamma\alpha$$

since $[w_1\beta w_2^{-1}] = [\beta\beta^{-1}] = [\varepsilon_1]$ and $[w_1\alpha\beta w_2^{-1}] = [\alpha\beta\beta^{-1}] = [\alpha]$, respectively, $[w_2\gamma w_1^{-1}] = [\beta\gamma] = [\alpha]$ and $[w_2\gamma\alpha w_1^{-1}] = [\beta\gamma\alpha] = [\alpha^2]$. Moreover, $F_{b,c}(\tilde{I}) = 0$ if and only if

$$F_{b,c}(\tilde{\beta}_n\tilde{\gamma}_{n+1}) - F_{b,c}(\tilde{\alpha}_n\tilde{\alpha}_{n+1}) = 0 \qquad \text{and} \qquad F_{b,c}(\tilde{\gamma}_n\tilde{\beta}_{n+1}) = 0,$$

for all $n \in \mathbb{Z}$, since $F_{b,c}(\tilde{\alpha}_n\tilde{\alpha}_{n+1}\tilde{\beta}_{n+2}) = 0$ and $F_{b,c}(\tilde{\gamma}_{n-1}\tilde{\alpha}_n\tilde{\alpha}_{n+1}) = 0$. Observe that the first equality is equivalent to the equality

$$(\mathrm{i})_n \qquad b_n + c_{n+1} = 0$$

since

$$(b_n\alpha\beta + I')(c_{n+1}\gamma\alpha + I') - (\alpha + I')^2 = (\beta\gamma + b_n\alpha\beta\gamma + c_{n+1}\beta\gamma\alpha - \alpha^2) + I' = (b_n + c_{n+1})(\alpha^3 + I'),$$

and the second one to the equality

$$(\mathrm{ii})_n \qquad c_n + b_{n+1} = 1$$

since

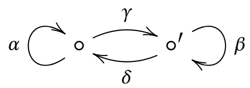
$$(\gamma + c_n\gamma\alpha + I')(\beta + b_{n+1}\alpha\beta + I') = (\gamma\beta + (c_n + b_{n+1})\gamma\alpha\beta) + I' = (c_n + b_{n+1} + 1)(\gamma\alpha\beta + I').$$

Consequently, $F_{b,c}(\tilde{I}) = 0$ if and only if (b,c) belongs to the subspace of $k^{\mathbb{Z}} \times k^{\mathbb{Z}}$ consisting of all solution of the system $\{(\mathrm{i})_n, (\mathrm{ii})_n : n \in \mathbb{Z}\}$ of linear equations. This subspace is 2-dimensional and its elements are parameterized by pairs $(b_0, c_0) \in k^2$ as follows:

$$b_n = \begin{cases} i + b_0 & \text{if } n = 2i, \\ i - c_0 & \text{if } n = 2i - 1, \end{cases} \qquad c_n = \begin{cases} -i + c_0 & \text{if } n = 2i, \\ -i - b_0 & \text{if } n = 2i + 1. \end{cases}$$

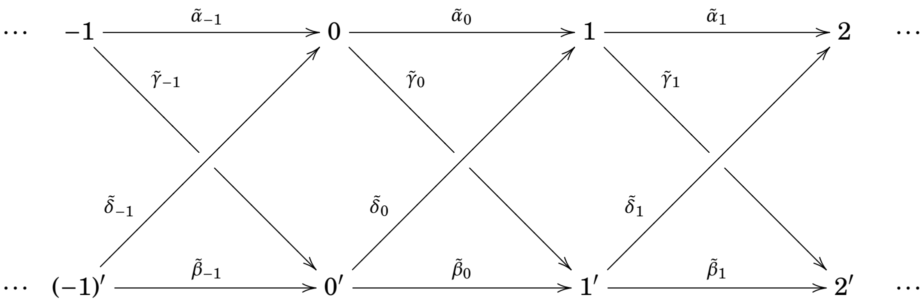
In conclusion, for any $(b_0, c_0) \in k^2$ the functor $F'_{b_0,c_0} : \tilde{R} \to R'$ induced by $F_{b,c}$, where (b,c) are as above and $\tilde{R} = R(\tilde{Q}, \tilde{I})$, is an almost Galois covering functor of integral type.

EXAMPLE (2.3.7). *Let $u \in k \setminus \{0,1\}$ be fixed parameter and (Q,I), (Q,I') a bounded quivers, where*

$Q :$ 

$I = \langle \alpha^4, \alpha^3\gamma, \delta\alpha^3, \alpha^2 - \gamma\delta, u\beta^2 - \delta\gamma, \alpha\gamma - \gamma\beta, \beta\delta - \delta\alpha \rangle$ and $I' = \langle \alpha^4, \alpha^3\gamma, \delta\alpha^3, \alpha^2 - \gamma\delta - \alpha^3, u\beta^2 - \delta\gamma, \alpha\gamma - \gamma\beta, \beta\delta - \delta\alpha \rangle$. Similarly as in the previous example we have $\underline{\dim} R = \underline{\dim} R'$, where $R = R(Q,I)$, $R' = R(Q,I')$, and also $J(R)^4 = 0 = J(R')^4$. Here the basis of corresponding morphism spaces for R and R' are given again by cosets of paths from the same sets $\{\varepsilon_\circ, \alpha, \alpha^2, \alpha^3\}$, $\{\gamma, \alpha\gamma\}$, $\{\varepsilon_{\circ'}, \beta, \beta^2, \beta^3\}$, $\{\delta, \beta\delta\}$, respectively. Note that the algebra $A(R)$ associated to R is a quotient of the algebra $\Lambda_{1,1}$ from [19].

Set $a_\bullet = \circ$. Analogously as in Example 2.3.6 $G = \Pi_1(Q,I)$ is an infinite cyclic group generated by $[\alpha]$, but here $(\tilde{Q}, \tilde{I})$ is given by the quiver

$\tilde{Q} :$ 

and the ideal $\tilde{I} = \langle \{\tilde{\alpha}_n\tilde{\alpha}_{n+1}\tilde{\alpha}_{n+2}\tilde{\alpha}_{n+3}, \tilde{\alpha}_n\tilde{\alpha}_{n+1}\tilde{\alpha}_{n+2}\tilde{\gamma}_{n+3}, \tilde{\delta}_n\tilde{\alpha}_{n+1}\tilde{\alpha}_{n+2}\tilde{\alpha}_{n+3}, \tilde{\alpha}_n\tilde{\alpha}_{n+1} - \tilde{\gamma}_n\tilde{\delta}_{n+1}, u\tilde{\beta}_n\tilde{\beta}_{n+1} - \tilde{\delta}_n\tilde{\gamma}_{n+1}, \tilde{\alpha}_n\tilde{\gamma}_{n+1} - \tilde{\gamma}_n\tilde{\beta}_{n+1}, \tilde{\beta}_n\tilde{\delta}_{n+1} - \tilde{\delta}_n\tilde{\alpha}_{n+1} : n \in \mathbb{Z}\} \rangle$, where $w_\circ = \alpha$, $w_{\circ'} = \gamma$, $n = [\alpha^{n+1}]$, $n' = [\alpha^n\gamma]$, $\tilde{\alpha}_n = (n,\alpha) = (\alpha, n+1)$, $\tilde{\beta}_n = (n', \beta) = (\beta, (n+1)')$, $\tilde{\gamma}_n = (n,\gamma) = (\gamma, (n+1)')$, $\tilde{\delta}_n = (n', \delta) = (\delta, n+1)$ for $n \in \mathbb{Z}$. In this way we again obtain a family of order relations $\prec'_{(a_1,a_2)}$ in $\mathscr{W}(a_1,a_2)$, for $a_1, a_2 \in Q_0 = \{\circ, \circ'\}$.

For any $(c,d) \in k^{\mathbb{Z}} \times k^{\mathbb{Z}}$, $c = (c_n)$, $d = (d_n)$, we define a functor

$$F_{c,d} : R(\tilde{Q}) \to R(Q,I')$$

setting $F_{c,d}(\tilde{\alpha}_n) = \alpha + I'$, $F_{c,d}(\tilde{\beta}_n) = \beta + I'$, $F_{c,d}(\tilde{\gamma}_n) = \gamma + c_n\alpha\gamma + I'$ and $F_{c,d}(\tilde{\delta}_n) = \delta + d_n\beta\delta + I'$, for $n \in \mathbb{Z}$. Here we have the inequalities

$$\gamma \prec' \alpha\gamma \quad \text{and} \quad \delta \prec' \beta\delta.$$

Moreover $F_{c,d}(\tilde{I}) = 0$ if and only if

$$F_{c,d}(\tilde{\gamma}_n\tilde{\delta}_{n+1}) - F_{c,d}(\tilde{\alpha}_n\tilde{\alpha}_{n+1}) = 0 \qquad \text{and} \qquad F_{c,d}(\tilde{\delta}_n\tilde{\gamma}_{n+1}) - uF_{c,d}(\tilde{\beta}_n\tilde{\beta}_{n+1}) = 0$$

for all $n \in \mathbb{Z}$. The first equality is equivalent to the equality

$$(\mathrm{i})_n \qquad c_n + d_{n+1} - 1 = 0$$

since

$$(\gamma + c_n\alpha\gamma + I')(\delta + d_{n+1}\beta\delta + I') - (\alpha^2 + I') = ((c_n + d_{n+1})\alpha^3 - \alpha^3) + I' = (c_n + d_{n+1} - 1)(\alpha^3 + I'),$$

and the second one to the equality

$$(\mathrm{ii})_n \qquad c_{n+1} + d_n = 0$$

since

$$(\delta + d_n\beta\delta + I')(\gamma + c_{n+1}\alpha\gamma) - u(\beta^2 + I') = (d_n + c_{n+1})(\beta^3 + I').$$

Consequently, $F_{c,d}(\tilde{I}) = 0$ if and only if (c,d) belongs to the 2-dimensional subspace of $k^{\mathbb{Z}} \times k^{\mathbb{Z}}$ consisting of all solution of the system $\{(\mathrm{i})_n, (\mathrm{ii})_n : n \in \mathbb{Z}\}$ of linear equations. Elements of that subspace are parameterized by pairs $(c_0, d_0) \in k^2$ as follows:

$$c_n = \begin{cases} -i + c_0 & \text{if } n = 2i, \\ -i - d_0 & \text{if } n = 2i+1, \end{cases} \qquad d_n = \begin{cases} i + d_0 & \text{if } n = 2i, \\ i - c_0 & \text{if } n = 2i-1. \end{cases}$$

Similarly as in previous example, for any $(c_0, d_0) \in k^2$ the functor $F'_{c_0,d_0} : \tilde{R} \to R'$ induced by $F_{c,d}$, where (c,d) are as above and $\tilde{R} = R(\tilde{Q}, \tilde{I})$, is an almost Galois covering functor of integral type. Finally notice that we deal here with (representation infinite) tame algebras in contrast to Example 2.3.6, where the considered algebras are representation finite.

(2.4) Now we formulate one of the most important results of this paper, which extends [16, Theorem 3.6] and [12, Theorem] (resp. [11, Theorem]) to the almost Galois coverings case. Throughout the remaining part of this section k is assumed to be an algebraically closed field.

THEOREM (2.4.1). *Let $G \subseteq \mathrm{Aut}_{k-\mathrm{cat}}(R)$ be a subgroup such that $G_N = \{\mathrm{id}_R\}$ for all indecomposable R-modules N and $\bar{R} = R/G$ is finite, and $F : R \to R'$ be an almost Galois G-covering functor of integral type. Then R' is tame* (*resp. representation-finite*), *if so is $\bar{R}$. In particular, if R is locally-support finite and tame* (*resp. locally representation-finite*) *then R' is tame* (*resp. representation-finite*).

In the proof we use a concept of (geometric) degeneration of algebras. Recall that all d-dimensional algebra structures form the affine variety $\mathrm{alg}_d = \mathrm{alg}_d(k) \subseteq k^{d^3}$, which is equipped with a natural regular action of the connected affine algebraic group $G_d = \mathrm{Gl}_d(k)$. Fixing a base, each d-dimensional algebra A can be treated as a point in alg_d by means of the structure constants $c = (c^l_{i,j})$, being by definition the coordinates of consecutive products of the base elements. Given k-algebras A_0 and A_1 of dimension d, we say that A_0 is a *degeneration* of A_1 if structure constants $c^{(0)}, c^{(1)} \in \mathrm{alg}_d$ of A_0 and A_1, respectively, satisfy the inclusion $G_d \cdot c^{(0)} \subseteq \overline{G_d \cdot c^{(1)}}$. (The inclusion does not depend on the choice of $c^{(0)}, c^{(1)}$ for A_0, A_1 respectively, we refer for details i.e. [7]).

The following variant of classical result on degenerations of algebras plays a crucial role in the proof of our theorem.

LEMMA (2.4.2). *Let $A = (A,\cdot)$ be a d-dimensional algebra and $\{V_i\}_{i\in\mathbb{Z}}$ a family of k-subspaces of A satisfying the following conditions:*

- $V_i \supseteq V_{i+1}$ *for all* $i \in \mathbb{Z}$,
- $V_i \cdot V_j \subseteq V_{i+j}$ *for all* $i, j \in \mathbb{Z}$,
- $V_{i_0} = A$ *and* $V_{j_0} = \{0\}$ *for some* $i_0, j_0 \in \mathbb{Z}$, $i_0 < j_0$,
- $1 \in V_0$.

Then $\bar{A} := \bigoplus_{i\in\mathbb{Z}} V_i/V_{i+1}$ *with multiplication induced by* $\cdot$ *carries a structure of d-dimensional k-algebra and $\bar{A}$ is a degeneration of A.*

Proof. Note first that clearly $\bar{A} = \bigoplus_{i_0\le i<j_0} V_i/V_{i+1}$ and $\dim_k \bar{A} = \dim_k A = d$. Moreover, $1 \notin V_i$ for every $i > 0$, since otherwise $1 \in V_1$, $1 = 1\cdot 1 \in V_2$, and by induction $1 \in V_i$ for every $i > 0$, so $V_i \neq \{0\}$ for all $i \in \mathbb{Z}$, a contradiction. Consequently, we have inequality $i_0 \le 0 < j_0$.

To show first assertion observe that for any $i, j \in \mathbb{Z}$ the map

$$*_{i,j} : V_i/V_{i+1} \times V_j/V_{j+1} \to V_{i+j}/V_{i+j+1}$$

induced by $\cdot$ is well defined, since $V_{i+1}\cdot V_j, V_i\cdot V_{j+1} \subseteq V_{i+j+1}$. It is clear that the family $\{*_{i,j}\}_{i,j\in\mathbb{Z}}$ yields the multiplication $*$ on $\bar{A}$ such that $\bar{A} = (\bar{A}, *)$ is a k-algebra with unit element $1 + V_1 \in V_0/V_1$, and that $\dim_k \bar{A} = d$.

Now we prove the second assertion. We start by fixing for any $i \in \mathbb{Z}$, a k-subspace $W_i \subseteq A$ being a complement of V_{i+1} in V_i. Then we have a k-space decompositions

$$A = \bigoplus_{i\in\mathbb{Z}} W_i = W_{i_0} \oplus \cdots \oplus W_{j_0-1}$$

and

$$V_j = \bigoplus_{i\ge j} W_i$$

for any $j \in \mathbb{Z}$. Denote by $p_i : A \to W_i$, $i \in \mathbb{Z}$, the canonical projections. Then for any $a \in W_i$ and $a' \in W_j$ we have

$$a\cdot a' = \sum_{l\ge i+j} p_l(a\cdot a')$$

since $a \in V_i$ and $a' \in V_j$, so by our assumption $a\cdot a' \in V_{i+j} = \bigoplus_{l\ge i+j} W_l$. Moreover, under the k-space identification $A \cong \bar{A}$, defined by the k-isomorphisms $W_i \cong V_i/V_{i+1}$, $a \mapsto a + V_{i+1}$, for $i \in \mathbb{Z}$, the multiplication $*$ in $\bar{A}$ is given by the formula

$$a * a' = p_{i+j}(a\cdot a') \qquad (*)$$

since $a\cdot a' + V_{i+j} = \sum_{l\ge i+j} p_l(a\cdot a') + V_{i+j+1} = p_{i+j}(a\cdot a') + V_{i+j+1}$, where a, a' are as above.

Next for any $t \in k^*$ we denote by ξ_t the k-linear automorphism

$$\bigoplus_{i\in\mathbb{Z}} t^i\,\mathrm{id}_{W_i} : \bigoplus_{i\in\mathbb{Z}} W_i \to \bigoplus_{i\in\mathbb{Z}} W_i$$

of A. Note that the all subspaces W_i and V_i, $i \in \mathbb{Z}$, are ξ_t-invariant, for every $t \in k^*$. Consider the family $A_t = (A, \cdot_t)$, $t \in k^*$, of k-algebras, where $\cdot_t : A\times A \to A$ looks as follows

$$a \cdot_t a' = \xi_t(\xi_t^{-1}(a)\cdot \xi_t^{-1}(a'))$$

for $a, a' \in A$. Clearly, each $\cdot_t$ yields a k-algebra structure on A such that $\xi_t : A \to A_t$ is a k-algebra isomorphism, in particular, $A = A_1$. Observe that for any $a \in W_i$ and $a' \in W_j$, the product $a \cdot_t a'$ is given by the formula

$$(**) \qquad a \cdot_t a' = \sum_{l \geq i+j} t^{l-i-j} p_l(a \cdot a')$$

since we have $\xi_t^{-1}(a) \in W_i$, $\xi_t^{-1}(a') \in W_j$, so $\xi_t^{-1}(a) \cdot \xi_t^{-1}(a') \in V_{i+j}$, and

$$a \cdot_t a' = \xi_t\Big(\sum_{l \geq i+j} p_l(\xi_t^{-1}(a) \cdot \xi_t^{-1}(a'))\Big) = \sum_{l \geq i+j} t^l p_l(t^{-i}a \cdot t^{-j}a') = \sum_{l \geq i+j} t^{l-i-j} p_l(a \cdot a')$$

Consequently, our family A_t, $t \in k^*$, can be extended to the family $A_t = (A, \cdot_t)$, $t \in k$, of algebras, where $\cdot_t$ is defined by the formula $(**)$. Note that $(**)$ has sense also for $t = 0$ and it has the form

$$a \cdot_0 a' = p_{i+j}(a \cdot a')$$

Hence, $A_0 = (A, \cdot_0)$ is an algebra structure such that $A_0 \cong \bar{A}$ (see $(*)$).

To complete the proof we fix arbitrarily basis B_i of the spaces W_i, $i \in \mathbb{Z}$, and we set $B = \bigcup_{i \in \mathbb{Z}} B_i$. Now it is easily seen that, the structure constants $c(t)$ of algebras A_t in the base B of the space A, for $t \in k$, yields a regular map $c : k \to \mathrm{alg}_d(k)$ of affine varieties, and in consequence a degeneration of A to $\bar{A}$. □

REMARK (2.4.3). *On can formulate and prove a corresponding version of the lemma above for locally bounded categories with n objects of dimension vector* $d \in \mathbb{N}^{[n]^2}$ *and degenerations in the sense of the affine variety* $\mathrm{lbc}_d(k)$ *(see* [7]*).*

(2.5)

Proof of Theorem 2.4.1. Let G and F satisfy all assumption of the theorem. Note that R/G is well defined, since $G_x = G_{R(-,x)} = \{\mathrm{id}_R\}$ for any $x \in \mathrm{ob}R$. As always we fix a finite set $(\mathrm{ob}R)_0$ consisting of representatives of all G orbits in $\mathrm{ob}R$. Without loss of generality we can replace F by $\check{F}$, so then $R' = \bar{R}_F$. Recall that the map $\check{F}_{g_1\bar{x}, g_2\bar{y}} : R(g_1\bar{x}, g_2\bar{y}) \to \bar{R}_F(\bar{x}, \bar{y})$ coincides with gth component $(\varphi_{g_1}(\bar{x},\bar{y})^{(g',g)})_{g'}$ of $\varphi_{g_1}(\bar{x},\bar{y})$, where $g = g_2^{-1}g_1$, for any $g_1, g_2 \in G$ and $\bar{x}, \bar{y} \in (\mathrm{ob}R)_0$, and $(\bigoplus_{\bar{x},\bar{y} \in (\mathrm{ob}R)_0} \varphi_{g_1}(\bar{x},\bar{y}))_{g_1 \in G} = \varphi(F) (= \varphi(\check{F}))$. We define a family $\mathcal{V}$ consisting of subspaces $V_i(\bar{x},\bar{y}) \subseteq R'(\bar{x},\bar{y})$, for $\bar{x},\bar{y} \in (\mathrm{ob}R)_0$ and $i \in \mathbb{Z}$, and show that it satisfies the following two conditions:

(i) the family $V_i = \bigoplus_{\bar{x},\bar{y} \in (\mathrm{ob}R)_0} V_i(\bar{x},\bar{y})$, $i \in \mathbb{Z}$, of k-spaces satisfies the assumptions of Lemma 2.4.2 for the algebra $A = A(R')$.

(ii) $\bar{A} \cong A(R/G)$, where $\bar{A} := \bigoplus_{i \in \mathbb{Z}} V_i/V_{i+1}$ ($\dim_{\mathrm{k}} \bar{A} = \dim_{\mathrm{k}} A$ by (i), from Lemma 2.4.2).

We set

$$(*) \qquad V_i(\bar{x},\bar{y}) = \bigoplus_{g \in G^{\geq i}} \bar{R}(\bar{x},\bar{y})_g$$

for $\bar{x},\bar{y} \in (\mathrm{ob}R)_0$ and $i \in \mathbb{Z}$, where $\pi : G \to \mathbb{Z}$ is a surjective group homomorphism as in Definition 2.2.1.

To prove that the family $\mathcal{V}$ defined above satisfies (i), it suffices only to verify the inclusions $V_i \cdot V_j \subseteq V_{i+j}$, for all $i, j \in \mathbb{Z}$. The fact that $\{V_i\}_{i \in \mathbb{Z}}$ fulfill the remaining three conditions from the assumptions of Lemma 2.4.2 follows immediately from the definition of $\mathcal{V}$.

Fix $i,j \in \mathbb{Z}$, $\bar{x},\bar{y},\bar{z} \in (\mathrm{ob}R)_0$ and $\alpha \in \bar{R}(\bar{x},\bar{y})_g$, $\beta \in \bar{R}(\bar{y},\bar{z})_h$, where $\pi(g) = j$ and $\pi(h) = i$. It is enough to show that the composition $\beta \bullet \alpha$ belongs to $V_{i+j}(\bar{x},\bar{z}) = \bigoplus_{r \in G^{\geq i+j}} \bar{R}(\bar{x},\bar{y})_r$. Recall that by 1.5$(***)$ we have

$$\beta \bullet \alpha = \sum_{g_2 \in G} \tilde{\beta}_{g_2} \cdot \alpha$$

where $(\tilde{\beta}_{g_2})_{g_2} = (\varphi_{g^{-1}}(\bar{y},\bar{z}))^{-1}(\beta) \in \bigoplus_{g_2 \in G} R(g^{-1}\bar{y}, g_2\bar{z}) = \bigoplus_{g_2 \in G} \bar{R}(\bar{y},\bar{z})_{(gg_2)^{-1}}$ and $\cdot$ is the composition in $\bar{R}$. Since $(\varphi_{g^{-1}}(\bar{y},\bar{z}))^{-1}$ is $\preceq$-unitriangular, the subspace $V_i(\bar{y},\bar{z}) \subseteq \bar{R}(\bar{y},\bar{z})$ is $(\varphi_{g^{-1}}(\bar{y},\bar{z}))^{-1}$-invariant (see 2.1). More precisely, $\tilde{\beta}_{(hg)^{-1}} = \beta$ and $\tilde{\beta}_{g_2} = 0$, if $(gg_2)^{-1} \prec h$ or $(gg_2)^{-1}$ and h are incomparable. Consequently,

$$(**) \qquad \beta \bullet \alpha = \beta \cdot \alpha + \sum_{g_2 : h \prec (gg_2)^{-1}} \tilde{\beta}_{g_2} \cdot \alpha,$$

and hence $\beta \bullet \alpha$ belongs to $V_{i+j}(\bar{x},\bar{z})$, since we have $\tilde{\beta}_{g_2} \cdot \alpha \in \bar{R}(\bar{y},\bar{z})_{(gg_2)^{-1}} \cdot \bar{R}(\bar{x},\bar{y})_g \subseteq \bar{R}(\bar{x},\bar{z})_{g_2^{-1}}$ for all g_2 such that $h \prec (gg_2)^{-1}$ or $h = (gg_2)^{-1}$ (equivalently, $hg \prec g_2^{-1}$ or $hg = g_2^{-1}$), and in this case $\pi(g_2^{-1}) > i+j$ or $\pi(g_2^{-1}) = i+j$, respectively.

Now we show that (ii) holds for $\mathcal{V}$. Note first that by (i) and Lemma 2.4.2 the algebra $\bar{A}$ is well defined. We have also the canonical k-isomorphisms $A(R/G) \cong \bar{A}$ defined by k-isomorphisms

$$(***) \qquad \bigoplus_{g \in \pi^{-1}(i)} \bar{R}(\bar{x},\bar{y})_g \cong V_i(\bar{x},\bar{y})/V_{i+1}(\bar{x},\bar{y})$$

$\bar{x},\bar{y} \in (\mathrm{ob}R)_0$ and $i \in \mathbb{Z}$, which are given by $\alpha \mapsto \alpha + V_{i+1}(\bar{x},\bar{y})$, for $\alpha \in \bar{R}(\bar{x},\bar{y})_g$. Moreover, by the formula $(**)$, this isomorphism preserves multiplication in our algebras. Observe that $\sum_{g_2 : h \prec (gg_2)^{-1}} \tilde{\beta}_{g_2} \cdot \alpha$ belongs to $V_{i+j+1}(\bar{x},\bar{y})$, since $\pi(g_2^{-1}) > i+j$ for all g_2 such that $h \prec (gg_2)^{-1}$.

In this way the proof of the fact that $\mathcal{V}$ satisfies the two condition (i) and (ii) is complete. Now, from Lemma 2.4.2, we immediately infer that $A(R/G)$ is a degeneration of $A(R')$. Then the assertions of our theorem follow immediately from the well known classical results on degenerations and Galois coverings: [15, Theorem] and [16, Theorem 3.6] in the representation-finite case, [18, Theorem] and [12, Theorem] (or its generalization [11, Theorem]) in the tame case, respectively. □

REMARK (2.5.1). *(a) We may ask if it is possible at all (in context of* [20, Proposition 1.4]*) that the degeneration of $A := A(R')$ to $\bar{A} \cong A(R/G)$ as above is nontrivial in the sense that $\bar{A} \not\cong A$. (Recall that if for a d-dimensional algebra B the first Hochschild cohomology group $H^1(B) = H^1(B,B)$ of B with coefficients in the B-B-bimodule B vanishes, then $B' \cong B$ for every algebra B' such that B is a degeneration of B'). It occurs that in case $F : R \to R'$ is an almost Galois G-covering functor of type L such that $R = R(\tilde{Q},\tilde{I})$ and $G = \Pi_1(Q,I)$, where $R/G \cong R(Q,I)$, we always have $H^1(\bar{A}) \neq 0$, provided L is a nontrivial abelian group. This follows from the general fact saying that for an algebra $B = A(Q',I')$ there exists a group embedding* $\mathrm{Hom}(\Pi_1(Q',I'),k^+) \hookrightarrow H^1(B)$*, which is induced by the linear map* $s : \mathrm{Hom}(\Pi_1(Q',I'),k^+) \to \mathrm{der}(B)$ *defined by the formula* $s(f)(\alpha) = f([w_{s(\alpha)}\alpha w_{t(\alpha)}^{-1}])\alpha$*, for any homomorphism $f : \Pi_1(Q',I') \to k^+$ and arrow $\alpha \in Q'_1$, where $w_a \in \mathcal{W}(a_\bullet,a)$, $a \in Q'_0$, is a fixed collection of walks as in 2.3. (We interpret here $H^1(B)$ as a quotient* $H^1(B) = \mathrm{der}(B)/\mathrm{der}^0(B)$*, where* $\mathrm{der}(B)$ *and* $\mathrm{der}^0(B)$*, denote the k-spaces of all derivatives, resp. inner derivatives, of the algebra B with values in the bimodule*

B). Note that in our situation we have $\mathrm{Hom}(\Pi_1(Q,I),k^+) = \mathrm{Hom}(G/[G,G],k^+) \neq 0$, *since* $G_{\mathrm{ab}} := G/[G,G]$ *contains an infinite cyclic direct summand. (The surjective homomorphism* $\pi : G \to L$ *factors through* G_{ab} *and the abelian finitely generated group* $L = \pi(G)$ *is torsion free, as an ordered group, so* $L \cong \mathbb{Z}^s$*, for some* $s \geq 1$*).*

(b) The first part of argumentation from the proof can be repeated in a quite general situation of an almost Galois G-covering $F : R \to R'$ *of type L, where L is an arbitrary (abelian) totally ordered group. Defining, analogously as before, the family* $\mathcal{V}$ *of the subspaces* $V_i(\bar{x},\bar{y})$*,* $i \in L$*, by the formulas* $(*)$*, the formula* $(**)$ *is again valid and we obtain a nice filtration. Moreover, replacing* $V_{i+1}(\bar{x},\bar{y})$ *by* $\sum_{j>i} V_j(\bar{x},\bar{y})$ *in* $(***)$*, we conclude that the associated graded algebra* $\bar{A}$ *is isomorphic to* $A(R/G)$*. However, it is completely not clear what is now a suitable variety X, which should be used, as substitute of k, for deformation of the composition formula* $(**)$*.*

(c) The assertion of Theorem 2.4.1 remains valid, if we assume that all maps ${}_{\bar{y}}f_{\bar{x}}^{g_1}$ *and* ${}^{g_1}_{\bar{y}}f_{\bar{x}}$ *induced by F are only surjective (see 1.5).*

3. Properties of functors $(F_\lambda, F_\bullet)$ for almost Galois coverings

In some sense Theorem 2.4.1 can be treated as an extension of [16, Theorem 3.6], [12, Theorem] and [11, Theorem] on the case of almost Galois coverings $F : R \to R'$. It says, roughly speaking, that if covering category R is locally-support finite then the representation type of R' is not more complex than that of R. However, it yields no information about direct interrelations between structure of the categories mod R' and mod R, in particular, about a behaviour and properties of the push-down functor F_λ. In this section we just discuss some particular questions concerning this problem.

(3.1) The main result we prove is the following one.

THEOREM (3.1.1). *Let* $F : R \to R$ *be an almost Galois covering functor with group G of type L and N be an indecomposable module in mod R such that* $G_N = \{\mathrm{id}_N\}$.

(a) Then $F_\lambda(N)$ *is indecomposable, if* $\mathrm{Hom}_R(N, {}^gN) = 0$ *for all* $g \in G^{\geq 0}$ *such that* $g \neq e$.

(b) Assume additionally that L is abelian. Then $F_\bullet F_\lambda(N) \cong \bigoplus_{g\in G} {}^gN$*, if* $\mathrm{Ext}^1_R(N, {}^gN) = 0$ *for all* $g \in G^{<e}$.

REMARK (3.1.2). *(a) In both statements, (a) and (b), the conditions we impose on N concern only with finite subsets of G dependent on N, since* $\mathrm{Hom}_R(N, {}^gN) = 0$*, if* $g(\mathrm{supp}N) \cap \mathrm{supp}N = \emptyset$ *and* $\mathrm{Ext}^1_R(N, {}^gN) = 0$*, if* $g(\mathrm{supp}N) \cap \widehat{\mathrm{supp}N} = \emptyset$ *(cf. Theorem 4.3.1).*

(b) The condition from (b) is weaker than claiming the equality $\mathrm{Ext}^1_R(\hat{N},\hat{N}) = 0$*, which is in fact equivalent to the equalities* $\mathrm{Ext}^1_R(N, {}^gN) = 0$ *for all* $g \in G$*, where* $\hat{N} := \bigoplus_{g\in G} {}^gN$ *(G acts freely on* ob$R$*, so* $\hat{N}$ *belongs to* Mod$R$*, and consequently* $\hat{N} = \prod_{g\in G} {}^gN$*). In the proof we construct certain R-submodule filtration of* $F_\bullet F_\lambda(N)$ *such that the associated graded module is isomorphic just to* $\hat{N}$*, so in case* $G = \mathbb{Z}$*,* $\hat{N}$ *is a degeneration of* $F_\bullet F_\lambda(N)$ *(one can extend naturally the definition of degeneration for R-modules from mod R to* Mod$R$*). Recall that for a finite dimensional module X over an algebra B, which is equipped with a filtration of B-submodules given by a (finite) chain starting with* $\{0\}$ *and ending with X, the associated graded module*

$\bar{X}$ is always a degeneration of X and then the condition $\mathrm{Ext}^1_A(\bar{X},\bar{X}) = 0$ implies that $X \cong \bar{X}$ (the orbit of $\bar{X}$ in the module variety is open, see [28]*).*

The proof of the theorem needs some preparation. It will be completed in 3.4. and 3.7.

(3.2) A basic role in the proof of the theorem is played by certain families of natural submodules of the R-module $F_\bullet F_\lambda(N)$.

LEMMA (3.2.1). *Let F be as in Theorem 3.1.1 and N be an arbitrary R-module. Then for any $i \in L$ (resp. $g \in G$) the family $\widetilde{N}^{(i)}(x) = \bigoplus_{h\in G^{\le i}} {}^hN(x)$, $x \in \mathrm{ob}R$, (resp. $\widetilde{N}^{(g)}(x) = \bigoplus_{h\in G^{\preceq g}} {}^hN(x)$, $x \in \mathrm{ob}R$) of k-spaces forms an R-submodule $\widetilde{N}^{(i)}$ (resp. $\widetilde{N}^{(g)}$) of the R-module $\widetilde{N} = F_\bullet F_\lambda(N)$.*

Proof. Fix $i \in L$ (resp. $g \in G$). We have to prove that the inclusion $\widetilde{N}(\alpha)(\widetilde{N}^{(i)}(y)) \subseteq \widetilde{N}^{(i)}(x)$ (resp. $\widetilde{N}(\alpha)(\widetilde{N}^{(g)}(y)) \subseteq \widetilde{N}^{(g)}(x)$) holds for every $\alpha \in R(x,y)$.

Fix an arbitrary $\alpha \in R(x,y) = R(g_x\bar{x}, g_y\bar{y})$, where $x = g_x\bar{x}$ and $y = g_y\bar{y}$. Recall that $\widetilde{N}(\alpha) : \widetilde{N}(y) \to \widetilde{N}(x)$ has the form

$$[N({}_{g_2'y}F(\alpha)_{g_1'x})] : \bigoplus_{g_2'\in G} N(g_2'y) \to \bigoplus_{g_1'\in G} N(g_1'x)$$

We show that under our assumptions $N({}_{g_2'y}F(\alpha)_{g_1'x}) = 0$, if $g_1' \prec g_2'$, or g_1' and g_2' are incomparable. Note that for any $g_2' \in G$, $({}_{g_2'y}F(\alpha)_{g_1'x})_{g_1'} \in \bigoplus_{g_1'\in G} R(g_1'x, g_2'y) = \bigoplus_{g_1'\in G} R(g_1'g_x\bar{x}, g_2'g_y\bar{y})$ is equal to $(\varphi'_{g_2'g_y}(\bar{x},\bar{y})^{-1} \cdot \varphi'_{g_y}(\bar{x},\bar{y}))(\alpha)$. Therefore, since $\varphi'_{g_2'g_y}(\bar{x},\bar{y})^{-1} \cdot \varphi'_{g_y}(\bar{x},\bar{y})$ is $\preceq$-unitriangular, we have ${}_{g_2'y}F(\alpha)_{g_1'x} = 0$ if $(g_2'g_y)^{-1}(g_1'g_x) \prec g_y^{-1}g_x$, or $(g_2'g_y)^{-1}(g_1'g_x)$ and $g_y^{-1}g_x$ are incomparable, equivalently, if $g_1' \prec g_2'$, or g_1' and g_2' are incomparable. In this way our claim is proved. As consequence, we obtain that

$$\widetilde{N}(\alpha)(N(h_2^{-1}y)) \subseteq \bigoplus_{h_1 \prec h_2} N(h_1^{-1}x) \oplus N(h_2^{-1}x)$$

for any $h_2 \in G$. Now the required two inclusions follow easily from that above. □

PROPOSITION (3.2.2). *Let F and N be as above. The families of R-submodules $\widetilde{N}^{(i)}$, $i \in L$, and $\widetilde{N}^{(g)}$, $g \in G$, have the following properties:*

(a) $\widetilde{N}^{(i)} \subseteq \widetilde{N}^{(j)}$ for $i \le j$, $\widetilde{N}^{(g)} \subseteq \widetilde{N}^{(h)}$ for $g \preceq h$,

(b) $\widetilde{N}^{(j)} \subseteq \widetilde{N}^{(g)} \subseteq \widetilde{N}^{(\pi(g))}$ for any g and j such that $j < \pi(g)$,

(c) $\Sigma_{i\in L}\widetilde{N}^{(i)} = \bigcup_{i\in L}\widetilde{N}^{(i)} = \widetilde{N}$, $\Sigma_{g\in G}\widetilde{N}^{(g)} = \widetilde{N}$,

(d) $\widetilde{N}^{(i)}/\sum_{j<i}\widetilde{N}^{(j)} \cong \bigoplus_{h\in\pi^{-1}(i)} {}^hN$ for any i, $\widetilde{N}^{(g)}/\sum_{j<\pi(g)}\widetilde{N}^{(j)} \cong {}^gN$ for any g.

(e) Let N' be a finite dimensional R-submodule of $\widetilde{N}$ (resp. $\sum_{j<i}\widetilde{N}^{(j)}$, for $i \in L$). Then there exists j (resp. $j < i$) such that $N' \subseteq \widetilde{N}^{(j)}$, moreover, the set $\{j : N' \subseteq \widetilde{N}^{(j)}\}$ contains the smallest element $j(N')$, provided $N' \ne 0$.

Proof. Note that we have $\widetilde{N}^{(i)}(x) = \bigoplus_{j\le i}\bigoplus_{h\in\pi^{-1}(j)} {}^hN(x) = (\bigoplus_{j<i}\bigoplus_{h\in\pi^{-1}(j)} {}^hN(x)) \oplus \bigoplus_{h\in\pi^{-1}(i)} {}^hN(x)$ and $\widetilde{N}^{(g)}(x) = (\bigoplus_{j<\pi(g)}\bigoplus_{h\in\pi^{-1}(j)} {}^hN(x)) \oplus {}^gN(x)$, for any $x \in \mathrm{ob}R$. Now (a) - (c) are straightforward. The assertion (d) follows easily from description of the maps $\widetilde{N}(\alpha)$, for $\alpha \in R(x,y)$, in the prof above. Note that ${}_{g_2'y}F(\alpha)_{g_2'x} = g_2'\alpha$, by $\preceq$-unitrianularity of the map $\varphi'_{g_2'g_y}(\bar{x},\bar{y})^{-1} \cdot \varphi'_{g_y}(\bar{x},\bar{y})$.

To prove (e) fix a finite dimensional submodule N' of $\widetilde{N}$. Because for any $x \in \mathrm{ob}R$, the family $\widetilde{N}^{(i)}(x)$, $i \in L$, forms an ascending subspace chain such that

$\bigcup_{i\in L}\widetilde{N}^{(i)}(x)=\widetilde{N}(x)$, and $\dim_k N'(x)$ is finite, so there exists $j_x\in L$ such that $N'(x)\subseteq \widetilde{N}^{(j_x)}(x)$. Now it is clear that we can take for j the greatest element of the finite set $\{j_x : x\in \operatorname{supp} N'\}$.

Similarly, we show the case $N'\subseteq \sum_{j<i}\widetilde{N}^{(j)}$.

Now assume $N'\neq 0$. To prove our assertion, we consider the sets $T_x=\{h\in G : p_h(N'(x))\neq 0\}$ for $x\in \operatorname{supp} N'$, where $p_h:\tilde{N}=\bigoplus_{h'\in G}{}^{h'}N(x)\to {}^{h}N(x)$ denotes the canonical projection, for $h\in G$. Note that all the sets T_x are nonempty and finite, since N' is finite dimensional.

Let $\{j_1,\ldots,j_s\}=\pi(T_x)$, for a fixed $x\in \operatorname{supp} N'$, where $j_1<\cdots<j_s$. Then we have $N'(x)\subseteq\widetilde{N}^{(j_s)}$ and $N'(x)\subsetneq\widetilde{N}^{(j)}$, for any $j<j_s$, since $N'(x)\subseteq\bigoplus_{h\in T_x}{}^{h}N(x)$ and $N'(x)\subsetneq\bigoplus_{h\in T'}{}^{h}N(x)$, for any $T'\supsetneq T_x$. Consequently, $j_x(N')=j_s$, is the smallest element in the set $\{j_x : N'(x)\subseteq \widetilde{N}^{(j_x)}(x)\}$. Now it is clear that the greatest among elements of the finite nonempty set $\{j_x(N') : x\in\operatorname{supp} N'\}$ satisfies the condition, we require from $j(N)$, and the proof is complete. □

(3.3) Now, applying the families of submodules introduced above, we study structure of the endomorphism algebra $\operatorname{End}_{R'}(F_\lambda(N))$ for N in mod R.

Recall that $(F_\lambda, F_\bullet)$ is a pair of adjoint functors so for any R-module N we have the k-space isomorphism

$$(*)\qquad \operatorname{End}_{R'}(F_\lambda(N))\cong \operatorname{Hom}_R(N, F_\bullet F_\lambda(N))$$

It is given by the mapping

$$f=([f(a)_{y,x}]_{y,x\in F^{-1}(a)})_{a\in \operatorname{ob}R}\mapsto f'=((f(Fx)_{y,x})_{y\in\operatorname{ob}R})_{x\in\operatorname{ob}R},$$

where $F_\lambda(N)(a)=\bigoplus_{x\in F^{-1}(a)}N(x)$, for $a\in\operatorname{ob}R'$, and $F_\bullet F_\lambda(N)(x)=\bigoplus_{y\in F^{-1}(Fx)}N(y)$, for $x\in\operatorname{ob}R'$. We fix the following notation. For any R-submodule N' of $\widetilde{N}=F_\bullet F_\lambda(N)$ we denote by (N,N') the subspace of $\operatorname{End}_{R'}(F_\lambda(N))$ corresponding via $(*)$ to the subspace $\{f'\in\operatorname{Hom}_R(N,\widetilde{N}) : \operatorname{Im} f'\subseteq N'\}$ of $\operatorname{Hom}_R(N,\widetilde{N})$.

LEMMA (3.3.1). *Let N be an R-module. Then for any $f\in\operatorname{End}_{R'}(F_\lambda(N))$ we have the following:*

(a) f belongs to $(N,\widetilde{N}^{(0)})$ (resp. to $(N,\sum_{j<0}\widetilde{N}^{(j)})$) if and only if the map $f(a):\bigoplus_{g\in G}N(g\bar{x}_a)\to\bigoplus_{g\in G}N(g\bar{x}_a)$ is $\preceq$-triangular (resp. $\prec$-triangular), for every $a\in\operatorname{ob}R$, where $\bar{x}_a\in F^{-1}(a)\cap(\operatorname{ob}R')_0$.

(b) f belongs to $(N,\widetilde{N}^{(e)})$ if and only if the map $f(a):\bigoplus_{g\in G}N(g\bar{x}_a)\to\bigoplus_{g\in G}N(g\bar{x}_a)$ is $\preceq$-triangular and the family

$$\bar{f}_i(a)=[f(a)_{g'\bar{x}_a,g\bar{x}_a}]:\bigoplus_{g\in\pi^{-1}(i)}N(g\bar{x}_a)\to\bigoplus_{g'\in\pi^{-1}(i)}N(g'\bar{x}_a),\ i\in L,$$

consists only of diagonal maps, for every $a\in\operatorname{ob}R$.

(c) if f belongs to $(N,\widetilde{N}^{(e)})$ then

$$\bar{f}=\left(\bigoplus_{i\in L}\bar{f}_i(a):\bigoplus_{i\in L}\bigoplus_{g\in\pi^{-1}(i)}N(g\bar{x}_a)\to\bigoplus_{i\in L}\bigoplus_{g\in\pi^{-1}(i)}N(g\bar{x}_a)\right)_{a\in\operatorname{ob}R'}$$

belongs to $F_\lambda(\operatorname{End}_R(N))$; moreover, $F_\lambda(\operatorname{End}_R(N))\subseteq(N,\widetilde{N}^{(e)})$ and $\bar{f}=f$, for any $f\in F_\lambda(\operatorname{End}_R(N))$.

Proof. (a) Set $\widetilde{N}(x)=\bigoplus_{h\in G}N(h^{-1}x)$ and $F_\lambda(N)(a)=\bigoplus_{g\in G}N(g\bar{x}_a)$, $x\in\operatorname{ob}R$ and $a\in\operatorname{ob}R'$, respectively. Then, if $F(x)=a$, the components of the maps $f'(x)=$

$(f'(x)_h)_h : N(x) \to \widetilde{N}(x) = \bigoplus_{h\in G} N(h^{-1}x)$ and $f(a) = [f(a)_{(g',g)}] : \bigoplus_{g\in G} N(g\bar{x}_a) \to \bigoplus_{g'\in G} N(g'\bar{x}_a)$ satisfies the equality

$$f'(x)_h = f(a)_{(g',g)}$$

where $x = g\bar{x}_a$ and $g' = h^{-1}g$, since $f'(x)_h = f(Fx)_{h^{-1}x,x}$ and $f(a)_{(g',g)} = f(a)_{g'\bar{x}_a,g\bar{x}_a}$. Consequently, for $j \in L$, the inclusion $\operatorname{Im} f' \subseteq \widetilde{N}^{(j)}$ holds if and only if $f'(x)_h = 0$ provided $j < \pi(h)$, for any x; equivalently, $f(a)_{(g',g)} = 0$, provided $\pi(g') + j < \pi(g)$, for any a.

Now the first assertion of (a) follows immediately, if we set $j = 0$. To prove the second one, we assume that $\operatorname{Im} f' \subseteq \sum_{j<0} \widetilde{N}^{(j)}$. By Proposition 3.2.2(e), $\operatorname{Im} f' \subseteq \widetilde{N}^{(j)}$ for some $j < 0$, so for any a, $f(a)_{(g',g)} = 0$ if $\pi(g') \le \pi(g)$, since $\pi(g') + j < \pi(g') \le \pi(g)$. Thus, all $f(a)$ are $\prec$-triangular. Conversely, if we assume that for any a, $f(a)_{(g',g)} = 0$ provided $\pi(g') \le \pi(g)$, then for any x, $f'(x)_h = f(Fx)_{h^{-1}x,x} = 0$ provided $0 \le \pi(h)$. Consequently, $\operatorname{Im} f'(x) \subseteq \bigoplus_{h:\pi(h)<0} N(h^{-1}x) = \sum_{j<0} \widetilde{N}^{(j)}(x)$ and the proof of (a) is complete.

(b) By Proposition 3.2.2(b), f belongs to $(N, \widetilde{N}^{(e)})$ if and only if f belongs to $(N, \widetilde{N}^{(0)})$, and for any x, $f'(x)_h = 0$, for all $h \neq e$ such that $\pi(h) = 0$, or equivalently, for any a, $f(a)_{(g',g)} = 0$ for all $g \neq g'$ such that $\pi(g) = \pi(g')$. In this way we are done.

(c) From (b), we have $\bar{f}(a) = \bigoplus_{g\in G} f(a)_{g\bar{x}_a,g\bar{x}_a}$, for all a. To prove that $\bar{f}$ belongs to $F_\lambda(\operatorname{End}_R(N))$ it suffices to note that $\bar{f} = F_\lambda(pf')$, where $p : \widetilde{N}^{(e)} \to N$ is a canonical projection (see Proposition 3.2.2(d)). The last equality holds, since $f'(x)_e = f(a)_{(g,g)} = f(a)_{g\bar{x}_a,g\bar{x}_a}$ for all x and $g \in G$ such that $a = Fx$ and $x = g\bar{x}_a$.

The inclusion $F_\lambda(\operatorname{End}_R(N)) \subseteq (N, \widetilde{N}^{(e)})$ and the last assertion follow now immediately from (b) and the definition of F_λ. □

PROPOSITION (3.3.2). *Let N be indecomposable module in mod R. Then* $(N, \widetilde{N}^{(e)})$ *is a subalgebra of the k-algebra* $\operatorname{End}_{R'}(F_\lambda(N))$ *and* $(N, \sum_{j<0} \widetilde{N}^{(j)})$ *is a two-sided nilpotent ideal in* $(N, \widetilde{N}^{(e)})$ *such that* $(N, \widetilde{N}^{(e)})/(N, \sum_{j<0} \widetilde{N}^{(j)}) \cong \operatorname{End}_R(N)$*, more precisely,* $(N, \widetilde{N}^{(e)}) = F_\lambda(\operatorname{End}_R(N)) \oplus (N, \sum_{j<0} \widetilde{N}^{(j)})$ *and* $J(N, \widetilde{N}^{(e)}) = F_\lambda(J(\operatorname{End}_R(N))) \oplus (N, \sum_{j<0} \widetilde{N}^{(j)})$*. In consequence, if N is indecomposable then* $(N, \widetilde{N}^{(e)})$ *is a local k-algebra.*

Proof. We start by observing that

$$A = \{(f(a)) \in P := \prod_{a\in \operatorname{ob} R'} \operatorname{End}_k(\bigoplus_{g\in G} N(g\bar{x}_a)) : (f(a)) \text{ satisfies } (\mathrm{b})_{\mathrm{r}}\}$$

is a subalgebra of the k-algebra P, where $(\mathrm{b})_{\mathrm{r}}$ denotes "the condition on the right hand side" from the equivalence (b). It is clear that

$$I = \{(f(a)) \in A : \forall_{a\in \operatorname{ob} R'}\ f(a) \text{ is } \prec\text{-triangular}\}$$

forms a twosided ideal in A. Note that I is nilpotent, since N is finite-dimensional. Consequently, $(N, \widetilde{N}^{(e)})$ is an subalgebra of $\operatorname{End}_{R'}(F_\lambda(N))$ and $(N, \sum_{j<0} \widetilde{N}^{(j)})$ is a two-sided nilpotent ideal in $(N, \widetilde{N}^{(e)})$, since $(N, \widetilde{N}^{(e)}) = A \cap \operatorname{End}_{R'}(F_\lambda(N))$ and $(N, \sum_{j<0} \widetilde{N}^{(j)}) = I \cap \operatorname{End}_{R'}(F_\lambda(N))$ from Lemma 3.3.1(a),(b). To prove the remaining assertions consider the k-homomorphism $\bar{(\cdot)} : (N, \widetilde{N}^{(e)}) \to F_\lambda(\operatorname{End}_R(N))$ defined by the mapping $f \mapsto \bar{f}$. Observe that $\bar{(\cdot)}$ is well defined, by Lemma 3.3.1(c), and it is an algebra homomorphism, by the multiplication formula for $\preceq$-triangular matrices. From Lemma 3.3.1(b), we have Ker $(\bar{\cdot}) = (N, \sum_{j<0} \widetilde{N}^{(j)})$, and required decomposition follows easily by Lemma 3.3.1(c).

Assume that N is indecomposable. Then $\mathrm{End}_R(N)$ is local, $(N, \widetilde{N}^{(e)})/J((N, \widetilde{N}^{(e)}))$ is isomorphic to a factor algebra of $\mathrm{End}_R(N)$ since $(N, \sum_{j<0} \widetilde{N}^{(j)}) \subseteq J((N, \widetilde{N}^{(e)}))$, so the algebra $(N, \widetilde{N}^{(e)})/J((N, \widetilde{N}^{(e)}))$ is also local. Consequently, $(N, \widetilde{N}^{(e)})$ is a local algebra and the proof is complete. □

(3.4) Now we can complete the proof of the first part of our theorem.

Proof of Theorem 3.1.1(a). By Proposition 3.3.2, it suffices to prove that $\mathrm{End}_{R'}(F_\lambda(N)) = (N, \widetilde{N}^{(e)})$ under assumptions of (a), or equivalently, that $\mathrm{Im} f' \subseteq \widetilde{N}^{(e)}$ for every $f' \in \mathrm{Hom}_R(N, \widetilde{N})$.

Fix $f' \in \mathrm{Hom}_R(N, \widetilde{N})$. By Proposition 3.2.2(e), the set $\{j : \mathrm{Im} f' \subseteq \widetilde{N}^{(j)}\}$ contains the smallest element $j_0 = j(\mathrm{Im} f')$. Clearly, $\mathrm{Im} f' \nsubseteq \sum_{j<j_0} \widetilde{N}^{(j)}$, since otherwise $\mathrm{Im} f' \subseteq \widetilde{N}^{(j)}$ for some $j < j_0$, a contradiction. Consequently, the composite R-homomorphism

$$pf' : N \to \widetilde{N}^{(j_0)} / \sum_{j<j_0} \widetilde{N}^{(j)} \cong \bigoplus_{g \in \pi^{-1}(j_0)} {}^{g}N,$$

is a well defined and nonzero map, where $p : \widetilde{N}^{(j_0)} \to \widetilde{N}^{(j_0)} / \sum_{j<j_0} \widetilde{N}^{(j)}$ is the canonical projection (see Proposition 3.2.2(d)). By our assumptions we have: either $j_0 < 0$, or $j_0 = 0$ and $\mathrm{Im}\, pf' \subseteq N$. Hence, by Proposition 3.2.2(b), we infer $\mathrm{Im} f' \subseteq \widetilde{N}^{(e)}$, and the proof of (a) is complete. □

(3.5) For the proof of the assertion (b) of the theorem we construct another family of R-submodules of $\tilde{N}$, where N is an R-module. We start by fixing some extra notation.

For any $\bar{x}, \bar{y} \in (\mathrm{ob} R)_0$ we set $G(\bar{x}, \bar{y}) = \{g \in G : \bar{R}(\bar{x}, \bar{y})_g \neq 0\}$ and $G(\bar{x}) = \{g \in G : g\bar{x} \in S\}$, where $S = \mathrm{supp} N$, respectively, $G_0 = \bigcup_{\bar{x}, \bar{y} \in (\mathrm{ob} R)_0} G(\bar{x}, \bar{y})$ and $G(S) = \bigcup_{\bar{x} \in (\mathrm{ob} R)_0} G(\bar{x})$. Note that G_0 is finite since R/G is finite, and $G(S)$ is finite if and only if S is finite. Finally, we set

$$T(S) = \{g_1 g_2 g_3^{-1} g_1^{-1} : g_1 \in G(S), g_2, g_3 \in G_0\} \cap G^{\prec e}$$

Clearly, $T(S)$ is finite, if S is finite.

Let $T_0, T \subseteq G$ be a pair of subsets of G such that $T(S) \subseteq T$. Then for any $m \in \mathbb{N}$ we set

$$T_{(m)} = T_0 \cdot T^m$$

and

$$T_{[m,\infty)} = \bigcup_{p \geq m} T_{(p)}.$$

LEMMA (3.5.1). *(a) For any $m \in \mathbb{N}$, the family $\tilde{N}_m(x) = \bigoplus_{h \in T_{[m,\infty)}} {}^{h}N(x)$, $x \in \mathrm{ob} R$, of k-spaces forms an R-submodule $\tilde{N}_m = \tilde{N}(T_0, T)_m$ of $\tilde{N}$.*

(b) The R-submodules $\tilde{N}_m$, $m \in \mathbb{N}$, form a descending chain

$$\tilde{N}_0 \supseteq \tilde{N}_1 \supseteq \cdots \supseteq \tilde{N}_m \supseteq \tilde{N}_{m+1} \supseteq \ldots$$

such that $\tilde{N}_{[m_1,m_2)}(x) = \bigoplus_{h \in T_{[m_1,m_2)}} {}^{h}N(x)$ for all $x \in \mathrm{ob} R$, where $\tilde{N}_{[m_1,m_2)} = \tilde{N}_{m_1} / \tilde{N}_{m_2}$ and $T_{[m_1,m_2)} = T_{[m_1,\infty)} \setminus T_{[m_2,\infty)}$, for any $0 \leq m_1 \leq m_2$.

Proof. (a) Observe that it suffices to show that given $h_2 \in G$, the inclusion $\tilde{N}(\alpha)({}^{h_2}N(y)) \subseteq (\bigoplus_{h \in T(S)} {}^{h_2 h}N(x)) \oplus {}^{h_2}N(x)$ holds for every $\alpha \in R(x,y)$. Recall, that $\tilde{N}(\alpha)_{(h_1,h_2)} = 0$, if $h_2 \prec h_1$ or h_1, h_2 are incomparable, where

$$\tilde{N}(\alpha)_{(h_1,h_2)} = N({}_{h_2^{-1}y}\dot{F}(\alpha)_{h_1^{-1}x})$$

is the component of the k-linear map

$$\tilde{N}(\alpha) : \bigoplus_{h_2 \in G} N(h_2^{-1}y) \to \bigoplus_{h_1 \in G} N(h_1^{-1}x)$$

(see 3.2). Therefore, we have only to prove that for any $h_1 \in G$ such that $h_1 \prec h_2$, h_1 belongs to $h_2 \cdot T(S)$, provided $\tilde{N}(\alpha)_{(h_1,h_2)} \neq 0$.

Let h_1, $h_1 \prec h_2$, be such that $\tilde{N}(\alpha)_{(h_1,h_2)} \neq 0$ for some $\alpha \in R(x,y)$, where $x = g_x\bar{x}$, $y = g_y\bar{y}$ and $\bar{x}, \bar{y} \in (\mathrm{ob}R)_0$. Then we have $h_2^{-1}y \in \mathrm{supp}N$, $\alpha \neq 0$ and $R(h_1^{-1}x, h_2^{-1}y) \neq 0$. Consequently,

- $g_y = h_2 g_1$ for some $g_1 \in G(\bar{y})$, since $h_2^{-1}g_y \in G(\bar{y})$,
- $g_x = g_y g_2$ for some $g_2 \in G(\bar{x},\bar{y}))$, since $0 \neq R(x,y) \cong R(\bar{x}, g_x^{-1}g_y\bar{y})$, so $(g_x^{-1}g_y)^{-1} \in G(\bar{x},\bar{y})$,
- $h_1 = g_x g_3^{-1} g_y^{-1} h_2$ for some $g_3 \in G(\bar{x},\bar{y}))$, since $0 \neq R(h_1^{-1}x, h_2^{-1}y) \cong R(\bar{x}, g_x^{-1} h_1 h_2^{-1} g_y \bar{y})$ so $(g_x^{-1}h_1h_2^{-1}g_y)^{-1} \in G(\bar{x},\bar{y})$.

Combining three formulas above we obtain

$$h_1 = g_y g_2 g_3^{-1} g_y^{-1} h_2 = h_2 g_1 g_2 g_3^{-1} (h_2 g_1)^{-1} h_2 = h_2 (g_1 g_2 g_3^{-1} g_1^{-1})$$

Note that $g_1 g_2 g_3^{-1} g_1^{-1} \prec e$ (equivalently, $g_2 g_3^{-1} \prec e$), since $h_1 \prec h_2$, so we infer $h_1 \in h_2 \cdot T(S)$. In this way the proof of (a) is complete.

(b) Follows immediately from definition of the modules $\tilde{N}_m$. □

(3.6) Let H be an abstract group and $T_0, T \subseteq H$ a pair of nonempty subsets of G such that $e \notin T$. We shortly discuss the problem when there exists $m_0 \in \mathbb{N}$ such that $T_0 \cap T_{(m)} = \emptyset$ for all $m \geq m_0$, where $T_{(m)} = T_0 \cdot T^m$. (Note that for $H = \mathbb{Z}^n$ this is always the case, if T_0 is finite and T consists only of elements with strictly positive coordinates).

PROPOSITION (3.6.1). *Let H be an L-totally ordered group, where L is abelian. If the sets $\pi(T_0)$, $\pi(T)$ are finite and $\pi(T) \subseteq \{i \in L : 0 < i\}$ (resp. $\pi(T) \subseteq \{i \in L : i < 0\}$) then there exits $m_0 \in \mathbb{N}$ such that $T_0 \cap T_{(m)} = \emptyset$ for all $m \geq m_0$.*

For the proof we need some more detail properties of abelian totally ordered groups.

LEMMA (3.6.2). *Let $L = (L, \leq)$ be a totally ordered abelian group and $i^{(1)} \dots, i^{(r)} \in L$ a collection o elements such that $0 < i^{(s)}$ for all s. The for any $i \in L$ there exists only finitely many $(d_s) \in \mathbb{N}^r$ such that $\sum_{s=1}^{r} d_s i^{(s)} = i$.*

Proof. We apply induction on r. The case $r = 0$ is trivial.

Assume $r \geq 1$. Consider the subgroup L' of L generated by $i^{(1)} \dots, i^{(r)}$. Since L' is finitely generated, so by classical Hölder's theorem on Archimedean groups [22, Theorem II.2.1] and [22, Corolary VII.2.1], the ordered group $(L', \leq_{|L' \times L'})$ is isomorphic to a finite lexicographic product $L_1 \times \cdots \times L_{r'}$, $r' \leq r$, where all L_s are finitely generated subgroups of the additive group $(\mathbb{R}^+, \leq)$ of real numbers equipped with natural ordering. From now on we identify L' with $L_1 \times \cdots \times L_{r'}$.

Let $i' = (i'_{s'})$ be a coordinate presentation of $i' \in L'$. We set $l(i') = \min\{s' : i'_{s'} \neq 0\}$ for any $i' \neq 0$. Note that if $0 < i' \leq i''$ then $0 < l(i'), l(i'')$ and $l(i'') \leq l(i')$, so $0 \leq i'_{l(i'')} \leq i''_{l(i'')}$.

Fix $i \in L$. It is clear that we have to prove the assertion only for $i \in L'$. Let $\sum_{s=1}^{r} d_s i^{(s)} = i$, where $(d_s) \in \mathbb{N}^r$. Without loss of generality we can assume that $i^{(1)} \leq \cdots \leq i^{(r)}$. Then, by previous observation, we have $0 < i_l^{(r)}$ and $0 \leq i_l^{(s)}$ for all $s = 1, \ldots r-1$, where $l = l(i^{(r)})$. Hence, $0 \leq d_r \leq \frac{i_l}{i_l^{(r)}}$, since $0 \leq d_r i_l^{(r)} \leq \sum_{s=1}^{r} d_s i_l^{(s)}$. From the inductive assumption for $i^{(1)} \ldots, i^{(r-1)}$, the set of all $(d'_s) \in \mathbb{N}^{r-1}$ such that $\sum_{s=1}^{r-1} d'_s i^{(s)} = i - d' i_r$ for some $0 \leq d' \leq \frac{i_l}{i_l^{(r)}}$, is finite. Consequently, the set of all $(d_s) \in \mathbb{N}^r$ such that $\sum_{s=1}^{r} d_s i^{(s)} = i$ is finite for any $i \in L'$, and the proof is complete. □

Proof of Proposition 3.6.1. We prove only the case $\pi(T) \subseteq \{i \in L : 0 < i\}$, where $T_0, T \subseteq H$ are as in the statement of the proposition. The case $\pi(T) \subseteq \{i \in L : i < 0\}$ follows immediately from the first one.

Let $\pi(T) = \{i^{(1)} \ldots, i^{(r)}\}$. Then, by the lemma, the set

$$D = \{(d_s) \in \mathbb{N}^r : \sum_{s=1}^{r} d_s i^{(s)} \in \pi(T_0) - \pi(T_0)\}$$

is finite since the set $\pi(T_0) - \pi(T_0)$ is finite due to finiteness of $\pi(T_0)$. Set $m_0 = \max\{\sum_{s=1}^{r} d_s : (d_s) \in D\} + 1$. We show that m_0 satisfies our assertion.

Fix an arbitrary $m \geq m_0$ and suppose that $t'_0 = t_0 t_1 \ldots t_m$, for $t_0, t'_0 \in T_0$ and $t_1, \ldots, t_m \in T$. Then

$$\pi(t'_0) - \pi(t_0) = \sum_{p=1}^{m} \pi(t_p) = \sum_{s=1}^{r} d_s i^{(s)}$$

where $d_s = |\{p : \pi(t_p) = i^{(s)}\}|$, and $\sum_{s=1}^{r} d_s = m$, a contradiction. Consequently, $T_0 \cap T_{(m)} = \emptyset$ for all $m \geq m_0$, and the proof is complete. □

COROLLARY (3.6.3). *Let G be as in 3.5, $T_0 \subseteq G$ an arbitrary finite subset and $T = T(S)$, where $S = \operatorname{supp} N$ for some R-module N. Then there exists $m_0 \in \mathbb{N}$ such that $T_0 \cap T_{(m)} = \emptyset$ for all $m \geq m_0$.*

Proof. Follows immediately form the proposition, since by the inclusion $\pi(T(S)) \subseteq \pi(G_0 \cdot G_0^{-1})$ the set $\pi(T)$ is finite. □

(3.7) The proof of Theorem 3.1.1 (b) is based on the following general fact.

LEMMA (3.7.1). *Let U and U' be a pair of a locally finite dimensional R-modules such that* $\dim_k U(x) = dim_k U'(x)$ *for all $x \in \operatorname{ob} R$. Assume that $U = \bigoplus_{p \in P} U_p$ is a decomposition of U into a direct sum of nonisomorphic indecomposable finite-dimensional R-modules. Then $U \cong U'$ if and only if U_p is isomorphic to a direct summand of U', for every $p \in P$.*

Proof. We have only to prove the implication "⇐". Assume that U_p is isomorphic to a direct summand of U', for every $p \in P$. Let $U' = \bigoplus_{p' \in P'} U'_{p'}$ be a decomposition of U' into a direct sum of indecomposable modules (always exists, see [6, Lemma 2.1], [13, Lemma 2.1]). Then there exists a function $(\cdot)' : P \to P'$ such that $U_p \cong U'_{p'}$, for $p \in P$, since endomorphism rings of the modules U_p are local. This function is injective, since modules U_p are pairwise nonisomorphic. Hence,

$U' \cong (\bigoplus_{p\in P} U_p) \oplus U'' = U \oplus U''$, for some direct summand U'' of U'. Consequently, $U'' = 0$ and $U \cong U'$, since $\dim_k U(x) = \dim_k U'(x)$ for all $x \in \operatorname{ob} R$. □

Proof of Theorem 3.1.1 (b). To prove our assertion we show that ${}^gN$ is isomorphic to a direct summand of $\tilde{N}$, for every $g \in G$. Since $\dim_k \tilde{N}(x) = \dim_k(\bigoplus_{g\in G} {}^gN(x))$ for all $x \in \operatorname{ob} R$, we obtain immediately the required the isomorphism $\tilde{N} \cong \bigoplus_{g\in G} {}^gN$ from the lemma above.

Fix $g \in G$. We set $T_0 = \{h \in G : hS \cap \widehat{gS} \neq \emptyset\}$, $T = T(S)$ and $\tilde{N}_m = \tilde{N}(T_0, T)_m$, for $m \in \mathbb{N}$, where $S = \operatorname{supp} {}^gN$. The set T_0 is finite, since $T_0 = \bigcup_{x\in\widehat{gS}} T_0(x)$, where $T_0(x) = \{h \in G : {}^hN(x) \neq 0\}$.

Let m_0 be as in Corollary 3.6.3. Then we have $T_0 \subseteq T'$, where $T' = T_{[0,m_0)} = T_{[0,\infty)} \setminus T_{[m_0,\infty)}$. Consequently, the canonical embedding $\tilde{N}_0 \hookrightarrow \tilde{N}$ induces the isomorphism

$$(\tilde{N}_{[0,m_0)})_{|\widehat{gS}} \cong \tilde{N}_{|\widehat{gS}}$$

since $\tilde{N}_{[0,m_0)}(x) = \bigoplus_{h\in T'} {}^hN(x)$ and $\tilde{N}(x) = \bigoplus_{h\in T_0(x)} {}^hN(x)$, for any $x \in \widehat{gS}$ (see Lemma 3.5.1(b)). We show that under our assumptions $\tilde{N}_{[0,m_0)}$ has a decomposition

$$(*) \qquad \tilde{N}_{[0,m_0)} \cong \bigoplus_{h\in T'} {}^hN$$

For this aim consider the family of R-submodules $\tilde{N}'^{(j)} = (\tilde{N}^{(j)} + \tilde{N}_{m_0}) \cap \tilde{N}_0$, $j \in L$, of $\tilde{N}_0$ containing $\tilde{N}_{m_0}$. It is clear that $\tilde{N}'^{(j)}(x) = \bigoplus_{h\in T^{(j)}} {}^hN(x)$, where $T^{(j)} = (T_{[m_0,\infty)} \cup G^{\leq j}) \cap T_{[0,\infty)} = T_{[m_0,\infty)} \cup (T_{[0,\infty)} \cap G^{\leq j})$.

Let $\pi(T') = \{j_1, \ldots, j_r\}$, where $j_1 < \cdots < j_r$. (The set T' is finite since $T' \subseteq \bigcup_{0\leq m<m_0} T_{(m)}$). Then we get an ascending chain

$$\tilde{N}'^{(j_0)} \subseteq \tilde{N}'^{(j_1)} \subseteq \cdots \subseteq \tilde{N}'^{(j_r)}$$

of submodules of $\tilde{N}_0$, where $\tilde{N}'^{(j_0)} = \tilde{N}_{m_0}$. Observe that $\tilde{N}'^{(j_r)} = \tilde{N}_0$, since $T' \subseteq T_{[0,\infty)} \cap G^{\leq j_r}$, and that $(\tilde{N}'^{(j_s)}/\tilde{N}'^{(j_{s-1})})(x) = \bigoplus_{h\in T'_s} {}^hN(x)$ for all $x \in \operatorname{ob} R$, where $T'_s = T \cap \pi^{-1}(j_s)$ for any $s = 1, \ldots, r$. Moreover, $\tilde{N}'^{(j_s)}/\tilde{N}'^{(j_{s-1})} \cong \bigoplus_{h\in T'_s} {}^hN$, by the arguments as in the proof of Proposition 3.2.2(b).

Consider the submodule filtration

$$(**) \qquad 0 \subseteq \tilde{N}'^{(j_1)}/\tilde{N}'^{(j_0)} \subseteq \cdots \subseteq \tilde{N}'^{(j_r)}/\tilde{N}'^{(j_0)} = \tilde{N}_{[0,m_0)}$$

of the R-module $\tilde{N}_{[0,m_0)}$. Applying the canonical exact sequences

$$0 \to \tilde{N}'^{(j_{s-1})}/\tilde{N}'^{(j_0)} \to \tilde{N}'^{(j_s)}/\tilde{N}'^{(j_0)} \to \tilde{N}'^{(j_s)}/\tilde{N}'^{(j_{s-1})} \to 0$$

associated to $(**)$, the isomorphisms above and the Ext-vanishing assumption, one easily proves by induction on $s = 1, \ldots, r$, that $\tilde{N}'^{(j_s)}/\tilde{N}'^{(j_0)} \cong \bigoplus_{h\in\bigcup_{s'=1}^{s} T'_{s'}} {}^hN$. In particular, we obtain the required isomorphisms $(*)$, since $T' = \bigcup_{s'=1}^{r} T'_{s'}$.

Now we complete our proof. Note that ${}^gN_{|\widehat{gS}}$ is a direct summand of $(\tilde{N}_{[0,m_0)})_{|\widehat{gS}}$ ($\cong \tilde{N}_{|\widehat{gS}}$), since g belongs to T_0. Hence, by [12, Lemma 2], ${}^gN$ is isomorphic to a direct summand of $\tilde{N}$, so our claim and the assertion (b) are proved. □

REMARK (3.7.2). *In fact we proved that if N is an arbitrary R-module then for any finite $T_0 \subseteq G$ ($T = T(S)$, $S = \operatorname{supp} N$) there exists m_0 such that:*

- $\tilde{N}_{[0,m_0)}(x) = \bigoplus_{h\in T'} {}^hN(x)$, *for any* $x \in \operatorname{ob} R$,

- $\tilde{N}_{[0,m_0)}$ *admits an R-submodule filtration* $\tilde{N}^{(0)}_{[0,m_0)} \subseteq \tilde{N}^{(1)}_{[0,m_0)} \subseteq \cdots \subseteq \tilde{N}^{(r)}_{[0,m_0)}$ *with* $\tilde{N}^{(0)}_{[0,m_0)} = \{0\}$, $\tilde{N}^{(r)}_{[0,m_0)} = \tilde{N}_{[0,m_0)}$ *and* $\tilde{N}^{(s)}_{[0,m_0)}/\tilde{N}^{(s-1)}_{[0,m_0)} \cong \bigoplus_{h\in T'\cap\pi^{-1}(j_s)} {}^{h}N$ *for* $1 \le s \le r$,

where $T' \subseteq G$ is a subset containing T_0 such that $\pi(T')$ is finite and consists of $j_1 < \cdots < j_r$.

(3.8) Next we discuss the problem of preserving Auslander Reiten structure by the push-down functor F_λ associated with an almost Galois coverings F. We start by the following rather general fact.

LEMMA (3.8.1). *Let $F : R \to R$ be a G-covering. Assume that there exists $g \in G$ such that $g({}_x\alpha^{\cdot}_y) = {}^{\cdot}_{gx}\alpha_{gy}$, for all $\alpha \in R'(b,a)$, $a,b \in \mathrm{ob}R$, then:*
(a) the functors $F_\lambda \circ {}^{g}(-), F_\rho : mod\ R \to mod\ R'$ are isomorphic,
(b) $F_\lambda(D_R(R(x,-)) \cong D_{R'}(R'(Fx,-))$ for any $x \in \mathrm{ob}R$,
(c) $\tau_{R'}(F_\lambda(N)) \cong F_\lambda(\tau_R({}^{g}N))$ (resp. $\tau^-_{R'}(F_\lambda(N)) \cong F_\lambda(\tau^-_R({}^{g^{-1}}N))$) for any non-projective (resp. non-injective) indecomposable N in mod R.

Proof. (a) Let N be an arbitrary module in mod R. Then, by our assumption, for any $\alpha \in R'(b,a)$ the map $(F_\rho({}^{g^{-1}}N))(\alpha) = [({}^{g^{-1}}N)({}_y\alpha^{\cdot}_x)]$ has the form

$$[N(g_x\alpha^{\cdot}_y)] = [N({}^{\cdot}_{gx}\alpha_{gy})] : \bigoplus_{x\in F^{-1}(a)} N(gx) \to \bigoplus_{y\in F^{-1}(b)} N(gy)$$

Applying the standard identifications $\eta_N(c) : \bigoplus_{z'\in F^{-1}(c)} N(z') \to \bigoplus_{z\in F^{-1}(c)} N(gz)$, $c \in \mathrm{ob}R'$, induced by the bijection $z' \mapsto g^{-1}z'$, $F_\rho({}^{g^{-1}}N)(\alpha)$ is equal to

$$F_\lambda(N)(\alpha) = [N({}^{\cdot}_x\alpha_y)] : \bigoplus_{x\in F^{-1}(a)} N(x) \to \bigoplus_{y\in F^{-1}(b)} N(y)$$

Now it is clear that the family $\eta_N = (\eta_N(c))_{c\in\mathrm{ob}R'}$, $N \in \mathrm{mod}\,R$, yields an isomorphism of the functors $F_\rho \circ {}^{g^{-1}}(-), F_\lambda : \mathrm{mod}\,R \to \mathrm{mod}\,R'$, hence $F_\lambda \circ {}^{g}(-) \cong F_\rho$.

(b) Applying (a), Corollary 1.2.3 and the isomorphisms ${}^{g^{-1}}R(x,-) \cong R(g^{-1}x,-)$, we have the following sequence of R-isomorphisms

$$F_\lambda(D_R(R(x,-))) \cong F_\rho({}^{g^{-1}}D_R(R(x,-)))$$

$$\cong F_\rho(D_R(R(g^{-1}x,-))) \cong D_{R'}(R'(F(g^{-1}x),-)) = D_{R'}(R'(Fx,-))$$

In this way the proof of (b) is complete. Note that $\theta_y \cdot F_\lambda(D_R(R(\alpha,-))) = D_{R'}(R'(F(g^{-1}\alpha),-)) \cdot \theta_x$, for any $\alpha \in R(x,y)$, where $\theta_x : F_\lambda(D_R(R(x,-))) \to D_{R'}(R'(Fx,-))$ denotes the composite isomorphism above, since $\sigma'_{x'} \circ R'(F\alpha',-) = F^{\mathrm{op}}_\lambda(R(\alpha',-)) \circ \sigma_{y'}$ for any $\alpha' \in R(x',y')$ (see 1.2).

(c) Let N be an indecomposable non-projective R-module in mod R. Fix a minimal projective presentation

$$(*) \qquad \bigoplus_{i=1}^{n} R(-,x_i) \xrightarrow{f} \bigoplus_{j=1}^{m} R(-,y_j) \to N \to 0$$

of N, where $f = [R(-,\alpha_{j,i})]$ for some $\alpha_{j,i} \in R(x_i,y_j)$, $i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,m\}$. Then the sequence $F_\lambda(*)'$ obtained from $(*)$ by applying F_λ and identifications induced by R-isomorphisms $\sigma'_z : F_\lambda(R(-,z)) \to R'(-,Fz)$, which are defined by the family $({}^z f_c)_{c\in \mathrm{ob}R}$, has the form

$$F_\lambda(*)' \qquad \bigoplus_{i=1}^{n} R'(-,Fx_i) \xrightarrow{f'} \bigoplus_{j=1}^{m} R'(-,Fy_j) \to F_\lambda(N) \to 0$$

where $f' = [R'(-,F\alpha_{j,i})]$. Clearly, the sequence $F_\lambda(*)'$ yields a projective presentation of the R'-module $F_\lambda(N)$. Consequently,

$$\tau_{R'}(F_\lambda(N)) \cong \mathrm{Ker}\ \nu_{R'}(f') = \mathrm{Ker}\ ([D_{R'}R'(F\alpha_{j,i},-)])$$

since

$$\nu_{R'}(f') : \bigoplus_{i=1}^{n} D_{R'}(R'(Fx_i,-)) \longrightarrow \bigoplus_{j=1}^{m} D_{R'}(R'(Fy_j,-))$$

has the form $\nu_{R'}(f') = [D_{R'}R'(F\alpha_{j,i},-)]$.

Now we compute $F_\lambda(\tau_R({}^gN))$. We apply for this aim the projective presentation

$${}^g(*) \qquad \bigoplus_{i=1}^{n} R(-,gx_i) \xrightarrow{{}^gf} \bigoplus_{j=1}^{m} R(-,gy_j) \to {}^gN \to 0$$

of the R-module ${}^gN$. Then $\tau_R({}^gN) \cong \mathrm{Ker}\ \nu_R({}^gf)$, where

$$\nu_R({}^gf) : \bigoplus_{i=1}^{n} D_R(R(gx_i,-)) \longrightarrow \bigoplus_{j=1}^{m} D_R(R(gy_j,-))$$

has the form $\nu_R({}^gf) = [D_R(R(g\alpha_{j,i},-))]$. Hence, we have

$$F_\lambda(\tau_R({}^gN)) \cong \mathrm{Ker}\ F_\lambda(\nu_R({}^gf)) \cong \mathrm{Ker}\ ([D_{R'}R'(F\alpha_{j,i},-)])$$

since $\theta_{gy} \cdot F_\lambda(D_R(R(g\alpha_{j,i},-))) = D_{R'}(R'(F(g^{-1}g\alpha_{j,i},-))) \cdot \theta_{gx}$ for any $\alpha_{j,i}$. In this way the proof of the isomorphism $F_\lambda(\tau_R({}^gN)) \cong \tau_{R'}(F_\lambda(N))$ is complete.

The proof of the dual assertion is similar. □

COROLLARY (3.8.2). *For any $x \in \mathrm{ob}R$, $F_\lambda(p_x)$ is a left almost split map in mod R' (see Corollary 1.2.3).*

PROPOSITION (3.8.3). *Let $F : R \to R$ be an almost Galois G-covering functor of type L. Assume that there exists $g \in G$ such that $g({}_x\dot{\alpha_y}) = {}_{gx}\dot{\alpha}_{gy}$, for all $\alpha \in R'(b,a)$, $a,b \in \mathrm{ob}R$. If the exact sequence*

$$e: \quad 0 \to N \xrightarrow{u} E \xrightarrow{v} \tau_R^-(N) \to 0$$

is an Auslander-Reiten sequence in mod R, and

(a) $\mathrm{Hom}_R(N, {}^hN) = 0$ *for all $e \neq h \in G$ such that $0 \leq \pi(g)$,*

(b) $F_\lambda(\tau_R^-({}^{g^{-1}}N)) \cong F_\lambda(\tau_R^-(N))$*,*

(c) $F_\lambda(E_i)$ is an indecomposable R'-module for every i, where $E = \bigoplus_{i=1}^n E_i$ is a decomposition of E into a direct sum of indecomposable submodules, then

$$F_\lambda(e): \quad 0 \to F_\lambda(N) \xrightarrow{F_\lambda(u)} F_\lambda(E) \xrightarrow{F_\lambda(v)} F_\lambda(\tau_R^-(N)) \to 0$$

is an Auslander-Reiten sequence in mod R'.

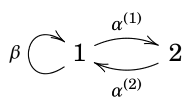
Proof. By our assumptions, the modules $F_\lambda(N)$ and $F_\lambda(\tau_R^-(N))$ are indecomposable, moreover, $F_\lambda(\tau_R^-(N)) \cong \tau_{R'}(F_\lambda(N))$ (see Theorem 3.1.1(a) and Lemma 3.8.1(c)). Consequently, the sequence $F_\lambda(e)$ does not split, since each of component maps $u_i : N \to E_i$, $i = 1,\dots,n$, is either proper monomorphism or proper epimorphism, so the same hold for the maps $F_\lambda(u_i)$. Therefore, to prove that $F_\lambda(e)$ is an Auslander-Reiten sequence in mod R' it suffices to show that for any $f \in J(\mathrm{End}_{R'}(F_\lambda(N)))$ there exists $\tilde{f} : F_\lambda(E) \to F_\lambda(N)$ such that $f = \tilde{f} \cdot F_\lambda(u)$. By Proposition 3.3.2 we have to consider to cases: $f \in J(F_\lambda(\mathrm{End}_R(N)))$ and $f \in (N, \sum_{j<0} \tilde{N}^{(j)})$. In the first case the claim follows immediately from the fact that e is an Auslander-Reiten sequence in mod R. To consider the second one fix $f \in (N, \sum_{j<0} \tilde{N}^{(j)})$. Note first that $\mathrm{supp}(\mathrm{Im} f') \subseteq \bigcup_{h:\pi(h)<0} \mathrm{supp}^h N$, since $\mathrm{Im} f' \subseteq \sum_{j<0} \tilde{N}^{(j)})$ and $\mathrm{supp} \tilde{N}^{(j)} = \bigcup_{h:\pi(h)\leq j} \mathrm{supp}^h N$, for any j, where $f' \in \mathrm{Hom}_R(N, \tilde{N})$ in the R-homomorphism corresponding to f via the isomorphism 3.3(∗). Now we set $S = \mathrm{supp} N$ and denote by N' the R-submodule of $\tilde{N}$ generated by $\tilde{N}_{|S}$. Clearly, N' is finite dimensional since $\mathrm{supp} N' \subseteq \widehat{S}$. Moreover, $\tilde{N}_{|S} = N'_{|S}$ and $\mathrm{Hom}_R(N, \tilde{N}) = \mathrm{Hom}_R(N, N')$. We show $f' \in \mathrm{Hom}_R(N, \tilde{N})$ regarded as an element of $\mathrm{Hom}_R(N, N')$ belongs to $\mathscr{J}_R(N, N')$. Suppose that this is not the case then f' has to be a split monomorphism, since $\dim_k N \leq \dim_{\mathrm{k}} N'$. Then we have $\mathrm{supp} N \subseteq \bigcup_{h:\pi(h)<0} \mathrm{supp}^h N$, but on other hand for any $x \in \mathrm{supp} N$, $h_0 x$ does not belongs to $\bigcup_{h:\pi(h)<0} \mathrm{supp}^h N$, where h_0 is such that $N(h_0 x) \neq 0$ and $\pi(h) \leq \pi(h_0)$ if $N(hx) \neq 0$, a contradiction. Consequently, f' belongs to $\mathscr{J}_R(N, N')$ and there exists $\tilde{f}' \in \mathrm{Hom}_R(E, N') \subseteq \mathrm{Hom}_R(E, \tilde{N})$ such that $\tilde{f}' \cdot u = f'$, since e is an Auslander-Reiten sequence in mod R. From the adjointness formula for the pair $(F_\lambda, F_\bullet)$ of functors we infer that $\tilde{f} \cdot F_\lambda(u) = f$ and the proof is complete. □

4. An interesting class of coverings

In this section we briefly present certain interesting (also with respect to properties of the fuctors $(F_\lambda, F_\bullet)$) family of almost Galois G-coverings of integral type between bounded quiver categories of the form as in 2.3, where G is an infinite cyclic group.

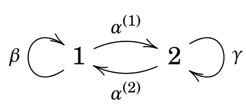
(4.1) Let $R_t^{(i)} = R(Q^{(i)}, I_t^{(i)})$, $t \in k$, for $i = 1, \ldots, 7$, be a list of algebras (with some additional data) defined as follows:

$Q^{(1)}$:



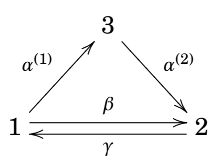
$I_t^{(1)} = (\beta^2\alpha^{(1)}, \alpha^{(2)}\beta^2, \alpha^{(1)}\alpha^{(2)}\alpha^{(1)}, \alpha^{(2)}\alpha^{(1)}\alpha^{(2)}, \alpha^{(1)}\alpha^{(2)} - t\alpha^{(1)}\beta\alpha^{(2)}, \alpha^{(2)}\alpha^{(1)} - \beta^3)$, (we also set $w_1 := \varepsilon_1, w_2 := \alpha^{(1)}$, $\omega := \beta$ and $r := 3$);

$Q^{(2)}$:



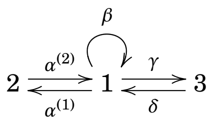
$I_t^{(2)} = I^{(2)}(u)_t = (\beta^4, \beta^2\alpha^{(1)}, \alpha^{(2)}\beta^2, \alpha^{(1)}\alpha^{(2)} - \beta^2 + t\beta^3, \alpha^{(2)}\alpha^{(1)} - s\gamma^2, \beta\alpha^{(1)} - \alpha^{(1)}\gamma, \alpha^{(2)}\beta - \gamma\alpha^{(2)})$, where $s \in k \setminus \{0,1\}$ is fixed, $(w_1 := \beta, w_2 := \alpha^{(1)}, \omega := \beta, r := 2)$;

$Q^{(3)}$:



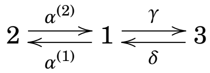
$I_t^{(3)} = (\alpha^{(1)}\alpha^{(2)} - \beta\gamma\beta, (\gamma\beta)^3\gamma, \alpha^{(2)}\gamma\alpha^{(1)}\alpha^{(2)}, \alpha^{(1)}\alpha^{(2)}\gamma\alpha^{(1)}, \alpha^{(2)}\gamma\alpha^{(1)} - t\alpha^{(2)}\gamma\beta\gamma\alpha^{(1)})$, $(w_1 := \varepsilon_1, w_2 := \beta, w_3 := \alpha^{(1)}, \omega := \beta\gamma, r := 2)$;

$Q^{(4)}:$



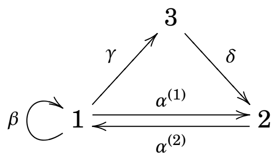
$I_t^{(4)} = (\alpha^{(1)}\alpha^{(2)} - \beta^2, \beta^3 - \gamma\delta, \alpha^{(2)}\gamma, \delta\alpha^{(1)}, \beta\gamma, \delta\beta, \alpha^{(1)}\alpha^{(2)}\alpha^{(1)}, \alpha^{(2)}\alpha^{(1)}\alpha^{(2)}, \alpha^{(2)}\alpha^{(1)} - t\alpha^{(2)}\beta\alpha^{(1)})$, $(w_1 := \beta, w_2 := \alpha^{(1)}, w_3 := \gamma, \omega := \beta, r := 2)$;

$Q^{(5)}:$



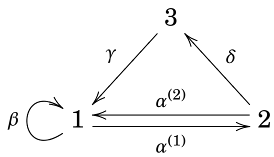
$I_t^{(5)} = (\alpha^{(2)}\gamma\delta\gamma, \delta\gamma\delta\alpha^{(1)}, \alpha^{(1)}\alpha^{(2)}\alpha^{(1)}, \alpha^{(2)}\alpha^{(1)}\alpha^{(2)}, \alpha^{(2)}\alpha^{(1)} - t\alpha^{(2)}\gamma\delta\alpha^{(1)}, \alpha^{(1)}\alpha^{(2)} - (\gamma\delta)^2)$, $(w_1 := \varepsilon_1, w_2 := \alpha^{(1)}, w_3 := \gamma, \omega := \gamma\delta, r := 2)$;

$Q^{(6)}:$



$I^{(6)} = (\alpha^{(2)}\alpha^{(1)} - t\alpha^{(2)}\beta\alpha^{(1)}, \beta\gamma, \beta\alpha^{(1)} - \gamma\delta, \delta\alpha^{(2)}\beta, \alpha^{(1)}\alpha^{(2)} - \beta^2, \delta\alpha^{(2)}\alpha^{(1)}, \alpha^{(1)}\alpha^{(2)}\alpha^{(1)}, \alpha^{(2)}\alpha^{(1)}\alpha^{(2)})$, $(w_1 := \varepsilon_1, w_2 := \alpha^{(1)}, w_3 := \gamma, \omega := \beta, r := 2)$;

$Q^{(7)}:$



$I_t^{(7)} = (\alpha^{(2)}\alpha^{(1)} - t\alpha^{(2)}\beta\alpha^{(1)}, \gamma\beta, \alpha^{(2)}\beta - \delta\gamma, \beta\alpha^{(1)}\delta, \alpha^{(1)}\alpha^{(2)} - \beta^2, \alpha^{(2)}\alpha^{(1)}\delta, \alpha^{(1)}\alpha^{(2)}\alpha^{(1)}, \alpha^{(2)}\alpha^{(1)}\alpha^{(2)})$, $(w_1 := \varepsilon_1, w_2 := \alpha^{(1)}, w_3 := \alpha^{(1)}\delta, \omega := \beta, r := 2)$.

Set $\bar{R}_{(i)} = R_0^{(i)}$, $R'_{(i)} = R_1^{(i)}$ and $\tilde{R}_{(i)} = R(\tilde{Q}^{(i)}, \tilde{I}^{(i)})$, where $(\tilde{Q}^{(i)}, \tilde{I}^{(i)})$ is a universal covering of $(Q^{(i)}, I_0^{(i)})$. Now we can formulate the following result extending previous examples (see 2.3).

THEOREM (4.1.1). *For each $i = 1, \ldots 7$, there exists an almost Galois G-covering $F'_{(i)} : \tilde{R}_{(i)} \to R'_{(i)}$ of integral type, where $G = \Pi(Q^{(i)}, I_0^{(i)})$ is an infinite cyclic group.*

To prove this result we first establish some notation referring to that from 2.3.

(4.2) Let (Q, I) be a connected bounded quiver with a fundamental group $G = \Pi(Q, I)$ and universal covering $(\tilde{Q}, \tilde{I})$, where $\Pi(Q, I) = \Pi((Q, I), a_\bullet)$ and $\tilde{Q} = \tilde{Q}(a_\bullet)$, for some fixed $a_\bullet \in Q_0$. Suppose that $\{w_a\}$, $a \in Q_0$, is a fixed collection of paths such that $w_a \in \mathscr{W}(a_\bullet, a)$. Then $[w_a]$, $a \in Q_0$, forms a set of representatives of fibers of the canonical Galois functor $\bar{F} : \tilde{R} \to \bar{R}$ and we have $\bar{F}^{-1}(a) = G[w_a]$, for any $a \in Q_0$, where $\bar{R} = R(Q, I)$ and $\tilde{R} = R(\tilde{Q}, \tilde{I})$.

For any $g=[u]\in G$ and vertex $a\in Q_0$ we set $a_g = g[w_a] = [uw_a]\in Q_0$. If $\delta\in \mathscr{P}(b,a)$ is a path in Q then the lifting

$$\tilde{\delta} = ([uw_b],\delta) : [uw_b] \to [uw_b\delta]$$

of δ to $\tilde{Q}$ we denote by $\tilde{\delta}_g$. Note that we have $\tilde{\delta}_g \in \tilde{\mathscr{P}}(b_g, a_{g\,\deg(\delta)})$.

Suppose that G is an infinite cyclic group and $g=[u]$ a fixed generator of G, for some $u\in \mathscr{W}(a_\bullet,a_\bullet)$. Then we have the identification $\mathbb{Z}=G$, given by $i\mapsto g^i$ for $i\in\mathbb{Z}$, and we write a_i (resp. $\tilde{\delta}_i$) instead of a_{g^i} (resp. $\tilde{\delta}_{g^i}$); moreover, $\tilde{\delta}_i \in \tilde{\mathscr{P}}(s(\delta)_i, t(\delta)_{i+\deg(\delta)})$.

Proof of Theorem 4.1.1. We will apply Theorem 2.3.3, proceeding case by case.

We start by observing that for each i, the group $\Pi(Q^{(i)}, I_0^{(i)}) = \Pi((Q^{(i)}, I_0^{(i)}), a_\bullet)$ is an infinite cyclic group with generator $[\omega]$, where for $a_\bullet = a_\bullet(i)$ we always take the vertex 1 and $\omega=\omega(i)$ as in 4.1. Next to construct the required functors $F'_{(i)}$ we define the functors $F_{(i)} : R(\tilde{Q}^{(i)}) \to R'_{(i)}$. We use for this aim the notation established above, which refers to the sets $\{[w_a]\}_{a\in Q_0}$ and the generators ω fixed in 4.1. For any arrow κ in $Q^{(i)}$, different from $\alpha^{(1)}$ and $\alpha^{(2)}$, we set $F_{(i)}(\tilde{\kappa}) = \kappa + I_1^{(i)}$; the values $F_{(i)}(\tilde{\kappa})$ for the remaining cases are given in the table below

i	$F'_{(i)}(\tilde{\alpha}_n^{(1)})$	$F'_{(i)}(\tilde{\alpha}_n^{(2)})$
1	$\alpha^{(1)} - \lfloor\frac{n+1}{3}\rfloor\beta\alpha^{(1)}$	$\alpha^{(2)} + \lfloor\frac{n}{3}\rfloor\alpha^{(2)}\beta$
2	$\alpha^{(1)} - \lfloor\frac{n}{2}\rfloor\beta\alpha^{(1)}$	$\alpha^{(2)} + \lfloor\frac{n+1}{2}\rfloor\alpha^{(2)}\gamma$
3	$\alpha^{(1)} - \lfloor\frac{n}{2}\rfloor\beta\gamma\alpha^{(1)}$	$\alpha^{(2)} + \lfloor\frac{n}{2}\rfloor\alpha^{(2)}\gamma\beta$
4	$\alpha^{(1)} - \lfloor\frac{n}{2}\rfloor\beta\alpha^{(1)}$	$\alpha^{(2)} + \lfloor\frac{n-1}{2}\rfloor\alpha^{(2)}\beta$
5	$\alpha^{(1)} - \lfloor\frac{n}{2}\rfloor\gamma\delta\alpha^{(1)}$	$\alpha^{(2)} + \lfloor\frac{n}{2}\rfloor\alpha^{(2)}\gamma\delta$
6	$\alpha^{(1)} - \lfloor\frac{n}{2}\rfloor\beta\alpha^{(1)}$	$\alpha^{(2)} + \lfloor\frac{n}{2}\rfloor\alpha^{(2)}\beta$
7	$\alpha^{(1)} - \lfloor\frac{n}{2}\rfloor\alpha^{(1)}\beta$	$\alpha^{(2)} + \lfloor\frac{n}{2}\rfloor\beta\alpha^{(2)}$

(more precisely, we have to take cosets modulo $I_1^{(i)}$ of the elements from $R(Q^{(i)})$, which are indicated in the tabele). Now one has to verify the assumptions of Theorem 2.3.3. We leave all necessary computations to the reader. □

(4.3) The almost Galois covering functors from the family considered above behave in a more regular way as usually (cf. Theorem 3.1.1).

THEOREM (4.3.1). *Let $F' : \tilde{R} \to R'$ be an almost Galois covering of the form $F' = F'_{(i)}$, for $i=1,\dots 7$. Then the functor F' has following properties:*

(a) For any $x,y \in \mathrm{ob}R'$, $\alpha\in R'(y,x)$, and $x_i\in F'^{-1}(x)$, $y_j \in F'^{-1}(y)$ we have $g({}_{x_i}\dot{\alpha}_{y_j}) = {}_{gx_i}\dot{\alpha}_{gy_j}$, where $g=[\omega]^{-1} (=-1)$; in consequence, $F'_\rho \circ {}^1(-) \cong F'_\lambda$ (cf. 3.8).

(b) $F'_\bullet \cong {}^rF'_\bullet$, where $r=r(i)$ is as in 4.1.

(c) For N in mod $\tilde{R}$, the R-isomorphism $F'_\bullet F'_\lambda(N) \cong \bigoplus_{j\in\mathbb{Z}} {}^jN$ holds, provided $\mathrm{Ext}^1_{\tilde{R}}({}^jN, N) = 0$ for all $j=1,\dots,r-1$.

This result is a specialization of the other one dealing with some more general situation described by certain rather complicated system of combinatorial axioms. The proof, even in the cases considered above, is very technical; therefore we do not present it here.

REMARK (4.3.2). *(a) The considered algebras* $R'_{(i)}$, $i = 1,\dots,7$, *belong to ten element list of basic (nonstandard) selfinjective algebras socle equivalent to selfinjective algebras of tubular type, which themselves are not of this form given in* [1]. *One of the remaining three admits some other kind of covering which is close to these discussed above. The other two seems to behave in a quite different way.*

(b) The theorem remains valid, if for $R'_{(i)}$ *we admit the algebras* $R_t^{(i)}$, *for* $t \in k \setminus \{0\}$. *The phenomena related to this fact has a deeper meaning (see* [9]*).*

(c) The general situation announced in Theorem 4.3.1 together with some other interesting examples of almost Galois coverings will be discussed in the separate forthcoming publication [10].

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THE CONVOLUTION INDUCED TOPOLOGY ON DUAL OF HYPERGROUP ALGEBRAS

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ABSTRACT. Let K be a hypergroup with Haar measure. Let $L^{\infty}(K)$ be the Banach algebra of essentially bounded measurable complex valued functions on K. The purpose of this paper is to initiate a systematic study of a locally convex topology on $L^{\infty}(K)$, called the τ_c-topology. We study the τ_c-topology and compare it with the norm topology and weak*-topology. For a hypergroup K, the τ_c-topology is not weaker than the weak*-topology and not stronger than the norm topology on $L^{\infty}(K)$.

1. Introduction

The theory of hypergroups was initiated by Dunkl [7], Jewett [13] and Spector [27] and has received a good deal of attention from harmonic analysts. A good deal of attention was paid to study amenability of hypergroups (see [22], [24] and [26]). Almost periodic functions and weakly almost periodic functions were investigated by Lasser [15] and Wolfenstetter [28].

A hypergroup K is roughly a locally compact space whose space of complex regular Borel measures forms an algebra similar to the convolution algebra of a locally compact group G. It is still unknown whether an arbitrary hypergroup admits a left Haar measure, but all known examples such as commutative hypergroups and central hypergroups do [1].

In [9] the author studied the τ_c-topology on the dual $M_a(S)^*$ of the semigroup algebra $M_a(S)$ of a locally compact foundation semigroup S, that is, the weak topology under all right multipliers induced by measures in $M_a(S)$ (for more on τ_c-topology, see [3]). During this investigation, it is shown that $f \in M_a(S)^*$ is τ_w-almost periodic if and only if $f\mu \in \mathrm{wap}(M_a(S))$ for every $\mu \in M_a(S)$ (see Theorem 4 in [9]). The aim of this paper is to study a locally convex topology (called the τ_c-topology) on a locally compact hypergroup. The question as to whether there are neighborhoods in the τ_c-topology which are also neighborhoods in the weak*-topology leads us to consider the apparently completely different problem of determining those functions $\phi \neq 0$ in $L^1(K)$ such that all conjugates of ϕ are in a finite dimensional subspace of $L^1(K)$. We investigate the relation between τ_c-topology and weak*-topology, and show that if K is a compact hypergroup, then $L^1(K)$ is the dual of $(L^{\infty}(K), \tau_c)$. Finally, we show that a τ_c-τ_c continuous operator on $L^{\infty}(G)$ commutes with the conjugate operators if and only if it commutes with the convolution operators.

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2. Preliminaries and Notations

Let K be a hypergroup with a left Haar measure λ, and a modular function Δ. For basic definitions and results on hypergroups we shall follow [1]. Symbols such as $\int \cdots dx$ will always denote integration with respect to λ. For a measurable subset A, we denote by $|A|$ the Haar measure of A and by χ_A the characteristic function on A. The involution of K is denoted by $x \mapsto \bar{x}$.

We begin by recalling, of course, the most intimately related objects: the hypergroup algebra $L^1(K)$ and the measure algebra $M(K)$. The first one is the Banach space of all measurable complex-valued functions ϕ on K satisfying $\int |\phi(x)| dx < \infty$. Let $L^\infty(K)$ be the Banach algebra of the measurable complex-valued functions that are bounded almost everywhere with respect to λ. We denote by $C_0(K)$ the subspace of those continuous functions vanishing at infinity. Following Li and Pier [20], we define the convolution of two functions $\phi, \psi \in L^1(K)$ as follows:

$$\psi \circledast \phi(x) = \int \phi(\bar{y} * x * y)\Delta(y)\psi(y) dy$$

where $x \in K$. The Banach space $L^1(K)$ is a Banach algebra under this product. Let $L^p(K)$ $(1 \le p < \infty)$ denote the Banach space of measurable functions f on K such that $|f|^p$ is integrable. Then $L^p(K)$ is a Banach left $L^1(K)$-module with norm $\|f\|_p = \int |f(x)|^p dx$ and module multiplication defined by

$$\phi \circledast f(x) = \int f(\bar{y} * x * y)\Delta(y)^{\frac{1}{p}} \phi(y) dy.$$

For $1 \le p < \infty$ and $x, y, z \in K$, we also put

$$A_{(y,z)} f(x) = f(\bar{y} * x * z)\Delta(z)^{\frac{1}{p}} \quad \text{and} \quad {}_y f_y(x) = f(\bar{y} * x * y).$$

Other Banach algebras (usually larger than $L^1(K)$ and $M(K)$) can also be associated with K. For instance, the second dual, $L^1(K)^{**}$, of $L^1(K)$ is a Banach algebra with an Arens product (for more information see [4], [5] and [18]). This product is obtained by letting first

$$\langle f\psi, \phi\rangle = \langle f, \psi \circledast \phi\rangle \quad \text{for all } \phi, \psi \in L^1(K) \text{ and } f \in L^\infty(K).$$

Then, for F and E in $L^1(K)^{**}$,

$$\langle Ff, \psi\rangle = \langle F, f\psi\rangle, \ \langle EF, f\rangle = \langle E, Ff\rangle \text{ for all } \psi \in L^1(K), \text{ and } f \in L^\infty(K).$$

As far as possible, we follow [1], [14] and [21] in our notation and refer to [25] for basic functional analysis.

3. Functions in $L^1(K)$ with finite-dimensional span

Our starting point of this section is the following lemma whose proof is straightforward.

LEMMA (3.1). *Let $f \in L^\infty(K)$ and $\phi, \psi \in L^1(K)$. Then*

(1) *for every fixed $y \in K$, $\psi \circledast {}_y\phi_y = \Delta(\bar{y})^2 \psi_{\bar{y}} \circledast \phi$;*
(2) *the mapping $y \mapsto A_{(y,y)}\phi$ is continuous;*
(3) $\langle f, \psi \circledast \phi\rangle = \int \langle f, A_{(y,y)}\phi\rangle d\psi(y)$.

Proof. The proof of 1) is easy, so we omit it.

2) This follows directly from Lemma 2.2B and Lemma 5.4H in [13].

3) Let $f \in L^\infty(K)$, $\psi,\phi \in L^1(K)$, $\psi \geq 0$ and we may assume that $C = \mathrm{supp}\psi$ is compact. Then $h : C \to L^1(K)$ defined by $h(x) = A_{(x,x)}\phi$ is continuous. So by Theorem 3.20 and Theorem 3.27 in [25], we can write $\int_C h(x)d\psi(x) \in L^1(K)$, i.e.; $\int_C A_{(x,x)}\phi d\psi(x) \in L^1(K)$. Now, let $g \in C_0(K)$. Then

$$\begin{aligned}
\langle g, \psi \circledast \phi \rangle &= \int g(x)\psi \circledast \phi(x)dx \\
&= \int g(x) \int_C \phi(\bar{y} * x * y)\psi(y)\Delta(y)dydx \\
&= \int_C \int \Delta(y)\phi(\bar{y} * x * y)g(x)dxd\psi(y) \\
&= \int_C \langle g, A_{(y,y)}\phi\rangle d\psi(y) = \langle g, \int_C A_{(y,y)}\phi d\psi(y)\rangle.
\end{aligned}$$

It follows that $\psi \circledast \phi = \int A_{(y,y)}\phi d\psi(y)$. By 3.26 in [25], we have

$$\begin{aligned}
\int_C \langle f, A_{(y,y)}\phi\rangle d\psi(y) &= \langle f, \int_C A_{(y,y)}\phi d\psi(y)\rangle \\
&= \langle f, \psi \circledast \phi\rangle.
\end{aligned}$$

This completes our proof. □

Let K be a hypergroup and for each ϕ in $L^1(K)$ define $\rho_\phi : L^\infty(K) \to [0,\infty)$ by $\rho_\phi(f) = \|\phi f\|_\infty$. Then ρ_ϕ is a seminorm. Let τ_c be the topology on $L^\infty(K)$ that has as a subbase the sets $\{h \in L^\infty(K);\ \rho_\phi(h - f) < \epsilon\}$, where $\phi \in L^1(K)$, $f \in L^\infty(K)$, and $\epsilon > 0$. Thus a subset U of $L^\infty(K)$ is open if and only if for every $f \in U$ there are $\phi_1,...,\phi_n$ in $L^1(K)$ and $\epsilon_1,...,\epsilon_n > 0$ such that

$$\bigcap_{i=1}^{n} \left\{h \in L^\infty(K);\ \rho_{\phi_i}(h - f) < \epsilon_i\right\} \subseteq U.$$

It is easy to see that

$$\bigcap_{\phi \in L^1(K)} \left\{f \in L^\infty(K);\ \rho_\phi(f) = 0\right\} = \{0\}.$$

Let $\mathscr{B}$ be the collection of all finite intersections of the sets $\{f \in L^\infty(K);\ \rho_\phi(f) < \epsilon\}$. By Theorem 1.37 in [25], $\mathscr{B}$ is a convex balanced local base for topology τ_c on $L^\infty(K)$, where turns $L^\infty(K)$ into a locally convex space. Let $\phi \in L^1(K)$. Since $f \mapsto \phi f$ is a continuous linear mapping of $L^\infty(K)$ into $L^\infty(K)$, so $\{f \in L^\infty(K);\ \rho_\phi(f) < \epsilon\}$ is open. This says that $\tau_c \leq \|.\|$. Since every weak*-neighborhood of 0 contains a neighborhood of the form $\left\{f;\ |\langle\phi_i, f\rangle| < \epsilon_i \text{ for all } 1 \leq i \leq n\right\}$ where $\phi_i \in L^1(K)$ and $\epsilon_i > 0$, it is easy to see that weak*-topology is weaker than τ_c. Indeed, let $\mathscr{U}$ be the collection of all compact neighborhoods of e and order $\mathscr{U}$ by reverse inclusion. Let $\mathscr{U} = \{U_i;\ i \in I\}$ where $i \leq j$ if and only if $U_j \subseteq U_i$. For each i in I put $e_i = \frac{\chi_{U_i}}{|U_i|}$, so $e_i \geq 0$ and $\int e_i(x)dx = 1$. Let $f \in \{f \in L^\infty(K);\ \|\phi f\|_\infty < \frac{\epsilon}{4}\}$. For every i,

$$\begin{aligned}\big|\langle f, e_i \circledast \phi\rangle - \langle f,\phi\rangle\big| &\leq \frac{1}{|U_i|}\int\int_{U_i} |\phi(\bar{y} * x * y)\Delta(y) - \phi(x)|dy|f(x)|dx \\ &= \frac{1}{|U_i|}\int_{U_i}\int |\phi(\bar{y} * x * y)\Delta(y) - \phi(x)||f(x)|dxdy \\ &\leq \frac{1}{|U_i|}\int_{U_i} \|{}_y\phi_y\Delta(y) - \phi\Delta(y)\|_1\|f\|_\infty dy \\ &+ \frac{1}{|U_i|}\int_{U_i} \|\phi\Delta(y) - \phi\|_1\|f\|_\infty dy.\end{aligned}$$

Since both mappings $y \mapsto \|{}_y\phi_y\Delta(y) - \phi\Delta(y)\|_1$ and $y \mapsto \|\phi\Delta(y) - \phi\|_1$ from K into $\mathbb{C}$ are continuous (see Lemma 2.2B and Lemma 5.4H in [13]), we have $|\langle f,\phi\rangle| < \epsilon$. This shows that

$$\{f \in L^\infty(K);\ \|\phi f\|_\infty < \frac{\epsilon}{4}\} \subseteq \{f \in L^\infty(K);\ |\langle f,\phi\rangle| < \epsilon\}.$$

Consequently the weak*-topology is weaker than τ_c.

THEOREM (3.2). *Let ϕ be a nonzero element in $L^1(K)$. The following properties are equivalent:*

(1) *there exists an $\epsilon > 0$ such that $\{f \in L^\infty(K);\ \rho_\phi(f) < \epsilon\}$ is a weak*-neighborhood of zero;*
(2) *$\{A_{(x,x)}\phi;\ x \in K\}$ is part of a finite-dimensional subspace of $L^1(K)$;*
(3) *given $\epsilon > 0$, there exists $x_1,...,x_n$ in K and $\delta > 0$ such that for $f \in L^\infty(K)$, the inequality $|\langle f, A_{(x_i,x_i)}\phi\rangle| < \delta$ for $i = 1,...,n$ implies that $\|\phi f\|_\infty < \epsilon$.*

Proof. Let $\{f \in L^\infty(K);\ \rho_\phi(f) < \epsilon\}$ be a weak *-neighborhood of zero. Then we may find functions $\phi_1,...,\phi_m$ in $L^1(K)$ and $\delta > 0$ such that, whenever $f \in L^\infty(K)$ and $|\int f(y)\phi_i(y)dy| < \delta$ for all i, then $\|\phi f\|_\infty < \epsilon$. Each ϕ_i determines a linear functional on $L^\infty(K)$; call N the intersection of their kernels. For each positive integer n, $nf \in N$ whenever $f \in N$. It follows that $\|\phi f\|_\infty < \frac{\epsilon}{n}$ for any $n \in \mathbb{N}$. This shows that $\phi f = 0$ for $f \in N$.

Let $x \in K$ and $f \in N$. Then $\mathcal{U}$, the collection of all symmetric compact neighborhoods of e, ordered by inclusion (i.e., for $U_1, U_2 \in \mathcal{U}$, we write $U_1 \leq U_2$ if and only if $U_2 \subseteq U_1$) form a directed set. For each $U \in \mathcal{U}$, put $e_U = \frac{\chi_U}{|U|}$. By Lemma (3.1), for every $U \in \mathcal{U}$, we have

$$\begin{aligned}|\langle f, e_U \circledast A_{(x,x)}\phi\rangle| &= |\langle f, \Delta(\bar{x})(e_U)_{\bar{x}} \circledast \phi\rangle| = |\langle \phi f, \Delta(\bar{x})(e_U)_{\bar{x}}\rangle| \\ &\leq \|\phi f\|_\infty \|\Delta(\bar{x})(e_U)_{\bar{x}}\|_1 = 0.\end{aligned}$$

Thus $\langle f, A_{(x,x)}\phi\rangle = 0$. Consequently

$$N \subseteq \{f \in L^\infty(K);\ \langle f, A_{(x,x)}\phi\rangle = 0\}.$$

By Lemma 3.9 in [25], there are scalars $c_1,...,c_m$ such that $A_{(x,x)}\phi = c_1\phi_1 + \cdots + c_m\phi_m$. Thus (1) implies (2).

Now, assume that 2 holds. The function $x \mapsto A_{(x,x)}\phi$ from K into $L^1(K)$ is continuous (see Lemma (3.1)), and its range is part of a finite dimensional subspace V of $L^1(K)$. Let $\{A_{(x_1,x_1)}\phi,...,A_{(x_n,x_n)}\phi\}$ be a basis of V. The function $T\big(\sum_{i=1}^n c_i A_{(x_i,x_i)}\phi\big) = (c_1,...,c_n)$ from V to the n-dimensional complex space $\mathbb{C}^n$ is a continuous linear map. By Theorem 1.21 in [25], T is a homeomorphism. Note that, $\{A_{(x,x)}\phi;\ x \in K\}$ is a norm-bounded subset of a finite dimensional subspace of $L^1(K)$ and so this set is relatively compact with respect to the norm-topology. So $T\big(\{A_{(x,x)}\phi;\ x\in K\}\big)$ is bounded. This means that there exists $M>0$ such that $|T(A_{(x,x)}\phi)|\le M$ for all $x\in K$. If $x\in K$, then there are complex numbers $c_1,\dots,c_n$ such that

$$A_{(x,x)}\phi=\sum_{i=1}^{n}c_iA_{(x_i,x_i)}\phi,\ \text{ and }\ \sum_{i=1}^{n}|c_i|\le nM.$$

Now, let $\epsilon>0$ be given. Choose $\delta>0$ such that $nM\delta<\frac{\epsilon}{2}$. Let $f\in L^\infty(K)$ and $|\langle f,A_{(x_i,x_i)}\phi\rangle|<\delta$ for all $i=1,\dots,n$. Then for $x\in K$, we have

$$\begin{aligned}|\langle f,A_{(x,x)}\phi\rangle| &= |\langle f,\sum_{i=1}^{n}c_iA_{(x_i,x_i)}\phi\rangle|\\ &\le \sum_{i=1}^{n}|c_i||\langle f,A_{(x_i,x_i)}\phi\rangle|<nM\delta<\frac{\epsilon}{2}.\end{aligned}$$

By Lemma (3.1), for every $\psi\in L^1(K)$ with $\|\psi\|_1\le 1$, we have

$$\begin{aligned}|\langle f,\psi\circledast\phi\rangle| &= |\int\langle f,A_{(x,x)}\phi\rangle d\psi(x)|\\ &\le \int|\langle f,A_{(x,x)}\phi\rangle|d|\psi|(x)\le\frac{\epsilon}{2}.\end{aligned}$$

This shows that $\|\phi f\|_\infty<\epsilon$. Thus (2) implies (3).

Let $\epsilon>0$ be given. Suppose that (3) holds and $x_1,\dots,x_n$, $\delta>0$ be as in (3). Since every weak* neighborhood of zero contains a neighborhood of the form

$$\{f\in L^\infty(K);\ |\langle f,\phi_i\rangle|<r_i\ \text{ for }\ 1\le i\le m\}$$

where $\phi_i\in L^1(K)$ and $r_i>0$, hence $\bigcap_{i=1}^{n}\{f\in L^\infty(K);\ |\langle f,A_{(x_i,x_i)}\phi\rangle|<\delta\}$ is a weak*-neighborhood of zero. By assumption,

$$\bigcap_{i=1}^{n}\{f\in L^\infty(K);\ |\langle f,A_{(x_i,x_i)}\phi\rangle|<\delta\}\subseteq\{f\in L^\infty(K);\ \|\phi f\|_\infty<\epsilon\}.$$

This completes the proof that (3) implies (1). □

REMARK (3.3). *Since the hypergroup algebra $L^1(K)$ is an F-algebra as introduced by Lau in* [16]*, left amenability of $L^1(K)$ is characterized in terms of the finite-dimensional invariant subspace property of Fan* [8] *as studied in* [19].

4. Compare τ_c-topology with other topologies

For some hypergroups, the τ_c-topology and weak*-topology are different. They induce the same topology on every norm bounded subset of $L^\infty(K)$ when K is compact, as the following theorem shows.

THEOREM (4.1). *If K is a compact hypergroup and $B=\{f\in L^\infty(K);\ \|f\|_\infty\le 1\}$, then the τ_c-topology and the weak*-topology coincide on B.*

Proof. Let $f_0\in B$. Suppose

$$\{f\in L^\infty(K);\ \|\phi_i f-\phi_i f_0\|_\infty<\epsilon \text{ for } 1\le i\le n\}\cap B$$

is a neighborhood of f_0 in B. By Lemma 3.1, $x\mapsto A_{(x,x)}\phi_i$ is a continuous mapping of K into $L^1(K)$. For every $x\in K$, there is a neighborhood V_x of x in K such that $\|A_{(x,x)}\phi_i-A_{(y,y)}\phi_i\|_1<\frac{\epsilon}{8}$ whenever $y\in V_x$ and $i\in\{1,\dots,n\}$. Since $K\subseteq\bigcup_{x\in K}V_x$, there exist $x_1,...,x_m \in K$ such that $K \subseteq \bigcup_{k=1}^m V_{x_k}$ and $\|A_{(x_k,x_k)}\phi_i - A_{(x,x)}\phi_i\|_1 < \frac{\epsilon}{8}$ for all $x \in V_{x_k}$ and $i \in \{1,...,n\}$. If

$$W = \{f \in L^\infty(K);\ |\langle f, A_{(x_k,x_k)}\phi_i\rangle - \langle f_0, A_{(x_k,x_k)}\phi_i\rangle| < \frac{\epsilon}{4} \text{ for all } i,k\},$$

then W is a weak*-neighborhood of f_0. Suppose f is in $W \cap B$ and $x \in K$. There exists some k such that $x \in V_{x_k}$. For every $1 \le i \le n$, we have

$$\begin{aligned} |\langle f - f_0, A_{(x,x)}\phi_i\rangle| &\le |\langle f - f_0, A_{(x,x)}\phi_i - A_{(x_k,x_k)}\phi_i\rangle| + |\langle f - f_0, A_{(x_k,x_k)}\phi_i\rangle| \\ &\le \|f - f_0\|_\infty \|A_{(x,x)}\phi_i - A_{(x_k,x_k)}\phi_i\|_1 + \frac{\epsilon}{4} \le \frac{\epsilon}{2}. \end{aligned}$$

An argument similar to the proof of Theorem 3.2 shows that $|\langle \phi_i f - \phi_i f_0, \psi\rangle| < \epsilon$ whenever $\psi \in L^1(K)$ with $\|\psi\|_1 \le 1$ and $i \in \{1,...,n\}$. It follows that $\|\phi_i f - \phi_i f_0\|_\infty < \epsilon$ for all $i \in \{1,...,n\}$. From this result we derive that the τ_c-topology is weaker than the weak* topology on B. On the other hand, from the definitions we immediately derive that the weak* topology is not stronger than the τ_c-topology on B. This completes the proof. □

Let K be a compact hypergroup. Let $I : (L^\infty(K), \|.\|) \to (L^\infty(K), \tau_c)$ be the linear map defined by $I(f) = f$. Then I is continuous. The linear map I is a homeomorphism if and only if the τ_c-topology and the norm topology coincide on $L^\infty(K)$. Let I be a homeomorphism. Then $\{f \in L^\infty(K);\ \|f\|_\infty < 1\}$ is open in the τ_c-topology. By Alaoglu's Theorem [6], $B = \{f \in L^\infty(K);\ \|f\|_\infty \le 1\}$ is weak*-compact. By Theorem 2, B is τ_c-compact, and so B is compact in the norm topology. By Theorem in [25], $L^\infty(K)$ is finite dimensional. Consequently K is finite. The converse is obvious, that is, if K is a finite hypergroup, then the τ_c-topology and the norm topology coincide on $L^\infty(K)$. In particular, we derive that if K is an infinite compact hypergroup, then the τ_c-topology is different from the norm topology.

COROLLARY (4.2). *If K is a compact hypergroup, then $(L^\infty(K), \tau_c)^* = L^1(K)$.*

Proof. Note that the bounded $L^1(K)$ topology for $L^\infty(K)$ is the strongest topology which coincides with the weak* topology on each set $B_r = \{f \in L^\infty(K);\ \|f\|_\infty \le r\}$ where $r > 0$. Thus a set $U \subseteq L^\infty(K)$ is open in the bounded $L^1(K)$ topology if and only if $U \cap B_r$ is a relatively weak* open subset of B_r for every $r > 0$. By Theorem 4.1, $U \subseteq L^\infty(K)$ is open in the bounded $L^1(K)$ topology if and only if $U \cap B_r$ is a relatively τ_c open subset of B_r. Now, let the linear functional F on $L^\infty(K)$ be continuous in the τ_c-topology. Then F is continuous in the bounded $L^1(K)$ topology. By Theorem V.5.5 in [6], F is weak* continuous. The result then follows from the fact that $L^1(K)$ is the dual of $(L^\infty(K), w^*)^*$ [6]. □

COROLLARY (4.3). *Suppose K is a compact hypergroup. If $\{f_n\}$ is a sequence in $L^\infty(K)$ that converges to some $f \in L^\infty(K)$ in the weak*-topology, then $f_n \to f$ in the τ_c-topology.*

Proof. Let $\{f_n\}$ converges to f in the weak*-topology. $\{f_n;\ n \in \mathbb{N}\} \cup \{f\}$ is weak*-compact, hence norm bounded. By Theorem 2, f_n converges to f in the τ_c-topology. □

THEOREM (4.4). *Let K be a unimodular locally compact hypergroup. Every norm-closed ball in $L^\infty(K)$ is τ_c-complete.*

Proof. Suppose $\{f_\alpha\}$ is a Cauchy net in a ball in $L^\infty(K)$, relative to the τ_c-topology. The net $\{f_\alpha\}$ admits a subnet $\{f_\beta\}$ converging to an element f in $L^\infty(K)$ in the weak*-topology (see Theorem 3.15 in [25]). Let $\mathscr{U}$ denote the family of compact neighborhoods of e and regard $\mathscr{U}$ as a directed set in the usual way: $U \geq V$ if $U \subseteq V$. For each $U \in \mathscr{U}$, let $e_U = \frac{\chi_U}{|U|}$. Suppose now that $\phi \in L^1(K)$ and $x \in K$. By Lemma 3.1, we have

$$\begin{aligned} |\langle f_\beta - f_{\beta'}, A_{(x,x)}\phi\rangle| &= \lim_U |\langle f_\beta - f_{\beta'}, e_U \circledast A_{(x,x)}\phi\rangle| \\ &= \lim_U |\langle f_\beta - f_{\beta'}, e_{U\bar{x}} \circledast \phi\rangle| \\ &= \lim_U |\langle \phi f_\beta - \phi f_{\beta'}, e_{U\bar{x}}\rangle| \\ &\leq \|\phi f_\beta - \phi f_{\beta'}\|_\infty. \end{aligned}$$

Given $\epsilon > 0$ and $\phi \in L^1(K)$, there exists β_ϵ such that $\|\phi f_\beta - \phi f_{\beta'}\|_\infty < \epsilon$ for all $\beta, \beta' \geq \beta_\epsilon$. In this last inequality, we take $\beta' = \gamma$ and let γ recede to infinity; then this leads to $|\langle f_\beta - f, A_{(x,x)}\phi\rangle| \leq \epsilon$ for all $\beta \geq \beta_\epsilon$ and all $x \in K$. Let $\psi \in L^1(K)$ and $\beta \geq \beta_\epsilon$. By Lemma 1, we have

$$\begin{aligned} |\langle f_\beta - f, \psi \circledast \phi\rangle| &= \left| \int \langle f_\beta - f, A_{(x,x)}\phi\rangle d\psi(x) \right| \\ &\leq \int \left| \langle f_\beta - f, A_{(x,x)}\phi\rangle \right| d|\psi|(x) \leq \epsilon \|\psi\|_1. \end{aligned}$$

This shows that $\|\phi f_\beta - \phi f\|_\infty \leq \epsilon$ for every $\beta \geq \beta_\epsilon$. Consequently $f_\beta \to f$ in the τ_c-topology and so $f_\alpha \to f$ in the τ_c-topology. This completes our proof. □

Let K be a unimodular locally compact hypergroup. In particular, we derive from Theorem 4.4 that a set in $L^\infty(K)$ is τ_c-relatively compact if and only if it is τ_c-totally bounded. Now, let A be a τ_c-relatively compact set in $L^\infty(K)$. Then τ_c-closure $\overline{A}$ of A is compact, and so totally bounded. Therefore the absolutely convex hull of $\overline{A}$ is totally bounded. It follows that the τ_c-closure of this set is totally bounded. On the other hand, any τ_c-compact subset in $L^\infty(K)$ is weak*-compact, and hence norm bounded. Since $\overline{A}$ is compact, so $\overline{A}$ is contained in some closed ball and the same is true for the τ_c-closure of the absolutely convex hull of $\overline{A}$. Consequently the absolutely convex hull of A is τ_c-relatively compact.

DEFINITION (4.5). *We denote by $U^\infty(K)$ the space consisting of the functions f in $L^\infty(K)$ that are τ_c-continuous with respect to conjugate action, that is, for every $x \in K$, $\epsilon > 0$ and $\phi \in L^1(K)$, there exists a neighborhood U_x of x such that $\|\phi\ {}_x f_x - \phi\ {}_y f_y\|_\infty < \epsilon$ for all $y \in U_x$. A subspace $\mathscr{X}$ is conjugate invariant if ${}_x f_x \in \mathscr{X}$ whenever $f \in \mathscr{X}$ and $x \in K$. We say that $\mathscr{X}$ is introverted if $\phi f \in \mathscr{X}$ whenever $f \in \mathscr{X}$ and $\phi \in L^1(K)$.*

Following Li and Pier [20], for each $\phi \in L^1(K)$ and $f \in L^\infty(K)$ we define $\phi \circledast f(x) = \int \phi(y) f(\overline{y} * x * y) dy$. It is easy to see that $\|\phi \circledast f\|_\infty \leq \|\phi\|_1 \|f\|_\infty$. Lau [17] has studied the bounded linear maps $T : L^p(G) \to L^q(G)$ which commute with translations and convolutions. In [23], Pavel extended these results to hypergroup algebras. We have studied left multipliers on hypergroup algebras [10] (see also [2], [11] and [12]). The following theorem is one of the main results of this paper.

THEOREM (4.6). *Let K be a hypergroup and let $\mathscr{X}$ be a conjugate invariant subspace of $U^\infty(K)$ that is introverted. If T is a τ_c-τ_c continuous linear operator on $\mathscr{X}$, then the following statements are equivalent:*

1. *T commutes with the conjugation operators, that is, $T({}_yf_y) = {}_yT(f)_y$ for all $f \in \mathscr{X}$ and $y \in K$.*
2. *T commutes with convolution, i.e., $T(\phi \circledast f) = \phi \circledast T(f)$ for all $f \in \mathscr{X}$ and $\phi \in L^1(K)$.*

Proof. Let $T : \mathscr{X} \to \mathscr{X}$ be a τ_c-τ_c continuous linear operator and $T({}_yf_y) = {}_yT(f)_y$ for all $f \in \mathscr{X}$ and $y \in K$. Suppose that $\phi \in L^1(K)$, $f \in \mathscr{X}$. Without loss of generality, we may assume that $\phi \geq 0$. We further assume that ϕ has a compact support, say C. Let

$$V := \bigcap_{j=1}^{l} \{h \in L^\infty(K);\ \|\psi_j T(\phi \circledast f) - \psi_j h\|_\infty < \epsilon\}$$

be a τ_c-neighborhood of $T(\phi \circledast f)$. Since T is τ_c-τ_c continuous, there exists an open neighborhood $U := \bigcap_{k=1}^{m}\{g \in L^\infty(K);\ \|\phi_k \phi \circledast f - \phi_k g\|_\infty < \gamma\}$ of $\phi \circledast f$ such that $T(U) \subseteq \bigcap_{j=1}^{l}\{h \in L^\infty(K);\ \|\psi_j T(\phi \circledast f) - \psi_j h\|_\infty < \frac{\epsilon}{2}\}$. Put $2\delta \int \phi(z)dz = \min\{\gamma, \epsilon\}$. Let $x \in K$. As $f \in U^\infty(K)$ and $T(f) \in U^\infty(K)$, there exists an open neighborhood U_x of x in K such that

$$\|\phi_{kz} f_z - \phi_{kx} f_x\|_\infty < \delta \quad \text{and} \quad \|\psi_{jz} T(f)_z - \psi_{jx} T(f)_x\|_\infty < \delta,$$

for all $z \in U_x$, $k \in \{1, ..., m\}$ and $j \in \{1, ..., l\}$. Since C is compact in K, we may determine a subset $\{x_1, ..., x_n\}$ in C such that $C \subseteq \bigcup_{i=1}^{n} U_{x_i}$,

$$\|\phi_{kz} f_z - \phi_{kx_i} f_{x_i}\|_\infty < \delta \text{ and } \|\psi_{jz} T(f)_z - \psi_{jx_i} T(f)_{x_i}\|_\infty < \delta$$

whenever $z \in U_{x_i} \bigcap C$; $i = 1, ..., n$, $k = 1, ..., m$ and $j = 1, ..., l$.

We put $A_1 = U_{x_1} \bigcap C$ and define inductively $A_i = U_{x_i} \bigcap (C \setminus \bigcup_{j=1}^{i-1} A_j)$ for $i = 2, ..., n$. If $i = 1, ..., n$, we also put $\alpha_i = \int_{A_i} \phi(z)dz$. For every $h \in C_C(K)$ (the space of complex-valued bounded continuous functions on K with compact support), with $\|h\|_1 = 1$, we have

$$\begin{aligned}
\langle \phi_k(\phi \circledast f - \sum_{i=1}^{n} \alpha_{ix_i} f_{x_i}), h\rangle &= \int h \circledast \phi_k(x)(\phi \circledast f(x) - \sum_{i=1}^{n} \alpha_{ix_i} f_{x_i}(x))dx \\
&= \int h \circledast \phi_k(x) \sum_{i=1}^{n} \int_{A_i} \phi(z)\left({}_zf_z - {}_{x_i}f_{x_i}\right)(x)dzdx \\
&= \sum_{i=1}^{n} \int_{A_i} \int \phi(z) h \circledast \phi_k(x)\left({}_zf_z - {}_{x_i}f_{x_i}\right)(x)dxdz \\
&= \sum_{i=1}^{n} \int_{A_i} \phi(z)\left\langle \phi_{kz} f_z - \phi_{kx_i} f_{x_i}, h\right\rangle dz,
\end{aligned}$$

and so

$$\begin{aligned}
|\langle \phi_k(\phi \circledast f - \sum_{i=1}^{n} \alpha_{ix_i} f_{x_i}), h\rangle| &\leq \sum_{i=1}^{n} \int_{A_i} \phi(z) \left\|\phi_{kz} f_z - \phi_{kx_i} f_{x_i}\right\|_\infty \|h\|_1 dz \\
&\leq \delta \int_C \phi(z) < \gamma.
\end{aligned}$$

Since $C_c(K)$ is dense in $L^1(K)$, we have $\|\phi_k(\phi\circledast f-\sum_{i=1}^n\alpha_{ix_i}f_{x_i})\|_\infty<\gamma$ for all $k=1,...,m$. It follows that

$$\|\psi_j(T(\phi\circledast f)-\sum_{i=1}^n\alpha_{ix_i}T(f)_{x_i})\|_\infty<\frac{\epsilon}{2}. \tag{4.7}$$

Similarly, for every $h\in C_C(K)$ with $\|h\|_1=1$,

$$\begin{aligned}|\langle\psi_j(\phi\circledast T(f)-\sum_{i=1}^n\alpha_{ix_i}T(f)_{x_i}),h\rangle| &\le \sum_{i=1}^n\int_{A_i}\phi(z)\left\|\psi_{jz}T(f)_z-\psi_{jx_i}T(f)_{x_i}\right\|\|h\|_1dz\\ &\le \delta\int_C\phi(z)<\frac{\epsilon}{2},\end{aligned}$$

whenever $j=1,...,l$. It follows that

$$\|\psi_j(\phi\circledast T(f)-\sum_{i=1}^n\alpha_{ix_i}T(f)_{x_i})\|_\infty<\frac{\epsilon}{2}. \tag{4.8}$$

Thus (4.1) and (4.2) implies that $\|\psi_j(T(\phi\circledast f)-\phi\circledast T(f))\|_\infty<\epsilon$. This shows that $T(\phi\circledast f)=\phi\circledast T(f)$.

To prove the converse, suppose that $f\in\mathscr{X}$, $y\in K$. Let

$$V:=\bigcap_{j=1}^l\{h\in L^\infty(K);\ \|\psi_jT({}_yf_y)-\psi_jh\|_\infty<\epsilon\}$$

be a τ_c-neighborhood of $T({}_yf_y)$. Since T is τ_c-τ_c continuous, there exists an open neighborhood $U:=\bigcap_{k=1}^m\{g\in L^\infty(K);\ \|\phi_{k\,y}f_y-\phi_kg\|_\infty<\gamma\}$ of ${}_yf_y$ such that

$$T(U)\subseteq\bigcap_{j=1}^l\{h\in L^\infty(K);\ \|\psi_jT({}_yf_y)-\psi_jh\|_\infty<\frac{\epsilon}{2}\}.$$

Put $2\delta=\min\{\gamma,\epsilon\}$. Let W be a compact neighborhood of e in K such that $z\in W$ implies

$$\|\psi_{jzy}T(f)_{y_z}-\psi_{jy}T(f)_y\|_\infty<\delta,\quad\text{and}\quad|\phi_{kzy}f_{y_z}-\phi_{ky}f_y\|_\infty<\delta,$$

for every $j=1,...,l$ and $k=1,...,m$. Then for $\phi=\frac{\chi_W}{|W|}$ and every $h\in C_C(K)$ with $\|h\|_1=1$,

$$\begin{aligned}|\langle\phi_k(\phi\circledast{}_yf_y-{}_yf_y),h\rangle| &= |\int h\circledast\phi_k(x)(\phi\circledast{}_yf_y(x)-{}_yf_y(x))dx|\\ &= |\int h\circledast\phi_k(x)\int\phi(z)({}_{zy}f_{y_z}(x)-{}_yf_y(x))dzdx|\\ &= |\int\phi(z)\int h\circledast\phi_k(x)({}_{zy}f_{y_z}(x)-{}_yf_y(x))dxdz|\\ &= |\int\phi(z)\int h(x)\phi_{kzy}f_{y_z}(x)-\phi_{ky}f_y(x)dxdz|\\ &\le \int\phi(z)\|h\|_1\|\phi_{kzy}f_{y_z}-\phi_{ky}f_y\|_\infty dz\\ &= \frac{1}{|W|}\int_W\|h\|_1\|\phi_{kzy}f_{y_z}-\phi_{ky}f_y\|_\infty dz<\delta,\end{aligned}$$

for every $k\in\{1,...,m\}$. This shows that $\|\phi_k\phi\circledast{}_yf_y-\phi_{ky}f_y\|_\infty<\gamma$. It follows that $\|\psi_j\phi\circledast T({}_yf_y)-\psi_jT({}_yf_y)\|_\infty<\frac{\epsilon}{2}$ for $j=1,...,l$. Similarly, we can prove that $\|\psi_j\phi\circledast{}_yT(f)_y-\psi_{jy}T(f)_y\|_\infty\le\frac{\epsilon}{2}$ for $j=1,...,l$.

Now, we consider the mapping $\psi(z) = \phi(z * \bar{y})\Delta(\overline{y})$ $(z \in K)$. For every $x \in K$, we have

$$\begin{aligned} \phi \circledast_y f_y(x) &= \int \phi(z)_y f_y(\bar{z} * x * z) dz \\ &= \int \Delta(\overline{y})\phi(z * \bar{y}) f(\bar{z} * x * z) dz \\ &= \int \psi(z) f(\bar{z} * x * z) dz = \psi \circledast f(x). \end{aligned}$$

Hence, $\|\phi_k \psi \circledast f - \phi_{k\,y} f_y\|_\infty = \|\phi_k \phi \circledast_y f_y - \phi_{k\,y} f_y\|_\infty < \gamma$ for $k = 1, ..., m$. Therefore $\|\psi_j \psi \circledast T(f) - \psi_j T({}_y f_y)\|_\infty < \frac{\epsilon}{2}$ for $j = 1, ..., l$. On the other hand, we can write $\psi \circledast T(f) = \phi \circledast_y T(f)_y$. So, we have

$$\begin{aligned} \|\psi_j(T({}_y f_y) - {}_y T(f)_y)\|_\infty &\leq \|\psi_j(T({}_y f_y) - \psi \circledast T(f))\|_\infty \\ &+ \|\psi_j(\psi \circledast T(f) - \phi \circledast_y T(f)_y)\|_\infty \\ &+ \|\psi_j(\phi \circledast_y T(f)_y - {}_y T(f)_y)\|_\infty < \epsilon, \end{aligned}$$

for $j = 1, ..., l$. Since V may be arbitrary, we have

$${}_y T(f)_y = T({}_y f_y).$$

This completes the proof. □

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THE HYPERSPACE OF INDECOMPOSABLE SUBCONTINUA OF A CUBE

ALICJA SAMULEWICZ

ABSTRACT. It is shown that if $3 \le n \le \infty$ then the set of all decomposable subcontinua of I^n is an F_σ-absorber in the Hilbert cube $C(I^n)$. Therefore the hyperspace of all indecomposable continua in the cube of dimension greater than 2 is homeomorphic to the separable Hilbert space l_2. This answers a question from [3] by the author.

A *continuum* is a compact connected metric space. By the Curtis-Schori Theorem [1], the hyperspace $C(I^n)$ of all nonempty subcontinua of the cube I^n equipped with the Hausdorff metric is homeomorphic to the Hilbert cube I^∞, where $I = [0,1]$, $2 \le n \le \infty$. A continuum is *decomposable* provided that it is the union of two proper subcontinua; otherwise it is called *indecomposable*. Denote by $\mathscr{D}(I^n)$ and $\mathscr{I}\mathscr{D}(I^n)$ the hyperspaces of all decomposable and all indecomposable subcontinua of I^n, respectively. It is well-known that if $n \ge 2$ then $\mathscr{I}\mathscr{D}(I^n)$ is a dense G_δ-set in $C(I^n)$.

Recall that an F_σ-subset A of a Hilbert cube M is a *σZ-set* if for every $\epsilon > 0$ there exists a map $\eta : M \to M \setminus A$ with $d(\eta, \mathrm{id}) < \epsilon$. A σZ-set A is called an *F_σ-absorber* in M provided that there is a homeomorphism $h : M \to I^\infty$ which maps A onto the pseudoboundary $B(Q) = I^\infty \setminus (0,1)^\infty$. Since the pseudointerior $(0,1)^\infty$ is homeomorphic to the separable Hilbert space l_2, the complement of an F_σ-absorber in M is homeomorphic to l_2.

The following theorem answers [3, Questions 4.7 and 4.8] for $n \ge 3$.

THEOREM (1.1). *If $3 \le n \le \infty$ then the hyperspace $\mathscr{D}(I^n)$ of all decomposable subcontinua of the cube I^n is an F_σ-absorber in the Hilbert cube $C(I^n)$.*

COROLLARY (1.2). *The hyperspace $\mathscr{I}\mathscr{D}(I^n)$ of all indecomposable subcontinua of the cube I^n is homeomorphic to l_2 for $3 \le n \le \infty$.*

A subset A of a space X is *strictly homotopy dense* in X provided that there is a homotopy $H : X \times I \to X$ such that

1. $H(\cdot, 0) = \mathrm{id}$;
2. $H(x,t) \in A$ for all $x \in X$ and $t > 0$;
3. if $H(x,t) = H(x',t')$ and $t > 0$, $t' > 0$ then $x = x'$.

The homotopy H is called a *strict deformation of X through A*.

LEMMA (1.3). [2, Theorem 4.2] *Assume that A is a σZ-subset of a Hilbert cube M which is strictly homotopy dense in M. Then A is an F_σ-absorber in M.*

LEMMA (1.4). *If $n \ge 2$ then $\mathscr{D}(I^n)$ is strictly homotopy dense in $C(I^n)$.*

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Proof. All the functions that appear in the construction are known and can be found in [3]. As usual, 2^{I^n} denotes the hyperspace of all nonempty closed subsets of I^n endowed with the Hausdorff metric.

Let $H_0 : C(I^n) \times I \to 2^{I^n}$ and $H_1 : C(I^n) \times I \to C(I^n)$ be homotopies satisfying the following conditions:

1. $H_0(\cdot, 0) = \mathrm{id}$ and $H_1(\cdot, 0) = \mathrm{id}$;
2. $H_0(A,t) \subset H_1(A,t) \subset [\frac{t}{2}, 1 - \frac{t}{2}]^n$ for all $A \in C(I^n)$ and $t \in [0,1]$;
3. if t is positive then $H_0(A,t)$ is a finite set and $H_1(A,t)$ is a straight-edge graph.

To obtain a strict deformation of $C(I^n)$ through $\mathscr{D}(I^n)$ we assign with every continuum $A \subset I^n$ a "marker" $\phi(A)$.

Let $\theta : I^\infty \to \mathscr{D}(I^n)$ be an embedding described in [3, §3.1]. The values of θ consist of a segment and countably many circles whose radii code the coordinates of arguments. By the Curtis-Schori Theorem there is a homeomorphism $h : C(I^n) \to I^\infty$. Put $\phi(A) = \theta(h(A))$ and define the homotopy $H : C(I^n) \times I \to C(I^n)$ by

$$H(A,t) = H_1(A,t) \cup \bigcup_{v \in H_0(A,t)} v + \frac{t}{4}\phi(A),$$

where addition and multiplication are usual linear operations in $\mathbb{R}^n$.

Recall that, for positive values of t, the only circles in $H(A,t)$ are contained in the copies of $\phi(A)$ and the radius of the biggest one is equal to $\frac{t}{2}$. Suppose that $H(A,t) = H(B,t')$ and $t, t' \in (0,1]$. It is easy to see that $t = t'$ and $\phi(A) = \phi(B)$. Since ϕ is an embedding then $A = B$. □

LEMMA (1.5). [3, Corollary 4.4] *If $n \geq 3$ then $\mathscr{D}(I^n)$ is a σZ-set in $C(I^n)$.*

Combining all the above facts we get our theorem.

QUESTION (1.6). *Is $\mathscr{D}(I^2)$ a σZ-set in $C(I^2)$?*

If the answer is positive, then – by Lemmas 1.3 and 1.4 – $\mathscr{D}(I^2)$ is an F_σ-absorber in $C(I^2)$ and its complement is homeomorphic to l_2. In the opposite case, the problem concerning the topological structure of $\mathscr{I}\mathscr{D}(I^2)$ remains open.

QUESTION (1.7). *Is $\mathscr{I}\mathscr{D}(I^2)$ homeomorphic to l_2? Is it an ANR?*

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