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ON ESTIMATES OF THE MAXIMAL OPERATOR ASSOCIATED TO NONDOUBLING MEASURES

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ABSTRACT. In this paper we show the boundedness of the non-centered Hardy-Littlewood maximal operator M_μ associated to certain rotational invariant measures. We prove that this maximal operator satisfies the modular inequality

$$\mu\left(\{x\in\mathbb{R}^n : M_\mu f(x)>\lambda\}\right)\le C\int_{\mathbb{R}^n}\frac{|f|}{\lambda}\left(1+\log^+\frac{|f|}{\lambda}\right)^m d\mu,$$

for $\lambda>0$ and $m>0$. We prove the modular inequality for the maximal operator associated to rotated squares from $\mathbb{R}^2$ and with a radial and decreasing measure. The technique used in the proof for cubes suggests extending this result to cones whose axes of symmetry pass through the origin. In both cases it is proved that the exponent of the modular inequality is $m=n$, which we prove is sharp.

1. Introduction

Let μ be a non negative measure in $\mathbb{R}^n$, finite on compact sets. Given a function $f\in L^1_{loc}(d\mu)$, we define the Hardy-Littlewood maximal operator

$$M_\mu f(x)=\sup_{x\in Q}\frac{1}{\mu(Q)}\int_Q|f(y)|\,d\mu(y), \tag{1.1}$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ containing x and $\mu(Q)>0$.

In this paper we will state properties over measures μ so that the following modular inequality can be satisfied

$$\mu\left(\{x\in\mathbb{R}^n : M_\mu f(x)>\lambda\}\right)\le C\int_{\mathbb{R}^n}\frac{|f|}{\lambda}\left(1+\log^+\frac{|f|}{\lambda}\right)^m d\mu, \tag{1.2}$$

for all $\lambda>0$, for any exponent m that depends only on the dimension.

The first result is reminiscent of the Jessen, Marcinkiewicz and Zygmund theorem (see [2] and [5]) on the boundedness of the strong maximal operator.

A set in $\mathbb{R}^n$ defined as

$$I=\{x : a_i\le x\le a_i+h_i,\ i=1,\dots,n\}\quad\text{with}\quad h_i>0,$$

is called an interval of $\mathbb{R}^n$. When $h_1=h_2=\cdots=h_n$, the interval I is called a cube. Let μ be a positive Borel measure on $\mathbb{R}^n$. Associated to this measure, we define the maximal operator

$$M_\mu^\square f(x)=\sup_{x\in I}\frac{1}{\mu(I)}\int_I|f(y)|\,d\mu(y),$$

where the supremum is taken over the intervals I, containing the point x and $\mu(I)>0$.

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THEOREM (1.3) (Product measures). *If $\mu_1, \mu_2, \dots, \mu_n$ are non-negative Borel measures on $\mathbb{R}$ and we define the measure*

$$d\mu(x_1, x_2, \dots, x_n) = d\mu_1(x_1)d\mu_2(x_2)\dots d\mu_n(x_n),$$

then $M_\mu^\square$ satisfies the modular inequality with exponent $n-1$, namely

$$\mu\left(\left\{x \in \mathbb{R}^n : M_\mu^\square f(x) > \lambda\right\}\right) \le C \int_{\mathbb{R}^n} \frac{|f|}{\lambda}\left(1 + \log^+ \frac{|f|}{\lambda}\right)^{n-1} d\mu, \tag{1.4}$$

for all $\lambda > 0$.

The Gaussian measure is an example with this product structure, since $e^{-|x|^2}dx = e^{-|x_1|^2}dx_1 e^{-|x_2|^2}dx_2 \cdots e^{-|x_n|^2}dx_n$. Our goal is to study the behavior of the measures of the type $e^{-|x|^\delta}dx$ for $\delta \neq 2$. We prove the modular inequality for the maximal operator defined with rotated squares of $\mathbb{R}^2$ (i.e. cubes whose sides are not necessarily parallel to the coordinate axes) and associated to a radial and decreasing measure. These measures are contained in a family of measures defined by P. Sjögren and F. Soria in [4]. In fact, the authors proved that a modular inequality can be obtained on the Orlicz space $L(\log L)_\mu^{\frac{n+1}{2}}(\mathbb{R}^n)$ with sharp exponent $\frac{n+1}{2}$, for the case in which the maximal operator is defined over the Euclidean balls of $\mathbb{R}^n$. In the case when the measure is doubling, the boundedness properties of the respective operator defined with balls or cubes are the same. However, if the measure is not doubling then the geometry associated to cubes is crucial, for example the Calderon-Zygmund decomposition or the Whitney decomposition.

Before beginning our main result, we first recall some definitions and notations given in [4]. Set $d\mu(x) = \gamma(x)dx$ a measure such that $\gamma(x)$ is a radial function, where $\gamma(x) = \gamma_0(|x|)$, with $\gamma_0(t)$ being a continuous function, strictly decreasing and $\lim_{t\to 0}\gamma_0(t) < \infty$. We define the function $\phi : (0,\infty) \to (0,\infty)$ by

$$\gamma_0(t + \phi(t)) = \frac{1}{2}\gamma_0(t),$$

and the function

$$\tau(t) = \frac{\phi(t)}{t}.$$

In the following theorem we consider the operator defined by equation (1.1), but the supremum is taken over rotated cubes in $\mathbb{R}^2$, and we denote it by M_μ^2.

THEOREM (1.5). *Let $d\mu(x) = \gamma_0(|x|)dx$, with γ_0, ϕ and τ as just described. If τ is decreasing in $(0,\infty)$ and*

$$\lim_{t\to\infty} \tau(t) = 0,$$

then there is a constant C such that the non-centered maximal operator M_μ^2, defined on rotated cubes in $\mathbb{R}^2$ and associated to the measure μ, verifies the modular inequality

$$\mu\left(\left\{x \in \mathbb{R}^2 : M_\mu^2 f(x) > \lambda\right\}\right) \le C \int_{\mathbb{R}^2} \frac{|f|}{\lambda}\left(1 + \log^+ \frac{|f|}{\lambda}\right)^2 d\mu, \ \forall \lambda > 0. \tag{1.6}$$

A consequence of the modular inequality is the following corollary, whose proof is a standard argument of interpolation, see Corollary 10 of [4].

COROLLARY (1.7). *Under the same conditions and notations of the Theorem (1.5), the maximal operator M_μ^2 associated to the measure $d\mu(x) = \gamma(x)dx$ is always bounded on $L_\mu^p(\mathbb{R}^2)$ for every $1 < p \le \infty$.*

Throughout this paper, the symbols C, $c, \ldots$ denote constants that may change from line to line, and $A \sim B$ mean that A is equivalent to B, in the sense that $CB \le A \le cB$, for some constants C, c.

2. Proof of Theorem (1.3) and (1.5)

Proof of Theorem (1.3). Let I be an interval of $\mathbb{R}^n$. We can write I as the cartesian product $I_1 \times I_2 \times \cdots \times I_n$ for certain intervals $I_1, I_2, \ldots, I_n$. Clearly, $\mu(I) = \mu_1(I_1)\mu_2(I_2)\cdots\mu_n(I_n)$ and besides

$$\begin{aligned}\frac{1}{\mu(I)}\int_I |f|\,d\mu &= \frac{1}{\mu_n(I_n)}\int_{I_n}\cdots\frac{1}{\mu_2(I_2)}\int_{I_2}\left(\frac{1}{\mu_1(I_1)}\int_{I_1}|f|\,d\mu_1\right)d\mu_2\cdots d\mu_n\\ &\le T_n\circ\cdots\circ T_2\circ T_1 f(x),\end{aligned}$$

where T_j acts only over the variable x_j and it is defined for each $j = 1, 2, \ldots, n$ by

$$T_j f(x) = \sup_{x_j \in I_j}\frac{1}{\mu_j(I_j)}\int_{I_j}\left|f(x_1,\ldots,x_{j-1},t,x_{j+1},\ldots,x_n)\right|d\mu_j(t).$$

For $n = 1$, see [2] and [3], it is proved that the maximal operator on $\mathbb{R}$ is always of weak type $(1,1)$, so each T_j is of weak type $(1,1)$ respect to the measure $d\mu$ as

$$\begin{aligned}\mu\{x : T_j f(x) > \lambda\} &= \int_{\mathbb{R}^{n-1}} d\mu_1\cdots\widehat{d\mu_j}\cdots d\mu_n\,\left(\mu_j\{x_j : T_j f(x) > \lambda\}\right)\\ &\le C\int_{\mathbb{R}^{n-1}} d\mu_1\cdots\widehat{d\mu_j}\cdots d\mu_n\int_{\mathbb{R}}\frac{|f|}{\lambda}d\mu_j\\ &= \frac{C}{\lambda}\int |f|\,d\mu.\end{aligned}$$

However, this does not suffice for what we are looking for.

In [1] M. de Guzmán proposed an easy proof of the Theorem of Jessen, Marcinkiewicz and Zygmund based on induction on the dimension, which can be adapted to our case, taking into account the following observations. To simplify, we assume $n = 2$. Using that each T_j is bounded on L^∞, with the standard arguments of truncation we obtain the inequality

$$\mu_2\{x_2 :\, T_2F(x_1,x_2) > \lambda\} \le \frac{C}{\lambda}\int_{\left\{x_2:\,|F(x_1,x_2)|>\frac{\lambda}{2}\right\}}|F(x_1,x_2)|\,d\mu_2(x_2),$$

uniformly in x_1. Therefore,

$$\begin{aligned}
&\mu\{(x_1,x_2): T_2\circ T_1 f(x_1,x_2)>\lambda\} = \\
&\quad= \int_{\mathbb{R}} d\mu_1(x_1)\mu_2\{x_2 : T_2(T_1 f)(x_1,x_2)>\lambda\} \\
&\quad\le \frac{C}{\lambda}\int_{\mathbb{R}} d\mu_1(x_1)\int_{\{x_2:T_1 f(x_1,x_2)>\lambda/2\}} T_1 f(x_1,x_2)d\mu_2(x_2) \\
&\quad= \frac{C}{\lambda}\int_{\mathbb{R}} d\mu_2(x_2)\int_{\{x_1:T_1 f(x_1,x_2)>\lambda/2\}} T_1 f(x_1,x_2)d\mu_1(x_1) \\
&\quad= \frac{C}{\lambda}\int_{\mathbb{R}} d\mu_2(x_2)\Big(\frac{\lambda}{2}\mu_1\{x_1 : T_1 f(x_1,x_2)>\lambda/2\} \\
&\qquad + \int_{\lambda/2}^{\infty}\mu_1\{x_1 : T_1 f(x_1,x_2)>t\}\,dt\Big) \\
&\quad\le \frac{C}{\lambda}\int_{\mathbb{R}} d\mu_2(x_2)\Big(\int_{\mathbb{R}} |f(x_1,x_2)|\,d\mu_1(x_1) \\
&\qquad + \int_{\lambda/2}^{\infty}\frac{1}{t}\int_{\{x_1:|f(x_1,x_2)|>t/2\}} |f(x_1,x_2)|\,d\mu_1(x_1)\,dt\Big) \\
&\quad\le \frac{C}{\lambda}\int\int |f|\,d\mu_1 d\mu_2 \\
&\qquad + \frac{C}{\lambda}\int_{\mathbb{R}} d\mu_2(x_2)\Big(\int_{\{|f|>\lambda/4\}}\frac{|f(x_1,x_2)|}{\lambda}\,d\mu_1(x_1)\int_{\lambda/2}^{2|f|}\frac{1}{t}\,dt\Big) \\
&\quad\le C\int\frac{|f|}{\lambda}\Big(1+\log^{+}\frac{|f|}{\lambda}\Big)d\mu.
\end{aligned}$$

This finishes the proof of Theorem (1.3). □

Basic results for the proof of Theorem (1.5). The following estimation will allow us to prove that if the maximal operator is defined only with cubes containing the origin, then it verifies the inequality of weak type $(1,1)$ in any dimension.

LEMMA (2.1). *Set $d\mu(x)=\gamma_0(|x|)dx$ where $\gamma_0(t)$ is a decreasing function. Given a cube Q containing the origin, consider B^0 the smallest ball centered at the origin containing Q. Then there is a constant C, which depends only on the dimension, such that*

$$\mu(B^0)\le C\mu(Q).$$

Proof. Let Q be a cube containing the origin 0. We denote the side of Q by ℓ. Observe that if Q' is one of the cubes with side ℓ, such that 0 is one of its vertices, then $\mu(Q')\le\mu(Q)$, since μ is radial and decreasing. Also, if Q^0 is the smallest cube centered at 0 containing Q, then we can cover the cube Q^0 with 2^n cubes Q' of this sort. So, we have

$$\mu(Q^0)\le 2^n\mu(Q')\le 2^n\mu(Q).$$

Finally, if we translate 3^n cubes from Q^0, B^0 can be covered with them, and therefore, of smaller measure. So, we obtain

$$\mu(B^0)\le 2^n 3^n\mu(Q). \qquad \square$$

For the class of cubes containing the origin we may use the following result.

LEMMA (2.2). *Let μ be a Borel measure. Consider the maximal operator*

$$A_\mu^0 f(x) = \sup_{\ell \ge |x|} \frac{1}{\mu(B_\ell^0)} \int_{B_\ell^0} |f(y)|\, d\mu(y),$$

where B_ℓ^0 denotes the ball with center at the origin 0 and radius ℓ. Then, A_μ^0 is of weak type (1,1).

Proof. We use a standard argument. For every $x \in E_\lambda := \left\{y : A_\mu^0 f(y) > \lambda\right\}$ there is a ball $\widetilde{B}_\ell^0$, centered at the origin and with $x \in \widetilde{B}_\ell^0$, such that

$$\frac{1}{\lambda} \int_{\widetilde{B}_\ell^0} |f(y)|\, d\mu > \mu(\widetilde{B}_\ell^0).$$

Therefore

$$\mu(\{y : \ |y| \le |x|\}) \le \mu(\widetilde{B}_\ell^0) \le \frac{1}{\lambda} \int |f|\, d\mu.$$

We have,

$$\mu\left(\left\{y : A_\mu^0 f(y) > \lambda\right\}\right) \le \sup_{x \in E_\lambda} \mu(\{y : \ |y| \le |x|\}) \le \frac{1}{\lambda} \int |f|\, d\mu. \qquad \square$$

LEMMA (2.3). *In the former conditions, the maximal operator associated to cubes containing the origin is of weak type* (1,1).

Proof. It is a consequence of the Lemmas (2.1) and (2.2). $\square$

For the proof of the theorem we need to introduce some notations. Given a cube Q with side ℓ and not containing the origin there exists an unique point Δ_Q in the boundary of Q, nearest to the origin. The distance of Q to the origin is denoted by $|\Delta_Q| = q_Q = q$. Let Γ_Q be the interior of the smallest cone containing Q, with vertex at Δ_Q and which central axis contains the segment $\overrightarrow{0\Delta_Q}$. If Δ_Q is not a vertex of Q then Γ_Q is the semi-space $\{y : \langle y - \Delta_Q, \Delta_Q\rangle > 0\}$. Let S_Q be the interior of the smallest cone containing Q, with vertex at the origin and which central axis contains the segment $\overrightarrow{0\Delta_Q}$. Let t_q be the distance from $\left(\partial S_Q \cap \partial \Gamma_Q\right)$ to the origin.

We can show that

$$t_q \sim \sqrt{q^2 \sin^2\theta + q^2 \cos^2\theta + 2q\ell\cos\theta + \ell^2} \sim (\ell + q),$$

where θ denotes the angle of aperture of the cone Γ_Q. Observe that $\pi/4 \le \theta < \pi/2$.

We define for each $t > 0$

$$\Sigma_Q(t) = \left\{\omega \in S^{n-1} : t\omega \in \Gamma_Q \cap S_Q\right\}.$$

Observe that the Lebesgue measure of $\Sigma_Q(t)$ on S^{n-1}, denoted by $\sigma(\Sigma_Q(t))$, is 0 for $0 < t < q$. For $q < t < t_q$, we have

$$\sigma(\Sigma_Q(t)) \sim \left(\frac{t^2 - q^2}{t\sqrt{t^2 - q^2 \sin^2\theta}}\right)^{n-1}. \tag{2.4}$$

For $t > t_q$,

$$\sigma(\Sigma_Q(t)) \sim \left(\frac{\ell \sin\theta}{t_q}\right)^{n-1} \sim \left(\frac{\ell}{\ell + q}\right)^{n-1}. \tag{2.5}$$

Now we are able to establish the basic result of this section, which in principle is valid in any dimension.

PROPOSITION (2.6). *If for some $m > 0$ there are constants C and ϵ such that the radial measure $d\mu(x) = \gamma_0(|x|)dx$ satisfies the inequality*

$$\int_q^\infty \exp\left(\epsilon\sigma\left(\Sigma_Q(t)\right)\frac{\mu(A_q)}{\mu(Q)}\right)^{\frac{1}{m}} \gamma_0(t)\,dt \le C\mu(A_q), \tag{2.7}$$

for any cube Q not containing the origin, $q = q_Q$ and side greater than $\phi(q)$, then the maximal operator associated to μ satisfies the modular inequality

$$\mu\left(\{x : M_\mu f(x) > \lambda\}\right) \le C\int_{\mathbb{R}^n} \frac{|f|}{\lambda}\left(1 + \log^+\frac{|f|}{\lambda}\right)^{m+1} d\mu,\ \forall\lambda > 0. \tag{2.8}$$

Proof. The proof is similar to the proof of Proposition 6 of [4], once it is observed that $\sigma(\Sigma_Q(t))$ increases when t increases. □

In [4] we can see a proof of the following technical lemmas.

LEMMA (2.9). *Consider the infinite annulus $A_q = \{y : |y| > q\}$. If τ is decreasing and* $\lim_{t\to\infty}\tau(t) = 0$, *then μ is finite. Besides, there exists q_0 such that*

$$\mu\left(A_q\right) \sim \gamma_0(q)q^{n-1}\phi(q),\ q > q_0.$$

LEMMA (2.10). *Under the same conditions and notation of the Theorem (1.5), if $t > q$ then*

$$\gamma_0(t) \le \gamma_0(q)\left(\frac{1}{2}\right)^{\frac{t-q}{\phi(q)}\frac{q}{t}}.$$

For $q > 0$, we define $\Psi_0(q) = q$, $\Psi_1(q) = q + \phi(q)$ and

$$\Psi_{k+1}(q) = \Psi_1\left(\Psi_k(q)\right) = \Psi_k(q) + \phi(\Psi_k(q)),\ k = 1, 2, \dots \tag{2.11}$$

For $k \ge 0$, we denote $a_k = \Psi_k(q_0)$ to define the level sets associated to γ as

$$S_0 = \{y : 0 < |y| < a_0\}, \qquad S_k = \{y : a_{k-1} < |y| < a_k\}, \qquad k = 1, 2, \dots$$

Note that γ_0 is essentially constant on each S_k. If $k \ge 1$ and $y \in S_k$ then

$$\gamma(a_k) \le \gamma(y) \le \gamma(a_{k-1}) = 2\gamma(a_k).$$

In the case where $y \in S_0$, $\gamma(0^+) \le \gamma(y) \le \gamma(a_0) = 2\gamma(0^+)$. The following result states that the maximal operator associated to the class of cubes that intersect no more than a fixed number of level sets S_k is weak type $(1,1)$.

LEMMA (2.12). *Let μ, γ_0 and S_k as before. Given an integer N, we define the class*

$$\mathcal{Q}_N = \{Q \quad cubes : card\{k : S_k \cap Q \ne \emptyset\} \le N\}.$$

Then the respective maximal operator associated to μ and $\mathcal{Q}_N$, denoted by M_μ^0, is weak type $(1,1)$.

Proof. Let $Q \in \mathcal{Q}_N$ a cube of center x_Q. We know that $\gamma(y) \sim \gamma(x_Q)$ for all $y \in Q$. So

$$\mu(Q) \sim \gamma(x_Q)|Q|,$$

where $|Q|$ is the Lebesgue measure of Q on $\mathbb{R}^n$. Therefore,

$$\frac{1}{\mu(Q)}\int_Q |f(y)|\,d\mu(y) \sim \frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

For $k = 1, 2, \ldots$, we consider

$$S_k^* = \bigcup_{j=k-N}^{k+N} S_j, \qquad (S_j = \emptyset, \text{ if } j < 0)$$

and $f_k = f\chi_{S_k^*}$. Note that if $x \in S_k$, we have

$$\mathcal{M}_\mu^0 f(x) = \mathcal{M}_\mu^0 f_k(x).$$

So,

$$\begin{aligned}
\mu\left\{x : \mathcal{M}_\mu^0 f(x) > \lambda\right\} &\le \sum_{k=0}^{\infty} \mu\left\{x \in S_k : \mathcal{M}_\mu^0 f(x) > \lambda\right\} \\
&\sim \sum_{k=0}^{\infty} \gamma_0(a_k) \left|\left\{x \in S_k : \mathcal{M}_\mu^0 f(x) > \lambda\right\}\right| \\
&\le \frac{C}{\lambda} \sum_{k=0}^{\infty} \int_{S_k^*} |f(y)| \gamma_0(a_k) dy \\
&\sim \frac{1}{\lambda} \sum_{k=0}^{\infty} \int_{S_k^*} |f(y)| \gamma_0(y) dy \\
&\sim \frac{1}{\lambda} \int |f(y)| d\mu(y).
\end{aligned}$$ □

Proof of the Theorem (1.5). The maximal operator associated to the measure μ and defined on the class formed by cubes Q of side smaller than $\phi(q)$ is of weak type (1,1), see Lemma (2.12). The same conclusion can be obtained for the maximal operator associated to the cubes containing the origin, see Lemma (2.3). The cubes Q with $q_Q \le q_0$ satisfy $\mu(Q) \ge C$, then the operator associated to these cubes is majorized by the norm of the function in L^1_μ. Hence, we have that it suffices to consider the squares (from now on $n = 2$) in the set

$$\mathcal{Q} = \left\{Q \text{ cubes} : 0 \notin Q,\ q > q_0,\ \ell > \phi(q)\right\}.$$

Let $Q \in \mathcal{Q}$, we have

$$\begin{aligned}
\mu(Q) &\ge \int_q^{q+\phi(q)} \int_{S^1} \chi_Q(t\omega) d\sigma(\omega) t\gamma_0(t) dt \\
&\ge \frac{1}{2}\gamma_0(q) \left|Q \cap \{y : q \le |y| \le q + \phi(q)\}\right| = \gamma_0(q) m_Q.
\end{aligned}$$

Observe that

$$m_Q \sim (\ell \wedge \ell_Q)^{m-1} \phi(q), \tag{2.13}$$

where ℓ_Q denotes the length of the part of the edge of the square Q corresponding to the angle θ and contained at the annulus centered at the origin and radius $q + \phi(q)$. If we denote $T = q + \ell_Q \cos\theta$, then $T^2 + \ell_Q^2 \sin^2\theta = (q - \phi(q))^2$, and therefore

$$\ell_Q \sim \frac{q\phi(q)}{\sqrt{q^2 \cos^2\theta + q\phi(q)}}.$$

So we deduce that

$$m_Q \sim \ell_Q \phi(q) \sim \begin{cases} q^{\frac{1}{2}} \phi(q)^{\frac{1}{2}} & \text{if } q\cos^2\theta < \phi(q) \\ \frac{\phi(q)^2}{(\cos\theta)} & \text{if } q\cos^2\theta \ge \phi(q) \end{cases} \tag{2.14}$$

Using estimates (2.5) and (2.13), and the Lemma (2.9), we obtain

$$\begin{aligned} Z_Q(t) &\equiv \sigma(\Sigma_Q(t))\frac{\mu(A_q)}{\mu(Q)} \\ &\leq C\left(\frac{(t^2-q^2)}{t\sqrt{t^2-q^2\sin^2\theta}} \wedge \frac{\ell}{\ell+q}\right)\frac{\gamma_0(q)q\phi(q)}{\gamma_0(q)m_Q}. \end{aligned}$$

We will distinguish three cases.

Case 1, $\ell \wedge \ell_q = \ell$.

$$Z_Q(t) \leq C\left(\frac{\ell}{\ell+q}\right)\left(\frac{q}{\ell}\right) \leq C.$$

Case 2, $\ell \wedge \ell_Q = \ell_Q$ and $q\cos^2\theta \leq \phi(q)$.

$$Z_Q(t) \leq C\left(\frac{t^2-q^2}{t\sqrt{t^2-q^2\sin^2\theta}}\right)\frac{q^1}{q^{\frac{1}{2}}\phi(q)^{\frac{1}{2}}} \leq C\left(\frac{t-q}{\phi(q)}\frac{q}{t}\right)^{\frac{1}{2}}.$$

Case 3, $\ell \wedge \ell_Q = \ell_Q$ and $q\cos^2\theta > \phi(q)$.

$$Z_Q(t) \leq C\left(\frac{t^2-q^2}{t\sqrt{t^2-q^2\sin^2\theta}}\frac{q\cos\theta}{\phi(q)}\right) \leq C\left(\frac{t-q}{\phi(q)}\frac{q}{t}\right),$$

where the last inequality is a direct calculation, first for $t \geq 2q$ and second for $q < t < 2q$.

Reorganizing the estimates, we get

$$Z_Q(t) \leq C\max\left(1, \left(\frac{t-q}{\phi(t)}\frac{q}{t}\right)\right). \tag{2.15}$$

Suppose Q is in the class $\mathcal{Q}$. Let ϵ and ϵ' be two positives constants to be determined later (the value of ϵ' will depend implicitly on ϵ), and set $\Psi(v) = e^{v^{1/(n-1)}}$. The way which we have estimated $Z_Q(t)$ suggests us considering two cases. In the first case, when

$$\max\left(1, \left(\frac{t-q}{\phi(t)}\frac{q}{t}\right)\right) = \left(\frac{t-q}{\phi(t)}\frac{q}{t}\right),$$

we have

$$\begin{aligned} \int_q^\infty \Psi\left(\epsilon'\sigma(\Sigma_Q(t))\frac{\mu(A_q)}{\mu(Q)}\right)\gamma_0(t)t^{n-1}dt \quad &\leq \int_q^\infty \exp\left(\epsilon\frac{t-q}{\phi(q)}\frac{q}{t}\right)t^{n-1}\gamma_0(t)dt \\ &= \int_q^{2q}\ldots + \int_{2q}^\infty \ldots = I + II. \end{aligned}$$

To estimate I we use Lemma (2.10), taking $\epsilon = \frac{\log 2}{2}$

$$\begin{aligned} I &\leq 2^2\gamma_0(q)q\int_q^{2q}\exp\left(-\epsilon\frac{t-q}{\phi(q)}\right)dt \\ &\leq 2^2\gamma_0(q)q\phi(q)\int_0^\infty \exp(-\epsilon s)ds \sim \mu(A_q). \end{aligned}$$

To estimate II we make the change of variables $t = 2s$, and we obtain

$$\begin{aligned} II &\leq \int_{2q}^{\infty} \exp\left(\epsilon \frac{q}{\phi(q)}\right) t\gamma_0(t)dt \\ &= 2^n e^{\epsilon \frac{1}{\tau(q)}} \int_q^{\infty} \gamma_0(2s)s^{n-1}ds. \end{aligned}$$

Using Lemma (2.10) with $t = 2s$ and $q = s$, we have

$$\begin{aligned} II &\leq 2^2 e^{\epsilon \frac{1}{\tau(q)}} \int_q^{\infty} \gamma_0(s)\exp\left(-\frac{s}{2\phi(s)}\log 2\right) s\, ds \\ &\leq 2^2 e^{\epsilon \frac{1}{\tau(q)}} e^{-\frac{\log 2}{2}\frac{1}{\tau(q)}} \int_q^{\infty} 2\gamma_0(s)s\, ds \\ &\sim \mu(A_q). \end{aligned}$$

In the second case, when

$$\max\left(1, \left(\frac{t-q}{\phi(t)}\frac{q}{t}\right)\right) = 1,$$

we have

$$\begin{aligned} \int_q^{\infty} \Psi\left(\epsilon' \sigma(\Sigma_R(t))\frac{\mu(A_q)}{\mu(Q)}\right)\gamma_0(t)t\, dt &\leq \int_q^{\infty} \exp(\epsilon)t\gamma_0(t)dt \\ &\sim \mu(A_q). \end{aligned}$$

these estimates together with Proposition (2.6) finishes the proof. □

3. Maximal operator defined for exterior regular cones

The first part of this work justifies the study of the maximal operator on exterior regular cones, as deduced from the notation previous to Proposition (2.6), since these are with which we compare the rotated cubes. These cones are those that have vertex at a point P, the symmetry axis that passes through 0 and P, and arbitrary opening. This section shows that the maximal operator defined by these cones and associated to a radial and decreasing measure verifies the same estimate as in the case of the maximal operator on cubes, with exponent n whatever the dimension.

Given $P \in \mathbb{R}^n \setminus \{0\}$, with $|P| = q_P = q$, and $0 \leq \theta \leq \frac{\pi}{2}$, we define the *exterior regular cone* associated to P and θ, as the cone with vertex P, aperture θ and symmetry axis the line that passes through 0 and P. We will denote this cone by $\Gamma^P_{q,\theta}$, or simply by $\Gamma_{q,\theta}$, this is

$$\Gamma^P_{q,\theta} = \Gamma_{q,\theta} = \{x : \text{ang}(x-P, P) \leq \theta\},$$

where ang(x,y) denotes the angle between the vectors $\overrightarrow{0x}$ and $\overrightarrow{0y}$.

Let μ be a positive Borel measure in $\mathbb{R}^n$, we define the maximal operator on exterior cones and associated to the measure μ as

$$M^3_{\mu} f(x) = \sup_{\Gamma_{q,\theta} \ni x} \frac{1}{\mu(\Gamma_{q,\theta})} \int_{\Gamma_{q,\theta}} |f(y)|\, d\mu(y),$$

where the supremum is taken over all exterior regular cones containing the point x.

THEOREM (3.1). *Under the same conditions of the former section, let* $d\mu(x) = \gamma_0(|x|)dx$ *be a radial and decreasing measure. If* $\tau(t)$ *is decreasing in* $(0,\infty)$ *and*

$$\lim_{t\to\infty} \tau(t) = 0,$$

then there is a constant C *such that the maximal operator* M^3_μ *associated to the measure* μ *satisfies the inequality*

$$\mu\left(\left\{x \in \mathbb{R}^n : M^3_\mu f(x) > \lambda\right\}\right) \le C \int_{\mathbb{R}^n} \frac{|f|}{\lambda}\left(1+\log^+ \frac{|f|}{\lambda}\right)^n d\mu,\ \forall \lambda > 0. \tag{3.2}$$

For the measures $d\mu_\delta = e^{-|x|^\delta}dx$ we have the following result,

THEOREM (3.3). *If* $n \ge 2$ *and* $\delta > 0$*, then* $M^3_{\mu_\delta}$ *satisfies the inequality*

$$\mu_\delta\left\{x \in \mathbb{R}^n : M^3_{\mu_\delta} f(x) > \lambda\right\} \le C\frac{(\log^+\lambda)^{n-1}}{\lambda}\int_{\mathbb{R}^n} |f|\log|f|\,d\mu_\delta + \frac{C}{\lambda}, \quad \lambda > 0. \tag{3.4}$$

In order to prove Theorem (3.1), we shall need the following technical results. We begin studying the cones whose distance to the origin is smaller than q_0, where q_0 is defined in Lemma (2.9).

LEMMA (3.5). *Consider* $d\mu(x) = \gamma_0(|x|)dx$ *with* γ_0 *decreasing and we set* $q_0 > 0$*. Then there is a constant* C*, which depends only on the dimension and* q_0*, such that if* $\Gamma^P_{q,\theta}$ *is an exterior regular cone with* $q < q_0$ *and* $\Gamma^P_{0,\theta}$ *is the cone with vertex at* 0*, aperture* θ *and central axis that passes through* 0 *and* P*, then* $\Gamma^P_{q,\theta} \subset \Gamma^P_{0,\theta}$ *and besides*

$$\mu\left(\Gamma^P_{0,\theta}\right) \le C\mu\left(\Gamma^P_{q,\theta}\right).$$

Proof. We will prove the two following equivalences:

$$\mu\left(\Gamma^P_{q,\theta} \cap \{y : |y| > 2q_0\}\right) \sim \mu\left(\Gamma^P_{0,\theta} \cap \{y : |y| > 2q_0\}\right) \tag{3.6}$$

and

$$\mu\left(\Gamma^P_{q,\theta} \cap \{y : |y| \le 2q_0\}\right) \sim \mu\left(\Gamma^P_{0,\theta} \cap \{y : |y| \le 2q_0\}\right). \tag{3.7}$$

In order to prove (3.6), observe that

$$\mu\left(\Gamma^P_{j,\theta} \cap \{y : |y| > 2q_0\}\right) \sim \int_{2q_0}^\infty \sigma\left(\Sigma_{\Gamma^P_{j,\theta}}(t)\right)\gamma_0(t)t^{n-1}dt,$$

for $j = 0$ and $j = q_0$. It suffices then to prove that

$$\sigma\left(\Sigma_{\Gamma^P_{q,\theta}}(t)\right) \sim \sigma\left(\Sigma_{\Gamma^P_{0,\theta}}(t)\right), \qquad t > q_0.$$

Observe that $\sin\theta \sim \sin\theta_t$, where θ_t is the aperture of the smallest cone with vertex at the origin containing $\partial(\Gamma^P_{q,\theta} \cap B_t(0))$. For $t > 2q$, we have

$$\sigma\left(\Sigma_{\Gamma^P_{q,\theta}}(t)\right) \sim (\sin\theta_t)^{n-1} \sim (\sin\theta)^{n-1} \sim \sigma\left(\Sigma_{\Gamma^P_{0,\theta}}(t)\right).$$

This proves (3.6).

In order to prove (3.7), we observe that μ is essentially constant in $\{y: |y| \le 2q_0\}$ and

$$\left|\Gamma^p_{q,\theta} \cap \{y: |y| > 2q_0\}\right| \sim \left|\Gamma^p_{0,\theta} \cap \{y: |y| > 2q_0\}\right|.$$

This finishes the proof. $\square$

For this class of cones we can use the following lemma.

LEMMA (3.8). *For any positive Borel measure μ, the maximal operator given by*

$$\widetilde{A}^0_\mu f(x) = \sup_{x\in\Gamma_{0,\theta}} \frac{1}{\mu\left(\Gamma_{0,\theta}\right)} \int_{\Gamma_{0,\theta}} |f(y)|\, d\mu(y),$$

is of weak type (1,1).

Proof. It suffices to observe that the Vitali lemma is satisfied, because given $\Gamma^P_{0,\theta}$ and $\Gamma^{P'}_{0,\theta'}$, such that

$$\Gamma^P_{0,\theta} \cap \Gamma^{P'}_{0,\theta'} \neq \emptyset,$$

and with $\theta' > \theta$, then

$$\Gamma^P_{0,\theta} \subset \Gamma^{P'}_{0,3\theta'},$$

where $\Gamma^{P'}_{0,3\theta'}$ is a cone with the same axis that $\Gamma^{P'}_{0,\theta'}$ and aperture $3\theta'$. Also, $\mu\left(\Gamma^{P'}_{0,3\theta}\right) \sim \mu\left(\Gamma^P_{0,\theta}\right)$, because $\mu\left(\Gamma^P_{0,\theta}\right) \sim (\sin\theta)^{n-1} \sim (\sin 3\theta)^{n-1}$ for small values of θ.

The set $E_\lambda = \left\{x \in \mathbb{R}^n : A^0_\mu f(x) > \lambda\right\}$, $\lambda > 0$, is the union of the cones with vertex in the origin $\{\Gamma_{0,\theta_i}\}_{i\in T}$ such that

$$\frac{1}{\mu(\Gamma_{0,\theta})} \int_{\Gamma_{0,\theta}} |f|\, d\mu > \lambda.$$

As μ is regular, then it suffices to prove that

$$\mu(K) \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(y)|\, d\mu(y),$$

for any compact set $K \subset E_\lambda$. If K is one of these compact sets then there are finite number of cones $\Gamma_1, \Gamma_2, \ldots, \Gamma_N$, of the class $\{\Gamma_{0,\theta_i}\}_{i\in T}$ covering K. We order them by the size of the angle, such that, $\theta_1 \ge \theta_2 \ge \ldots \ge \theta_N$.

By induction, we proceed to choose a sequence of disjoint cones $\{\widetilde{\Gamma}_k\}$. Set $\widetilde{\Gamma}_1 = \Gamma_1$. If $1 \le k < N$ and $\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \ldots, \widetilde{\Gamma}_{k-1}$ are chosen, let $\widetilde{\Gamma}_k$ be the first cone with the preceding order, disjoint with $\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \ldots, \widetilde{\Gamma}_{k-1}$, and so on. We obtain the family $\{\widetilde{\Gamma}_j\}_{j=1}^J$. If $\Gamma_\alpha \in \{\Gamma_i\} \setminus \{\widetilde{\Gamma}_j\}$ then there is $j_0 \in \{1, 2, \ldots, J\}$ such that $\theta_{j_0} \ge \alpha$, $\Gamma_\alpha \cap \widetilde{\Gamma}_{\theta_{j_0}} \neq \emptyset$, and we have $\Gamma_\alpha \subset \widetilde{\Gamma}_{3\theta_{j_0}}$. Namely,

$$\bigcup_{k=1}^N \Gamma_k \subset \bigcup_{j=1}^J \widetilde{\Gamma}_{3\theta_j}.$$

So

$$\mu(K) \le \mu\left(\bigcup_{k=1}^N \Gamma_k\right) \le \mu\left(\bigcup_{j=1}^J \widetilde{\Gamma}_{3\theta_j}\right) \le \frac{1}{\lambda} \int_{\bigcup_{j=1}^J \widetilde{\Gamma}_{3\theta_j}} |f|\, d\mu \le \frac{1}{\lambda} \int |f|\, d\mu.$$

This finishes the proof. $\square$

LEMMA (3.9). *The maximal operator defined over the class of cones $\Gamma_{q,\theta}$, with $q < q_0$, is of weak type* $(1,1)$.

Proof. The Lemma (3.5) implies that if $\Gamma^p_{q,\theta}$ is a cone with $q < q_0$, then

$$\frac{1}{\mu\left(\Gamma^p_{q,\theta}\right)}\int_{\Gamma^p_{q,\theta}}|f(y)|\,d\mu(y) \le C\frac{1}{\mu\left(\Gamma^p_{0,\theta}\right)}\int_{\Gamma^p_{0,\theta}}|f(y)|\,d\mu(y).$$

This and Lemma (3.8) it suffices to conclude the proof. □

For the cones $\Gamma_{q,\theta}$ with $q > 0$ we want to use an analog result to the Proposition (2.6). For this, we need to define the set $\Sigma_{\Gamma_{q,\theta}}(t)$ as in the former section. Namely,

$$\Sigma_{\Gamma_{q,\theta}}(t) = \left\{\omega \in S^{n-1} : \, t\omega \in \Gamma_{q,\theta}\right\}$$

We have that $\sigma\left(\Sigma_{\Gamma_{q,\theta}}(t)\right) = 0$, if $0 < t < q$, and in other case

$$\sigma\left(\Sigma_{\Gamma_{q,\theta}}(t)\right) \sim \left(\frac{(t^2-q^2)\sin\theta}{t\sqrt{t^2-q^2\sin^2\theta}}\right)^{n-1}. \tag{3.10}$$

We need a result that relates the measure of a certain given cone $\Gamma_{q,\theta}$ with the measure of the infinite annulus $A_q := \{y : \, |y| > q\}$, $q > 0$.

PROPOSITION (3.11). *Set $\gamma(x) = \gamma_0(|x|)$, where $\gamma_0(t)$ is decreasing and continuous on $(0,\infty)$. If for some $m > 0$ there are two constants C and ϵ, such that the measure $d\mu(x) = \gamma_0(|x|)dx$ satisfies the inequality*

$$\int_q^\infty \exp\left(\epsilon\sigma\left(\Sigma_{q,\theta}(t)\right)\frac{\mu(A_q)}{\mu(\Gamma_{q,\theta})}\right)^{\frac{1}{m}}\gamma_0(t)t^{n-1}dt \le C\mu(A_q) \tag{3.12}$$

for any exterior regular cone $\Gamma_{q,\theta}$, with $q > q_0$, then the maximal operator associated to μ, satisfies the modular inequality

$$\mu\left(\{x : M_\mu f(x) > \lambda\}\right) \le C\int_{\mathbb{R}^n}\frac{|f|}{\lambda}\left(1+\log^+\frac{|f|}{\lambda}\right)^{m+1}d\mu, \qquad \lambda > 0.$$

The proof of this Proposition is similar to the proof of the Proposition 6 in [4], if we notice that $\sigma\left(\Sigma_{\Gamma_{q,\theta}}(t)\right)$ increases when t increases too.

Proof of Theorem (3.1). From Lemma (3.9) we know that the maximal operator associated to those cones $\Gamma_{q,\theta}$ with $q \le q_0$ is of weak type (1,1). So, it suffices to consider the maximal operator defined over the class

$$\Gamma^* = \left\{\Gamma_{q,\theta} : \, q > q_0\right\}.$$

For proving the modular inequality for this class, it suffices to prove that the hypothesis of the Proposition (3.11) are satisfied, with $m = n-1$. Let $\Gamma_{q,\theta}$ be a cone of the class Γ^*. We have

$$\mu(\Gamma_{q,\theta}) \ge \mu\left(\Gamma_{q,\theta}\cap B_{q+\phi(q)}(0)\right) \ge \frac{1}{2}\gamma_0(q)(\ell_q\sin\theta)^{n-1}\phi(q), \tag{3.13}$$

where

$$\ell_q \sim \frac{q\phi(q)}{\sqrt{q\phi(q)+q^2\cos^2\theta}}.$$

For $t > q$ we have,

$$
\begin{aligned}
&\sigma\left(\Sigma_{\Gamma_{q,\theta}}\right)\frac{\mu(A_q)}{\mu\left(\Gamma_{q,\theta}\right)} \leq \\
&\quad\leq C\left(\frac{t^2-q^2}{t\sqrt{(t^2-q^2)+q^2\cos^2\theta}}\right)^{n-1}\frac{q^{n-1}}{\ell_q^{n-1}}\\
&\quad\leq C\left(\frac{t-q}{\sqrt{(t^2-q^2)+q^2\cos^2\theta}}\left(\left(\frac{q}{\phi(q)}\right)^{\frac{1}{2}}+\frac{q\cos\theta}{\phi(q)}\right)\right)^{n-1}\\
&\quad\leq C\left(\left(\frac{t-q}{t}\right)^{\frac{1}{2}}\left(\frac{q}{\phi(q)}\right)^{\frac{1}{2}}+\left(\frac{t-q}{\sqrt{(t^2-q^q)+q^2\cos 2\theta}}\frac{q\cos\theta}{\phi(q)}\right)\right)^{n-1}\\
&\quad\leq C\left(\frac{t-q}{\phi(q)}\frac{q}{t}\right)^{\frac{n-1}{2}}+\left(\frac{t-q}{\phi(q)}\frac{q}{t}\right)^{n-1}\\
&\quad\leq C\left(1+\frac{t-q}{\phi(q)}\frac{q}{t}\right)^{n-1}.
\end{aligned}
$$

Reorganizing the inequality, we have

$$
\sigma\left(\Sigma_{\Gamma_{q,\theta}}\right)\frac{\mu(A_q)}{\mu\left(\Gamma_{q,\theta}\right)} \leq C\left(\max\left\{1,\frac{t-q}{\phi(q)}\frac{q}{t}\right\}\right)^{n-1}.
$$

Let ϵ and ϵ' be two constants to be determined later, and set $\Psi(v) = e^{v^{1/(n-1)}}$. According to the previous maximum value, in the case when

$$
\max\left\{1,\frac{t-q}{\phi(q)}\frac{q}{t}\right\} = \left(\frac{t-q}{\phi(q)}\frac{q}{t}\right),
$$

we have

$$
\begin{aligned}
\int_q^\infty \Psi\left(\epsilon'\sigma(\Sigma_{\Gamma_{q,\theta}}(t))\frac{\mu(A_q)}{\mu(\Gamma_{q,\theta})}\right)\gamma_0(t)t^{n-1}dt &\leq \int_q^\infty \exp\left(\epsilon\frac{t-q}{\phi(q)}\frac{q}{t}\right)t^{n-1}\gamma_0(t)dt\\
&= \int_q^{2q}\dots+\int_{2q}^\infty\dots = I+II.
\end{aligned}
$$

To estimate I we use Lemma (2.10), with $\epsilon = \frac{\log 2}{2}$

$$
\begin{aligned}
I &\leq 2^n\gamma_0(q)q^{n-1}\int_q^{2q}\exp\left(-\epsilon\frac{t-q}{\phi(q)}\right)dt\\
&\leq 2^n\gamma_0(q)q^{n-1}\phi(q)\int_0^\infty \exp(-\epsilon s)ds \sim \mu(A_q).
\end{aligned}
$$

To estimate II we make the change of variables $t = 2s$,

$$
\begin{aligned}
II &\leq \int_{2q}^\infty \exp\left(\epsilon\frac{q}{\phi(q)}\right)t^{n-1}\gamma_0(t)dt\\
&= 2^n e^{\epsilon\frac{1}{\tau(q)}}\int_q^\infty \gamma_0(2s)s^{n-1}ds.
\end{aligned}
$$

From Lemma (2.10) with $t = 2s$ and $q = s$, we have

$$\begin{aligned} II &\leq 2^n e^{\epsilon \frac{1}{\tau(q)}} \int_q^\infty \gamma_0(s) \exp\left(-\frac{s}{2\phi(s)}\log 2\right) s^{n-1} ds \\ &\leq 2^n e^{\epsilon \frac{1}{\tau(q)}} e^{-\frac{\log 2}{2}\frac{1}{\tau(q)}} \int_q^\infty 2\gamma_0(s) s^{n-1} ds \\ &\sim \mu(A_q). \end{aligned}$$

In the case when

$$\left(\max\left\{1, \frac{t-q}{\phi(q)}\frac{q}{t}\right\}\right) = 1,$$

we have

$$\begin{aligned} \int_q^\infty \Psi\left(\epsilon' \sigma(\Sigma_{\Gamma_{q,\theta}})(t) \frac{\mu(A_q)}{\mu(\Gamma_{q,\theta})}\right) \gamma_0(t) t^{n-1} dt &\leq \int_q^\infty \exp(\epsilon) t^{n-1} \gamma_0(t) dt \\ &\sim \mu(A_q). \end{aligned}$$

This estimation shows that we can apply Proposition (3.11) to the class Γ^* with $m = n-1$ and obtain the modular inequality, which is the conclusion of Theorem (3.1). □

Proof of the Theorem (3.3). To prove the inequality (3.4) it suffices to study the operator associated to the class

$$\Gamma^* = \{\Gamma_{q,\theta} : q > q_0\}.$$

So, as we have seen, the maximal operator out of this set is of weak type $(1,1)$ (see Lemma (3.8)).

Consider the operator

$$\widetilde{M}_{\mu_\delta} f(x) = \sup_{q<\rho; 0<\theta} \frac{1}{\mu(A_q)} \int_q^\infty \left(\frac{1}{\theta^{n-1}} \int_{|x'-\omega|<\theta} |f(t\omega)| \, d\sigma(\omega)\right) d\mu_0(t),$$

where $x = \rho x' \in R_+ \times S^{n-1}$. So, $\widetilde{M}_\mu$ is majorized by the composition of two operators, one of them acting upon the angular variable and the second acting on the radial variable, both of the weak type $(1,1)$. So, we have that $\widetilde{M}_\mu$ satisfies the inequality $L\log L$, namely

$$\mu\{x : \widetilde{M}_\mu f(x) > \lambda\} \leq C_0 \int_{\mathbb{R}^n} \frac{|f|}{\lambda}\left(1 + \log^+ \frac{|f|}{\lambda}\right) d\mu,$$

for any $\lambda > 0$.

If $\Gamma_{q,\theta} \in \Gamma^*$, we know from (3.13) that

$$\mu(\Gamma_{q,\theta}) \geq C\gamma_0(q)\phi(q)\left(\phi(q)\sin\theta\right)^{n-1},$$

and $\mu(A_q) \sim \gamma_0(q) q^{n-1}\phi(q)$ (see Lemma (2.9)). Besides, if $t\omega, \rho x' \in \Gamma_{q,\theta}$, with $t, \rho \in R_+$ and $\omega, x' \in S^{n-1}$, then

$$\left|x' - \omega\right| < \theta \sim \sin\theta.$$

Therefore, if $x = \rho x' \in \Gamma_{q,\theta}$ we have,

$$\begin{aligned}\frac{1}{\mu\left(\Gamma_{q,\theta}\right)} & \int_{\Gamma_{q,\theta}} |f|\, d\mu \leq \\ & \leq \frac{1}{\mu\left(\Gamma_{q,\theta}\right)} \int_q^\infty \int_{S^{n-1}} \chi_{\Gamma_{q,\theta}}(t\omega)|f(t\omega)|\, d\sigma(\omega) d\mu_0(t) \\ & \leq \frac{1}{\mu(A_q)} \int_q^\infty \left(\frac{\mu(A_q)(\sin\theta)^{n-1}}{\mu\left(\Gamma_{q,\theta}\right)}\right) \int_{\Sigma_{\Gamma_{q,\theta}}(t)} \frac{1}{|x'-\omega|^{n-1}} |f(t\omega)|\, d\sigma(\omega) d\mu_0(t) \\ & \leq C\left(\frac{q}{\phi(q)}\right)^{n-1} \widetilde{M}_\mu f(x) = C(\widetilde{\tau}(q))^{n-1} \widetilde{M}_\mu f(x) \\ & \leq C(\widetilde{\tau}(|x|))^{n-1} \widetilde{M}_\mu f(x),\end{aligned}$$

where $\widetilde{\tau}(q) = 1/\tau(q)$ is an increasing function. So we obtain,

$$M^3_{\mu_\delta} f(x) \equiv \sup_{x\in\Gamma_{q,\theta}\in\Gamma^*} \frac{1}{\mu\left(\Gamma_{q,\theta}\right)} \int_{\Gamma_{q,\theta}} |f|\, d\mu_\delta \leq C(\widetilde{\tau}(|x|))^{n-1} \widetilde{M}_{\mu_\delta} f(x). \tag{3.14}$$

As the measure is $\mu_\delta = e^{-|x|^\delta}$, we know that $\phi(q) \sim q^{1-\delta}$ and $\widetilde{\tau}(q) \sim q^\delta$.

Set $\Phi = \Phi_{n-1}(u) = u(1+\log^+ u)^{n-1}$, $u > 0$ and $\Psi(v) = e^{v^{\frac{1}{n-1}}}$. Using the Young inequality, $uv \leq C(\Phi(u)+\Psi(v))$, in the right side of (3.14), we obtain

$$M^3_{\mu_\delta} f(x) \leq C\left(\widetilde{M}_{\mu_\delta} f(x)\right) + Ce^{\frac{|x|^\delta}{2}}.$$

We have,

$$\mu_\delta\left\{x :\, M^3_{\mu_\delta} f(x) > \lambda\right\} \leq \mu_\delta\left\{x :\, \widetilde{M}^3_{\mu_\delta} f(x) > \Phi^{-1}\left(\frac{\lambda}{2C}\right)\right\} + \frac{C}{\lambda}.$$

Using that $\Phi^{-1}(\lambda) \sim \frac{\lambda}{(\log\lambda)^{n-1}}$ for $\lambda \gg 1$, we finally obtain

$$\mu_\delta\left\{x :\, M^3_{\mu_\delta} f(x) > \lambda\right\} \leq C\frac{(\log^+\lambda)^{n-1}}{\lambda} \int_{\mathbb{R}^n} |f|\,|\log|f||\, d\mu_\delta + \frac{C}{\lambda}. \qquad \square$$

4. Counterexample: the exponent n of the modular inequality for M^3_μ is sharp

In this part we will prove the sharpness of the exponent of the modular inequality for the Gaussian case.

THEOREM (4.1). *Let $M^3_{\mu_2}$ be the maximal operator associated to the Gaussian measure, $d\mu_2(x)dx = e^{-|x|^2}dx$, and defined over exterior regular cones. Let $\Phi(u)$ be an increasing function with $\Phi(0) = 0$ and such that $\Phi(u) = uG(u)$, where G satisfies that*

$$\lim_{u\to\infty} \frac{G(u)}{(\log^+ u)^n} = 0.$$

Then, given any constant C, it is always possible to find out a function f and $\lambda > 0$ such that

$$\mu_\delta\left\{x :\, M^3_{\mu_2} f(x) > \lambda\right\} > C\int_{\mathbb{R}^n} \Phi\left(\frac{|f|}{\lambda}\right) d\mu_2.$$

For proving this theorem, we will need the following lemma.

LEMMA (4.2). *If μ is a measure that verifies the hypothesis of the Theorem (3.1), and $\Gamma_{q,\theta}$ is an exterior regular cone with $q > q_0$, then*

$$\mu(\Gamma_{q,\theta}) \sim \gamma_0(q)\left(\frac{q\sin\theta}{\sqrt{q\phi(q)+q^2\cos^2\theta}}\right)^{n-1}(\phi(q))^n.$$

Proof. Using the definition of $\Sigma_{\Gamma_{q,\theta}}$ we have

$$\begin{aligned}\mu(\Gamma_{q,\theta}) &\sim \int_q^\infty\int_{S^{n-1}}\chi_{\Gamma_{q,\theta}}(t\omega)d\sigma(\omega)t^{n-1}\gamma_0(t)dt\\ &\sim \int_q^\infty \sigma\left(\Sigma_{\Gamma_{q,\theta}}\right)t^{n-1}\gamma_0(t)dt,\end{aligned}$$

where, as we already said,

$$\sigma\left(\Sigma_{\Gamma_{q,\theta}}\right)\sim\left(\frac{(t^2-q^2)\sin\theta}{t\sqrt{t^2-q^2\sin^2\theta}}\right)^{n-1}.$$

Therefore, we have

$$\begin{aligned}\mu(\Gamma_{q,\theta}) &= C\int_q^\infty\left(\frac{(t^2-q^2)\sin\theta}{t\sqrt{t^2-q^2\sin^2\theta}}\right)^{n-1}t^{n-1}\gamma_0(t)dt\\ &= \left(\frac{\sin\theta}{\sqrt{q\phi(q)+q^2\cos^2\theta}}\right)^{n-1}\\ &\qquad \int_q^\infty (t-q)^{n-1}\left(\frac{\sqrt{q\phi(q)+q^2\cos^2\theta}}{\sqrt{t^2-q^2\sin^2\theta}}\right)^{n-1}t^{n-1}\gamma_0(t)dt\\ &\le C\left(\frac{\phi(q)\sin\theta}{\sqrt{q\phi(q)+q^2\cos^2\theta}}\right)^{n-1}\int_q^\infty\left(1+\left(\frac{t-q}{\phi(q)}\frac{q}{t}\right)^{n-1}\right)t^{n-1}\gamma_0(t)dt\\ &\le C\left(\frac{\sin\theta}{\sqrt{q\phi(q)+q^2\cos^2\theta}}\right)^{n-1}(\phi(q))^{n-1}\mu(A_q)\\ &\le C\gamma_0(q)\left(\frac{q\sin\theta}{\sqrt{q\phi(q)+q^2\cos^2\theta}}\right)^{n-1}(\phi(q))^n.\end{aligned}$$

This inequality together with the equation (3.13) finishes the proof. □

Proof of the Theorem (4.1). Let $R > 0$ be a number given later. We denote B^* as a ball of center $(R,0,\dots,0)$ and radius $\phi(R)$. We define the function f by

$$f(x)=\frac{\chi_{B^*}(x)}{\mu_2(B^*)}.$$

We take $\epsilon > 0$ and let Γ_0 be an exterior regular cone with aperture $\theta = \pi/4$, whose closer point to the origin is $Q_0 = (R_0,0,\dots,0)$, where $R_0 = R-\ell_0\cos\theta$, and $\ell_0 = \epsilon\phi(R)\log R$. Observe that $\phi(q) \ll \ell_0 \ll R$ and $R \sim R_0$.

For $j = 0,1,\dots,k$, set $R_j = \psi_j(R_0)$ (see (2.11)), where k is given by

$$\psi_k(R_0) < R_0+\frac{\ell_0\cos\theta}{2} < \psi_{k+1}(R_0).$$

From this inequality we can deduce

$$\psi_k(R_0) - R_0 \le \frac{\ell_0 \cos\theta}{2} \le \psi_{k+1}(R_0) - R_0.$$

Using the definition of $\psi_k(R_0)$ we know that $\psi_k(R_0) = R_0 + \sum_{j=0}^{k-1} \phi(R_j) \sim R_0 + k\phi(R_0)$, the last equivalence is obtained since $R_j \sim R_0$ implies $\phi(R_j) \sim \phi(R) \sim \frac{1}{R}$. Hence,

$$\frac{\ell_0}{2} \sim k\phi(R),$$

namely, $k \sim \frac{\ell_0}{\phi(R)} = \epsilon \log R$.

We denote the exterior regular cone by Γ_j, $j = 0,1,\ldots,k$, whose vertex is $Q_j = \Delta_j = (R_j, R_j, \ldots, R_j)$, and its aperture angle θ_j is given by

$$\ell_j \sin\theta_j = \left(2^{\frac{2}{n-1}}\right)^j \ell_0 \sin\theta.$$

Set $\lambda = \min\left\{\frac{1}{\mu_2(\Gamma_j)} : j = 0,1,\ldots,k\right\}$. As $B^* \subset \Gamma_j$, for every j, we have

$$\frac{1}{\mu_2(\Gamma_j)} \int_{\Gamma_j} |f(y)|\, d\mu_2(y) = \frac{1}{\mu_2(\Gamma_j)} \ge \lambda.$$

For every $j = 1,2,\ldots,k$, T_j denotes the biggest number that satisfies

$$S_j \left\{ t\omega \in \mathbb{R} \times S^{n-1} : \frac{R + R_j}{2} < t < R_j,\ |\omega - (1,0,\ldots,0)| < T_j \right\} \subset \bigcup_{\alpha} \Gamma_j^{\alpha},$$

where Γ_j^{α} form the class of exterior regular cones containing B^*, with the same distance respect to the origin and the same aperture that the cone Γ_j.

So, it can be deduced that

$$\mu_2(S_j) \ge C\gamma_0(R_j)\phi(R_j)R_j^{n-1}\left(\frac{R - R_j}{R}\right)^{n-1}.$$

For the junction, it is verified

$$\mu_2\left(\bigcup_{j=0}^{k} S_j\right) = \sum_{j=1}^{k} \mu_2(S_j - S_{j-1}) + \mu_2(S_0) \sim \sum_{j=1}^{k} \mu_2(S_j),$$

and $\bigcup_{j=0}^{k} S_j \subset \left\{x\colon\ M_{\mu_2}^3 f(x) \ge \lambda\right\}$. So we obtain

$$\begin{aligned}
\mu_2\left\{x\colon M_{\mu_2} f(x) \ge \lambda\right\} &\ge c\sum_{j=0}^{k} \gamma_0(R_j)(\phi(R_j))^n \left(\frac{R - R_j}{\phi(R_j)}\right)^{n-1} \\
&\sim \sum_{j=0}^{k} \gamma_0(R_j)(\phi(R_j))^n \left(\frac{\ell_0}{\phi(R)}\right)^{n-1} \\
&\sim \sum_{j=0}^{k} \mu_2(Q_j)(\log R)^{n-1} \\
&\ge \frac{1}{\lambda} k(\log R)^{n-1} \ge c\frac{1}{\lambda}(\log R)^n.
\end{aligned}$$

As

$$\int \Phi\left(\frac{f(x)}{\lambda}\right) d\mu_2 = \frac{1}{\lambda} G\left(\frac{\mu_2(Q_k)}{\mu_2(Q^*)}\right),$$

we only need to prove that, for sufficiently large R, it is verified

$$C\frac{1}{\lambda}G\left(\frac{\mu_2(Q_k)}{\mu_2(Q^*)}\right) < \frac{1}{\lambda}(\log R)^n.$$

Observe that

$$u(R) \equiv \frac{1}{\lambda\mu_2(Q^*)} \sim \frac{\mu_2(Q_k)}{\mu_2(Q^*)} \sim \frac{\gamma_0(R_k)(\phi(R_k))^n}{\gamma_0(R)(\phi(R)^n} \sim e^{R^2-R_k^2},$$

and that

$$\log u(R) \sim (R^2-R_k^2) \sim R(R-R_0) \sim \frac{\ell_0}{\phi(R)} \sim \log R.$$

From the hypothesis on G, from $\lim_{R\to\infty} u(R) = \infty$ and

$$\lim_{R\to\infty} G\left(\frac{1}{\lambda\mu_2(B^*)}\right)(\log R)^{-n} \le C \lim_{R\to\infty} G(u(R))(\log u(R))^{-n} = 0.$$

finishes the proof. □

5. Counterexample: the exponent $n-1$ for the estimation of M^3_μ in quasinorm is sharp

In this section we will prove that for measures of type $e^{-|x|^\delta}dx$, $\delta > 0$, the inequality in quasinorm (3.4) can not be improved.

THEOREM (5.1). *For $\delta > 0$ fixed, we define $d\mu_\delta(x) = e^{-|x|^\delta}dx$. Let $\Phi(u)$ be an increasing function with $\Phi(0) = 0$ such that $\Phi(u) = uG(u)$ where G verifies*

$$\lim_{u\to\infty} \frac{G(u)}{(\log u)^{n-1}} = 0.$$

Then, for every constant C there exists a function f and $\lambda > 0$ such that

$$\mu_\delta\left\{x : M^3_{\mu_\delta} f(x) > \lambda\right\} > \frac{C}{\lambda}\left(\int_{\mathbb{R}^n} \Phi(|f|)d\mu_\delta + 1\right).$$

Proof. For sufficiently large $R > 0$, whose value will be given later we define

$$f_R(x) = \frac{1}{\mu_\delta(A_{2R})}\chi_{A_{2R}}(x).$$

For $|x| > R$ we denote $\Gamma_{R,\theta}(x)$ as the regular exterior cone containing x and whose distance of $\Gamma_{R,\theta}(x)$ to the origin is R. If $\theta = \frac{\pi}{4}$ and $x = (R,0,\dots,0)$ we simply denote it by Γ_R. For $|x| \ge R$ we have

$$\begin{aligned} M^3_{\mu_\delta} f(x) &\ge \frac{1}{\mu_\delta(\Gamma_{R,\theta}(x))}\int_{\Gamma_{R,\theta}} |f_R(y)|\,d\mu_\delta(y) \\ &= \frac{1}{\mu_\delta(\Gamma_{R,\theta})}\int_{\Gamma_{R,\theta}} |f_R(y)|\,d\mu_\delta(y) \equiv \lambda_R. \end{aligned}$$

Therefore, $\mu_\delta\left\{x : M^3_{\mu_\delta} f_R(x) \ge \lambda_R\right\} \ge \mu_\delta(A_R)$. It is clear that $\lambda_R \sim \frac{1}{\mu_\delta(\Gamma_R)}$. On the other hand, as

$$\begin{aligned} \frac{1}{\lambda_R}\left(\int \Phi(f_R)d\mu_\delta\right) &\sim \mu_\delta(\Gamma_R)\Phi\left(\frac{1}{\mu_\delta(A_{2R})}\right)\mu_\delta(A_{2R}) \\ &= \mu_\delta(\Gamma_R)G\left(\frac{1}{\mu_\delta(A_{2R})}\right), \end{aligned}$$

it suffices to prove that

$$\mu_\delta(A_R) \geq C\mu_\delta(\Gamma_R) G\left(\frac{1}{\mu_\delta(A_{2R})}\right),$$

for sufficiently large R. The measure of the cone Γ_R is

$$\begin{aligned}\mu_\delta(\Gamma_R) \quad &\sim \quad \gamma_0(R)\left(\frac{R\sin\frac{\pi}{4}}{\sqrt{R\phi(R)+R^2\cos^2\frac{\pi}{4}}}\right)^{n-1}(\phi(R))^n \\ &\sim \quad \gamma_0(R)R^{n-1}\phi(R)\tau(R)^{n-1} \sim \mu_\delta(A_R)\tau^{n-1}.\end{aligned}$$

Therefore, it suffices to prove that

$$\lim_{R\to\infty} G\left(\frac{1}{\mu_\delta(A_{2R})}\right)\tau(R)^{n-1} = 0.$$

As

$$\left(\log\frac{1}{\mu_\delta(A_{2R})}\right)^{n-1} \sim \left(\log\frac{1}{\gamma_0(2R)}\right)^{n-1} \sim R^{\delta(n-1)} \sim \left(\frac{1}{\tau(R)}\right)^{n-1},$$

we obtain

$$\begin{aligned}&\lim_{R\to\infty} G\left(\frac{1}{\mu_\delta(A_{2R})}\right)\tau(R)^{n-1} \leq \\ &\qquad \leq C\lim_{R\to\infty} G\left(\frac{1}{\mu_\delta(A_{2R})}\right)\left(\log\frac{1}{\mu_\delta(A_{2R})}\right)^{-(n-1)} = 0,\end{aligned}$$

this concludes the proof. □

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REFERENCES

[1] M. DE GUZMÁN, *An inequality for the Hardy-Littlewood maximal operator with respect to a product of differentiation bases*, Studia Math, **49**, (1972), 185–194.

[2] M. DE GUZMÁN, Differentiation of Integrals in R^n, Lecture Notes in Mathematics, 481, Springer, 1975.

[3] P. SJÖGREN, *A remark on the maximal function for measures in* $\mathbb{R}^n$, Amer. J. Math, **105**, (1983), 1231–1233.

[4] P. SJÖGREN AND F. SORIA, *Sharp estimates for the non centered maximal operator associated to gaussian and other radial measures*, Adv. Math., **181 (2)**, (2004), 251–275.

[5] A. ZYGMUND, Trigonometric Series, Cambridge Univ. Press, 1968.

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PROPER LATTICES AND SPECTRALLY INVARIANT SUBSPACES OF A LINEAR OPERATOR

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ABSTRACT. A proper lattice of X is a pair (A,L) composed of a bounded linear operator A on X and its invariant finite-dimensional subspace L. The set of all proper lattices of X is denoted by $\mathcal{P}l(X)$. For $(A,L) \in \mathcal{P}l(X)$, the operator A induces two operators, the restriction operator $A_{|L}$ and the operator $\widehat{A_L}$ from the quotient X/L into itself, i.e. $\widehat{A_L}(\pi(y)) = \pi(A(y))$, where π is the natural homoeomorphism between X and the quotient space X/L.

In this note its shown that (A,L) is a proper lattice if and only if there are a finite sequence of eigenvalues $\{\lambda_1,\dots,\lambda_n\} \in \sigma_p(A)$ and an appropriate set of linearly independent eigenvectors $\{x_1,\dots,x_n\}$ such that $L = \mathcal{L}(x_1,\dots,x_n)$. Moreover, λ_i is a simple pole of A if and only if $\lambda_i \notin \sigma(\widehat{A_L})$.

Following this concept we can define a spectrally invariant (finite dimensional) subspace of linear operator T as an invariant subspace E such that $\sigma(T_{|E}) \cap \sigma(\widehat{T_E}) = \emptyset$. Also, we give some properties of stability of spectrally invariant subspaces.

1. Introduction

Let X be a Banach space, and $\mathcal{B}(X)$ denotes the space of all bounded linear operators from X to X. For $T \in \mathcal{B}(X)$, let $N(T)$, $R(T)$, $\sigma(T)$ and $\sigma_p(T)$ denote respectively the null space, the range, the spectrum and the point spectrum of T. Let $n(T)$ and $d(T)$ denote the nullity and the deficiency of T defined by

$$n(T) = \dim N(T), \text{ and } d(T) = \operatorname{codim} R(T).$$

Let $\pi_0(T)$ denote the set of Riesz points of T (i.e., the set of isolated eigenvalues of T of finite algebraic multiplicity). Then $\lambda \in \pi_0(T)$ is called a simple eigenvalue (pole) of T if its algebraic multiplicity is 1. Let $\pi_{00}(T)$ denote the set of all isolated eigenvalues of T of finite geometric multiplicity (i.e. $0 < n(T-\lambda) < \infty$).

The *ascent*, notated by $\operatorname{asc}(T)$, and the *descent*, notated by $\operatorname{dsc}(T)$, of T are given by

$$\operatorname{asc}(T) = \inf\{n : N(T^n) = N(T^{n+1})\},\ \operatorname{dsc}(T) = \inf\{n : R(T^n) = R(T^{n+1})\};$$

if no such n exists, then $\operatorname{asc}(T) = \infty$, respectively $\operatorname{dsc}(T) = \infty$.

An operator $T \in B(X)$ is said to be Drazin invertible if there exists an operator $D \in B(X)$ such that

$$T^{d+1}D = T^d, \quad DTD = D \text{ and } TD = DT$$

for some nonnegative integer d. It is known that T is Drazin invertible if and only if T has finite ascent and descent.

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Let Λ be an $n \times n$ complex matrix. Following [7], we say that Λ is an eigenmatrix for $T \in \mathcal{B}(X)$ if there exists an n-tuple $\mathbf{h} = (h_1, \dots, h_n)^T$ (which we call the corresponding eigenvector) of independent vectors in X such that

$$(T \oplus \cdots \oplus T)\mathbf{h} = \Lambda \mathbf{h}.$$

It easy to see that for every eigenmatrix Λ of $T \in \mathcal{B}(X)$ we have $\sigma(\Lambda) = \sigma_p(\Lambda) \subset \sigma_p(T)$. Moreover, for any invertible $n \times n$ matrix Γ, the matrix $\Gamma^{-1}\Lambda\Gamma$ is an eigenmatrix of T as well.

2. Manifold of proper lattices and pols of a linear operator

Let $P_0(X)$ denote the collection of all finite dimensional subspaces of X. The *manifold of proper lattices of X* is the set

$$\mathcal{P}l(X) = \{(A, L) \in \mathcal{B}(X) \times P_0(X) : A(L) \subset L\}.$$

In another words, a proper lattice of X is a pair (A, L) composed of a bounded linear operator A on X and its invariant finite-dimensional subspace L.

Let $S \subset X$ be an arbitrary subset. Then $\mathcal{L}(S)$ denotes the subspace of X generated by the set S.

PROPOSITION (2.1). *Let $(A, L) \in \mathcal{P}l(X)$. Then there are a finite sequence of eigenvalues $\{\lambda_1, \dots, \lambda_n\} \in \sigma_p(A)$ and an appropriate set of linearly independent eigenvectors $\{x_1, \dots, x_n\}$ such that $L = \mathcal{L}(x_1, \dots, x_n)$. Moreover, every sequence of linearly independent eigenvectors $\{x_1, \dots, x_n\}$ corresponding to a sequence of eigenvalues of an operator $A \in \mathcal{B}(X)$ generates an invariant subspace L such that $(A, L) \in \mathcal{P}l(X)$.*

Proof. Let $\dim(L) = n$. By [7], p. 740, the existence of an invariant subspace of dimension n is equivalent to the existence of an of $n \times n$ eigenmatrix Λ of A. Moreover, every eigenvalue of the matrix Λ is an eigenvalue of A (see ([7]) Proposition (2.8)). Let $\{\lambda_1, \dots, \lambda_n\} = \sigma_p(\Lambda) \subset \sigma_p(A)$ and $\{x_1, \dots, x_n\}$ be sequences of linearly independent appropriate eigenvectors. Obviously, $L = \mathcal{L}(x_1, \dots, x_n)$.

For the second part of the proof it is easy to see that for every $x \in L = \mathcal{L}(x_1, \dots, x_n)$, $Ax \in L$. □

Let $(A, L) \in \mathcal{P}l(X)$, where $L = \mathcal{L}(\{x_1, \dots, x_n\})$, $\{x_1, \dots, x_n\}$ is a set of linearly independent eigenvectors for the sequence of eigenvalues $\{\lambda_1, \dots, \lambda_n\} \in \sigma_p(A)$. Then the operator A induces two operators, the restriction operator $A_{|L}$ and the operator $\widehat{A_L}$ from the quotient X/L into itself, i.e. $\widehat{A_L}$ is the operator $\widehat{A_L}(\pi(y)) = \pi(A(y))$ where π is the natural homoeomorphism between X and X/L. It is known that the spectrum of any of the operators A, $A_{|L}$ and $\widehat{A_L}$ is contained in the union of the spectrum of rest of them (see [4]). The special interest is when equality holds and Theorem (2.9) give us an answer for it.

In the following, we will always connect the finite dimensional invariant subspace L for an operator $A \in \mathcal{B}(X)$ with the sets of eigenvalues $\{\lambda_1, \dots, \lambda_n\} \in \sigma_p(A)$ and with the set of linearly independent corresponding eigenvectors $\{x_1, \dots, x_n\}$ that generate L.

PROPOSITION (2.2). *Let $(A, L) \in Pl(X)$. Then, for any $i \in \{1, 2, \dots, n\}$, $\lambda_i \notin \sigma_p(\widehat{A_L})$ if and only if $n(A - \lambda_i) = n(A_{|L} - \lambda_i)$ and $\operatorname{asc}(A - \lambda_i) = 1$.*

Proof. ($\Rightarrow$:) Let $\lambda_i \notin \sigma_p(\widehat{A_L})$. Suppose that $n(A-\lambda_i) > n(A_{|L}-\lambda_i)$. Then there exists $x \notin L$ such that $(A-\lambda_i)x = 0$ implies

$$(\widehat{A_L}-\lambda_i)[x+L] = [(A-\lambda_i)x] = [0].$$

Hence, $\lambda_i \in \sigma_p(\widehat{A_L})$.

Next, suppose that there exists $x \in N(A-\lambda_i)^2 \setminus N(A-\lambda_i)$. Then $(A-\lambda_i)^2(x) = 0$ and $(A-\lambda_i)x \neq 0$. Clearly, $(\widehat{A_L}-\lambda_i)^2[x+L] = [0]$. If we suppose that $(\widehat{A_L}-\lambda_i)[x+L] \neq [0]$, then $\lambda_i \in \sigma_p(\widehat{A_L})$. Hence, suppose that $(\widehat{A_L}-\lambda_i)[x+L] = [0]$.

If we suppose that $[x+L] \neq [0]$, we have again that $\lambda_i \in \sigma_p(\widehat{A_L})$. Let $[x+L] = [0]$, or equivalently $x \in L$, so x is a linear combination of linearly independent eigenvectors x_j (corresponding to eigenvalues λ_j). Let $x = \sum_{j=1}^n \alpha_j x_j$. Then

$$0 = (A-\lambda_i)^2(x) = \sum_{j=1}^n \alpha_j(\lambda_j-\lambda_i)^2 x_j$$

and, by linear independence of the vectors $x_1,\dots,x_n$, we have $\alpha_j = 0$, for $i \neq j$. Hence, $x = \alpha_i x_i$ and, in this case, we have $(A-\lambda_i)x = 0$. This last contradiction implies that $N(A-\lambda_i)^2 = N(A-\lambda_i)$.

($\Leftarrow$:) First, we will show that $(A-\lambda_i)^{-1}(L) = N(A-\lambda_i)+L$. Since $N(A-\lambda_i)+L \subset (A-\lambda_i)^{-1}(L)$, we have to show only the opposite inclusion. Fix $i = n$ and let $x \in (A-\lambda_n)^{-1}(L)$. Then there exists $x \in X$ such that $L \ni (A-\lambda_n)x = \sum_{i=1}^n \alpha_i x_i$ and $(A-\lambda_n)^2 x = \sum_{i=1}^{n-1} \alpha_i(\lambda_i-\lambda_n)x_i$. Then for $w = \sum_{i=1}^{n-1} \beta_i x_i \in L$, where $\beta_i = \frac{\alpha_i}{\lambda_i-\lambda_n}$, if $\lambda_i-\lambda_n \neq 0$, and $\beta_i = 0$ if $\lambda_i-\lambda_n = 0$, we have $(A-\lambda_n)^2(x-w) = 0$. Since $\operatorname{asc}(A-\lambda_i) = 1$, it follows that $x-w = n \in N(A-\lambda_n)$. Hence, $x \in L+N(A-\lambda_n)$.

By [2], Proposition 7, we have

$$n(A-\lambda_n) = n(A_{|L}-\lambda_n) + n(\widehat{A_L}-\lambda_n)$$

and, since $n(A-\lambda_n) = n(A_{|L}-\lambda_n)$, we have $n(\widehat{A_L}-\lambda_n) = 0$ and $\lambda_n \notin \sigma_p(\widehat{A_L})$. □

For many practical reasons, it is important that the eigenvalues connected with chosen proper lattices be (simple) poles of A. For example, it is known that if $\lambda_0 \in \pi_0(A)$, then for any sequence $\{A_n\}$ in $\mathcal{B}(X)$ that converges in norm to A, there exists a sequence $\{\lambda_n\}$ such that $\lambda_n \in \pi_0(A_n)$ and $\lambda_n \to \lambda_0$. Moreover, if λ_0 is a simple pole, then for almost all positive integers n, λ_n is a simple pole of A_n, and the corresponding eigenvectors x_n converge to x_0. (For the previous see [1], Theorem (2.17)). Following this, our interest is that λ_i are simple poles, and we will give necessary and sufficient conditions to obtain that λ_i are such points in $\sigma_p(A)$.

THEOREM (2.3). *Let $(A,L) \in \mathcal{P}l(X)$. Then $\lambda_i \notin \sigma(\widehat{A_L})$ if and only if the following conditions hold:*

(i) $n(A-\lambda_i) = n(A_{|L}-\lambda_i)$;

(ii) $\operatorname{asc}(A-\lambda_i) = \operatorname{dsc}(A-\lambda_i) = 1$.

Proof. ($\Leftarrow$:) Using the proof of the previous proposition, if $\operatorname{asc}(A-\lambda_i) = 1$, then

$$(A-\lambda_i)^{-1}(L) = N(A-\lambda_i)+L.$$

By [2], Proposition 7, we have

$$n(A-\lambda_i) = n(A_{|L}-\lambda_i) + n(\widehat{A_L}-\lambda_i). \tag{2.4}$$

and

$$d(A-\lambda_i)=d(A_{|L}-\lambda_i)+d(\widehat{A_L}-\lambda_i). \tag{2.5}$$

Then by applying the condition (ii) we have that $A-\lambda_i$ is Drazin invertible and, consequently, $A-\lambda_i$ is a Fredholm operator with $\lambda_i\in \operatorname{iso}\sigma(A)$. The continuity of the index implies

$$n(A-\lambda_i)=d(A-\lambda_i). \tag{2.6}$$

Since any finite dimensional operator is a Fredholm operator with index zero, we have

$$n(A-\lambda_i)=n(A_{|L}-\lambda_i)=d(A_{|L}-\lambda_i). \tag{2.7}$$

Now, by the equations (2.4) to (2.7) we have $d(\widehat{A_0}-\lambda_0)=n(\widehat{A_0}-\lambda_0)=0$, i.e. $\lambda_0\notin\sigma(\widehat{A_0})$.

($\Rightarrow$:) Let $\lambda_i\notin\sigma(\widehat{A_L})$, then by Proposition (2.2) the conditions (i) and (ii) hold and by [5], Proposition 2.2,

$$\operatorname{asc}(A-\lambda_i)=\operatorname{asc}(A_{|L}-\lambda_i)=1 \text{ and } \operatorname{dsc}(A-\lambda_i)=\operatorname{dsc}(A_{|L}-\lambda_i).$$

Since $A_{|L}-\lambda_i$ is a finite-dimensional operator, then $\operatorname{asc}(A_{|L}-\lambda_i)=\operatorname{dsc}(A_{|L}-\lambda_i)$ and the proof is completed. □

COROLLARY (2.8). *Let $(A,L)\in\mathcal{P}l(X)$. Then $\lambda_i\notin\sigma(\widehat{A_L})$ if and only if λ_i is a simple pole of A and $n(A-\lambda_i)=n(A_{|L}-\lambda_i)$.*

THEOREM (2.9). *Let $(A,L)\in\mathcal{P}l(X)$. Then $\sigma(A_{|L})\cap\sigma(\widehat{A_L})=\emptyset$ if and only if $L=\bigvee_{i=1}^n N(A-\lambda_i)$, where λ_i is a simple pole of A, $i=1,\dots,n$. Moreover, $\sigma(A)=\sigma(A_{|L})\cup\sigma(\widehat{A_L})$.*

Proof. Let the finite dimensional invariant subspace L of $A\in B(X)$ be generated by a finite sequence of eigenvectors connected with an appropriate sequence of eigenvalues $\{\lambda_1,\dots,\lambda_n\}$. Then $\sigma(A_{|L})=\{\lambda_1,\dots,\lambda_n\}$ and $\{\lambda_1,\dots,\lambda_n\}\cap\sigma(\widehat{A_L})=\emptyset$. Then, by Theorem (2.3) and Corollary (2.8), we have that $L=\bigvee_{i=1}^n N(A-\lambda_i)$, where λ_i is a simple pole of A, for every $i=1,\dots,n$. The opposite implication is a direct consequence of Theorem (2.3).

In the case when $\sigma(A_{|L})\cap\sigma(\widehat{A_L})=\emptyset$, Corollary (2.2) of [4] implies that $\sigma(A)=\sigma(A_{|L})\cup\sigma(\widehat{A_L})$. □

3. Spectrally invariant subspaces of a linear operator

Following notations from [6], we say that the closed T-invariant subspace $E\subset X$ is spectrally invariant if $\sigma(T_{|E})\cap\sigma(\widehat{T_E})=\emptyset$. By [6], Theorem 2, every spectrally invariant subspace is a reducing subspace in the sense that there exists a projection $P\in B(X)$ commuting with T such that $P(X)=E$. In this case there exists a subset F such that (E,F) is a spectrally invariant pair satisfying $X=E\oplus F$,

$$T=\begin{pmatrix} T_{11} & 0\\ 0 & T_{22}\end{pmatrix}:\begin{pmatrix} E\\ F\end{pmatrix}\to\begin{pmatrix} E\\ F\end{pmatrix}$$

with $\sigma(T_{11})=\sigma(T_{|E})$, $\sigma(T_{22})=\sigma(\widehat{T_E})$ and $\sigma(T)=\sigma(T_{11})\cup\sigma(T_{22})$.

THEOREM (3.1). *Let $(A,L)\in\mathcal{P}l(X)$ where L is spectrally invariant for A. Then there exists a $\delta>0$ such that for any operator $B\in B(X)$, with $\parallel A-B\parallel<\delta$, there exists a spectrally invariant finite dimensional subspace M and $(B,M)\in\mathcal{P}l(X)$.*

Proof. By Theorem (2.9) there exists a finite tuple of distinct simple poles $(\lambda_1,\dots,\lambda_n)$, such that $L = \bigvee_{i=1}^n N(A-\lambda_i)$. Then, by [1], Theorem 2.18, for any $B \in B(X)$ close enough to A there exists a finite tuple of distinct simple poles $(\mu_1,\dots,\mu_n)$ of B close to $(\lambda_1,\dots,\lambda_n)$ (in $\mathbb{C}^n$) such that

$$\dim \bigvee_{i=1}^n N(A-\lambda_i) = \dim \bigvee_{i=1}^n N(B-\mu_i).$$

It is easy to see that $M = \bigvee_{i=1}^n N(B-\mu_i)$ is a spectrally invariant finite dimensional subspace of B. □

Let $(A,L) \in \mathcal{P}l(X)$, where L is spectrally invariant for A. Then the proof of Theorem (3.1) and the proof of [1], Theorem 2.18, (c), claim that for any sequence $\{A_j\} \subset B(X)$ that converge in norm to A there exists a sequence $\{L_j\} \subset P_0(X)$ such that $(A_j,L_j) \in \mathcal{P}l(X)$ and $L_j \to L$. Of course, the sense of $L_j \to L$ we take is taken from $\mathcal{P}l(X)$; let $\dim L = \dim L_j (= n)$ and let $\{\mu_{i,j} : i = 1,\dots n\}$ be a set of eigenvalues of A_j such that its appropriate vectors $x_{i,j}$ generate L_j and $\mu_{i,j} \to \lambda_i$ together with $x_{i,j} \to x_i$, $j \to \infty$ (for more details see [1], p. 98). Hence, in the particular case when L is spectrally invariant for A, we can claim that there exists a sequence $\{(A_j,L_j)\} \subset \mathcal{P}l(X)$ such that $(A_j,L_j) \to (A,L)$. In the general case, for an arbitrary $(A,L) \in \mathcal{P}l(X)$ we cannot claim that for every sequence of operators that converges to A we will find a sequence of eigenvalues and eigenvectors such that $(A_j,L_j) \to (A,L)$. Moreover, the following theorems give us a method to construct a sequence of proper lattices that converges to (A,L).

THEOREM (3.2). *Let $(A_0,L_0) \in \mathcal{P}l(X)$ with $\dim L_0 = n$. Then $A \in B(X)$ has an $n \times n$ eigenmatrix Λ with eigenvector $\mathbf{x}_1 \in X^n$ if and only if the following system of equations*

$$(A_{12} \oplus \cdots \oplus A_{12})\mathbf{h}_1 = (\Lambda - (A_{11} \oplus \cdots \oplus A_{11}))\mathbf{x}_0$$
$$(A_{21} \oplus \cdots \oplus A_{21})\mathbf{x}_0 = (\Lambda - (A_{22} \oplus \cdots \oplus A_{22}))\mathbf{h}_1$$

holds, where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is the matrix representation of the operator A with respect to the direct sum $L_0 \oplus X_0 = X$ where $L_0 = \mathcal{L}(\{x_1^0,\dots,x_n^0\})$, $\mathbf{x}_0 = (x_1^0 \cdots x_n^0)^T$, $\mathbf{h}_1 = (h_1 \cdots h_n)^T$ and $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{h}_1$.

Proof. Let $L_0 = \mathcal{L}(\{x_1^0,\dots,x_n^0\})$. Since $\dim L_0 = n$, there exists a closed subspace X_0 of X such that $X = L_0 \oplus X_0$. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in B(L_0 \oplus X_0)$ have eigenmatrix Λ and eigenvector $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{h}_1$. Then

$$\Lambda(\mathbf{x}_0 + \mathbf{h}_1) = (A \oplus \cdots \oplus A)(\mathbf{x}_0 + \mathbf{h}_1) =$$

$$= ((A_{11} \oplus \cdots \oplus A_{11})\mathbf{x}_0 + (A_{12} \oplus \cdots \oplus A_{12})\mathbf{h}_1) + ((A_{21} \oplus \cdots \oplus A_{21})\mathbf{x}_0 + (A_{22} \oplus \cdots \oplus A_{22})\mathbf{h}_1)$$

$$\iff \begin{array}{l} (A_{12} \oplus \cdots \oplus A_{12})\mathbf{h}_1 = (\Lambda - (A_{11} \oplus \cdots \oplus A_{11}))\mathbf{x}_0 \\ (A_{21} \oplus \cdots \oplus A_{21})\mathbf{x}_0 = (\Lambda - (A_{22} \oplus \cdots \oplus A_{22}))\mathbf{h}_1. \end{array}$$

For the opposite implication, suppose that the equations hold for some $\mathbf{h}_1 \in X_0^n$ and an $n \times n$ complex matrix Λ. It is easy to see that Λ is an eigenmatrix of A with eigenvector $\mathbf{x}_0 + \mathbf{h}_1$. □

REMARK (3.3). If $\dim L_0 = 1$, using that the 1×1 matrix $\Lambda = (\lambda)$ is an eigenmatrix of A if and only if λ is an eigenvalue of A, the previous theorem becomes a more general case of [3], Theorem 6. Moreover, in [3] the case is excluded when $h_1 = 0$, but we can to see that then $A_{21} = 0$, or L_0 is an invariant subspace of A. This implies that x_0 is an eigenvector for the eigenvalue λ_1 of A.

THEOREM (3.4). *Let* $(A_0, L_0) \in \mathcal{P}l(X)$ *with* $L_0 = \mathcal{L}(\{x_1^0, \ldots, x_n^0\})$. *Then there exists a transformation* $F : \mathcal{M}_{n\times n} \times X^n \to B(X)$ *defined in a neighborhood* U *of* $(\Lambda_0, (x_1^0, \ldots, x_n^0))$ *such that*

$$F(\Lambda_0, (x_1^0, \ldots, x_n^0)) = A_0,\ (F(\Lambda, \mathcal{L}(x_1, \ldots, x_n)), \mathcal{L}(x_1, \ldots, x_n)) \in \mathcal{P}l(X)$$

for every $(\Lambda, (x_1, \ldots, x_n)) \in U$, *and* F *is continuous at* $(\Lambda_0, (x_1^0, \ldots, x_n^0))$.

Proof. Let $X = L_0 \oplus X_0$, $L_0 = \mathcal{L}(\{x_1^0, \ldots, x_n^0\})$, $\mathbf{x}_0 = (x_1^0 \cdots x_n^0)^T$ and let A_0 have matrix representation

$$A_0 = \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ 0 & A_{22}^0 \end{bmatrix}$$

with respect to the decomposition of the space X. Let $(\tilde\Lambda, (\tilde x_1, \ldots, \tilde x_n)) \in U$, $\tilde L = \mathcal{L}(\{\tilde x_1, \ldots, \tilde x_n\})$ and $\tilde\Lambda = (\tilde\lambda_{ij})_{i,j=1}^n$. Using the decomposition of X and appropriate notation, let $\tilde{\mathbf{x}} = \mathbf{x}_0 + \tilde{\mathbf{h}}_0$ ($\tilde{\mathbf{x}} = (\tilde x_1 \cdots \tilde x_n)^T$, $\tilde{\mathbf{h}}_0 = (\tilde h_1 \cdots \tilde h_n)^T$), we define $F(\tilde\Lambda, (\tilde x_1 \cdots \tilde x_n)) \in B(X)$ by the operator matrix

$$F(\tilde\Lambda, (\tilde x_1, \ldots, \tilde x_n)) = \begin{bmatrix} \tilde A_{11} & A_{12}^0 \\ \tilde A_{21} & A_{22}^0 \end{bmatrix} : L_0 \oplus X_0 \to L_0 \oplus X_0,$$

with

$$\begin{aligned} \tilde{\mathbb{A}}_{11}(\mathbf{x}_0) &= \tilde\Lambda \mathbf{x}_0 - \mathbb{A}_{12}^0 \tilde{\mathbf{h}}_0 \quad \text{and} \\ \tilde{\mathbb{A}}_{21}(\mathbf{x}_0) &= \tilde\Lambda \tilde{\mathbf{h}}_0 - \mathbb{A}_{22}^0 \tilde{\mathbf{h}}_0, \end{aligned} \tag{3.5}$$

where for $T \in \mathcal{B}(K)$, $K \in \{L_0, X_0, X\}$, we use the notation $\mathbb{T} = T \oplus \cdots \oplus T \in \mathcal{B}(K^n)$. By Theorem 3.4, it is easy to see that $F(\tilde\Lambda, (\tilde x_1, \ldots, \tilde x_n))(\tilde L) \subseteq \tilde L$, i.e. $(F(\tilde\Lambda, (\tilde x_1, \ldots, \tilde x_n)), \tilde L) \in \mathcal{P}l(X)$.

Let $x = l_x + h_x \in X (= L_0 \oplus X_0)$ be an arbitrary norm one vector. Then

$$F(\tilde\Lambda, (\tilde x_1, \ldots, \tilde x_n))x - A_0 x = (\tilde A_{11} - A_{11}^0) l_x + \tilde A_{21} l_x = (\text{ by } (3.5))$$

$$= (\tilde\Lambda - \Lambda_0) l_x + Z(\tilde{\mathbf{h}}_0),$$

where Z is defined using $\tilde\Lambda$, A_{12}^0 and A_{22}^0. Now it easy to see that $\|F(\tilde\Lambda, (\tilde x_1, \ldots, \tilde x_n)) - A_0\|$ converge to zero when $\tilde\Lambda \to \Lambda_0$ and $\tilde{\mathbf{h}}_0 \to 0$. □

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References

[1] M. Ahues, A. Largiller, and B. V. Limaye, Spectral Computations for Bounded Operators, Chapman-Hall/CRC, Boca Raton, FL, 2001.

[2] B. A. Barnes, *Spectral and Fredholm theory involving the diagonal of a bounded linear operator*, Acta Sci. Math. (Szeged), **73**, (2007), 237–250.

[3] M. Chaperon and S. L. de Medrano, *Some regularities and singularities appearing in the study of polinomials and operators*, Équations différentielles et singularités en l'honneur de J.M. Aroca. Astérisque, **57**, (2008), 123–160.

[4] S. V. Djordjević and B. P. Duggal, *Spectral properties of linear operator through invariant subspaces*, Funct. Anal. Approx. Comput., **1:1**, (2009), 19–29.

[5] S. V. Djordjević and B. P. Duggal, *Drazin invertibility of the diagonal of an operator*, Linear and Multilinear Algebra, **60**, (2012), 65–71.

[6] S. V. Djordjević, R. Harte, and D. Larson, *Partially hyperinvariant subspaces*, Oper. Matrices, **6**, (2012), 97–106.

[7] R. A. Martínez-Avendaño, *Eigenmatrices and operators commuting with finite-rank operators*, Linear Algebra Appl., **419** (2006), no. 2-3, 739–749.

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CLASSIFICATION OF CONSTANT ANGLE HYPERSURFACES IN WARPED PRODUCTS VIA EIKONAL FUNCTIONS

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ABSTRACT. Given a warped product of the real line with a Riemannian manifold of arbitrary dimension, we classify the hypersurfaces whose tangent spaces make a constant angle with the vector field tangent to the real direction. We show that this is a natural setting in which to extend previous results in this direction made by several authors. Moreover, when the constant angle hypersurface is a graph over the Riemannian manifold, we show that the function involved satisfies a generalized eikonal equation, which we solve via a geometric method. In the final part of this paper we prove that minimal constant angle hypersurfaces are cylinders over minimal submanifolds.

Introduction

Several classical, well-known geometric objects are defined in terms of making a constant angle with a given, distinguished direction. Firstly, classical helices are curves making a constant angle with a fixed direction. A second example is the logarithmic spiral, the *spira mirabilis* studied by Jacob Bernoulli, which makes a constant angle with the radial direction. In a third famous example which had applications to navigation, the *loxodromes* or *rhumb lines* are those curves in the sphere making a constant angle with the sphere meridians.

Recently, several authors had established and investigated some generalizations of the above situation. In 2007, F. Dillen et al. characterized those surfaces M in $\mathbb{S}^2 \times \mathbb{R}$ whose normal vector ξ makes a constant angle θ with the direction tangent to $\mathbb{R}$ (see [7]). Two years later, F. Dillen and M. I. Munteanu gave in [8] a similar characterization theorem for constant angle surfaces in the product $\mathbb{H}^2 \times \mathbb{R}$, using the hyperboloid model for the hyperbolic plane $\mathbb{H}^2$. In the final part of the paper they classified the constant angle surfaces with constant mean curvature in this Riemannian product.

Another nice paper in this direction is [13], where M. I. Munteanu made a review of some applications of constant angle surfaces and gave a complete classification of the so-called constant slope surfaces in $\mathbb{R}^3$, that is, those surfaces making a constant angle with the radial position vector field. He showed that a surface $S \subset \mathbb{R}^3$ is a constant slope surface iff either it is an Euclidean 2-sphere centered at the origin or it can be parameterized by

$$r(u,v) = u\sin\theta(\cos\xi f(v) + \sin\xi f(v) \times f'(v)),$$

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where θ is a constant different from 0, $\xi = \xi(u) = \cot\theta \log u$ and f is a unit speed curve on the Euclidean sphere $\mathbb{S}^2$.

It is also worth mentioning the recent paper [10], where Dillen, Munteanu, Van der Veken and Vrancken classified the constant angle surfaces in the warped product $I \times_\rho \mathbb{R}^2$. We will discuss the relation of this and other works with ours in Section 3. This class of surfaces or curves making a constant angle with respect to some direction have been also investigated in Minkowski space, see [1] and [12] for details.

Using another approach, A. Di Scala and the third named author studied in [4] the helix submanifolds of Euclidean spaces, i. e., submanifolds making a constant angle with a constant direction. They builded constant angle hypersurfaces of $\mathbb{R}^{n+1}$, as follows: Given an orientable hypersurface L of $\mathbb{R}^n$ with a unit normal vector field η, let $r : L \times \mathbb{R} \to \mathbb{R}^{n+1}$ be defined by

$$r(x,s) = x + s\left((\sin\theta)\eta(x) + (\cos\theta)d\right),$$

where θ is constant and $d = (0,\ldots,0,1)$. Then f parameterize a hypersurface making a constant angle θ with the fixed direction determined by d. Moreover, they showed that, except for some trivial cases, any helix hypersurface admits locally such a parametrization. They also showed that these non-trivial constant angle submanifolds are given locally as graphs of functions whose gradient has constant length (that is, solutions of the so-called eikonal equation). In [5], they showed further that any function satisfying the eikonal equation may be characterized as a distance function relative to an embedded hypersurface in the ambient space.

All of the above results suggest the existence of a general framework in which it is natural to consider the study of constant angle submanifolds. As it will turn out along this paper, a natural choice for that purpose is an ambient space $\bar{M}$ given as a warped product of the form $I \times_\rho \mathbb{P}^n$, where I is an open interval and $\rho : I \to \mathbb{R}^+$ is a smooth positive function. We consider those submanifolds making a constant angle with the vector field ∂_t tangent to the $\mathbb{R}$-direction. Of course, the case of the Euclidean ambient space is obtained by considering $\mathbb{P}^n = \mathbb{R}^n$ and the constant warping function $\rho \equiv 1$.

The plan of this paper is the following. Section 1 gives the basic geometric properties of constant angle hypersurfaces in a warped product, showing that they have a rich extrinsic and intrinsic geometry. In Theorem (1.5) we prove that if the projection of ∂_t to the tangent space of a constant angle hypersurface does not vanish, it determines a principal direction on the hypersurface. In the terminology of the recent works [6], [9] and [14], the hypersurface has a *canonical principal direction* relative to the distinguished vector field ∂_t. Also, we prove that the integral lines of this tangential component are lines of curvature and geodesics of the hypersurface.

In Section 2 we state our main result giving a complete characterization of constant angle hypersurfaces in $I \times_\rho \mathbb{P}^n$ (see Theorem (2.3)):

Let $\overline{M}^{n+1}$ be the warped product $I \times_\rho \mathbb{P}^n$. A connected hypersurface M of $\overline{M}$ is a constant angle hypersurface in $\bar{M}$ if and only if it is an open subset of either

- *A cylinder of the form $I \times L^{n-1}$, where L is a hypersurface of $\mathbb{P}$; or*
- *The graph of a function $f : \mathbb{P} \to \mathbb{R}$ satisfying the* generalized eikonal equation

$$|\nabla f| = C \cdot (\rho \circ f), \tag{0.1}$$

where C is a constant, ρ stands for the warping function and the graph of f is defined as the set of points $(f(p),p)$ with $p \in \mathbb{P}$.

We also give a geometric method to build the solutions of the generalized eikonal equation, by generalizing the technique given in [5] for the case of the classical eikonal equation. Our result in the context of constant angle hypersurfaces is the following (see Corollary (2.8)):

Let $\overline{M}^{n+1}$ be the warped product $I \times_\rho \mathbb{P}^n$. A connected hypersurface in $\bar{M}$ is a constant angle hypersurface with $\theta \in (0,\pi/2)$ if and only if it is the graph of a function $f : \mathbb{P} \to \mathbb{R}$ of the form $f = h \circ d$, where d measures the distance to a fixed orientable hypersurface $L \subset \mathbb{P}$ and h satisfies

$$h^{-1}(s) = \int_{s_0}^{s} \frac{d\sigma}{C\rho(\sigma)},$$

with $C = \tan\theta$.

In Section 3 we show the relation between the parametrizations of constant angle surfaces obtained by the authors already mentioned in this Introduction and our language. Note that our setting includes all codimension 1 cases, and in particular, the case of surfaces in every 3-dimensional warped product of the form $I \times_\rho \mathbb{P}^2$.

Finally, in Section 4 we prove that minimal constant angle hypersurfaces are cylinders over a minimal submanifold of codimension two. We deduce this result from the following nice property:
Let $f : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function with connected open domain Ω. If f is harmonic and eikonal then f is linear in Ω.

1. A canonical principal direction

Throughout this paper, we will use the following notations:

- $\overline{M}^{n+1}$ will denote a warped product of the form $I \times_\rho \mathbb{P}^n$, where I is an open interval, $\mathbb{P}$ is a Riemannian manifold and $\rho : I \to \mathbb{R}^+$.
- $\overline{\nabla}$ is the Riemannian connection on $\overline{M}$ relative to the warped product metric.
- ∂_t will denote the unit vector field tangent to the $\mathbb{R}$-direction in $\overline{M}$.
- M will be a connected orientable hypersurface in $\overline{M}$.
- ∇ will denote the induced Riemannian connection on M.
- $\xi \in \mathfrak{X}(M)$ will be a unit vector field, everywhere normal to M.
- θ will denote the function on M measuring the angle between ∂_t and ξ.

Definition (1.1). We say that M is a *constant angle hypersurface* iff the angle function θ is constant along M.

Remark (1.2). Given a constant angle hypersurface, we may choose the orientation of M so that $\theta \in [0,\pi/2]$, as we will do.

Our aim here is to classify all constant angle hypersurfaces M of the warped product $I \times_\rho \mathbb{P}^n$. A trivial case occurs when $\theta \equiv 0$. In the language of the warped product structure, $\xi = \partial_t$ and then a connected constant angle hypersurface is contained in a slice $\{t_0\} \times \mathbb{P}$. So, we suppose in this section that $\theta \in (0,\pi/2]$.

Let us fix some additional notation. As usual, we have the Gauss and Weingarten equations for hypersurfaces:

$$\overline{\nabla}_Y Z = \nabla_Y Z + II(Y,Z), \quad \overline{\nabla}_Y \xi = -A_\xi Y,$$

where $Y,Z \in \mathfrak{X}(M)$, II is the second fundamental form of M and A_ξ is the shape operator associated to ξ. Recall also that II and A_ξ are related by the formula

$$\langle II(Y,Z),\xi\rangle = \langle A_\xi Y, Z\rangle.$$

Let $\partial_t^\top$ be the component of ∂_t tangent to M, that is,

$$\partial_t^\top = \partial_t - \langle \partial_t, \xi\rangle \xi,$$

Note that $\theta \in (0,\pi/2]$ implies $\partial_t^\top \neq 0$ and we may define

$$T = \frac{\partial_t^\top}{|\partial_t^\top|}. \tag{1.3}$$

Hence we may write

$$\partial_t = (\sin\theta)T + (\cos\theta)\xi. \tag{1.4}$$

Now we are ready to give some basic geometric properties of the constant angle hypersurfaces.

THEOREM (1.5). *Let M be a constant angle hypersurface of $\overline{M}^{n+1}$ such that $\theta \in (0,\pi/2]$. Then the integral lines of the vector field T defined in* (1.3) *are lines of curvature of M; in fact,*

$$A_\xi T = -\cos\theta \frac{\rho'}{\rho} T.$$

In other words, T is a principal direction of M. Moreover, these lines are geodesics of M, that is, $\nabla_T T = 0$.

Additionally, the integral lines of T are geodesics of $\overline{M}$ iff either ∂_t is parallel or $\theta = \pi/2$.

Proof. Suppose first that $\theta \in (0,\pi/2)$, which implies $\cos\theta \neq 0$. Differentiating (1.4) with respect to a vector field $W \in \mathfrak{X}(M)$, we obtain

$$\overline{\nabla}_W \partial_t = (\sin\theta)\overline{\nabla}_W T + (\cos\theta)\overline{\nabla}_W \xi. \tag{1.6}$$

Suppose additionally that $\langle W, T\rangle = 0$ or, equivalently, $\langle W, \partial_t\rangle = 0$. To calculate $\overline{\nabla}_W \partial_t$, we may suppose that W is given as a lifting of a vector field on $\mathbb{P}$ and use standard derivation formulas in warped products (see [15], p. 296, Prop. 35, for example) to obtain that $\overline{\nabla}_W \partial_t = (\rho'/\rho)W$. Taking the components tangent and normal to M in the above formula and using that $\cos\theta \neq 0$, we have

$$A_\xi W = -\frac{\rho'}{\rho\cos\theta} W + (\tan\theta)\nabla_W T,$$

and $II(W,T) = 0$, which implies that

$$\langle A_\xi T, W\rangle = \langle A_\xi W, T\rangle = \langle II(W,T),\xi\rangle = 0$$

for every $W \in \mathfrak{X}(M)$ such that $\langle W, T\rangle = 0$. In turn, this fact implies that $A_\xi T$ is a scalar multiple of T, i.e., T is a principal direction of M.

We return to the general expression (1.6) and take $W = T$. In order to use the derivation formulas for warped products again, we write

$$T = (\sin\theta)\partial_t + (\cos\theta)[(\cos\theta)T - (\sin\theta)\xi],$$

and note that the vector field $(\cos\theta)T - (\sin\theta)\xi$ is orthogonal to ∂_t. Hence,

$$\begin{aligned}\overline{\nabla}_T\partial_t &= (\sin\theta)\overline{\nabla}_{\partial_t}\partial_t + (\cos\theta)\overline{\nabla}_{[(\cos\theta)T-(\sin\theta)\xi]}\partial_t \\ &= (\cos\theta)\frac{\rho'}{\rho}[(\cos\theta)T - (\sin\theta)\xi];\end{aligned}$$

so that the tangent and normal components of (1.6) are

$$\cos^2\theta\frac{\rho'}{\rho}T = (\sin\theta)\nabla_T T - (\cos\theta)A_\xi T$$

and

$$-\sin\theta\cos\theta\frac{\rho'}{\rho}\xi = (\sin\theta)II(T,T).$$

From the first of these expressions, since $A_\xi T$ is a scalar multiple of T (and $\sin\theta \neq 0$), we deduce that the same happens with $\nabla_T T$; but as T is a unit vector field, we have $\nabla_T T = 0$; i.e., the integral lines of T are geodesics in M. Also,

$$A_\xi T = -\cos\theta\frac{\rho'}{\rho}T,$$

meaning that T is a principal direction. In the case of the second fundamental form, we have

$$II(T,T) = -\cos\theta\frac{\rho'}{\rho}\xi.$$

Since we are analyzing the case $\cos\theta \neq 0$, $II(T,T) = 0$ if and only if $\rho' = 0$; i.e., ρ is constant. In this case, $\overline{\nabla}_W\partial_t = 0$ for every vector field $W \in \mathfrak{X}(\overline{M})$. That is, the integral lines of T are geodesics of $\overline{M}$ if and only if ∂_t is parallel.

The analysis in the case $\theta = \pi/2$ is similar, but easier, since in this case equation (1.4) reduces to $T = \partial_t$. We have that $\overline{\nabla}_T T = \overline{\nabla}_{\partial_t}\partial_t = 0$ and then the integral lines of T are geodesics of $\bar{M}$, thus they also are geodesics of M. If $W \in \mathfrak{X}(M)$ is orthogonal to T we have on one hand

$$\overline{\nabla}_W T = \overline{\nabla}_W\partial_t = \frac{\rho'}{\rho}W,$$

and on the other hand, $\overline{\nabla}_W T = \nabla_W T + II(W,T)$, which implies that $II(W,T) = 0$. As in the previous case, this in turn implies that $A_\xi T$ is a scalar multiple of T and T is a principal direction. In fact, since $\overline{\nabla}_T T = \nabla_T T + II(T,T) = 0$, we have

$$\langle A_\xi T, T\rangle = \langle II(T,T),\xi\rangle = 0,$$

and then $A_\xi T = 0$ and T is a principal direction. □

Theorem 1.5 says that the constant angle hypersurfaces with $\theta \in (0,\pi/2]$ are examples of hypersurfaces with a canonical principal direction, which means that there exists a vector field in the ambient such that the component of this vector field tangent to the surface is a principal direction for the shape operator of the surface. This notion has been studied recently by several authors; see, for example [6], [9] and [14], where the authors classify surfaces with a canonical principal direction in $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{R}^2 \times \mathbb{R}$, respectively.

2. Construction and characterization of constant angle hypersurfaces

In this section we prove our main results, classifying the constant angle hypersurfaces in any warped product of the form $I \times_\rho \mathbb{P}^n$. First we consider the case of $\theta = \pi/2$:

PROPOSITION (2.1). *Let M be a connected hypersurface of $I \times_\rho \mathbb{P}^n$. M is a constant angle hypersurface with $\theta = \pi/2$ if and only if M is an open subset of a cylinder $I \times L^{n-1}$, where L is a $(n-1)$-dimensional hypersurface of $\mathbb{P}$.*

Proof. Suppose M is a constant angle hypersurface with $\theta = \pi/2$. By transversality, the intersection of M with a fixed slice $\{t_0\} \times \mathbb{P}^n$ is (isometric to) a hypersurface L of $\mathbb{P}^n$. Since ∂_t is everywhere tangent to M in this case, we reconstruct M by departing from this intersection and following the flow of ∂_t, obtaining the aforementioned cylinder. The converse is clear. □

In view of this result, we may suppose from now on that $\theta \in [0, \pi/2)$. Using transversality, we may suppose additionally that M is given locally as a graph of a real function $f : \mathbb{P} \to I$. We will prove that such a graph is a constant angle hypersurface if and only if f satisfies a condition on the norm of its gradient (see equation (0.1)). In the following definition we fix the classical terminology for this kind of functions.

Definition (2.2). Let $\mathbb{P}^n$ be a Riemannian manifold and $f : \mathbb{P} \to I$ a differentiable function, where I is a real interval. We say that f is *eikonal* if it is a solution of the *eikonal equation*

$$|\nabla f| = C,$$

where ∇f denotes the gradient of f and C is a given constant. More generally, let $\rho : I \to \mathbb{R}^+$ be a differentiable positive function. We say that f is a *transnormal function* if it satisfies the generalized eikonal equation (0.1), namely,

$$|\nabla f| = C \cdot (\rho \circ f).$$

The concept of transnormal function is related to the class of submanifolds called isoparametric submanifolds which are level hypersurfaces of isoparametric functions. According to [18], a transnormal function is a smooth function f satisfying the equation $|\nabla f|^2 = b \circ f$, where b is a smooth function which can be zero at some points. In our case $b = C\rho > 0$. An isoparametric function is a transnormal function that also satisfies the condition $\Delta f = a \circ f$, where a is a smooth function. It is well known that Cartan investigated such functions on space forms; see [2] and [18] for more details. An interesting result in [18], is that a transnormal function in $\mathbb{S}^n$ or in $\mathbb{R}^n$ is isoparametric.

The next theorem is our main result, giving the precise relation between the transnormal functions and the constant angle hypersurfaces.

THEOREM (2.3). *Let $\overline{M}^{n+1}$ be the warped product $I \times_\rho \mathbb{P}^n$. A connected hypersurface M of $\overline{M}$ is a constant angle hypersurface in $\bar{M}$ if and only if it is an open subset of either*

- *A cylinder of the form $I \times L^{n-1}$, where L is a hypersurface of $\mathbb{P}$; or*

- *The graph of a transnormal function $f : \mathbb{P} \to I$ satisfying equation (0.1) for the warping function ρ. Here the graph of f is defined as the set of points $(f(p), p)$ with $p \in \mathbb{P}$.*

Proof. Let M be a constant angle hypersurface in $\overline{M}$. By Proposition (2.1), we may suppose that $\theta \in [0, \pi/2)$ and that M is a graph of a function f. Let us denote by ∇f the lift to $\overline{M}$ of the gradient of f. Then it is easy to see that a vector field ξ everywhere normal to the graph of f may be chosen as

$$\xi = (\rho \circ f)^2 \partial_t - \nabla f.$$

Using the definition of the warped product metric and the fact that ∂_t and ∇f are orthogonal, we have that the square of the norm of ξ is given by

$$\langle \xi, \xi \rangle = (\rho \circ f)^4 + (\rho \circ f)^2 |\nabla f|^2 = (\rho \circ f)^2((\rho \circ f)^2 + |\nabla f|^2),$$

and consequently the angle θ between ξ and ∂_t satisfies

$$\cos\theta = \left\langle \frac{\xi}{|\xi|}, \partial_t \right\rangle = \frac{\rho \circ f}{\sqrt{(\rho \circ f)^2 + |\nabla f|^2}}.$$

Note that $\cos\theta \neq 0$ for $\theta \in [0, \pi/2)$. Hence we may express $|\nabla f|$ in terms of $\rho \circ f$ as

$$|\nabla f| = (\tan\theta)(\rho \circ f),$$

which means that f is transnormal with $C = \tan\theta$.

Conversely, if we consider the graph of a transnormal function satisfying equation (0.1), the angle θ between its normal ξ and ∂_t is such that

$$\cos\theta = \left\langle \frac{\xi}{|\xi|}, \partial_t \right\rangle = \frac{\rho \circ f}{\sqrt{(\rho \circ f)^2 + |\nabla f|^2}} = \frac{1}{\sqrt{1 + C^2}}; \tag{2.4}$$

meaning that the graph of f is a constant angle hypersurface. □

In short, Theorem 2.3 proves that every constant angle hypersurface is locally the graph of a function satisfying a partial differential equation on a Riemannian manifold $\mathbb{P}^n$, the generalized eikonal equation (0.1). In the final part of this section we will solve this equation explicitly by a geometric method using the distance function to an arbitrary hypersurface in $\mathbb{P}^n$.

As a first step, in our next Proposition we prove the (local) existence of solutions using a constructive method.

PROPOSITION (2.5). *Let $\mathbb{P}^n$ be a Riemannian manifold and $\rho : I \to \mathbb{R}^+$ a differentiable positive function. Fix an orientable hypersurface $L \subset \mathbb{P}$ and a tubular neighborhood L_ϵ of L such that the distance function d to L is well-defined in L_ϵ and is differentiable in $L_\epsilon \setminus L$. Also, define a real valued and invertible function $h : I \to \mathbb{R}^+$ by*

$$h^{-1}(s) = \int_{s_0}^{s} \frac{d\sigma}{C\rho(\sigma)}, \tag{2.6}$$

where $C \neq 0$. Then $f = h \circ d$ is transnormal in $L_\epsilon \setminus L$.

Proof. It is well-known that $|\nabla d| = 1$ in $L_\epsilon \setminus L$; then,

$$\begin{aligned} |\nabla f| &= |\nabla(h \circ d)| = (h' \circ d)|\nabla d| = h' \circ d \\ &= \frac{1}{(h^{-1})'(h \circ d)} = C \cdot (\rho \circ h \circ d) = C \cdot (\rho \circ f), \end{aligned}$$

which proves the claim. □

Now we analyze the (local) uniqueness of solutions of the generalized eikonal equation. We will use the results proved by Di Scala and the third named author in [5], where they studied the local uniqueness of the solutions of an eikonal equation.

PROPOSITION (2.7). *Let* $f : \mathbb{P} \to I$ *satisfy* $|\nabla f| = C \cdot (\rho \circ f)$ *for* $C \neq 0$. *Then* f *is given locally as in Proposition (2.5).*

Proof. Let $d = h^{-1} \circ f$, where h^{-1} is defined in equation (2.6). Let us calculate the gradient of d in $\mathbb{P}$:

$$\nabla d = \nabla(h^{-1} \circ f) = ((h^{-1})' \circ f)\nabla f = \frac{1}{C \cdot (\rho \circ f)}\nabla f.$$

Therefore, $|\nabla d| = 1$. Theorem (5.3) in [5] implies then that for every point $p \in \mathbb{P}$ there exists a neighborhood U of p in $\mathbb{P}$ and a hypersurface $L \subset \mathbb{P}$ such that $d|_U$ measures the distance from a point in U to the hypersurface L. This proves that $f = h \circ d$ has the form given in Proposition (2.5). □

We are ready to translate the above results to our constant angle hypersurfaces setting.

COROLLARY (2.8). *Let* $\overline{M}^{n+1}$ *be the warped product* $I \times_\rho \mathbb{P}^n$. *A connected hypersurface in* $\bar{M}$ *is a constant angle hypersurface with* $\theta \in (0, \pi/2)$ *if and only if it is the graph of a function* $f : \mathbb{P} \to \mathbb{R}$ *of the form* $f = h \circ d$, *where* d *measures the distance to a fixed orientable hypersurface* $L \subset \mathbb{P}$ *and* h *satisfies*

$$h^{-1}(s) = \int_{s_0}^{s} \frac{d\sigma}{C\rho(\sigma)}.$$

with $C = \tan\theta$.

3. Applications and Examples

In this section we will construct some examples of constant angle hypersurfaces and will show the relation of our construction with those made in the papers already mentioned in the Introduction.

Example (3.1). Let us consider the upper-half space model for the hyperbolic space $\mathbb{H}^{n+1}$, which can be expressed as the warped product $(0, \infty) \times_\rho \mathbb{R}^n$, where $\rho(t) = 1/t$. Then, taking $s_0 = 1$,

$$r = h^{-1}(s) = \int_1^s \frac{d\sigma}{C\rho(\sigma)} = \frac{1}{C}\int_1^s \sigma\, d\sigma = \frac{s^2 - 1}{2C}.$$

Hence, $s = h(r) = \sqrt{2Cr + 1}$. The hypersurface we consider is $L = \mathbb{R}^{n-1}$, identified as usual with the points $(x_1, \ldots, x_{n-1}, 0)$ so that the (oriented) distance function to L is x_n, the n-th coordinate function on $\mathbb{R}^n$.

Therefore, the explicit expression of the function $f = h \circ d$ is

$$f(x_1, \ldots, x_n) = h \circ d(x_1, \ldots, x_n) = h(x_n) = \sqrt{2Cx_n + 1}.$$

We calculate the gradient of f as

$$\nabla f(x_1, \ldots, x_n) = \frac{C}{\sqrt{2Cx_n + 1}}\, \partial_n,$$

where $\partial_n = \partial_{x_n}$. Note that

$$|\nabla f(x_1,\ldots,x_n)|^2 = \frac{C^2}{2Cx_n+1} = C^2(\rho\circ f)^2(x_1,\ldots,x_n).$$

Application (3.2). In [13], Munteanu studied the surfaces in three-dimensional Euclidean space whose normal vector at a point makes a constant angle with the position vector of that point, showing (Theorem 1 in [13]) that a constant angle surface is an open part of the Euclidean 2-sphere or it can be parameterized by

$$r(u,v) = u\{\sin\theta[\cos(\cot\theta\ln u)\,\alpha(v)+\sin(\cot\theta\ln u)\cdot\alpha(v)\times\alpha'(v)]\}, \tag{3.3}$$

where $\theta\neq 0$ and α is a unit speed curve $\alpha: I\to\mathbb{S}^2$.

To translate Munteanu's analysis to our context, note that the Euclidean 3-space minus the origin is isometric to the warped product

$$(0,\infty)\times_\rho\mathbb{S}^2(\sin\theta),\quad \rho(t)=\frac{t}{\sin\theta};$$

here $\mathbb{S}^2(\sin\theta)$ denotes a 2-dimensional sphere with radius $\sin\theta$. Of course, the natural isometry of this warped product with $\mathbb{R}^3\setminus\{0\}$ is given explicitly by $(t,p)\mapsto tp$.

To be able to compare Munteanu's result with our Corollary (2.8), we note that the function h given by equation (2.6) is given by

$$h^{-1}(s)=\int_1^s\frac{d\sigma}{C\rho(\sigma)}=\frac{\sin\theta}{C}\ln s.$$

Also, we will obtain an expression for the distance function in $\mathbb{S}^2$ to the curve α that appears in (3.3). Note that the expression in braces in (3.3) gives a point $\varphi(u,v)$ in $\mathbb{S}^2(\sin\theta)$ and that its distance $d=d(\varphi(u,v))$ to $\alpha(v)$ is precisely the product of the radius and the angle between the two vectors; i.e.,

$$d(\varphi(u,v))=\sin\theta\cdot\cot\theta\cdot\ln u=\cos\theta\cdot\ln u;$$

recalling that C may be seen as $\tan\theta$, we have

$$d(\varphi(u,v))=h^{-1}(u),$$

which gives

$$f(\varphi(u,v))=h\circ d(\varphi(u,v))=u.$$

This fact means that a constant angle surface in $(0,\infty)\times_\rho\mathbb{S}^2(\sin\theta)$ is given by the graph $(f(\varphi(u,v)),\varphi(u,v))$ of f, i. e., by

$$(u,\varphi(u,v))=(u,\sin\theta[\cos(\cot\theta\ln u)\,\alpha(v)+\sin(\cot\theta\ln u)\cdot\alpha(v)\times\alpha'(v)]);$$

but this expression corresponds precisely to equation (3.3) via the aforementioned isometry of $(0,\infty)\times_\rho\mathbb{S}^2(\sin\theta)$ with the Euclidean space. Thus, we recover Munteanu's result.

Application (3.4). In our last comparison we consider the work [10], where Dillen et al. analyzed the hypersurfaces in the warped product $I\times_\rho\mathbb{R}^2$ making a constant angle with the vector field ∂_t. Theorem 1 in [10] states that an isometric immersion $r: M^2\to\overline{M}=I\times_\rho\mathbb{R}^2$ defines a surface with constant angle $\theta\in[0,\pi/2]$ if and only if, up to rigid motions of $\overline{M}$, one of the following holds locally:

1. There exist parameters (u,v) of M, with respect to which the immersion r is given by

$$r(u,v) = \left(u\sin\theta, \cot\theta\left(\int^{u\sin\theta} \frac{d\sigma}{\rho(\sigma)}\right)\cos v - \int^{v} g(\sigma)\sin\sigma\, d\sigma,\right.$$
$$\left.\cot\theta\left(\int^{u\sin\theta} \frac{d\sigma}{\rho(\sigma)}\right)\sin v + \int^{v} g(\sigma)\cos\sigma\, d\sigma\right) \tag{3.5}$$

   for some smooth function g.
2. $r(M)$ is an open part of the cylinder $x - G(t) = 0$ for the real function G given by

$$G(t) = \cot\theta \int^{t} \frac{d\sigma}{\rho(\sigma)}.$$

   (Here (x,y) are the standard coordinates in $\mathbb{R}^2$.)
3. $r(M)$ is an open part of the surface $t = t_0$ for some real number t_0, and $\theta = 0$.

We will discuss items (1) and (2) of this theorem. In relation with item (2) and in analogy with our previous discussion of Munteanu's work, we see that the function G may be written in our terminology as

$$G(t) = \cot\theta \int^{t} \frac{d\sigma}{\rho(\sigma)} = \int^{t} \frac{d\sigma}{C\rho(\sigma)} = h^{-1}(t).$$

To obtain the cylinder $x - G(t) = 0$, we proceed as follows: We build a constant angle curve in the (t,x)-plane, that is, a curve making a constant angle with the vertical vector field ∂_t. Note that this plane is a warped product $I \times_\rho \mathbb{R}$.

By Corollary 2.8, we may build this curve by first taking a codimension one manifold in $\mathbb{R}$, i.e., fixing a point in the real axis, which we may take as the origin. Next, we calculate the distance function d in $\mathbb{R}$ to this point, which obviously gives $d(x) = x$. Hence, the graph of $f = h \circ d = h = G^{-1}$ is the constant angle curve we were looking for. By taking the cylinder over this curve in the 3-dimensional space, we obtain the constant angle surface given in item (2).

To analyze item (1), we define the following curve $\alpha(v)$ in the (x,y)-plane:

$$\alpha(v) = \left(-\int^{v} g(\sigma)\sin\sigma\, d\sigma, \int^{v} g(\sigma)\cos\sigma\, d\sigma\right);$$

which may be obtained from the second and third coordinates in (3.5) making $u = 0$.
Note that $\alpha'(v) = g(v)(-\sin v, \cos v)$, so that $(\cos v, \sin v)$ is a unit vector field everywhere normal to this curve. An easy calculation shows that the second and third coordinates in (3.5) give a parametrization $\varphi(u,v)$ of a neighborhood of α by Fermi coordinates; in fact, the distance of a point in this neighborhood to the curve α is precisely

$$d(\varphi(u,v)) = \cot\theta\left(\int^{u\sin\theta} \frac{d\sigma}{\rho(\sigma)}\right),$$

which is equal to $h^{-1}(u\sin\theta)$ in our terminology. From this we have that the eikonal function f given in Corollary 2.8 is

$$f(\varphi(u,v)) = h \circ d(\varphi(u,v)) = u\sin\theta;$$

that is, equation (3.5) is the expression of the graph of f in $I \times_\rho \mathbb{R}^2$.

Remark (3.6). Note that instead of $u\sin\theta$ we may use a function $\psi(u)$ in the upper limit of the integrals appearing in (3.5) to obtain a point $\varphi(u,v)$ in the plane whose distance to the curve α is

$$d(u,v)=\cot\theta\left(\int^{\psi(u)}\frac{d\sigma}{\rho(\sigma)}\right),$$

so that $f(\varphi(u,v))=\psi(u)$.

4. Minimal constant angle hypersurfaces

Let us recall that a function in Euclidean space is called eikonal if its gradient has constant length.

LEMMA (4.1). *Let $f:U\subset\mathbb{R}^n\longrightarrow\mathbb{R}$ be a smooth function defined on the connected open subset U. If f is a non constant harmonic and eikonal function then f is linear.*

Proof. The idea is to prove that f is locally linear and then to use that U is connected. So in our argument we can take smaller open neighbourhoods if it were necessary. Without loss of generality we can assume that $|\nabla f|^2=1$. Then the level hypersurfaces of f are equidistant embedded hypersurfaces in $\mathbb{R}^n$ because the distance between two level hypersurfaces is measured along the integral curves of the vector field ∇f, which has constant length. Since f is harmonic and eikonal every level hypersurface $f^{-1}(t)$ of f is minimal in $\mathbb{R}^n$ because the mean curvature vector field H of the level hypersurfaces is given by

$$H=-\frac{1}{|\nabla f|}\triangle f+\frac{1}{|\nabla f|^2}\nabla|\nabla f|, \tag{4.2}$$

see [17] for details. As we said before, in our case we can conclude that $H\equiv 0$, i.e. every level hypersurface is minimal. So, $\{f^{-1}(t)\}_{t\in f(U)}$ is a family of equidistant minimal hypersurfaces of $\mathbb{R}^n$. We will prove that this is possible if and only if every level hypersurface in the family is a hyperplane.
Let $\lambda_1,\lambda_2,\ldots\lambda_{n-1}$ be the principal curvatures of $f^{-1}(t_0)$. It is known that for every $t\in F(U)$ close to t_0, the principal curvatures of $f^{-1}(t)$ are given by

$$\frac{\lambda_1}{1-(t-t_0)\lambda_1},\frac{\lambda_2}{1-(t-t_0)\lambda_2},\ldots,\frac{\lambda_{n-1}}{1-(t-t_0)\lambda_{n-1}}.$$

This is a consequence of the relation between the shape operator A of $f^{-1}(t_0)$ and the shape operator A_t of $f^{-1}(t)$: $A_t=(I-tA)^{-1}A$. See [3] page 38.
Since every level hypersurface $f^{-1}(t)$ of f is minimal, the mean curvature of $f^{-1}(t)$ is zero:

$$\frac{\lambda_1}{1-(t-t_0)\lambda_1}+\frac{\lambda_2}{1-(t-t_0)\lambda_2}+\ldots+\frac{\lambda_{n-1}}{1-(t-t_0)\lambda_{n-1}}=0.$$

Taking the derivative with respect to t and evaluating in $t=t_0$ we obtain that $\lambda_1^2+\lambda_2^2+\ldots+\lambda_{n-1}^2=0$, which implies that $\lambda_1=\lambda_2=\ldots=\lambda_{n-1}=0$. Therefore $f^{-1}(t)$ is totally geodesic, i.e. it is part of a hyperplane. This proves that f is linear. □

Remark (4.3). As noted by the referee, Lemma 4.1 is a consequence of classical results obtained by Levi-Civita and Segre (see [11] and [16]) in the context of isoparametric hypersurfaces; we included the above proof for the sake of completeness. The referee also pointed out to us that we may prove the lemma in a shorter way using the well-known Bochner's formula, valid for any smooth function f over a Riemannian manifold:

$$\frac{1}{2}\Delta|\nabla f|^2 = \langle \nabla f, \nabla \Delta f\rangle - \mathrm{Ric}(\nabla f, \nabla f) - |\mathrm{Hess}\, f|^2. \tag{4.4}$$

In order to prove the lemma, note that $\mathbb{R}^n$ is Ricci flat and f is harmonic and eikonal; hence Bochner's formula implies that $\mathrm{Hess}\, f = 0$. So, the second order partial derivatives of f are zero. This proves that f is linear.

The next Corollary (4.5), improves Theorem (2.8) in [4] which says that a constant angle hypersurface M in Euclidean space is minimal if and only if every slice of M is also minimal. Our Corollary here gives a complete, explicit classification of these hypersurfaces.

COROLLARY (4.5). *Let M be a connected constant angle hypersurface in $\mathbb{R}^n$ with respect to a constant direction X. If M is minimal then either M is part of a cylinder, over a minimal hypersurface in $\mathbb{R}^{n-1}$ or M is part of a hyperplane.*

Proof. We can assume that X is a unit vector field. If X is tangent to M, then it is clear that M is part of a cylinder over a hypersurface L in a $\mathbb{R}^{n-1}$ orthogonal to X. Moreover, L should be minimal because M is minimal.
If X is transversal to M then M if the graph of a smooth function f, the height function in direction X. Since M is minimal, every slice of M with hyperplanes orthogonal to X is minimal in the Euclidean ambient, which follows from Theorem 2.8 of [4]. Equivalently, every level hypersurface of f is minimal. Under the hypothesis that f is eikonal and using relation (4.2), the latter condition holds if and only if f is a harmonic function. So, f is an eikonal and harmonic function. By Lemma (4.1), f is linear. Therefore, M is part of a hyperplane. □

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REFERENCES

[1] A. T. ALI AND R. LÓPEZ, *Slant helices in Minkowski space* $\mathbf{E}_1^3$, J. Korean Math. Soc., **48** (1), (2011), 159–167.
[2] J. BERNDT, S. CONSOLE, AND C. OLMOS, Submanifolds and holonomy, vol. 434 of Chapman & Hall/CRC Research Notes in Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2003.
[3] M. DAJCZER, *Submanifolds and isometric immersions*, vol. 13 of Mathematics Lecture Series, Publish or Perish Inc., Houston, TX, 1990.
[4] A. J. DI SCALA AND G. RUIZ-HERNÁNDEZ, *Helix submanifolds of Euclidean spaces*, Monatsh. Math., **157** (3), (2009), 205–215.
[5] A. J. DI SCALA AND G. RUIZ-HERNÁNDEZ, *Higher codimensional Euclidean helix submanifolds*, Kodai Math. J., **33** (2), (2010), 192–210.
[6] F. DILLEN, J. FASTENAKELS, AND J. VAN DER VEKEN, *Surfaces in* $\mathbb{S}^2 \times \mathbb{R}$ *with a canonical principal direction*, Ann. Global Anal. Geom., **35** (4), (2009), 381–396.
[7] F. DILLEN, J. FASTENAKELS, J. VAN DER VEKEN, AND L. VRANCKEN, *Constant angle surfaces in* $\mathbb{S}^2 \times \mathbb{R}$, Monatsh. Math., **152** (2), (2007), 89–96.
[8] F. DILLEN AND M. I. MUNTEANU, *Constant angle surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Bull. Braz. Math. Soc. (N.S.), **40** (1), (2009), 85–97.
[9] F. DILLEN, M. I. MUNTEANU, AND A.-I. NISTOR, *Canonical coordinates and principal directions for surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Taiwanese Math. J., **15** (5), (2011), 2265–2289.
[10] F. DILLEN, M. I. MUNTEANU, J. VAN DER VEKEN, AND L. VRANCKEN, *Classification of constant angle surfaces in a warped product*, Balkan J. of Geom. and its Appl., **16** (2), (2011), 35–47.
[11] T. LEVI-CIVITA, *Famiglie di superficie isoparametriche nellordinario spazio euclideo*, Rend. Acc. Naz. Lincei, **XXVI**, (1937), 355–362. Also in: Opere matematici, V, 657-664.
[12] R. LÓPEZ AND M. I. MUNTEANU, *Constant angle surfaces in Minkowski space*, Bull. Belg. Math. Soc. Simon Stevin, **18** (2), (2011), 271–286.
[13] M. I. MUNTEANU, *From golden spirals to constant slope surfaces*, J. Math. Phys., **51** (7), (2010), 073507, 9.
[14] M. I. MUNTEANU AND A. I. NISTOR, *Complete classification of surfaces with a canonical principal direction in the Euclidean space* $\mathbb{E}^3$, Cent. Eur. J. Math., **9** (2), (2011), 378–389.
[15] B. O'NEILL, Semi-Riemannian geometry, with applications to relativity, vol. 103 of Pure and Applied Mathematics, Academic Press Inc., New York, 1983.
[16] B. SEGRE, *Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni*, Rend. Acc. Naz. Lincei, **XXVII**, (1938), 203–207.
[17] P. TONDEUR, *Foliations on Riemannian manifolds*, Universitext, Springer-Verlag, New York, 1988.
[18] Q. M. WANG, *Isoparametric functions on Riemannian manifolds. I*, Math. Ann., **277** (4), (1987), 639–646.

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CONSTRAINING EXTENT BY DENSITY: ON GENERALIZATIONS OF NORMALITY AND COUNTABLE PARACOMPACTNESS

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ABSTRACT. It is well known that in the class of separable spaces both normality and countable paracompactness imply certain constraints on the sizes of closed discrete subsets. In this paper we show that analogous constraints hold for generalizations of such properties. We also discuss the limitations one has to deal with when trying to find similar constraints for separable, countably metacompact spaces.

1. Introduction

Determining upper bounds – in terms of the size of dense sets – on the sizes of closed discrete subsets of topological spaces satisfying particular properties is a longstanding procedure in General Topology. It goes back, at least, to the fourth decade of the twentieth century, with the seminal work of Jones ([10]). There the classical result that is nowadays referred to as "Jones' Lemma" – that normal separable spaces cannot include closed discrete subsets of size $\mathfrak{c}$ – was established. Jones, moreover, also showed that under $2^{\omega} < 2^{\omega_1}$ such spaces cannot include uncountable closed discrete subsets.

Using the language of cardinal functions, we say that normality is a topological property that "constrains the extent in terms of the density." Recall that the *extent* of a topological space X, $e(X)$, is the supremum of the cardinalities of all closed discrete subsets of X and the *density* of X, $d(X)$, is the smallest cardinality of a dense subset of X (in each case subject to the proviso that if the supremum/minimum, resp., is not infinite the value is set as ω). In these terms the statement "the cardinality of the closed discrete subsets of X is constrained by the minimal cardinality of a dense subset of X" reduces to "$e(X) \leqslant d(X)$". The "separable case" is to look at X with $d(X) = \omega$, and "countable extent" is an abbreviation for "non-existence of uncountable closed discrete subsets".

Another topological property under which extent is constrained by density (in the separable case) is countable paracompactness. Fleissner [6] has shown that countably paracompact, separable spaces cannot include closed discrete subsets of size $\mathfrak{c}$.

In this paper, we show that similar constraints hold for generalizations of the properties so far mentioned.

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In §2 we work with *semi-normal spaces*, which generalize normal spaces in exactly the same way that semi-regular spaces generalize regular spaces. For this class of spaces we obtain restrictions on extents in terms of densities which are entirely similar to those that hold for normal spaces, but by a different argument: the proof of Jones' Lemma is based on a careful construction of a injective function from the family of subsets of a given closed and discrete subset into the family of the subsets of a given dense set, while our results are based on a notion of *separation of families*.

In §3 we consider countable paracompactness[1] and introduce a wide-reaching notion of *μ-semi-paracompactness* – where μ is supposed to be an infinite cardinal. This concept generalizes the notion of μ-paracompactness due to M. E. Rudin [18]. Our definition is based on decreasing sequences of closed sets and decreasing sequences of *semi-open sets* – and this definition seems to be broader, as we discuss, than the expected version based on semi-open refinements of open covers. We present a number of constraints on the size of closed discrete subsets for this very general class of spaces, again by different arguments: the previously obtained constraints (as well as [6] see also [24] and [21]) were established (for case $\mu = \omega$) by constructing a suitable countable open cover and considering a locally finite open refinement, while our results deal directly with the decreasing sequences mentioned above.

We remark that whenever one considers uncountable closed discrete subsets of separable countably paracompact spaces one has to deal with *small dominating families*. These combinatorial structures are related to large cardinals (see below) – and it follows that the problem of comparing extent and density in the class of separable countably paracompact spaces involves intrinsic set-theoretic difficulties.

Returning to our general problem of constraining extent by density, a related theme is to find restrictions on the cardinalities of closed discrete subsets satisfying *relative* versions of normality and countable paracompactness – or even relative versions of their generalizations. We do not go into the details of these results here, but limit ourselves in the main to statements of some of them and giving references to the relevant literature.[2]

It would be highly desirable to find constraints for extent in terms of density similar to those discussed in §2, §3 for (non-trivial) classes of separable, countably metacompact spaces. In §4 we give examples of specific classes of such spaces where instances of those constraints do not necessarily pertain and discuss the limitations with which one therefore has to deal when addressing this problem. We conclude, in §5, with some notes and questions.

We end this introduction with some standard references and definitions.

For small cardinals such as $\mathfrak{p}$, $\mathfrak{b}$ and $\mathfrak{d}$, and for basic information on spaces from almost disjoint families (the so-called *Isbell-Mrówka spaces*), see [4]. For information on cardinal functions we refer to [7].

For $f, g \in {}^{\omega_1}\omega$ write $f \leqslant g$, $f \leqslant^* g$, when the set $\{\alpha < \omega_1 : g(\alpha) < f(\alpha)\}$ is empty, countable, respectively. (The latter is the *mod countable order*).

[1]A space X is [countably] metacompact (resp. paracompact) if every [countable] open cover of X has an open refinement which is point finite (resp. locally finite).

[2]Specific information on relative versions of normality and countable paracompactness may be found in [21], [16] and [22].

$D \subseteq {}^{\omega_1}\omega$ is a *dominating family* if it is cofinal in $\langle {}^{\omega_1}\omega, \leqslant^* \rangle$, meaning $(\forall f \in {}^{\omega_1}\omega)$ $(\exists g \in D)[f \leqslant^* g]$. The size of the smallest dominating family in ${}^{\omega_1}\omega$ is denoted by $cf\langle {}^{\omega_1}\omega, \leqslant^* \rangle$. We recall that $cf\langle {}^{\omega_1}\omega, \leqslant^* \rangle = cf\langle {}^{\omega_1}\omega, \leqslant \rangle$ (see [2]).

A *small dominating family* is a dominating family of functions in ${}^{\omega_1}\omega$ of size not larger than the continuum. (Of course, if $2^\omega = 2^{\omega_1}$ all families of functions are "small".)

Jech and Prikry showed that "$2^\omega < 2^{\omega_1}$ + 2^ω regular + There is a small dominating family" implies that "There is an inner model with a measurable cardinal" ([9]). On the other hand, Watson has shown that the existence of a small dominating family is equivalent to the existence of a separable countably paracompact space with an uncountable closed discrete subset ([24]). It follows that, under "$cf(2^\omega) = 2^\omega < 2^{\omega_1}$" and "There are no inner models with measurable cardinals", countably paracompact separable spaces have, necessarily, countable extent.

2. On semi-normal spaces

We start by giving a couple of combinatorial definitions and remarks.

Definition (2.1). (i) If A, B and C are sets such that $A \subseteq B$ and $B \cap C = \emptyset$ we say that B *separates* A *from* C.

(ii) Let X be a set. If $\mathcal{A} \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ is a set of pairs of disjoint sets we say that $\mathcal{B} \subseteq \mathcal{P}(X)$ is a *separating family for* $\mathcal{A}$, or simply that $\mathcal{B}$ *separates* $\mathcal{A}$, if for all $\langle A, C \rangle \in \mathcal{A}$ there is some $B \in \mathcal{B}$ which separates A from C.

This notion of separation has immediate consequences when it comes to giving upper bounds for the cardinalities of certain specific sets.

PROPOSITION (2.2). *Let X be a set and suppose $\mathcal{B}$ separates $\mathcal{A}$, where $\mathcal{A}$ is a set of pairs of disjoint subsets of X. Let H be a subset of X and suppose $\mathcal{H}$ is a subfamily of $\{\langle A, H \setminus A \rangle : A \in \mathcal{P}(H)\}$. Then $\mathcal{H} \subseteq \mathcal{A}$ implies $|\mathcal{H}| \leqslant |\mathcal{B}|$. In particular, if $\mathcal{H} = \{\langle A, H \setminus A \rangle : A \in \mathcal{P}(H)\} \subseteq \mathcal{A}$ then $2^{|H|} \leqslant |\mathcal{B}|$.*

Proof. For each $\langle A, H \setminus A \rangle \in \mathcal{H}$ let $B_A \in \mathcal{B}$ separate A from $H \setminus A$. Then $B_A \cap H = A$ and so the function $\langle A, H \setminus A \rangle \mapsto B_A$ is an injection from $\mathcal{H}$ to $\mathcal{B}$. □

It is easy to see that if X is a topological space and $H \subseteq X$ then H is a closed discrete subset of X if, and only if, every subset of H is closed. It follows that the preceding proposition has the following corollary:

COROLLARY (2.3). *Let X be a topological space and suppose that $\mathcal{B}$ separates the family $\mathcal{A}$ of all pairs of disjoint closed subsets of X. If H is a closed discrete subset of X then $2^{|H|} \leqslant |\mathcal{B}|$.*

Proof. Under the assumptions, $\{\langle A, H \setminus A \rangle : A \subseteq H\} \subseteq \mathcal{A}$. □

As usual, let $\mathrm{RO}(X)$ denote the family of all *regular open sets* of a topological space X – *i.e.*, the open sets O satisfying $O = int\overline{O}$. Recall that, if D is a dense subset of X, then $\overline{V} = \overline{V \cap D}$ for any open set V, and therefore there are at most $2^{|D|}$ sets of the form the closure of an open set, and similarly for sets of the form the interior of a closed set. This gives us the well-known inequality $|\mathrm{RO}(X)| \leqslant 2^{d(X)}$ for every topological space X.

The following notion of semi-normal spaces generalizes normal spaces analogously to the way that semi-regular spaces generalize regular spaces. Recall that a topological space is *semi-regular* if every point has a base of regular open sets.

Definition (2.4) ([23]). A topological space X is *semi-normal* if $\mathrm{RO}(X)$ separates the family of all pairs of disjoint closed subsets of X.

Indeed, spaces satisfying the above definition are those in which every closed set has a base of regular open sets.

For a semi-normal space X, $\mathrm{RO}(X)$ is a separating family, of size not larger than $2^{d(X)}$, for the family $\mathcal{A}$ of all disjoint closed subsets of X. So, applying Corollary 2.3 for $\mathcal{B} = \mathrm{RO}(X)$, we establish the following version of Jones' Lemma for semi-normal spaces:

THEOREM (2.5). *If X is a semi-normal topological space and H is a closed discrete subset of X then $2^{|H|} \leqslant 2^{d(X)}$, hence $|H| < 2^{d(X)}$. Moreover, if $2^{d(X)} < 2^{d(X)^+}$ then $|H| \leqslant d(X)$, and thus $e(X) \leqslant d(X)$.*

Proof. Indeed, Corollary 2.3 gives us $2^{|H|} \leqslant |\mathcal{B}| = |\mathrm{RO}(X)| \leqslant 2^{d(X)}$. The remaining claims follow by elementary cardinal arithmetic. $\square$

In particular, semi-normal separable spaces behave just like normal separable spaces when it comes to comparing extent and density.

3. On countably semi-paracompact spaces

Throughout this section, μ is always supposed to be an infinite cardinal.

M. E. Rudin noted some time ago that one of the usual equivalents of countable paracompactness (due to Ichikawa [8], see [5], Theorem 5.2.1) is more appropriate for generalization than the formulation in terms of locally-finite refinements of open coverings. She defines μ-paracompact spaces, integral to her definition of μ-Dowker spaces, as follows.

Definition (3.1) ([18], §4.3). A space X is *μ-paracompact* if for every decreasing sequence of closed sets $\langle C_\alpha : \alpha < \mu \rangle$ such that $\bigcap_{\alpha<\mu} C_\alpha = \emptyset$ there is a decreasing sequence of open sets $\langle A_\alpha : \alpha < \mu \rangle$ such that $C_\alpha \subseteq A_\alpha$ for all $\alpha < \mu$ and $\bigcap_{\alpha<\mu} \overline{A_\alpha} = \emptyset$.

Weakenings of the notion of an "open" set have frequently been considered. One can consult [17] for a survey up to the early 2000s, and there has been a noticeable spurt of work in the area since then. Among these several "weak versions of open sets", we call the reader's atention to Levine's notion of *semi-open sets* (1963), which will be very useful here.

Definition (3.2) ([11]). Let X be a topological space. A set $A \subseteq X$ is *semi-open* if $A \subseteq \overline{intA}$. $C \subseteq X$ is *semi-closed* if its complement is semi-open.

We remarked above that, for any topological space X, $\overline{O} = \overline{V \cap D}$ whenever O is an open set and D is a dense set. Levine's notion gives us characterizations for this kind of phenomena: he proved that U is a semi-open set if, and only if, $\overline{U} = \overline{U \cap D}$ for all dense sets D and D is a dense set if, and only if, $\overline{U} = \overline{U \cap D}$ for all semi-open sets U (see [12]).

For the sake of completeness, we give a proposition with some basic facts on semi-open sets.

PROPOSITION (3.3). *Let X be topological space. The following statements hold:*
(1) *$A \subseteq X$ is semi-open if, and only if, there is an open set O such that $O \subseteq A \subseteq \overline{O}$.*
(2) *The intersection of an open set with a semi-open set is semi-open.*
(3) *If $\{A_i : i \in I\}$ is a family of semi-open subsets of X, then $\bigcup_{i \in I} A_i$ is also semi-open.*

Before the proof we remark that, regarding (2), it is easy to see that the intersections of a pair of semi-open sets need not to be a semi-open set.

Proof. (1): Notice that, if A is a semi-open set, one has just to take $O = int(A)$ and we get the "only if" part. Conversely, if there is an open set O as in the statement then $O \subseteq int(A) \subseteq A \subseteq \overline{A} \subseteq \overline{O}$ and therefore $\overline{int(A)} = \overline{A}$, thus the desired holds.

(2): Let U be open and A be semi-open. With respect to A, consider an open set O as in (1). We have

$$U \cap O \subseteq U \cap A \subseteq U \cap \overline{O} \subseteq \overline{U \cap O},$$

the last inclusion valid because $U \cap \overline{Y} \subseteq \overline{U \cap Y}$ for any set Y, provided U is open. Then the open set $U \cap O$ testifies that $U \cap A$ is semi-open, by (1).

(3): We choose an open set O_i for each A_i, under the conditions of item (1). It follows that $\bigcup_{i \in I} O_i \subseteq \bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} \overline{O_i} \subseteq \overline{\bigcup_{i \in I} O_i}$, the latter since $\bigcup_{i \in I} \overline{B_i} \subseteq \overline{\bigcup_{i \in I} B_i}$ for any family of B_i's. Hence $\bigcup_{i \in I} O_i$ witnesses that $\bigcup_{i \in I} A_i$ is semi-open. $\square$

In the last 25 years, many generalized topological properties, stated in terms of weakenings of the notion of open set, have been introduced and investigated. Works in this line of research have been written by Di Maio, Noiri, Dontchev and Ganster, amongst others; we refer to Noiri's survey [17] for more details and references. Accordingly, given the theme of this paper, we generalize Rudin's definition as follows.

Definition (3.4). A space X is *μ-semi-paracompact* if for every decreasing sequence of closed sets $\langle C_\alpha : \alpha < \mu \rangle$ such that $\bigcap_{\alpha < \mu} C_\alpha = \emptyset$ there is a decreasing sequence of semi-open sets $\langle A_\alpha : \alpha < \mu \rangle$ such that $C_\alpha \subseteq A_\alpha$ for all $\alpha < \mu$ and $\bigcap_{\alpha < \mu} \overline{A_\alpha} = \emptyset$. If $\mu = \omega$ we say X is *countably semi-paracompact*.

Equivalently: for every increasing open cover $\langle O_\alpha : \alpha < \mu \rangle$ there is an increasing sequence of semi-closed sets $\langle F_\alpha : \alpha < \mu \rangle$ with $F_\alpha \subseteq O_\alpha$ for each $\alpha < \mu$ and such that $\bigcup_{\alpha < \mu} intF_\alpha = X$.

For the case $\mu = \omega$ we have equivalences mirroring the usual ones for countably paracompact spaces, as we will see in the following result. The statement corresponding to (ii) in the following proposition was taken in [1] as the definition of countably S-paracompact.

PROPOSITION (3.5). *Let X be a topological space. The following statements are equivalent:*

(i) *X is countably semi-paracompact.*

(ii) *Every countable open cover of X has a locally finite semi-open refinement.*

(iii) *Every open cover $\mathcal{U} = \{U_i : i < \omega\}$ of X has a locally finite semi-open refinement $\mathcal{V} = \{V_i : i < \omega\}$ with $V_i \subseteq U_i$ for each $i < \omega$.*

Proof. First we prove the equivalence between (ii) and (iii). It is clear that (iii) implies (ii). On the other hand, if $\mathcal{W}$ is any locally finite semi-open refinement of $\mathcal{U}$, consider for every $W \in \mathcal{W}$ the natural number $i(W) = \min\{i < \omega : W \subseteq U_i\}$ and let $V_i = \bigcup\{W \in \mathcal{W} : i(W) = i\}$. $\mathcal{V} = \langle V_i : i < \omega\rangle$ is a locally finite refinement of $\mathcal{W}$ and the V_i's are semi-open by (3) of Proposition 3.3.

We proceed by proving that (iii) implies (i). For this, suppose $\langle C_i : i < \omega\rangle$ is a decreasing sequence of closed subsets of X satisfying $\bigcap_{i<\omega} C_i = \emptyset$. By (iii), let $\mathcal{V} = \langle V_i : i < \omega\rangle$ be a locally finite semi-open cover such that $V_i \subseteq X \setminus C_i$ for all $i < \omega$. Let $A_i = \bigcup_{j>i} V_j$ for $i < \omega$. Then each A_i is semi-open (again by (3) of Proposition 3.3), the family of the A_i's is decreasing and, as $\bigcup_{j\leqslant i} V_j \subseteq \bigcup_{j\leqslant i} X \setminus C_j = X \setminus C_i$, we have

$$C_i \subseteq X \setminus \bigcup_{j\leqslant i} V_j \subseteq A_i$$

for each $i < \omega$; and $\bigcap_{i<\omega} \overline{A_i} = \emptyset$ holds as, by the local finiteness of $\mathcal{V}$, each $x \in X$ has a neighbourhood contained in some $X \setminus A_k$.

Finally, we prove that (i) implies (ii). As already remarked in the definition, (i) is equivalent to the following statement:

(*) For every increasing open cover $\langle O_i : i < \omega\rangle$ there is an increasing sequence of semi-closed sets $\langle F_i : i < \omega\rangle$ with $F_i \subseteq O_i$ for each $i < \omega$ and such that $\bigcup_{i<\omega} int(F_i) = X$.

To see that $(*) \Rightarrow (ii)$, let $\mathcal{U} = \langle U_i : i < \omega\rangle$ be any countable open cover of X and consider the increasing open cover $\langle O_i : i < \omega\rangle$ defined by putting $O_i = \bigcup_{j\leqslant i} U_j$ for all $i < \omega$. Consider the increasing sequence of semi-closed sets $\langle F_i : i < \omega\rangle$ given by (*). For every $i < \omega$, let $V_i = U_i \setminus \bigcup_{j<i} F_j$. As the sequence of the F_i's is increasing, each one of the V_i's is semi-open, by (2) of Proposition 3.3 – since they all can be written as the intersection of an open set with a semi-open set. Notice that, for every $i < \omega$,

$$\bigcup_{j<i} F_j \subseteq \bigcup_{j<i} O_j \subseteq \bigcup_{j<i} U_j$$

and therefore $U_i \setminus \bigcup_{j<i} U_j \subseteq U_i \setminus \bigcup_{j<i} F_j = V_i$, and this clearly implies that the family $\mathcal{V} = \langle V_i : i < \omega\rangle$ is a cover of X (thus a semi-open refinement of $\mathcal{U}$). What is left to prove is the local finiteness of $\mathcal{V}$. But, as $X = \bigcup_{i<\omega} int(F_i)$, every $x \in X$ has an open neighbourhood of the form $int(F_k)$ for some $k < \omega$, and this neighbourhood is disjoint from the semi-open set V_j for $j \geqslant k$ – and this completes the proof. $\square$

For uncountable cardinals μ, it is easy to adapt the proof of implication $(ii) \Rightarrow (i)$ of the preceding proposition and show that if every open cover of X with size not larger than μ has a semi-open refinement which is locally less than μ, then X is μ-semi-paracompact. However, the proof of the converse does not similarly admit an easy adaptation, and indeed it is not clear to the authors if such a converse holds without some additional assumption on the nature of the space. This seems to indicate that our notion of μ-semi-paracompactness is preferable to the obvious alternative formulations.

Now we present the main theorem of this section.

THEOREM (3.6). *Let κ be an uncountable cardinal. If X is a countably semi-paracompact separable space and $H \subseteq X$ is a closed discrete subset of size κ then there is a dominating family of size at most 2^ω in ${}^\kappa\omega$.*

Proof. Let X be countably semi-paracompact, ω a dense set in X and $\kappa \setminus \omega$ a closed discrete subset of X. Let $\{G_\alpha : \alpha < \lambda\}$, for some $\lambda \leqslant \mathfrak{c}$, be an enumeration of all decreasing sequences of subsets of ω, say $G_\alpha = \langle G_{\alpha,n} : n < \omega\rangle$, which satisfy $\bigcap_{n<\omega} \overline{G_{\alpha,n}} = \emptyset$.[3]

For each $\alpha < \lambda$ define a function $f_\alpha : (\kappa \setminus \omega) \to \omega$ such that, for every $\beta \in \kappa \setminus \omega$,

$$f_\alpha(\beta) = \begin{cases} \max(\{n : \beta \in \overline{G_{\alpha,n}}\}) & \text{if } \{n : \beta \in \overline{G_{\alpha,n}}\} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{F} = \{f_\alpha : \alpha < \lambda\}$. We claim that $\mathcal{F}$ is a dominating family in ${}^{\kappa\setminus\omega}\omega$ – and this clearly suffices for us.

Indeed: let $g \in {}^{\kappa\setminus\omega}\omega$ be arbitrary. For every $n < \omega$ let $H_n = g^{-1}``\{n\}$ and let $C_n = \bigcup_{m\geqslant n} H_m$. As $\kappa \setminus \omega$ is closed and discrete and $\{H_n : n < \omega\}$ is a partition of $\kappa \setminus \omega$, $\mathcal{C} = \langle C_n : n < \omega\rangle$ is an decreasing sequence of closed subsets of X with empty intersection.

By countable semi-paracompactness, let $\mathcal{A} = \langle A_n : n < \omega\rangle$ be a decreasing sequence of semi-open sets with $C_n \subseteq A_n$ for each $n < \omega$ such that $\cap_{n<\omega}\overline{A_n} = \emptyset$. We have $\overline{A_n} = \overline{A_n \cap \omega}$ for each $n < \omega$, since A_n is semi-open (by Levine's characterization, [12]). So, there is some $\xi < \lambda$ such that $\langle A_n \cap \omega : n < \omega\rangle = \langle G_{\xi,n} : n < \omega\rangle$.

It follows that g is dominated by f_ξ, because if $\beta \in \kappa \setminus \omega$ and $m = g(\beta)$ then $\beta \in H_m \subseteq C_m \subseteq A_m \subseteq \overline{A_m} = \overline{A_m \cap \omega} = \overline{G_{\xi,m}}$ and therefore $g(\beta) = m \leqslant \max\{n : \beta \in \overline{G_{\xi,n}}\} = f_\xi(\beta)$. □

We proceed by presenting some immediate consequences of the preceding theorem.

In the view of the results of Jech and Prikry on small dominating families in ${}^{\omega_1}\omega$ already remarked (see Section 1), we have the following:

COROLLARY (3.7). *Under "$cf(2^\omega) = 2^\omega < 2^{\omega_1}$" and "There are no inner models with measurable cardinals", countably semi-paracompact separable spaces have countable extent.*

We next present a statement which is a strengthening of Fleissner's result on the size of the closed discrete subsets of countably paracompact spaces ([6]). Our argument is purely combinatorial: by a standard diagonal argument, families of size κ in $\langle {}^\kappa\omega, \leqslant\rangle$ are not dominating – in particular, there are no dominating families of size $\mathfrak{c}$ in ${}^\mathfrak{c}\omega$. So, Theorem 3.6 gives us the following:

COROLLARY (3.8). *A countably semi-paracompact separable space cannot include a closed discrete subset of size $\mathfrak{c}$.*

The following result is a corollary to the proof of Theorem 3.6. We emphasise that, in light of the discussion immediately before the statement of the theorem, the hypothesis is, as far as our knowledge goes and in comparision with all previous results of this type, the weakest presented in the literature.

[3] Note that these are precisely the decreasing locally finite sequences of subsets of ω.

COROLLARY (3.9). *If X is a μ-semi-paracompact space for any $\mu \leqslant d(X)$ and $H \subseteq X$ is a closed discrete set of size $\kappa > d(X)$ then there is a dominating family of size at most $2^{d(X)}$ in ${}^{\kappa}\mu$.*

Proof. Immediate. □

By the above mentioned diagonal argument on dominating families, one can similarly easily deduce from this last corollary that a space X which is μ-semi-paracompact, for $\mu \leqslant d(X)$, cannot include a closed discrete subset of size $2^{d(X)}$.

This makes plain that, even in this very general context, we have to deal with constraints on the extents of these spaces – and that these constraints are stated in terms of their densities.

Defining the notion of *relative countable semi-paracompactness* in the natural way,[4] it is straightforward to check that obvious adaptations of what we have just done also give:

COROLLARY (3.10). *The existence of a separable space with an uncountable closed discrete subset which is also relatively countably semi-paracompact implies the existence of a small dominating family.*

COROLLARY (3.11). *Separable spaces cannot include closed discrete subsets which have size $\mathfrak{c}$ and are countably semi-paracompact in them.*

4. Obstructions: on what cannot be extended for separable, countably metacompact spaces

Countably paracompact spaces have constraints on their extents in terms of their densities, in the separable case – and, moreover, we have just proved that analogous constraints hold for certain generalizations of countable paracompactness. It is very natural to ask: what about countably metacompact, separable spaces? In what follows we show that, even supposing additional properties for the space, there are easy examples of countably metacompact, separable spaces for which extent is not constrained by density.

There are additional properties which it is very natural to consider in this case. Let us review some of these properties. It is well known that countably metacompact normal spaces are countably paracompact (see e.g. [5], 5.2.6), and therefore one should investigate what happens when one adds hypotheses weaker than normality – for instance, *pseudonormality*.

A topological space X is *pseudonormal* if countable closed sets have arbitrarily small closed neighbourhoods, or, equivalently, disjoint closed sets are separated by disjoint open sets whenever at least one of the closed sets is countable.

Let us turn in another direction. A topological property which is commonly considered together with countable metacompactness is *orthocompactness*. Indeed, in the literature the relationship between countable metacompactness and orthocompactness is usually compared with the relationship between countable paracompactness and normality, because of obvious similarities found in several results involving finite topological products (see, e.g., [19]).

[4] $Y \subseteq X$ is *countably semi-paracompact in X* if for every decreasing sequence $\langle C_i : i < \omega \rangle$ of closed subsets of X with $\bigcap_{i<\omega} C_i = \emptyset$ there is a decreasing sequence of semi-open sets $\langle A_i : i < \omega \rangle$, such that $C_i \subseteq A_i$ for all $i < \omega$ and $\bigcap_{i<\omega} \overline{A_i} \cap Y = \emptyset$.

A topological space X is *orthocompact* if every open cover has an open refinement which is interior preserving, i.e., the intersection of any subfamily of the open refinement is an open set.

In what follows, we show that considering both topological properties together with countable metacompactness (even simultaneously), one may exhibit easy examples for which density does not constrain extent – either absolute examples or, at least, consistent ones.

PROPOSITION (4.1). *The following statement is a theorem of ZFC:*

"There is a countably metacompact, orthocompact, separable space with a closed discrete subset of size $\mathfrak{c}$"

The following statement is consistent with ZFC + $2^{\omega} < 2^{\omega_1}$:

"There is a countably metacompact, orthocompact, pseudonormal, separable space with uncountable extent"

Proof. Here we use Isbell-Mrówka spaces $\Psi(\mathcal{A})$, constructed from almost disjoint families of subsets of ω. Every such space is separable, countably metacompact (see [20] for a proof) and also orthocompact: every space $\Psi(\mathcal{A})$ has an open refinement on which every $A \in \mathcal{A}$ is covered by only one open set of the form $\{A\} \cup (A \setminus n_A)$, where $n_A < \omega$. The intersection of any subfamily of this refinement is a subset of ω and therefore is open in $\Psi(\mathcal{A})$. So, any almost disjoint family of size $\mathfrak{c}$ shows that the first statement is a theorem of ZFC.

For the second statement, we use the fact that any almost disjoint family of size less than $\mathfrak{b}$ is pseudonormal (see [20] or [4] for more details), and therefore it suffices to exhibit a model of ZFC on which $\mathfrak{b} > \omega_1$ and $2^{\omega} < 2^{\omega_1}$: in such a model, any almost disjoint family of size ω_1 will give us the desired consistency.

In order to get a model satisfying these requirements, we can, e.g., use a very general theorem due to Cummings and Shelah ([3], Theorem 2), which asserts, as a particular case, that the cardinals in the triple $\langle \mathfrak{b}, \mathfrak{d}, \mathfrak{c} \rangle$ may assume any "reasonable" values. More precisely: suppose that θ, κ and μ are cardinals of a model of GCH satisfying $\omega_1 \leqslant \theta = cf(\theta) \leqslant cf(\kappa)$, $\kappa \leqslant \mu$ and $cf(\mu) > \omega$. Then there is a forcing such that in the generic extension we have $\mathfrak{b} = \theta$, $\mathfrak{d} = \kappa$ and $\mathfrak{c} = \mu$. Applying this theorem for $\theta = \kappa = \omega_2 = \aleph_2$ and $\mu = \aleph_{\omega_1}$, then in the extension we have $\mathfrak{b} = \mathfrak{d} = \omega_2$ and $2^{\omega} = \aleph_{\omega_1}$, and it follows that $2^{\omega} < 2^{\omega_1}$ also holds in the extension (for if $2^{\omega} = 2^{\omega_1}$ in the extension then $\omega_1 = cf(\aleph_{\omega_1}) = cf(2^{\omega_1})$ – but this contradicts König's theorem, which states that $\lambda < cf(2^{\lambda})$ for every infinite cardinal λ). □

5. Notes and Questions

First of all, we would like to call the reader's attention to the following subtlety: there are models of $2^{\omega} = 2^{\omega_1}$ in which it is easy to find examples of separable, normal, countably paracompact (a)-spaces for which extent *is not* constrained by density. It suffices to get a model of $\omega_1 < \mathfrak{p}$ and, taking a subset Y of $\mathbb{R}$ of size ω_1 in this model, consider the well-known space $M(Y)$, the Moore space derived from

Y. If $|Y| = \omega_1 < \mathfrak{p}$, then $2^\omega = 2^{\omega_1}$ and $M(Y)$ is a separable, normal, countably paracompact space with uncountable extent (see [20] for more details[5]). Therefore, it is reasonable to suppose that the general problem of finding topological and set-theoretical conditions implying $e(X) \leqslant d(X)$ should be investigated in models of $2^\omega < 2^{\omega_1}$. This motivates and justifies our interest in the consistency of the second statement of Proposition 4.1.

We also remark that, by previous results of the authors (see [15]), topological (or any kind of) properties that imply the existence of small dominating families cannot hold under a certain weak parametrized diamond principle, $\Phi(\omega,<)$ (one of the family of such principles introduced by Moore, Hrušák and Džamonja in [13]). In view of Theorem 3.6, we have the following:

THEOREM (5.1). $\Phi(\omega,<)$ *implies countable extent for separable countably semi-paracompact spaces.*

There are some questions and problems which are naturally suggested by our work. In particular, there are questions which may be regarded as generalizations of questions formerly presented by the authors. For instance, on the one hand, in the view of the preceding theorem – since all weak parametrized diamond principles similar to $\Phi(\omega,<)$ imply $2^\omega < 2^{\omega_1}$, see [13], it is quite natural to ask the following question which, on the other hand, generalizes a question asked by the authors in [15].

QUESTION (5.2). *Does* $2^\omega < 2^{\omega_1}$ *alone imply countable extent for separable countably semi-paracompact spaces?*

Despite the obstructions presented in Section 4, the authors are still very interested in investigating the borders between countable paracompactness and countable metacompactness – specifically, with regard to the general problem under consideration: when does density constrain extent ?

PROBLEM (5.3). *Find topological properties* $\mathcal{P}$ *such that countably metacompact spaces which satisfy* $\mathcal{P}$ *have their extents constrained in terms of their densities. Obviously, we are interested in properties* $\mathcal{P}$ *which neither imply countable paracompactness when combined with countable metacompactness, nor are satisfied by every Isbell-Mrówka space.*

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[5]In fact, if $|Y| < \mathfrak{p}$ then $M(Y)$ also satisfies a related topological property: Matveev's *property* (a). In previous works of the authors, this property was extensively studied, side-by-side with normality and countable paracompactness (see [14], [15], [16], [20] and [22] for results and references).

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REFERENCES

[1] K. AL-ZOUBI, *S-paracompact spaces*, Acta Math. Hungar., **110**, 1-2 (2006), 165-174.
[2] W. W. COMFORT, *Cofinal families in certain function spaces*, Comment. Math. Univ. Carolin., **29**, 4 (1988), 665-675.
[3] J. CUMMINGS, AND S. SHELAH, *Cardinal invariants above the continuum*, Ann. Pure Appl. Logic, **75**, 3 (1995), 251-268.
[4] E. K. VAN DOUWEN, The integers and topology, in: Kunen, K. and Vaughan, J. E., (eds.), *Handbook of Set Theoretic Topology* North Holland, Amsterdam, 1984, 111-167.
[5] R. ENGELKING, *General topology*. Helderman, Berlin, 1989.
[6] W. FLEISSNER, *Separation properties in Moore spaces*, Fundamenta Mathematicae **98**, (1978), 279-286.
[7] R. HODEL, *Cardinal functions, I*. In: Kunen, K. and Vaughan, J.E., (eds.), *Handbook of Set Theoretic Topology* North Holland, Amsterdam, 1984, 1-61.
[8] F. ICHIKAWA, *On countably paracompact spaces*, Proc. Japan Acad. **31**, (1954), 686-687.
[9] T. JECH, K. PRIKRY, *Cofinality of the partial ordering of functions from ω_1 into ω under eventual domination*, Mathematical Proceedings of the Cambridge Philosophical Society, **95**, (1984), 25-32.
[10] F. B. JONES, *Concerning normal and completely normal spaces*, Bulletin of the American Mathematical Society **43**, (1937), 671-677.
[11] N. LEVINE, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math Monthly **70**, 1 (1963), 36-41.
[12] N. LEVINE, *Some remarks on the closure operator in topological spaces*, Amer. Math Monthly **70**, 5 (1963), 553.
[13] J. T. MOORE, M. HRUŠÁK, AND M. DŽAMONJA, *Parametrized $\diamondsuit$ principles*. Trans. Amer. Math. Soc. **356**, (2004), 2281-2306.
[14] C. J. G. MORGAN, S. G. DA SILVA, *Almost disjoint families and "never" cardinal invariants*, Comment. Math. Univ. Carolin., **50**, 3 (2009), 433-444.
[15] C. J. G. MORGAN, S. G. DA SILVA, *Covering properties which, under weak diamond principles, constrain the extents of separable spaces*, Acta Mathematica Hungarica, **128**, 4 (2010), 358-368.
[16] C. J. G. MORGAN, S. G. DA SILVA, *A note on closed discrete subsets of separable (a)-spaces*, to appear in Houston Journal of Mathematics, 2012.
[17] T. NOIRI, *a-2: Modified open and closed sets (semi-open set etc.)*, in: Hart, K. P., Nagata, J., and Vaughan, J. E., (eds.), Encyclopedia of General Topology, Elsevier, Amsterdam, 2004, 8-10.
[18] M. E. RUDIN, *Dowker spaces*, in: Kunen, K. and Vaughan, J.E., (eds.), Handbook of Set Theoretic Topology North Holland, Amsterdam, 1984, 761-780.
[19] B. M. SCOTT, Towards a product theory for orthocompactness, in: *Studies in Topology*, Stavrakas N. M. and Allen K. R. (eds.), Academic Press, New York, 1975, 517-537.
[20] S. G. DA SILVA, *On the presence of countable paracompactness, normality and property (a) in spaces from almost disjoint families*, Questions & Answers Gen. Topology **25**, 1 (2007), 1-18.
[21] S. G. DA SILVA, *Large cardinals and topology: a short retrospective and some new results*, Logic Journal of the IGPL, **15**, 5-6 (2007), 433-443.

[22] S. G. DA SILVA, *Closed discrete subsets of separable spaces and relative versions of normality, countable paracompactness and property* (a), Commentationes Mathematicae Universitatis Carolinae, **52**, 3 (2011), 435–444.

[23] G. VIGLINO, *Seminormal and C-compact spaces*. Duke Math. Journal **38**, 1 (1971), 57-61.

[24] W. S. WATSON, *Separation in countably paracompact spaces*, Trans. Amer. Math. Soc., **290**, (1985), 831-842.

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RIGID GEMS IN DIMENSION N

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ABSTRACT. We extend to dimension $n \geq 3$ the concept of ρ-pair in a coloured graph and we prove the existence theorem for minimal rigid crystallizations of handle-free, closed n-manifolds.

1. Introduction

The concept of *ρ-pair* in a 4-coloured graph was introduced for the first time by Sostenes Lins in [12]. Roughly speaking, it consists of two equally coloured edges, which belong to two or three bicoloured cycles. A graph with no ρ-pairs was then called *rigid* in the same paper, where the following basic result was proved:

Every handle-free, closed 3-manifold admits a rigid crystallization of minimal order.

The proof is based on the definition of a particular move, called *switching* of a ρ-pair. Starting from any gem Γ of a closed, irreducible 3-manifold M, a finite sequence of such moves, together with the cancelling of a suitable number of 1-dipoles, produces a rigid crystallization Γ' of the same manifold M, whose order is strictly less than the order of Γ.

The above existence theorem plays a fundamental rôle in the problem of generating automatically essential catalogues of 3-manifolds, with "small" Heegaard genus and/or graph order (see, e.g., [12], [3], [5], [4], [14]).

In the present paper, we extend the concepts of ρ-pair, switching and rigidity to $(n+1)$-coloured graphs, for $n > 3$.

Our main result is the proof of the existence of a rigid crystallization of minimal order, for every handle-free n-dimensional, closed manifold. It will be used in a subsequent paper to generate the catalogue of all 4-dimensional, closed manifolds, represented by (rigid) crystallizations of "small" order.

2. Notations

In the following all manifolds will be piecewise linear (PL), closed and, when not otherwise stated, connected. For the basic notions of PL topology, we refer to [17] and to [8]; "$\cong$" will mean "*PL-homeomorphic*". For graph theory, see [9] and [18].

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We will use the term "graph" instead of "multigraph". Hence multiple edges are allowed, but loops are forbidden. As usual, $V(\Gamma)$ and $E(\Gamma)$ will denote the vertex-set and the edge-set of the graph Γ.

If Γ is an oriented graph, then each edge $\mathbf{e}$ is directed from its first endpoint $\mathbf{e}(0)$ (also called *tail*) to its second endpoint $\mathbf{e}(1)$ (called *head*).

An $(n+1)$-*coloured graph* is a pair (Γ,γ), where Γ is a graph, regular of degree $n+1$, and $\gamma : E(\Gamma) \to \Delta_n = \{0,\dots,n\}$ is a map with the property that, if $\mathbf{e}$ and $\mathbf{f}$ are adjacent edges of $E(\Gamma)$, then $\gamma(\mathbf{e}) \neq \gamma(\mathbf{f})$. We shall often write Γ instead of (Γ,γ).

Let B be a subset of Δ_n. Then, the connected components of the graph $\Gamma_B = (V(\Gamma),\gamma^{-1}(B))$ are called B-*residues* of (Γ,γ). Moreover, for each $c \in \Delta_n$, we set $\hat{c} = \Delta_n \setminus \{c\}$. If B is a subset of Δ_n, we define g_B to be the number of B-residues of Γ; in particular, given any colour $c \in \Delta_n$, $g_{\hat{c}}$ denotes the number of components of the graph $\Gamma_{\hat{c}}$, obtained by deleting all edges coloured c from Γ. If $i,j \in \Delta_n, i \neq j$, then g_{ij} denotes the number of cycles of Γ, alternatively coloured i and j, i.e. $g_{ij} = g_{\{i,j\}}$.

An isomorphism $\phi : \Gamma \to \Gamma'$ is called a *coloured isomorphism* between the $(n+1)$-coloured graphs (Γ,γ) and (Γ',γ') if there exists a permutation φ of Δ_n such that $\varphi \circ \gamma = \gamma' \circ \phi$.

A pseudocomplex K of dimension n [11] with a labelling on its vertices by $\Delta_n = \{0,\dots,n\}$, which is injective on the vertex-set of each simplex of K is called a *coloured n-complex* .

It is easy to associate a coloured n-complex $K(\Gamma)$ to each $(n+1)$-coloured graph Γ, as follows:

- for each vertex $\mathbf{v}$ of Γ, take an n-simplex $\sigma(\mathbf{v})$ and label its vertices by Δ_n;
- if $\mathbf{v}$ and $\mathbf{w}$ are vertices of Γ joined by a c-coloured edge ($c \in \Delta_n$), then identify the $(n-1)$-faces of $\sigma(\mathbf{v})$ and $\sigma(\mathbf{w})$ opposite to the vertices labelled c.

If M is a manifold of dimension n and Γ an $(n+1)$-coloured graph such that $|K(\Gamma)| \cong M$, then, following Lins [12], we say that Γ is a *gem* (*graph-encoded-manifold*) *representing* M.

Note that Γ *is a gem of an n-manifold M iff, for every colour* $c \in \Delta_n$, *each* $\hat{c}$-*residue represents* $\mathbb{S}^{n-1}$. Moreover, M *is orientable iff* Γ *is bipartite.*

If, for each $c \in \Delta_n$, $\Gamma_{\hat{c}}$ is connected , then the corresponding coloured complex $K(\Gamma)$ has exactly $(n+1)$ vertices (one for each colour $c \in \Delta_n$); in this case both Γ and $K(\Gamma)$ are called *contracted*. A contracted gem Γ, representing an n-manifold M, is called a *crystallization* of M.

The *existence theorem* of crystallizations for every n-manifold M was proved by Pezzana [15], [16]. Surveys on crystallizations theory can be found in [7], [2].

Let $\mathbf{x},\mathbf{y}$ be two vertices of an $(n+1)$-coloured graph Γ joined by k edges $\{\mathbf{e}_1,\dots,\mathbf{e}_k\}$ with $\gamma(\mathbf{e}_h) = i_h$, for $h = 1,\dots,k$. We call $\Theta = \{\mathbf{x},\mathbf{y}\}$ a *dipole of type* k, *involving colours* $i_1,\dots,i_k$, iff $\mathbf{x}$ and $\mathbf{y}$ belong to different $(\Delta_n \setminus \{i_1,\dots,i_k\})$-residues of Γ.

In this case a new $(n+1)$-coloured graph Γ' can be obtained by deleting $\mathbf{x},\mathbf{y}$ and all their incident edges from Γ and then joining, for each $i \in \Delta_n \setminus \{i_1,\dots,i_k\}$, the vertex i-adjacent to $\mathbf{x}$ to the vertex i-adjacent to $\mathbf{y}$. Γ' is said to be obtained from Γ by *cancelling* (or *deleting*) *the k-dipole* Θ. Conversely Γ is said to be obtained from Γ' by *adding the k-dipole* Θ.

By a *dipole move*, we mean either the *adding* or the *cancelling* of a dipole from a gem Γ.

As proved in [6], *two gems Γ and Γ' represent PL-homeomorphic manifolds iff they can be obtained from each other by a finite sequence of dipole moves.*

An n-dipole $\Theta = (\mathbf{x},\mathbf{y})$ is often called a *blob* (see [13], where a different calculus for gems is introduced). If c is the (only) colour not involved in the blob Θ, and $\mathbf{x}',\mathbf{y}'$ are the vertices c-adjacent to $\mathbf{x}$ and $\mathbf{y}$ respectively, then the cancelling of Θ from Γ produces (in Γ') a new c-coloured edge $\mathbf{e}'$, joining $\mathbf{x}'$ with $\mathbf{y}'$. Following Lins, we call the inverse procedure the *adding of a blob on the edge* $\mathbf{e}'$.

Two vertices $\mathbf{x},\mathbf{y}$ of an $(n+1)$-coloured graph Γ are called *completely separated* if, for each colour $c \in \Delta_n$, $\mathbf{x}$ and $\mathbf{y}$ belong to two different $\hat{c}$-residues. The *fusion graph* $\Gamma\mathrm{fus}(\mathbf{x},\mathbf{y})$ is obtained simply by deleting $\mathbf{x}$ and $\mathbf{y}$ from Γ and then by gluing together the "hanging edges" with the same colours.

The following result was first proved, for case (a), in [12] and, for case (b), in [12] ($n=3$) and in [10].

LEMMA (2.1). *Let $\mathbf{x},\mathbf{y}$ be two completely separated vertices of a (possibly disconnected) graph Γ.*

(a) *If $\mathbf{x}$ and $\mathbf{y}$ belong to the (only) two different components Γ' and Γ'' of Γ, representing two n-dimensional manifolds M' and M'' respectively, then $\Gamma fus(\mathbf{x},\mathbf{y})$ is a gem of a connected sum $M'\#M''$.*

(b) *If Γ is a gem of a (connected) n-manifold M, then $\Gamma fus(\mathbf{x},\mathbf{y})$ is a gem of $M\#\mathbb{H}$, where $\mathbb{H}$ is either $\mathbb{S}^{n-1}\times\mathbb{S}^1$ or $\mathbb{S}^{n-1}\tilde{\times}\mathbb{S}^1$ (i. e. the orientable or non-orientable $(n-1)$-sphere bundle over $\mathbb{S}^1$).*

Note that such a manifold $\mathbb{H}$ is often called a *handle* (of dimension n). A manifold M is called *handle-free* if it admits no handles as connected summands (i.e. if M is not homeomorphic to $M'\#\mathbb{H}, M'$ being any n-manifold).

3. Switching of ρ-pairs

Let (Γ,γ) be an $(n+1)$-coloured graph. Let further $(\mathbf{e},\mathbf{f})$ be any pair of edges, both coloured c, of Γ.

If we delete $\mathbf{e},\mathbf{f}$ from Γ, we obtain an edge-coloured graph $\overline{\Gamma}$, with exactly four vertices of degree n (namely, the endpoints $\mathbf{u},\mathbf{v}$ of $\mathbf{e}$ and the endpoints $\mathbf{w},\mathbf{z}$ of $\mathbf{f}$).

Now, there are exactly two $(n+1)$-coloured graphs $(\Gamma_1,\gamma_1),(\Gamma_2,\gamma_2)$ (different from (Γ,γ)) which can be obtained by adding two new edges (both coloured c) to $\overline{\Gamma}$: such edges are either $\mathbf{e}_1,\mathbf{f}_1$, joining $\mathbf{u}$ with $\mathbf{w}$ and $\mathbf{v}$ with $\mathbf{z}$ respectively, or $\mathbf{e}_2,\mathbf{f}_2$, joining $\mathbf{u}$ with $\mathbf{z}$ and $\mathbf{v}$ with $\mathbf{w}$ respectively. (See Figure 1, Figure 1a and Figure 1b, where, without loss of generality, we consider $c=0$)

We will say that (Γ_1,γ_1) and (Γ_2,γ_2) are obtained from (Γ,γ) by a *switching* on the pair $(\mathbf{e},\mathbf{f})$.

Actually, we are interested in particular pair of equally coloured edges of Γ. More precisely:

Definition (3.1). A pair $R = (\mathbf{e},\mathbf{f})$ of edges of Γ, with $\gamma(\mathbf{e}) = \gamma(\mathbf{f}) = c$, will be called:

(a) a ρ_n-pair involving colour c if, for each colour $i \in \Delta_n \setminus \{c\}$, we have $\Gamma_{\{c,i\}}(\mathbf{e}) = \Gamma_{\{c,i\}}(\mathbf{f})$;

(b) a ρ_{n-1}-pair $(n>1)$, involving colour c, if there exists a colour $d \neq c$, such that:

(b_1) $\Gamma_{\{c,d\}}(\mathbf{e}) \neq \Gamma_{\{c,d\}}(\mathbf{f})$, and

(b_2) for each colour $j \in \Delta_n \setminus \{c,d\}, \Gamma_{\{c,j\}}(\mathbf{e}) = \Gamma_{\{c,j\}}(\mathbf{f})$.

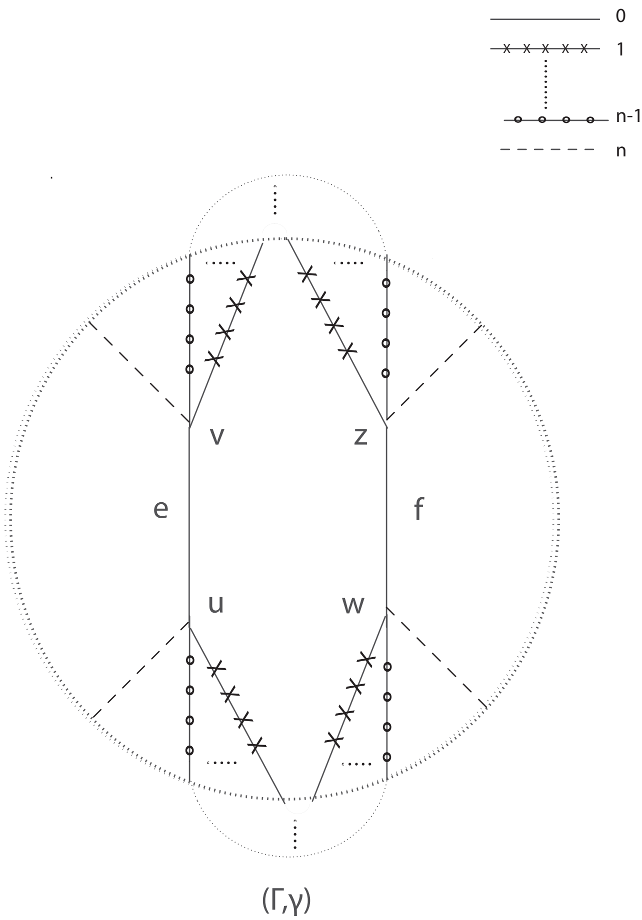
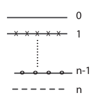


Figure 1.



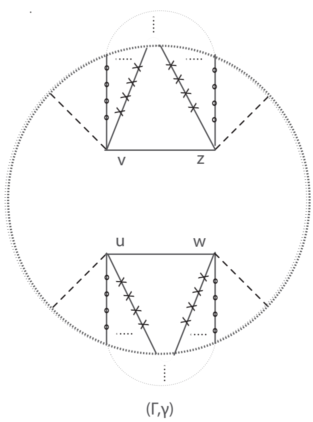


Figure 1a.

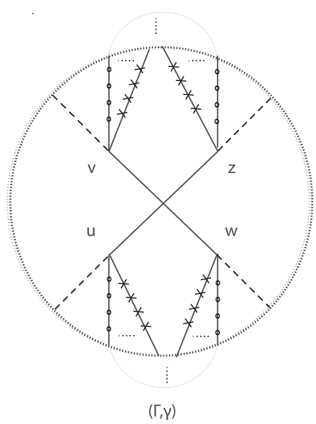


Figure 1b.

The colour d of above will be said to be *not involved* in the ρ_{n-1}-pair R.

By a *ρ-pair*, we will mean for short either a ρ_n-pair or a ρ_{n-1}-pair.

THEOREM (3.2). *Let (Γ,γ) be an $(n+1)$-coloured graph, $R = (\mathbf{e},\mathbf{f})$ be a ρ-pair of Γ and let (Γ_1,γ_1) be obtained from (Γ,γ) by any switching of R. Then:*

(a) *if R is a ρ_{n-1}- pair, then Γ and Γ_1 have the same number of components;*

(b) *if R is a ρ_n- pair, then Γ_1 has at most one more component than Γ.*

Proof. As before, let us call $\mathbf{u},\mathbf{v}$ the endpoints of $\mathbf{e}$ and $\mathbf{w},\mathbf{z}$ the endpoints of $\mathbf{f}$. Let further $\overline{\Gamma}$ be the graph obtained by deleting $\mathbf{e}$ and $\mathbf{f}$ from Γ.

As it is easy to check, $\mathbf{u},\mathbf{v},\mathbf{w}$ and $\mathbf{z}$ lie in the same component of Γ.

(a) If R is a ρ_{n-1}- pair, then $\mathbf{u},\mathbf{v},\mathbf{w}$ and $\mathbf{z}$ also lie in the same component of $\overline{\Gamma}$ (and therefore of Γ_1).

For, let d be the colour not involved in R. By definition of ρ_{n-1}- pair, $\Gamma_{\{c,d\}}(\mathbf{e})$ and $\Gamma_{\{c,d\}}(\mathbf{f})$ are two different bicoloured cycles of Γ, the first one containing $\mathbf{e}$ and the second one containing $\mathbf{f}$.

Hence there are two paths of $\overline{\Gamma}$, which join $\mathbf{u}$ with $\mathbf{v}$ and $\mathbf{w}$ with $\mathbf{z}$, respectively.

On the other hand, if j is any colour, $j \neq c,d$, then $\Gamma_{\{c,j\}}(\mathbf{e}) = \Gamma_{\{c,j\}}(\mathbf{f})$ is a single bicoloured cycle, containing both $\mathbf{e}$ and $\mathbf{f}$.

This proves that there is a path of $\overline{\Gamma}$, which joins $\mathbf{u}$ with either $\mathbf{w}$ or $\mathbf{z}$.

This completes the proof of (a).

(b) If $i \in \Delta_n \setminus \{c\}$, then by definition of ρ_n-pair, $\Gamma_{\{c,i\}}(\mathbf{e}) = \Gamma_{\{c,i\}}(\mathbf{f})$. This proves that there are two paths of $\overline{\Gamma}$, the first one joining $\mathbf{u}$ with either endpoint of $\mathbf{f}$, $\mathbf{w}$ say, and the second one joining $\mathbf{z}$ with $\mathbf{v}$.

This shows that $\overline{\Gamma}$ (hence also Γ_1) has at most one more component than Γ. □

In the following, we will show that in some particular, but geometrically relevant cases, it is possible to choose a "preferred" way to switch a pair of equally coloured edges of (Γ,γ).

CASE (A): Γ *bipartite.*

If $R = (\mathbf{e},\mathbf{f})$ is any pair of edges, both coloured c (in particular, if R is a ρ-pair) of a bipartite $(n+1)$-coloured graph (Γ,γ), then it is easy to see that only one of the two possible switching of R is again bipartite.

If, further, V_0,V_1 are the two bipartition classes of Γ and we orient $\mathbf{e},\mathbf{f}$ from V_0 to V_1, so that their tails $\mathbf{x_0} = \mathbf{e(0)},\mathbf{y_0} = \mathbf{f(0)}$ belong to V_0, and their heads $\mathbf{x_1} = \mathbf{e(1)},\mathbf{y_1} = \mathbf{f(1)}$ belong to V_1, the (bipartite) switching (Γ',γ') of R is obtained as follows:

(I) delete $\mathbf{e}$ and $\mathbf{f}$ from Γ (thus obtaining $\overline{\Gamma}$);

(II) join $\mathbf{x_0}$ with $\mathbf{y_1}$ and $\mathbf{x_1}$ with $\mathbf{y_0}$ by two new edges $\mathbf{e}',\mathbf{f}'$, both coloured c.

CASE (B): Γ *non bipartite, with bipartite residues.*

If Γ is a non bipartite graph, but for each colour i, $\Gamma_{\hat{\imath}}$ has bipartite components (residues), then we shall consider two subcases.

Subcase (B$_1$): *$R = (\mathbf{e},\mathbf{f})$ is a ρ_{n-1}-pair of Γ, involving colour c and not involving colour d.*

Let Ξ be the residue of $\Gamma_{\hat{d}}$ containing both $\mathbf{e}$ and $\mathbf{f}$ (note that $\mathbf{e}$ and $\mathbf{f}$ belong to the same $\hat{\imath}$-residue, because for every colour $i \neq c,d$, $\Gamma_{\{c,i\}}(\mathbf{e}) = \Gamma_{\{\mathbf{c},\mathbf{i}\}}(\mathbf{f})$.)

Let V_0,V_1 be the two bipartition classes of Ξ (recall that Ξ is bipartite), As in Case (A), let us orient $\mathbf{e}$ from V_0 to V_1. Now, the switching of $R = (\mathbf{e},\mathbf{f})$ is the $(n+1)$-coloured graph (Γ',γ') , obtained as before (Case (A)):

(I) delete $\mathbf{e}$ and $\mathbf{f}$ from Γ;
(II) join $\mathbf{x_0} = \mathbf{e(0)}$ with $\mathbf{y_1} = \mathbf{f(1)}$ and $\mathbf{x_1} = \mathbf{e(1)}$ with $\mathbf{y_0} = \mathbf{f(0)}$ by two new edges $\mathbf{e}', \mathbf{f}'$, both coloured c.

Subcase (B_2): *$R = (\mathbf{e}, \mathbf{f})$ is a ρ_n-pair (involving colour c) of Γ and $n \geq 3$.*

Let us orient arbitrarily the edge $\mathbf{e}$, as before, let us call $\mathbf{x_0} = \mathbf{e(0)}$ and $\mathbf{x_1} = \mathbf{e(1)}$. Let now i be any colour different from c. The orientation on $\mathbf{e}$ induces a coherent orientation on all edges of the cycle $\Gamma_{\{c,i\}}(\mathbf{e})$ and, in particular, on the edge $\mathbf{f}$ (with the induced orientation).

Now, we shall prove that the orientation on $\mathbf{f}$ (and hence its tail and its head) is independent from the choice of colour i ($i \neq c$).

For, let h be any colour of $\Delta_n, h \neq c$, and let $\mathbf{y_0^h}, \mathbf{y_1^h}$ be the tail and the head of the edge $\mathbf{f}$, with the orientations induced by the cycle $\Gamma_{\{c,h\}}(\mathbf{e})$ ($\mathbf{e}$ being oriented as before).

Let now $j \in \Delta_n, j \neq i, c$. In order to prove that $\mathbf{y_0^i} = \mathbf{y_0^j}$ (and, as a consequence $\mathbf{y_1^i} = \mathbf{y_1^j}$), let us consider a further colour k, with $k \neq i, j, c$.

Note that such a colour k must exist, since $n \geq 3$ and therefore Δ_n contains at least four colours.

Let now Ξ be the $\hat{k}$-residue of Γ, which contains $\mathbf{e}$. Ξ is bipartite and contains both the cycles $\Gamma_{\{c,i\}}(\mathbf{e})$ and $\Gamma_{\{c,j\}}(\mathbf{e})$. As a consequence, $\mathbf{y_0^i} = \mathbf{y_0^j}$. In fact, supposing on the contrary, $\mathbf{y_0^i} = \mathbf{y_1^j}$, we could construct an odd cycle of Ξ.

The construction of the switching (Γ', γ') of the ρ_n-pair $R = (\mathbf{e}, \mathbf{f})$ can now be performed as in the above cases:

(I) delete $\mathbf{e}$ and $\mathbf{f}$ from (Γ, γ);
(II) join $\mathbf{x_0}$ with $\mathbf{y_1^i}$ and $\mathbf{x_1}$ with $\mathbf{y_0^i}$ by two new edges $\mathbf{e}', \mathbf{f}'$, both coloured c.

Remark (3.3). The above cases include all ρ-pairs of gems representing orientable n-manifolds (Case (A)), all ρ_{n-1}-pairs of gems representing non orientable n-manifolds (Case (B_1)) and all ρ_n-pairs of gems representing non orientable n-manifolds, with $n \geq 3$ (Case (B_2)).

The only remaining case is that of a ρ_2-pair of a gem Γ representing a non orientable surface, for which it is not always possible the choice of a standard switching.

In fact, for $n = 2$, the procedure described in Case (B_2) doesn't work, as it depends on the choice of the colour i. [1]

4. Main results

The present section is devoted to prove the following Theorems (4.1) and (4.5), which concern the geometrical meaning of switching ρ-pairs in gems of n-dimensional manifolds.

As in Section 2, let $\mathbb{H}$ be a handle, i.e. either $(\mathbb{S}^{n-1} \times \mathbb{S}^1)$ or $(\mathbb{S}^{n-1} \tilde{\times} \mathbb{S}^1)$.

THEOREM (4.1). *Let (Γ, γ) be a gem of a (connected) n-manifold M, $n \geqslant 3$, $R = (\mathbf{e}, \mathbf{f})$ be a ρ_n-pair in Γ and let (Γ', γ') be the $(n+1)$-coloured graph, obtained by switching R. Then:*

[1] The case $n = 2$ is completely analyzed in [1], also for graphs representing surfaces with non-empy boundary.

(a) *if* (Γ',γ') *splits into two connected components,* (Γ'_1,γ'_1) *and* $(\Gamma'_2\gamma'_2)$ *say, then they are gems of two n-manifolds* M'_1 *and* M'_2 *respectively, and* $M \cong M'_1 \# M'_2$;
(b) *if* (Γ',γ') *is connected, then it is a gem of an n-manifold* M' *such that* $M \cong M' \# \mathbb{H}$.

Moreover, if (Γ,γ) *is a crystallization of M, then* (Γ',γ') *must be connected, and only case (b) may occur.*

In order to prove Theorem 4.1, we need some further constructions and a double sequence of Lemmas, which will be proved by induction on n.

LEMMA (4.2). – **step n** *Let* (Σ,σ) *be a gem of the n-sphere* $\mathbb{S}^n$, $n \geqslant 2$, $R = (\mathbf{e},\mathbf{f})$ *be a* ρ_n*-pair of* Σ *and let* (Σ',σ') *be obtained by switching* R*. Then* Σ' *splits into two connected components, both representing* $\mathbb{S}^n$.

Let now $\Gamma, R = (\mathbf{e},\mathbf{f}), \Gamma'$ be as in the statement of Theorem 4.1. Recall that (n being ≥ 3) any orientation of $\mathbf{e}$ induces a coherent orientation on $\mathbf{f}$. As in Section 2, let $\mathbf{e}(\mathbf{0}), \mathbf{f}(\mathbf{0}), \mathbf{e}(\mathbf{1})$ and $\mathbf{f}(\mathbf{1})$, be the ends of $\mathbf{e}$ and $\mathbf{f}$, so that $\mathbf{e}$ is directed from $\mathbf{e}(\mathbf{0})$ to $\mathbf{e}(\mathbf{1})$ and $\mathbf{f}$ is directed from $\mathbf{f}(\mathbf{0})$ to $\mathbf{f}(\mathbf{1})$. Furthermore, after the switching, the new edges $\mathbf{e}', \mathbf{f}'$ of Γ' join $\mathbf{e}(\mathbf{0})$ with $\mathbf{f}(\mathbf{1})$ and $\mathbf{e}(\mathbf{1})$ with $\mathbf{f}(\mathbf{0})$ respectively. Denote by $\tilde{\Gamma}$ the $(n+1)$-coloured graph obtained by adding a blob (i.e. an n-dipole), with vertices $\mathbf{A}$ and $\mathbf{B}$ on the edge $\mathbf{f}'$ of Γ' (see Figure 2)

LEMMA (4.3). – **step n** *With the above notations, if* Γ *is a gem of a (connected) n-manifold M,* $n \geq 3$*, then:*

(i) Γ' *(hence also* $\tilde{\Gamma}$*) is a gem of a (possibly disconnected) n-manifold* M';
(ii) $\mathbf{e}(\mathbf{0})$ *and* $\mathbf{B}$ *are two completely separated vertices of* $\tilde{\Gamma}$*; moreover* Γ *coincides with* $\tilde{\Gamma} fus(\mathbf{e}(\mathbf{0}),\mathbf{B})$.

Proof. First of all, we repeat here the proof of Lemma 4.2, step 2, which is exactly Corollary 13 of [1].

Let (Σ,σ) a 3-coloured, bipartite graph representing $\mathbb{S}^2$. Let R be a ρ_2- pair in Σ involving colour $c \in \Delta_2$. Then, by switching R in the only possible way, we obtain a new graph (Σ',σ'), either connected or with two connected components. Moreover, if we denote by d,k the further two colours of Δ_2, then Σ' has the same number of (d,k)- coloured cycles ($\hat{c}$-residues) and one more (c,h)- coloured cycle ($\hat{h}$-residue), for $h = d,k$.

Hence $\chi(\Sigma') = \chi(\Sigma)+2 = 4$. This implies that Σ' must have two connected components, both representing $\mathbb{S}^2$.

Now, assuming Lemma (4.2), step $n-1$, we prove Lemma (4.3), step n.

For, let us suppose Γ to be a gem of the n-manifold M. As a consequence, for each colour $i \in \Delta_n$, all $\hat{\imath}$-residues are gems of $\mathbb{S}^{n-1}$. Now, suppose $R = (\mathbf{e},\mathbf{f})$ to be a ρ_n-pair of Γ, involving color c, whose switching produces the graph Γ'.

Of course, the switching of R has no effects on the $\hat{c}$-residues of Γ. Hence, each $\hat{c}$-residue of Γ' is colour-isomorphic to the corresponding one of Γ, and therefore represents $\mathbb{S}^{n-1}$. Let now i be any colour different from c and let Ξ be the $\hat{\imath}$-residue containing R. Of course, R is a ρ_{n-1}-pair of Ξ (where Ξ is a gem of $\mathbb{S}^{n-1}$). Hence, by Lemma 4.2, step $n-1$, the switching of R splits Ξ into two new $\hat{\imath}$-residues of Γ', both representing $\mathbb{S}^{n-1}$.

Since all $\hat{\imath}$-residues of Γ, different from Ξ, are left unaltered by the switching of R, Γ' is again a gem of a n-manifold M' (with either one or two connected

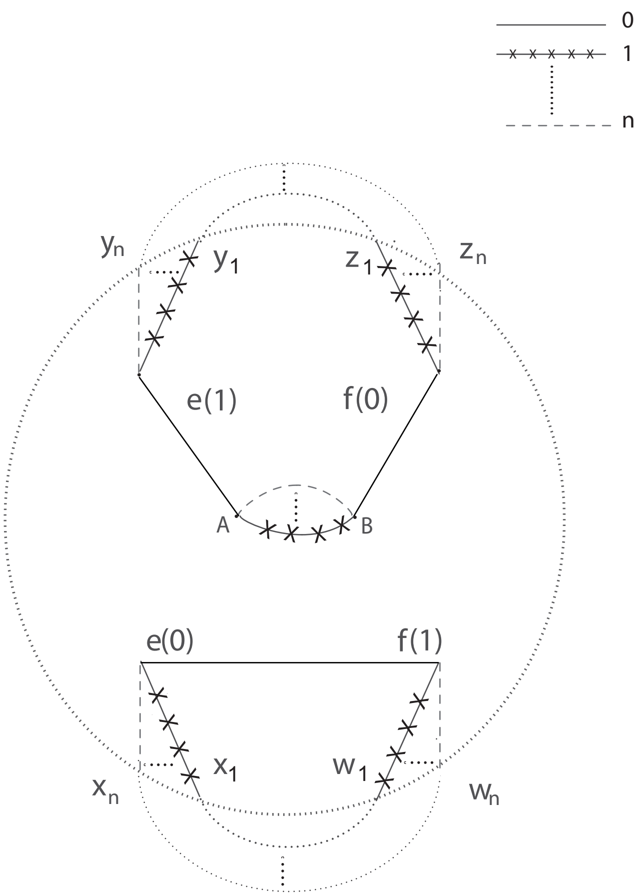


Figure 2.

components). Let now $\tilde{\Gamma}$ be obtained from Γ' by adding a blob (i.e. an n-dipole $\Theta = (\mathbf{A},\mathbf{B})$) on the edge $\mathbf{f'}$, with endpoints $\mathbf{e(1)},\mathbf{f(0)}$. Of course, $\tilde{\Gamma}$ is again a gem of M' and, as it is easy to check, $\tilde{\Gamma}\text{fus}(\mathbf{e(0)},\mathbf{B})$ is colour-isomorphic to Γ (where the vertex $\mathbf{A}$ plays the role of $\mathbf{e(0)}$).

Now, assuming Lemma (4.3), step n, we prove Lemma (4.2), step n. Let $\Sigma, R = (\mathbf{e},\mathbf{f}), \Sigma'$ be as in the statement of Lemma (4.2). Let further $\tilde{\Sigma}$ be obtained by adding a blob $\Theta = (\mathbf{A},\mathbf{B})$ on the edge $\mathbf{f'}$ of Σ'. Hence, by Lemma (4.2), step n, Σ' and $\tilde{\Sigma}$ are both gems of an n-manifold M'; moreover $\mathbf{e}(0)$ and $\mathbf{B}$ are completely separated vertices of $\tilde{\Sigma}$, and Σ is isomorphic to $\tilde{\Sigma}\text{fus}(\mathbf{e(0)},\mathbf{B})$. If Σ' (hence also $\tilde{\Sigma}$) is connected, then, by Lemma (2.1), the manifold represented by Σ must have a handle $\mathbb{H}$ as a direct summand, but this is impossible, since Σ represents $\mathbb{S}^n$, by hypothesis. Hence Σ' (and $\tilde{\Sigma}$) must split into two components Σ'_1, Σ'_2 say, representing two connected n-manifolds M'_1, M'_2 respectively, so that $\mathbb{S}^n \cong M'_1 \# M'_2$. But this implies that both M'_1, M'_2 are gems of $\mathbb{S}^n$, too.

This concludes the proof of Lemmas (4.2) and (4.3). □

Proof of Theorem (4.1). The proof of Theorem (4.1), (a) and (b), is now a direct consequence of Lemma (4.3), Step n, and Lemma (2.1).

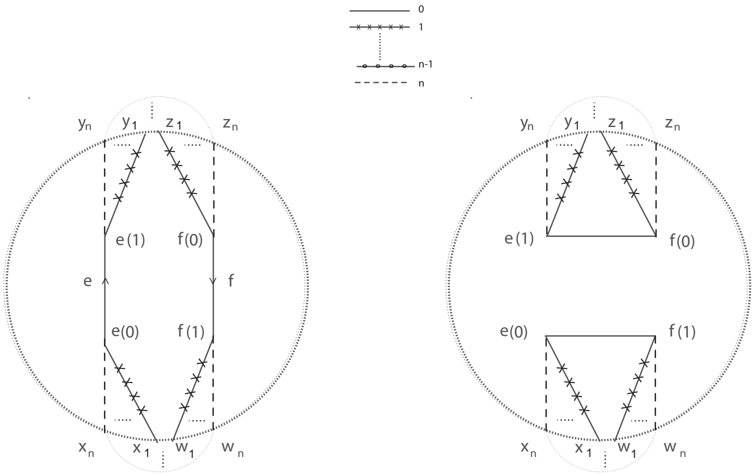


Figure 2a. Figure 2b.

If, further, $\Gamma_{\hat{c}}$ is connected, c being the colour involved in R (in particular, if Γ is a crystallization of M), then Γ' must be connected, too, and therefore $M \cong M'\#\mathbb{H}$. □

As a consequence of Theorem (4.1) and of Corollary 13 of [1], we have the following

COROLLARY (4.4). *If* (Σ, σ) *is a crystallization of the n-sphere* $\mathbb{S}^n$*,* $n \geq 2$ *then it cannot contain any* ρ_n*-pair.*

THEOREM (4.5). *Let* (Γ, γ) *be a gem of a (connected) n-manifold M,* $R = (\mathbf{e}, \mathbf{f})$ *be a* ρ_{n-1}*-pair of* Γ *and let* (Γ', γ') *be obtained by switching R. Then* Γ' *is a gem of the same manifold M.*

Proof. W.l.o.g., let us suppose $c = 0$ to be the colour involved and $d = n$ the one not involved in R. By Theorem (3.2), Γ' has the same number of connected components as Γ and, by performing the switching, it is bipartite (resp. non- bipartite) iff Γ is.

Consider the graph $\tilde{\Gamma}$, obtained by replacing the neighborhood of R in Γ (Figure 3a), with the graph of Figure 3b. The switching of R can be thought as the replacing of the neighborhood of $R = (\mathbf{e}, \mathbf{f})$ by the neighborhood of $R' = (\mathbf{e}', \mathbf{f}')$ (see Figure 1a). Consider now the graph $\tilde{\Gamma}$ obtained by replacing the above neighborhood by the graph of Figure 3b, where Θ_1 (Θ_2 resp.) is formed by two vertices $\mathbf{A}', \mathbf{e(1)}$ ($\mathbf{B}', \mathbf{f(0)}$ resp.) joined by $n-1$ edges coloured $1, \ldots, n-1$.

We will describe two sequences of dipole moves, joining $\tilde{\Gamma}$ with Γ and Γ' respectively, thus proving that Γ, Γ' are gems of PL-homeomorphic manifolds.

The first sequence starts by considering $\delta_1 = (\mathbf{A}, \mathbf{A}')$, which is a 1-dipole. In fact, $\tilde{\Gamma}_{\hat{n}}(\mathbf{A}') = \Theta_1$, whose further end is $\mathbf{e(1)}$; hence the $\hat{n}$-residue $\tilde{\Gamma}_{\hat{n}}(\mathbf{A})$ is different from Θ_1. By deleting the 1-dipole δ_1 from $\tilde{\Gamma}$, yields a 2-dipole δ_2 with ends $\mathbf{B}, \mathbf{B}'$; in fact $\tilde{\Gamma}_{\hat{n}}(\mathbf{B}')$ consists of exactly n multiple edges, whose further common endpoint is $\mathbf{f(0)}$ and which differs from the $\hat{n}$-residue $\tilde{\Gamma}_{\hat{n}}(\mathbf{B})$. By cancelling δ_2, too, we obtain Γ (Fig. 3c and 3d).

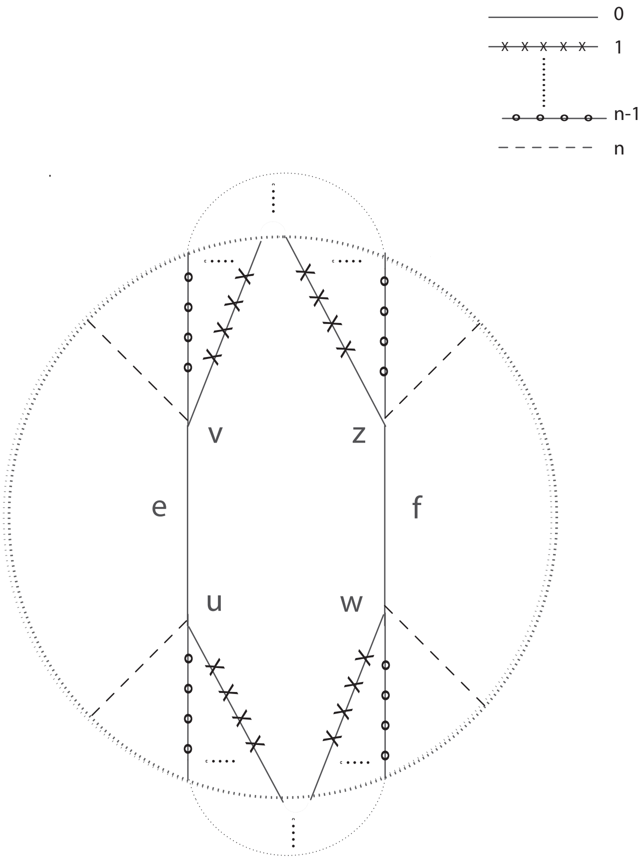


Figure 3a.

Θ_1 and Θ_2 are $(n-1)$-dipoles, since the $(0,n)$-residue containing $\mathbf{A}',\mathbf{B}'$ is a quadrilateral cycle whose vertices are $\mathbf{A},\mathbf{B},\mathbf{A}',\mathbf{B}'$ only. By deleting them from $\tilde{\Gamma}$ (Figg. 3e and 3f), we obtain Γ'. □

5. Rigid gems

Definition (5.1). An $(n+1)$-coloured graph (Γ,γ), $n \geq 3$, is called *rigid* iff it has no ρ-pairs. [2]

THEOREM (5.2). *The* $(n+1)$*-coloured graph* (Γ,γ)*,* $n \geq 3$*, is rigid iff for each* $i \in \Delta_n$*, the graph* $\Gamma_{\hat{\imath}}$ *has no* ρ_{n-1}*-pairs.*

[2] Note that, for $n = 2$, the concept of rigidity has no interest at all. In fact, if Γ is a 3-coloured graph representing a closed surface, then it contains ρ-pairs: ρ_2-pairs, if Γ is a crystallization, either ρ_1-pairs or ρ_2-pairs, otherwise. Hence, given any closed surface M^2, it cannot exist any rigid crystallization of M^2

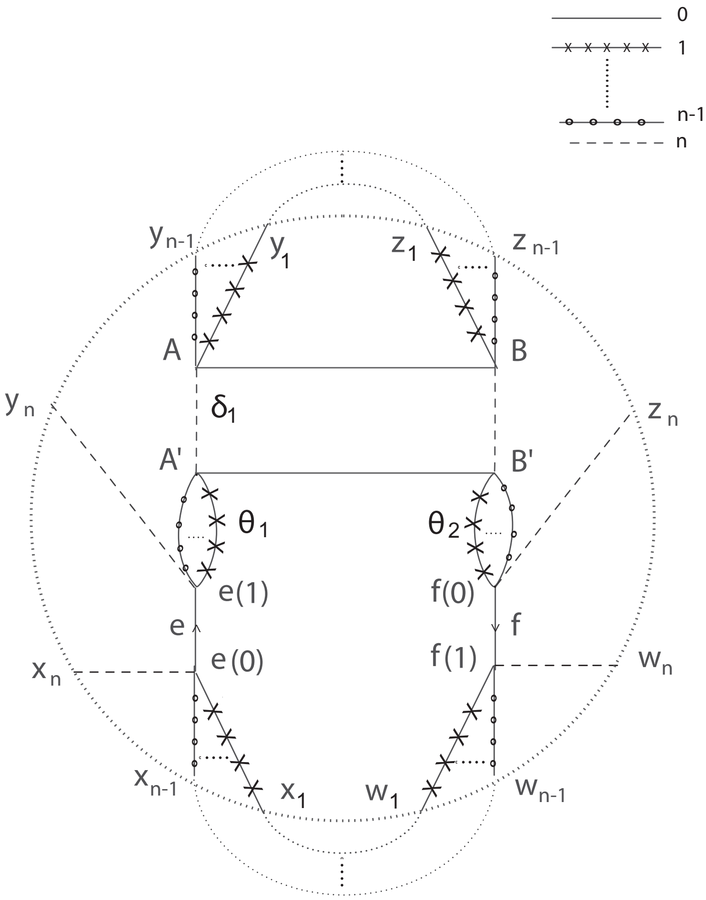


Figure 3b.

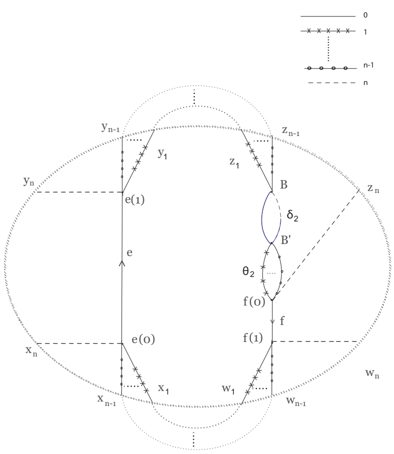


Figure 3c.

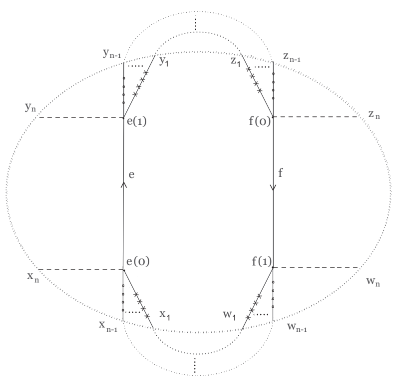


Figure 3d.

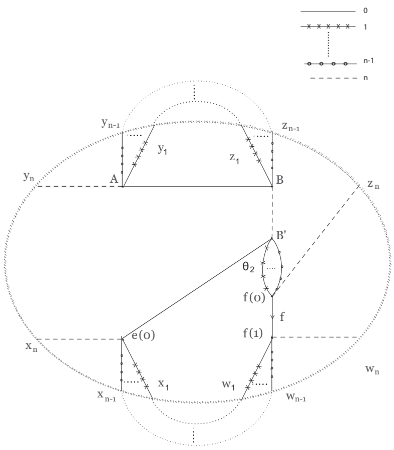


Figure 3e.

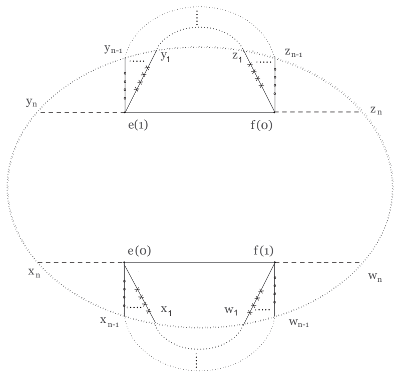


Figure 3f.

Proof. Suppose that (Γ,γ) is rigid and that there is a colour $i \in \Delta_n$ such that $\Gamma_{\hat{\imath}}$ has a ρ_{n-1}-pair $R = (\mathbf{e},\mathbf{f})$ of colour $c \in \Delta_n - \{i\}$. Then R is a ρ-pair in Γ too, and Γ cannot be rigid.

Conversely, if for each $i \in \Delta_n$, $(\Gamma)_{\hat{\imath}}$ contains no ρ_{n-1}-pairs, but (Γ,γ) is not rigid, then Γ contains at least a ρ-pair $R = (\mathbf{e},\mathbf{f})$.

If R is a ρ_n-pair, then R is a ρ_{n-1}-pair in $(\Gamma)_{\hat{\imath}}$, for each $i \in \Delta_n$.

If R is a ρ_{n-1}-pair, and d is the non-involved colour, then R is a ρ_{n-1}-pair in $\Gamma_{\hat{d}}$. □

THEOREM (5.3). *Every closed, connected, handle-free n-manifold M^n, $n \geq 3$, admits a rigid crystallization.*

Moreover, if (Γ,γ) is a crystallization of a closed, connected, handle-free n-manifold M^n of order p, then there exists a rigid crystallization of M^n of order $\leq p$.

Proof. Starting from any gem of M^n by cancelling a suitable number of 1-dipoles, we always can obtain a crystallization of M^n (see [7]). Suppose now that Γ is a crystallization of M^n; if Γ is rigid, then it is the requested crystallization.

If Γ has some ρ_{n-1}-pair $R = (\mathbf{e},\mathbf{f})$, of colour $c \in \Delta_n$ and non involving colour $d \in \Delta_n \setminus \{c\}$, then consider the connected component Ξ of $\Gamma_{\hat{d}}$ containing both $\mathbf{e}$ and $\mathbf{f}$. Since M^n is a manifold, Ξ represents $\mathbb{S}^{n-1}$ and R is a ρ_{n-1}-pair in $\Gamma_{\hat{d}}$, again. For Lemma 4.2, by switching R in $\Gamma_{\hat{d}}$, we obtain two connected components, both representing $\mathbb{S}^{n-1}$; since $\Gamma_{\hat{d}}$ is connected (Theorem 3.2), then there is at least a 1-dipole in $\Gamma_{\hat{d}}$. Hence, $\Gamma_{\hat{d}}$ is not contracted; the cancellation of such 1-dipole reduces the vertex-number.

If Γ has some ρ_n-pair $R = (\mathbf{e},\mathbf{f})$, of colour $c \in \Delta_n$, then, for each colour $i \in \Delta_n \setminus \{c\}$, the connected component of $\Gamma_{\hat{\imath}}$ containing $\mathbf{e}$ and $\mathbf{f}$, represents $\mathbb{S}^{n-1}$ and R is a ρ_{n-1}-pair in $\Gamma_{\hat{\imath}}$, as before, by switching R in $\Gamma_{\hat{\imath}}$, we obtain two connected components, both representing $\mathbb{S}^{n-1}$; since $\Gamma_{\hat{\imath}}$ is connected (Theorem (3.2)), then there is at least a 1-dipole in $\Gamma_{\hat{\imath}}$. Hence, $\Gamma_{\hat{\imath}}$ is not contracted; the cancellation of such 1-dipole reduces the vertex-number, for each $i \in \Delta_n \setminus \{c\}$. □

Note that the minimal crystallizations of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and $\mathbb{S}^{n-1} \tilde{\times} \mathbb{S}^1$ are not rigid (see, e.g., [10]). Hence the second statement of Theorem 5.2 is false for handles.

In dimension 3, there exist rigid crystallizations for $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$. The minimal ones have order 20 for $\mathbb{S}^2 \times \mathbb{S}^1$ and order 14 for $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$.

For $n > 3$, it is easy to construct a rigid crystallization of $\mathbb{S}^{n-1} \times \mathbb{S}^1$, if n is even, and of $\mathbb{S}^{n-1} \tilde{\times} \mathbb{S}^1$, if n is odd, both of order $2(2^n - 1)$.

The remaining cases are still open.

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REFERENCES

[1] P. BANDIERI, *ρ-pairs in graphs representing surfaces* (submitted).

[2] P. BANDIERI, M. R. CASALI, AND C. GAGLIARDI, *Representing manifolds by crystallization theory: foundations, improvements and related results*, Atti Sem. Mat. Fis. Univ. Modena. Suppl. **49** (2001), 283–337.

[3] P. BANDIERI, P. CRISTOFORI, AND C. GAGLIARDI, *Nonorientable 3-manifolds admitting coloured triangulations with at most 30 tetrahedra*, J. Knot Theory Ramifications **18** (3), (2009) 381–395.

[4] P. BANDIERI, P. CRISTOFORI, AND C. GAGLIARDI, *A census of genus-two 3-manifolds up to 42 coloured tetrahedra*, Discrete Mathematics **310**, (2010), 2469–2481.

[5] M. R. CASALI AND P. CRISTOFORI, *A catalogue of orientable 3-manifolds triangulated by* 30 *coloured tethraedra*, J. Knot Theory Ramifications **17** (5), (2008), 1–23.

[6] M. FERRI AND C. GAGLIARDI, *Crystallization moves*, Pacific J. Math. **100**, (1982), 85–103.

[7] M. FERRI, C. GAGLIARDI, AND L. GRASSELLI, *A graph-theoretical representation of PL-manifolds - A survey on crystallizations*, Aeq. Math. **31**, (1986), 121–141.

[8] L. C. GLASER, Geometrical combinatorial topology I, II, Van Nostrand Reinhold Co., Math. Studies, New York, 1970, 1972.

[9] J. L. GROSS AND T. W. TUCKER, *Topological graph theory*, John Wiley & Sons, New York, 1987.

[10] C. GAGLIARDI AND G. VOLZONE, *Handles in graphs and sphere bundles over* $\mathbb{S}^1$, Europ. J. Combinatorics **8**, (1987), 151–158.

[11] P. J. HILTON AND S. WYLIE, An introduction to algebraic topology-homology theory, Cambridge Univ. Press, Cambridge, 1960.

[12] S. LINS, *Gems, computers and attractors for* 3*-manifolds* Knots and Everything **5**, 1995.

[13] S. LINS AND M. MULAZZANI, *Blobs and flips on gems*, J. Knot Theory Ramifications **15**, (2006) (8), 1001–1035.

[14] S. V. MATVEEV, Algorithmic Topology and Classification of 3-Manifolds, Springer-Verlag, Berlin and Heidelberg, 2010.

[15] M. PEZZANA, *Sulla struttura topologica delle varietà compatte*, Atti Sem. Mat. Fis. Univ. Modena **23**, (1974), 269–277.

[16] M. PEZZANA, *Diagrammi di Heegaard e triangolazione contratta*, Boll. Un. Mat. Ital. **12 (4)**, (1975), 98–105.

[17] C. ROURKE AND B. SANDERSON, Introduction to piecewise-linear topology, Springer-Verlag, New York-Heidelberg, 1972.

[18] A. T. WHITE, Graphs, groups and surfaces, North Holland, revised edition, Amsterdam, 1984.

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HYPERBOLIC WEIGHTED HARMONIC CLASSES

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ABSTRACT. In this paper we introduce hyperbolic weighted harmonic classes for harmonic functions from the open complex unit disk to the interval $(-1,1)$.

Dedicated to Prof. Enrique Ramírez de Arellano, *for his remarkable labour as Managing Editor of the Boletín de la Sociedad Matemática Mexicana*

1. Introduction

Denote by $\mathbb{D} = \mathbb{D}_1$ the open unit disk in the complex plane $\mathbb{C}$ and T its boundary. Let $\phi_a : \mathbb{C} \to \mathbb{C}$ be the Möbius transformation

$$\phi_a(z) = \frac{a-z}{1-\overline{a}z}, \qquad |a| < 1,$$

with pole at $z = 1/\bar{a}$ and satisfying $\phi_a^{-1} = \phi_a$. We observe that

$$1 - |\phi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2} = (1-|z|^2)|\phi_a'(z)| \; . \tag{1.1}$$

Let $0 < r$. Define $D_r(0) = \{z \in \mathbb{D} : |z| < r\}$ and $D(a,r) := \phi_a(D_r(0))$. For $z,\ a \in \mathbb{D}$, we denote the Green's function of $\mathbb{D}$, with logarithmic singularity at a, by

$$g(z,a) = \ln \frac{|1-\overline{a}z|}{|a-z|} = \ln \frac{1}{|\phi_a(z)|} \; . \tag{1.2}$$

En 1994 R. Aulaskari and P. Lappan introduced in [1] the $\mathcal{Q}_p$ spaces for $1 \leq p < \infty$ as the familiy of analytic functions $f : \mathbb{D} \to \mathbb{C}$ satisfying the condition

$$\sup_{a\in\mathbb{D}} \int\int_{\mathbb{D}} |f'(z)|^2 g^p(z,a)\, dx\, dy < \infty \; .$$

The $\mathcal{Q}_p$ spaces for $0 < p < 1$ were introduced and studied by R. Aulaskari, J. Xiao and R. Zhao in [4].

Motivated by the recently emerged $\mathcal{Q}_p$ theory, in 1996 Ruhan Zhao introduced in [19] the $F(p,q,s)$ spaces of analytic functions $f : \mathbb{D} \to \mathbb{C}$ satisfying

$$\sup_{a\in\mathbb{D}} \int\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q g^s(z,a)\, dx\, dy < \infty \; ,$$

where $0 < p < \infty$, $-2 < q < \infty$ and $0 \leq s < \infty$.

For an analytic function $f : \mathbb{D} \to \mathbb{D}$, its hyperbolic derivative is defined by

$$f^*(z) = \frac{|f'(z)|}{1-|f(z)|^2}. \tag{1.3}$$

Considering this special type of derivative, in the eighties and nineties of the last century, Yamashita wrote a series of papers on hyperbolic classes [13], [14], [15]

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and [16]. In 2005 Xianon Li [7] introduced, for $0<p<\infty$, the so called hyperbolic $\mathcal{Q}_p^*$ class as the set of analytic functions $f:\mathbb{D}\to\mathbb{D}$ such that

$$\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}} f^*(z)^2 g^p(z,a)\,dx\,dy<\infty\,.$$

Note that the definition of hyperbolic $\mathcal{Q}_p^*$ classes is obtained from the definition of $\mathcal{Q}_p$ spaces when the derivative is replaced by the hyperbolic derivative.

Recently Aulaskari, Reséndis and Tovar introduced in [2] the hyperbolic weighted Bergman classes. More precisely, for $0<p<\infty$, $-2<q<\infty$, $0\le s<\infty$, define the q,s-weighted hyperbolic p-Bergman class $A^*(p,q,s)$ as the set of functions $f:\mathbb{D}\to\mathbb{D}$ such that

$$\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}} \frac{|f(z)|^p}{(1-|f(z)|^2)^2}(1-|z|^2)^q g^s(z,a)\,dx\,dy<\infty\,.$$

The weighted harmonic spaces introduced in [9] are comprised of harmonic functions $u:\mathbb{D}\to\mathbb{R}$ that satisfy

$$\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}} |\nabla u(z)|^p(1-|z|^2)^q g^s(z,a)dx\,dy<\infty,$$

and are the real valued harmonic counterpart of the spaces $F(p,q,s)$.

The aim of this paper is to obtain explicitly properties of the weighted hyperbolic harmonic classes of harmonic functions $u:\mathbb{D}\to(-1,1)$ satisfying

$$\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^q g^s(z,a)dx\,dy<\infty.$$

where $0<p<\infty$, $-2<q<\infty$ and $0<s<\infty$.

To introduce the hyperbolic gradient in these classes of functions, consider first the *hyperbolic distance* d defined as

$$d(z,w)=\frac{1}{2}\log\frac{1+|\phi_w(z)|}{1-|\phi_w(z)|},$$

where $z,w\in\mathbb{D}$. Observe that $d(z,w)\approx 2|\phi_w(z)|=2\left|\dfrac{w-z}{1-\overline{w}z}\right|$ when $d(z,w)$ is small.

For a function $u\colon(\mathbb{D},|\cdot|_{Euc})\to((-1,1),|\cdot|_{Euc})$ its gradient is determined by the partial derivative with respect to x and y. However for a function $u\colon(\mathbb{D},|\cdot|_{Euc})\to\big((-1,1),|\cdot|_{Hyp}\big)$ its hyperbolic gradient is determined in the following way, for $h\in\mathbb{R}$:

$$\begin{aligned}\frac{\partial^* u}{\partial x}(z) &= \lim_{h\to 0}\frac{d\left(u(x+h,y),u(x,y)\right)}{h}=\lim_{h\to 0}\frac{2\left|\dfrac{u(x+h,y)-u(x,y)}{1-u(x+h,y)u(x,y)}\right|}{|h|}\\ &= 2\lim_{h\to 0}\frac{|u(x+h,y)-u(x,y)|}{|h|\,|1-u(x+h,y)u(x,y)|}=2\frac{\left|\frac{\partial u}{\partial x}(z)\right|}{1-u(z)^2}.\end{aligned}$$

Similarly for $\dfrac{\partial^* u}{\partial y}(z)$. In this way

$$\nabla^* u(z)=\frac{1}{1-u(z)^2}\left(\left|\frac{\partial u}{\partial x}(z)\right|,\left|\frac{\partial u}{\partial y}(z)\right|\right).$$

The next expression initiates the motivation of this work:

$$\frac{|\nabla u(z)|^p}{(1-|u(z)|^2)^p} = \frac{\left(|\nabla u(z)|^2\right)^{p/2}}{(1-|u(z)|^2)^p} = \frac{\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]^{p/2}}{(1-|u(z)|^2)^p}$$
$$= \frac{\left(\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2\right)^{p/2}}{(1-|u(z)|^2)^p} = |\nabla^* u(z)|^p .$$

Let $\mathcal{H}(\mathbb{D})$ be the class of harmonic functions $u\colon \mathbb{D} \to (-1,1)$. We define for $0 < p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$ and $u \in \mathcal{H}(\mathbb{D})$ the functions $I^*_{p,q,s}(u)(a)\colon \mathbb{D} \to [0,\infty)$ as

$$I^*_{p,q,s}(u)(a) := \iint_{\mathbb{D}} |\nabla^* u(z)|^p (1-|z|^2)^q g^s(z,a)\, dx\, dy; \tag{1.4}$$

the hyperbolic harmonic class $\mathcal{H}I^*(p,q,s)$ as

$$\mathcal{H}I^*(p,q,s) := \{\, u \in \mathcal{H}(\mathbb{D}) \;:\; \sup_{a\in\mathbb{D}} I^*_{p,q,s}(u)(a) < \infty \,\};$$

and the little hyperbolic harmonic class $\mathcal{H}I^*_0(p,q,s)$ as

$$\mathcal{H}I^*_0(p,q,s) := \{\, u \in \mathcal{H}(\mathbb{D}) \;:\; \lim_{|a|\to 1^-} I^*_{p,q,s}(u)(a) = 0 \,\}.$$

For $0 < p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$ and $u \in \mathcal{H}(\mathbb{D})$ define the functions $J^*_{p,q,s}\colon \mathbb{D} \to [0,\infty)$ as

$$J^*_{p,q,s}(u)(a) := \iint_{\mathbb{D}} |\nabla^* u(z)|^p (1-|z|^2)^q (1-|\phi_a(z)|^2)^s\, dx\, dy; \tag{1.5}$$

the hyperbolic harmonic class $\mathcal{H}J^*(p,q,s)$ as

$$\mathcal{H}J^*(p,q,s) := \{\, u \in \mathcal{H}(\mathbb{D}) \;:\; \sup_{a\in\mathbb{D}} J^*_{p,q,s}(u)(a) < \infty \,\};$$

and the little hyperbolic harmonic class $\mathcal{H}J^*_0(p,q,s)$ as

$$\mathcal{H}J^*_0(p,q,s) = \{\, u \in \mathcal{H}(\mathbb{D}) \;:\; \lim_{|a|\to 1^-} J^*_{p,q,s}(u)(a) = 0 \,\}.$$

In section 3 we will show that $\mathcal{H}I^*(p,q,s) = \mathcal{H}J^*(p,q,s)$ and $\mathcal{H}I^*_0(p,q,s) = \mathcal{H}J^*_0(p,q,s)$.

We write $\mathcal{H}J^*_p = \mathcal{H}J^*(p,0,0)$ and observe that $\mathcal{H}I^*(2,0,0) = \mathcal{H}J^*(2,0,0) = \mathcal{H}J^*_2$ is the hyperbolic Dirichlet class of harmonic functions.

We say that $u \in \mathcal{H}(\mathbb{D})$ belongs to the hyperbolic harmonic Bloch class $\mathcal{H}B^*(\mathbb{D})$ if

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\frac{|\nabla u(z)|}{1-u(z)^2} = \sup_{z\in\mathbb{D}}(1-|z|^2)\,|\nabla^* u(z)| < \infty$$

and to the little hyperbolic harmonic Bloch class $\mathcal{H}B^*_0(\mathbb{D})$ if

$$\lim_{|z|\to 1^-}(1-|z|^2)\frac{|\nabla u(z)|}{1-u(z)^2} = \lim_{|z|\to 1^-}(1-|z|^2)\,|\nabla^* u(z)| = 0 \,.$$

In a similar way, for $-1 < q < \infty$ and $0 < p$, we say that $u \in \mathcal{H}(\mathbb{D})$ belongs to the hyperbolic q-Dirichlet class $\mathcal{H}J^*(p,q,0)$ if

$$J^*_{p,q,0}(u)(a) = \iint_{\mathbb{D}} |\nabla^* u(z)|^p (1-|z|^2)^q \, dx\, dy < \infty \,.$$

The main references for this work are R. Aulaskari et al [3], X. Li [7], Jie Xiao [12], Ruhan Zhao [18], [19] and Reséndis, Tovar [9].

Although this work is inspired in X. Li [7], the techniques and properties are in general different.

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2. Basic properties of $\mathcal{H}J^*(p,q,s)$ and $\mathcal{H}I^*(p,q,s)$

In this part we clarify some elementary aspects of our functions. The following two results are fundamental for the development of this paper.

LEMMA (2.1) ([17], Theorem 1.12). *Let $t > -1$, $c \in \mathbb{R}$ and define $I_{t,c} : \mathbb{D} \to [0,\infty)$ by*

$$I_{t,c}(a) = \iint_{\mathbb{D}} \frac{(1-|z|^2)^t}{|1-\overline{a}z|^{2+t+c}}\, dxdy.$$

Then

(a) *If $c < 0$ then $I_{t,c}(a)$ is bounded in a.*

(b) *If $c = 0$, then*

$$I_{t,c}(a) \approx \ln \frac{1}{1-|a|^2}, \qquad (\,|a| \to 1^-)\,.$$

(c) *If $c > 0$, then*

$$I_{t,c}(a) \approx \frac{1}{(1-|a|^2)^c}, \qquad (\,|a| \to 1^-)\,.$$

We need the following result.

PROPOSITION (2.2). *Let $0 < p < \infty$ and $u \in \mathcal{H}(\mathbb{D})$. Then the function $|\nabla^* u(z)|^p : \mathbb{D} \to [0,\infty)$ defined by*

$$|\nabla^* u(z)|^p = \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p},$$

is subharmonic.

Proof. Let $v : \mathbb{D} \to \mathbb{R}$ be a harmonic conjugate of u. If $f(z) = u(x,y) + iv(x,y)$ is the analytic completion of u, then $\ln|\nabla u(z)| = \ln|f'(z)|$ is a subharmonic function.

Define $g(z) = -\ln(1-u(z)^2)$. Thus

$$\frac{\partial g}{\partial x}(z) = \frac{2u(z)}{1-u(z)^2}\frac{\partial}{\partial x}u(z) = \frac{2u(z)}{1-u(z)^2}u_x(z)$$

and

$$\frac{\partial^2 g}{\partial x^2}(z) = \frac{2(1+u(z)^2)}{(1-u(z)^2)^2}u_x(z)^2 + \frac{2u(z)}{1-u(z)^2}u_{xx}(z)\,.$$

A similar result is obtained for $g_{yy}(z)$. From the harmonicity of $u(z)$ we have

$$\Delta g(z) = \frac{\partial^2 g}{\partial x^2}(z) + \frac{\partial^2 g}{\partial y^2}(z) = \frac{2(1+u(z)^2)}{(1-u(z)^2)^2}(u_x(z)^2 + u_y(z)^2) \geq 0\,.$$

Then $g(z)$ is a subharmonic function. As $0 < p < \infty$, the function $\ln|\nabla^* u(z)|^p = p\ln|\nabla u(z)| - p\ln(1-u(z)^2)$ is subharmonic. We now compose with the convex function e^α and this concludes the proof. $\square$

The following elementary estimate is well known: let $q \in \mathbb{R}$ and $a \in \mathbb{D}$. Then for all $z \in \mathbb{D}$,

$$\frac{1}{\rho(a,q)}(1-|z|^2)^q \le (1-|\phi_a(z)|^2)^q \le \rho(a,q)(1-|z|^2)^q\,, \tag{2.3}$$

where

$$\rho(a,q) = \left(\frac{1+|a|}{1-|a|}\right)^{|q|}.$$

Let $q \in \mathbb{R}$ and $a \in \mathbb{D}$. Then for all $z \in \mathbb{D}$,

$$\frac{1}{\rho(a,q)} \le |\phi_a'(z)|^q \le \rho(a,q)\,. \tag{2.4}$$

The class $\mathcal{H}I^*(p,q,s)$ is Möbius invariant in the classical sense:

PROPOSITION (2.5). *Let $a \in \mathbb{C}$ with $|a| < 1$, $0 < p < \infty$, $-2 < q < \infty$ and $0 \le s < \infty$. If $u \in \mathcal{H}I^*(p,q,s)$, then*

$$\sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} \frac{|\nabla(u\circ\phi_a)(w)|^p}{(1-u(\phi_a(w))^2)^p}(1-|w|^2)^q g^s(w,b)\,d\xi\,d\eta$$
$$= \sup_{c\in\mathbb{D}} \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^q g^s(z,c)\,dx\,dy < \infty,$$

if $q = p-2$.

Proof. Let $a \in \mathbb{D}$ be fixed. Denote $w = \phi_a(z)$ and $b = \phi_a(c)$. Since the Green function is conformally invariant, then $g(\phi_a(z),\phi_a(c)) = g(z,c)$. Thus

$$\iint_{\mathbb{D}} \frac{|\nabla(u\circ\phi_a)(w)|^p}{(1-u(\phi_a(w))^2)^p}(1-|w|^2)^q g^s(w,b)\,d\xi\,d\eta$$
$$= \iint_{\mathbb{D}} \frac{|\nabla(u(\phi_a(\phi_a(z)))|^p}{(1-u(\phi_a(\phi_a(z)))^2)^p}|\phi_a'(\phi_a(z))|^p|\phi_a'(z)|^2(1-|\phi_a(z)|^2)^q g^s(\phi_a(z),\phi_a(c))\,dx\,dy$$
$$= \iint_{\mathbb{D}} \frac{|\nabla(u(z))|^p}{(1-u(z)^2)^p}|\phi_a'(z)|^{2-p}(1-|\phi_a(z)|^2)^q g^s(z,c)\,dx\,dy.$$

Therefore if $q = p-2$, by (1.1)

$$\sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} \frac{|\nabla(u\circ\phi_a)(w)|^p}{(1-u(\phi_a(w))^2)^p}(1-|w|^2)^{p-2} g^s(w,b)\,d\xi\,d\eta$$
$$= \sup_{c\in\mathbb{D}} \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^{p-2} g^s(z,c)\,dx\,dy < \infty. \qquad \square$$

In addition the class $\mathcal{H}I^*(p,q,s)$ is Möbius invariant in the following sense.

PROPOSITION (2.6). *Let $a \in \mathbb{C}$ with $|a| < 1$, $-2 < q < \infty$, $0 < p < \infty$, $0 \le s < \infty$. If $u \in \mathcal{H}I^*(p,q,s)$, then $u\circ\phi_a \in \mathcal{H}I^*(p,q,s)$, that is,*

$$\sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} \frac{|\nabla(u\circ\phi_a)(w)|^p}{(1-u(\phi_a(w))^2)^p}(1-|w|^2)^q g^s(w,b)\,d\xi\,d\eta < \infty\,.$$

Proof. Let $a \in \mathbb{D}$ be fixed. Denote $w = \phi_a(z)$, $b = \phi_a(c)$. Since the Green function is conformally invariant, then $g(\phi_a(z), \phi_a(c)) = g(z,c)$. Thus by (2.3) and (2.4)

$$\begin{aligned}
&\iint_{\mathbb{D}} \frac{|\nabla(u\circ\phi_a)(w)|^p}{(1-u(\phi_a(w))^2)^p}(1-|w|^2)^q g^s(w,b)\,d\xi\,d\eta \\
&= \iint_{\mathbb{D}} \frac{|\nabla(u(\phi_a(\phi_a(z)))|^p}{(1-u(\phi_a(\phi_a(z)))^2)^p}|\phi_a'(\phi_a(z))|^p|\phi_a'(z)|^2(1-|\phi_a(z)|^2)^q g^s(\phi_a(z),\phi_a(c))\,dx\,dy \\
&\le \rho(a,p+|q|+2)\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^q g^s(z,c)\,dx\,dy\,.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sup_{b\in\mathbb{D}} \iint_{\mathbb{D}} \frac{|\nabla(u\circ\phi_a)(w)|^p}{(1-u(\phi_a(w))^2)^p}(1-|w|^2)^q g^s(w,b)\,d\xi\,d\eta \\
&\qquad \le \rho(a,p+|q|+2)\sup_{c\in\mathbb{D}}\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^q g^s(z,c)\,dx\,dy < \infty. \qquad \square
\end{aligned}$$

As corollary of this proof we have the following result.

COROLLARY (2.7). *Let $a \in \mathbb{C}$ be with $|a| < 1$ and $0 < p < \infty$, $-2 < q < \infty$, $0 \le s < \infty$. If $f \in \mathcal{H}I_0^*(p,q,s)$, then $f\circ\phi_a \in \mathcal{H}I_0^*(p,q,s)$.*

Proof. With the notation of the previous proposition, $b = \phi_a(c)$ if and only if $c = \phi_a(b)$, so by (1.1)

$$1-|c|^2 = 1-|\phi_a(b)|^2 = \frac{(1-|a|^2)(1-|b|^2)}{|1-\overline{a}b|^2}\,.$$

Therefore $|c| \to 1$ if and only if $|b| \to 1$. $\square$

The next proposition is an important tool.

THEOREM (2.8) ([5], Schwarz-Pick Theorem). *Let $f\colon \mathbb{D} \to \mathbb{D}$ be holomorphic. Then, for all $z_1, z_2 \in \mathbb{D}$*

$$\left|\frac{f(z_1)-f(z_2)}{1-\overline{f(z_1)}f(z_2)}\right| \le \left|\frac{z_1-z_2}{1-\overline{z_1}z_2}\right|$$

and, for all $z \in \mathbb{D}$,

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}.$$

From this, we can extend the inequality for our harmonic functions:

PROPOSITION (2.9). *Let $u, v \in \mathcal{H}(\mathbb{D})$. If $f(z) = u(z) + i\,v(z)$ is the analytic completion of u with $f\colon \mathbb{D} \to \mathbb{D}$, it follows that*

$$\frac{|\nabla u(z)|}{1-u(z)^2} \le \frac{1}{1-|z|^2}\,.$$

Proof. Consider a harmonic function $u\colon \mathbb{D} \to (-1,1)$, and let f be as in the statement. As $|f'(z)| = |\nabla u(z)|$ and $|u(z)| = |\mathrm{Re} f(z)| \le |f(z)|$, then as

$$\frac{1}{1-u(z)^2} \le \frac{1}{1-|f(z)|^2}.$$

Finally

$$\frac{|\nabla u(z)|}{1-u(z)^2} \le \frac{1}{1-|z|^2}. \qquad \square$$

Observe that $|f'(z)| = |\nabla u(z)| = |\nabla v(z)|$. So if u belongs to $\mathcal{H}I^*(p,q,s)$, then as $|f'(z)| \le \dfrac{|\nabla u(z)|}{1-u(z)^2}$ we have $f \in F(p,q,s)$ and from [9] $f \in F(p,q,s)$ if and only if $u, v \in \mathcal{H}F(p,q,s)$, where $\mathcal{H}F(p,q,s)$ means the harmonic $F(p,q,s)$ class.

THEOREM (2.10). *Let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$ with $q+s \le -1$. Then the class $\mathcal{H}J^*(p,q,s)$ consists only of constant functions.*

Proof. Let $u \in \mathcal{H}J^*(p,q,s)$ be a nonconstant function. Then there exists $0 < b < 1$ such that $\nabla u(z_0) \neq 0$ with $|z_0| = b$. Since ∇u is a continuous function, there exists $0 < \delta < 1$ with $\nabla u(z) \neq 0$ for all $z \in D_\delta(z_0)$. By Proposition (2.2)

$$\begin{aligned}
\infty > J^*_{p,q,s}(u)(0) &= \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^{q+s}\, dx\, dy \\
&\ge \int_b^1 \int_0^{2\pi} \frac{|\nabla u(re^{i\theta})|^p}{(1-u(re^{i\theta})^2)^p}(1-r^2)^{q+s} r\, d\theta\, dr \\
&\ge \int_0^{2\pi} \frac{|\nabla u(be^{i\theta})|^p}{(1-u(be^{i\theta})^2)^p}\, d\theta \int_b^1 (1-r^2)^{q+s} r\, dr.
\end{aligned}$$

Since

$$0 < \int_0^{2\pi} \frac{|\nabla u(be^{i\theta})|^p}{(1-u(be^{i\theta})^2)^p}\, d\theta$$

and as

$$\int_b^1 (1-r^2)^{q+s} r\, dr = \infty$$

for $q+s \le -1$, we get a contradiction. □

Example (2.11). Let $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$ and let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function given by $f(z) = u(z) + iv(z)$; thus $u : \mathbb{D} \to (-1,1)$ is a harmonic function (for instance $\mathrm{Re}(\phi_a(z))$). Since $|f'(z)| = |\nabla u(z)|$ and $|u(z)| = |\mathrm{Re} f(z)| \le |f(z)|$, then by Proposition (2.9) we have

$$\frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p} \le \frac{|f'(z)|^p}{\left(1-|f(z)|^2\right)^p} \le \frac{1}{\left(1-|z|^2\right)^p}\,. \tag{2.12}$$

Therefore

$$\begin{aligned}
&\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p} \left(1-|z|^2\right)^q \left(1-|\phi_a(z)|^2\right)^s dx\, dy \\
&\qquad\qquad \le \iint_{\mathbb{D}} \left(1-|z|^2\right)^{q-p} \left(1-|\phi_a(z)|^2\right)^s dx\, dy.
\end{aligned}$$

From (1.1), we have

$$\iint_{\mathbb{D}} \left(1-|z|^2\right)^{q-p} \left(1-|\phi_a(z)|^2\right)^s dx\, dy \le \left(1-|a|^2\right)^s \iint_{\mathbb{D}} \frac{\left(1-|z|^2\right)^{q-p+s}}{|1-\overline{a}z|^{2s}}\, dx\, dy.$$

By Lemma (2.1) with $q-p+s = t$ and $2s = 2+t+c$, we have $c = s-q+p-2$:

- If $c > 0$, there exists $C > 0$ such that

$$\left(1-|a|^2\right)^s \iint_{\mathbb{D}} \frac{\left(1-|z|^2\right)^{q-p+s}}{|1-\overline{a}z|^{2s}}\, dx\, dy \le \left(1-|a|^2\right)^s \frac{C}{\left(1-|a|^2\right)^{s-q+p-2}}$$

and $\lim_{|a|\to 1^-} \left(1-|a|^2\right)^{q-p+2} = 0$ when $p < 2+q$. Hence $u \in \mathcal{H}J_0^*(p,q,s)$.

- If $c = 0$, there exists $C > 0$ such that

$$\left(1-|a|^2\right)^s \iint_{\mathbb{D}} \frac{\left(1-|z|^2\right)^{q-p+s}}{|1-\overline{a}z|^{2s}}\, dx\, dy \le C\left(1-|a|^2\right)^s \log\frac{1}{1-|a|^2}$$

and $\lim_{|a|\to 1^-}\left(1-|a|^2\right)^s \log\dfrac{1}{1-|a|^2} = 0$ for the parameter $0 < s$. Hence $u \in \mathcal{H}J_0^*(p,q,s)$.

- If $c < 0$,

$$\left(1-|a|^2\right)^s \iint_{\mathbb{D}} \frac{\left(1-|z|^2\right)^{q-p+s}}{|1-\overline{a}z|^{2s}}\, dx\, dy \le \left(1-|a|^2\right)^s M$$

and $\lim_{|a|\to 1^-}\left(1-|a|^2\right)^s M = 0$ for the parameter $0 < s$. Hence $u \in \mathcal{H}J_0^*(p,q,s)$.

Remark (2.13). The same estimates are obtained for $\mathrm{Im}(f(z)) = v(z)$, so $v \in \mathcal{H}J_0^*(p,q,s)$.

Example (2.14). Let G a domain such that $\overline{\mathbb{D}} \subset G$ and $f : G \to [-\delta,\delta]\times\mathbb{R}$ an analytic function with $|\delta| < 1$. Then $\mathrm{Re} f(z) \in \mathcal{H}J_0^*(p,q,s)$.

As the real and imaginary parts of an analytic function belonging to $\mathcal{Q}_s^*$ are harmonic functions from $\mathbb{D}$ to $[-1,1]$, it is good to compare some results of the $\mathcal{Q}_s^*$ classes with the corresponding ones for hyperbolic weighted harmonic classes (see [7]).

Example (2.15). For the restriction to the disk $\mathbb{D}$ of the function $f(z) = z$, Li has shown that $f \notin \mathcal{Q}_s^*$ for $0 < s \le 1$. For $u(z) = \mathrm{Re}\, z = x$ we have some similarities but also some differences respect to $f(z) = z$. Let $0 < R < \sqrt{2}-1$ and define $\Omega = \{1+\rho e^{i\theta} \in \mathbb{D} : 0 < \rho < R, \frac{3}{4}\pi < \theta < \frac{5}{4}\pi\}$. Applying the Fatou's Lemma and after a straightforward calculation and estimations in polar coordinates we obtain, for $q-s-p \ne -2$,

$$\begin{aligned}
&\iint_{\mathbb{D}} \frac{1}{(1-x^2)^p}(1-|z|^2)^q\,(1-|\phi_a(z)|^2)^s\, dx\, dy \\
&\ge\; (1-|a|^2)^s \iint_{\Omega} \frac{1}{(1-x^2)^p}\frac{\left(1-|z|^2\right)^{q+s}}{|1-z|^{2s}}\, dx\, dy \\
&\ge\; \frac{1}{2^p}(1-|a|^2)^s \int_0^R \int_{\frac{3}{4}\pi}^{\frac{5}{4}\pi} \rho^{q-s-p+1}\, dr\, d\theta = \frac{\pi}{2^{p+1}}(1-|a|^2)^s \rho^{q-s-p+2}\Big|_0^R .
\end{aligned}$$

For $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$, if $q-s-p+2 < 0$ then $u \notin \mathcal{H}J^*(p,q,s)$. Integrating, a similar result is obtained if $q-s-p+2 = 0$. In particular for $p = 2$, $q = 0$ and $0 < s \le 1$, the function $u(z) = \mathrm{Re}(z) \notin \mathcal{H}J^*(2,0,s)$ since $-s < 0$.

However, if $0 < s \le 1$, $q = 0$ and taking in account Lemma (2.1), for $\{0 < p < 1, 0 < s < 2-p\} \bigcup \{1 \le p \le \dfrac{3}{2}, p-1 < s < 2-p\}$ we have that $u \in \mathcal{H}J^*(p,0,s)$.

Example (2.16). If $f \in \mathcal{Q}_s^*$, by the first inequality in (2.12) we have $u = \mathrm{Re} f \in \mathcal{H}J^*(2,0,s)$.

In particular, for the restriction to $\mathbb{D}$ of the function $f(z) = 1-(1-z)^\alpha$ with $0 < \alpha < 1$, from [7] we know that $f \in \mathcal{Q}_s^*$ for all $0 < s < 1$, therefore,

$$\mathrm{Re}\Big(1-(1-z)^\alpha\Big) = 1-|1-x-y|^\alpha \cos\Big(\alpha \arctan\frac{y}{x-1}\Big) \in \mathcal{H}J^*(2,0,s)$$

and

$$\operatorname{Im}\big(1-(1-z)^{\alpha}\big) = |1-x-y|^{\alpha}\sin\Big(\alpha \arctan\frac{y}{x-1}\Big) \in \mathcal{H}J^{*}(2,0,s).$$

Example (2.17). Consider the function $H\colon \mathbb{D}\to\mathbb{D}$ defined by

$$H(z) = \frac{1-2^{\frac{1-s_1}{2}}}{2}\cdot f(z),$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^{2^n}$ with $a_n = \dfrac{1}{2^{\frac{n(1-s_1)}{2}}}$. From [7], we know that for $p=2$, $q=0$ and $0<s_1<s<1$ we have $H\in \mathcal{Q}_s^*$ but $H\notin \mathcal{Q}_{s_1}^*$, in particular $|H(z)|<\dfrac{1}{2}$. Then if $u=\operatorname{Re}(H)$, $u\in \mathcal{H}J^*(2,0,s)$, however $u\notin \mathcal{H}J^*(2,0,s_1)$ since

$$\begin{aligned}\frac{9}{16}\iint_{\mathbb{D}}&\frac{|H'(z)|^2}{(1-|H(z)|^2)^2}(1-|\phi_a(z)|^2)^s\,dx\,dy\\ &\le \iint_{\mathbb{D}}\frac{|\nabla u(z)|^2}{(1-u(z)^2)^2}(1-|\phi_a(z)|^2)^s\,dx\,dy\\ &\qquad\le \frac{16}{9}\iint_{\mathbb{D}}\frac{|H'(z)|^2}{(1-|H(z)|^2)^2}(1-|\phi_a(z)|^2)^s\,dx\,dy\,.\end{aligned}$$

Thus this example shows that the inclusion $\mathcal{H}J^*(2,0,s_1)\subsetneq \mathcal{H}J^*(2,0,s)$ $(0<s_1<s<1)$ is strict.

PROPOSITION (2.18). *For $0<p<\infty$, we have $\mathcal{H}J^*(p,0,0)\subset \mathcal{H}B_0^*(\mathbb{D})$.*

Proof. Take $u\in\mathcal{H}J^*(p,0,0)$. By Proposition (2.2) we have

$$\frac{|\nabla u(0)|^p}{(1-u(0)^2)^p} \le \frac{1}{2\pi}\int_0^{2\pi}\frac{|\nabla u(re^{i\theta})|^p}{(1-u(re^{i\theta})^2)^p}\,d\theta\,.$$

Let $0<R<1$ be fixed. After multiplying by R and integrating from 0 to R we have

$$\frac{|\nabla u(0)|^p}{(1-u(0)^2)^p}\le\frac{1}{\pi R^2}\iint_{D_R(0)}\frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}\,dx\,dy\,.$$

As $|\nabla u|\circ\phi_a$ is also subharmonic, with the change of variable $w=\phi_a(z)$ and by (2.4)

$$\begin{aligned}\frac{|\nabla u(a)|^p}{(1-u(a)^2)^p} &\le \frac{1}{\pi R^2}\iint_{D(a,R)}\frac{|\nabla u(w)|^p}{(1-u(w)^2)^p}|\phi_a'(w)|^2\,du\,dv\\ &\le \frac{1}{\pi R^2}\frac{(1+|a|)^2}{(1-|a|)^2}\iint_{D(a,R)}\frac{|\nabla u(w)|^p}{(1-u(w)^2)^p}\,du\,dv.\end{aligned}$$

Thus

$$(1-|a|^2)^2\frac{|\nabla u(a)|^p}{(1-u(a)^2)^p} \le \frac{(1+|a|)^4}{\pi R^2}\iint_{D(a,R)}\frac{|\nabla u(w)|^p}{(1-u(w)^2)^p}\,du\,dv\,.$$

If $|a|\to 1^-$ then $|D(a,R)|\to 0$ and $u\in\mathcal{H}B_0^*(A)$. □

We require the following result.

LEMMA (2.19). *Let $-2<q<\infty$, $0<p<\infty$, $0<s<\infty$ and $|a|<1$. Then*

$$\frac{1}{\rho(a,p+|q|+2)}J^*_{p,q,s}(u)(a)\le J^*_{p,q+s,0}(u\circ\phi_a)(0)\le \rho(a,p+|q|+2)J^*_{p,q,s}(u)(a)\,.$$

Proof. Suppose that $J^*_{p,q,s}(u)(a)<\infty$. If $u\circ\phi_a\in\mathcal{H}J^*(p,q+s,0)$, then by the change of variable formula with $z=\phi_a(w)$, we have by inequalities (2.3) and (2.4)

$$J^*_{p,q,s}(u\circ\phi_a)(0)=\iint_{\mathbb{D}}\frac{|\nabla(u\circ\phi_a)(z)|^p}{(1-u(\phi_a(z))^2)^p}(1-|z|^2)^{q+s}\,dx\,dy$$

$$\begin{aligned}J^*_{p,q+s,0}(u\circ\phi_a)(0)&=\iint_{\mathbb{D}}\frac{|\nabla u(\phi_a(z))\phi_a'(z)|^p}{(1-u(\phi_a(z))^2)^p}(1-|z|^2)^{q+s}\,dx\,dy\\&=\iint_{\mathbb{D}}\frac{|\nabla u(w)|^p}{(1-u(w)^2)^p}|\phi_a'(\phi_a(w))|^p|\phi_a'(w)|^2(1-|\phi_a(w)|^2)^{q+s}\,d\xi\,d\eta\\&\le\rho(a,p+|q|+2)\iint_{\mathbb{D}}\frac{|\nabla u(w)|^p}{(1-u(w)^2)^p}(1-|w|^2)^q(1-|\phi_a(w)|^2)^s\,d\xi\,d\eta\\&=\rho(a,p+|q|+2)\,J^*_{p,q,s}(u)(a).\end{aligned}$$

Conversely, if $J^*_{p,q+s,0}(u\circ\phi_a)(0)<\infty$ then with $z=\phi_a(w)$, it is enough to consider

$$\begin{aligned}J^*_{p,q+s,0}(u\circ\phi_a)(0)&=\iint_{\mathbb{D}}\frac{|\nabla(u\circ\phi_a)(z)|^p}{(1-u(\phi_a(z))^2)^p}(1-|z|^2)^{q+s}dxdy\\&=\iint_{\mathbb{D}}\frac{|\nabla u(\phi_a(z))\phi_a'(z)|^p}{(1-u(\phi_a(z))^2)^p}(1-|z|^2)^{q+s}dxdy\\&=\iint_{\mathbb{D}}\frac{|\nabla u(w)|^p}{(1-u(w)^2)^p}|\phi_a'(\phi_a(w))|^p|\phi_a'(w)|^2(1-|\phi_a(w)|^2)^{q+s}dudv\\&\ge\frac{1}{\rho(a,p+|q|+2)}\iint_{\mathbb{D}}\frac{|\nabla u(w)|^p}{(1-u(w)^2)^p}(1-|w|^2)^q(1-|\phi_a(w)|^2)^s dudv\\&=\frac{1}{\rho(a,p+|q|+2)}J^*_{p,q,s}(u)(a).\end{aligned}$$

□

Let $0<s<s'<\infty$. Then it is immediate that

$$\mathcal{H}J^*(p,q,s)\subset\mathcal{H}J^*(p,q,s')\qquad\text{and}\qquad\mathcal{H}J^*_0(p,q,s)\subset\mathcal{H}J^*_0(p,q,s')\,.$$

The following results clarifies the relation between $\mathcal{H}J^*_0(p,q,s)$ and $\mathcal{H}J^*(p,q,s)$.

PROPOSITION (2.20). *Let $-2<q<\infty$, $0<p<\infty$, $0\le s<\infty$ and $u\in\mathcal{H}J^*(0,q,s)$. Then $J^*_{p,q,s}(u)(a)$ is a continuous function, as a function of $a\in\mathbb{D}$.*

Proof. If u is constant on $\mathbb{D}$, it is clear that $J^*_{p,q,s}(u)(a)$ is continuous for all $a\in\mathbb{D}$. Therefore suppose that u is not constant, in particular $J^*_{p,q,s}(u)(0)\neq 0$. Let $a\in\mathbb{D}$ be fixed and let $\delta>0$ be such that $\overline{\mathbb{D}}(a,\delta)\subset\mathbb{D}$. The function $l\colon\overline{\mathbb{D}}\times\overline{\mathbb{D}}(a,\delta)\to\mathbb{R}$ defined by

$$(z,\zeta)\to\frac{(1-|\zeta|^2)^s}{|1-\overline{\zeta}z|^{2s}}$$

is uniformly continuous on $\overline{\mathbb{D}}\times\overline{\mathbb{D}}(a,\delta)$. Then given $\epsilon>0$, there exists $\rho>0$ such that if $|z'-z|<\rho$ and $|\zeta'-\zeta|<\rho$ then

$$|l(z',\zeta')-l(z,\zeta)|<\frac{\epsilon}{J^*_{p,q,s}(u)(0)}\,.$$

Note that $J^*_{p,q,s}(u)(0)<\infty$ since $u\in\mathcal{H}J^*(p,q,s)$. Then if $|a-b|<\rho$,

$$|J^*_{p,q,s}(u)(a)-J^*_{p,q,s}(u)(b)|\le$$

$$\le\iint_{\mathbb{D}}\frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}\left(1-|z|^2\right)^{q+s}|l(z,a)-l(z,b)|\,dx\,dy<\epsilon.$$

□

COROLLARY (2.21). *Let* $-2 < q < \infty$, $0 < p < \infty$, $0 \le s < \infty$. *Then* $\mathcal{H}J_0^*(p,q,s) \subset \mathcal{H}J^*(p,q,s)$.

Proof. If u is a function with $|\nabla(z)| = 0$ for all $z \in \mathbb{D}$, the statement is clear. Suppose u is not a constant and $u \in \mathcal{H}J_0^*(p,q,s)$. Then there exists $0 < R < 1$ such that $J^*_{p,q,s}(u)(a) < J^*_{p,q,s}(u)(0)$ for all $R < |a| < 1$. By Proposition (2.20), $J^*_{p,q,s}(u)$ attains its finite maximum on $\mathbb{D}_R$, so we have $u \in \mathcal{H}J^*(p,q,s)$. □

3. Equivalence between $\mathcal{H}I^*(p,q,s)$ and $\mathcal{H}J^*(p,q,s)$

In this section we obtain basic properties for the hyperbolic weighted harmonic classes $\mathcal{H}I^*(p,q,s)$ and $\mathcal{H}J^*(p,q,s)$. In particular we will see that they coincide.

THEOREM (3.1). *Let* $-2 < q < \infty$, $0 < p < \infty$ *and* $u \in \mathcal{H}(\mathbb{D})$. *If* $J^*_{p,q,s}(u)(0) < \infty$ *then for* $0 < s < \infty$

$$\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^q \ln^s \frac{1}{|z|}\, dx\, dy \le t \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^{q+s}\, dx\, dy \tag{3.2}$$

where $t = t(q,s,R)$ *for some fixed* $0 < R < 1$.

If $J^*_{p,q,s}(u)(0) < \infty$ *then for* $0 < s < 1$

$$\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}|z|^{-2s}(1-|z|^2)^{q+s}\, dx\, dy \le \tilde{t} \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^{q+s}\, dx\, dy \tag{3.3}$$

where $\tilde{t} = \tilde{t}(q,s,R)$ *for some fixed* $0 < R < 1$.

Proof. We follow the idea of the proof of Theorem 2.2 in [2]. Let $c = .0183403$ be the root of $-\ln x = 4(1-x^2)$. Let R be fixed with $c < R < 1$. Define

$$\begin{aligned} 0 < \frac{1}{\tau(q,s,R)} = \int_c^R (1-r^2)^{q+s} r\, dr &= \frac{1}{2(1+q+s)}((1-c^2)^{1+q+s} - (1-R^2)^{1+q+s}) \\ &= \frac{1}{2(1+q+s)}(.999664^{1+q+s} - (1-R^2)^{1+q+s}) . \end{aligned}$$

Since R is fixed, $\tau(q,s,R) = \tau(q,s)$. By Proposition (2.2), $\dfrac{|\nabla u(z)|^p}{(1-u(z)^2)^p}$ is subharmonic. Then

$$\begin{aligned} \frac{1}{\tau(q,s,R)} \int_0^{2\pi} \frac{|\nabla u(ce^{i\theta})|^p}{(1-u(ce^{i\theta})^2)^p}\, d\theta &= \int_c^R (1-r^2)^{q+s} r\, dr \int_0^{2\pi} \frac{|\nabla u(ce^{i\theta})|^p}{(1-u(ce^{i\theta})^2)^p}\, d\theta \\ &\le \int_c^R (1-r^2)^{q+s} r\, dr \int_0^{2\pi} \frac{|\nabla u(re^{i\theta})|^p}{(1-u(re^{i\theta})^2)^p}\, d\theta \\ &= \iint\limits_{D_R(0)\setminus D_c(0)} \frac{|\nabla u(re^{i\theta})|^p}{(1-u(re^{i\theta})^2)^p}(1-|z|^2)^{q+s}\, dx\, dy \\ &\le \iint\limits_{D_R(0)} \frac{|\nabla u(re^{i\theta})|^p}{(1-u(re^{i\theta})^2)^p}(1-|z|^2)^{q+s}\, dx\, dy . \end{aligned}$$

Thus

$$\int_0^{2\pi} \frac{|\nabla u(ce^{i\theta})|^p}{(1-u(ce^{i\theta})^2)^p}\, d\theta \le \tau(q,s,R) \iint_{D_R(0)} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^{q+s}\, dx\, dy . \tag{3.4}$$

Define

$$0 < \tilde{\tau}(q,s) = \int_0^c r(1-r^2)^q \ln^s \frac{1}{r}\, dr < \infty. \tag{3.5}$$

By subharmonicity and (3.4) we have the estimate

$$\begin{aligned}
&\iint_{D_c(0)} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^q \ln^s \frac{1}{|z|}\, dx\,dy \\
&= \int_0^c \int_0^{2\pi} \frac{|\nabla u(re^{i\theta})|^p}{(1-u(re^{i\theta})^2)^p}(1-r^2)^q \ln^s \frac{1}{r}\, r\, d\theta\, dr \\
&\le \int_0^c r(1-r^2)^q \ln^s \frac{1}{r}\, dr \int_0^{2\pi} \frac{|\nabla u(ce^{i\theta})|^p}{(1-u(ce^{i\theta})^2)^p}\, d\theta \\
&\le \tilde{\tau}(q,s)\tau(q,s,R) \iint_{D_r(0)} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^{q+s}\, dx\,dy \\
&\le \tau(q,s,R)\tilde{\tau}(q,s) \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^{q+s}\, dx\,dy. \qquad (3.6)
\end{aligned}$$

From the inequality

$$-\ln x \le 4(1-x^2) \quad \text{for each } x \in (c,1] \qquad (3.7)$$

we have

$$\begin{aligned}
&\iint_{\mathbb{D}\setminus D_c(0)} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^q \ln^s \frac{1}{|z|}\, dx\,dy \\
&\le 4^s \iint_{\mathbb{D}\setminus D_c(0)} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^{q+s}\, dx\,dy \\
&\le 4^s \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^{q+s}\, dx\,dy\,. \qquad (3.8)
\end{aligned}$$

Let $t(q,s,R) = \tau(q,s,R)\tilde{\tau}(q,s) + 4^s$. Combining (3.6) and (3.8) we have

$$\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^q \ln^s \frac{1}{|z|}\, dx\,dy \le t(q,s,R) \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^{q+s}\, dx\,dy\,.$$

For $0 < s < 1$ we need to consider instead of (3.5) the following expression,

$$0 < \int_0^c r^{1-2s}(1-r^2)^{q+s}\, dr = \frac{1}{2} B[c^2, 1-s, 1+q+s].$$

Here B is the incomplete Beta function, and we prove (3.3) in a similar way. □

THEOREM (3.9). *Let $-2 < q < \infty$, $0 < p < \infty$ and $0 \le s < \infty$. Then for $u \in \mathcal{H}(\mathbb{D})$*

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^q g^s(z,a)\, dx\,dy < \infty \qquad (3.10)$$

if and only if

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p}(1-|z|^2)^q (1-|\phi_a(z)|^2)^s\, dx\,dy < \infty\,. \qquad (3.11)$$

Proof. We have

$$1-x^2 \le -2\ln x \quad \text{for each } x \in (0,1]\,.$$

Taking $x = |\phi_a(z)|$ we have $1-|\phi_a(z)|^2 \le 2g(z,a)$, thus

$$\begin{aligned}
&\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p} \left(1-|z|^2\right)^q \left(1-|\phi_a(z)|^2\right)^s dx\,dy \\
&\qquad \le \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{(1-u(z)^2)^p} \left(1-|z|^2\right)^q g^s(z,a)\, dx\,dy,
\end{aligned}$$

that is,

$$J^*_{p,q,s}(u)(a) \le 2I^*_{p,q,s}(u)(a) \quad \text{for each } a \in \mathbb{D} .$$

Then (3.10) implies (3.11).

By the hypothesis and Lemma (2.19), $J^*_{p,q,s}(u\circ\phi_a)(0) < \infty$. Since $u\circ\phi_a : \mathbb{D} \to (-1,1)$ is a harmonic function and $0 < p < \infty$, then by Proposition (2.2)

$$\frac{|\nabla(u\circ\phi_a)(z)|^p}{(1-u(\phi_a(z))^2)^p}$$

is a subharmonic function. Therefore the inequality (3.2) implies

$$\iint_{\mathbb{D}} \frac{|\nabla(u\circ\phi_a)(z)|^p}{(1-u(\phi_a(z))^2)^p}(1-|z|^2)^q \ln^s \frac{1}{|z|}\,dx\,dy$$
$$\le t(q,s,R)\iint_{\mathbb{D}} \frac{|\nabla(u\circ\phi_a)(z)|^p}{(1-u(\phi_a(z))^2)^p}(1-|z|^2)^{q+s}\,dx\,dy .$$

Consider the change of variable $z = \phi_a(w)$ to obtain

$$\iint_{\mathbb{D}} \frac{|\nabla(w)|^p}{(1-u(w)^2)^p}|\phi_a'(\phi_a(w))|^p|\phi_a'(w)|^2(1-|\phi_a(w)|^2)^q \ln^s \frac{1}{|\phi_a(w)|}\,d\xi\,d\eta$$
$$\le t(q,s,R)\iint_{\mathbb{D}} \frac{|\nabla(w)|^p}{(1-u(w)^2)^p}|\phi_a'(\phi_a(w))|^p|\phi_a'(w)|^2(1-|\phi_a(w)|^2)^{q+s}\,d\xi\,d\eta$$

or equivalently

$$\begin{aligned} 0 \quad &\le \quad \iint_{\mathbb{D}} \frac{|\nabla(w)|^p}{(1-u(w)^2)^p}|\phi_a'(\phi_a(w))|^p|\phi_a'(w)|^2(1-|\phi_a(w)|^2)^q \cdot \\ &\quad\ \left(t(q,s,R)(1-|\phi_a(w)|^2)^s - \ln^s \frac{1}{|\phi_a(w)|}\right)d\xi\,d\eta \\ &\le \quad \rho(a,p+|q|+2)\iint_{\mathbb{D}} \frac{|\nabla(w)|^p}{(1-u(w)^2)^p}(1-|w|^2)^q \cdot \\ &\quad\ \left(t(q,s,R)(1-|\phi_a(w)|^2)^s - \ln^s \frac{1}{|\phi_a(w)|}\right)d\xi\,d\eta. \end{aligned}$$

As $0 < \rho(a,p+|q|+2)$ we obtain

$$\iint_{\mathbb{D}} \frac{|\nabla(w)|^p}{(1-u(w)^2)^p}(1-|w|^2)^q \ln^s \frac{1}{|\phi_a(w)|}\,d\xi\,d\eta$$
$$\le \quad t(q,s,R)\iint_{\mathbb{D}} \frac{|\nabla(w)|^p}{(1-u(w)^2)^p}(1-|w|^2)^q(1-|\phi_a(w)|^2)^s\,d\xi\,d\eta \tag{3.12}$$

and the theorem follows. □

From Theorem (3.9) we have

COROLLARY (3.13). *Let $-2 < q < \infty$, $0 < p < \infty$, $0 \le s < \infty$. Then $\mathcal{H}I^*(p,q,s) = \mathcal{H}J^*(p,q,s)$.*

COROLLARY (3.14). *Let $-2 < q < \infty$, $0 < p < \infty$ and $0 \le s < \infty$. Then $\mathcal{H}I^*_0(p,q,s) = \mathcal{H}J^*_0(p,q,s)$.*

Now we give a characterization of the classes $\mathcal{H}J^*(p,q,s)$ in terms of Carleson squares:

For $0 < s < \infty$, we say that a positive measure μ defined on $\mathbb{D}$ is a bounded s-Carleson measure provided

$$\sup_{I\subset T} \frac{\mu(S(I))}{|I|^s} < \infty$$

where $|I|$ denotes the arc length of $I \subset T$ and $S(I)$ denotes the Carleson box based on I, that is

$$S(I) = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I, 1-|z| \leq \frac{|I|}{2\pi} \right\}.$$

If

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^s} = 0$$

for $I \subset T$, we say that is a compact $s-$Carleson measure. By Lemma 2.1 in [3] we obtain

THEOREM (3.15). *Let $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$. Consider $u \in \mathcal{H}(\mathbb{D})$ and*

$$d\mu_u = \frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p} \left(1-|z|^2\right)^{q+s} dx\,dy.$$

Then

- $u \in \mathcal{H}J^*(p,q,s)$ *if and only if* $d\mu_u(z)$ *is a bounded* s-Carleson measure.
- $u \in \mathcal{H}J_0^*(p,q,s)$ *if and only if* $d\mu_u(z)$ *is a compact* s-Carleson measure.

4. Several properties of weighted harmonic classes

It is clear that the sets $\mathcal{H}J^*(p,q,s)$ and $\mathcal{H}J_0^*(p,q,s)$ are not vector spaces. However they have some other interesting properties.

Although $uv\colon \mathbb{D} \to (-1,1)$ is not necessarily a harmonic function when $u,v \in \mathcal{H}(\mathbb{D})$, we have

THEOREM (4.1). *Let $0 < p < \infty$, $-2 < q < \infty$, $0 \leq s < \infty$. If $u,v \in \mathcal{H}J^*(p,q,s)$, then*

$$\sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} \frac{|\nabla(uv)(z)|^p}{\left(1-((uv)(z))^2\right)^p} (1-|z|^2)^q (1-|\phi_a(z)|^2)^s dx\,dy < \infty;$$

and if $u,v \in \mathcal{H}J_0^*(p,q,s)$, then

$$\lim_{|a|\to 1^-} \iint_{\mathbb{D}} \frac{|\nabla(uv)(z)|^p}{\left(1-((uv)(z))^2\right)^p} (1-|z|^2)^q (1-|\phi_a(z)|^2)^s dx\,dy = 0.$$

Proof. Let $u,v \in \mathcal{H}J^*(p,q,s)$. Note that

$$\nabla(u\,v)(z) = v(z)\nabla u(z) + u(z)\nabla v(z) \qquad \text{for all } z \in \mathbb{D}$$

and

$$\begin{aligned} |\nabla(u\,v)(z)|^p &= |v(z)\nabla u(z) + u(z)\nabla v(z)|^p \leq \Big[|v(z)\nabla u(z)| + |u(z)\nabla v(z)|\Big]^p \\ &\leq 2^p \Big[|u(z)\nabla v(z)|^p + |v(z)\nabla u(z)|^p\Big]. \end{aligned}$$

Since $u\colon \mathbb{D} \to (-1,1)$ and $v\colon \mathbb{D} \to (-1,1)$, we have $u(z)^2 v(z)^2 < 1$ for all $z \in \mathbb{D}$. Also $u(z)^2 v(z)^2 < v(z)^2$, then $1-v(z)^2 < 1-u(z)^2 v(z)^2$ and $1-u(z)^2 < 1-u(z)^2 v(z)^2$.

Therefore

$$\begin{aligned}\frac{|\nabla(u\,v)(z)|^p}{\left(1-((u\,v)(z))^2\right)^p} &\le 2^p\frac{|u(z)|^p|\nabla v(z)|^p}{\left(1-v(z)^2\right)^p}+2^p\frac{|v(z)|^p|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p}\\ &\le 2^p\frac{|\nabla v(z)|^p}{\left(1-v(z)^2\right)^p}+2^p\frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p}\end{aligned}$$

and the theorem follows. □

Remember that for $u,v\in\mathcal{H}(\mathbb{D})$, $uv\colon\mathbb{D}\to(-1,1)$ is a harmonic function if and only if $\nabla u\cdot\nabla v\equiv 0$, that is, if u y v are harmonic conjugate functions. In this case we have the following.

COROLLARY (4.2). *Let $0<p<\infty$, $-2<q<\infty$, $0\le s<\infty$ and $u,v\in\mathcal{H}J^*(p,q,s)$ such that $\nabla u\cdot\nabla v\equiv 0$. Then $uv\in\mathcal{H}J^*(p,q,s)$ and for $u,v\in\mathcal{H}J_0^*(p,q,s)$, $uv\in\mathcal{H}J_0^*(p,q,s)$.*

Now we will make our classes into convex metric spaces: for $0<p<\infty$, $-2<q<\infty$, $0\le s<\infty$ and $u,w\in\mathcal{H}(\mathbb{D})$ define the metric d in the class $\mathcal{H}I^*(p,q,s)$ as

$$d(u,w):=|u(0)-w(0)|+\sup_{a\in\mathbb{D}}\left[\iint_{\mathbb{D}}\left|\frac{\nabla u(z)}{1-u(z)^2}-\frac{\nabla w(z)}{1-w(z)^2}\right|^p d\mu\right]^{\frac{1}{p}}\ \text{for } 1<p<\infty$$

and

$$d(u,w):=|u(0)-w(0)|+\sup_{a\in\mathbb{D}}\left[\iint_{\mathbb{D}}\left|\frac{\nabla u(z)}{1-u(z)^2}-\frac{\nabla w(z)}{1-w(z)^2}\right|^p d\mu\right]\quad \text{for } 0<p\le 1,$$

where $d\mu=(1-|z|^2)^q g^s(z,a)\,dx\,dy$.

- $1<p<\infty$

Observe that d satisfies the metric properties which are inherited from the Lebesgue spaces $L^p(d\mu)$.

- $0<p\le 1$

To prove the triangle inequality we observe that is enough to show that

$$(\alpha+\beta)^p\le\alpha^p+\beta^p$$

for $0<p\le 1$ and $\alpha>0$, $\beta>0$; and this is equivalent to the inequality

$$1\le a^p+b^p\qquad \text{with } a+b=1, a>0, b>0.$$

It is easy to see that

$$0\le a^p+(1-a)^p-1\qquad \text{with } 0<a<1.$$

Finally, we only prove that if $d(u,w)=0$ then $u=v$, since the other properties follow easily from the properties of the complex module. In this way

$$\left|\frac{\nabla u(z)}{1-u(z)^2}-\frac{\nabla w(z)}{1-w(z)^2}\right|=0 \tag{4.3}$$

and

$$|u(0)-w(0)|=0. \tag{4.4}$$

From (4.3)

$$\frac{\nabla u(z)}{1-u(z)^2}=\frac{\nabla w(z)}{1-w(z)^2}.$$

It follows that

$$\left(\frac{\frac{\partial u(z)}{\partial x}}{1-u(z)^2}, \frac{\frac{\partial u(z)}{\partial y}}{1-u(z)^2}\right) = \left(\frac{\frac{\partial w(z)}{\partial x}}{1-w(z)^2}, \frac{\frac{\partial w(z)}{\partial y}}{1-w(z)^2}\right)$$

for $z = x + iy$. Now

$$\int \frac{\frac{\partial u(z)}{\partial x}}{1-u(z)^2}\, dx = \int \frac{\frac{\partial w(z)}{\partial x}}{1-w(z)^2}\, dx$$

implies

$$\ln \frac{1+u(z)}{1-u(z)} = \ln \frac{1+w(z)}{1-w(z)} + C(y),$$

then

$$\frac{1+u(z)}{1-u(z)} = e^{C(y)} \frac{1+w(z)}{1-w(z)};$$

and from (4.4) it follows that $e^{C(y)} = 1$, which implies that $C(y) = 0$.

Hence

$$\begin{aligned} (1+u(z))(1-w(z)) &= (1-u(z))(1+w(z)) \\ 1-w(z)+u(z)-u(z)w(z) &= 1+w(z)-u(z)-u(z)w(z) \\ u(z) &= w(z). \end{aligned}$$

PROPOSITION (4.5). *Let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. Then the class $(\mathcal{H}I^*(p,q,s), d)$ equipped with the metric d is a complete metric space. Moreover, $\mathcal{H}I_0^*(p,q,s)$ is a closed (and therefore complete) subspace of $\mathcal{H}I^*(p,q,s)$.*

Proof. The family $\mathcal{H}(\mathbb{D})$ is locally bounded and therefore is normal. Let $\{u_n\}$ be a Cauchy sequence in $\mathcal{H}I^*(p,q,s)$ which converges to u uniformly on compact subsets of $\mathbb{D}$. Let $\epsilon > 0$. Then there exists N such that if $n \geq m > N$,

$$\int_{\mathbb{D}} \left| \frac{\nabla u_n(z)}{1-u_n(z)^2} - \frac{\nabla u_m(z)}{1-u_m(z)^2} \right|^p (1-|z|^2)^q\, g^s(z,a)\, dx\, dy < \frac{\epsilon^p}{2}\ .$$

Then Fatou's lemma yields

$$\begin{aligned} &\int_{\mathbb{D}} \left| \frac{\nabla u(z)}{1-u(z)^2} - \frac{\nabla u_m(z)}{1-u_m(z)^2} \right|^p (1-|z|^2)^q\, g^s(z,a)\, dx\, dy \\ &\leq \int_{\mathbb{D}} \lim_{n\to\infty} \left| \frac{\nabla u_n(z)}{1-u_n(z)^2} - \frac{\nabla u_m(z)}{1-u_m(z)^2} \right|^p (1-|z|^2)^q\, g^s(z,a)\, dx\, dy\ < \epsilon^p. \end{aligned}$$

It follows that

- For $0 < p < 1$, the estimates

$$\left| \frac{\nabla u(z)}{1-u(z)^2} \right|^p \leq \left| \frac{\nabla u(z)}{1-u(z)^2} - \frac{\nabla u_m(z)}{1-u_m(z)^2} \right|^p + \left| \frac{\nabla u_m(z)}{1-u_m(z)^2} \right|^p .$$

$$\begin{aligned} \int_{\mathbb{D}} &\frac{|\nabla u(z)|^p}{(1-u(z)^2)^p} (1-|z|^2)^q\, g^s(z,a)\, dx\, dy \\ &\leq \epsilon^p + \int_{\mathbb{D}} \frac{|\nabla u_m(z)|^p}{(1-u_m(z)^2)^p} (1-|z|^2)^q\, g^s(z,a)\, dx\, dy . \end{aligned} \tag{4.6}$$

Thus $u \in \mathcal{H}I^*(p,q,s)$.

- for $1 < p < \infty$

$$
\begin{aligned}
(4.7)\quad &\left(\int_{\mathbb{D}}\left|\frac{\nabla u(z)}{1-u(z)^2}\right|^p (1-|z|^2)^q\, g^s(z,a)\,dx\,dy\right)^{\frac{1}{p}}\\
&\le\left(\int_{\mathbb{D}}\left|\frac{\nabla u(z)}{1-u(z)^2}-\frac{\nabla u_m(z)}{1-u_m(z)^2}\right|^p (1-|z|^2)^q\, g^s(z,a)\,dx\,dy\right)^{\frac{1}{p}}\\
&\qquad+\left(\int_{\mathbb{D}}\left|\frac{\nabla u_m(z)}{1-u_m(z)^2}\right|^p (1-|z|^2)^q\, g^s(z,a)\,dx\,dy\right)^{\frac{1}{p}}.
\end{aligned}
$$

Thus $u \in \mathcal{H}I^*(p,q,s)$.

The second part of the assertion follows by (4.6) and (4.7). □

THEOREM (4.8). *Let $0<p<\infty$, $-2<q<\infty$, $0\le s<\infty$. Then for all $0\le t\le 1$*

$$(1-t)\cdot\mathcal{H}J^*(p,q,s)+t\mathcal{H}J^*(p,q,s)=\mathcal{H}J^*(p,q,s)$$

and

$$(1-t)\cdot\mathcal{H}J_0^*(p,q,s)+t\mathcal{H}J_0^*(p,q,s)=\mathcal{H}J_0^*(p,q,s)\,.$$

Proof. For $t=0$, $t=1$ the result is immediate, thus we are going to consider $t\in(0,1)$. Let $0<t<1$ be fixed and $u,\ v\in\mathcal{H}J^*(p,q,s)$. Note that

$$|\nabla((1-t)u(z)+tv(z))|=|(1-t)\nabla u(z)+t\nabla v(z)|\le(1-t)|\nabla u(z)|+t|\nabla v(z)|$$

and

$$|\nabla((1-t)u(z)+tv(z))|^p\le 2^p\left[(1-t)^p|\nabla u(z)|^p+t^p|\nabla v(z)|^p\right]$$

for all $z\in\mathbb{D}$. Then

$$
\begin{aligned}
&\iint_{\mathbb{D}}\frac{|\nabla((1-t)u(z)+tv(z))|^p}{\left(1-\left((1-t)u(z)+tv(z)\right)^2\right)^p}(1-|z|^2)^q\left(1-|\phi_a(z)|^2\right)^s dx\,dy\\
&\qquad\le 2^p\iint_{\mathbb{D}}\frac{(1-t)^p|\nabla u(z)|^p+t^p|\nabla v(z)|^p}{\left(1-\left((1-t)u(z)+tv(z)\right)^2\right)^p}(1-|z|^2)^q\left(1-|\phi_a(z)|^2\right)^s dx\,dy.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\left((1-t)u(z)+tv(z)\right)^2 &\le \left((1-t)|u(z)|+t|v(z)|\right)^2\\
&= (1-t)^2u(z)^2+2t(1-t)|u(z)|\,|v(z)|+t^2v(z)^2\\
&\le (1-t)^2+2t(1-t)+t^2v(z)^2
\end{aligned}
$$

and

$$
\begin{aligned}
1-\left((1-t)u(z)+tv(z)\right)^2 &\ge 1-\left((1-t)^2+2t(1-t)+t^2v(z)^2\right)\\
&= t^2\left(1-v(z)^2\right).
\end{aligned}
$$

For the second term in the sum of the numerator,

$$
\begin{aligned}
&\iint_{\mathbb{D}}\frac{t^p|\nabla v(z)|^p}{\left(1-\left((1-t)u(z)+tv(z)\right)^2\right)^p}(1-|z|^2)^q\left(1-|\phi_a(z)|^2\right)^s dx\,dy.\\
&\qquad\le\frac{1}{t^{2p}}\iint_{\mathbb{D}}\frac{|\nabla v(z)|^p}{\left(1-v(z)^2\right)^p}(1-|z|^2)^q\left(1-|\phi_a(z)|^2\right)^s dx\,dy.
\end{aligned}
$$

Now, for the first term we get

$$\begin{aligned}
\Big((1-t)u(z)+tv(z)\Big)^2 &\le \Big((1-t)|u(z)|+t|v(z)|\Big)^2 \\
&= (1-t)^2u(z)^2+2t(1-t)|u(z)|\,|v(z)|+t^2v(z)^2 \\
&\le (1-t)^2u(z)^2+2t(1-t)+t^2 \\
&\le (1-t)^2u(z)^2+2t-t^2
\end{aligned}$$

and

$$\begin{aligned}
1-\Big((1-t)u(z)+tv(z)\Big)^2 &\ge 1-\Big((1-t)^2u(z)^2+2t-t^2\Big) \\
&= (1-t^2)\left(1-u(z)^2\right)
\end{aligned}$$

$$\begin{aligned}
&\iint_{\mathbb{D}} \frac{(1-t)^p|\nabla u(z)|^p}{\left(1-\Big((1-t)u(z)+tv(z)\Big)^2\right)^p}(1-|z|^2)^q\left(1-|\phi_a(z)|^2\right)^s\,dx\,dy \\
&\qquad\le \frac{1}{(1+t)^p}\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p}(1-|z|^2)^q\left(1-|\phi_a(z)|^2\right)^s\,dx\,dy,
\end{aligned}$$

and the theorem follows from this estimate. □

It is possible to replace the weight $(1-|\phi_a(z)|^2)^s$ by its reflection $(|\phi_a(z)|^{-2}-1)^s$, as the following theorem shows.

THEOREM (4.9). *Let $0<p<\infty$, $-2<q<\infty$, $0<s<1$ be. Then $u\in\mathcal{H}J^*(p,q,s)$ if and only if*

$$\sup_{a\in\mathbb{D}}\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p}(1-|z|^2)^q\left(|\phi_a(z)|^{-2}-1\right)^s\,dx\,dy<\infty\,.$$

Proof. Since $|\phi_a(z)|^2<1$ for all $z\in\mathbb{D}$, then $\left(1-|\phi_a(z)|^2\right)^s<\left(|\phi_a(z)|^{-2}-1\right)^s$. Therefore

$$\begin{aligned}
&\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p}(1-|z|^2)^q\left(1-|\phi_a(z)|^2\right)^s\,dx\,dy \\
&\qquad\le \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p}(1-|z|^2)^q\left(|\phi_a(z)|^{-2}-1\right)^s\,dx\,dy.
\end{aligned}$$

Conversely, if

$$\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p}(1-|z|^2)^q\left(1-|\phi_a(z)|^2\right)^s\,dx\,dy<\infty,$$

then consider

$$\begin{aligned}
&\iint_{\mathbb{D}} \frac{|\nabla u\left(\phi_a(w)\right)|^p}{\left(1-u\left(\phi_a(w)\right)^2\right)^p}\,\frac{\left(1-|w|^2\right)^s}{|w|^{2s}}(1-|w|^2)^q\,d\xi\,d\eta \\
&\qquad\le \tilde{t}\iint_{\mathbb{D}} \frac{|\nabla u\left(\phi_a(w)\right)|^p}{\left(1-u\left(\phi_a(w)\right)^2\right)^p}\left(1-|w|^2\right)^{q+s}\,d\xi\,d\eta;
\end{aligned}$$

where $\tilde{t} = \tilde{t}(q,s,R)$ as in Theorem (3.1). Consider the change of variable $z = \phi_a(w)$ to obtain

$$\iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p} |\phi_a'(z)|^2 \frac{\left(1-|\phi_a(z)|^2\right)^s}{|\phi_a(z)|^{2s}} (1-|\phi_a(z)|^2)^q \, dx\, dy$$
$$\leq \tilde{t} \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p} |\phi_a'(z)|^2 \left(1-|\phi_a(z)|^2\right)^{q+s} dx\, dy;$$

then

$$0 \leq \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p} |\phi_a'(z)|^2 (1-|\phi_a(z)|^2)^q \left[\tilde{t}(1-|\phi_a(z)|^2)^s - \frac{(1-|\phi_a(z)|^2)^s}{|\phi_a(z)|^2}\right] dx\, dy,$$

by the estimates for $(1-|\phi_a(z)|^2)^q$ and $|\phi_a'(z)|^2$ in (2.3) and (2.4) respectively we get

$$0 \leq \rho(a,|q|+2) \iint_{\mathbb{D}} \frac{|\nabla u(z)|^p}{\left(1-u(z)^2\right)^p} (1-|z|^2)^q \left[\tilde{t}(1-|\phi_a(z)|^2)^s - \frac{(1-|\phi_a(z)|^2)^s}{|\phi_a(z)|^2}\right] dx\, dy$$

and the result follows from this estimate. □

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References

[1] R. Aulaskari, P. Lappan, *Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal*, Complex analysis and its applications, Pitman Research Notes in Mathematics, **305**, Longman Scientific and Technical, Harlow, 1994, 136–146.

[2] R. Aulaskari, L. F. Reséndis O. and L.M. Tovar S., *Hyperbolic weighted Bergman classes*, Bol. Soc. Mat. Mexicana (3), **14** (2) (2008), 217–232.

[3] R. Aulaskari, D. Stegenga, and J. Xiao, *Some subclasses of BMOA and their characterization in terms of Carleson measures*, Rocky Mountain J. Math. **26** (2) (1996), 485–506.

[4] R. Aulaskari, J. Xiao and R. Zhao, *On subspaces and subsets of BMOA and UBC*, Analysis **15** (1995), 101–121.

[5] J. B. CONWAY, Functions of one complex variable (1st. ed.), Grad. Texts in Math., **11**, Springer-Verlag, New York-Heidelberg, 1973.

[6] J. LAWRYNOWICZ, L. F. RESÉNDIS O. AND L.M. TOVAR S., *Like-Hyperbolic Bloch-Bergman classes*, Contemp. Math. **509** (2010), 103–118.

[7] X. LI, *On Hyperbolic Q Classes*, Ann. Acad. Sci. Fenn. Math. Diss. **145** (2005).

[8] E. RAMÍREZ DE ARELLANO, L. F. RESÉNDIS O. AND L.M. TOVAR S., *Zhao $F(p,q,s)$ function spaces and harmonic majorants*, Bol. Soc. Mat. Mexicana (3), **11** (2) (2005), 241–258.

[9] L. F. RESÉNDIS O. AND L.M. TOVAR S., *Weighted harmonic spaces*, Function Spaces in Complex and Clifford Analysis, Int. Soc. Anal. Appl. Comput., 2008, 36–45.

[10] W. SMITH, *Inner Functions in the Hyperbolic Little Bloch Class*, Michigan Math. J. **45** (1998), 103–114.

[11] W. SMITH AND R. ZHAO, *Composition operators mapping into the Q_p spaces*, Analysis **17** (1997), 239–263.

[12] J. XIAO, Holomorphic Q classes, Lecture Notes in Math.**1767**, Springer, New York, 2001.

[13] S. YAMASHITA, *Hyperbolic Hardy Class H^1*, Math. Scand. **45** (1979), 261–266.

[14] S. YAMASHITA, *Hyperbolic Hardy Classes and hyperbolically Dirichlet-finite functions*, Hokkaido Math. J. **10** (1981), 709–722.

[15] S. YAMASHITA, *Functions of uniformly bounded characteristic*, Ann. Acad. Sci. Fenn. Math., Series A. I. **7** (1982), 349–367.

[16] S. YAMASHITA, *Holomorphic Functions of Hyperbolically Bounded Mean Oscillation*, Boll. Un. Mat. Ital. B **(6)** 5 (1986), no. 3, 983–1000.

[17] K. ZHU, Spaces of Holomorphic Functions in the Unit Ball, Springer, New York, 2005.

[18] R. ZHAO, *On α-Bloch functions and $VMOA$*, Acta Math. Sci. **16** (3) (1996), 349–360.

[19] R. ZHAO, *On a general family of functions spaces*, Ann. Acad. Sci. Fenn. Math. Diss., **105** (1996).