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STABILITY OF THE EULER OBSTRUCTION OF A FUNCTION

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ABSTRACT. We study in this paper properties of the Euler obstruction of a function and its stability for families of functions with isolated singularities on weighted homogeneous hypersurfaces with isolated singularities and on holonomic free divisors.

Introduction

The local Euler obstruction was first introduced by R. MacPherson in [14] as a key ingredient for his construction of characteristic classes of singular complex algebraic varieties. This invariant was then interpreted in [5] as an index of vector fields. For an overview on the Euler obstruction see [1, 2]. The recent book [6] is also a very good reference, since the point of view in this book is quite close to the point of view of the present work.

In the paper [4], J.-P. Brasselet, Lê D. T. and J. Seade gave a Lefschetz type formula for the local Euler obstruction, which shows that the local Euler obstruction, as a constructible function, satisfies the Euler condition relatively to generic linear forms. A natural continuation of the result is the paper by J.-P. Brasselet, D. Massey, A. J. Parameswaran and J. Seade [3], whose aim is to understand what the obstacle is for the local Euler obstruction to satisfy the Euler condition relatively to analytic functions with isolated singularity at the point considered. This is the role of the so-called local Euler obstruction of f, denoted by $\mathrm{Eu}_{f,V}(0)$.

The relation between the local Euler obstruction of f and the number of Morse points of a Morsification of f has been described in [19] by J. Seade, M. Tibar and A. Verjovsky. They compare $\mathrm{Eu}_{f,V}(0)$ with two different generalizations of the Milnor number for functions with isolated singularities on singular varieties. In the case where $(V,0)$ is a complete intersection with isolated singularities they also study the GSV index [11, 17].

In [12] we present some relations between the Euler obstruction of f and the notion of Milnor number of functions with an isolated critical point on singular spaces introduced by J. W. Bruce and R. M. Roberts in [7]. The originality of the main formula in [12] is that we use the information of all strata, while in [19] Seade, Tibar and Verjovsky use only the information in the regular stratum.

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From the relations between the Euler obstruction of f and the Bruce and Roberts' Milnor number we obtain in [12] relations between the Bruce and Roberts' Milnor number and Lê's Milnor number and Goryunov-Mond's Milnor number.

Let $V \subset \mathbb{C}^n$ be a hypersurface with isolated singularity and $F : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to \mathbb{C}$ a family of functions with isolated singularity on V at 0. In this context we can ask about the constancy of the Euler obstruction of f_t for this family.

In [12], we have discussed the stability of the Euler obstruction and its consequences when the *characteristic logarithmic variety* associated to V, denoted by $LC(V)$, is Cohen-Macaulay. Here we study the stability of the Euler obstruction of f for families of functions with isolated singularities on the germ of a weighted homogeneous hypersurface with isolated singularities and on holonomic free divisors $(V,0)$.

The notion of free divisors was introduced by Saito in [16]. In a series of papers Damon has discussed the importance of free divisors [8, 9, 10] and how common they are. For example, discriminants of versal unfoldings of isolated hypersurface and complete intersection singularities are free divisors; the bifurcation sets associated to the versal unfoldings of isolated hypersurface singularities are also free divisors; the discriminant of a versal deformation of a space curve singularity is a free divisor. One way to investigate these objects is to compute and understand the behavior of some invariants on them: for example, we can study the local Euler obstruction.

A key tool to determine the stability of the Euler obstruction of f in [12] was the Cohen-Macaulayness property of the characteristic logarithmic variety associated to V. In the cases of weighted homogeneous hypersurfaces and for free divisors, we have this property by results of [7] and [15].

As we said before, we know from [19] and [12] that the Euler obstruction of f is a generalization of the notion of Milnor number to the case of functions on singular varieties. In this way we can expect that the study of the constancy of the Euler obstruction of f can be useful to get a Lê-Ramanujan type result on singular varieties [21], which is an open problem in this theory.

1. Euler obstruction and Morsification of a function

Let $(V,0)$ be the germ of a reduced complex analytic space at the origin, embedded in $\mathbb{C}^n$. In this work we will always consider $(V,0)$ with pure dimension. Let $\{V_\alpha\}$ be a Whitney stratification of a sufficiently small representative V of the germ. We may suppose that V has only a finite number of strata V_α, $\alpha \in \{0,1,2,\cdots,d\}$, for some $d \in \mathbb{N}$, such that $0 \in \overline{V_\alpha}$.

To define a stratified Morse function we need the definition of a general point of a function (or general function at a point).

Definition (1.1). Let $(V,0)$ be the germ of an analytic variety in $\mathbb{C}^n$, endowed with a Whitney stratification, and let $f : (V,0) \to (\mathbb{C},0)$ be a germ of analytic function, restriction of an analytic function $F : U \to (\mathbb{C},0)$, where U is an open ball around 0. We say that 0 is a general point of f (or $f : (V,0) \to (\mathbb{C},0)$ is general at 0) if the hyperplane $ker(dF(0))$ is transverse in $\mathbb{C}^n$ to every generalized tangent space at 0; that is, for every sequence $\{x_n\}$ of points in some stratum V_α such that the sequence converges to 0 and the sequence of tangent spaces $\{T_{x_n}V_\alpha\}$ has a limit T, one has that $ker(dF(0))$ is transverse to T.

Definition (1.2). Let $(V,0)$ be the germ of an analytic variety in $\mathbb{C}^n$, endowed with a Whitney stratification, and let $f:(V,0)\to(\mathbb{C},0)$ be a germ of an analytic function. Let V be a sufficiently small representative of the germ $(V,0)$. We say that $f:V\to\mathbb{C}$ is a *stratified Morse function* if the following holds:

(a) If $0\in V_\alpha$, $\dim V_\alpha\geq 1$, the restriction of f to the stratum V_α has a Morse point at 0.
(b) If $0\notin V_\alpha$, f is general at 0 with respect to the strata V_α.

Note that in the case where the stratum containing 0 has dimension zero, the point 0 is a stratified Morse point of f (by definition) if f is general at 0.

The Euler obstruction of a function is a "hard to work" concept, but using Morsifications we can find an interesting way to study this concept. Let us first give an idea of the definition of the Euler obstruction of a function.

Let $(V,0)$ be the germ of an analytic variety in $\mathbb{C}^n$, endowed with a Whitney stratification, and let $f:(V,0)\to(\mathbb{C},0)$ be a germ of analytic function. Let V be a sufficiently small representative of the germ $(V,0)$. The complex conjugate of the gradient of the extension F of f in the ambient space projects to the tangent spaces of the strata of V into a vector field, which may not be continuous. One can make it continuous as in [3]. One gets a well-defined continuous stratified vector field, up to stratified homotopy, which we denote by $\overline{\nabla}_V f$. If f is a function on V with an isolated singularity at 0, with respect to the stratification, then $\overline{\nabla}_V f$ has an isolated zero at 0. If $\nu:\widetilde{V}\to V$ is the Nash modification of V and $\widetilde{T}$ is the Nash bundle over $\widetilde{V}$, then $\overline{\nabla}_V f$ lifts canonically to a never-zero section $\widetilde{\overline{\nabla}} f_V$ of $\widetilde{T}$ restricted to $\widetilde{V}\cap\nu^{-1}(V\cap\mathbb{S}_\varepsilon)$, where $\mathbb{S}_\varepsilon$ is a small enough sphere around 0. Following [3], the obstruction to extend $\widetilde{\overline{\nabla}}_V f$ without zeros over $\widetilde{V}\cap\nu^{-1}(V\cap\mathbb{B}_\varepsilon)$, where $\mathbb{B}_\varepsilon$ is a small ball around 0 is denoted by $\mathrm{Eu}_{f,V}(0)$ and is called the local Euler obstruction of f at 0.

LEMMA (1.3). *(Lemma 4.1,* [18]*) Let f be the germ of an analytic function on $(V,0)$ with an isolated Morse singularity at the origin in the stratified sense, V_α the stratum that contains the origin. Then*

1. $\mathrm{Eu}_{f,V}(0)=0$ *if* $\dim V_\alpha<\dim V$*;*
2. $\mathrm{Eu}_{f,V}(0)=(-1)^{\dim_{\mathbb{C}}(V,0)}$ *if V_α is the regular stratum* V_{reg}.

The next result characterizes the Euler obstruction of f on V by the number of the Morse points of a Morsification of f on V.

PROPOSITION (1.4) (Proposition 2.3, [19]). *Let $f:(V,0)\to(\mathbb{C},0)$ be the germ of an analytic function with isolated singularity at the origin. Then*

$$\mathrm{Eu}_{f,V}(0)=(-1)^{\dim_{\mathbb{C}}(V,0)}n_{\mathrm{reg}},$$

where n_{reg} is the number of Morse points in V_{reg} in a stratified Morsification of f.

2. Generalizations of the Milnor Number

Let us recall that $\mu(f)$ denotes the Milnor number of a germ of an analytic function $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ with an isolated critical point at the origin, defined as

$\dim_{\mathbb{C}} \mathcal{O}_n/J(f)$, where $\mathcal{O}_n$ is the ring of germs of analytic functions at the origin, and $J(f)$ is the Jacobian ideal of f.

Let (V,x) be the germ of a complex analytic space. For simplicity let us take $x=0$. Bruce and Roberts in [7] defined a Milnor number for functions on singular spaces.

Let V be a sufficiently small representative of the germ $(V,0)$ and let $\mathcal{I}(V)$ denote the ideal in $\mathcal{O}_n$ consisting of the germs of functions vanishing on V.

One of the main goals in [7] is to characterize germs of diffeomorphisms preserving V. The usual technique is the integration of germs of vector fields tangent to V.

Definition (2.1). For $x \in \mathbb{C}^n$, let $Der_x\mathbb{C}^n$ denote the $\mathcal{O}_n$-module of germs of analytic vector fields on $\mathbb{C}^n$ at x. A vector field δ in $Der_x\mathbb{C}^n$ is said to be logarithmic for (V,x) if, when considered as a derivation $\delta : \mathcal{O}_n \to \mathcal{O}_n$, we have $\delta(h) \in \mathcal{I}(V)$ for all $h \in \mathcal{I}(V)$. The $\mathcal{O}_n$-module of such vector fields is denoted by $\Theta_{(V,x)}$. When $x=0$, we denote it by Θ_V.

LEMMA (2.2) (Lemma 1.5, [7]). *Let V be a sufficiently small representative of the germ $(V,0)$ and U an open neighborhood of the origin. There is a unique stratification $\{V_\alpha\}$ of U with the following properties:*

(i) *Each stratum V_α is a connected manifold embedded in U, and U is the union $\bigcup V_\alpha$;*
(ii) *if V_α and V_β are two different strata, such that $V_\alpha \cap \overline{V}_\beta \neq \emptyset$, then $V_\alpha \subset \partial V_\beta$, where ∂V_β denotes the boundary of V_β and $\overline{V}_\beta$ denotes the closure of V_β.*
(iii) *if $x \in V_\alpha$ then the tangent space T_xV_α of V_α at x coincide with $\Theta_{(V,x)}$;*

Definition (2.3). The decomposition $\{V_\alpha\}$ as above is called a logarithmic stratification of V, and each V_α is called the logarithmic stratum.

We would like to clarify that we call $\{V_\alpha\}$ a logarithmic stratification, but it is not always a usual stratification; in some cases this decomposition is not locally finite. For the locally finite case we have the next definition.

Definition (2.4). The germ $(V,0)$ is holonomic for some neighborhood U at 0 in $\mathbb{C}^n$ if the logarithmic stratification of U has only a finite number of strata.

PROPOSITION (2.5) (Proposition 1.10, [7]). *Let $(V,0)$ be a holonomic germ. Then in a sufficiently small neighborhood of the origin, the logarithmic stratification is Whitney regular.*

Definition (2.6). Let $(V,0)$ be the germ of an analytic variety in $\mathbb{C}^n$ endowed with a Whitney stratification $\{V_\alpha\}$. Let $f:(V,0)\to(\mathbb{C},0)$ be the restriction of a analytic function $F:(U,0)\to(\mathbb{C},0)$. Let V be a sufficiently small representative of the germ $(V,0)$. We call a critical point of f a point $x \in V$ such that, if $x \in V_\alpha$ we have $dF(x)(T_x(V_\alpha))=0$. We say that f has an isolated singularity at $0 \in V$ relative to the given Whitney stratification, if f has no critical points in a punctured neighborhood of 0 in V.

Definition (2.7). Let $J_V(f)$ be the ideal $\{\delta f : \delta \in \Theta_V\}$ in $\mathcal{O}_{n,0}$. If the germ $f : (\mathbb{C}^n,0)\to(\mathbb{C},0)$ has an isolated singularity on V at the origin, then $\dim_{\mathbb{C}} \mathcal{O}_n/J_V(f)$ is finite and is called the multiplicity of f over V at 0, or the Bruce-Roberts' Milnor number, and is denoted by $\mu_{BR}(f)$.

The *characteristic logarithmic variety* associated to V, denoted by $LC(V)$, is an important tool in [7], and we use the information of this variety to obtain the main results of this work.

Definition (2.8). [7] Let us suppose that the vector fields $\delta_1,\dots,\delta_m$ generate Θ_V for some neighborhood U at $0 \in \mathbb{C}^n$. Then if $T^*_U\mathbb{C}^n$ is the restriction of the cotangent bundle of $\mathbb{C}^n$ in U, we define

$$LC_U(V) = \{(x,\xi) \in T^*_U\mathbb{C}^n : \xi(\delta_i(x)) = 0, i = 1\dots,m\}.$$

$LC(V)$ is the germ of $LC_U(V)$ in $T^*_0\mathbb{C}^n$, and it can be shown that it is independent of the choice of generators of Θ_V.

Let V_α be a stratum of the logarithmic stratification of V. The conormal space of V_α is the subspace of $T^*_0\mathbb{C}^n$ given by all forms vanishing on the tangent bundle TV_α. We denote it by $C(V_\alpha)$.

Then we have,

$$LC(V) = \bigcup_\alpha \overline{C(V_\alpha)}.$$

PROPOSITION (2.9) (Proposition 1.14 (ii), [7]). *Let V be a holonomic space with stratification $\{V_\alpha\}$. Then the sets $\overline{C(V_\alpha)}$ are the irreducible components of $LC(V)$.*

THEOREM (2.10) (Corollary 5.8, [7]). *If $f : (V,0) \to \mathbb{C}$ has an isolated singularity at the origin and n_α is the number of critical points of a stratified Morsification of f on V_α, and m_α denotes the multiplicity of $C(V_\alpha)$ in $LC(V)$, then*

$$\sum_\alpha m_\alpha n_\alpha \le \dim_{\mathbb{C}} \mathcal{O}_n / J_V(f).$$

with equality if and only if $LC(V)$ is Cohen-Macaulay at $(0, df(0))$.

THEOREM (2.11) (Theorem 3.14, [12]). *Let $(V,0)$ be as above such that the $LC(V)$ is Cohen-Macaulay. Let $f : (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a function with an isolated singularity at the origin and such that $f : V \to \mathbb{C}$ has also a stratified isolated singularity at the origin. Then*

$$\mu_{BR}(f) = \sum_\alpha m_\alpha (-1)^{\dim_{\mathbb{C}} V_\alpha} Eu_{f,\overline{V}_\alpha}(0),$$

where m_α denotes the multiplicity of $C(V_\alpha)$ in $LC(V)$.

As a consequence of the last theorem, we state here the following result:

COROLLARY (2.12). *Let $(V,0)$ be a weighted homogeneous hypersurface with isolated singularities and $f : (\mathbb{C}^n,0) \to (\mathbb{C},0)$ a function with an isolated singularity at the origin and such that $f : V \to \mathbb{C}$ has also a stratified isolated singularity at the origin. Then*

$$\mu_{BR}(f) = \mu(f) + \mu^{(1)}(f) + (-1)^{\dim_{\mathbb{C}} V} Eu_{f,V}(0),$$

where $\mu^{(1)}(f)$ is the Milnor number of a generic hyperplane section of $(V,0)$.

Proof. This follows from the last theorem and by the fact that since V is weighted homogeneous hypersurface with isolated singularity, then $LC(V)$ is Cohen-Macaulay [7, 15] and from [7] the multiplicities m_α, $0 < \alpha \le d$, are 1 and $m_0 = \mu^{(1)}(f)$, where $\mu^{(1)}(f)$ is the Milnor number of a generic hyperplane section of $(V,0)$. □

3. Stability of the Euler obstruction of f and the Bruce-Roberts' Milnor number

In [20], another notion of Milnor number arises, a generalization of the Milnor number for analytic functions defined on singular analytic spaces such that the rectified homotopical depth of V at 0, denoted $\mathrm{rhd}(V,0)$ satisfies $\mathrm{rhd}(V,0) = \dim_{\mathbb{C}}(V,0)$.

Let V be a sufficiently small representative of the germ $(V,0)$. The Milnor fiber of the complex analytic function f, with an isolated singularity at 0, defined on V (in the stratified way), has the homotopy type of a bouquet of spheres. Lê's Milnor number, denoted by $\mu_L(f)$, is defined as the number of spheres in the bouquet.

The relations between this invariant and the local Euler obstruction of f were obtained in [19], in particular for a complete intersection with isolated singularity (ICIS). Since in this case we have $\mathrm{rhd}(V,0) = \dim(V,0)$, the following holds (Section 3.1, [19]):

THEOREM (3.1). *Let V be a sufficiently small representative of an ICIS germ, $0 \in V$, f an analytic function on V with stratified isolated singularity at* 0, *and l a generic linear form. Then we have*

$$Eu_{f,V}(0) = (-1)^{\dim_{\mathbb{C}}(V,0)}[\mu_L(f) - \mu_L(l)].$$

The next result relates these different notions of Milnor number above (and the classical one) and the Euler obstruction of a function. This theorem was presented in [12], but the assumption that $LC(V)$ is Cohen-Macaulay was missing. In fact, this is an open problem. Another version of this result was presented in [13].

THEOREM (3.2) (Theorem 4.18, [12], see also [13]). *Let $V \subset \mathbb{C}^n$ be a hypersurface with isolated singularity such that its $LC(V)$ is Cohen-Macaulay and $F : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to \mathbb{C}$ a family of functions with isolated singularity on V at* 0. *Then:*

(a) *$\mu_{BR}(f_u)$ constant for the family implies $\mu(f_u)$, $\mu_L(f_u)$ and $Eu_{f_u,V}(0)$ constant for the family.*
(b) *When $\mu(f_u)$ is constant for the family, we have that $Eu_{f_u,V}(0)$ or $\mu_L(f_u)$ constant for the family implies $\mu_{BR}(f_u)$ is constant for the family.*

In the weighted homogeneous case we immediately get the following result:

COROLLARY (3.3). *Let $V \subset \mathbb{C}^n$ be a weighted homogeneous hypersurface with isolated singularity and $F : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to \mathbb{C}$ a family of functions with isolated singularity on V at* 0. *Then:*

(a) *$\mu_{BR}(f_u)$ constant for the family implies $\mu(f_u)$, $\mu_L(f_u)$ and $\mathrm{Eu}_{f_u,V}(0)$ constant for the family.*
(b) *When $\mu(f_u)$ is constant for the family, we have that $\mathrm{Eu}_{f_u,V}(0)$ or $\mu_L(f_u)$ constant for the family implies $\mu_{BR}(f_u)$ is constant for the family.*

Let us now give the definition of a free divisor introduced by Saito [16].

Definition (3.4). A reduced hypersurface $(V,0) \subset (\mathbb{C}^n, 0)$ is said to be a *free divisor* if $\Theta_{V,0}$ is a free $\mathcal{O}_n$-module.

Examples of free divisors are discriminants of versal unfoldings of isolated hypersurfaces and complete intersection singularities; the bifurcation sets associated to the versal unfoldings of isolate hypersurfaces singularities are also free divisors; and the discriminant of versal deformation of a space curve singularity.

REMARK (3.5). *If* $(V,0)\subset(\mathbb{C}^n,0)$ *is a free divisor, then* $\Theta_{V,0}$ *is necessarily generated by n elements.*

THEOREM (3.6) (Proposition 6.3, [7]). *If* $(V,0)\subset(\mathbb{C}^n,0)$ *is a reduced analytic subvariety, any two of the following properties imply the third.*

(i) V *is holonomic;*
(ii) $LC(V)$ *is a complete intersection;*
(iii) $(V,0)$ *is a free divisor.*

In particular when V is holonomic and a free divisor, $LC(V)$ is a complete intersection, therefore Cohen-Macaulay, and in this case we have the equality in the formula of Theorem (2.10),

$$\mu_{BR}(h)=\sum_{\alpha} m_\alpha n_\alpha,$$

where n_α is the number of Morse points of a Morsification f_t on V_α.

Let $(V,0)\subset(\mathbb{C}^n,0)$ be the germ of a holonomic analytic variety. Let us take V as a sufficiently small representative of the germ, such that the logarithmic stratification $\{V_\alpha\}$, with $\alpha\in\{0,1,2,\cdots,d\}$ for some $d\in\mathbb{N}$, be Whitney. The closure of each stratum V_α is itself an analytic space, with regular part V_α, so it makes sense to define the invariant $\mathrm{Eu}_{f,\overline{V}_\alpha}(0)$.

THEOREM (3.7). *Let* $V\subset\mathbb{C}^n$ *be a holonomic free divisor, and*

$$F:(\mathbb{C}^n\times\mathbb{C}^r,0)\to\mathbb{C}$$

a family of functions with isolated singularity on at 0 *on* V. *Then* $\mu_{BR}(f_u)$ *constant for the family implies* $\mu(f_u)$, $\mu_L(f_u)$ *and* $Eu_{f_u,V}(0)$ *constant for the family.*

Proof. Since V is a holonomic free divisor, by Proposition 6.3 of [7] the $LC(V)$ is a complete intersection, therefore Cohen-Macaulay, and in this case we have by Theorem 2.12, the following,

$$\mu_{BR}(f)=\sum_{\alpha} m_\alpha(-1)^{\dim_{\mathbb{C}} V_\alpha}\mathrm{Eu}_{f,\overline{V}_\alpha}(0),$$

where m_α denotes the multiplicity of $C(V_\alpha)$ in $LC(V)$.

Since $g_\alpha(u)=(-1)^{\dim_{\mathbb{C}} V_\alpha}\mathrm{Eu}_{f_u,\overline{V}_\alpha}(0)$ are upper semicontinuous, because it is counting the number of Morse points, we have by the relations

$$\mu_{BR}(f)=\sum_{\alpha} m_\alpha g_\alpha(u),$$

that $\mu_{BR}(f_u)$ constant for the family implies that all terms $g_\alpha(u)$ and $\mu(f_u)$ are constant for the family. In particular it follows that $\mu(f)$ and $Eu_{f_u,V}(0)$ are constant for the family. Therefore, by Proposition 3.16 of [12] $\mu_L(f_u)$ is also constant. □

The motivation to study the constancy of theses invariants above (in particular Euler obstruction of f) is based on their relations with the Milnor number. We expect that the constancy of them can be an important tool to prove a Lê-Ramanujan type result on singular varieties [21], which is an open problem in this theory.

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EXTENSIONS OF HOM-LIE ALGEBRAS ON LIE ALGEBRAS

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ABSTRACT. Hom-Lie algebras can be considered as a deformation of Lie algebras, including Lie algebras as a subclass. The central extension theory of hom-Lie algebras has been given. In this note, we develop the extension theory of hom-Lie algebras on Lie algebras based on the cohomology of Lie algebras.

1. Introduction

A new approach to the deformation theory of Witt and Virasoro algebras using σ-derivations was introduced by Jonas T. Hartwig, Daniel Larsson and Sergei D. Silvestrov in [4]. They also introduced the notion of a hom-Lie algebra, which is a non-associative algebra satisfying the skew symmetry and the σ-twisted Jacobi identity. Here σ is called twisting homomorphism. When the twisting homomorphism is identity, the hom-Lie algebras degenerate to exactly the Lie algebras. Deformation theory and cohomology of Hom-Lie algebras were considered in [8] and earlier precursors of Hom-Lie algebras can be found in [2, 6]. In [5], Daniel Larsson and Sergei D. Silvestrov introduced the notion of a quasi-hom-Lie algebra, which is a natural generalization of hom-Lie algebras. Quasi-hom-Lie algebras include also as special cases (color) Lie algebras and super-algebras, and can be seen as deformations of these by maps, twisting the Jacobi identity and skew-symmetry.

For the class of hom-Lie algebras (or quasi-hom-Lie algebras), the central extension theory was developed in [4] (or [5]). Also it is of key importance to develop the abelian extension theory. In order to do this, it is necessary to restrict the hom-Lie algebras on certain class of algebras satisfying the skew symmetry. Thus it is natural to study the hom-Lie algebras on Lie algebras. The goal of this paper is to develop the extension theory of hom-Lie algebras on Lie algebras following the notions of [4], in particular, for the case when the twisting homomorphisms are automorphisms.

The paper is organized as follows. In Section 2, we list some notions. In Section 3, we give the central extension theory firstly introduced in [4]. In Section 4, we develop the extension theory of hom-Lie algebras on Lie algebras. In Section 5, assume that the twisting homomorphisms are automorphisms. Then we have another σ-twisted Jacobi identity, which give a new form of the extension theory of hom-Lie algebras on Lie algebras.

2. Preliminary

First let us recall some notions in [4].

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Definition (2.1). A hom-Lie algebra (L,σ) is a non-associative algebra L over the complex numbers field $\mathbb{C}$ together with an algebra homomorphism $\sigma : L \to L$, such that

$$[x,y] = -[y,x], \tag{2.2}$$

$$[(id+\sigma)(x),[y,z]]+[(id+\sigma)(y),[z,x]]+[(id+\sigma)(z),[x,y]] = 0, \tag{2.3}$$

for any $x,y,z \in L$, where $[\cdot,\cdot]$ denotes the product in L.

REMARK (2.4). *We call (2.2) the skew symmetry and (2.3) the σ-twisted Jacobi identity and σ the twisting homomorphism. The condition that σ is a homomorphism of algebras in the definition of hom-Lie algebra has been relaxed in* [7], *to simply being a linear map. But in this paper, we follow the definition in* [4].

EXAMPLE (2.5). *Taking $\sigma = id$ in the above definition gives us the definition of a Lie algebra. Hence hom-Lie algebras include Lie algebras as a subclass.*

EXAMPLE (2.6). *For any vector space V, if we put $[x,y] = 0$ for any $x,y \in V$, then (V,σ) is obviously a hom-Lie algebra for any linear map $\sigma : V \to V$, since the conditions are trivially satisfied. We call these algebras abelian hom-Lie algebras.*

EXAMPLE (2.7). *For hom-Lie algebras (L_1,σ_1) and (L_2,σ_2), as in the Lie algebra case, we can define a hom-Lie algebra structure on the space $L_1 \oplus L_2$ by defining $[x_1+x_2, y_1+y_2] = [x_1,y_1]+[x_2,y_2]$ and $\sigma_1 \oplus \sigma_2(x_1+x_2) = \sigma_1(x_1)+\sigma_2(x_2)$ for any $x_1,y_1 \in L_1, x_2,y_2 \in L_2$. We call this hom-Lie algebra the direct sum of (L_1,σ_1) and (L_2,σ_2) and denote it by $(L_1 \oplus L_2, \sigma_1 \oplus \sigma_2)$.*

Definition (2.8). A homomorphism (respectively isomorphism) of hom-Lie algebra $\phi : (L_1,\sigma_1) \to (L_2,\sigma_2)$ is an algebra homomorphism (respectively isomorphism) from L_1 to L_2 such that $\phi \circ \sigma_1 = \sigma_2 \circ \phi$. In other words, the following diagram

$$\begin{tikzcd} L_1 \arrow[r, "\phi"] \arrow[d, "\sigma_1"] & L_2 \arrow[d, "\sigma_2"] \\ L_1 \arrow[r, "\phi"] & L_2 \end{tikzcd}$$

commutes.

PROPOSITION (2.9) ([4]). *Let (L,σ) be a hom-Lie algebra, and let N be any non-associative algebra. Let $\phi : L \to N$ be an algebra homomorphism. Then the following two conditions are equivalent:*

(1) *There exists a linear subspace $U \subseteq N$ containing $\phi(L)$ and a linear map $k : U \to N$ such that $\phi \circ \sigma = k \circ \phi$.*

(2) $\ker\phi \subseteq \ker(\phi \circ \sigma)$.

Moreover, if these conditions are satisfied, then

(i) *k is uniquely determined on $\phi(L)$ by ϕ and σ,*

(ii) *$k|_{\phi(L)}$ is a homomorphism,*

(iii) *$(\phi(L), k|_{\phi(L)})$ is a hom-Lie algebra, and*

(iv) *ϕ is a homomorphism of hom-Lie algebras.*

3. Central extensions of hom-Lie algebras

If U and V are vector spaces, let $Alt^2(U,V)$ denote the space of skew-symmetric forms (alternating mappings) $U \times U \to V$.

Definition (3.1). An extension of a hom-Lie algebra (L,σ) by an abelian hom-Lie algebra (H,σ_H) is a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "\iota"] \arrow[d, "\sigma_H"] & \hat{L} \arrow[r, "pr"] \arrow[d, "\hat{\sigma}"] & L \arrow[r] \arrow[d, "\sigma"] & 0 \\ 0 \arrow[r] & H \arrow[r, "\iota"] & \hat{L} \arrow[r, "pr"] & L \arrow[r] & 0, \end{tikzcd}$$

where $(\hat{L},\hat{\sigma})$ is a hom-Lie algebra. We say that the extension is central if $\iota(H) \subseteq ann(\hat{L}) = \{x \in \hat{L} \mid [x,y] = 0, \forall y \in \hat{L}\}$.

The sequence above splits (as vector spaces) just as in the Lie algebra case, meaning that there is a (linear) section $s : L \to \hat{L}$, i.e., a linear map such that $pr \circ s = id_L$.

THEOREM (3.2) ([4]). *Suppose that (L,σ) and (H,σ_H) are hom-Lie algebras with H abelian. If there exists a central extension $(\hat{L},\hat{\sigma})$ of (L,σ) by (H,σ_H), then for every section $s : L \to \hat{L}$ there is a $g_s \in Alt^2(L,H)$ and a linear map $f_s : \hat{L} \to H$ such that*

$$f_s \circ \iota = \sigma_H, \tag{3.3}$$

$$g_s(\sigma(x),\sigma(y)) = f_s([s(x),s(y)]_{\hat{L}}), \tag{3.4}$$

$$g_s((id+\sigma)(x),[y,z]_L) + g_s((id+\sigma)(y),[z,x]_L) + g_s((id+\sigma)(z),[x,y]_L) = 0 \tag{3.5}$$

for any $x,y,z \in L$. Moreover, the third equation is independent of the choice of section s.

Definition (3.6). A central extension $(\hat{L},\hat{\sigma})$ of (L,σ) by (H,σ_H) is called trivial if there exists a linear section $s : L \to \hat{L}$ such that $g_s(x,y) = 0$ for any $x,y \in L$.

PROPOSITION (3.7). *A central extension is trivial if and only if for any section $s : L \to \hat{L}$ there is a linear map $s_1 : L \to \hat{L}$ such that $(s+s_1)$ is a section and $\iota g_s(x,y) = s_1([x,y]_L)$ for any $x,y \in L$.*

THEOREM (3.8) ([4]). *Suppose that (L,σ) and (H,σ_H) are hom-Lie algebras with H abelian. Then for every $g \in Alt^2(L,H)$ and every linear map $f : L \oplus H \to H$ such that*

$$f(0,a) = \sigma_H(a), \forall a \in H, \tag{3.9}$$

$$g(\sigma(x),\sigma(y)) = f([x,y]_L, g(x+y)), \tag{3.10}$$

$$g((id+\sigma)(x),[y,z]_L) + g((id+\sigma)(y),[z,x]_L) + g((id+\sigma)(z),[x,y]_L) = 0 \tag{3.11}$$

for any $x,y,z \in L$, there exists a hom-Lie algebra $(\hat{L},\hat{\sigma})$, which is a central extension of (L,σ) by (H,σ_H).

4. Extensions of hom-Lie algebras on Lie algebras

It is well known that the classical algebras satisfying the skew symmetry are Lie algebras. Then it is natural to study the hom-Lie algebras on Lie algebras, such as the hom-Lie algebras on semisimple Lie algebras [3]. We mainly develop the extension theory of hom-Lie algebras on Lie algebras.

Definition (4.1). If A is a Lie algebra with the products $[\cdot,\cdot]$ and $\sigma : A \to A$ is a homomorphism of Lie algebras such that the identity

$$[(id+\sigma)(x),[y,z]]+[(id+\sigma)(y),[z,x]]+[(id+\sigma)(z),[x,y]]=0$$

holds for any $x,y,z \in A$, then (A,σ) is called a hom-Lie algebra on a Lie algebra.

Let (L,σ) and (H,σ_H) be hom-Lie algebras on Lie algebras with H abelian. The extension $(\hat{L},\hat{\sigma})$ of (L,σ) by (H,σ_H) is of a hom-Lie algebra in the sense of Definition (3.1) such that $(\hat{L},\hat{\sigma})$ is a hom-Lie algebra on a Lie algebra. To construct an extension $(\hat{L}=L\oplus H,\hat{\sigma})$ of (L,σ) by (H,σ_H), we have two things to do:

1. define the hom-Lie algebra homomorphism $\hat{\sigma}$, and
2. construct the bracket $[\cdot,\cdot]_{\hat{L}}$ with the desired properties.

Note first of all that

$$pr\circ\hat{\sigma}(x)=\sigma\circ pr(x), \forall x\in\hat{L}.$$

This means that

$$pr(\hat{\sigma}(x)-s\circ\sigma\circ pr(x))=0$$

and this leads to, by the exactness,

$$\hat{\sigma}(x)=s\circ\sigma\circ pr(x)+\iota\circ f_s(x), \tag{4.2}$$

where $f_s:\hat{L}\to H$ is a linear map dependent on s. Note that combining (4.2) with the commutativity of the left square in Definition (3.1) we get for $a\in H$ that

$$\iota\circ\sigma_H(a)=\hat{\sigma}\circ\iota(a)=s\circ\sigma\circ pr\circ\iota(a)+\iota\circ f_s\circ\iota(a)=\iota\circ f_s\circ\iota(a),$$

and since ι is an injective,

$$\sigma_H(a)=f_s\circ\iota(a). \tag{4.3}$$

Since for any $x,y\in L$,

$$pr([s(x),s(y)]_{\hat{L}}-s[x,y]_L)=0,$$

we have that

$$[s(x),s(y)]_{\hat{L}}=s[x,y]_L+\iota\circ\omega_s(x,y), \tag{4.4}$$

where $\omega_s\in Alt^2(L,H)$. Since $\hat{L}$ is a Lie algebra, we know that ω_s is a Lie algebra 2-cocycle in the sense of Chevalley-Eilenberg and (H,π) is a representation of L, where π is defined by

$$\iota(\pi(x)a)=[s(x),\iota(a)]_{\hat{L}}. \tag{4.5}$$

THEOREM (4.6). *Suppose that (L,σ) and (H,σ_H) are hom-Lie algebras on Lie algebras with H abelian. If there exists an extension $(\hat{L},\hat{\sigma})$ of (L,σ) by (H,σ_H), then* *for every section $s : L \to \hat{L}$ there is a Lie algebra 2-cocycle $\omega_s : L \times L \to H$ in the sense of Chevalley-Eilenberg and a linear map $f_s : \hat{L} \to H$ such that*

$$\sigma_H(a) = f_s \circ \iota(a), \tag{4.7}$$

$$\sigma_H(\pi(x)a) = \pi(\sigma(x))(\sigma_H(a)), \tag{4.8}$$

$$\sigma_H \circ \omega_s(x,y) - \omega_s(\sigma(x),\sigma(y)) = \pi(\sigma(x))\phi_s(y) - \pi(\sigma(y))\phi_s(x) - \phi_s([x,y]_L), \tag{4.9}$$

$$\pi([x,y]_L)\sigma_H(a) - \pi(\sigma(x))\pi(y)a + \pi(\sigma(y))\pi(x)a = 0, \tag{4.10}$$

$$\circlearrowleft_{x,y,z} (\omega_s(\sigma(x),[y,z]_L) - \pi([y,z]_L)\phi_s(x) + \pi(\sigma(x))\omega_s(y,z)) = 0, \tag{4.11}$$

for any $x,y,z \in L, a \in H$, where $\circlearrowleft_{x,y,z}$ denotes cyclic summation with respect to x,y,z and $\phi_s = f_s \circ s$ is a linear map from L to H.

Proof. Set $\phi_s = f_s \circ s$. Since $\hat{\sigma}$ is a homomorphism of $\hat{L}$, we know that for any $\hat{x},\hat{y} \in \hat{L}$, $\hat{\sigma}[\hat{x},\hat{y}]_{\hat{L}} = [\hat{\sigma}(\hat{x}),\hat{\sigma}(\hat{y})]_{\hat{L}}$. Then for any $x \in L, a \in H$,

$$\hat{\sigma}[s(x),\iota(a)]_{\hat{L}} = [\hat{\sigma} \circ s(x), \hat{\sigma} \circ \iota(a)]_{\hat{L}}.$$

Also we have

$$\begin{aligned}
&[\hat{\sigma} \circ s(x), \hat{\sigma} \circ \iota(a)]_{\hat{L}} \\
&= [s \circ \sigma \circ pr \circ s(x) + \iota \circ f_s \circ s(x), s \circ \sigma \circ pr \circ \iota(a) + \iota \circ f_s \circ \iota(a)]_{\hat{L}} \\
&= [s \circ \sigma(x) + \iota \circ \phi_s(x), \iota \circ \sigma_H(a)]_{\hat{L}} \\
&= [s \circ \sigma(x), \iota \circ \sigma_H(a)]_{\hat{L}} \\
&= \iota(\pi(\sigma(x))(\sigma_H(a))).
\end{aligned}$$

Since ι is injective, by (4.5),

$$\sigma_H(\pi(x)a) = \pi(\sigma(x))(\sigma_H(a)).$$

For any $x,y \in L$, we also have $\hat{\sigma}[s(x),s(y)]_{\hat{L}} = [\hat{\sigma} \circ s(x), \hat{\sigma} \circ s(y)]_{\hat{L}}$. By (4.2) and (4.4),

$$\begin{aligned}
&\hat{\sigma}[s(x),s(y)]_{\hat{L}} \\
&= \hat{\sigma}(s[x,y]_L + \iota \circ \omega_s(x,y)) \\
&= s \circ \sigma \circ pr \circ s[x,y]_L + \iota \circ f_s \circ s[x,y]_L + \iota \circ f_s \circ \iota \circ \omega_s(x,y) \\
&= s \circ \sigma([x,y]_L) + \iota \circ \phi_s([x,y]_L) + \iota \circ \sigma_H \circ \omega_s(x,y); \\
&[\hat{\sigma} \circ s(x), \hat{\sigma} \circ s(y)]_{\hat{L}} \\
&= [s \circ \sigma \circ pr \circ s(x) + \iota \circ f_s \circ s(x), s \circ \sigma \circ pr \circ s(y) + \iota \circ f_s \circ s(y)]_{\hat{L}} \\
&= [s \circ \sigma(x) + \iota \circ \phi_s(x), s \circ \sigma(y) + \iota \circ \phi_s(y)]_{\hat{L}} \\
&= [s \circ \sigma(x), s \circ \sigma(y)]_{\hat{L}} + [s \circ \sigma(x), \iota \circ \phi_s(y)]_{\hat{L}} + [\iota \circ \phi_s(x), s \circ \sigma(y)]_{\hat{L}} \\
&= s[\sigma(x),\sigma(y)]_L + \iota \circ \omega_s(\sigma(x),\sigma(y)) + \iota \circ \pi(\sigma(x))\phi_s(y) - \iota \circ \pi(\sigma(y))\phi_s(x).
\end{aligned}$$

It follows that

$$\sigma_H \circ \omega_s(x,y)) - \omega_s(\sigma(x),\sigma(y)) = \pi(\sigma(x))\phi_s(y) - \pi(\sigma(y))\phi_s(x) - \phi_s([x,y]_L).$$

Since $(\hat{L},\hat{\sigma})$ is a hom-Lie algebra, we know that $\circlearrowleft_{\hat{x},\hat{y},\hat{z}} [(\hat{\sigma} + id)(\hat{x}),[\hat{y},\hat{z}]_{\hat{L}}]_{\hat{L}} = 0$. Thus

$$\circlearrowleft_{\hat{x},\hat{y},\hat{z}} [\hat{\sigma}(\hat{x}),[\hat{y},\hat{z}]_{\hat{L}}]_{\hat{L}} = 0$$

for any $\hat{x},\hat{y},\hat{z} \in \hat{L}$. By (4.2), (4.4) and the linearity of the product, for any $x,y \in L, a \in H$,

$$\begin{aligned}
&[\hat{\sigma}\circ\iota(a),[s(x),s(y)]_{\hat{L}}]_{\hat{L}}\\
&= [s\circ\sigma\circ pr\circ\iota(a)+\iota\circ f_s\circ\iota(a), s[x,y]_L+\iota\circ\omega_s(x,y)]_{\hat{L}}\\
&= -[s[x,y]_L,\iota\circ\sigma_H(a)]_{\hat{L}}\\
&= -\iota(\pi([x,y]_L)\sigma_H(a));\\
&[\hat{\sigma}\circ s(x),[s(y),\iota(a)]_{\hat{L}}]_{\hat{L}}\\
&= [s\circ\sigma\circ pr\circ s(x)+\iota\circ f_s\circ s(x),\iota\circ\pi(y)a]_{\hat{L}}\\
&= [s\circ\sigma(x),\iota\circ\pi(y)a]_{\hat{L}}\\
&= \iota(\pi(\sigma(x))\pi(y)a).
\end{aligned}$$

Then we have that

$$\begin{aligned}
&[\hat{\sigma}\circ\iota(a),[s(x),s(y)]_{\hat{L}}]_{\hat{L}}+[\hat{\sigma}\circ s(x),[s(y),\iota(a)]_{\hat{L}}]_{\hat{L}}+[\hat{\sigma}\circ s(y),[\iota(a),s(x)]_{\hat{L}}]_{\hat{L}}\\
&= -\iota(\pi([x,y]_L)\sigma_H(a)-\pi(\sigma(x))\pi(y)a+\pi(\sigma(y))\pi(x)a),
\end{aligned}$$

which implies that

$$\pi([x,y]_L)\sigma_H(a)-\pi(\sigma(x))\pi(y)a+\pi(\sigma(y))\pi(x)a=0.$$

Also by (4.2), (4.4) and the linearity of the product, for any $x,y,z \in L$,

$$\begin{aligned}
&[\hat{\sigma}\circ s(x),[s(y),s(z)]_{\hat{L}}]_{\hat{L}}\\
&= [\hat{\sigma}\circ s(x), s[y,z]_L+\iota\circ\omega_s(y,z)]_{\hat{L}}\\
&= [\hat{\sigma}\circ s(x), s[y,z]_L]_{\hat{L}}+[\hat{\sigma}\circ s(x),\iota\circ\omega_s(y,z)]_{\hat{L}}\\
&= [s\circ\sigma\circ pr\circ s(x)+\iota\circ f_s\circ s(x), s[y,z]_L]_{\hat{L}}+[s\circ\sigma\circ pr\circ s(x)+\iota\circ f_s\circ s(x),\iota\circ\omega_s(y,z)]_{\hat{L}}\\
&= [s\circ\sigma(x)+\iota\circ\phi_s(x), s[y,z]_L]_{\hat{L}}+[s\circ\sigma(x)+\iota\circ\phi_s(x),\iota\circ\omega_s(y,z)]_{\hat{L}}\\
&= [s\circ\sigma(x), s[y,z]_L]_{\hat{L}}+[\iota\circ\phi_s(x), s[y,z]_L]_{\hat{L}}+[s\circ\sigma(x),\iota\circ\omega_s(y,z)]_{\hat{L}}\\
&\quad +[\iota\circ\phi_s(x),\iota\circ\omega_s(y,z)]_{\hat{L}}\\
&= s[\sigma(x),[y,z]_L]_L+\iota\circ\omega_s(\sigma(x),[y,z]_L)-\iota\circ\pi([y,z]_L)\phi_s(x)+\iota\circ\pi(\sigma(x))\omega_s(y,z)
\end{aligned}$$

Summing up cyclically we get

$$\circlearrowleft_{x,y,z}(\omega_s(\sigma(x),[y,z]_L)-\pi([y,z]_L)\phi_s(x)+\pi(\sigma(x))\omega_s(y,z))=0. \qquad \square$$

THEOREM (4.12). *Suppose that L is a Lie algebra and (H,σ_H) is a hom-Lie algebra on a Lie algebra with H abelian. If there exists a hom-Lie algebra extension $(\hat{L},\hat{\sigma})$ of (L,id) by (H,σ_H), then for every section $s: L\to\hat{L}$ there is a Lie algebra 2-cocycle $\omega_s: L\times L\to H$ in the sense of Chevalley-Eilenberg and a linear map $f_s:\hat{L}\to H$ such that*

$$\sigma_H(a)=f_s\circ\iota(a), \tag{4.13}$$

$$\sigma_H(\pi(x)a)=\pi(x)(\sigma_H(a)), \tag{4.14}$$

$$\sigma_H\circ\omega_s(x,y)-\omega_s(x,y)=\pi(x)\phi_s(y)-\pi(y)\phi_s(x)-\phi_s([x,y]_L), \tag{4.15}$$

$$\pi([x,y]_L)(\sigma_H-id)(a)=0, \tag{4.16}$$

$$\circlearrowleft_{x,y,z}(\pi([y,z]_L)\phi_s(x))=0, \tag{4.17}$$

for any $x,y,z\in L, a\in H$, where $\circlearrowleft_{x,y,z}$ denotes cyclic summation with respect to x,y,z and $\phi_s=f_s\circ s$ is a linear map from L to H.

Proof. The identity (4.17) follows the identity (4.11) and the fact that ω_s is a 2-cocycle. The others are clear. $\square$

For this case, the map σ_H involves cohomology of Lie algebras. First to recall the definition of cohomology of Lie algebras [1]. Let B be a Lie algebra and V be a module of B. For each non-negative integer k let $C^k(B,V)$ denote the space of alternating k-linear $B\times\cdots\times B$ into V where $C^0(B,V)$ is defined to be equal to V and $C^1(B,V)$ is a linear map from B into V. Denote by $C(B,V)$ the direct sum of all the spaces $C^k(B,V)(0\le k<\infty)$. The coboundary operator $d: C(B,V)\to C(B,V)$ is defined by

$$(df)(b_0,b_1,\cdots,b_k)$$
$$=\sum_{i=0}^{k}(-1)^i b_i f(b_0,\cdots,\hat{b}_i,\cdots,b_k)+\sum_{i<j}(-1)^{i+j}f([b_i,b_j],b_0,\cdots,\hat{b}_i,\cdots,\hat{b}_j,\cdots,b_k)$$

for any $f\in C^k(B,V)$ and $b_0,b_1,\cdots,b_k\in B$, where the hat ˆ over a symbol means that it should be omitted. It is known that $d^2=0$. Call any k-form $f\in C^k(B,V)$ a k-cocycle if and only if $df=0$ and denote the subspace of k-cocycles by $Z^k(B,V)$. The k-th cohomology group $H^k(B,V)$ is defined to be the factor space $Z^k(B,V)/dC^{k-1}(B,V)$ for $k\ge 1$ and $Z^0(B,V)$ for $k=0$.

Since H is an L-module defined by (4.5), we have that for any $x_0,x_1,\cdots,x_k\in L$ and $f\in C^k(L,H)$, by (4.14),

$$\begin{aligned}
\sigma_H(df(x_0,x_1,\cdots,x_k)) &= \sigma_H(\sum_{i=0}^{k}(-1)^i\pi(x_i)f(x_0,\cdots,\hat{x}_i,\cdots,x_k)\\
&+\sum_{i<j}(-1)^{i+j}f([x_i,x_j],x_0,\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots,x_k))\\
&=\sum_{i=0}^{k}(-1)^i\sigma_H(\pi(x_i)f(x_0,\cdots,\hat{x}_i,\cdots,x_k))\\
&+\sum_{i<j}(-1)^{i+j}\sigma_H(f([x_i,x_j],x_0,\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots,x_k))\\
&=\sum_{i=0}^{k}(-1)^i\pi(x_i)\sigma_H(f(x_0,\cdots,\hat{x}_i,\cdots,x_k))\\
&+\sum_{i<j}(-1)^{i+j}\sigma_H(f([x_i,x_j],x_0,\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots,x_k))\\
&=\sum_{i=0}^{k}(-1)^i\pi(x_i)(\sigma_H\circ f)(x_0,\cdots,\hat{x}_i,\cdots,x_k)\\
&+\sum_{i<j}(-1)^{i+j}(\sigma_H\circ f)([x_i,x_j],x_0,\cdots,\hat{x}_i,\cdots,\hat{x}_j,\cdots,x_k)\\
&= d(\sigma_H\circ f)(x_0,\cdots,x_k).
\end{aligned}$$

It follows that

$$\sigma_H(dC^{k-1}(L,H))\subseteq dC^{k-1}(L,H),$$
$$\sigma_H(Z^k(L,H))\subseteq Z^k(L,H).$$

Then there is a linear map σ_H^* on $H^k(L,H)$ induced by σ_H. In particular, by (4.15), for the 2-cocycle ω_s,

$$\sigma_H \circ \omega_s(x,y) - \omega_s(x,y) = d\phi_s(x,y), \tag{4.18}$$

where $\phi_s \in C^1(L,H)$. It follows that $\sigma_H^* \bar{\omega}_s = \bar{\omega}_s$. That is, we have the following theorem:

THEOREM (4.19). *Suppose that L is a Lie algebra and (H,σ_H) is a hom-Lie algebra on a Lie algebra with H abelian, $(\hat{L},\hat{\sigma})$ is an extension of (L,id) by (H,σ_H), ω_s and f_s are the 2-cocycle and the linear map satisfying the identities* (4.13) – (4.17) *corresponding to some section s. Then the element in the cohomology group $H^2(L,H)$ induced by ω_s is a fixed element of the linear map σ_H^*, where σ_H^* is the linear map on $H^k(L,H)$ induced by σ_H for any non-negative integer k.*

5. Extensions of hom-Lie algebras on Lie algebras with twisting automorphisms

It is natural to discuss a class of special hom-Lie algebras, hom-Lie algebras $(\mathfrak{g},\theta)$ on Lie algebras with θ automorphism of Lie algebras. For a given hom-Lie algebra $(\mathfrak{g},\theta)$ on a Lie algebra with θ automorphism, then θ-twisted Jacobi identity

$$\circlearrowleft_{x,y,z} [\theta(x),[y,z]] = 0$$

holds for any $x,y,z \in \mathfrak{g}$ if and only if

$$\circlearrowleft_{x,y,z} [x,\theta^{-1}[y,z]] = 0. \tag{5.1}$$

In this section, assume that (L,σ) and (H,σ_H) are hom-Lie algebras on Lie algebras with σ_H and σ automorphisms and H abelian and $(\hat{L},\hat{\sigma})$ is an extension of (L,σ) by (H,σ_H). Then $\hat{\sigma}$ is an automorphism. Furthermore we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & H \arrow[r, "\iota"] & \hat{L} \arrow[r, "pr"] & L \arrow[r] & 0 \\
0 \arrow[r] & H \arrow[r, "\iota"] \arrow[u, "\sigma_H^{-1}"'] & \hat{L} \arrow[r, "pr"] \arrow[u, "\hat{\sigma}^{-1}"'] & L \arrow[r] \arrow[u, "\sigma^{-1}"'] & 0.
\end{tikzcd}$$

Similar to the discussion in Section 4, note first of all that

$$pr \circ \hat{\sigma}^{-1}(x) = \sigma^{-1} \circ pr(x), \forall x \in \hat{L}.$$

This means that

$$pr(\hat{\sigma}^{-1}(x) - s \circ \sigma^{-1} \circ pr(x)) = 0$$

and this leads to, by the exactness,

$$\hat{\sigma}^{-1}(x) = s \circ \sigma^{-1} \circ pr(x) + \iota \circ g_s(x), \tag{5.2}$$

where $g_s : \hat{L} \to H$ is a linear map dependent on s. Combining (5.2) with the commutativity of the left square in the above digram we get for $a \in H$ that

$$\iota \circ \sigma_H^{-1}(a) = \hat{\sigma}^{-1} \circ \iota(a) = s \circ \sigma^{-1} \circ pr \circ \iota(a) + \iota \circ g_s \circ \iota(a) = \iota \circ g_s \circ \iota(a),$$

and since ι is an injective,

$$\sigma_H^{-1}(a) = g_s \circ \iota(a). \tag{5.3}$$

Also we have that for any $x, y \in L$

$$[s(x), s(y)]_{\hat{L}} = s[x,y]_L + \iota \circ \omega_s(x,y), \tag{5.4}$$

where ω_s is a Lie algebra 2-cocycle in the sense of Chevalley-Eilenberg and (H,π) is a representation of L, where π is defined by

$$\iota(\pi(x)a) = [s(x), \iota(a)]_{\hat{L}}. \tag{5.5}$$

THEOREM (5.6). *Suppose that (L,σ) and (H,σ_H) are hom-Lie algebras on Lie algebras with H abelian and σ_H and σ automorphism. If there exists an extension $(\hat{L},\hat{\sigma})$ of (L,σ) by (H,σ_H), then for every section $s : L \to \hat{L}$ there is a Lie algebra 2-cocycle $\omega_s : L \times L \to H$ in the sense of Chevalley-Eilenberg and a linear map $g_s : \hat{L} \to H$ such that*

$$\sigma_H^{-1}(a) = f_s \circ \iota(a), \tag{5.7}$$

$$\sigma_H^{-1}(\pi(x)a) = \pi(\sigma^{-1}(x))(\sigma_H^{-1}(a)), \tag{5.8}$$

(5.9)

$$\sigma_H^{-1} \circ \omega_s(x,y) - \omega_s(\sigma^{-1}(x), \sigma^{-1}(y)) = \pi(\sigma^{-1}(x))\phi_s(y) - \pi(\sigma^{-1}(y))\phi_s(x) - \phi_s([x,y]_L),$$

$$\pi(\sigma^{-1}[x,y]_L)a - \pi(x)(\sigma_H^{-1}(\pi(y)a)) + \pi(y)(\sigma_H^{-1}(\pi(x)a)) = 0, \tag{5.10}$$

$$\circlearrowleft_{x,y,z} (\omega_s(x, \sigma^{-1}[y,z]_L) + \pi(x)\phi_s([y,z]_L) + \pi(x)(\sigma_H^{-1} \circ \omega_s(y,z)) = 0, \tag{5.11}$$

for any $x,y,z \in L, a \in H$, where $\circlearrowleft_{x,y,z}$ denotes cyclic summation with respect to x,y,z and $\phi_s = g_s \circ s$ is a linear map from L to H.

Proof. The proof is similar to that of Theorem (4.6). □

THEOREM (5.12). *Suppose that $(\hat{L},\hat{\sigma})$ is an extension of (L,σ) by (H,σ_H), where (L,σ) and (H,σ_H) are hom-Lie algebras on Lie algebras with H abelian and σ_H and σ automorphisms, ω_s and g_s are the 2-cocycle and the linear map corresponding to some section s respectively and $\phi_s = g_s \circ s$. If s_1 is a section such that $\omega_{s_1} = \omega_s$, then $\Omega \in Z^2(L,H)$, where $\Omega(x,y) = (\phi_s - \phi_{s_1})[x,y]_L$.*

Proof. By Theorem (5.6), for sections s, s_1, we have that

$$\circlearrowleft_{x,y,z} (\omega_s(x, \sigma^{-1}[y,z]_L) + \pi(x)\phi_s([y,z]_L) + \pi(x)(\sigma_H^{-1} \circ \omega_s(y,z)) = 0,$$
$$\circlearrowleft_{x,y,z} (\omega_{s_1}(x, \sigma^{-1}[y,z]_L) + \pi(x)\phi_{s_1}([y,z]_L) + \pi(x)(\sigma_H^{-1} \circ \omega_{s_1}(y,z)) = 0.$$

Set $\Omega(x,y) = (\phi_s - \phi_{s_1})[x,y]_L$. Then $\Omega \in C^2(L,H)$. By the above two identities and the assumption $\omega_{s_1} = \omega_s$, we have that

$$\circlearrowleft_{x,y,z} \pi(x)\Omega(y,z) = \circlearrowleft_{x,y,z} \pi(x)(\phi_s - \phi_{s_1})[y,z]_L = 0.$$

Thus for any $x,y,z \in L$,

$$\begin{aligned} & d\Omega(x,y,z) \\ &= \pi(x)\Omega(y,z) - \pi(y)\Omega(x,z) + \pi(z)\Omega(x,y) - \Omega([x,y],z) - \Omega([y,z],x) + \Omega([x,z],y) \\ &= \pi(x)\Omega(y,z) + \pi(y)\Omega(z,x) + \pi(z)\Omega(x,y) - (\phi_s - \phi_{s_1})([[x,y],z] + [[y,z],x] + [[z,x],y]) \\ &= \circlearrowleft_{x,y,z} \pi(x)\Omega([y,z]_L) = 0. \end{aligned}$$

That is, the theorem follows. □

THEOREM (5.13). *Suppose that $(\hat{L},\hat{\sigma})$ is an extension of (L,id) by (H,σ_H), where (L,id) and (H,σ_H) are hom-Lie algebras on Lie algebras with H abelian and σ_H automorphism, ω_s and g_s are the 2-cocycle and the linear map corresponding to some section s respectively and $\phi_s = g_s \circ s$. If s_1 is a section such that $\omega_{s_1} = \omega_s$, then*

$$\Lambda \in Z^1(L,H) \quad and \quad \Omega \in Z^2(L,H),$$

where $\Lambda(x) = (\phi_s - \phi_{s_1})(x)$ and $\Omega(x,y) = (\phi_s - \phi_{s_1})[x,y]_L$.

Proof. By (5.9) for $\sigma = id$, we know that,

$$\sigma_H^{-1} \circ \omega_s(x,y) - \omega_s(x,y) = d\phi_s(x,y).$$

Similarly, for the section s_1, we have that

$$\sigma_H^{-1} \circ \omega_{s_1}(x,y) - \omega_{s_1}(x,y) = d\phi_{s_1}(x,y).$$

Set $\Lambda(x) = (\phi_s - \phi_{s_1})(x)$ and $\Omega(x,y) = (\phi_s - \phi_{s_1})[x,y]_L$. Then by the assumption $\omega_{s_1} = \omega_s$ and the above identities, we have that

$$d\Lambda(x,y) = 0.$$

That is, $\Lambda \in Z^1(L,H)$ and the theorem follows from Theorem (5.12). □

6. Examples

Let L be a non-abelian Lie algebra in dimension 2 over $\mathbb{C}$ and σ be a homomorphism of L. It is easy to see that (L,σ) is a hom-Lie algebra if and only if there exists a basis $\{x,y\}$ such that

1. $[x,y] = y$, $\sigma = 0$, or
2. $[x,y] = y$, $\sigma(x) = y$, $\sigma(y) = 0$, or
3. $[x,y] = y$, $\sigma(x) = ax$ for $a \neq 0$, $\sigma(y) = 0$or
4. $[x,y] = y$, $\sigma(x) = x$, $\sigma(y) = ay$ for $a \neq 0$, or
5. $[x,y] = y$, $\sigma(x) = x + y$, $\sigma(y) = y$;

Assume that V is a one-dimensional non-trivial module of L. Then there exists a basis $\{z\}$ of V such that

$$x \cdot z = bz, \quad y \cdot z = 0, \tag{6.1}$$

where $b \neq 0$. It is easy to check that $\dim H^2(L,V) = 1$. As a Lie algebra, the abelian extension $\hat{L}$ of L by V is defined by

$$[x,y] = y + cz, \quad [x,z] = bz. \tag{6.2}$$

Let $\hat{\sigma}$ be a homomorphism of $\hat{L}$ such that $\hat{\sigma}(z) = dz$. Then $(\hat{L},\hat{\sigma})$ is a hom-Lie algebra if and only if there exists a basis $\{x,y,z\}$ of $\hat{L}$ such that

1. $[x,y] = y + z$, $[x,z] = z$, $\hat{\sigma}(x) = x + k_1 y + k_2 z$, $\hat{\sigma}(y) = dy + k_3 z$, $\hat{\sigma}(z) = dz \neq 0$, or
2. $[x,y] = y + z$, $[x,z] = z$, $\hat{\sigma}(x) = kx$, $\hat{\sigma}(y) = \hat{\sigma}(z) = 0$, or
3. $[x,y] = y + z$, $[x,z] = z$, $\hat{\sigma}(x) = x + k_1 y$, $\hat{\sigma}(y) = k_2 y + k_3 z \neq 0$, $\hat{\sigma}(z) = 0$, or
4. $[x,y] = y$, $[x,z] = bz$, $\hat{\sigma}(x) = kx$, $\hat{\sigma}(y) = \hat{\sigma}(z) = 0$, or
5. $[x,y] = y$, $[x,z] = z$, $\hat{\sigma}(x) = x + k_1 y$, $\hat{\sigma}(y) = k_2 y + k_3 z \neq 0$, $\hat{\sigma}(z) = 0$, or
6. $[x,y] = y$, $[x,z] = bz$ for $b \neq 1$, $\hat{\sigma}(x) = bx$, $\hat{\sigma}(y) = kz \neq 0$, $\hat{\sigma}(z) = 0$, or
7. $[x,y] = y$, $[x,z] = bz$ for $b \neq 1$, $\hat{\sigma}(x) = x + k_1 y$, $\hat{\sigma}(y) = k_2 y \neq 0$, $\hat{\sigma}(z) = 0$, or
8. $[x,y] = y$, $[x,z] = bz$, $\hat{\sigma}(x) = bx + k_1 y + k_2 z$, $\hat{\sigma}(y) = 0$, $\hat{\sigma}(z) = dz \neq 0$, or
9. $[x,y] = y$, $[x,z] = z$, $\hat{\sigma}(x) = x + k_1 y + k_2 z$, $\hat{\sigma}(y) = k_3 y + k_4 z \neq 0$, $\hat{\sigma}(z) = dz \neq 0$, or
10. $[x,y] = y$, $[x,z] = -z$, $\hat{\sigma}(x) = -x + k_1 z$, $\hat{\sigma}(y) = k_2 z \neq 0$, $\hat{\sigma}(z) = dz \neq 0$.

In fact, this gives the classification of 1-dimensional abelian extensions of the 2-dimensional hom-Lie algebras on the non-abelian Lie algebra.

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ON SIMILAR MATRICES AND THEIR PRODUCTS

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ABSTRACT. Let $\mathrm{GL}(n,q)$ be the group of $n\times n$ invertible matrices over a field with q elements, and $\mathrm{SL}(n,q)$ be the group of $n\times n$ matrices with determinant 1 over a field with q elements. We prove that the product of any two non-central conjugacy classes in $\mathrm{GL}(n,q)$ is the union of at least $q-1$ distinct conjugacy classes, and that the product of any two non-central conjugacy classes in $\mathrm{SL}(n,q)$ is the union of at least $\lceil \frac{q}{2} \rceil$ distinct conjugacy classes.

1. Introduction

Let $\mathcal{G}$ be a finite group and $A \in \mathcal{G}$. Denote by $A^{\mathcal{G}} = \{A^B \mid B \in \mathcal{G}\}$ the conjugacy class of A in $\mathcal{G}$. Let $\mathcal{X}$ be a $\mathcal{G}$-invariant subset of $\mathcal{G}$, i.e. $\mathcal{X}^A = \{B^A \mid B \in \mathcal{X}\} = \mathcal{X}$ for all $A \in \mathcal{G}$. Then $\mathcal{X}$ can be expressed as a union of n distinct conjugacy classes of $\mathcal{G}$, for some integer $n > 0$. Set $\eta(\mathcal{X}) = n$.

Given any conjugacy classes $A^{\mathcal{G}}, B^{\mathcal{G}}$ in $\mathcal{G}$, we can check that the product $A^{\mathcal{G}}B^{\mathcal{G}} = \{XY \mid X \in A^{\mathcal{G}}, Y \in B^{\mathcal{G}}\}$ is a $\mathcal{G}$-invariant subset of $\mathcal{G}$ and thus $A^{\mathcal{G}}B^{\mathcal{G}}$ is the union of $\eta(A^{\mathcal{G}}B^{\mathcal{G}})$ distinct conjugacy classes of $\mathcal{G}$. For instance, if A or B is in the center $\mathbf{Z}(\mathcal{G})$ of $\mathcal{G}$, then $A^{\mathcal{G}}B^{\mathcal{G}} = (AB)^{\mathcal{G}}$ and thus $\eta(A^{\mathcal{G}}B^{\mathcal{G}}) = 1$. The number $\eta(A^{\mathcal{G}}B^{\mathcal{G}})$ is related to the (non-vanishing) structure constants for the center of the group algebra $C\mathcal{G}$ and can (at least in principle) be calculated from the character table for $\mathcal{G}$.

It is proved in [4] that for any integer $n > 5$, given any nontrivial conjugacy classes α^{S_n} and β^{S_n} of the symmetric group S_n of n letters, that is $\alpha, \beta \in S_n \setminus \{e\}$, if n is a multiple of two or of three, the product $\alpha^{S_n}\beta^{S_n}$ is the union of at least two distinct conjugacy classes, i.e. $\eta(\alpha^{S_n}\beta^{S_n}) \geq 2$, otherwise the product $\alpha^{S_n}\beta^{S_n}$ is the union of at least three distinct conjugacy classes, i.e. $\eta(\alpha^{S_n}\beta^{S_n}) \geq 3$. A similar result is proved for the alternating group A_n in [2]. If p is a prime number, in [1] it is proved that given any p-group P and any conjugacy classes a^P and b^P of P of size p, the product $a^P b^P$ is either a conjugacy class or the union of at least $\frac{p+1}{2}$ distinct conjugacy classes.

Fix a prime p and integers $m > 0$ and $n \geq 2$. Let $\mathcal{F} = \mathcal{F}(q)$ be a field with $q = p^m$ elements and $\mathcal{G} = \mathrm{GL}(n,q)$ be the general linear group of $n \times n$ invertible matrices over $\mathcal{F}$. Then a conjugacy class $A^{\mathcal{G}}$ in $\mathcal{G}$ is the set of all similar matrices to A. Given any two noncentral conjugacy classes $A^{\mathcal{G}}, B^{\mathcal{G}}$ of $\mathcal{G}$, is there any relationship among n, q and $\eta(A^{\mathcal{G}}B^{\mathcal{G}})$?

THEOREM (1.1). *Let A and B be matrices in $\mathcal{G} = \mathrm{GL}(n,q)$. Then one of the following holds:*
(i) $A^{\mathcal{G}}B^{\mathcal{G}} = (AB)^{\mathcal{G}}$ and at least one of A, B is a scalar matrix.

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(ii) $A^{\mathcal{G}}B^{\mathcal{G}}$ *is the union of at least* $q-1$ *distinct conjugacy classes, i.e.* $\eta(A^{\mathcal{G}}B^{\mathcal{G}}) \geq q-1$.

Given any group G, denote by $\min(G)$ the smallest integer in the set $\{\eta(a^G b^G) \mid a,b \in G \setminus \mathbf{Z}(G)\}$. We want to emphasize that the previous result is not an optimal result, that is, $\min(\mathrm{GL}(n,q)) > q-1$ for certain values of n and q. By Remark 2.15, we have that $\min(\mathrm{GL}(2,2^m)) = 2^m - 1$ for any integer $m > 0$. Also, using GAP [6], we can check that $\min(\mathrm{GL}(2,q)) = q-1$ for $q \in \{3,5,7,9,11,13\}$, but $\mathrm{GL}(3,3) = 4 > 2$. Hence, we suspect that $\min(\mathrm{GL}(n,q))$ should be a function of n as well as q.

We now turn our attention to the special linear group $\mathrm{SL}(n,q)$, the group of $n \times n$ matrices with determinant 1 over a finite field with q elements. For example, in the case of $\mathrm{SL}(2,q)$ with $q \geq 4$ even, by a conjecture of Arad and Herzog [5], since $\mathrm{SL}(2,q) = \mathrm{PSL}(2,q)$ is simple nonabelian, we must have that the product of two nontrivial conjugacy classes $A^{\mathrm{SL}(2,q)}$, $B^{\mathrm{SL}(2,q)}$ of $\mathrm{SL}(2,q)$ is never a conjugacy class, i.e $\eta(A^{\mathrm{SL}(2,q)}B^{\mathrm{SL}(2,q)}) > 1$. In [3], $\min(\mathrm{SL}(2,q))$ is given for any q. We now consider $\mathrm{SL}(n,q)$ for any integer $n \geq 2$.

THEOREM (1.2). *Let* A *and* B *be matrices in* $\mathcal{S} = \mathrm{SL}(n,q)$*. Then one of the following holds:*

(i) $A^{\mathcal{S}}B^{\mathcal{S}} = (AB)^{\mathcal{S}}$ *and at least one of* A, B *is a scalar matrix.*

(ii) $A^{\mathcal{S}}B^{\mathcal{S}}$ *is the union of at least* $\lceil \frac{q}{2} \rceil$ *distinct conjugacy classes, i.e.* $\eta(A^{\mathcal{S}}B^{\mathcal{S}}) \geq \lceil \frac{q}{2} \rceil$.

Given a group G, let $\max(G)$ denote the largest integer in the set $\{\eta(a^G b^G) \mid a,b \in G\}$. Our computations suggest that $\max(\mathrm{GL}(2,q)) = \max(\mathrm{SL}(2,q)) = q+1$ for q even, $\max(\mathrm{GL}(2,q)) = q+2$ for odd q, and $\max(\mathrm{SL}(2,q)) = q+3+\frac{(q+1) \bmod 4}{2}$ for odd $q > 3$. These do not hold for $n > 2$, however. For instance, $\max(\mathrm{GL}(3,3)) = \max(\mathrm{SL}(3,3)) = 12$. Still, at least for our limited data, $\max(\mathrm{SL}(n,q))$ is always $\nu - 1$ or ν, where ν is the number of conjugacy classes of $\mathrm{SL}(n,q)$, with the exception of $\max(\mathrm{SL}(2,3)) = 3$.

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2. Proofs

LEMMA (2.1). *Let* G *be a finite group,* a^G *and* b^G *be conjugacy classes of* G*. Then* $a^G b^G = b^G a^G$.

Proof. See Lemma 3 of [4]. □

LEMMA (2.2). *Let* $\mathcal{F}$ *be a finite field with* q *elements, and* $a,b,c \in \mathcal{F}$ *with* $a \neq 0$.

(i) If q *is even, then the set* $\{ai^2 + c \mid i \in \mathcal{F}\}$ *has* q *elements.*

(ii) The set $\{ai^2 + bi + c \mid i \in \mathcal{F}\}$ *has at least* $\lceil \frac{q}{2} \rceil$ *elements.*

Proof. (i) If the field $\mathcal{F}$ has even characteristic, the map $x \mapsto x^2$ is an automorphism of the field. Observe then that

$$|\{ai^2 + c \mid i \in \mathcal{F}\}| = |\{ai^2 \mid i \in \mathcal{F}\}| = |\{i^2 \mid i \in \mathcal{F}\}| = q.$$

(ii) Observe that $ai^2 + bi + c = aj^2 + bj + c$ for $i \neq j$ if and only if $a(i^2 - j^2) = a(i+j)(i-j) = -b(i-j)$, and thus if and only if $i + j = \frac{-b}{a}$, i.e $j = -i + \frac{-b}{a}$. Thus given any $i \in \mathcal{F}$, we can find at most one other element $j \in \mathcal{F}$ such that $j \neq i$ and

$ai^2+bi+c=aj^2+bj+c$, namely $j=-i+\frac{-b}{a}$. Hence, the set $\{ai^2+bi+c \mid i\in\mathcal{F}\}$ has at least $\lceil\frac{q}{2}\rceil$ elements. (In fact, the set has exactly $\lceil\frac{q}{2}\rceil$ elements, except when q even and $b=0$.) $\square$

LEMMA (2.3). *Let $\mathcal{F}$ be a finite field with q elements. Fix a,b,c,d and e in $\mathcal{F}$. Then for at least $q-1$ distinct values of f in $\mathcal{F}$, the equation*

$$ax^2-y^2+bxy+cy+(d-f)x+e=0 \tag{2.4}$$

has a solution $(x,y)\in\mathcal{F}\times\mathcal{F}$ with $x\neq 0$. In particular, the set $\{\frac{1}{x}(ax^2-y^2+bxy+cy+e)+d \mid (x,y)\in\mathcal{F}\times\mathcal{F}, x\neq 0\}$ has at least $q-1$ elements.

Proof. Suppose $a=0$. If $e\neq 0$, then $(\frac{-e}{d-f},0)$ is a solution for the equation as long as $f\neq d$. If $e=0$, choose $y\neq 0$. Then a solution for (2.4) is $(\frac{y^2-cy}{by+d-f},y)$, for all $f\neq by+d$. Therefore in each case, we exclude only one possible value of f, giving at least $q-1$ values.

We may assume then that $a\neq 0$.

Case 1. Assume that q is even and $q>2$. (The result follows trivially when $q=2$.)

Suppose $b=0$. If $f=d$, choose y such that $-y^2+cy+e\neq 0$. Then the set $\{ax^2-y^2+cy+e \mid x\in\mathcal{F}\}$ has q elements by Lemma 2.2 (i). Hence, (2.4) has a solution with $x\neq 0$.

Suppose $f\neq d$. For each y, choose $x=\frac{-cy}{d-f}$, so that $cy+(d-f)x=0$. By Lemma 2.2 (i), the set $\{ax^2-y^2+e \mid y\in\mathcal{F}, x=\frac{-cy}{d-f}\}=\{(\frac{-ac^2}{(d-f)^2}-1)y^2+e \mid y\in\mathcal{F}\}$ has q elements, as long as $-ac^2\neq(d-f)^2$. Then for some y, $(\frac{-cy}{d-f},y)$ is a solution for (2.4), and $x\neq 0$ as long as $c\neq 0$ and $y\neq 0$. If $c=0$, then the set $\{ax^2-y^2+(d-f)x+e \mid y\in\mathcal{F}, x=1\}$ has q elements by Lemma 2.2 (i), and so for some y, $(1,y)$ is a solution for (2.4). If $y=0$, since $(\frac{-cy}{d-f},y)$ was a solution, we must have that $e=0$, and so $(-\frac{d-f}{a},0)$ is a solution for (2.4) with $x\neq 0$.

We may assume then that $b\neq 0$. If $c\neq 0$, then consider $x=\frac{-c}{b}$ and so $bxy+cy=0$. As before, the set $\{ax^2-y^2+(d-f)x+e \mid y\in\mathcal{F}, x=\frac{-c}{b}\}$ has q elements and thus for some y, we have that $(\frac{-c}{b},y)$ is a solution for (2.4). We may assume then that $c=0$. Let $y=-\frac{d-f}{b}$ and thus $bxy+(d-f)x=0$. Then the set $\{ax^2-y^2+e \mid x\in\mathcal{F}\}$ has q elements. Thus, for some x, we have that (x,y) is a solution for (2.4). If $x=0$ then $-y^2+e=0$ and since the field is of characteristic 2, then there is a unique $f\in\mathcal{F}$ such that $-(-\frac{d-f}{b})^2+e=0$. We conclude that in each case, for at least $q-1$ values of f, there exists a solution for (2.4) with $x\neq 0$ when q is even.

Case 2. Assume that q is odd.

Solving for x with the quadratic formula, we get the discriminant

$$\Delta=(by+d-f)^2-4a(e+cy-y^2)=y^2(b^2+4a)+y(2b(d-f)-4ac)+((d-f)^2-4ae)$$

which takes on at least $(q+1)/2$ values as long as b^2+4a and $2b(d-f)-4ac$ are not both zero.

Suppose $b^2+4a=0$ and assume that c^2+4e is a square. If $2b(d-f)-4ac=0$, $(\frac{-c+2y+\sqrt{c^2+4e}}{b},y)$ is a solution for (2.4). Observe that if for some y, $(0,y)$ is a solution, then $-y^2+cy+e=0$. Thus $y=\frac{c\pm\sqrt{c^2+4e}}{2}$ and $(\frac{f-d-by}{a},y)$ is another solution for (2.4). Thus, for any value of f such that $f\neq d+by$, there exists some $x\neq 0$ such that (2.4) holds. If $2b(d-f)-4ac\neq 0$, the discriminant Δ must be a square for some y, since $\frac{q+1}{2}$ elements of $\mathcal{F}$ are squares and thus $(\frac{-(by+d-f)+\sqrt{\Delta}}{2a}, y)$ is a solution for (2.4). If $(0, y)$ is a solution, then as above, $y = \frac{c\pm\sqrt{c^2+4e}}{2}$ and $(\frac{f-d-by}{a}, y)$ is another solution. Thus, for $f \neq d + by$, there exists some $x \neq 0$ such that (2.4) holds.

Suppose $b^2 + 4a = 0$ and assume that $c^2 + 4e$ is not a square. For all f such that $2b(d-f) - 4ac \neq 0$, the discriminant Δ must be a square for some y, and thus $(\frac{-(by+d-f)+\sqrt{\Delta}}{2a}, y)$ is a solution for (2.4). If $(0, y)$ is a solution, then $-y^2 + cy + e = 0$ and so $c^2 + 4e$ is a square. But $c^2 + 4e$ is not a square by assumption. Hence, (x, y) is a solution for (2.4) with $x = \frac{-(by+d-f)+\sqrt{\Delta}}{2a} \neq 0$. Thus, for any f such that $2b(d-f) - 4ac \neq 0$, there exists some $x \neq 0$ such that (2.4) holds.

Now, suppose that $b^2 + 4a \neq 0$. Δ must be a square for some y, and so $(\frac{-(by+d-f)+\sqrt{\Delta}}{2a}, y)$ is a solution for (2.4). If $(0, y)$ is a solution, then $y = \frac{c\pm\sqrt{c^2+4e}}{2}$ and $(\frac{f-d-by}{a}, y)$ is another solution. Thus, for $f \neq d + by$, there exists some $x \neq 0$ such that (2.4) holds.

We conclude that in each case, for at least $q - 1$ values of f, there exists a solution (x, y) for (2.4) with $x \neq 0$.

Observe that $\frac{1}{x}(ax^2 - y^2 + bxy + cy + e) + d = f$ if and only if $ax^2 - y^2 + bxy + cy + (d-f)x + e = 0$ for some $(x, y) \in \mathcal{F} \times \mathcal{F}$ with $x \neq 0$. The last statement then follows. □

Notation. We will denote matrices with uppercase letters and elements in $\mathcal{F}$ with lowercase letters.

REMARK (2.5). *Let A be an $n \times n$ matrix over $\mathcal{F}$. It is well known that A is similar to a matrix M such that M is the direct sum of the companion matrices of a family of polynomials $p_1, \ldots, p_t$ in $\mathcal{F}[x]$.*

Recall that the companion matrix of a polynomial $x^r + \lambda_{r-1}x^{r-1} + \cdots + \lambda_0 \in \mathcal{F}[x]$ is

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\lambda_0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -\lambda_1 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & -\lambda_2 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & -\lambda_3 \\ \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -\lambda_{r-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda_{r-1} \end{pmatrix} \tag{2.6}$$

Theorem A and B follow from the following remark

REMARK (2.7). *Two matrices in the same conjugacy class have the same trace. Thus, if the matrices do not have the same trace, then they belong to distinct conjugacy classes.*

We clearly then have

LEMMA (2.8). *Let $\mathcal{H}$ be a subgroup of* GL(n, q) *and A, B in $\mathcal{H}$. Suppose that the set* $\{\mathrm{Trace}(XY) \mid X \in A^{\mathcal{H}}B^{\mathcal{H}}\}$ *has at least r elements. Then $A^{\mathcal{H}}B^{\mathcal{H}}$ is the union of at least r distinct conjugacy classes of $\mathcal{H}$, i.e. $\eta(A^{\mathcal{H}}B^{\mathcal{H}}) \geq r$.*

LEMMA (2.9). *Let R be an $r\times r$ matrix as in* (2.6) *with $r\geq 2$, I be the $(r-2)\times(r-2)$ identity matrix, $D=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with determinant $w=ad-bc\neq 0$. and 0 be a matrix of zeros with the appropriate size. Set $E=\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}$. Then for $r=2$,*

$$R^E=\frac{1}{w}\begin{pmatrix} -\lambda_0 cd-ab+\lambda_1 bc & -\lambda_0 d^2-b^2+\lambda_1 bd \\ \lambda_0 c^2+a^2-a\lambda_1 c & \lambda_0 cd+ab-a\lambda_1 d \end{pmatrix},$$

and for $r>2$, R^E is the $r\times r$ matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\lambda_0 c & -\lambda_0 d \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -\lambda_1 c & -\lambda_1 d \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & -\lambda_2 c & -\lambda_2 d \\ \vdots & & & & & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -\lambda_{r-3} c & -\lambda_{r-3} d \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{d}{w} & \frac{-\lambda_{r-2}cd-ab+\lambda_{r-1}bc}{w} & \frac{-\lambda_{r-2}d^2-b^2+\lambda_{r-1}bd}{w} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-c}{w} & \frac{\lambda_{r-2}c^2+a^2-a\lambda_{r-1}c}{w} & \frac{\lambda_{r-2}cd+ab-a\lambda_{r-1}d}{w} \end{pmatrix}.$$

Proof. Suppose $r=2$. Then

$$\begin{aligned} R^E &= \frac{1}{w}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 0 & -\lambda_0 \\ 1 & -\lambda_1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \frac{1}{w}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} -\lambda_0 c & -\lambda_0 d \\ a-\lambda_1 c & b-\lambda_1 d \end{pmatrix} \\ &= \frac{1}{w}\begin{pmatrix} d(-\lambda_0 c)-b(a-\lambda_1 c) & d(-\lambda_0 d)-b(b-\lambda_1 d) \\ -c(-\lambda_0 c)+a(a-\lambda_1 c) & -c(-\lambda_0 d)+a(b-\lambda_1 d) \end{pmatrix}. \end{aligned}$$

For $r>2$, $RE=\begin{pmatrix} R_{11} & R_{12} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}=\begin{pmatrix} R_{11} & R_{12}D \end{pmatrix}$, where R_{11} and R_{12} are $r\times(r-2)$ and $r\times 2$ submatrices of R, respectively. Hence,

$$RE=\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\lambda_0 c & -\lambda_0 d \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -\lambda_1 c & -\lambda_1 d \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & -\lambda_2 c & -\lambda_2 d \\ \vdots & & & & & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & -\lambda_{r-3} c & -\lambda_{r-3} d \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\lambda_{r-2} c & -\lambda_{r-2} d \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a-\lambda_{r-1} c & b-\lambda_{r-1} d \end{pmatrix}.$$

Observe that

$$R^E=E^{-1}(RE)=\begin{pmatrix} I & 0 \\ 0 & D^{-1} \end{pmatrix}\begin{pmatrix} (RE)_{11} \\ (RE)_{21} \end{pmatrix}=\begin{pmatrix} (RE)_{11} \\ D^{-1}(RE)_{21} \end{pmatrix},$$

where $(RE)_{11}$ and $(RE)_{21}$ are $(r-2)\times r$ and $2\times r$ submatrices of RE, respectively. Hence, the first $r-2$ rows of R^E are identical to those of RE, and the last two form the submatrix

$$\begin{pmatrix} 0 & \cdots & 0 & \frac{d}{w} & \frac{d(-\lambda_{r-2}c)-b(a-\lambda_{r-1}c)}{w} & \frac{d(-\lambda_{r-2}d)-b(b-\lambda_{r-1}d)}{w} \\ 0 & \cdots & 0 & \frac{-c}{w} & \frac{-c(-\lambda_{r-2}c)+a(a-\lambda_{r-1}c)}{w} & \frac{-c(-\lambda_{r-2}d)+a(b-\lambda_{r-1}d)}{w} \end{pmatrix}. \qquad \square$$

HYPOTHESIS (2.10). *Fix $n \geq 2$. Let M be a matrix similar to A such that M is the direct sum of the companion matrices of a family of polynomials $p_1, \ldots, p_t$. Also, let N be a matrix similar to B such that N is the direct sum of the companion matrices of a family of polynomials $q_1, \ldots, q_w$. Assume that R is the last direct summand in M, that is,*

$$M = \begin{pmatrix} M_{11} & 0 \\ 0 & R \end{pmatrix} \tag{2.11}$$

where M_{11} is an $(n-r)\times(n-r)$ matrix.

Let S be the companion matrix of the polynomial $x^s + \rho_{s-1}x^{s-1} + \cdots + \rho_0$. Assume that S is the last direct summand in N, that is

$$N = \begin{pmatrix} N_{11} & 0 \\ 0 & S \end{pmatrix}, \tag{2.12}$$

where N_{11} is an $(n-s)\times(n-s)$ matrix.

LEMMA (2.13). *Assume Hypothesis 2.10. Let $E_1 = \begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix}$, where E is as in Lemma 2.9. Given $x, y \in \mathcal{F}$ with $x \neq 0$, let $D(x,y) = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$, i.e. $a = x$, $b = y$, $c = 0$ and $d = 1$ in D, and $E(x,y) = \begin{pmatrix} I & 0 \\ 0 & D(x,y) \end{pmatrix}$.*

(i) If $r > 2$ and $s > 2$, then

$$\begin{aligned} \mathrm{Trace}(M^{E_1}N) &= \mathrm{Trace}(MN) - \lambda_{r-3}c + \lambda_{r-2} + \rho_{s-2} - \lambda_{r-1}\rho_{s-1} \\ &\quad + \tfrac{1}{w}(-\lambda_{r-2}d^2 - b^2 + \lambda_{r-1}bd + \rho_{s-3}c - \lambda_{r-2}\rho_{s-2}c^2 - a^2\rho_{s-2} \\ &\quad + a\lambda_{r-1}\rho_{s-2}c - \lambda_{r-2}\rho_{s-1}cd - ab\rho_{s-1} + a\lambda_{r-1}\rho_{s-1}d). \end{aligned}$$

In particular,

$$\begin{aligned} \mathrm{Trace}(M^{E(x,y)}N) &= \tfrac{1}{x}(-\rho_{s-2}x^2 - y^2 - \rho_{s-1}xy + \lambda_{r-1}y - \lambda_{r-2}) \\ &\quad + \lambda_{r-2} + \rho_{s-2} + \mathrm{Trace}(MN). \end{aligned}$$

Thus the set $\{\mathrm{Trace}(M^{E(x,y)}N) \mid (x,y) \in \mathcal{F}\times\mathcal{F}, x \neq 0\}$ has at least $q-1$ elements, and $\{\mathrm{Trace}(M^{E(1,y)}N) \mid y \in \mathcal{F}\}$ has at least $\lceil \frac{q}{2} \rceil$ elements.

(ii) If $r = 2$ and $s \geq r$, then

$$\begin{aligned} \mathrm{Trace}(M^{E_1}N) &= \mathrm{Trace}(MN) + \lambda_0 + \rho_{s-2} - \lambda_1\rho_{s-1} \\ &\quad + \tfrac{1}{w}(-\lambda_0 d^2 - b^2 + \lambda_1 bd - \rho_{s-2}(\lambda_0 c^2 + a^2 - a\lambda_1 c) \\ &\quad - \rho_{s-1}(\lambda_0 cd + ab - a\lambda_1 d)). \end{aligned}$$

In particular,

$$\begin{aligned} \mathrm{Trace}(M^{E(x,y)}N) &= \tfrac{1}{x}(-\rho_{s-2}x^2 - y^2 - \rho_{s-1}xy + \lambda_1 y - \lambda_0)) \\ &\quad + \lambda_0 + \rho_{s-2} + \mathrm{Trace}(MN). \end{aligned}$$

Thus the set $\{\mathrm{Trace}(M^{E(x,y)}N) \mid (x,y) \in \mathcal{F}\times\mathcal{F}, x \neq 0\}$ has at least $q-1$ elements, and $\{\mathrm{Trace}(M^{E(1,y)}N) \mid y \in \mathcal{F}\}$ has at least $\lceil \frac{q}{2} \rceil$ elements.

(iii) Assume that $r \geq 2$. Let $N_1 = \begin{pmatrix} I & 0 \\ 0 & D_1 \end{pmatrix}$, where $D_1 = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$ is a 2×2 matrix where $u \neq v$. Then

$$\begin{aligned} \mathrm{Trace}(M^{E_1}N_1) &= \mathrm{Trace}(MN_1) + \tfrac{u}{w}(-\lambda_{r-2}cd - ab + \lambda_{r-1}bc) \\ &\quad + \tfrac{v}{w}(\lambda_{r-2}cd + ab - a\lambda_{r-1}d + \lambda_{r-1}). \end{aligned}$$

Thus

$$\mathrm{Trace}(M^{E(x,y)}N_1) \;=\; \tfrac{1}{x}(xy(v-u)-\lambda_{r-1}x+\lambda_{r-1})+\mathrm{Trace}(MN_1).$$

Therefore the set $\{\mathrm{Trace}(M^{E(1,y)}N_1) \mid (x,y)\in\mathcal{F}\times\mathcal{F}, x\neq 0\}$ *has q elements.*

Proof. (i) Observe that all but the last three elements of the diagonal of the matrices MN and $M^{E_1}N$ have the same values. Hence, using the previous result, the last three diagonal values of $M^{E_1}N-MN=(M^{E_1}-M)N$ are

- $(0-0)0+(-\lambda_{r-3}c-0)1+(-\lambda_{r-3}d+\lambda_{r-3})0=-\lambda_{r-3}c$,
- $(\frac{d}{w}-1)0+(\frac{1}{w}(-\lambda_{r-2}cd-ab+\lambda_{r-1}bc)-0)0+(\frac{1}{w}(-\lambda_{r-2}d^2-b^2+\lambda_{r-1}bd)+\lambda_{r-2})1=\frac{1}{w}(-\lambda_{r-2}d^2-b^2+\lambda_{r-1}bd)+\lambda_{r-2}$, and
- $(\frac{-c}{w}-0)(-\rho_{s-3})+(\frac{1}{w}(\lambda_{r-2}c^2+a^2-a\lambda_{r-1}c)-1)(-\rho_{s-2})+(\frac{1}{w}(\lambda_{r-2}cd+ab-a\lambda_{r-1}d)+\lambda_{r-1})(-\rho_{s-1})=\frac{1}{w}(\rho_{s-3}c-\lambda_{r-2}\rho_{s-2}c^2-a^2\rho_{s-2}+a\lambda_{r-1}\rho_{s-2}c-\lambda_{r-2}\rho_{s-1}cd-ab\rho_{s-1}+a\lambda_{r-1}\rho_{s-1}d)+\rho_{s-2}-\lambda_{r-1}\rho_{s-1}$.

Hence,

$$\begin{aligned}
\mathrm{Trace}(M^{E_1}N)-\mathrm{Trace}(MN) &= \mathrm{Trace}(M^{E_1}N-MN)\\
&= -\lambda_{r-3}c+\lambda_{r-2}+\rho_{s-2}-\lambda_{r-1}\rho_{s-1}\\
&\quad +\frac{1}{w}(-\lambda_{r-2}d^2-b^2+\lambda_{r-1}bd+\rho_{s-3}c\\
&\quad -\lambda_{r-2}\rho_{s-2}c^2-a^2\rho_{s-2}+a\lambda_{r-1}\rho_{s-2}c\\
&\quad -\lambda_{r-2}\rho_{s-1}cd-ab\rho_{s-1}+a\lambda_{r-1}\rho_{s-1}d).
\end{aligned}$$

Thus, when $a=x$, $b=y$, $c=0$ and $d=1$,

$$\begin{aligned}
\mathrm{Trace}(M^{E_1}N)-\mathrm{Trace}(MN) &= \lambda_{r-2}+\rho_{s-2}-\lambda_{r-1}\rho_{s-1}\\
&\quad +\frac{1}{x}(-\lambda_{r-2}-y^2+\lambda_{r-1}y-\rho_{s-2}x^2\\
&\quad -\rho_{s-1}xy+\lambda_{r-1}\rho_{s-1}x)\\
&= \frac{1}{x}(-\rho_{s-2}x^2-y^2-\rho_{s-1}xy+\lambda_{r-1}y-\lambda_{r-2})\\
&\quad +\lambda_{r-2}+\rho_{s-2}.
\end{aligned}$$

By Lemma 2.3, the set $\{\frac{1}{x}(-bs-2x^2-y^2-\rho_{s-1}xy+\lambda_{r-1}y-\lambda_{r-2})+\lambda_{r-2}+\rho_{s-2}+\mathrm{Trace}(MN) \mid (x,y)\in\mathcal{F}\times\mathcal{F}, x\neq 0\}$ has at least $q-1$ elements.

By Lemma 2.2 (ii), the set $\{\mathrm{Trace}(M^{E(1,y)}N \mid y\in\mathcal{F}\}=\{-\rho_{s-2}-y^2+(-\rho_{s-1}+\lambda_{r-1})y-\lambda_{r-2}+\lambda_{r-2}+\rho_{s-2}+\mathrm{Trace}(MN)\mid y\in\mathcal{F}\}$ has at least $\lceil\frac{q}{2}\rceil$ elements.

(ii) In this case, the diagonals of MN and $M^{E_1}N$ have the same values, in all but the last two entries. Hence, using the previous result, the last two diagonal values of $M^{E_1}N-MN=(M^{E_1}-M)N$ are

- $(\frac{1}{w}(-\lambda_0cd-ab+\lambda_1bc)-0)0+(\frac{1}{w}(-\lambda_0d^2-b^2+\lambda_1bd)+\lambda_0)1$ and
- $(\frac{1}{w}(\lambda_0c^2+a^2-a\lambda_1c)-1)(-\rho_{s-2})+(\frac{1}{w}(\lambda_0cd+ab-a\lambda_1d)+\lambda_1)(-\rho_{s-1})$.

Hence, $\mathrm{Trace}(M^{E_1}N)-\mathrm{Trace}(MN)=\mathrm{Trace}(M^{E_1}N-MN)=$
$\lambda_0+\rho_{s-2}-\lambda_1\rho_{s-1}+\frac{1}{w}(-\lambda_0d^2-b^2+\lambda_1bd-\rho_{s-2}(\lambda_0c^2+a^2-a\lambda_1c)-\rho_{s-1}(\lambda_0cd+ab-a\lambda_1d))$.

When $a=x$, $b=y$, $c=0$ and $d=1$, $\mathrm{Trace}(M^{E_1}N)-\mathrm{Trace}(MN)=$
$\lambda_0+\rho_{s-2}-\lambda_1\rho_{s-1}+\frac{1}{x}(-\lambda_0-y^2+\lambda_1y-\rho_{s-2}x^2-\rho_{s-1}xy+\lambda_1\rho_{s-1}x)$. Thus, $\mathrm{Trace}(M^{E_1}N)$

$= \frac{1}{x}(-\rho_{s-2}x^2 - y^2 - \rho_{s-1}xy + \lambda_1 y - \lambda_0) + \lambda_0 + \rho_{s-2} + \mathrm{Trace}(MN)$. By Lemma 2.3 it follows that $\mathrm{Trace}(M^{E_1}N)$ can take $q-1$ values.

By Lemma 2.2 (ii), the set $\{\mathrm{Trace}(M^{E(1,y)}N) \mid y \in \mathcal{F}\} = \{-\rho_{s-2} - y^2 + (-\rho_{s-1} + \lambda_1)y - \lambda_0 + \lambda_0 + \rho_{s-2} + \mathrm{Trace}(MN) \mid y \in \mathcal{F}\}$ has at least $\lceil \frac{q}{2} \rceil$ elements.

(iii) In this case, the diagonals of MN_1 and $M^{E_1}N_1$ have the same values, in all but the last two entries. Hence, using the previous result, the last two diagonal values of $M^{E_1}N_1 - MN_1 = (M^{E_1} - M)N_1$ are

- $(\frac{1}{w}(-\lambda_{r-2}cd - ab + \lambda_{r-1}bc) - 0)u + (\frac{1}{w}(-\lambda_{r-2}d^2 - b^2 + \lambda_{r-1}bd) + \lambda_{r-2})0$ and
- $(\frac{1}{w}(\lambda_{r-2}c^2 + a^2 - a\lambda_{r-1}c) - 1)0 + (\frac{1}{w}(\lambda_{r-2}cd + ab - a\lambda_{r-1}d) + \lambda_{r-1})v$.

Hence, $\mathrm{Trace}(M^{E_1}N_1) - \mathrm{Trace}(MN_1) = \mathrm{Trace}(M^{E_1}N_1 - MN_1) =$
$\frac{u}{w}(-\lambda_{r-2}cd - ab + \lambda_{r-1}bc) + \frac{v}{w}(\lambda_{r-2}cd + ab - a\lambda_{r-1}d + \lambda_{r-1})$.

When $a = x$, $b = y$, $c = 0$ and $d = 1$, $\mathrm{Trace}(M^{E_1}N_1) - \mathrm{Trace}(MN_1) =$
$\frac{-uxy}{x} + \frac{v}{x}(xy - \lambda_{r-1}x + \lambda_{r-1})$. Thus, $\mathrm{Trace}(M^{E_1}N_1) = \frac{1}{x}(xy(v-u) - \lambda_{r-1}x + \lambda_{r-1}) + \mathrm{Trace}(MN_1)$. Since $v - u \neq 0$, the set $\{\frac{1}{x}(xy(v-u) - \lambda_{r-1}x + \lambda_{r-1}) + \mathrm{Trace}(MN_1) \mid x = 1, y \in \mathcal{F}\}$ has q elements. □

LEMMA (2.14). *Let* $C = \begin{pmatrix} C_{11} & 0 \\ 0 & D_1 \end{pmatrix}$ *be a* $n \times n$ *matrix, where* $D_1 = \begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix}$ *is a* 2×2 *matrix where* $u_1 \neq v_1$. *Let* $E = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}$ *be in* $\mathrm{GL}(n,q)$, *where* $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and* $ad - bc = 1$.

Then $C^{E_1} = \begin{pmatrix} C_{11} & 0 \\ 0 & D_1^D \end{pmatrix}$, *where* $D_1^D = \begin{pmatrix} adu_1 - bcv_1 & bd(u_1 - v_1) \\ -ac(u_1 - v_1) & adv_1 - bcu_1 \end{pmatrix}$.

Thus, given a matrix $N = \begin{pmatrix} N_{11} & 0 \\ 0 & D_2 \end{pmatrix}$ *in* $\mathrm{GL}(n,q)$, *where* $D_2 = \begin{pmatrix} u_2 & 0 \\ 0 & v_2 \end{pmatrix}$ *is a* 2×2 *matrix where* $u_2 \neq v_2$, *we have that*

$$\mathrm{Trace}(C^E N) = \mathrm{Trace}(CN) - u_1u_2 - v_1v_2 + u_2(adu_1 - bcv_1) + v_2(adv_1 - bcu_1).$$

In particular, if we fix $x \in \mathcal{F}$, $ad = x$, *and* $bc = x - 1$, *we have that*

$$\mathrm{Trace}(C^E N) = x(u_1 - v_1)(u_2 - v_2) + (\mathrm{Trace}(CN) - (u_1 - v_1)(u_2 - v_2)).$$

Therefore given any $f \in \mathcal{F}$, *we can find some* $x \in \mathcal{F}$ *such that* $\mathrm{Trace}(C^E N) = f$.

Proof. Observe that

$$\begin{aligned} D_1^D &= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} u_1 & 0 \\ 0 & v_1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} au_1 & bu_1 \\ cv_1 & dv_1 \end{pmatrix} \\ &= \begin{pmatrix} adu_1 - bcv_1 & bd(u_1 - v_1) \\ -ac(u_1 - v_1) & adv_1 - bcu_1 \end{pmatrix}. \end{aligned}$$

The diagonals of $C^E N$ and CN have the same values, in all but the last two entries. Hence, the last two diagonal values of $C^E N - CN = (C^E - C)N$ are

- $(adu_1 - bcv_1 - u_1)u_2 + (bd(u_1 - v_1) - 0)0$ and
- $(-ac(u_1 - v_1) - 0)0 + (adv_1 - bcu_1 - v_1)v_2$.

Hence,

$$\begin{aligned}\mathrm{Trace}(C^E N) - \mathrm{Trace}(CN) &= \mathrm{Trace}(C^E N - CN)\\ &= (adu_1 - bcv_1 - u_1)u_2 + (adv_1 - bcu_1 - v_1)v_2.\end{aligned}$$

When $ad = x$ and $bc = x-1$,

$$\begin{aligned}\mathrm{Trace}(M^E N) - \mathrm{Trace}(MN) &= (xu_1 - (x-1)v_1 - u_1)u_2\\ &\quad +(xv_1 - (x-1)u_1 - v_1)v_2\\ &= x((u_1 - v_1)u_2 + (v_1 - u_1)v_2)\\ &\quad +(v_1 - u_1)u_2 + (u_1 - v_1)v_2\\ &= x(u_1 - v_1)(u_2 - v_2) - (u_1 - v_1)(u_2 - v_2).\end{aligned}$$

Since $u_1 \neq v_1$ and $u_2 \neq v_2$, we have that $(u_1 - v_1)(u_2 - v_2) \neq 0$ and thus the set $\{x(u_1 - v_1)(u_2 - v_2) + (\mathrm{Trace}(CN) - (u_1 - v_1)(u_2 - v_2)) \mid x \in \mathcal{F}\} = \mathcal{F}$. □

Proof of Theorem A. If at least one of A, B is in the center $\mathbf{Z}(\mathcal{G})$ of $\mathcal{G}$, then $A^{\mathcal{G}}B^{\mathcal{G}} = (AB)^{\mathcal{G}}$. Thus we may assume that both matrices A and B are not in the center, that is we may assume that both A and B are non-scalar matrices.

Assume Hypothesis 2.10. If $r \geq 2$ or $s \geq 2$, then by Lemma 2.1, Lemma 2.8 and Lemma 2.13 we have that $\eta(A^{\mathcal{G}}B^{\mathcal{G}}) \geq q-1$. Without loss of generality, we may assume then that for $i = 1,\dots,t$ and $j = 1,\dots,w$, the polynomials p_i, q_j have degree 1, that is both A and B are diagonal matrices. Since both A and B are non-scalar matrices, we may assume that A is similar to C and B is similar to N, where C and N are as constructed in Lemma 2.14. Hence, by Lemma 2.14, $\eta(A^{\mathcal{G}}B^{\mathcal{G}}) \geq q$. □

REMARK (2.15). *Let $\mathcal{F}$ be a field with $q = 2^m$ elements, for some integer $m > 0$. Set $\mathcal{G} = \mathrm{GL}(2,q) = \mathrm{GL}(2,\mathcal{F})$. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ -1 & w \end{pmatrix}$ in $\mathcal{G}$, where $x^2 - wx + 1$ is an irreducible polynomial over $\mathcal{F}$. Observe that both A and B are in $\mathrm{SL}(2,q)$ and thus $A^{\mathrm{GL}(2,q)}B^{\mathrm{GL}(2,q)} \subseteq \mathrm{SL}(2,q)$.*

By Proposition 2.13 of [3], *$\eta(A^{\mathrm{SL}(2,q)}B^{\mathrm{SL}(2,q)}) = q-1$. Since $x \mapsto x^2$ is an automorphism of $\mathcal{F}$, two matrices C, D in $\mathrm{GL}(2,q)$ are similar if and only there exists some $H \in \mathrm{SL}(2,q)$ such that $C^H = D$. Thus $\eta(A^{\mathrm{SL}(2,q)}B^{\mathrm{SL}(2,q)}) = \eta(A^{\mathcal{G}}B^{\mathcal{G}}) = q-1$.*

Proof of Theorem B. As in the proof of Theorem A, we may assume that both matrices A and B in $\mathcal{S} = \mathrm{SL}(n,q)$ are not in the center, that is we may assume that both A and B are non-scalar matrices.

We may assume then Hypothesis 2.10. Observe that the matrix $E(1,y)$ in Lemma 2.13 is in $\mathcal{S}$. Thus if $r \geq 2$ or $s \geq 2$, then by Lemma 2.1, Lemma 2.8 and Lemma 2.13 we have that $\eta(A^{\mathcal{S}}B^{\mathcal{S}}) \geq \lceil \frac{q}{2} \rceil$. As in the proof of Theorem A, we may assume then that both A and B are diagonal matrices. Observe that since $ad = x$ and $bc = x-1$, then $ad - bc = 1$ and so the matrix E in Lemma 2.14 is in $\mathcal{S}$. Since A and B are non-scalar matrices, the result then follows by Lemma 2.8 and Lemma 2.14. □

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INTERPOLATION OF ENTIRE FUNCTIONS

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We establish an interpolation theorem for entire functions, in the case of an unbounded set of interpolation, which generalizes a series known results.

1. Introduction

Series of the form

$$\sum_{k=0}^{\infty} a_k P_k, \tag{1.1}$$

where the coefficients are complex numbers and P_k are polynomials are useful tools in different areas of mathematics. For example the expansion of a function into a series with real coefficients constructed by means of orthogonal polynomial with respect to a scalar product or interpolation problems of finding a function from its values on a given sequence are solved by means of series of this type. Also a proof of a well-known result of Lindemann on the transcendency of e^{γ}, when γ is an algebraic number (see [8], Theorem 6, Ch. 2, Sec. 3) is based on a series of the form (1.1). The case

$$P_k(z) = \prod_{j=1}^{k} \left(1 - \frac{z}{\alpha_j}\right),\ k \geq 1,\ P_0(z) = 1,$$

when the numbers α_k defines either a monotone increasing sequence of positive real numbers with $\lim_{k\to\infty} \alpha_k = \infty$ or a monotone decreasing sequence of positive real numbers with $\lim_{k\to\infty} \alpha_k = 0$ and the series $\sum_{k=1}^{\infty} \alpha_k$ converges is the subject of [7]. Applications of these series to approximate solutions of boundary value problems for differential equations are presented in [4] and [5].

Given an arbitrary sequence $S = \{\alpha_k\}_{k\geq 1}$ of complex numbers, in Section 2 we denote by $\mathbb{C}_S[[X]]$ the set of Newton interpolating series at S, constructed by means of Newton interpolation polynomials, and by $\mathcal{H}(\mathbb{C})$ the $\mathbb{C}$-algebra of all entire functions with coefficients in $\mathbb{C}$. For every $f \in \mathcal{H}(\mathbb{C})$ we consider $g = \varphi_S(f)$ from $\mathbb{C}_S[[X]]$, the Newton interpolating series at S associated to f. If $\varphi_S(f)$ converges absolutely for all $z \in \mathbb{C}$, we call it a Newton entire function at S.

This paper deals with entire functions which are represented as Newton entire functions. If S is a bounded set the result is known (see [2], Theorem 4.3.1).

In [6] Ismail and Stanton established q-analogues of Taylor series expansions for entire functions where $M(r, f)$ grows like $\ln^2 r$, so-called q-Taylor series. They solved the problem of constructing such entire functions from their values at

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$\alpha_n = \frac{aq^n + a^{-1}q^{-n}}{2}$, for a, q positive real numbers less than one and $n \in \mathbb{N}$. In [9] and [10] Welter generalized this result, when the set $\{\alpha_n\}_{n\geq 1}$ is a particular infinite discrete set called regular sparse. Theorem 2.4 establishes an analogous result for an arbitrary unbounded sequence of distinct complex numbers, not necessarily regular sparse or even discrete. Consequences of this result are Theorems 3.1 and 3.3 from [6] and Theorem 2.2 from [10] which we obtain in Corollary 2.6 and Remark 1.

Corollary 2.10 is a well known theorem of Carlson when $\alpha_k = k-1$ and f is of order one and type less $\ln 2$. An example of more general result obtained by this method is given in Corollary 2.11.

2. Interpolation at an unbounded sequence

Let $S = \{\alpha_k\}_{k\geq 1}$ be a sequence of complex numbers and the polynomials u_k defined by

$$u_0 = 1,\ u_k = \prod_{j=1}^{k} (X - \alpha_j),\ k \geq 1. \tag{2.1}$$

We consider the set of formal series

$$\mathbb{C}_S[[X]] = \left\{ f = \sum_{k=0}^{\infty} a_k u_k : a_k \in \mathbb{C} \right\},$$

two such expressions being regarded as equal if and only if they have the same coefficients. When $\alpha_k = 0$, for all k, $\mathbb{C}_S[[X]]$ becomes the usual $\mathbb{C}$-algebra of formal power series $\mathbb{C}[[X]]$. We call an element f from $\mathbb{C}_S[[X]]$ a (formal) *Newton interpolating series* at S with coefficients in $\mathbb{C}$.

If $\{f_k\}_{k\geq 1}$ is a sequence of complex numbers and $S = \{\alpha_k\}_{k\geq 1}$ is a sequence of distinct complex numbers, we denote by $f_{i_1,i_2,\ldots,i_s}$ the divided difference with respect to s distinct points $\alpha_{i_1}, \ldots, \alpha_{i_s}$. Thus $f_{j,k} = \frac{f_k - f_j}{\alpha_k - \alpha_j}$ and generally

$$f_{i_1,\ldots,i_s} = \frac{f_{i_2,\ldots,i_s} - f_{i_1,\ldots,i_{s-1}}}{\alpha_{i_s} - \alpha_{i_1}}.$$

Then (see (7) from [3], p. 7)

$$f_{1,2,\ldots,n} = \sum_{k=1}^{n} \frac{f_k}{\prod\limits_{j=1, j\neq k}^{n} (\alpha_k - \alpha_j)}. \tag{2.2}$$

If f is an entire function and S is a sequence of distinct complex numbers, we take $f_k = f(\alpha_k)$ and, for every non-negative integer s,

$$a_s = f_{1,2,\ldots,s+1}.$$

We consider

$$g = \sum_{k=0}^{\infty} a_k u_k \in \mathbb{C}_S[[X]]. \tag{2.3}$$

Then the series g given by (2.3) is called the *Newton interpolating series at S associated to f*.

If $f = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{H}(\mathbb{C})$, then we define a mapping φ_S from $\mathcal{H}(\mathbb{C})$ to $\mathbb{C}_S[[X]]$ such that $\varphi_S(f)$ is the Newton interpolating series at S associated to f.

A Newton interpolating series g given by (2.3) which converges absolutely for all complex numbers is called a *Newton entire function* at S. Let $\mathcal{H}_S(\mathbb{C})$ be the subset of $\mathbb{C}_S[[X]]$ of all Newton entire functions. For an entire function f, we are interested in choosing a particular sequence S such that the Newton interpolating series at S associated to f is a Newton entire function.

We study the problem when $S = \{\alpha_k\}_{k\geq 1}$ is an unbounded sequence of distinct complex numbers. There are sequences S, such that g, the Newton interpolating series at S associated with a given entire function f, does not define a complex function equal to f. Thus if we consider f a non constant entire function having infinitely many zeros α_k, $k = 1,2,...$, then by taking $S = \{\alpha_k\}_{k\geq 1}$ it is easy to see that g is the zero function.

Consider $R = \{r_n\}_{n\geq 1}$ a sequence of positive real numbers such that $r_n \geq \gamma B_n$, for all n, where $\gamma > 1$, B_1 is either equal to $|\alpha_1|$, if $\alpha_1 \neq 0$, or equal to $|\alpha_2|$, if $\alpha_1 = 0$, and for $n > 1$,

$$B_n := \max_{1\leq i\leq n} \{|\alpha_i|\}.$$

We set $M_n(f) := \max_{|z|\leq r_n} |f(z)|$,

$$e_{s,n,S,R}(f) := \left(\frac{M_{n+1}(f) \prod_{k=1}^{n+1} (s + B_k)}{\prod_{k=1}^{n+1} (r_{n+1} - |\alpha_k|)} \right)^{\frac{1}{n}}, \ s \leq n,$$

and

$$e_{S,R}(f) := \limsup_{s\to\infty} \limsup_{n\to\infty} e_{s,n,S,R}(f).$$

THEOREM (2.4). *Let f be an entire function. Consider $S = \{\alpha_k\}_{k\geq 1}$ an unbounded sequence of distinct complex numbers. If there exists a sequence of positive real numbers $R = \{r_n\}_{n\geq 1}$ such that $r_n \geq \gamma B_n$, for sufficiently large n, where $\gamma > 1$ and $e_{S,R}(f) < 1$, then $\varphi_S(f)$ belongs to $\mathcal{H}_S(\mathbb{C})$,*

$$f(z) = \sum_{n=0}^{\infty} a_n u_n(z), \tag{2.5}$$

where the series converges uniformly on every compact subset of $\mathbb{C}$, a_n are given by (2.2) and every such f is uniquely determined by its values on S.

Proof. By (123) from [3], Section 2.3.1, it follows that

$$R_n(z) = \frac{u_{n+1}(z)}{2\pi i} \int_{|\xi|=r_{n+1}} \frac{f(\xi)d\xi}{(\xi - z)u_{n+1}(\xi)}.$$

Let z be a fixed complex number. Since there exist $s \in \mathbb{N}^*$, a real number $\delta > 1$ and $n_1 = n_1(z)$ such that $|z| \leq s < B_{n_1+1}$, $\frac{\delta}{\delta-1} < \gamma$ and, for all $n \geq n_1$,

$$\frac{r_{n+1}}{r_{n+1} - s} < \delta,$$

it follows that

$$|u_n(z)| \leq \prod_{k=1}^{n} (|z| + |\alpha_k|) \leq \prod_{k=1}^{n} (s + B_k)$$

and

$$|R_n(z)| \le \frac{\prod_{k=1}^{n+1}(s+B_k)r_{n+1}M_{n+1}(f)}{(r_{n+1}-s)\prod_{k=1}^{n+1}(r_{n+1}-|\alpha_k|)} \le \delta e_{s,n,S,R}(f)^n.$$

Hence, because $e_{S,R}(f) < 1$, it follows that $\limsup_{n\to\infty}(|R_n(z)|)^{\frac{1}{n}} < 1$ which implies the theorem. □

COROLLARY (2.6). *([6], Theorems 3.1 and 3.3) If $f \in \mathcal{H}(\mathbb{C})$,*

$$x_n = \frac{aq^{\frac{n}{2}} + a^{-1}q^{-\frac{n}{2}}}{2},\ n \ge 0,\ a,q \in (0,1)$$

and

$$\limsup_{r\to\infty} \frac{\ln M(r,f)}{\ln^2 r} = c,$$

with $M(r,f) := \sup_{|z|\le r}\{|f(z)|\}$ and $c < \frac{1}{2\ln q^{-1}}$, then

$$f(z) = \sum_{n=0}^{\infty} c_n \varphi_n(z,a),$$

where

$$\varphi_n(z,a) = \prod_{k=0}^{n-1}(1 - 2azq^k + a^2q^{2k}),$$

$$c_n = \sum_{k=0}^{n} D_k f(x_{2k}),$$

and D_k are constants constructed by means of a, q and the q-shifted factorials. Moreover f is uniquely determined by its values on $\{x_{2n}\}_{n\ge 0}$.

Proof. We take, for every $k \ge 1$, $\alpha_k = x_{2(k-1)}$. Then $\varphi_n(z,a) = (-2a)^n q^{\frac{n(n-1)}{2}} u_n(z)$ and we choose $\gamma > 1$, $R = \{\gamma a^{-1}q^{1-n}\}_{n\ge 1}$. If $\varepsilon > 0$ is small enough, then

$$M_{n+1}(f) < \exp\left(\frac{(1-\varepsilon)\ln^2 r_{n+1}}{2\ln q^{-1}}\right) \le (q^{-1})^{\frac{(1-\varepsilon)n^2}{2}} + O((q^{-1})^n). \tag{2.7}$$

Since, for a fixed s and n large enough, there exists $C > 0$ such that

$$\prod_{k=1}^{n+1}(s+B_k) \le C2^{n+1}B_1B_2...B_{n+1} \le a^{-n-1}(q^{-1})^{\frac{n^2+n}{2}}C2^{n+1} \tag{2.8}$$

and

$$\prod_{k=1}^{n+1}(r_{n+1}-|\alpha_k|) \ge \prod_{k=1}^{n+1}(r_{n+1}-B_{n+1}) \ge \left(\frac{\gamma-1}{a}\right)^{n+1}(q^{-1})^{(n+1)^2}, \tag{2.9}$$

by (2.7)-(2.9) we obtain

$$e^n_{s,n,S,R}(f) \le C(s)2^n q^{\frac{n^2\varepsilon}{2}+O(n)}(\gamma-1)^{-n-1},$$

where $C(s)$ is a positive real number. Hence $e_{S,R}(f) = 0$ and by Theorem 2.4 it follows the corollary. □

COROLLARY (2.10). *([1], Theorem 9.10.7) Let f be an entire function of order one and type less than* $\ln 2$, *then*

$$f(z) = \sum_{j=0}^{\infty} \binom{z}{j} (\Delta^j f_0),$$

where $\Delta^0 f_0 = 1$, $\Delta^j f_0 = \sum_{k=0}^{j} (-1)^k \binom{j}{k} f(k)$, *and the series converges uniformly on compact subsets of the complex plane.*

Proof. We choose $\alpha_k = k-1$, $k \geq 1$, $r_k = 2k$. Since f is of order one and type less than $\ln 2$ there exists a positive real number ε such that

$$M_n(f) \leq \left(\frac{2}{e^{\varepsilon}}\right)^{r_n}.$$

Hence there exists $\varepsilon_1 < 1 - \frac{1}{e^{2\varepsilon}}$ such that

$$e_{s,n,S,R}(f) \leq \left(\frac{2}{e^{\varepsilon}}\right)^{\frac{2n+2}{n}} \left(\frac{(s+1)\ldots(s+n+1)}{(2n+1)\ldots(n+1)}\right)^{\frac{1}{n}} + \frac{\varepsilon_1}{2} \leq \frac{1}{e^{2\varepsilon}} + \varepsilon_1$$

which implies $e_{S,R}(f) < 1$. Now the result follows by Theorem 2.4. □

COROLLARY (2.11). *Let f be an entire function of order $\rho \in (0,\infty)$ and of finite type σ. If $\alpha_n = n^{\frac{\theta}{\rho}}$, with $\theta \in (0,1)$, then $\varphi_S(f)$ belongs to $\mathcal{H}_S(\mathbb{C})$,*

$$f(z) = \sum_{n=0}^{\infty} a_n u_n(z),$$

where the series converges uniformly on every compact subset of $\mathbb{C}$, where a_n are defined by (2.2) and every such f is uniquely determined by its values on S.

Proof. We choose $r_n = 2\alpha_n$. By hypothesis there exists $\varepsilon > 0$ such that

$$M_{n+1}(f) \leq e^{(\sigma+\varepsilon) r_{n+1}^{\rho}}.$$

Then

$$e_{s,n,S,R}(f) \leq e^{\frac{2^{\rho}(\sigma+\varepsilon)(n+1)^{\theta}}{n}} \left(\prod_{k=1}^{n+1} \frac{s+\alpha_k}{2\alpha_{n+1}-\alpha_k}\right)^{\frac{1}{n}}.$$

Hence there exists $\varepsilon_1 > 0$ small enough such that

$$e_{s,n,S,R}(f) \leq (1+\varepsilon_1) v_n^{\frac{1}{n}},$$

where

$$v_n = \prod_{k=1}^{n+1} \frac{s+\alpha_k}{2\alpha_{n+1}-\alpha_k}.$$

Since, for every $k \geq 2$,

$$\frac{2\alpha_{n+1}-\alpha_1}{2\alpha_{n+2}-\alpha_1} > \frac{2\alpha_{n+1}-\alpha_k}{2\alpha_{n+2}-\alpha_k}$$

it follows that

$$\lim_{n\to\infty} \frac{v_{n+1}}{v_n} \leq e^{-\frac{\theta}{\rho}}.$$

Thus $e_{S,R}(f) < 1$ and the result follows by Theorem 2.4. □

Remark 1. In [9] and [10], Welter call a subset $X \subset \mathbb{C}$ *regular sparse* if X is infinite, discrete and there exist $\theta \in (1, +\infty)$ and $T \in \mathbb{R}$ such that

$$\psi_X(r^\theta) \le T\psi_X(r) + o(\psi_X(r)), \text{ when } r \to +\infty, \tag{2.12}$$

where $\psi_X(r) = \text{card}\{x \in X : |x| \le r\}$.

For a fixed θ he set

$$T_X(\theta) := \limsup_{r \to +\infty} \frac{\psi_X(r^\theta)}{\psi_X(r)}$$

and

$$\bar{\Lambda}(X) := \limsup_{r \to +\infty} \frac{\sum\limits_{x \in X, 0 < x \le r} \ln|x|}{\psi_X(r)\ln r}.$$

If X is a regular sparse subset of $\mathbb{C}$ and $\{\alpha_n\}_{n\ge1}$ is the sequence of all distinct elements of X ordered by increasing modulus, then he proved that all entire functions f with

$$\sigma := \limsup_{r \to +\infty} \frac{\ln M(r,f)}{\psi_X(r)\ln r} < \sup_{\theta \in (1,+\infty)} \frac{\theta - \bar{\Lambda}(X)}{\theta T_X(\theta)} \tag{2.13}$$

have the series expansion of the form (2.5) with a_n given by (2.2). The result generalizes Theorem 3.3 from [6]. However the set of all nonnegative integers is not a regular sparse subset.

To obtain Theorem 2.2 of [10], from Theorem 2.4, as in [10], p. 401 we choose $\theta > 1$ such that $\sigma < \frac{\theta - \bar{\Lambda}(X)}{\theta T_X(\theta)}$ and $r_n = |\alpha_n|^\theta$. Then $r_n \ge 2B_n$ for sufficiently large n. Hence, because $\psi_X(|\alpha_n|) = n + O(1)$,

$$\prod_{k=1}^{n+1} (r_{n+1} - |\alpha_k|) \ge C_1 \frac{|\alpha_{n+1}^\theta|^{n+1}}{2^{n+1}} \ge 2^{-n-1} C_1 \exp\left(\theta\psi_X(|\alpha_{n+1}|)\ln|\alpha_{n+1}|\right)$$

$$+O(|\alpha_{n+1}^\theta|), \tag{2.14}$$

where C_1 is a positive constant independent on n. By (2.13) and (2.12) it follows that, for n sufficiently large,

$$\ln M_{n+1}(f) \le \sigma\psi_X(|\alpha_{n+1}^\theta|)\ln|\alpha_{n+1}|^\theta$$

$$\le \sigma T_X(\theta)\theta\psi_X(|\alpha_{n+1})|\ln|\alpha_{n+1}| + o(\psi_X(|\alpha_{n+1})|\ln|\alpha_{n+1}|). \tag{2.15}$$

Then, for each $\varepsilon > 0$,

$$B_1B_2...B_{n+1} = \exp\left(\sum_{x \in X, 0 < x \le |\alpha_{n+1}|} \ln|x|\right) \le \bar{\Lambda}(X)\psi_X(|\alpha_{n+1})|\ln|\alpha_{n+1}| + \varepsilon \tag{2.16}$$

and by (2.14)-(2.16)

$$e_{s,n,S,R}^n(f) \le 4^n C_2 \exp((\sigma\theta T_X(\theta) - \theta + \bar{\Lambda}(X))\psi_X(|\alpha_{n+1}|)\ln|\alpha_{n+1}|)$$

$$+2^n o(\psi_X(|\alpha_{n+1}|)\ln|\alpha_{n+1}|),$$

where C_2 is a positive constant independent on n. Since $\psi_X(|\alpha_{n+1}|) = n + O(1)$, $\sigma\theta T_X(\theta) - \theta + \bar{\Lambda}(X) < 0$ and for regular sparse sets there are positive constants C_3 and C_4 such that $\ln|\alpha_n| \ge C_3 n^{C_4}$, for all n, (see [9], Proposition 1) we obtain that $\lim\limits_{n\to\infty} e_{s,n,S,R}(f) = 0$. Then the result follows by Theorem 2.4.

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ASYMPTOTICALLY CENTRAL NETS IN SEMIGROUP ALGEBRAS OF LOCALLY COMPACT TOPOLOGICAL SEMIGROUPS

B. MOHAMMADZADEH

Let $\mathfrak{S}$ be a locally compact semigroup and $M_a(\mathfrak{S})$ be the semigroup algebra of all complex Radon measures on $\mathfrak{S}$ with continuous translations. In this paper, we study the existence of asymptotically central and quasi-central nets in $M_a(\mathfrak{S})$ and their relations with inner amenability of a locally compact semigroup $\mathfrak{S}$.

1. Introduction

Throughout this paper, $\mathfrak{S}$ denotes a *locally compact semigroup*; i.e., a semigroup with a locally compact Hausdorff topology whose binary operation is jointly continuous. As usual, we denote by $M(\mathfrak{S})$ the Banach algebra of all complex Radon measures on $\mathfrak{S}$ with the convolution product $*$ and the total variation norm. The space of all measures $\mu \in M(\mathfrak{S})$ for which the maps $x \longmapsto \delta_x * |\mu|$ and $x \longmapsto |\mu| * \delta_x$ from $\mathfrak{S}$ into $M(\mathfrak{S})$ are weakly continuous is denoted by $M_a(\mathfrak{S})$ (or $\widetilde{L}(\mathfrak{S})$ as in [2]), where δ_x denotes the Dirac measure at x. It is well-known that $M_a(\mathfrak{S})$ is a closed two-sided L-ideal of $M(\mathfrak{S})$; see [2] or [21]. $\mathfrak{S}$ is said a *foundation semigroup* if $\mathfrak{S}$ coincides with the closure of the set $\bigcup\{\mathrm{supp}(\mu) : \mu \in M_a(\mathfrak{S})\}$. Let us point out that the second dual $M_a(\mathfrak{S})^{**}$ of $M_a(\mathfrak{S})$ is a Banach algebra with the first Arens product $\odot$ defined by the equations

$$(F \odot H)(f) = F(Hf),\ (Hf)(\mu) = H(f\mu) \text{ and } (f\mu)(\nu) = f(\mu * \nu)$$

for all $F, H \in M_a(\mathfrak{S})^{**}, f \in M_a(\mathfrak{S})^*$, and $\mu, \nu \in M_a(\mathfrak{S})$.

For $\mathcal{M} \subseteq M(\mathfrak{S})$, we say that a net (μ_α) in $M_a(\mathfrak{S})$ is $\mathcal{M}$-*quasi-central* (resp. *weakly* $\mathcal{M}$-*quasi-central*) if

$$\mu * \mu_\alpha - \mu_\alpha * \mu \longrightarrow 0$$

for all $\mu \in \mathcal{M}$ in the norm (resp. weak) topology of $M_a(\mathfrak{S})$; let us remark that (μ_α) is simply called *quasi-central* (resp. *weakly quasi-central*) if it is $M(\mathfrak{S})$-quasi-central (resp. weakly $M(\mathfrak{S})$-quasi-central). The purpose of this paper is to initiate a study asymptotically central nets in $M_a(\mathfrak{S})$ to obtain some characterizations for inner amenability for certain class of locally compact semigroup of $\mathfrak{S}$.

2. Asymptotically central net and inner amenability

We commence with the following lemma which is needed in the sequel. First, let $\mathrm{co}(\mathcal{A})$ denote the convex hull of a subset $\mathcal{A}$ of $M_a(\mathfrak{S})$.

LEMMA (2.1). *Let $\mathfrak{S}$ be a locally compact semigroup and $\mathcal{M} \subseteq M(\mathfrak{S})$. If $(\mu_\alpha)_{\alpha\in A}$ is a weakly $\mathcal{M}$-quasi-central net in $M_a(\mathfrak{S})$, then there exists a $\mathcal{M}$-quasi-central net in* $\mathrm{co}(\{\mu_\alpha\}_{\alpha\in A})$.

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Proof. Let E be the locally convex space $\Pi\{M_a(\mathfrak{S}) : \mu \in \mathcal{M}\}$ under the product of the norm topology of $M_a(\mathfrak{S})$. Then the weak topology of E is the product of the weak topology of $M_a(\mathfrak{S})$. Following an idea due to Namioka [17], let $T : M_a(\mathfrak{S}) \longrightarrow E$ be defined by

$$T(\nu)(\mu) = \mu * \nu - \nu * \mu \qquad (\nu \in M_a(\mathfrak{S}), \mu \in \mathcal{M}).$$

Then T is well defined and linear. Let $(\mu_\alpha)_{\alpha\in A}$ be a weakly $\mathcal{M}$-quasi-central net in $M_a(\mathfrak{S})$. Since $T(\mu_\alpha) \longrightarrow 0$ in the weak topology of E, it follows that 0 lies in the weak closure of $T(\mathrm{co}(\{\mu_\alpha\}_{\alpha\in A}))$ in E. Now, the convexity of $T(\mathrm{co}(\{\mu_\alpha\}_{\alpha\in A}))$ implies that 0 lies in the closure of $T(\mathrm{co}(\{\mu_\alpha\}_{\alpha\in A}))$ in E with respect to the product of the norm topology of $M_a(\mathfrak{S})$. So there exists a net (ν_β) in $\mathrm{co}(\{\mu_\alpha\}_{\alpha\in A})$ such that $\|(T(\nu_\beta))(\mu)\| \longrightarrow 0$ for all $\mu \in \mathcal{M}$. That is $\|\mu * \nu_\beta - \nu_\beta * \mu\| \longrightarrow 0$ for all $\mu \in \mathcal{M}$. □

In the following, $P_1(M_a(\mathfrak{S}))$ denotes the set of all probability measures in $M_a(\mathfrak{S})$.

PROPOSITION (2.2). *Let $\mathfrak{S}$ be a locally compact semigroup and $\mathcal{M} \subseteq M(\mathfrak{S})$. Then there is a weakly $\mathcal{M}$-quasi-central net in $P_1(M_a(\mathfrak{S}))$ if and only if there is a $\mathcal{M}$-quasi-central net in $P_1(M_a(\mathfrak{S}))$.*

Proof. Since $P_1(M_a(\mathfrak{S}))$ is a convex set in $M_a(\mathfrak{S})$, this follows immediately from Lemma 2.1. □

An element m in the second dual $M_a(\mathfrak{S})^{**}$ of $M_a(\mathfrak{S})$ is said to be a *mean* on $M_a(\mathfrak{S})^*$ if $\|m\| = m(u) = 1$, where $u \in M_a(\mathfrak{S})^*$ is defined by $u(\mu) := \mu(\mathfrak{S})$ for all $\mu \in M_a(\mathfrak{S})$. The set of all means on $M_a(\mathfrak{S})^{**}$ is denoted by $P_1(M_a(\mathfrak{S})^{**})$. We say that a mean m on $M_a(\mathfrak{S})^*$ is $\mathcal{M}$-*inner invariant* if

$$m(f\mu) = m(\mu f) \qquad (\mu \in \mathcal{M},\ f \in M_a(\mathfrak{S})^*);$$

where $f\mu$ and μf in $M_a(\mathfrak{S})^*$ are defined by $f\mu(\nu) := f(\mu * \nu)$ and $\mu f(\nu) := f(\nu * \mu)$ for all $f \in M_a(\mathfrak{S})^*$ and $\mu, \nu \in \mathcal{M}$. We also say that $\mathfrak{S}$ is $\mathcal{M}$-*inner amenable* if there exists an $\mathcal{M}$-inner invariant mean on $M_a(\mathfrak{S})^*$.

PROPOSITION (2.3). *Let $\mathfrak{S}$ be a foundation semigroup with identity and $\mathcal{M} \subseteq M(\mathfrak{S})$. Then the following assertions are equivalent.*

(a) *$\mathfrak{S}$ is $\mathcal{M}$-inner amenable.*
(b) *There is a weakly $\mathcal{M}$-quasi-central net in $P_1(M_a(\mathfrak{S}))$.*
(c) *There is an $\mathcal{M}$-quasi-central net in $P_1(M_a(\mathfrak{S}))$.*

Proof. Suppose that (a) holds, and let m be an $\mathcal{M}$-inner invariant mean on $M_a(\mathfrak{S})^*$. Since $\mathfrak{S}$ is a foundation semigroup with identity, it follows from Proposition 3.6 of [21]. Thus $P_1(M_a(\mathfrak{S}))$ is weak* dense in $P_1(M_a(\mathfrak{S})^{**})$ [11, Lemma 2.1]. So, there is a net (μ_α) in $P_1(M_a(\mathfrak{S}))$ such that $\mu_\alpha \longrightarrow m$ in the weak* topology of $M_a(\mathfrak{S})^{**}$. For each $\mu \in \mathcal{M}$ and $f \in M_a(\mathfrak{S})^*$ we have

$$(\mu * \mu_\alpha - \mu_\alpha * \mu)(f) = \mu_\alpha(f\mu - \mu f) \longrightarrow m(f\mu - \mu f)$$

Therefore $\mu * \mu_\alpha - \mu_\alpha * \mu \longrightarrow 0$ in the weak topology of $M_a(\mathfrak{S})$ for all $\mu \in \mathcal{M}$. That is (b) holds. That (b) implies (c) follows from Lemma 2.2. In order to prove that (c) implies (a), we suppose that (μ_α) is an $\mathcal{M}$-quasi-central net in $P_1(M_a(\mathfrak{S}))$. Then it is clear that any weak* cluster point of (μ_α) defies an $\mathcal{M}$-inner invariant mean on $M_a(\mathfrak{S})^*$. This completes the proof. □

To prepare the setting for the next results, let $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ be the set of all complex-valued bounded functions g on $\mathfrak{S}$ that are μ-measurable for all $\mu \in M_a(\mathfrak{S})$. We identify functions in $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ that agree μ-almost everywhere for all $\mu \in M_a(\mathfrak{S})$. For every $g \in L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$, define

$$\| g \|_\infty := \sup\{ \|g\|_{\infty,|\mu|} : \mu \in M_a(\mathfrak{S}) \},$$

where $\|.\|_{\infty,|\mu|}$ denotes the essential supremum norm with respect to $|\mu|$. Observe that $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ with the complex conjugation as involution, the pointwise operations and the norm $\|.\|_\infty$ is a commutative C^*-algebra. Recall that a that a mean on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ is a positive functional m with norm one; for $\mathcal{M} \subseteq M(\mathfrak{S})$, we say that m is $\mathcal{M}$-*inner invariant* if

$$m(\mu \circ g) = m(g \circ \mu)$$

for all $\mu \in \mathcal{M}$ and $g \in L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$, where $\mu \circ g, g \circ \mu \in L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ are defined by

$$(\mu \circ g)(x) := \int_{\mathfrak{S}} f(yx)d\mu(y) \text{ and } (g \circ \mu)(x) := \int_{\mathfrak{S}} f(xy)d\mu(y)$$

for all $x \in \mathfrak{S}$. Let us recall that if $\mathfrak{S}$ is a foundation semigroup with identity, then the equation $\tau(g)(\mu) = \mu(g)$ defines an isometric isomorphism τ of $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ onto $M_a(\mathfrak{S})^*$. Therefore, the adjoint τ^* of τ defines an isometric isomorphism of $M_a(\mathfrak{S})^{**}$ onto $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))^*$.

PROPOSITION (2.4). *Let $\mathfrak{S}$ be a foundation semigroup with identity. Then τ^* maps $\mathcal{M}$-inner invariant means on $M_a(\mathfrak{S})^*$ onto $\mathcal{M}$-inner invariant means on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$.*

Proof. First note that $\tau(g)\mu = \tau(\mu \circ g)$ for all $\mu \in \mathcal{M}$ and $g \in L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$. Indeed; for any $\nu \in M_a(\mathfrak{S})$ we have

$$\tau(\mu \circ g)(\nu) = \int_{\mathfrak{S}} \int_{\mathfrak{S}} g(xy)\, d\mu(x)\, d\nu(y) = \int_{\mathfrak{S}} g(t)\, d(\mu * \nu)(t) = \tau(g)(\mu * \nu) = (\tau(g)\mu)(\nu).$$

Now, let $n \in M_a(\mathfrak{S})^{**}$ and $f \in M_a(\mathfrak{S})^*$. Then since τ is onto, there is $g \in L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ such that $f = \tau(g)$. Thus

$$\begin{aligned} \tau^*(n)(\mu \circ g) &= n(\tau(\mu \circ g)) \\ &= n(\tau(g)\mu) \\ &= n(f\mu). \end{aligned}$$

A similar argument shows that $\tau^*(n)(g \circ \mu) = n(\mu f)$. Also $\tau^*(1) = n(1)$ and $\|\tau^*(n)\| = \|n\|$. Therefore $\tau^*(n)$ is an $\mathcal{M}$-inner invariant mean if so is n. □

A mean on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ is called *inner invariant* if it is $\delta_{\mathfrak{S}}$-inner invariant.

COROLLARY (2.5). *Let $\mathfrak{S}$ be a foundation semigroup with identity. Then $\mathfrak{S}$ is $\mathcal{M}$-inner amenable if and only if there is an $\mathcal{M}$-inner invariant mean on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$.*

We say that $\mathfrak{S}$ is *inner amenable* if there exists an inner invariant mean on $M_a(\mathfrak{S})^* = L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$; that is, a $\delta_{\mathfrak{S}}$-inner invariant mean.

The study of inner amenability was initiated by Effros [8] and pursued by Akemann [1], H. Choda and M. Choda [5], M. Choda [6], Kaniuth and Markfort [9], Paschke [19], Pier [20], and Watatani [25] for discrete groups, Lau and Paterson

[11], Losert and Rindler [13], Stokke [23], Takahashi [24], Yuan [26] for locally compact groups, and recently by Ling [12] for discrete semigroups.

The following consequence of Proposition 2.6 is due to Losert and Rindler [13] and Yuan [26] for the case of locally compact groups.

COROLLARY (2.6). *Let $\mathfrak{S}$ be a foundation semigroup with identity. Then the following assertions are equivalent.*

(a) *$\mathfrak{S}$ is inner amenable.*

(b) *There is a weakly asymptotically central net in $P_1(M_a(\mathfrak{S}))$.*

(c) *There is an asymptotically central net in $P_1(M_a(\mathfrak{S}))$.*

3. Inner invariant extension of Dirac measures

Let $\mathfrak{S}$ be a foundation semigroup with identity e and let $C_b(\mathfrak{S})$ denote the closed subspace of $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ consisting of all bounded and continuous functions on $\mathfrak{S}$. Then δ_e is an inner invariant mean on $C_b(\mathfrak{S})$. The possibility of extension of δ_e to an inner invariant mean on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ has been studied in [3].

PROPOSITION (3.1). *Let S be foundation semigroup with identity and $\mu_n \subseteq P_1(M_a(\mathfrak{S}))$. Then (μ_n) is an asymptotically central sequence if and only if it is a quasi-central sequence.*

Proof. The “if” part is trivial. To prove the converse, suppose that (μ_n) is an an asymptotically central sequence in $P_1(M_a(\mathfrak{S}))$. Then for each $\nu \in M(\mathfrak{S})$ and $g \in L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$. Then we have

$$\begin{aligned}
\left|(\nu * \mu_n - \mu_n * \nu)(g)\right| &= \left|\int_{\mathfrak{S}} (\nu * \mu_n - \mu_n * \nu)(y) g(y)\, dy\right| \\
&= \left|\int_{\mathfrak{S}} (\int_{\mathfrak{S}} (\delta_x * \mu_n - \mu_n * \delta_x)(y)\, d\nu(x)) g(y)\, dy\right| \\
&= \left|\int_{\mathfrak{S}} (\int_{\mathfrak{S}} (\delta_x * \mu_n - \mu_n * \delta_x)(y) g(y)\, dy)\, d\nu(x)\right| \\
&= \left|\int_{\mathfrak{S}} (\delta_x * \mu_n - \mu_n * \delta_x)(g)\, d\nu(x)\right| \\
&\leq ||g||_\infty \int_{\mathfrak{S}} ||\delta_x * \mu_n - \mu_n * \delta_x||\, d|\nu|(x).
\end{aligned}$$

Since

$$||\delta_x * \mu_n - \mu_n * \delta_x|| \longrightarrow 0 \qquad (y \in \mathfrak{S}).$$

Lebesgue’s theorem implies that

$$(\nu * \mu_n - \mu_n * \nu)(g) \longrightarrow 0.$$

Since $\nu \in M(\mathfrak{S})$ and $g \in L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ is arbitrary, we have

$$(\nu * \mu_n - \mu_n * \nu)(g) \longrightarrow 0$$

in the weak topology of $M(\mathfrak{S})$. Now, invoke Proposition 2.2 to conclude that (μ_n) is a quasi-central sequence in $P_1(M_a(\mathfrak{S}))$. □

Let us recall from [10] that $\mathfrak{S}$ is said to be *left compactly cancelletive* if $C^{-1}D$ is a compact subset of S for all compact subsets C and D of $\mathfrak{S}$, where

$$C^{-1}D \quad = \quad \{x \in \mathfrak{S} : cx \in D \text{ for some } c \in C\}.$$

Right compactly cancellative locally compact semigroups are defined similarly. Moreover, let $C_0(\mathfrak{S})$ denote the space of all continuous functions on $\mathfrak{S}$ vanishing at infinity.

THEOREM (3.2). *Let $\mathfrak{S}$ be a left or right compactly cancelletive foundation semigroup with identity e and $(e_n) \subseteq P_1(M_a(\mathfrak{S}))$. Then (e_n) is an asymptotically central approximate identity for $M_a(\mathfrak{S})$ if and only if it is a quasi-central approximate identity for $M_a(\mathfrak{S})$.*

Proof. The "*if*" part follows from [9, Theorem 2.4]. The converse follows from Proposition 3.1. □

4. Strict inner amenability

Let $\mathfrak{S}$ be a foundation semigroup with identity e and recall that $E \in M_a(\mathfrak{S})^{**}$ is a mixed identity if $E \odot \mu = \mu \odot E = \mu$ for all$\mu \in P_1(M_a(\mathfrak{S}))$. By [18, Proposition 2.4], any mixed identity with norm one is a topological inner invariant mean on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$. Consequently, $M_a(\mathfrak{S})$ is always has a topological inner invariant mean on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$; that is, a $M_a(\mathfrak{S})$)-inner invariant mean on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$. Following [18], the measure algebra $M_a(\mathfrak{S})$ is called *strictly inner amenable* if there is a topological inner invariant mean m on $L^\infty(\mathfrak{S}; M_a(\mathfrak{S}))$ that is not a mixed identity.

Furthermore, recall from [3] that $\mathfrak{S}$ is called *strictly inner amenable* if there is a topological inner invariant mean m on $L^\infty(\mathfrak{S}; M_a(\mathfrak{S}))$ whose restriction to $C_b)\mathfrak{S})$ is not equal to δ_e.

In the case where $\mathfrak{S}$ is discrete, δ_e is the only mixed identity with norm one in $\ell^\infty(\mathfrak{S})^*$, and of course δ_e is an inner invariant mean. However, a mixed identity with norm one in $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))^*$ is not in general an inner invariant mean. In particular, inner amenability of $\mathfrak{S}$ is not equivalent to inner amenability of $M_a(\mathfrak{S})$.

It is shown in [15] that if a locally compact group G is strictly inner amenable, then $L^1(G)$ is strictly inner amenable. This resolves positively a question raised in [18]. We show that this result remains valid for certain foundation semigroups with identity.

PROPOSITION (4.1). *if $\mathfrak{S}$ be a compact foundation semigroup with identity e. Then strict inner amenability of S implies strict inner amenability of $M_a(\mathfrak{S})$.*

Proof. First, note that $C_b(\mathfrak{S})$ is equal to the space of $UC(\mathfrak{S})$ of all uniformly continuous functions on $\mathfrak{S}$. If $\mathfrak{S}$ is strict inner amenable and m is an inner invariant mean on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ not a mixed identity, then m is also a topological inner invariant mean on $UC(\mathfrak{S})$; [10, Corollary 2.3]. So, the result follows from this fact that any topological inner invariant mean on $C_b(\mathfrak{S})$ has an inner invariant extension to a topological inner invariant mean on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$; [9, Proposition 3.1]. □

THEOREM (4.2). *Let $\mathfrak{S}$ be a left or right compactly cancelletive foundation semigroup with identity e such that $M_a(\mathfrak{S})$ is separable, then strict inner amenability of $\mathfrak{S}$ implies strict inner amenability of $M_a(\mathfrak{S})$.*

Proof. Suppose that $\mathfrak{S}$ is strictly inner amenable. Then there there is an inner invariant mean m on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ not equal to δ_e on $C_b(\mathfrak{S})$. In view of [9, Theorem 2.3], the set of mixed identities with norm one on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ are exactly the extensions of δ_e to means on $L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$. In particular, m is not a mixed identity.

On the other hand, since $M_a(\mathfrak{S})$ is separable, there is a sequence $(\mu_n) \in P_1$ $(M_a(\mathfrak{S}))$ converging to m in the weak* topology of $M_a(\mathfrak{S})^{**}$ such that

$$(\delta_x * \mu_n - \mu_n * \delta_x)(g) \longrightarrow 0$$

for all $g \in L^\infty(\mathfrak{S}, M_a(\mathfrak{S}))$ and $x \in \mathfrak{S}$. So, $\mu_n)$ is a quasi-central sequence in $P_1(M_a$ $(\mathfrak{S}))$, and hence m as a weak* cluster point of $\mu_n)$ in $P_1(M_a(\mathfrak{S}))$ is a topological inner invariant mean. □

In the end, let $T = \{1, 2, ..., n\}$ ($n \in \mathbb{N}$). Define the multiplication on T by $1k = k1 = k$ for every $k \in T$ and $kl = k$ for $k \neq 1$ and $l \neq 1$. Let G be any locally compact group. Then $\mathfrak{S} = T \times G$ with the product topology and coordinatewise multiplication defines a foundation semigroup with identity such that $\mathfrak{S}$ is not a subset of any group.

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COMMUTATIVE ALGEBRAS OF TOEPLITZ OPERATORS ON THE SUPERSPHERE OF DIMENSION (2 | 2)

To my wife

ARMANDO SÁNCHEZ-NUNGARAY

ABSTRACT. In this paper, we introduce the Bergman theory and Toeplitz operator on the super-sphere $S^{(2|2)}$. We characterize the functions invariant under the action of the super-circle and finally we show that the C^* algebra of the Toeplitz operators with "radial-like" symbols (invariant under the action of the super-circle) is commutative.

1. Introduction

Commutative C^*-algebras generated by Toeplitz operators acting on the (weighted) Bergman space over the unit disc have been recently an important object of study.

In [15, 16] Vasilevski discovered a family of commutative C^*-algebras of Toeplitz operators on the unit disk. These algebras can be classified as follows. Each pencil of hyperbolic geodesics determines a set of symbols consisting of functions which are constant on the corresponding cycles, the orthogonal trajectories to geodesics forming a pencil. The C^*-algebra generated by all Toeplitz operators with such symbols turns out to be commutative.

In [6] Grudski, Quiroga, and Vasilevski proved that the *C^*-algebra generated by the Toeplitz operators is commutative on each weighted Bergman space if and only if there is a pencil of hyperbolic geodesics such that the symbols of the Toeplitz operators are constant on the cycles of this pencil*. All cycles are in fact the orbits of a one-parameter subgroup of isometries for the hyperbolic geometry on the unit disc. This provides us with the following scheme: *the C^*-algebra generated by Toeplitz operators is commutative on each weighted Bergman space if and only if there is a maximal commutative subgroup of Möbius transformations such that the symbols of the Toeplitz operators are invariant under the action of this subgroup.*

Other similar results on the sphere, the ball and on Reinhardt domains can be found in [8, 5, 9, 10, 11].

In [2, 3] Borthwick, Klimek, Lesniewski and Rinaldi introduced a general theory of the non-perturbative quantization of a class of hermitian symmetric supermanifolds. The quantization scheme is based on the notion of a Toeplitz superoperator on a suitable $\mathbb{Z}_2$-graded Hilbert space of superholomorphic functions. The quantized supermanifold arises as the $\mathbb{C}^*$-algebra generated by such operators.

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They carried out the quantization on the superplane, superdisc, and on Cartan superdomains.

The paper is organized as follows: In Section 2, we define the supersphere and the Lie supergroup $SU(2\mid 1)$. We construct a symplectic form on the supersphere invariant under the action of $SU(2\mid 1)$. We also define an $SU(2\mid 1)$-invariant Hermitian supermetric on it.

In Sections 3 and 4, we define the weighted Bergman superspace on the supersphere and construct the corresponding Bergman kernel. We study some properties of this kernel and define the Bergman supermetric on the supersphere. We define and describe Toeplitz superoperators on the supersphere as well.

In Section 5, we define the supercircle and prove that it is a super subgroup of isometries of the supersphere and describe the action of the supercircle on the supersphere. We determine those functions on the supersphere that are invariant under the action of the supercircle.

In the final section we determine the Toeplitz superoperators having supercircle-invariant symbols, and we prove that these operators are diagonal with respect to the base $\mathfrak{B}_0$. Therefore, the C^* algebra generated by the Toeplitz superoperators with "radial-like" symbols (ie, invariant under the action of the supercircle), is commutative.

2. The supersphere and the Lie supergroup $SU(2|1)$

In this paper, the supersphere $S^{(2|2)}$ is the superspace of equivalence classes defined on the set of two even and one odd complex coordinates (z_1,z_2,θ) taken from $\mathbb{C}^{(2|1)}$ by letting,

$$(z_1,z_2,\theta)\sim(z_1',z_2',\theta')\Leftrightarrow \exists\lambda\in\mathbb{C}-\{0\}:\ \text{such that } (z_1,z_2,\theta)=(\lambda z_1',\lambda z_2',\lambda\theta').$$

This yields the two usual charts on the sphere given by the complex local coordinates z, and z' , respectively, together with the real rank-2 vector bundle defined by the trivial complex line bundle having θ as a global non-vanishing section. The local charts are

$$(z,\theta)=\left(\frac{z_1}{z_2},\frac{\theta}{z_2}\right)\quad (z',\theta')=\left(\frac{z_2}{z_1},\frac{\theta}{z_1}\right).$$

and therefore $S^{(2|2)}$ can be covered by two open domains glued by

$$(z',\theta')=\left(\frac{1}{z},\frac{\theta}{z}\right).$$

More details concerning the construction of the superprojective plane can be found in [7].

In particular, any $f\in C^\infty(S^{(2|2)})$ can be written in local coordinates as

$$f(z,\theta,\bar\theta)=f_{00}(z)+f_{10}(z)\theta+f_{01}(z)\bar\theta+f_{11}(z)\theta\bar\theta$$

where $f_{ij}\in C^\infty(\mathbb{C})$.

Definition (2.1). A function $\Phi\in C^\infty(S^{(2|2)})$ is called *superholomorphic* if $\partial_{\bar z}\Phi=0$ and $\partial_{\bar\theta}\Phi=0$ or, equivalently, if

$$\Phi(z,\theta)=\varphi_0(z)+\varphi_1(z)\theta, \tag{2.2}$$

where φ_0 and φ_1 are holomorphic functions in $\mathbb{C}$. In what follows, we shall use the notation $Z=(z,\theta)$.

The Lie supergroup $SU(2|1)$ is defined as follows: Its base manifold is $SU(2)$, the group of unitary 2×2 matrices. Its real supermanifold structure sheaf is generated by the 3×3 matrices γ_{ij} and $\bar{\gamma}_{ij}$ satisfying the following conditions:

$$\gamma^*\gamma = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.3}$$

with $\gamma^*_{ij} = \bar{\gamma}_{ji}$;

$$\mathrm{Ber}\gamma = 1, \tag{2.4}$$

where Ber denotes the Berezinian (see, [1]); and, the parity assignments on γ_{ij} and $\bar{\gamma}_{ij}$ are given by:

$$|\gamma_{ij}| = |\bar{\gamma}_{ij}| = \begin{cases} 0, & \text{if } 1 \le i,j \le 2 \text{ and } i=j=3, \\ 1, & \text{otherwise.} \end{cases} \tag{2.5}$$

Conditions (2.3), (2.4), and (2.5) are the defining relations of the structure sheaf of $SU(2\mid 1)$, and its supergroup structure is given by matrix multpilication as usual. More details are given in [1].

We define the *action* of $SU(2\mid 1)$ on $S^{(2|2)}$ as follows:

$$z \to z' := \frac{\gamma_{11}z + \gamma_{12} + \gamma_{13}\theta}{\gamma_{21}z + \gamma_{22} + \gamma_{23}\theta},$$

$$\theta \to \theta' := \frac{\gamma_{31}z + \gamma_{32} + \gamma_{33}\theta}{\gamma_{21}z + \gamma_{22} + \gamma_{23}\theta}. \tag{2.6}$$

The expression $(\gamma_{21}z + \gamma_{22} + \gamma_{23}\theta)^{-1}$ is defined in terms of the Taylor series for superfunctions (see [1]) by

$$(\gamma_{21}z + \gamma_{22} + \gamma_{23}\theta)^{-1} = \frac{1}{\gamma_{21}z + \gamma_{22}} - \frac{\gamma_{23}}{(\gamma_{21}z + \gamma_{22})^2}\theta.$$

By a slight abuse of notation, we write (2.6) as $Z' = (z', \theta') = \gamma(Z)$.

We define

$$\gamma'(Z) = \mathrm{Ber}\begin{pmatrix} \frac{\partial z'}{\partial z} & \frac{\partial \theta'}{\partial z} \\ \frac{\partial z'}{\partial \theta} & \frac{\partial \theta'}{\partial \theta} \end{pmatrix} = \mathrm{Ber}\frac{\partial Z'}{\partial Z}. \tag{2.7}$$

PROPOSITION (2.8). *If $\gamma \in SU(2\mid 1)$, then*

$$\gamma'(Z) = \frac{1}{\gamma_{21}z + \gamma_{22} + \gamma_{23}\theta}. \tag{2.9}$$

(See [2]*).*

PROPOSITION (2.10). *Let $Z = (z,\theta)$ and $W = (w,\eta)$. If $\gamma(Z) = (z',\theta')$ and $\gamma(W) = (w',\eta')$ then*

$$1 + z'\bar{w'} - \theta'\bar{\eta'} = (1 + z\bar{w} - \theta\bar{\eta})\gamma'(Z)\overline{\gamma'(W)}. \tag{2.11}$$

Proof. The proof is by explicit computation. We start with

$$(1 + z'\bar{w'} - \theta'\bar{\eta'})(\gamma'(Z))^{-1}(\overline{\gamma'(W)})^{-1}.$$

If we substitute (2.9) and (2.6) in the above equation, we obtain

$$\left[1+\left(\frac{\gamma_{11}z+\gamma_{12}+\gamma_{13}\theta}{\gamma_{21}z+\gamma_{22}+\gamma_{23}\theta}\right)\overline{\left(\frac{\gamma_{11}w+\gamma_{12}+\gamma_{13}\eta}{\gamma_{21}w+\gamma_{22}+\gamma_{23}\eta}\right)}\right.$$
$$\left.-\left(\frac{\gamma_{31}z+\gamma_{32}+\gamma_{33}\theta}{\gamma_{21}z+\gamma_{22}+\gamma_{23}\theta}\right)\overline{\left(\frac{\gamma_{31}w+\gamma_{32}+\gamma_{33}\eta}{\gamma_{21}w+\gamma_{22}+\gamma_{23}\eta}\right)}\right](\gamma_{21}z+\gamma_{22}+\gamma_{23}\theta)\overline{(\gamma_{21}w+\gamma_{22}+\gamma_{23}\eta)},$$

or equivalently

$$(\gamma_{21}z+\gamma_{22}+\gamma_{23}\theta)(\bar\gamma_{21}\bar w+\bar\gamma_{22}-\bar\gamma_{23}\bar\eta)+(\gamma_{11}z+\gamma_{12}+\gamma_{13}\theta)(\bar\gamma_{11}\bar w+\bar\gamma_{12}-\bar\gamma_{13}\bar\eta)$$
$$-(\gamma_{31}z+\gamma_{32}+\gamma_{33}\theta)(\bar\gamma_{31}\bar w+\bar\gamma_{32}+\bar\gamma_{33}\bar\eta),$$

or

$$(\bar\gamma_{11}\gamma_{11}+\bar\gamma_{21}\gamma_{21}+\bar\gamma_{31}\gamma_{31})z\bar w+(\bar\gamma_{11}\gamma_{12}+\bar\gamma_{21}\gamma_{22}+\bar\gamma_{31}\gamma_{32})\bar w+(\bar\gamma_{11}\gamma_{13}+\bar\gamma_{21}\gamma_{23}+\bar\gamma_{31}\gamma_{33})\bar w\theta$$
$$+(\bar\gamma_{12}\gamma_{11}+\bar\gamma_{22}\gamma_{21}+\bar\gamma_{32}\gamma_{31})z+(\bar\gamma_{12}\gamma_{12}+\bar\gamma_{22}\gamma_{22}+\bar\gamma_{32}\gamma_{32})+(\bar\gamma_{12}\gamma_{13}+\bar\gamma_{22}\gamma_{23}+\bar\gamma_{32}\gamma_{33})\theta$$
$$-(\bar\gamma_{13}\gamma_{11}+\bar\gamma_{23}\gamma_{21}+\bar\gamma_{33}\gamma_{31})z\bar\eta-(\bar\gamma_{13}\gamma_{12}+\bar\gamma_{23}\gamma_{22}+\bar\gamma_{33}\gamma_{32})\bar\eta-(\bar\gamma_{13}\gamma_{13}+\bar\gamma_{23}\gamma_{23}+\bar\gamma_{33}\gamma_{33})\theta\bar\eta.$$

From (2.3) the identity follows. □

By an abuse of notation, we write

$$1+Z\bar W=1+z\bar w-\theta\bar\eta. \tag{2.12}$$

Let

$$D:=dz\otimes\frac{\partial}{\partial z}+d\theta\otimes\frac{\partial}{\partial\theta},$$
$$\bar D:=d\bar z\otimes\frac{\partial}{\partial\bar z}+d\bar\theta\otimes\frac{\partial}{\partial\bar\theta}.$$

Consider the following two-form

$$\omega:=iD\wedge\bar D\log(1+Z\bar Z)=iD\wedge\bar D\log(1+z\bar z-\theta\bar\theta). \tag{2.13}$$

PROPOSITION (2.14). *ω is an $SU(2|1)$-invariant supersymplectic form on $S^{(2|2)}$.*

Proof. To see that ω is $SU(2|1)$-invariant, we note that, as a consequence of (2.11) we have

$$\begin{aligned}\log(1+\gamma(Z)\overline{\gamma(Z)}) &= \log[(1+Z\bar Z)\gamma'(Z)\overline{\gamma'(Z)}]\\ &= \log(1+Z\bar Z)+\log(\gamma'(Z))+\log(\overline{\gamma'(Z)}).\end{aligned}$$

Since $\gamma'(Z)$ is holomorphic,

$$D\wedge\bar D\log\gamma'(Z)=0,$$

and thus $\gamma^*\omega=\omega$, where $\gamma^*\omega$ is the pullback.

To see that ω is supersymplectic, we write

$$\omega=i(\omega_{z\bar z}dz\wedge d\bar z+\omega_{z\bar\theta}dz\wedge d\bar\theta+\omega_{\theta\bar z}d\theta\wedge d\bar z+\omega_{\theta\bar\theta}d\theta\wedge d\bar\theta),$$

where

$$\omega_{z\bar z}=\frac{1}{(1+z\bar z)^2}+\frac{(1-z\bar z)}{(1+z\bar z)^3}\theta\bar\theta,\quad \omega_{z\bar\theta}=\frac{-\theta\bar z}{(1+z\bar z)^2},$$

$$\omega_{\theta\bar z}=\frac{z\bar\theta}{(1+z\bar z)^2}\quad\text{and}\quad \omega_{\theta\bar\theta}=\frac{-1}{1+z\bar z}. \tag{2.15}$$

Let

$$\Omega=\begin{pmatrix}\omega_{z\bar z} & \omega_{z\bar\theta}\\ \omega_{\theta\bar z} & \omega_{\theta\bar\theta}\end{pmatrix}. \tag{2.16}$$

We compute the Berezianian of Ω,

$$\left[\frac{1}{(1+z\bar{z})^2}+\frac{(1-z\bar{z})}{(1+z\bar{z})^3}\theta\bar{\theta}-\left(\frac{\theta\bar{z}}{(1+z\bar{z})^2}(-1-z\bar{z})\frac{z\bar{\theta}}{(1+z\bar{z})^2}\right)\right](-1-z\bar{z}).$$

As a consequence, we have

$$\mathrm{Ber}\Omega=\frac{-1}{1+z\bar{z}-\theta\bar{\theta}}. \tag{2.17}$$

It is clear from these explicit formulas that ω is non-degenerate and close, and thus it is supersymplectic. □

Observe that the inverse matrix of Ω is given by

$$\begin{pmatrix} (1+z\bar{z}-\theta\bar{\theta})(1+z\bar{z}) & (1+z\bar{z})\bar{z}\theta \\ (1+z\bar{z})z\bar{\theta} & -(1+z\bar{z}+\theta\bar{\theta}) \end{pmatrix}. \tag{2.18}$$

Definition (2.19). We define the *superspherical measure* by

$$d\mu(Z):=\frac{-1}{\pi}(1+z\bar{z}-\theta\bar{\theta})^{-1}dA(z)d\theta\wedge d\bar{\theta}, \tag{2.20}$$

where $dA(z)=(i/2)dz\wedge d\bar{z}$.

It is clear that the above form is $SU(2|1)$-invariant.

On the other hand, the natural almost complex structure (acs) on the supersphere is compatible with the symplectic form, moreover this acs is integrable since the projective plane is rigid (that is, all deformations of analytical geometry are trivial). See [13] and [14] for details.

Using the natural almost complex structure on the supersphere and the symplectic form given by (2.15), we obtain the Hermitian supermetric for the supersphere:

$$g=\omega_{z\bar{z}}dzd\bar{z}+\omega_{z\bar{\theta}}dzd\bar{\theta}+\omega_{\theta\bar{z}}d\theta d\bar{z}+\omega_{\theta\bar{\theta}}d\theta d\bar{\theta}. \tag{2.21}$$

Since $d\omega=0$, this supermetric is Kählerian.

3. Bergman superspace

We consider the perturbation of the measure (2.20) given by

$$d\mu_h(Z)=(1+z\bar{z}-\theta\bar{\theta})^{-1/h}d\mu(Z)=\frac{-1}{\pi}(1+z\bar{z}-\theta\bar{\theta})^{-1-N}dA(z)d\theta\wedge d\bar{\theta}, \tag{3.1}$$

where $h=1/2,1/3,\dots$ and $N=1/h\in\mathbb{N}$.

PROPOSITION (3.2). *The form defined by* (3.1) *has the following properties:*

$$\int_{S^{(2|2)}}d\mu_h(Z)=1,$$

and

$$d\mu_h(\gamma(Z))=\gamma'(Z)^{-1/h}\overline{\gamma'(Z)^{-1/h}}d\mu_r(Z),$$

where $\gamma'(Z)$ *is given by* (2.7).

Proof. The second statement is a consequence of (2.11). The first statement establishes that the integral is independent of h. For $N = 1/h$, we have the expansion

$$(1+z\bar{z}-\theta\bar{\theta})^{-N-1} = (1+z\bar{z})^{-N-1} + (N+1)(1+z\bar{z})^{-N-2}\theta\bar{\theta}, \tag{3.3}$$

and thus

$$\int_{S^{(2|2)}} d\mu_h(Z) = \frac{-N-1}{\pi}\int_{\mathbb{C}}(1+z\bar{z})^{-N-2}dA(z)\int\int\theta\bar{\theta}d\theta d\bar{\theta}$$

$$= \frac{N+1}{\pi}\int_0^{2\pi}\int_0^{\infty}(1+r^2)^{-N-2}rdrdt = (N+1)\int_0^{\infty}(1+u)^{-(N+2)}du = 1. \qquad \square$$

Let f, g be a functions defined on $S^{(2|2)}$, we define the *semi-inner product* by

$$(f,g)_h := \int_{S^{(2|2)}} f(Z)\overline{g(Z)}d\mu_h(Z), \tag{3.4}$$

where f, g have the form (5.5), and f_{ij} are measurable functions on S^2.

Expanding (3.4) we obtain

$$(f,g)_h = \frac{N+1}{\pi}\int_{\mathbb{C}}\frac{f_{00}(z)\overline{g_{00}(z)}dA(z)}{(1+z\bar{z})^{N+2}}$$

$$+\frac{1}{\pi}\int_{\mathbb{C}}\frac{(f_{11}(z)\overline{g_{00}} + f_{10}(z)\overline{g_{10}(z)} - f_{01}(z)\overline{g_{01}(z)} + f_{00}(z)\overline{g_{11}(z)})dA(z)}{(1+z\bar{z})^{N+1}}.$$

We note that the above semi-inner product is not positive definite. Now we consider the restriction of the semi-inner product to the set of superholomorphic functions. This semi-inner product turns out to be positive definite and therefore defines the inner product

$$(f,g)_h = \frac{N+1}{\pi}\int_{\mathbb{C}}\frac{f_0(z)\overline{g_0(z)}dA(z)}{(1+z\bar{z})^{N+2}} + \frac{1}{\pi}\int_{\mathbb{C}}\frac{f_1(z)\overline{g_1(z)}dA(z)}{(1+z\bar{z})^{N+1}}.$$

Using the above, we define the superspace

$$L_h^2(S^{(2|2)}) = \{f \ : \ f(Z) = f_0(z) + f_1(z)\theta, \ f_0, f_1 \text{are measurables and } (f,f)_h < \infty\}.$$

The completion of the set superholomorphic functions with respect to the norm $\|\cdot\|_h$ is a Hilbert space. This superspace is called the *weighted Bergman superspace* on the supersphere and is denoted by $\mathcal{A}_h^2(S^{(2|2)})$. It is clearly finite dimensional and closed.

For $f \in L^\infty(S^{(2|2)})$, we define the *weighted Bergman projection* by

$$P_h(f(Z)) = \int_{S^{(2|2)}} f(W)K^h(Z,W)d\mu_h(W),$$

where

$$K^h(Z,W) = (1+z\bar{w}-\theta\bar{\eta})^N.$$

THEOREM (3.5). *If $f \in L_h^\infty(S^{(2|2)})$, then $P(f) \in \mathcal{A}_h^2(S^{(2|2)})$ and $P(f) = f$ if $f \in \mathcal{A}_h^2(S^{(2|2)})$.*

Proof. To verify this statement we consider the functions:

$$\phi_{n,0}(Z) = \left[\binom{N}{n}\right]^{1/2} z^n, \text{ for } n = 0, 1, \ldots, N,$$

and

$$\phi_{n,1}(Z) = \left[N \begin{pmatrix} N-1 \\ n \end{pmatrix} \right]^{1/2} z^n \theta, \text{ for } n = 0, 1, ..., N-1, \tag{3.6}$$

where

$$\begin{pmatrix} N \\ n \end{pmatrix} = \frac{N!}{n!(N-n)!}.$$

The set of functions $\{\phi_{n,0}, \phi_{m,1},\ n = 0, ..., N,\ m = 0, ..., N-1\}$ is an orthonormal basis for $\mathcal{A}_h^2(S^{(2|2)})$.

In consequence,

$$\begin{aligned} K^h(Z,W) &= \sum_{n=0}^{N} \overline{\phi_{n,0}(W)} \phi_{n,0}(Z) + \sum_{n=0}^{N-1} \overline{\phi_{n,1}(W)} \phi_{n,1}(Z) \\ &= (1 + z\bar{w})^N - N(1 + z\bar{w})^{N-1} \theta \bar{\eta} \\ &= (1 + z\bar{w} - \theta\bar{\eta})^N, \end{aligned} \tag{3.7}$$

is the Bergman kernel for the space $\mathcal{A}_h^2(S^{(2|2)})$. □

PROPOSITION (3.8). *The Bergman kernel transforms under the action of the Lie supergroup $SU(2,|\,1)$ according to the rule*

$$K^h(\gamma(Z), \gamma(W)) = \gamma'(Z)^N \overline{\gamma'(W)}^N K^h(Z,W).$$

Proof. This is an immediate consequence of Proposition (2.11) with $Z = W$. □

Definition (3.9). We define the *Bergman supermetric* on the supersphere by

$$h = \sum_{\alpha,\beta} \frac{\partial^2 log K^1(Z,Z)}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \otimes d\bar{z}_\beta.$$

That is,

$$\begin{aligned} h &= \frac{\partial^2 log(1 + z\bar{z} - \theta\bar{\theta})}{\partial z \partial \bar{z}} dz d\bar{z} + \frac{\partial^2 log(1 + z\bar{z} - \theta\bar{\theta})}{\partial z \partial \bar{\theta}} dz d\bar{\theta} \\ &+ \frac{\partial^2 log(1 + z\bar{z} - \theta\bar{\theta})}{\partial \theta \partial \bar{z}} d\theta d\bar{z} + \frac{\partial^2 log(1 + z\bar{z} - \theta\bar{\theta})}{\partial \theta \partial \bar{\theta}} d\theta d\bar{\theta}. \end{aligned}$$

It is clear that the supermetric h is equal to the Hermitian supermetric g defined by (2.21).

4. Toeplitz superoperators

Definition (4.1). Let a be a function on $C^\infty(S^{(2|2)})$ or $L^\infty(S^{(2|2)})$. We define the *Toeplitz superoperator* with symbol a acting on the weighted Bergman superspace and $\mathcal{A}_h^2(S^{(2|2)})$ as

$$T_a^h(\varphi)(Z) = P_h(M_a(\varphi))(Z) = \int_{S^{(2|2)}} a(W)\Phi(W) K^h(Z,W) d\mu_h(W). \tag{4.2}$$

where $\Phi \in \mathcal{A}_h^2(S^{(2|2)})$ and M_a is a multiplication operator.

Definition (4.3). We define the *weighted Bergman space on the sphere* S^2 by

$$A_N^2(S^2) = A_h^2(S^2) = \{\phi \in L^2(S^2, (N+1)d\mu_h) \ : \ \phi \text{ is holomorphic}\},$$

where $d\mu_h = (1/\pi)(1 + z\bar{z})^{-N-2} dA(z)$ and $N = 1/h = 1, 2, 3, \ldots$.

Definition (4.4). Let b be a function on $L^\infty(S^2)$. We define the *Toeplitz operator with symbol a acting on* $A^2_h(S^2)$ as

$$T_h(b)(\phi)(z) = T_N(b)(\phi)(z) = B_h(b(w)\phi(w))(z);$$

where B_h is the Bergman projection onto the weighted Bergman space $A^2_h(S^2)$.

We can represent the superspace $\mathcal{A}^2_h(S^{(2|2)})$ as a direct sum of Bergman spaces on the sphere S^2. We have that if $\Phi = \varphi_0 + \varphi_1\theta \in \mathcal{A}^2_h(S^{(2|2)})$ then $\varphi_0 \in A^2_N(S^2)$ and $\varphi_1 \in A^2_{N-1}(S^2)$. Therefore we obtain that the Bergman superspace has the form

$$\mathcal{A}^2_h(S^{(2|2)}) = A^2_N(S^2) \oplus A^2_{N-1}(S^2)$$

where $A^2_N(S^2)$ and $A^2_{N-1}(S^2)\Theta$ are the even and odd parts of the Bergman superspace respectively.

On the other hand, we describe the Toeplitz operator acting on the superspace $A^2_N(S^2) \oplus A^2_{N-1}(S^2)$.

First, we know that $a(W)$ is of the form (5.5) and $\Phi(W) = \varphi_0(w) + \varphi_1(w)\eta$, then

$$\begin{aligned} a(W)\Phi(W) &= \varphi_0(w)[a_{00}(w) + a_{10}(w)\eta + a_{01}(w)\bar{\eta} + a_{11}(w)\eta\bar{\eta}] \\ &+ \varphi_1(w)[a_{00}(w)\eta - a_{01}(w)\eta\bar{\eta}], \end{aligned}$$

Using equations (3.3) and (3.7) we have

$$\begin{aligned} K^h(Z,W)d\mu_h(W) &= \frac{-1}{\pi}\left[\frac{(1+z\bar{w})^N}{(1+w\bar{w})^{N+1}}\right] dA(w)d\eta \wedge d\bar{\eta} \\ &+ \frac{-1}{\pi}\left[-\frac{N(1+z\bar{w})^{N-1}\theta\bar{\eta}}{(1+w\bar{w})^{N+1}}\right. \\ &+ \left.\frac{(N+1)(1+z\bar{w})^N}{(1+w\bar{w})^{N+2}}\eta\bar{\eta}\right] dA(w)d\eta \wedge d\bar{\eta}. \end{aligned}$$

The only elements in the integrals that do not vanish are the ones that contain the term $\eta\bar{\eta}$. In consequence we obtain

$$\begin{aligned} T^h_a\Phi(Z) &= \frac{-1}{\pi}\int_{\mathbb{C}} \frac{\varphi_1(w)a_{01}(w)(1+z\bar{w})^N}{(1+w\bar{w})^{N+1}} dA(w) \\ &+ \frac{N+1}{\pi}\int_{\mathbb{C}}\left(a_{00}(w) + \frac{a_{11}(w)(1+z\bar{z})}{N+1}\right)\frac{\varphi_0(w)(1+z\bar{w})^N}{(1+w\bar{w})^{N+2}} dA(w) \\ &+ \left[\frac{N}{\pi}\int_{\mathbb{C}} \frac{\varphi_0(w)a_{10}(w)(1+z\bar{w})^{N-1}}{(1+w\bar{w})^{N+1}} dA(w)\right]\theta \\ &+ \left[\frac{N}{\pi}\int_{\mathbb{C}} \frac{\varphi_1(w)a_{00}(w)(1+z\bar{w})^{N-1}}{(1+w\bar{w})^{N+1}} dA(w)\right]\theta. \end{aligned}$$

Therefore the Toeplitz superoperator T^h_a on $A^2_N(S^2)\bigoplus A^2_{N-1}(S^2)$ is given by

$$\begin{pmatrix} T_N(a_{00} + a_{11}(1+z\bar{z})/(N+1)) & -T^{N-1}_N(a_{01}) \\ T^N_{N-1}(a_{10}) & T_{N-1}(a_{00}) \end{pmatrix}\begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \tag{4.5}$$

where $T_N(a_{00} + a_{11}(1+z\bar{z})/(N+1))$ and $T_{N-1}(a_{00})$ are Toeplitz operators on the weighted Bergman spaces $A^2_N(S^2)$ and $A^2_{N-1}(S^2)$ respectively. On the other hand, we have that $T_N^{N-1}(a_{01})$ and $T_{N-1}^N(a_{01})$ are quasi-Toeplitz operators defined by

$$(4.6)\qquad \begin{array}{rcl} T_N^{N-1}(a_{01}): \ A^2_{N-1}(S^2) & \longrightarrow & A^2_N(S^2). \\ \varphi_1 & \mapsto & \dfrac{1}{\pi}\displaystyle\int_{\mathbb{C}} \frac{\varphi_1(w)a_{01}(w)(1+z\bar{w})^N}{(1+w\bar{w})^{N+1}}dA(w) \end{array}$$

and

$$(4.7)\qquad \begin{array}{rcl} T_{N-1}^N(a_{01}): \ A^2_N(S^2) & \longrightarrow & A^2_{N-1}(S^2) \\ \varphi_0 & \mapsto & \dfrac{N}{\pi}\displaystyle\int_{\mathbb{C}} \frac{\varphi_0(w)a_{10}(w)(1+z\bar{w})^{N-1}}{(1+w\bar{w})^{N+1}}dA(w). \end{array}$$

Note that the Toeplitz superoperator has the form

$$(4.8)\qquad T_a^h = \begin{pmatrix} T^h(a_{00}+a_{11}\theta\bar{\theta}) & T^h(a_{01}\bar{\theta}) \\ T^h(a_{10}\theta) & T^h(a_{00}) \end{pmatrix},$$

and we have that $a_{00}+a_{11}\theta\bar{\theta}, a_{00}$ are even functions and $a_{10}\theta, a_{01}\bar{\theta}$ are odd functions. In consequence we have that every Toeplitz superoperator is an even operator on the Bergman superspace.

5. $S^{(1|1)}$-invariant functions

The super circle is defined as the super manifold

$$S^{1|1} = \{(z = x+iy, \zeta = \zeta_1 + i\zeta_2)| \ x^2+y^2+\bar{\zeta}\zeta = 1, \ \ \zeta_1 x + \zeta_2 y = 0\}$$

where ζ_1, ζ_2 are real Grassmann varibles.

The definition of $S^{1|1}$ given above is equivalent to define the supercircle as the set of matrices

$$A = \begin{pmatrix} z & \zeta \\ \zeta & z \end{pmatrix},$$

such that $A^*A = I, \ \ BerA = 1$.

In local coordinates, $z = e^{it}$ and $\zeta = ie^{it}\xi$, where ξ is a real grassmann variable.

Now, we give a representation of the supercircle as $SU(2\,|\,1)$.

THEOREM (5.1). *The supercircle is a supergroup of isometries of the supersphere.*

Proof. First we define the map

$$A : S^{(1|1)} \longrightarrow SU(2\,|\,1),$$

given by

$$A\left[\begin{pmatrix} z & \zeta \\ \zeta & z \end{pmatrix}\right] = \begin{pmatrix} z & 0 & \zeta \\ 0 & 1 & 0 \\ \zeta & 0 & z \end{pmatrix},$$

where

$$
\begin{aligned}
A\left[\begin{pmatrix} z & \zeta \\ \zeta & z \end{pmatrix}\right]^{*} \cdot A\left[\begin{pmatrix} z & \zeta \\ \zeta & z \end{pmatrix}\right]
&= \begin{pmatrix} \bar{z} & 0 & \bar{\zeta} \\ 0 & 1 & 0 \\ \bar{\zeta} & 0 & \bar{z} \end{pmatrix}\begin{pmatrix} z & 0 & \zeta \\ 0 & 1 & 0 \\ \zeta & 0 & z \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Then,

$$
\operatorname{Ber}\begin{pmatrix} z & 0 & \zeta \\ 0 & 1 & 0 \\ \zeta & 0 & z \end{pmatrix}
$$

$$
= \operatorname{Det}\left(\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \zeta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} z \end{pmatrix} \cdot \begin{pmatrix} \zeta & 0 \end{pmatrix}\right) \cdot \operatorname{Det}\begin{pmatrix} \bar{z} \end{pmatrix} = 1.
$$

It is easy to see that A satisfy

$$
A[z_1,\xi_1)] \cdot A[(z_2,\xi_2)] = A[(z_1z_2, z_2\xi_1 + z_1\xi_2)];
$$

and therefore, it is a Lie supergroup homomorphism. Actually, it is a monomorphism.

We have proved that the supercircle is a subsupergroup of $SU(2|1)$. In particular, the Riemannian supermetric in $S^{(2|2)}$ given by (2.21) is $S^{(1|1)}$-invariant. Since $SU(2|1)$ acts by isometries on $S^{(2|2)}$, so does the supercircle. Moreover we have that the group $SU(2|1)$ is a group of isometrics of the supermanifold $S^{(2|2)}$. Thus we obtain that the supercircle is a supergroup of isometries of the supersphere. □

The action of the group $SU(2|1)$ on the supermanifold $S^{(2|2)}$ is given by (2.6), thus the action of the supercircle on the supersphere

$$
(e^{it},\xi)\cdot(z,\theta),
$$

is given by

$$
\begin{aligned}
z &\mapsto w = ze^{it} + ie^{it}\xi\theta, \\
\theta &\mapsto \eta = e^{it}\theta + ie^{it}z\xi. \qquad (5.2)
\end{aligned}
$$

THEOREM (5.3). *Let a be a smooth function on the supersphere. If a is invariant under the action of the supercircle (i.e. $a(z,\theta) = a((e^{it},\xi)\cdot(z,\theta))$), then a has the form*

$$
a(z,\theta) = a_0(r) + a_1(r)\bar{z}\theta + a_1(r)z\bar{\theta} - \frac{a_0'(r)}{2r}\theta\bar{\theta}, \qquad (5.4)
$$

where a_0 and a_1 are radial functions.

Proof. An element $a \in C^\infty(S^{(2|2)})$ is of the form

$$
a(z,\theta) = a_{00}(z) + a_{10}(z)\theta + a_{01}(z)\bar{\theta} + a_{11}(z)\theta\bar{\theta}, \qquad (5.5)
$$

where $a_{ij} \in C^\infty(S^2)$.

Now, we want to find the $S^{(1|1)}$-invariant functions, i.e. $a \in C^\infty(S^{(2|2)})$ such that

$$
a(z,\theta) = a(ze^{it} + ie^{it}\xi\theta, e^{it}\theta + ie^{it}z\xi), \qquad (5.6)
$$

where $(e^{it}, \xi) \in S^{(1|1)}$.

First we take the elements of the form $(e^{it}, 0)$, then a is invariant under the action of these elements, if this function satisfies the equation

$$\begin{aligned} & a_{00}(z) + a_{10}(z)\theta + a_{01}(z)\bar{\theta} + a_{11}(z)\theta\bar{\theta} \\ = & a_{00}(ze^{it}) + a_{10}(ze^{it})\theta e^{it} + a_{01}(ze^{it})\bar{\theta}e^{-it} + a_{11}(ze^{it})\theta\bar{\theta}. \end{aligned}$$

By the above equation, we obtain that the function a_{00} and a_{11} are radial, while the function $a_{10}(z)$ is of the form $\bar{z}\tilde{a}_{10}$ where $\tilde{a}_{10}$ is a radial function, analogously $a_{01}(z) = z\tilde{a}_{01}$.

Therefore we have that the function a is of the form

$$a(z,\theta) = a_{00}(r) + \bar{z}\tilde{a}_{10}(r)\theta + z\tilde{a}_{01}(r)\bar{\theta} + a_{11}(r)\theta\bar{\theta}. \tag{5.7}$$

Now we consider the action of elements of the form $(1, \xi)$,

$$w = z + i\xi\theta \quad \text{and} \quad \eta = \theta + iz\xi.$$

We note that

$$s = (w\bar{w})^{1/2} = ((z + i\xi\theta)(\bar{z} + i\xi\bar{\theta}))^{1/2} = r + \frac{i\xi(\bar{z}\theta + z\bar{\theta})}{2r}.$$

Now, we take a radial function $h(r)$ then $h(s)$ is defined in terms of the Taylor series for superfunctions (see, [1]), thus

$$h(s) = h(r) + \frac{i\xi(\bar{z}\theta + z\bar{\theta})h'(r)}{2r}. \tag{5.8}$$

On the other hand,

$$h(s)\bar{w}\eta = \left(h(r) + \frac{i\xi(\bar{z}\theta + z\bar{\theta})h'(r)}{2r}\right)(\bar{z} + i\xi\bar{\theta})(\theta + iz\xi) = h(r)\bar{z}\theta + ir^2h(r)\xi, \tag{5.9}$$

or equivalently

$$h(s)w\bar{\eta} = h(r)\bar{z}\theta - ir^2h(r)\xi. \tag{5.10}$$

Analogously

$$h(s)\eta\bar{\eta} = \left(h(r) + \frac{\xi(\bar{z}\theta - z\bar{\theta})h'(r)}{2r}\right)(\theta + iz\xi)(\bar{\theta} - i\bar{z}\xi) = h(r)\theta\theta + h(r)i\xi(\bar{z}\theta + z\bar{\theta}). \tag{5.11}$$

As a consequence of (5.8)-(5.11), we have

$$\begin{aligned} a(w,\eta) &= a_{00}(s) + \tilde{a}_{10}(s)\bar{w}\eta + \tilde{a}_{01}(s)w\bar{\eta} + a_{11}(s)\eta\bar{\eta} \\ &= a_{00}(r) + \frac{i\xi(\bar{z}\theta + z\bar{\theta})a'_{00}(r)}{2r} + \tilde{a}_{10}(r)\bar{z}\theta + ir^2\tilde{a}_{10}(r)\xi \\ &+ \tilde{a}_{01}(r)\bar{z}\theta - ir^2\tilde{a}_{01}(r)\xi + a_{11}(r)\theta\theta + a_{11}(r)i\xi(\bar{z}\theta + z\bar{\theta}). \end{aligned}$$

Therefore, a function a is $S^{(1|1)}$-invariant if this function satisfies (5.7) and the following condition:

$$a_{11}(r) = \frac{-a'_{00}(r)}{2r} \quad \text{and} \quad \tilde{a}_{10}(r) = \tilde{a}_{01}(r). \tag{5.12}$$

□

6. Algebra of Toeplitz operator with $S^{(1|1)}$-invariant symbols

Now we study Toeplitz operators whose symbols are invariant under the action of the supergroup $S^{(1|1)}$; this functions are of the form (5.4).

The Toeplitz superoperator with symbol a acting on the Bergman superspace $\mathcal{A}_h^2(S^{(2|2)})$ where a is $S^{(1|1)}$-invariant function, has the form

$$\begin{aligned} T_a^h(\varphi_0(z)+\varphi_1(z)\theta) &= \frac{-1}{\pi}\int_{\mathbb{C}} \frac{\varphi_1(w)a_1(r)w(1+z\bar{w})^N}{(1+w\bar{w})^{N+1}}dA(w) \\ &+ \frac{N+1}{\pi}\int_{\mathbb{C}} \frac{\varphi_0(w)a_0(r)(1+z\bar{w})^N}{(1+w\bar{w})^{N+2}}dA(w) \\ &- \frac{1}{\pi}\int_{\mathbb{C}} \frac{\varphi_0(w)a_0'(r)(1+z\bar{w})^N}{2r(1+w\bar{w})^{N+1}}dA(w) \\ &+ \left[\frac{N}{\pi}\int_{\mathbb{C}} \frac{\varphi_0(w)a_1(r)\bar{w}(1+z\bar{w})^{N-1}}{(1+w\bar{w})^{N+1}}dA(w)\right]\theta \\ &= \left[\frac{N}{\pi}\int_{\mathbb{C}} \frac{\varphi_1(w)a_0(r)(1+z\bar{w})^{N-1}}{(1+w\bar{w})^{N+1}}dA(w)\right]\theta. \end{aligned}$$

We define the coefficients $\hat{a^N}$ of the function a by the equation

$$\hat{a^N}(n) = \int_0^\infty \frac{a(r)}{(1+r^2)^{N+1}} r^{2n+1} dr \quad \text{for} \quad n = 0,1,\dots,N. \tag{6.1}$$

We consider the base $\mathfrak{B}$ of $\mathcal{A}_h^2(S^{(2|2)})$ given by

$$\mathfrak{B} = \{z^n, \text{ for } n = 0,1,\dots,N, \quad \text{and} \quad z^n\theta, \text{ for } n = 0,1,\dots,N-1\}. \tag{6.2}$$

Now we evaluate the Toeplitz superoperator T_a^h on the elements of $\mathfrak{B}$.
First, we evaluate on z^n for $n = 0$

$$\begin{aligned} T_a^h(1) &= \frac{N+1}{\pi}\int_{\mathbb{C}} \frac{a_0(r)(1+z\bar{w})^N}{(1+w\bar{w})^{N+2}}dA(w) \\ &- \frac{1}{\pi}\int_{\mathbb{C}} \frac{a_0'(r)(1+z\bar{w})^N}{2r(1+w\bar{w})^{N+1}}dA(w) \\ &+ \left[\frac{N}{\pi}\int_{\mathbb{C}} \frac{a_1(r)\bar{w}(1+z\bar{w})^{N-1}}{(1+w\bar{w})^{N+1}}dA(w)\right]\theta \\ &= 2(N+1)\int_0^\infty \frac{a_0(r)r\,dr}{(1+r^2)^{N+2}} - \int_0^\infty \frac{a_0'(r)dr}{(1+r^2)^{N+1}} = \tilde{a}_0^N(0). \end{aligned} \tag{6.3}$$

Now, we evaluate on z^n for $n = 1,2,...,N$

$$\begin{aligned}
T_a^h(z^n) &= \frac{N+1}{\pi}\int_{\mathbb{C}} \frac{w^n a_0(r)(1+z\bar{w})^N}{(1+w\bar{w})^{N+2}} dA(w) \\
&- \frac{1}{\pi}\int_{\mathbb{C}} \frac{w^n a_0'(r)(1+z\bar{w})^N}{2r(1+w\bar{w})^{N+1}} dA(w) \\
&+ \left[\frac{N}{\pi}\int_{\mathbb{C}} \frac{w^n a_1(r)\bar{w}(1+z\bar{w})^{N-1}}{(1+w\bar{w})^{N+1}} dA(w)\right]\theta \\
&= 2\binom{N}{n}\left[(N+1)\int_0^\infty \frac{a_0(r)r^{2n+1}dr}{(1+r^2)^{N+2}} - \int_0^\infty \frac{a_0'(r)r^{2n+1}dr}{2r(1+r^2)^{N+1}}\right] z^n \\
&+ 2N\binom{N-1}{n-1}\left[\int_0^\infty \frac{a_1(r)r^{2n+1}dr}{(1+r^2)^{N+1}}\right] z^{n-1}\theta. \qquad (6.4)
\end{aligned}$$

Integrating by parts we have

$$\begin{aligned}
&\int_0^\infty \frac{a_0(r)r^{2n+1}dr}{(1+r^2)^{N+2}} \\
&= \frac{n}{N+1}\int_0^\infty \frac{a_0(r)r^{2(n-1)+1}dr}{(1+r^2)^{N+1}} + \frac{1}{N+1}\int_0^\infty \frac{a_0'(r)r^{2n+1}dr}{2r(1+r^2)^{N+1}}. \qquad (6.5)
\end{aligned}$$

We substitute (6.5) on (6.4) and use the coefficients of equation (6.1), then

$$T_a^h(z^n) = 2n\binom{N}{n}\left[\widehat{a_0^N}(n-1)\right] z^n + 2N\binom{N-1}{n-1}\left[\widehat{a_1^N}(n)\right] z^{n-1}\theta. \qquad (6.6)$$

Finally, we evaluate $z^n\theta$ for $n = 0,1,...,N-1$

$$\begin{aligned}
T_a^h(z^n\theta) &= \frac{-1}{\pi}\int_{\mathbb{C}} \frac{w^n a_1(r)w(1+z\bar{w})^N}{(1+w\bar{w})^{N+1}} dA(w) \\
&+ \left[\frac{N}{\pi}\int_{\mathbb{C}} \frac{w^n a_0(r)(1+z\bar{w})^{N-1}}{(1+w\bar{w})^{N+1}} dA(w)\right]\theta \\
&= -2\binom{N}{n+1}\left[\int_0^\infty \frac{a_1(r)r^{2(n+1)+1}dr}{(1+r^2)^{N+1}}\right] z^{n+1} \\
&+ 2N\binom{N-1}{n}\left[\int_0^\infty \frac{a_0(r)r^{2(n)+1}dr}{(1+r^2)^{N+1}}\right] z^n\theta.
\end{aligned}$$

In consequence we have that

$$T_a^h(z^n\theta) = -2\binom{N}{n+1}\left[\widehat{a_1^N}(n+1)\right] z^{n+1} + 2N\binom{N-1}{n}\left[\widehat{a_0^N}(n)\right] z^n\theta. \qquad (6.7)$$

7. Spectrum of Toeplitz operators

Consider the basis $\mathfrak{B}_0$ of the Bergman space $\mathcal{A}_h^2(S^{(2|2)})$ given by

$$\varphi_0 = 1,\ \varphi_n = \frac{-iz^n}{\sqrt{n}} + z^{n-1}\theta,\ \phi_n = \frac{iz^n}{\sqrt{n}} + z^{n-1}\theta \ \text{ for } n = 1,\dots,N$$

LEMMA (7.1). *The basis $\mathfrak{B}_0$ is orthogonal.*

Proof. It is sufficient to prove that φ_n and ϕ_n are orthogonal, since the basis $\mathfrak{B}$ generated by z^n and $z^n\theta$ is orthogonal. Calculate:

$$(\varphi_n,\phi_n)_h = \frac{N+1}{\pi}\int_{\mathbb{C}} \frac{-z^n\overline{z^n}dA(z)}{n(1+z\bar{z})^{N+2}} + \frac{1}{\pi}\int_{\mathbb{C}} \frac{z^{n-1}\overline{z^{n-1}}dA(z)}{(1+z\bar{z})^{N+1}}.$$
$$= \frac{-1}{n}\begin{pmatrix} N \\ n \end{pmatrix}^{-1} + \frac{1}{N}\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}^{-1} = 0. \qquad \square$$

THEOREM (7.2). *Let a be a function invariant under the action of supercircle. Then the Toeplitz operator T_a^h is diagonal with respect to the basis $\mathfrak{B}_0$.*

Proof. By (6.3) that $T_a^h(1)$ belong to the space generated by 1. That is, 1 is a eigenfunction of Toeplitz operator with eigenvalue $\tilde{a}_0^N(0)$.

On the other hand, By (6.6) and (6.7) we know that $T_a^h(z^n)$ and $T_a^h(z^{n-1}\theta)$ belong to the space generated by z^n and $z^{n-1}\theta$, where $n = 1,\ldots,N$. Therefore the Toeplitz superoperator has the form

$$\begin{pmatrix} 2n\begin{pmatrix} N \\ n \end{pmatrix}\widehat{a_0^N}(n-1) & -2\begin{pmatrix} N \\ n \end{pmatrix}\widehat{a_1^N}(n) \\ 2N\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}\widehat{a_1^N}(n) & 2N\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}\widehat{a_0^N}(n-1) \end{pmatrix}$$
$$= 2N\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}\begin{pmatrix} \widehat{a_0^N}(n-1) & \frac{-\widehat{a_1^N}(n)}{n} \\ \widehat{a_1^N}(n) & \widehat{a_0^N}(n-1) \end{pmatrix}$$

on the superspace generated by $\{z^n, z^{n-1}\theta\}$, where $n = 1,\ldots,N$.

We calculate the eigenvalue corresponding to the above matrix, which are given by

$$2N\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}\left(\widehat{a_0^N}(n-1) - \frac{i\widehat{a_1^N}(n)}{\sqrt{n}}\right) \quad \text{and} \quad 2N\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}\left(\widehat{a_0^N}(n-1) + \frac{i\widehat{a_1^N}(n)}{\sqrt{n}}\right)$$

this engenvalues corresponding to eigenfunction φ_n, ϕ_n for $n = 1,\ldots N$ respectively. $\square$

COROLLARY (7.3). *Let a be invariant under the action of supercircle, then the spectrum of the Toeplitz operator T_a^h is given by*

$$spT_a^h = \left\{ \tilde{a}_0^N(0), \quad 2N\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}\frac{\pm i\widehat{a_1^N}(n) + \sqrt{n}\widehat{a_0^N}(n-1)}{\sqrt{n}} \ \textit{for}\ n = 1,\ldots N \right\}.$$

COROLLARY (7.4). *Let a be invariant under the action of supercircle. Then the elements of the basis $\mathfrak{B}_0$ are eigenfunctions of the Toeplitz operator T_a^h, equivalently*

$$T_a^h(1) = \tilde{a}_0^N(0)\cdot 1$$
$$T_a^h(\varphi_n) = 2N\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}\left(\frac{-i\widehat{a_1^N}(n) + \sqrt{n}\widehat{a_0^N}(n-1)}{\sqrt{n}}\right)\cdot\varphi_n \ \textit{for}\ n = 1,\ldots N.$$
$$T_a^h(\phi_n) = 2N\begin{pmatrix} N-1 \\ n-1 \end{pmatrix}\left(\frac{i\widehat{a_1^N}(n) + \sqrt{n}\widehat{a_0^N}(n-1)}{\sqrt{n}}\right)\cdot\phi_n \ \textit{for}\ n = 1,\ldots N.$$

COROLLARY (7.5). *The algebra of Toeplitz operators with invariant symbols under the action of the supercircle is commutative.*

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SOME NEW NONIMMERSION RESULTS FOR REAL PROJECTIVE SPACES

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ABSTRACT. We use the spectrum tmf to obtain new nonimmersion results for many real projective spaces RP^n for n as small as 113. The only new ingredient is some new calculations of tmf-cohomology groups. We present an expanded table of nonimmersion results. Our theorem is new for 17% of the values of n between 2^i and $2^i + 2^{14}$ for $i \geq 15$.

1. Introduction

We use the spectrum tmf to prove the following new nonimmersion theorem for real projective spaces P^n.

THEOREM (1.1). *Let $\alpha(n)$ denote the number of 1's in the binary expansion of n.*

(a) *If $\alpha(M) = 3$, then P^{8M+9} does not immerse in ($\not\subseteq$) $\mathbb{R}^{16M-1}$.*
(b) *If $\alpha(M) = 6$, then $P^{8M+9} \not\subseteq \mathbb{R}^{16M-11}$.*
(c) *If $\alpha(M) = 7$, then $P^{16M+16} \not\subseteq \mathbb{R}^{32M-7}$ and $P^{16M+17} \not\subseteq \mathbb{R}^{32M-6}$.*
(d) *If $\alpha(M) = 9$, then $P^{32M+25} \not\subseteq \mathbb{R}^{64M-4}$ and $P^{32M+26} \not\subseteq \mathbb{R}^{64M-3}$.*
(e) *If $\alpha(M) = 10$, then $P^{16M+17} \not\subseteq \mathbb{R}^{32M-20}$ and $P^{16M+18} \not\subseteq \mathbb{R}^{32M-19}$.*

We apply the same method that was used in [4], using $\mathrm{tmf}^*(-)$ to detect nonexistence of axial maps. The novelty here is that we compute and utilize groups $\mathrm{tmf}^*(P^m \wedge P^n)$ when m and/or n is odd. In [4], only even values of m and n were considered. There is, however, no significant difference or complication in using the odd values. We prove Theorem 1.1 in Section 2.

For many years, the author has maintained a website ([5]) which listed all known immersion, nonimmersion, embedding, and nonembedding results for P^n and tabulated them for $n = 2^i + d$ with $2^i > d$ and $0 \leq d \leq 63$. In [12], W. Stephen Wilson acknowledged how this table motivated him to try (and succeed) to prove nonimmersions for small P^n. Our Theorem 1.1(a) includes $P^{2^i+49} \not\subseteq \mathbb{R}^{2^{i+1}+79}$ and $P^{2^i+57} \not\subseteq \mathbb{R}^{2^{i+1}+95}$ for $i \geq 6$, which improve on previous best results (of [12]) by 1 and 2 dimensions, respectively, and hence enter the table [5].

To facilitate checking whether results are new, the author has greatly expanded his table of nonimmersion results at `www.lehigh.edu/~dmd1/imms.html`. We have listed there the best known nonimmersions of P^{2^i+d} for $2^i > d+1$ and $0 \leq d \leq 16{,}383$ together with the first acknowledged source. A listing of and link to the `Maple` program that generated this table is also included there. This table gives all known nonimmersion results for P^n with $7 < n < 49{,}152$ except for James' nonimmersions ([11]) of P^{2^e-1} in dimension $2^{e+1} - 2e - \langle 3,2,2,4\rangle$ if $e \equiv \langle 0,1,2,3\rangle$ mod 4.

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Theorem 1.1 appears 2796 times in this table, thus giving new results for 17% of the projective spaces of dimension between 2^i and 2^i+2^{14} for $i\ge 15$. The seminal result of [6],

$$P^{2(m+\alpha(m)-1)}\not\subseteq\mathbb{R}^{4m-2\alpha(m)}, \tag{1.2}$$

appears 7063 times in the table, but is divided among four references. The first 4361 of them appeared in [1], which obtained a result equivalent to (1.2) for P^n with n satisfying a very complicated condition. The statement (1.2) was first conjectured in [2] and proved there for $\alpha(m)\le 6$, which yielded 168 new results in this table. It was extended to $\alpha(m)=7$ in [13], and this still applies to 700 values. This left 1834 values which were covered by the general result (1.2) and not by any of the three preceding references, and have not been bettered in subsequent work.

The first tmf-paper, ([4]), appears 2866 times in the table; there are 110 additional values for small $\alpha(-)$ of tmf-implied nonimmersions which were overlooked in [4] and noted in [8]. The other big collection of nonimmersion results is those obtained in [12] using $ER(2)$-cohomology, which appears 2092 times. Both $ER(2)$ and tmf can be considered as real versions of $BP\langle 2\rangle$. Using $ER(2)$ is advantageous because $ER(2)^*(P^n)$ has a 2-dimensional class, while $\text{tmf}^*(P^n)$ only has an 8-dimensional class. Also $ER(2)$ is more closely related to $BP\langle 2\rangle$, and so, as W. Stephen Wilson says, it can "mooch" off the result (1.2). The advantage of tmf is that some of its groups are one 2-power larger than those of $ER(2)$.

In [6], it was stated that (1.2) was within 2 dimensions of all known nonimmersion results, in the sense that the two dimensions could come from the Euclidean space, the projective space, or a combination. In other words, if $D(n)$ denotes the nonimmersion dimension for P^n obtained from (1.2), and $K(n)$ the best known nonimmersion dimension for P^n, then, at the time, it was true that

$$K(n)\le\max(D(n)+2,\ D(n+1)+1,\ D(n+2)). \tag{1.3}$$

This is no longer true. There are 10 values of n in the table for which the result of [9], which states that if $\alpha(n)=4$ and $n\equiv 10 \bmod 16$ then $P^n\not\subseteq\mathbb{R}^{2n-9}$, does not satisfy (1.3), and there are 418 values of n in the table for which Theorem 1.1(c) does not satisfy (1.3). These are the only results which are more than 2 stronger than (1.2) in the sense of (1.3), and it is still true that (1.2) is within 3 dimensions of all known results in the same sense. That is, the following statement is currently true:

$$K(n)\le\max(D(n)+3,\ D(n+1)+2,\ D(n+2)+1,\ D(n+3)).$$

The first example of (1.3) not being satisfied occurs for $n=58$; we have $K(58)=107$ due to [9] (which used modified Postnikov towers) while $D(58)=D(59)=98$ and $D(60)=D(61)=106$. The first example of our 1.1(c) causing (1.3) to be not satisfied occurs from $K(3584)=7129$ (due to 1.1(c)) while $D(3584)=D(3585)=7124$ and $D(3586)=D(3587)=7128$.

Some of our new results improve on previous best results by large amounts. For example, from 1.1(b), we obtain $P^{32265}\not\subseteq\mathbb{R}^{64501}$. Prior to this paper, the best result was $P^{32265}\not\subseteq\mathbb{R}^{64466}$, due to (1.2), and so we improve by 35 dimensions. However, (1.3) holds here because (1.2) also implies that $P^{32266}\not\subseteq\mathbb{R}^{64500}$, which is weaker than our new result by 1 dimension in the projective space plus 1 in the Euclidean space.

Theorem 1.1 can be extended to larger values of $\alpha(M)$ similarly to what was done in [4]. We have emphasized the results for small values of $\alpha(M)$ for clarity of exposition. The extension, whose proof we sketch in Section 3, is as follows. The lettering of the parts corresponds to the parts of Theorem 1.1.

THEOREM (1.4). *Let $p(h)$ denote the smallest 2-power $\geq h$.*

(b,e) *Suppose $\alpha(M) = 4h+2$ and $h \leq 2^{e_1} - 2^{e_0}$ if $M \equiv 2^{e_0} + 2^{e_1} \mod 2^{e_1+1}$ with $e_0 < e_1$. Then*
  (b) *If h is odd, $P^{8M+8h+1} \not\subseteq \mathbb{R}^{16M-8h-3}$, and*
  (e) *If h is even, then $P^{8M+8h+1} \not\subseteq \mathbb{R}^{16M-8h-4}$ and $P^{8M+8h+2} \not\subseteq \mathbb{R}^{16M-8h-3}$.*

(c) *If $\alpha(M) = 4h+3$ with h odd and $M \equiv 0 \mod p(h+1)$, then $P^{8M+8h+8} \not\subseteq \mathbb{R}^{16M-8h+1}$ and $P^{8M+8h+9} \not\subseteq \mathbb{R}^{16M-8h+2}$.*

(d) *If $\alpha(M) = 4h+1$ with h even and $M \equiv 0 \mod p(h+1)$, then $P^{8M+8h+9} \not\subseteq \mathbb{R}^{16M-8h+12}$ and $P^{8M+8h+10} \not\subseteq \mathbb{R}^{16M-8h+13}$.*

2. Proof of Theorem 1.1

Let tmf denote the 2-local connective spectrum introduced in [10], whose mod-2 cohomology is the quotient of the mod-2 Steenrod algebra A by the left ideal generated by Sq^1, Sq^2, and Sq^4. Thus $\mathrm{tmf}_*(X)$ may be computed by the Adams spectral sequence (ASS) with $E_2 = \mathrm{Ext}_{A_2}(H^*X, \mathbb{Z}_2)$, where A_2 is the subalgebra of A generated by Sq^1, Sq^2, and Sq^4. We rely on Bob Bruner's software ([3]) for our calculations of these Ext groups. It was proved in [8, p.167] that there are 8-dimensional classes X, X_1, and X_2 such that the homomorphism in $\mathrm{tmf}^*(-)$ induced by an axial map $P^m \times P^n \to P^k$ effectively sends X to $u(X_1 + X_2)$, where u is a unit in $\mathrm{tmf}^0(P^m \times P^n)$ which will be omitted from our exposition.

We will often use duality isomorphisms $\mathrm{tmf}^i(P^n) \approx \mathrm{tmf}_{-i-1}(P_{-n-1})$ for $i > 2$, and $\mathrm{tmf}^i(P^m \wedge P^n) \approx \mathrm{tmf}_{-i-2}(P_{-m-1} \wedge P_{-n-1})$ for $i > \max(m,n)+2$. For any integer m, P_m denotes the spectrum P_m^∞. We make frequent use of the periodicity $P_{b+8}^{t+8} \wedge \mathrm{tmf} \simeq \Sigma^8 P_b^t \wedge \mathrm{tmf}$ proved in [4, Prop 2.6]. Other aspects of the proof in [4] will be noted when needed.

We let $\nu(-)$ denote the exponent of 2 in an integer, and use $\nu\left(\binom{m}{n}\right) = \alpha(n) + \alpha(m-n) - \alpha(m)$. Also, if L is large, $\nu\left(\binom{2^L-k}{n}\right) = \nu\left(\binom{-k}{n}\right) = \nu\left(\binom{n+k-1}{n}\right)$. We will never be interested in the values of odd factors of coefficients, and will not list them.

Proof of (a). If the immersion exists, there is an axial map $P^{8M+9} \times P^{8M+9} \to P^{16M-1}$. The induced homomorphism in $\mathrm{tmf}^*(-)$ sends $0 = X^{2M}$ to

$$\sum \binom{2M}{i} X_1^i X_2^{2M-i} \tag{2.1}$$

in $\mathrm{tmf}^{16M}(P^{8M+9} \wedge P^{8M+9})$. This group is isomorphic to $\mathrm{tmf}_{-2}(P_{-10} \wedge P_{-10}) \approx \mathrm{tmf}_{30}(P_6 \wedge P_6)$. The portion of the ASS for $\mathrm{tmf}_{30}(P_6 \wedge P_6)$ arising from filtration 0 by h_0-extensions appears in Figure 1.

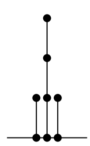

Figure 1. Portion of $\mathrm{tmf}_{30}(P_6 \wedge P_6)$.

There are also several elements in higher filtration in $\mathrm{tmf}_{30}(P_6 \wedge P_6)$ which are not relevant to our argument. The elements pictured in Figure 1 cannot be hit by differentials in the ASS because in dimension 31 there is only one tower in low enough filtration and it cannot support a differential by the argument of [4, p.54], namely that its generator is a constructible homotopy class. The filtration-0 elements must correspond to $X_1^{M-1}X_2^{M+1}$, $X_1^M X_2^M$, and $X_1^{M+1}X_2^{M-1}$ in $\mathrm{tmf}^{16M}(P^{8M+9} \wedge P^{8M+9})$. Since

$$2^2 X^{M+1} = 0 \text{ in } \mathrm{tmf}^*(P^{8M+9}), \tag{2.2}$$

the two $\mathbb{Z}/4$'s in Figure 1 must represent $X_1^{M\pm1}X_2^{M\mp1}$, and multiples of these are 0 in all filtrations > 1. Thus $X_1^M X_2^M$ generates the $\mathbb{Z}/2^4$ in $\mathrm{tmf}^{16M}(P^{8M+9} \wedge P^{8M+9})$. Since $\nu\left(\binom{2M}{M}\right) = \alpha(M) = 3$, we obtain that (2.1) is nonzero, contradicting the existence of the immersion. $\square$

Proof of (b). If the immersion exists, there is an axial map $P^{8M+9} \times P^{2^{L+3}-16M+9} \to P^{2^{L+3}-8M-11}$ for sufficiently large L. Hence

$$\sum \binom{-M-1}{i} X_1^i X_2^{2^L-M-1-i} = 0 \in \mathrm{tmf}^{2^{L+3}-8M-8}(P^{8M+9} \wedge P^{2^{L+3}-16M+9}). \tag{2.3}$$

This group is isomorphic to $\mathrm{tmf}_{38}(P_6 \wedge P_6)$, and the relevant part of it is given in Figure 2. Similarly to case (a), and continuing in all remaining cases, it cannot be hit by a differential in the ASS.

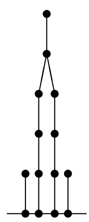

Figure 2. Portion of $\mathrm{tmf}_{38}(P_6 \wedge P_6)$.

The outer ($\mathbb{Z}/4$) generators must correspond to the classes $X_1^{M-2}X_2^{2^L-2M+1}$ and $X_1^{M+1}X_2^{2^L-2M-2}$. (Note that 4 times each of these classes is 0 by (2.2), and so they cannot produce a higher-filtration component impacting the middle summands. This will be the case also for the outer summands in subsequent diagrams.) The inner generators must be $X_1^{M-1}X_2^{2^L-2M}$ and $X_1^M X_2^{2^L-2M-1}$. By [4, Thm 2.7], the class $2^4(X_1^{M-1}X_2^{2^L-2M} + X_1^M X_2^{2^L-2M-1})$ has filtration ≥ 5. This is depicted by the behavior of the chart between filtration 3 and 4. Since $\alpha(M) = 6$, the component of these terms in (2.3) is

$$\binom{-M-1}{M-1} X_1^{M-1}X_2^{2^L-2M} + \binom{-M-1}{M} X_1^M X_2^{2^L-2M-1} = 2^5 X_1^{M-1}X_2^{2^L-2M} + 2^6 X_1^M X_2^{2^L-2M-1},$$

which is nonzero in the group depicted by Figure 2, contradicting the existence of the immersion. $\square$

Proof of (c). If the first immersion exists, there is an axial map

$$P^{16M+16} \times P^{2^{L+3}-32M+5} \to P^{2^{L+3}-16M-18}.$$

Hence

$$\sum \binom{-2M-2}{i} X_1^i X_2^{2^L-2M-2-i} = 0 \in \mathrm{tmf}^*(P^{16M+16} \wedge P^{2^{L+3}-32M+5}). \tag{2.4}$$

This group is isomorphic to $\mathrm{tmf}_{46}(P_7 \wedge P_2)$, and the relevant part of it is given in the left side of Figure 3.

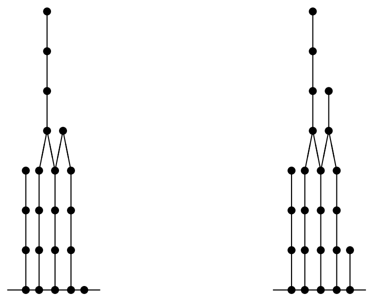

Figure 3. Portion of $\mathrm{tmf}_{46}(P_7 \wedge P_2)$ and $\mathrm{tmf}_{46}(P_6 \wedge P_3)$.

The generators, from left to right, correspond to

$$X_1^{2M-2} X_2^{2^L-4M}, \ldots, X_1^{2M+2} X_2^{2^L-4M-4},$$

with the sum relation in filtration 4 similar to that of the previous (and future) parts. Since $\alpha(M) = 7$, the component of the middle terms in (2.4) is

$$2^{8+\nu(M)} X_1^{2M-1} X_2^{2^L-4M-1} + 2^7 X_1^{2M} X_2^{2^L-4M-2} + 2^8 X_1^{2M+1} X_2^{2^L-4M-3},$$

which is nonzero in the group depicted by Figure 3. The argument for the second nonimmersion involves the same sum in a group isomorphic to $\mathrm{tmf}_{46}(P_6 \wedge P_3)$, which is pictured on the right side of Figure 3. □

Proof of (d). The proof is similar to those of parts (b) and (c). The first nonimmersion is proved by showing if $\alpha(M) = 9$, then

$$\sum \binom{-4M-3}{i} X_1^i X_2^{2^L-4M-3-i} \neq 0 \in \mathrm{tmf}^{2^{L+3}-32M-24}(P^{32M+25} \wedge P^{2^{L+3}-64M+2}). \tag{2.5}$$

This group is isomorphic to $\mathrm{tmf}_{62}(P_6 \wedge P_5)$, the relevant part of which is depicted in Diagram 4, with generators corresponding to $i = 4M-3, \ldots, 4M+3$ in (2.5). The sum relation in filtration 8 follows from [4, Thm 2.7]. The middle components of our class are

$$2^{10+\nu(M)} X_1^{4M-1} X_2^{2^L-8M-2} + 2^9 X_1^{4M} X_2^{2^L-8M-3} + 2^9 X_1^{4M+1} X_2^{2^L-8M-4},$$

which is nonzero in filtration 9. Note that $2^9 X_1^{4M} X_2^{2^L-8M-3}$ is 0 in filtration 9, as can be seen from Diagram 4 or from [4, 2.7], which says that if g_1, g_2, g_3 denote the middle three generators, then there are relations that both $2^8(g_1+g_2+g_3)$ and $2^8(g_1+g_3)$ have filtration > 8.

The argument for the second nonimmersion is virtually identical. Its obstruction is the same sum in a group isomorphic to $\mathrm{tmf}_{62}(P_5 \wedge P_6)$, so just the reverse of Figure 4. □

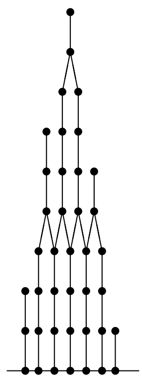

Figure 4. Portion of $\mathrm{tmf}_{62}(P_6 \wedge P_5)$.

Proof of (e). The obstruction this time is $\sum \binom{-2M-2}{i} X_1^i X_2^{2^L-2M-2-i}$ in a group isomorphic to the one depicted in Figure 4. The middle terms are

$$2^9 X_1^{2M-2} X_2^{2^L-4M} + 2^{11+\nu(M)} X_1^{2M-1} X_2^{2^L-4M-1} + 2^{10} X_1^{2M} X_2^{2^L-4M-2},$$

which is nonzero. $\square$

3. Sketch of proof of Theorem 1.4

We use the v_2^8-periodicity of Ext_{A_2} proved in [7, p.299,Thm 5.9] to see that, if one of the diagrams of Section 2 depicts a portion of $\mathrm{tmf}_i(P_a \wedge P_b)$, then the top part of the portion of $\mathrm{tmf}_{i+48j}(P_a \wedge P_b)$ generated by filtration-0 classes has the same form $8j$ units higher. We also use the arguments on [4, p.54] to see that, when this portion is interpreted as a quotient of a $\mathrm{tmf}^k(P^c \wedge P^d)$ group, the relations are of the same sort as those in [4, Thm 2.7]. The relation [4, (2.10)] is especially important and will be noted specifically below. We use cofiber sequences such as $S^a \wedge P_b \to P_a \wedge P_b \to P_{a+1} \wedge P_b$ to deduce results for our spaces, in which at least one of the bottom dimensions is even, from those of [4], which dealt with the situation when both bottom dimensions are odd. The nice form of $\mathrm{Ext}_{A_2}(H^*P_b)$ below a certain line of slope 1/6 is important here. As noted on [4, p.54], it is just a sum of copies of $\mathrm{Ext}_{A_1}(\mathbb{Z}_2)$, suitably placed.

Proof of 1.4(b,e). If the immersion in (b) exists, there is an axial map

$$P^{8M+8h+1} \times P^{2^{L+3}-16M+8h+1} \to P^{2^{L+3}-8M-8h-3}.$$

We obtain a contradiction to this by showing

$$\sum \binom{-M-h}{i} X_1^i X_2^{2^L-M-h-i} \neq 0 \in \mathrm{tmf}^*(P^{8M+8h+1} \wedge P^{2^{L+3}-16M+8h+1}). \tag{3.1}$$

Our obstruction will be in filtration $4h+1$, where there is a nonzero class by v_2^8-periodicity from Figure 2, which is the case $h = 1$. Note that the group in which (3.1) lies is isomorphic to $\mathrm{tmf}_{24h+14}(P_6 \wedge P_6)$. The terms in (3.1) with $i > M$ cannot interfere in this filtration because for such i, $2^{4h-2}X_1^i = 0$ in $\mathrm{tmf}^*(P^{8M+8h+1})$. The same holds for terms with $i < M-h$ due to the second factor. By [4, 3.12], the coefficients of the terms in (3.1) with $M-h \le i \le M$ are all divisible by $2^{\alpha(M)-1} = 2^{4h+1}$. This is where the strange hypothesis comes into play. Next we note that

$$\nu\Big(\sum_{j=0}^{h} \tbinom{h}{j}\tbinom{-M-h}{M-j}\Big) = \nu\big(\tbinom{-M}{M}\big) = \alpha(M)-1.$$

By a variant on [4, Cor 2.13.3], this implies that (3.1) is nonzero. There are four things that are required to make this work. (a) No interference from the outer terms because they are precisely 0 in a lower filtration. (b) All the $h+1$ intermediate terms have filtration at least $4h+1$. (c) The chart is nonzero in filtration $4h+1$. (d) An odd number of the intermediate terms which have $\binom{h}{j}$ odd, $0 \le j \le h$, are nonzero in filtration $4h+1$. This latter is a version of [4, (2.10)]. It is a consequence of a relation in every fourth filtration that the sum of the basic classes in the previous filtration is 0 in that filtration. By "basic," we mean those obtained from canonical classes in filtration 0 or 4 by v_2^8 periodicity.

The proof of (e) is virtually identical. □

Proof of 1.4(c,d). The proof of (d) is virtually identical to that of (c), and this is similar to that of (b) with the main difference being that the obstruction is due to $\binom{-M-1}{M}$ instead of $\binom{-M}{M}$, which causes a very different-looking hypothesis. The contradiction to the first result of (c) is obtained by showing

$$\sum \tbinom{-M-h-1}{i} X_1^i X_2^{2^L-M-h-1-i} \ne 0 \in \mathrm{tmf}^*(P^{8M+8h+8} \wedge P^{2^{L+3}-16M+8h-3}). \tag{3.2}$$

The obstruction will be in filtration $\alpha(M) = 4h+3$. The terms with $i > M$ or $i < M-h$ are precisely 0 in filtration less than $4h+3$ due to their first or second factor. By our hypothesis and [4, 3.8], the intermediate terms are all divisible by $2^{\alpha(M)}$. Since

$$\nu\Big(\sum_{j=0}^{h} \tbinom{h}{j}\tbinom{-M-h-1}{M-j}\Big) = \nu\big(\tbinom{-M-1}{M}\big) = \alpha(M),$$

and, by v_2^8-periodicity from Figure 3, the obstruction group is nonzero in filtration $\alpha(M) = 4h+3$. □

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