

Boletín de la

SOCIEDAD MATEMÁTICA MEXICANA

Tercera Serie

Volumen 18

Número 2

Octubre de 2012

Contenido

ARTÍCULOS PANORÁMICOS

Jorge Ize: A tribute to his mathematical work

Z. Balanov, W. Krawcewicz, and J. Pejsachowicz 89

The work of Jorge Ize regarding the n -body problem

C. García-Azpetitia 113

The academic legacy of Ernesto A. Lacomba

E. Pérez Chavela 129

ARTÍCULOS DE INVESTIGACIÓN

On the isosceles solutions of the three-body problem

H. E. Cabral 135

Radó's theorem for factorisations of the Laplace Operator

C. Gonzalez-Flores and E. S. Zeron 143

The strong matrix Stieltjes moment problem

A. E. Choque Rivero and S. M. Zagorodnyuk 151

The almost sure central limit theorem for randomly indexed sums of associated random variables

M. Dudziński and P. Górká 171



JORGE IZE: A TRIBUTE TO HIS MATHEMATICAL WORK

Z. BALANOV, W. KRAWCEWICZ, AND J. PEJSACHOWICZ

Foreword

We were all shocked and devastated by the news of Jorge's passing. His mathematical ideas and personal relationship have permanently influenced our lives. Jorge constructively affected our professional attitudes, gave us a strong sense of responsibility as researchers and professionals, and was a close personal friend. We believe that nobody is better suited to express the feelings confronting Jorge's premature departure than Alfonso Vignoli, his dearest friend and close collaborator for forty years.

I met Jorge in June 1973 at a conference in Canada. Our encounter could have not been more explosive. I found myself talking to a person very different from me: reserved and of few, albeit essential words. After each conference I was astounded by the depth in analysis of the results submitted. He often put me in a difficult spot by asking my opinion on a newly announced theorem. He was not, I underscore, a mathematics 'crank'. He adored art, food, traveling and knowing the history of nations. He was abreast of politics, even Italian, and his readings were always lucid. I remember our visits to the church of Santa Maria Maggiore in Rome, with Jorge equipped with small, powerful binoculars to probe the mosaics, looking for precious particulars. I think some of the best results we achieved upon return in our Roman home with a good glass of wine that was never missing. Coming to Italy relieved him from the headaches that assailed him after a glass of wine in Mexico City, probably, due to altitude. I believe one of the reasons our friendship grew stronger over the years is tied to (quite) a few hearty drinks and laughs, and to his subtle irony that let transpire a deep affection for me and my family. Of his extended scientific production I only want to highlight the book Equivariant Degree Theory—a book Jorge did not want to write. I had to insist manifold, and almost court him, to convince him. The result was a much cleaner version of the theory and a text filled with his ideas that I believe will give several cues to the coming schools of researchers. Many were the research projects upon which we fantasized, dreamed and ironized of. Many were the meetings we had planned for the coming months and years, even 'just' to drink our glass of wine, to laugh and see him raise his eyebrow to my free and odd stories.

Alfonso Vignoli.

1. Introduction

Without doubt the name of Jorge Ize will ultimately be attached to two related areas of nonlinear functional analysis. The first is the topological approach to bifurcation of solutions of a parametrized family of nonlinear equations, termed by him “topological bifurcation”. The second is the degree theory for maps which are equivariant with respect to the action of a compact Lie group. Roughly speaking, he worked mainly on topological bifurcation for the twenty years between 1972 to 1992, and devoted the next twenty years mainly to the equivariant degree and its applications to bifurcation theory.

Our aim here is to give a non-technical presentation of the main achievements of Jorge Ize in both of these areas and of the influence of his ideas on the work of other researchers in this field, and in particular on our own research. The third named author will deal with topological bifurcation in Part I, while the other two authors will present the material related to the equivariant degree in Part II.

In Part I, after a short introduction to topological methods in bifurcation theory (sections 2.1, 2.2), we will discuss what are, in our opinion, the main contributions to one-parameter bifurcation of Ize’s brilliant PhD thesis *Bifurcation theory for Fredholm operators* and of his paper [50] which appeared in *Memoires of AMS* with the same title (section 2.3). Then we will review his work on several parameter bifurcation (section 2.4) and present some recent developments. Section 2.5 is devoted to Jorge’s work on the global structure and dimension of branches of solutions to multiparameter nonlinear equations. We finish this part by discussing the use of the equivariant obstruction theory in multiparameter Hopf bifurcation (section 2.6) which paved the way to the equivariant degree theory.

Part II starts with the motivation (section 3.1) behind the equivariant degree theory. Then, in section 3.2, we present the construction of the equivariant degree due to Ize *et al.* Next, in section 3.3, we discuss its range of values (i.e. the equivariant homotopy groups of spheres). Computability and various versions of the equivariant degree are considered in section 3.4. Applications of the equivariant degree are outlined in section 3.5. Finally, in section 3.6, we give some comments on the monograph “*Equivariant Degree Theory*” by Ize and Vignoli.

Let us point out that our presentation of Ize’s work is very far from being comprehensive. It leaves aside all of his applied research related to his activity as coordinator of Fenomec, an interdisciplinary group of scientists from UNAM devoted to the study of nonlinear phenomena in natural sciences. It also leaves aside several mathematical textbooks, articles about the history of mathematics and academic politics, and several surveys of bifurcation theory and equivariant degree [52, 55, 56] which he wrote.

2. Part I Topological bifurcation

(2.1) Bifurcation from the trivial branch. Bifurcation is one of those ill-defined concepts in the mathematical literature. The study of bifurcation, vaguely understood as “change in morphology”, arises in many different areas of mathematics as well as in the natural sciences and engineering. Each of these fields has given his own imprint to the concept, which ultimately led to its indeterminacy.

However, one particular aspect of bifurcation theory, namely bifurcation of solutions of a parametrized family of equations from a trivial branch of solutions,

is one of the oldest and best understood notions of bifurcation in mathematics. Indeed, the earliest example of a specific bifurcation phenomenon of this type can be traced back to Leonard Euler who, in 1757, studied the deviation of a loaded vertical elastic column as it buckled from its position of equilibrium, which he modeled as a boundary value problem parametrized by the load. By increasing the load on the top of the column the vertical equilibrium position becomes unstable and the column changes its configuration, acquiring a new configuration each time the amount of load, considered as a parameter, crosses some critical values. The boundary value problem always has a solution corresponding to the vertical configuration, but new solutions appear at critical values of the load. Euler proved that those critical values, when suitably normalized, coincide with the eigenvalues of the linearization of the problem at the trivial solution.

Another bifurcation phenomenon was studied in 1834 by Jacobi, who discovered new gyrostatic equilibria taking shapes of asymmetric ellipsoids, bifurcating from the known branch of MacLaurin spheroids. Prior to his discovery the MacLaurin spheroids were considered as the only possible shape of a liquid body in gyrostatic equilibrium [48]. We remark in passing that Jacobi's last geometric theorem, which characterizes conjugate points along an extremal of a variational integral as intersection points of the given extremal with the envelope of the family of extremals through the initial point, can be understood as yet another manifestation of bifurcation from a trivial branch.

In his article *L'Équilibre d'une masse fluide animée d'un mouvement de rotation*, published in 1885 in *Acta Mathematica*, Henri Poincaré conjectured the existence of another branch of gyrostatic equilibria exiting from the branch of Jacobi ellipsoids, using, for the first time, the word "bifurcation" to designate the phenomenon.

The above classical examples belong to a much wider variety of bifurcation phenomena which arise in geometry, analysis, mathematical elasticity, hydrodynamics, elementary particle physics, engineering, mathematical biology and many other fields of knowledge. They provide sufficient motivation for the formulation of a general mathematical theory of bifurcation from the trivial branch.

The leitmotiv of such a theory, as seen from the above discussion, can be schematized as follows: assuming that there is a known (trivial) branch of solutions of a parametrized family of nonlinear equations, find necessary and sufficient conditions for the appearance of nontrivial solutions arbitrarily close to some points of the trivial branch. The corresponding values of the parameter are called bifurcation points. In many cases one has to deal with one parameter, either real or complex, but several parameter bifurcation is also of considerable interest.

Toward the end of the nineteenth century, motivated by the study of periodic orbits of small amplitude near equilibrium points of an autonomous differential equation, Henri Poincaré laid the foundations of bifurcation theory. However, in the framework of his qualitative theory of dynamical systems, the term bifurcation acquired a broader meaning than the one discussed in this work.

An important tool for the analysis of bifurcation was invented by Lyapunov and Schmidt at the beginning of the past century. It is known as the Lyapunov-Schmidt reduction. It permits a given bifurcation problem for a family of integral or differential equations to be recast as a locally equivalent bifurcation problem for a finite number of nonlinear equations in a finite number of unknowns.

Granting enough smoothness, by the implicit function theorem, bifurcation can only arise at a point of the branch of trivial solutions corresponding to a parameter λ at which the corresponding linearization fails to be invertible: such values of the parameter are called singular points. This gives a necessary condition for bifurcation and the starting point for the search of sufficient conditions via the Lyapunov-Schmidt reduction.

Assuming that the singular points are isolated, there is a large variety of methods which, combined with the Lyapunov-Schmidt reduction, provide sufficient conditions for the appearance of nontrivial solutions close to the singular point. See for instance: [18, 49, 67, 65, 87, 19, 85]. The two most prominent methods use either elementary singularity theory or topological invariants. In the former, whether the singular point under consideration is a bifurcation point or not is determined by investigating higher order jets of the Lyapunov-Schmidt reduction. In the latter, the presence of bifurcation is determined from topological invariants associated to the family of linearizations at points of the trivial branch.

(2.2) Topological Methods in Bifurcation Theory. The origin of topological bifurcation can be traced back to two classical bifurcation theorems of M. A. Krasnosel'skii, his collaborator P. Zabreiko and others. Both appeared in Krasnosel'skii's book *“Topological Methods in Nonlinear Integral Equations”* [66], which deeply influenced the nonlinear analysis of the sixties. As was then customary in the Soviet scientific literature, both results were quoted without any reference to the original papers in which they were proven.

The two theorems are as follows: Let g be compact map from a Banach space X into itself, such that $g(0) = 0$. Let $f(\lambda, x) = x - \lambda g(x)$. Consider $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ as the trivial branch of solutions of the equation $f(\lambda, x) = 0$. If g is differentiable, then every characteristic value of the Fréchet derivative $A \equiv Dg(0)$ of odd algebraic multiplicity is a bifurcation point from the trivial branch. Moreover, if X is a Hilbert space and g is the gradient of a functional, then all characteristic values of A , irrespective of their multiplicity, are bifurcation points. The proofs of both theorems use topological tools in order to draw conclusions about the nonlinear equation from assumptions about the linearization at a potential bifurcation point. The first is based on the classical formula for the Leray-Schauder degree of a map in terms of its linearization. The second combines Lagrange multiplier techniques with the existence of critical points of weakly continuous functionals.

Another result known as the “Global Rabinowitz Alternative” [79] gave a strong impulse to the development of topological methods in bifurcation theory, showing that one can draw very strong global conclusions about the structure and behavior of bifurcating branches of solutions, based solely on local data about the linearization at bifurcation points. Using his global alternative, Paul Rabinowitz proved that the set of solutions of certain nonlinear Sturm-Liouville eigenvalue problems presented a topological pattern analogous to the linear ones.

The Rabinowitz bifurcation theorem was widely recognized by nonlinear analysts, and this renewed interest in bifurcation theory prompted four PhD theses: E.N. Dancer (Cambridge 1972), D. Westreich (Yeshiva 1972), W. Magnus (Sussex 1974) and J. Ize (Courant 1974), which have greatly contributed to further progress of this theory. We will review the Global Rabinowitz Alternative later, in the improved form due to Jorge Ize.

(2.3) Bifurcation Theory for Fredholm Operators. First of all, let us describe the general setting of Ize’s thesis: let X, Y be Banach spaces and let $f: \mathbb{R}^k \times X \rightarrow Y$ be a continuously differentiable map such that $f(\lambda, 0) = 0$ for all λ in \mathbb{R}^k . Solutions of the equation $f(\lambda, x) = 0$ of the form $(\lambda, 0)$ are called *trivial* and the set $\mathbb{R}^k \times \{0\}$ is called the *trivial branch*. In what follows we will identify the parameter space \mathbb{R}^k with the set of trivial solutions and we will frequently write the parameter variable as a subscript. Accordingly we denote by $f_\lambda: X \rightarrow Y$ the map defined by $f_\lambda(x) = f(\lambda, x)$.

A *bifurcation point* for solutions of the equation $f(\lambda, x) = 0$ is a point λ_* in \mathbb{R}^k such that every neighborhood of $(\lambda_*, 0)$ contains nontrivial solutions of this equation.

Let $L_\lambda \equiv Df_\lambda(0)$ be the Fréchet derivative of the map f_λ at the point 0. The map L which sends $\lambda \in \mathbb{R}^k$ to L_λ is called the *family of linearizations along the trivial branch*. By the Implicit Function Theorem, bifurcation cannot occur at points where the operator L_λ is an isomorphism. Therefore, bifurcation can occur only at points belonging to the “generalized spectrum” of $L \quad \Sigma(L) \equiv \{\lambda \in \mathbb{R}^k \mid L_\lambda \text{ is singular}\}$. However not every singular point of L is a bifurcation point. For example, if $f_\lambda(x) = x - \lambda Kx + o(\|x\|)$ with K a linear compact operator, then characteristic values of K of even multiplicity may or may not be bifurcation points, depending on the higher order terms of the Taylor expansion at 0.

Following Jorge Ize we will assume that $A = L_0$ is a Fredholm operator, that is, $\ker A$ is finite dimensional and $\text{Im} A$ is a finite codimensional subspace of Y . Moreover, in order to simplify the presentation, we also assume that the Fredholm index of A , $\text{ind} A \equiv \dim \ker A - \dim \text{coker} A$, is zero and that $\lambda = 0$ is an isolated point of $\Sigma(L)$. Thus $L_\lambda = Df_\lambda(0)$ is invertible for $0 < \|\lambda\| < \epsilon$, provided ϵ is sufficiently small. The more general case of positive Fredholm index, which was also considered by Ize, can be reduced to the zero-index case by the methods explained in his thesis.

We will also assume all the smoothness needed in order to make our arguments valid. This deviates considerably from his approach. Indeed Ize used minimal Lipschitz assumptions on the nonlinear perturbation $g_\lambda = f_\lambda - L_\lambda$ and gave rather precise growth estimates on the nonlinearity g needed for his results.

Under our assumptions, each f_λ is a Fredholm map of index zero defined on a neighborhood U of 0, that is, $Df_\lambda(x)$ is Fredholm of index zero for λ and x small enough. Let Q' and Q be projections of Y onto $Y_1 = \text{Im} A$ and of X onto $E_0 = \ker A$, respectively. Let $F_0 \equiv \ker Q' \simeq \text{coker} A$. Under the splitting of Y and X into direct sums $Y_1 \oplus F_0$ and $X_1 \oplus E_0$, the Frechet derivative $D_{x_1} Q' f(0, 0)$ in the direction of X_1 is an isomorphism.

By the implicit function theorem, there is a map ρ defined on a neighborhood of $(0, 0)$ in $\mathbb{R}^k \times E_0$ with values in X_1 such that, close enough to $(0, 0) \in \mathbb{R}^k \times X$, we have $Q' f(\lambda, x_1 + x_0) = 0$ if and only if $x_1 = \rho(\lambda, x_0)$. Let us define a map b on a product neighborhood of $(0, 0)$ in $\mathbb{R}^k \times E_0$ by

$$(2.1) \quad b(\lambda, x_0) = (\text{Id} - Q')f(\lambda, \rho(\lambda, x_0) + x_0).$$

From its definition it follows that, for small (λ, x) , the solutions of $f(\lambda, x) = 0$ are in one to one correspondence with the solutions of the finite dimensional reduced system $b(\lambda, x_0) = 0$, called the bifurcation equation. This is in essence the Lyapunov-Schmidt reduction. Clearly $b(\lambda, 0) = 0$. Identifying $E_0 \simeq F_0 \simeq \mathbb{R}^n$ via

an isomorphism we are left with the problem of finding sufficient conditions for the bifurcation of solutions of the finite dimensional system $b(\lambda, x) = 0$, where $b: \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Let $B_\lambda = D b_\lambda(0)$ be the linearization of b at the trivial branch. We can write the map b in the form

$$(2.2) \quad b(\lambda, x) = B_\lambda x + g(\lambda, x),$$

with $g(\lambda, x) = o(\|x\|)$ uniformly in λ .

For any small enough closed disk $D_k = D(0, \delta)$ centered at 0, the restriction of B to the boundary $\partial D_k \simeq S^{k-1}$ defines a map

$$(2.3) \quad B: S^{k-1} \rightarrow GL(n),$$

where as usual $GL(n) = \{A \in \mathbb{R}^{n \times n} \mid \det A \neq 0\}$. Clearly the homotopy class $\gamma \equiv [B|_{\partial D_k}] \in \pi_{k-1}(GL(n))$ of this map is independent of the choice $\delta > 0$ if δ is sufficiently small. The topological approach to (linearized) bifurcation consist in the study of topological invariants attached to the homotopy class γ whose nontriviality forces bifurcation of zeroes of f from the trivial branch.

For $k = 1$, the homotopy class γ is completely determined by the change in sign of the determinant of B as λ crosses $\lambda = 0$. Moreover, using the homotopy invariance of the Brouwer degree, one easily shows that the change of sign of the determinant forces the appearance of nontrivial solution of $b(\lambda, x) = 0$ (and hence of $f(\lambda, x) = 0$) arbitrary close to $(0, 0)$.

After discussing the general setting of Ize's dissertation let us describe his one-parameter bifurcation results. Chapters regarding the several-parameter case will be discussed in the next section together with his other papers in this area. Among the many topics considered in the thesis we choose: bifurcation criteria at eigenvalues of finite multiplicity, the definition of generalized multiplicity, and the proof of an improved version of the Global Rabinowitz Alternative.

2.3.1. Bifurcation criteria at eigenvalues of finite multiplicity. The first chapter of Ize's thesis deals with the case $k = 1$. He begins by looking at maps of the special form

$$(2.4) \quad f(\lambda, x) = Ax - \lambda x + g(\lambda, x),$$

with $g(\lambda, x) = o(\|x\|)$ uniformly in λ . Here $X \subset Y$ and A is viewed both as a closed unbounded Fredholm operator with domain $\mathcal{D}(A) \subset Y$ and as a bounded operator from X to Y , where $X = \mathcal{D}(A)$ is endowed with the graph norm of A . Using a variant of the Lyapunov-Schmidt reduction in which $\text{Ker } A$ is substituted by the generalized eigenspace $E(0) = \ker A^m$, he relates the change of $\text{sgn det } B$ with the (finite) multiplicity of the eigenvalue $\lambda = 0$ of the unbounded operator A , showing in this way that bifurcation arises whenever the multiplicity is odd. In doing so he also gives a partial answer to a question raised by A.E. Taylor in [86], showing that isolated points in the spectrum are precisely the ones having a finite algebraic multiplicity. More precisely, he proves that the following statements are equivalent:

- i) 0 is an eigenvalue of A of finite algebraic multiplicity,
- ii) $Y = \text{Im } A^m \oplus \ker A^m$, for some,
- iii) 0 is isolated in the spectrum of A .

2.3.2. Definition of the Generalized Multiplicity. In the next section he considers more general maps of the form

$$(2.5) \quad f(\lambda, x) = Ax - T(\lambda)x + g(\lambda, x),$$

where $T(\lambda) = \lambda T_1 + \lambda^2 T_2 + \dots + \lambda^k T_k$ is a polynomial in λ with values in the space of bounded operators.

In the more general case (2.5), the change of sign of the determinant is no longer related to the algebraic multiplicity of 0 as an eigenvalue of A . However, there is a natural way to define a notion of generalized multiplicity of an isolated point in $\Sigma(L)$. Namely, the generalized multiplicity m is the smallest integer such that $\det B_\lambda = a\lambda^m +$ higher order terms, with $a \neq 0$. Clearly, when $T_\lambda = \lambda \text{Id}$, the generalized multiplicity coincides with the algebraic multiplicity of 0 as an eigenvalue of A .

Ize proves that the parity of m is independent of the choice of projections used in the Lyapunov-Schmidt reduction and that bifurcation arises whenever the generalized multiplicity m is odd.

More or less at the same time, two other definitions of generalized multiplicity were given, by Westreich in [88] and by Magnus in [70], and later several more were found. For the comparison between various definitions see [81, 34, 30].

Note that the generalized multiplicity is not a topological invariant of the linearization, and as a consequence only its parity is inherent to topological bifurcation. Indeed, Jorge Ize proved that if the generalized multiplicity of 0 is even then one can find a higher order perturbation g such that the map f of (2.5) has no bifurcation points.

2.3.3. Global Rabinowitz Alternative. A new proof of this famous result is given in Chapter III of Ize’s thesis; see also [74]. He considers maps f of the form (2.5) with $A = \text{Id}$, T a polynomial in λ with values in the linear compact operators and a compact nonlinear perturbation $g = o(\|x\|)$. Here, the fact that T is a polynomial is used only in order to ensure that $\Sigma(\text{Id} - T)$ discrete. What is essential is the compactness of the perturbation, since the main tool used in the proof is the Leray-Schauder degree for compact perturbations of identity. The Global Rabinowitz Alternative for general Fredholm maps was proved by Fitzpatrick, Pejsachowicz and Rabier 18 years later, once a special degree theory for Fredholm maps of index 0 was constructed [35].

In what follows we will refer to elements of $\Sigma(\text{Id} - T)$ as the “generalized characteristic values” and we will use “multiplicity” for the generalized multiplicity of a characteristic value.

Theorem 2.10 of the thesis states that every generalized characteristic value λ_0 of odd algebraic multiplicity is a bifurcation point. Moreover, if \mathcal{C} is the connected component of $(\lambda_0, 0)$ in the closure of the set \mathcal{S} of nontrivial solutions of the equation $f(\lambda, x) = 0$, we have:

- i) either \mathcal{C} is unbounded
- ii) or \mathcal{C} is bounded and if $\lambda_1 < \lambda_2 < \dots < \lambda_{p+1}$ are such that $(\lambda_i, 0) \in \mathcal{C}$; then, denoting by n_i the number of characteristic values strictly between λ_i and λ_{i+1} , one has

$$(2.6) \quad 1 = (-1)^{n_1} - (-1)^{n_1+n_2} + \dots + (-1)^p (-1)^{\sum_{i=1}^p n_i}.$$

In particular, it follows from (2.6) that \mathcal{C} connects λ_0 to an odd number of characteristic numbers of odd multiplicity.

The above result improves the main theorem of [79] in providing some extra information about the behavior of the bifurcating branch. However, what is interesting is Ize's proof of this theorem. We will only indicate here how he proves that a characteristic value of odd multiplicity is a bifurcation point. Instead of using the homotopy invariance of the degree, his approach is based on the following device which has far reaching consequences since it works in the multiparameter case as well.

The device consists in "complementing" the map f with the real valued function $\|x\|^2 - \epsilon^2$ in the following sense: Consider the compact perturbation of the identity $F: \mathbb{R} \times X \rightarrow \mathbb{R} \times X$ defined by

$$(2.7) \quad F(\lambda, x) = (\|x\|^2 - \epsilon^2, f(\lambda, x)).$$

It is easy to see that one can choose a $\delta > 0$ small enough such that for all small $\epsilon > 0$ the map F does not vanish on the boundary of $\Omega = D(\lambda_0, 2\delta) \times D(0, 2\epsilon)$. Hence, the Leray-Schauder degree $\deg(F, \Omega, 0)$ is defined. Any zero of F has the second component of norm ϵ . Thus, if λ_0 is not a bifurcation point, then $\deg(F, \Omega, 0) = 0$.

On the other hand, on $\bar{\Omega}$ there is an admissible homotopy between F and the map $G(\lambda, x) = (\delta^2 - (\lambda - \lambda_0)^2, f(\lambda, x))$ which, for δ small enough, has two non degenerate zeroes $(\lambda_0 \pm \delta, 0)$. Denoting with $i(f, x)$ the index of an isolated zero of f , it follows from the product property of the Leray-Schauder degree that

$$(2.8) \quad 0 = \deg(F, \Omega, 0) = i(f_{\lambda_0 - \delta}, 0) - i(f_{\lambda_0 + \delta}, 0).$$

Since the index of an isolated zero remain (up to sign) unchanged under the Lyapunov-Schmidt reduction, we have

$$i(f_{\lambda_0 \pm \delta}, 0) = i(b_{\lambda_0 \pm \delta}, 0) = \pm \operatorname{sgn} \det B_{\lambda_0 \pm \delta},$$

where B is the linearization of the bifurcation equation arising in (2.2). But this contradicts our assumptions since an odd generalized multiplicity produces a change in sign of the determinant of B_λ .

The formula (2.6) follows from a similar calculation, applied to an open subset isolating a bounded component \mathcal{C} .

An extension of the above result to compact perturbations of the identity depending on one complex parameter λ was also obtained. The proof used the additivity property of a generalized cohomology theory, introduced by Geba and Granas in [44]. Another version of this proof can be found in [52]. All the remaining proofs in the thesis used only elementary homotopy theory. Indeed, Jorge Ize mastered the art of deforming families of matrices depending on parameters, using the Jordan canonical form.

Ize's complementing device and its generalizations have been applied by many people; see, for example, [46, 8, 90, 82, 69]. But its importance will be better appreciated in the next section.

(2.4) Several Parameter Bifurcation and the J -Homomorphism. James Alexander and Jorge Ize can be considered as the cofounders of topological several parameter bifurcation theory. The role of Whitehead's J -homomorphism as the fundamental topological invariant in linearized several parameter bifurcation

was discovered independently by Ize and Alexander more or less at the same time. While Alexander, motivated by his work with Jim Yorke on global Hopf bifurcation, used the stable J -homomorphism, Ize considered the unstable version as well.

The J -homomorphism arises in bifurcation theory as follows: let us recall that by the Lyapunov-Schmidt reduction (2.1) the solutions of the equation $f(\lambda, x) = 0$ close to the point $(0, 0)$ are in one to one correspondence with the solutions of the bifurcation equation $b(\lambda, x) = 0$, where $b: \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $b(\lambda, 0) = 0$.

For $k = 1$, bifurcation is determined directly by the homotopy class of the linearization B defined in (2.2), restricted to $S^0 = \partial D_1$, that is, by the change in sign of the determinant of B_λ .

For $k > 1$, the nonvanishing of the homotopy class $\gamma = [B|_{\partial D_k}]$ is no longer sufficient in order to guarantee bifurcation and one has to look at the image of γ under Whitehead's J -homomorphism.

Whitehead's J -homomorphism [89] is the group homomorphism

$$J: \pi_{k-1}(GL(n)) \rightarrow \pi_{k-1}(S^n)$$

which associates to $\theta \in \pi_{k-1}(GL(n))$ the element obtained by applying the Hopf construction to the map $(\lambda, x) \rightarrow T(\lambda)x$, where $T: S^{k-1} \rightarrow O(n) \simeq GL(n)$ is any representative of the homotopy class θ .

If $J(\gamma) \neq 0$, then $(0, 0)$ is a bifurcation point for any nonlinear perturbation b of B as in (2.2) and hence also a bifurcation point of f . The bifurcation is global, but in a weaker sense than the one discussed in the previous section. Indeed, the connected component of $(0, 0)$ is either unbounded or it connects $(0, 0)$ to another bifurcation point, or it reaches the boundary of the open set where the Lyapunov-Schmidt reduction is valid. If f is a compact perturbation of the identity then one has a full global alternative whenever $J^s(\gamma) \neq 0$. Here J^s is the stable J -homomorphism of [1].

We will see below that $J(\gamma) \neq 0$ entails bifurcation by interpreting this class as an obstruction to the extension of a map. It follows from classical obstruction theory that there is yet another topological invariant which detects bifurcation. Namely, the image of γ under the homomorphism

$$P: \pi_{k-1}(GL(n)) \rightarrow \pi_{k-1}(R^n - \{0\}),$$

defined by $P(\theta) = [\lambda \rightarrow T(\lambda)x_0]$, where $x_0 \in \partial D(0, \epsilon)$ is fixed. In this case, there is bifurcation in every direction. However, from the point of view of nonlinear analysis this invariant is less interesting, since it vanishes for $k < n$, and in general one cannot ensure that $n = \dim \text{ker } A$ is smaller than the number of parameters of the problem.

In proving that the nonvanishing of $J^s(\gamma)$ causes bifurcation, Alexander uses heavy machinery from algebraic topology. He relates global bifurcation to nonvanishing of an element belonging to the framed cobordism group of a topological space which arise in the problem and then computes this element as the image under the stable J -homomorphism of γ . In contrast, the approach of Ize is completely elementary and strictly tied to his complementing device. We briefly outline it.

Let $D_n = D(0, 2\epsilon)$ and let $D_{k+n} = D_k \times D_n$. As in the previous section, the appearance of nontrivial solutions is related to the nonexistence of a nonvanishing

extension of the map

$$F(\lambda, x) = (b(\lambda, x), \|x\|^2 - \epsilon^2)$$

sending the boundary of the disk $\partial D_{k+n} \simeq S^{k+n-1}$ into $R^{n+1} - \{0\} \simeq S^n$. On ∂D_{k+n} , the map F is clearly homotopic to the map G defined by

$$G(\lambda, x) = (B_\lambda x, \|x\|^2 - \epsilon^2).$$

Another simple homotopy leads to the map

$$H(\lambda, x) = (B_\lambda x, \|x\|^2 - \|\lambda\|^2)$$

whose homotopy class in $\pi_{k+n-1}(S^n)$ is precisely $J(\gamma)$. A proof of this fact can be found in [54] and [46].

Using his elementary approach, Jorge Ize gives in [50] a simpler proof of the global Hopf bifurcation theorem first proved by Alexander and Yorke in the paper [3], which circulated in preprint form at that time.

In the long paper [54], using obstruction theory, Jorge Ize proves that the two invariants discussed above are the only homotopy invariants of the linearization whose nonvanishing entails bifurcation. If they vanish one can find a higher order nonlinear perturbation g of the linearization L at points of the trivial branch such that $L + g$ has no bifurcation points. Ize works with parametrized families of maps from \mathbb{R}^N into itself only, but his results can be easily extended to infinite dimensions. Analogous result holds for global bifurcation; see also [1].

It is appropriate at this point to say two words about a personal characteristic of Jorge which endured through all of his mathematical activity. His intellectual curiosity always pushed him to find a proof of any result of his interest based on his own, mainly elementary, methods. In this way, he produced an almost infinite number of examples and counterexamples in bifurcation theory and equivariant degree which led to an intense professional correspondence with many researchers in his field. As a personal remembrance, let us mention that the third author was delighted when, to the surprise of all present, Ize found a disappointing orbit of period two in an iteration algorithm, invented at the Computer Center of the University of Calabria, which was supposed to converge to a fixed point irrelevant of the choice of the starting point.

There has been a great amount of work related to multiparameter bifurcation. Most of the literature deals with various forms of Hopf bifurcation. This aspect will be thoroughly covered in the second part of the paper. We only mention here a few of the papers which appeared prior the development of equivariant degree theory. Jorge Ize in [51] and Fiedler in [31] extended the topological approach to bifurcation of periodic solutions of parabolic partial differential equations. Fitzpatrick's paper [32] is a good introduction to the computation of the Alexander-Yorke invariant arising in Hopf bifurcation. Many multiparameter continuation and bifurcation problems arise in mathematical elasticity. Pioneering applications to this field were obtained by S. Antman and J. Alexander. Some of them are described in Antman's book "Nonlinear Problems of Elasticity" [4]. Alexander-Ize's discovery of the role of the J -homomorphism in bifurcation theory prompted further theoretical research by Thomas Bartsch [12]. An extension of bifurcation theory to more general parameter spaces can be found in [11] and [14]. Global bifurcation of fixed points using related methods was considered in [15]. In [16] thanks to a combined use of J -homomorphism and the fiberwise Conley index,

introduced in that paper, Bartsch obtained the first, to the best of our knowledge, result on topological bifurcation for critical points of functionals parametrized by general spaces.

After completing his Ph.D. thesis Jorge Ize never came back to the study of bifurcation for general Fredholm maps. He devoted most of his research to the Hopf bifurcation, where general Fredholm maps are not strictly needed, since working with periodic orbits of systems of differential equations in \mathbb{R}^n one has at hand a rather natural finite-dimensional approximation obtained by cutting the Fourier expansion of the periodic solution after the n -th coefficient. A similar disregard to the use of infinite dimensional function spaces in dealing with Morse theory of geodesics on Riemannian manifolds was manifested more than once by Raoul Bott. He considered the use of broken geodesics, as a finite-dimensional approximation to the problem, to be a more flexible and robust tool.

Nevertheless, there are several reasons for investigating bifurcation in the framework of nonlinear Fredholm maps. On the one hand, the Lyapunov-Schmidt reduction shows that locally a Fredholm map is equivalent to a map between two finite dimensional spaces, and hence the local bifurcation invariants for Fredholm maps do not differ from the ones for maps between finite dimensional vector spaces. However, on the other hand, the space of all linear Fredholm operators between infinite dimensional spaces has many nonvanishing homotopy groups. As a consequence, the linearized bifurcation theory for Fredholm maps possesses global topological invariants that are not present neither in finite dimensions nor in the case of compact perturbations of the identity.

This type of invariants were studied in [75] using K-theoretical methods. In that paper the construction of an index of bifurcation points for nonlinear compact perturbations of linear Fredholm operators was sketched. Global continuation and bifurcation theorems for maps of this class were proved by Bartsch in [13] using a similar approach. Both [75] and [13] contain two different versions of an index of bifurcation points. The one from [75] was extended to an index of bifurcation for continuous families of C^1 -Fredholm maps parametrized by general spaces in [76]. At an isolated bifurcation point, the index is related to the Alexander-Ize invariant, but the nonvanishing of the total index arise only in the infinite dimensional Fredholm setting. Applications of these ideas to elliptic boundary value problems for systems of partial differential equations can be summarized as follows: substituting Whithead's J -homomorphism with the generalized J -homomorphism of Atiyah-Adams, and the matrix family B of this paper with the principal symbol of the linearization at the trivial branch, sufficient conditions for bifurcation are obtained directly from the coefficients of the top order derivatives of the linearized operator using the family version of the Atiyah-Singer index theorem [76, 77]. These results complement the work of Alexander and Ize, since criteria of the above type cannot be obtained via Lyapunov-Schmidt reduction, which depends in an essential way on the lower order terms.

(2.5) Structure and Dimension of Branches of Solutions to Multiparameter Nonlinear Equations. The work of Alexander and Antman [2] and its applications to mathematical elasticity motivated many extensions of Ize's complementing technique. In [33] more general complementing maps were defined in order to obtain a unified approach to multiparameter continuation and bifurcation

problems and also provide estimates on the topological dimension of the branch. Assuming that $f(\lambda, x)$ is a compact perturbation of the identity, results about the global behavior and dimension of the branch were obtained using cohomological methods.

During a visit of Jorge Ize to the University of Calabria the above approach was changed into one of a homotopic nature. This provided not only simpler proofs for the earlier results but also permitted an extension of the theory to a more general class of maps. The maps under consideration were the 0-epi maps introduced by Furi, Martelli and Vignoli in [37]. The results were published in [57]. We won't discuss this paper in details. Chapter III of Ize's survey [55] is an excellent review of its contents. We only point out a few aspects of the article. First of all, in order to obtain from the main result (Theorem 3.1) a global version of the implicit function theorem (Theorem 4.1) was used a scaling property of 0-epi maps discovered by Jorge Ize. To some extent, it compensated for the lack of an additive degree theory in this setting. In addition, the relation of 0-epi maps with the J -homomorphism of the previous section is thoroughly described in Proposition 4.5 of that paper. Finally, Propositions 4.3 and 4.5 can be considered as a partial exception to what we said in the previous section. In fact, they deal with the nonlinear k -set of contractive perturbations of linear Fredholm operators of positive index.

(2.6) Multiparameter Hopf Bifurcation. The Poincaré-Andronov-Hopf bifurcation provides one of the most illuminating examples of the power of topological methods in nonlinear functional analysis. The use of function spaces allows one to formulate this classical problem, belonging to the qualitative theory of dynamical systems, in the abstract framework of bifurcation from the trivial branch. The use of topological invariants permits the replacement of some rather strong, albeit generic, assumptions with considerably weaker ones which moreover provide some information of global nature. The same approach can be used in order to study other bifurcation phenomena of dynamical systems, e.g., bifurcation of homoclinic and heteroclinic orbits. Of course, in the topological approach, the very precise local information generated by the singularity theory is lost.

To some extent, the Hopf bifurcation can be considered as a motivating example for the evolution of Ize's research interests. Indeed, bifurcation of periodic orbits of a one parameter family of vector fields requires a second parameter, namely the unknown period of the bifurcating orbit. This stimulated the development of a bifurcation theory for families of maps depending on several parameters. Furthermore, the natural action of S^1 on the space of periodic maps led to equivariant bifurcation theory [53], and ultimately to a degree theory for maps equivariant under the action of S^1 , as a tool which permitted to improve Alexander-Yorke's earlier results on global Hopf bifurcation. Dealing with various types of symmetries arising in dynamics, equivariance under more general group actions had to be considered.

Ize's paper [53] is an inspiring example of an application of obstruction theory to a concrete problem in dynamics. Motivated by the results of [20], Ize uses obstruction theory for the extension of S^1 -equivariant maps in order to study local and global bifurcation of periodic orbits of an ordinary differential system of the form

$$(2.9) \quad \dot{x} = L(\lambda)x + g(\lambda, x),$$

where $\lambda \in \mathbb{R}^k$, $L(\lambda) \in \mathbb{R}^{n \times n}$, $k < 2n$, and g is a higher order perturbation. Under the usual transversality assumption in Hopf bifurcation, after numbering the multiples $m_1 = 1 < m_2 < \dots < m_l$, such that $im_j\beta$ belongs to the spectrum of the complexification L^c of L , he proves that a global branch of period orbits bifurcates from $(0, 0, 2\pi/m_j\beta)$ provided a number $n_j(k)$, associated to the eigenvalue $im_j\beta$, does not vanish. Moreover, if the branch is bounded, then, under appropriate nondegeneracy assumptions, he obtains a formula of the same type as (2.6). Not surprisingly, for $k = 1$, $n_j(1)$ coincides with the spectral flow of L^c through $im_j\beta$, i.e., the number of eigenvalues of L^c crossing the imaginary axis through $im_j\beta$ in one direction, minus the number of those crossing the axis in the opposite direction, as λ traverses 0. However, what in our opinion is the most interesting result of Ize in [53] is the identification of $n_j(k)$ for $k > 1$. The number $n_j(k)$, whose nonvanishing is shown by Ize to force bifurcation, is the Bott degree of the family of matrices $B(\lambda, \mu) = im_j\mu Id - L^c(\lambda)$ restricted to the boundary of a small disk centered at $(0, \beta)$.

This completes our review of the first part of Ize’s research activity. However it would be unfair to finish this chapter without acknowledging how much this present chapter owes to Ize’s comprehensive survey of 25 years of topological bifurcation theory [55].

3. Part II: Equivariant Degree

(3.1) Symmetries and Equivariant Degree. Complexity of the natural world gave rise to fundamental problems of modern science dealing with the impact of symmetries on physical and biological systems. These real world problems are expressed as mathematical models usually exhibiting nonlinear character accompany by the presence of *symmetries*, which may be related to some physical or geometric regularities. For such systems, the existence of multiple solutions is not just a possibility but it is a fact. Getting knowledge of such solution sets and their symmetric classification constitutes an important problem for a complete analysis of these mathematical models. For instance, phase transitions in crystals correspond exactly to the changes of their symmetries; in a mechanical system, the presence of symmetries allows one to decrease the number of its degrees of freedom; breaking spherical symmetry of a hydrogen atom by introducing a magnetic field gives rise to the elimination of its degeneracy resulting in splitting energy levels (Zeeman’s phenomena); in biological systems symmetry provides an explanation for pattern generation and synchrony, etc. Related mathematical models usually inherit symmetries of the prototypal real life phenomena in the form of the so-called *equivariance* of the corresponding maps. To be more specific, given two representations W and V of a (compact Lie) group G , a continuous map $f : W \rightarrow V$ is called equivariant if $f gx = g f x$ for all $x \in W$ and $g \in G$.

Unfortunately, studying nonlinear symmetric problems is not a simple task and involves advanced mathematical methods and technicalities that make this area difficult. The equivariant degree theory which emerged in the late 80s, was a paramount contribution to the development of the new mathematical tools for studying symmetric models. In short, the equivariant degree is a topological tool allowing “counting” orbits of solutions to symmetric equations in the same way as the usual Brouwer degree does, but according to their symmetry properties.

Jorge Ize, who was one of the principal founders of the equivariant degree, played a fundamental role in the creation and development of the concepts and methods in this theory.

The equivariant degree theory is both powerful and difficult. It emanated from the intersection of several mathematical fields and its ideas can be traced back to many classical premises: Borsuk-Ulam theorems, fundamental domains in Riemannian geometry/invariant theory, equivariant retract theory, equivariant homotopy groups of spheres, equivariant general position theorems, topological invariants of equivariant gradient maps, geometric obstruction theory, J -homomorphism in multiparameter bifurcation, and many others.

(3.2) Construction. The elegant construction of the equivariant degree was presented by Jorge Ize *et al.* in [59] (see also [58, 64]). This construction is essentially parallel to the homotopy definition of the Brouwer degree. To be more specific, let $\Omega \subset \mathbb{R}^l$ be an open bounded subset and $f : (\Omega, \partial\Omega) \rightarrow (\mathbb{R}^l, \mathbb{R}^l \setminus \{0\})$ a continuous map. The homotopical definition of the Brouwer degree $\deg(f, \Omega)$ contains three ingredients: (i) assigning to f an element of the group π_{l+1} of homotopy classes of maps of pairs $(\overline{B(\mathbb{R}^{l+1})}, \partial(B(\mathbb{R}^{l+1}))) \rightarrow (\mathbb{R}^{l+1}, \mathbb{R}^{l+1} \setminus \{0\})$, (ii) usage of the canonical isomorphism $\pi_{l+1} \simeq \pi_l(S^l)$, and (iii) usage of an isomorphism $\pi_l(S^l) \simeq \mathbb{Z}$.

Following the same paradigm, Jorge Ize considered two orthogonal representations $W := \mathbb{R}^k \oplus U$ and V of a compact Lie group G (\mathbb{R}^k equipped with the trivial G -action can be thought of as a space of “free” parameters) and a G -equivariant map $f : W \rightarrow V$ such that $f(x) \neq 0$ for $x \in \partial\Omega$, where Ω is an open bounded G -invariant subset of W , and assigned to f the equivariant degree $\deg_G(f, \Omega)$ being an element of the corresponding equivariant homotopy group of spheres. More precisely, since the map f can have zeros outside Ω , one introduces an auxiliary invariant function $\varphi : W \rightarrow [0, 1]$ with value 1 outside Ω and 0 on an invariant neighborhood of $f^{-1}(0) \cap \Omega$. Then the map

$$\hat{f}(t, x) = (2t + 2\varphi(x) - 1, f(x)), \quad \hat{f} : [0, 1] \times B \rightarrow \mathbb{R} \oplus V$$

is non-zero on $\partial([0, 1] \times B)$, where $B \subset W$ is an open ball centred at the origin and containing $\overline{\Omega}$ (G acts trivially on \mathbb{R} and $[0, 1]$). Then the G -equivariant degree $\deg_G(f, \Omega)$ is defined as the class of the map \hat{f} from $\partial([0, 1] \times B)$ to $(\mathbb{R} \oplus V) \setminus \{0\}$ in the abelian group $\Pi_{S^W}^G(S^V)$ of equivariant homotopy classes of maps from S^W to S^V , where S^W and S^V stand for the corresponding (invariant) one-point compactifications.

The constructed above equivariant degree satisfies the usual properties expected from a degree theory, like *Existence*, *Equivariant Homotopy Invariance*, *Additivity* (up to one suspension), etc. Roughly speaking, the equivariant degree “measures” (equivariant) homotopy obstructions for $f|_{\partial\Omega}$ to have an equivariant extension without zeros over Ω (composed of several orbit types).

For the case $G = S^1$, the construction of the equivariant degree was outlined in [58]. It was shown in [59, 60] that the well-known Fuller index for autonomous differential equations (cf. [36]) and the S^1 -degree of S^1 -equivariant gradient maps defined by N. Dancer (cf. [21]) can be viewed as particular cases of the S^1 -equivariant degree. Also, a connection of the S^1 -equivariant degree to the Dancer-Toland invariant (cf. [26]) (associated to systems with first integral) and

to a number of indices due to J. Mallet-Paret and J. A. Yorke (cf. [73]), G. Dylaw-
 erski (cf. [27]), and K. Gęba *et al.* (cf. [28]) was established.

(3.3) Range of Values. The computations of the groups $\Pi_{S^W}^G(S^V)$ is crucial for an effective usage of the equivariant degree theory. Combining the concept of *fundamental cell* with the so-called *complementing map* approach, Ize *et al.* analyzed these groups in the case G is compact and *abelian* (see [61]; see also [60], [59] for $G = S^1$). The notion of a fundamental cell can be traced back to many classical mathematical disciplines: (i) fundamental domain for isometry groups of Riemannian manifolds, (ii) Weierstrass section in the invariant theory, (iii) Poincaré section in ODEs, to mention a few. A detailed exposition of the concept of a fundamental domain for arbitrary (in general, non-abelian) compact Lie group action on a metric space in the context relevant to the equivariant extension problems can be found in [68]. The complementing maps approach is relevant to a kind of general position results in the equivariant setting (cf. equivariant transversality and invariant foliation techniques developed in [68] and normal maps based techniques suggested in [45], see also [8]).

To simplify our exposition of the main results from [61], assume $U = V$. The following direct sum decomposition obtained in [61] plays an important role in applications:

$$(3.1) \quad \Pi_{S^W}^G(S^V) = \Pi_{k-1} \oplus \bigoplus_{(H)} \Pi(H)$$

where $\Pi(H) \simeq \mathbb{Z}$ and the summation is taken over all orbit types (H) occurring in S^W and satisfying the “maximality” condition $\dim G/H = k$ (and, therefore, called *primary components*). The component Π_{k-1} is relevant to the orbit types (H) with $\dim G/H < k$ (and, therefore, called *secondary*). In a parallel way, one can speak on the *primary* and *secondary* components of the equivariant degree. This terminology is determined by a different nature of the (equivariant) topological obstructions appearing in both cases (as well as different techniques utilized in their computations; Brouwer degree methods in the “primary case” and higher obstruction tools in the “secondary case”). In general, when $k > 1$, the part Π_{k-1} is very difficult to analyze. In [61], a complete characterization of Π_{k-1} (including explicit generators) was obtained for the case $k = 1$. In particular, under certain natural simply-connectedness conditions,

$$(3.2) \quad \Pi_{k-1} = \bigoplus_{(H) : \dim G/H=0} (\mathbb{Z}_2 \oplus G/H).$$

After intensive personal discussions with Jorge Ize, Wieslaw came up with the idea to extending decompositions (3.1) and (3.2) to arbitrary compact Lie groups. This was done by Z. Balanov and W. Krawcewicz in [9] (see also [71] and [78]). Other interesting results related to the secondary groups for $G = SO(3) \times S^1$ were obtained by H. Steinlein and J. Arpe (see [5]).

(3.4) Computability and Different Faces. The equivariant degree has different faces reflecting a diversity of symmetric equations related to applications: (i) (quasi)-periodic solutions (in particular, Hopf bifurcation phenomenon) appeal to the primary degree, (ii) steady-state bifurcation is related to the secondary degree, (iii) Hamiltonian and Newtonian systems give rise to the equivariant gradient/orthogonal degree, etc.

Up to certain technicalities related to the so-called *bi-orientability* property of compact Lie groups (cf. [78]), the *primary* equivariant degree is a projection of the general equivariant degree onto the primary components of decomposition (3.1). Using a different approach, the orthogonal degree was introduced in [45]. For $k = 1$ and G being an abelian group, the computational formulae for the primary degree (reduction to the Brouwer degree on slices to the orbits) have been worked out in [62] (see also [8] for the case of an arbitrary compact Lie group). The multi-parameter primary degree still awaits the development of effective computational techniques.

Being highly inspired by Jorge's ideas, Z. Balanov and W. Krawcewicz proposed the concept of the so-called *twisted degree* for $G = \Gamma \times S^1$ for studying periodic solutions to Γ -symmetric dynamical systems (see [7]).

The well-known (functorial) *product* property of the Brouwer degree is reflected in similar product properties of different versions of the equivariant degree. In the case of abelian group action, this property was extensively explored in [64] (for the case of non-abelian twisted degree, see [8]).

One should also mention important contributions of Ize in establishing Borsuk-Ulam type results/congruences for Brouwer degrees of equivariant maps $W \rightarrow V$ with $\dim W = \dim V$. Motivated by his exquisite examples, a general version of the congruence principle was suggested in [68]. In fact, it was Jorge Ize who pointed out a gap in a preliminary version of [68] (this gap was eliminated after a careful analysis of one of Ize's examples).

In contrast to the primary degree, the computational techniques for the secondary degree (i.e. the projection of the general equivariant degree onto Π_{k-1} (see (3.2)) allowing its effective usage, are still needed. For G being abelian and $k = 1$, Jorge Ize has established interesting computational results in [62] (see also [8]).

The problem of classifying symmetric properties of solutions to the equation $\nabla f(x) = 0$, $x \in \Omega$, where $f : W \rightarrow \mathbb{R}$ is a smooth invariant function, has been studied by many authors using various methods: Lusternik-Schnirelman theory (cf. [72, 80]), equivariant Conley index theory (cf. [17]), Morse-Floer techniques (cf. [6]), to mention a few. The degree-theoretic treatment of this problem (for $G = S^1$) was initiated in [21], where a rational-valued gradient S^1 -homotopy invariant was introduced (see also [23], where a similar invariant was considered in the context of systems with first integral). K. Gęba (cf. [43]) suggested a method to study the above problem using the so-called equivariant gradient degree (for more information on the equivariant gradient degree, we refer to [84]). Under reasonable conditions, the equivariant gradient degree turns out to be the full equivariant gradient homotopy invariant (cf. [22]).

It is a standard fact of undergraduate multivariable analysis that the gradient of a smooth function is orthogonal to the corresponding level surface. In the case of a G -invariant function, the above fact translates to the orthogonality of the gradient to the corresponding G -orbit (in particular, the gradient field is G -equivariant). This simple observation leads to the general concept of *G -orthogonal* maps, to which one can associate the so-called *orthogonal* equivariant degree. Roughly speaking, a map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be *G -orthogonal* if $f(x)$ is orthogonal to the orbit $G(x)$. For $G = S^1$, this degree was introduced by S. Rybicki (cf. [83]). The setting when the acting group is different from S^1 is rather involved since one has to deal with orbits of different dimensions. An elegant construction

of the orthogonal degree when G is an arbitrary (infinite) compact abelian Lie group, was carried out by J. Ize in [63]. The equivariant degree for orthogonal maps is an element of the equivariant group of sphere (for G -orthogonal maps), which is denoted by $\Pi_{\perp S^{n-1}}^G(S^{n-1})$. Jorge Ize was able to explicitly evaluate these homotopy groups and establish computational formulae for the *equivariant degree of G -orthogonal maps* for the most important *generic* cases.

(3.5) Applicability. As any abstract theory “survives” only by having a wide spectrum of applications, Jorge Ize developed several important techniques allowing effective usage of the equivariant degree methods for qualitative study of various types of differential equations. His original ideas led to many interesting examples in differential equations, which could be used as models for further applications. His innovative techniques cover a large number of situations including autonomous differential equations, differential equations with fixed period, or with first integrals, time dependent equations, symmetry breaking for differential equations, existence of twisted orbits, Hopf bifurcation with symmetries and Hopf bifurcation for time dependent systems, other bifurcation problems, existence of periodic solutions for Hamiltonian systems, time reversible systems, and spring-pendulum system, provide the reader with a solid basis for an independent research in the case of even more sophisticated problems. This impressive list of possible applications of the equivariant degree methods will probably have a long lasting effect on their further applications in real life mathematical models. Very often, Jorge Ize modestly called the obtained results just examples. However, as we clearly see today, many of these “examples” opened interesting and quite non-trivial possibilities for the further applications of the equivariant degree theory.

Existence of Periodic Solutions. In [64], Jorge Ize showed how the primary S^1 -equivariant degree could be applied to study the averaged system of van der Pol-like equations (in fact, this is just a particular case of integro-differential equations). To be more specific, in the setting considered, he established a formula for the computation of the local index (i.e. the equivariant degree in a small neighborhood around an isolated periodic orbit). His approach is based on a systematic usage of some (canonical) maps which, to some extent, are parallel to the concept of the so-called “basic maps” exploited, for example, in [8]. The results related to the index computations can be effectively applied for studying, for example, cascades of period-doubling bifurcations.

In order to illustrate another stream of Ize’s ideas, let us consider a parametrized system of autonomous equations

$$(3.3) \quad \dot{x} = g(x, \nu), \quad x \in \mathbb{R}^n.$$

Assume $x_0 = x_0(t)$ is a given 2π -periodic solution with minimal period $\frac{2\pi}{m}$ ($1 < m \in \mathbb{N}$). To establish the existence of other periodic solutions (for example, by means of the bifurcation theory), it is natural to reformulate system (3.3) as an operator equation in an appropriate Sobolev space $W := H^1(S^1; \mathbb{R}^n)$, and further to compute the local index around the known orbit x_0 in terms of the operators. Jorge suggested an effective algorithm for this kind of computations. With this information in hand, the standard degree theoretic methodology provides options for finding additional periodic solutions.

Let us observe that the so-called Newtonian systems coming from celestial mechanics normally do not have stationary solutions (a configuration of celestial bodies must be in motion in order to be “stable”). However, several periodic orbits (the so-called relative equilibria) can serve as reasonable analogues of stationary solutions. Finding less obvious periodic solutions (exhibiting more sophisticated behavior) is a significant challenge in celestial mechanics. Such systems clearly admit first integrals. To be more specific, let us elaborate further Ize’s ideas related to this kind of systems.

Consider a system

$$(3.4) \quad \dot{x} = g(x), \quad x \in \mathbb{R}^n$$

with first integral $V(x)$, i.e.

$$(3.5) \quad \nabla V(x) \bullet \dot{x} = \nabla V(x) \bullet g(x) = 0.$$

Suppose that a $\frac{2\pi}{m}$ -periodic solution to (3.4) is given and one is interested in finding new periodic solutions to (3.4). To this end, Ize considered an auxiliary parametrized system (cf. [23], [25])

$$(3.6) \quad \dot{x} = g(x) + v\nabla V(x), \quad x \in \mathbb{R}^n,$$

(v is an additional parameter) for which he looked for another 2π -periodic solution $x_1 = x_1(t)$ satisfying the condition $\nabla V(x_1(t)) \neq 0$. Clearly, the expression

$$(3.7) \quad \dot{x}_1 \bullet \nabla V(x) = v \|\nabla V(x_1)\|^2 = \frac{d}{dt} V(x_1(t))$$

integrated from 0 to 2π yields

$$(3.8) \quad v \|\nabla V(x_1)\|_{L^2}^2 = 0,$$

thus $v = 0$ and, therefore, x_1 is in fact a solution to (3.4). This simple observation allows one to employ the equivariant degree techniques to study multiple solutions systems with first integral (see [64]).

Symmetric Hopf Bifurcation. The (symmetric) Hopf bifurcation, with no doubt, was Ize’s main motivation for develop equivariant degree theory. Roughly speaking, in a parametrized dynamical system, a Hopf bifurcation phenomenon is related to the appearance of small amplitude periodic solutions, bifurcating from a stationary point when it changes its stability. To be more specific consider an autonomous system of ODEs

$$(3.9) \quad \dot{x} = L(\mu) + f(\mu, x), \quad x \in V = \mathbb{R}^N,$$

where $L(\mu) : V \rightarrow V$ is a linear operator (continuously depending on $\mu \in \mathbb{R}$) and f is continuous and such that $f(\mu, x) = O(\|x\|^2)$, for which one is interested to find p -periodic solutions (for some unknown $p > 0$). By introducing the frequency v as an additional parameter, system (3.9) can be rewritten as

$$(3.10) \quad v\dot{x} = L(\mu) + f(\mu, x), \quad x \in V = \mathbb{R}^N,$$

and the problem is reduced to finding 2π -periodic solutions. The Hopf bifurcation occurs when μ crosses some critical value μ_0 (for which the linearization admits a purely imaginary eigenvalue iv_0) and results in appearance of small amplitude periodic solutions near $(\mu_0, 0)$. Problem (3.10) can be reformulated as an operator equation $F(\mu, v, u) = 0$, $u \in W := H^1(S^1; V)$, with two parameters (μ, v) .

Next, by applying a complementing function φ , one can associate to (μ_0, ν_0) an S^1 -equivariant map $F_\varphi : \mathbb{R}^2 \oplus W \rightarrow \mathbb{R} \oplus W$, $F_\varphi(\mu, \nu, u) = (\varphi(\mu, \nu, u), F(\mu, \nu, u))$, for which the one-parameter S^1 -degree $\text{deg}_{S^1}(F_\varphi, \Omega)$ is correctly defined in a neighborhood Ω of $(\mu_0, \nu_0, 0)$. The nontriviality of the degree $\text{deg}_{S^1}(F_\varphi, \Omega)$ guarantees the occurrence of the Hopf bifurcation.

Assuming, in addition, that V is a representation of a compact Lie group Γ and the right hand side of (3.9) is Γ -equivariant, by following the same constructions one can associate to the system considered a Γ -symmetric Hopf bifurcation invariant $\text{deg}_G(F_\varphi, \Omega)$, $G = \Gamma \times S^1$, allowing (in addition to the occurrence of Hopf bifurcation) a complete classification of symmetric properties of the bifurcation branches. For Γ being an abelian group, Jorge Ize provided full computation of $\text{deg}_G(F_\varphi, \Omega) \in \Pi_{S^{\mathbb{R}^2 \oplus W}}^G(S^{\mathbb{R} \oplus W})$, including both primary and secondary components. The secondary part of $\text{deg}_G(F_\varphi, \Omega)$ is useful for studying the Hopf bifurcation phenomenon when $L(\mu_0)$ is singular, as well as in the case of symmetry breaking by a time dependent periodic forcing. For the case of non-abelian Γ , we refer to [8]. It is our belief that this method is simple enough to be understood by a large spectrum of applied mathematicians, and effective enough to be applied in a standard way to various types of symmetric dynamical systems (including ODEs, FDEs, PDEs, FPDEs, etc), and allows a computerization. Moreover, this method is general enough to treat many kinds of “pathological” systems exhibiting, for example, lack of smoothness, equivariantly multiple and resonant purely imaginary characteristic roots.

Symmetric Variational Systems. The equivariant orthogonal G -degree techniques (for a compact abelian group G) can be effectively applied to determine the existence of 2π -periodic solutions in symmetric Hamiltonian and Newtonian systems. To be more specific, consider the following Hamiltonian system

$$(3.11) \quad J\dot{x} + \nabla H(x) = 0, \quad x = (y, z) \in \mathbb{R}^{2N},$$

where J is the standard symplectic matrix and H is a C^2 -function on \mathbb{R}^{2N} . We assume that a compact abelian group Γ acts symplectically on $V := \mathbb{R}^{2N}$ (i.e. the Γ -action commutes with J). The group $G := \Gamma \times S^1$ acts on spaces of 2π -periodic V -valued functions, thus (3.11) can be rewritten as $F(x) = 0$, where $F : H^{\frac{1}{2}}(S^1; V) \rightarrow H^{\frac{1}{2}}(S^1; V)$ is a G -orthogonal map. Similarly, a Newtonian system

$$(3.12) \quad -\ddot{x} + \nabla V(x) = 0, \quad x \in V := \mathbb{R}^N,$$

where V is again a Γ -invariant C^2 -function, can be reformulated as a variational G -equivariant problem $F(x) = 0$, with a G -gradient map (i.e. also G -orthogonal) $F : H^1(S^1; V) \rightarrow H^1(S^1; V)$.

Using this framework, Ize *et al.* applied the T^2 -orthogonal degree to study symmetric bifurcation in two spring-pendulum system, where he established a bifurcation from an S^1 -orbit to T^2 -orbit (see [64]).

In his last papers ([38, 39, 40, 41, 42]), Ize’s interests evolved to applications of the equivariant orthogonal degree in celestial mechanics and fluid dynamics (vortex and filament problems). Let us mention his very interesting results related to a (polygonal) Z_n -symmetric n -body configuration in a plane (see [38]). Symmetric periodic solution bifurcating from this relative equilibrium are described in details. Historically, this model was suggested by Maxwell to explain the stability of

Saturn's rings. In paper [41], Ize *et al.* considered the movement of $n + 1$ almost parallel vortices/coupled anharmonic oscillators, for which the authors provided a description of a global symmetric bifurcation of relative equilibria.

(3.6) Monograph “Equivariant Degree Theory”. Special consideration should be given to the monograph *Equivariant Degree Theory* by Ize and Vignoli –the first book written on the topic of the equivariant degree theory and its applications to differential equations with symmetries. This pioneering work constitutes a significant contribution to the area of nonlinear analysis. Although it requires only minimal mathematical background, it is a serious work which is not easy to read. One should remember that this was just the first attempt to open the stream of ideas related to the equivariant degree theory to a wide public. Therefore, one should not be surprised to find out that many important topics were only briefly outlined or presented in very technical way. Nevertheless, the reader will discover there a multitude of interesting ideas and new approaches that will give an inspiration to conduct further research. We believe that all specialists in the field of nonlinear analysis should appreciate this book, which is an excellent source of information and ideas related to the equivariant degree and its applications.

Received September 21, 2012

Final version received October 26, 2012

Z. BALANOV
DEPARTMENT OF MATHEMATICAL SCIENCES,
UNIVERSITY OF TEXAS AT DALLAS,
RICHARDSON, TEXAS, 75080, USA
balanov@utdallas.edu

W. KRAWCEWICZ
DEPARTMENT OF MATHEMATICAL SCIENCES,
UNIVERSITY OF TEXAS AT DALLAS,
RICHARDSON, TEXAS, 75080, USA
wieslaw@utdallas.edu

J. PEJSACHOWICZ
DIPARTIMENTO DI MATEMATICA POLITECNICO DI TORINO,
CORSO DUCA DEGLI ABRUZZI 24,
10129 TORINO, ITALY.
jacobopejsachowicz@polito.it

REFERENCES

- [1] J. C. ALEXANDER, *Bifurcation of zeros of parametrized functions*, J. Funct. Anal., **29**, (1978), 37–53.
- [2] J. C. ALEXANDER AND S. S. ANTMAN, *Global and local behavior of bifurcating multidimensional continua of solutions for multiparameter nonlinear eigenvalue problems*, Arch. Ration. Mech. Anal., **76**, (1981), 339–354.
- [3] J. C. ALEXANDER AND J. A. YORKE, *Global bifurcation of periodic orbits*, Amer. J. Math., **100**, (1978), 263–292.
- [4] S. S. ANTMAN, *Nonlinear problems of elasticity*. Second edition. Appl. Math. Sci., **107**, (2005), Springer.

- [5] J. ARPE, Berechnung sekundärer Koeffizientengruppen des $SO(3) \times S^1$ -äquivarianten Abbildungsgrades, Diploma thesis, University of Munich, 2001.
- [6] Z. BALANOV AND E. SCHWARTZMAN, *Morse complex, even functionals and asymptotically linear differential equations with resonance at infinity*, *Topol. Methods Nonlinear Anal.*, **12**, (1998), 323–366.
- [7] Z. BALANOV, W. KRAWCEWICZ, AND H. RUAN, *Applied Equivariant Degree. Part I. An Axiomatic Approach to Primary Degree*, *Discrete Contin. Dyn. Syst.*, **15**, (2006), 983–1016.
- [8] Z. BALANOV, W. KRAWCEWICZ, AND H. STEINLEIN, *Applied Equivariant Degree*. AIMS Ser. Differ. Equ. Dyn. Syst., (2006) **Vol. 1**.
- [9] Z. BALANOV AND W. KRAWCEWICZ, *Remarks on the equivariant degree theory*, *Topol. Methods Nonlinear Anal.*, **13**, (1999), 91–103.
- [10] Z. BALANOV, A. KUSHKULEY, AND P. ZABREJKO, *A degree theory for equivariant maps: geometric approach*, Preprint, (1993).
- [11] T. BARTSCH, *Global bifurcation from a manifold of trivial solutions*, *Univ. of Heidelberg Math. Inst.*, **13**, (1987).
- [12] T. BARTSCH, *The role of the J -homomorphism in multiparameter bifurcation theory*, *Bull. Sei. Math.*, **112**, (1988), 177–184.
- [13] T. BARTSCH, *The global structure of the zero set of a family of semilinear Fredholm maps*, *Nonlinear Anal. T.M.A.*, **17**, (1991), 313–332.
- [14] T. BARTSCH, *Bifurcation theory for metric parameter space*, in *Topological fixed point theory and applications* (Tianjin, 1988), 1–8, *Lecture Notes in Math.*, (1989) **1411**, Springer.
- [15] T. BARTSCH, *A global index for bifurcation of fixed points*, *J. Reine Angew. Math.*, **391**, (1988), 181–197.
- [16] T. BARTSCH, *The Conley index over a space*, *Math. Z.*, **209**, (1992), 167–177.
- [17] T. BARTSCH, *Topological methods for variational problems with symmetries*, *Lecture Notes in Math.*, **1560**, (1993), Springer-Verlag.
- [18] S.N. CHOW AND J. K. HALE, *Methods of Bifurcation theory*, *Grundlehren Math. Wiss.*, **251**, (1982), Springer-Verlag.
- [19] B. BUFFONI AND J. TOLAND, *Analytic theory of global bifurcation. An introduction*, Princeton University Press, Princeton, NJ, (2003).
- [20] S. N. CHOW, J. MALLET-PARET, AND J. A. YORKE, *Global Hopf bifurcation from a multiple eigenvalue*, *Nonlinear Anal. TMA*, **2**, (1978), 753–763.
- [21] E. N. DANCER, *A new degree for S^1 -invariant gradient mappings and applications*, *Annal. Inst. H. Poincaré, Anal. Non Linéaire*, **2**, (1985), 329–370.
- [22] E. N. DANCER, K. GEBA, AND S. RYBICKI, *Classification of homotopy classes of G -equivariant gradient maps*, *Fund. Math.*, **185**, (2005), 1–18.
- [23] E. N. DANCER AND J. F. TOLAND, *Degree theory for orbits of prescribed period of flows with a first integral*, *Proc. London Math. Soc.*, **60**, (1990), 549–580.
- [24] K. DEIMLING, *Nonlinear Functional Analysis*, Springer Verlag, (1985).
- [25] E. N. DANCER AND J. F. TOLAND, *Equilibrium states in the degree theory of periodic orbits with a first integral*, *Proc. London Math. Soc.*, **61**, (1991), 564–594.
- [26] E. N. DANCER AND J. F. TOLAND, *The index change and global bifurcation for flows with first integrals*, in *Proc. London Math. Soc.*, **66**, (1993), 539–567.
- [27] G. DYLAWEWSKI, *An S^1 -degree and S^1 -maps between representation spheres*, in *Algebraic Topology and Transformation Groups* (ed. T. tom Dieck), 14–28, *Lecture Notes in Math.*, 1361, Springer-Verlag, Berlin-Heidelberg-New York-London-Paris-Tokyo (1988).
- [28] G. DYLAWEWSKI, K. GEBA, J. JODEL, AND W. MARZANTOWICZ, *An S^1 -equivariant degree and the Fuller index*, *Ann. Polon. Math.*, **52**, (1991), 243–280.
- [29] L. ERBE, K. GEBA, W. KRAWCEWICZ, AND J. WU, *S^1 -degree and global Hopf bifurcation theory of functional differential equations*, *J. Differential Equations*, **98**, (1992), 277–298.
- [30] J. ESQUINAS AND J. LÓPEZ GÓMEZ, *Optimal multiplicity in local bifurcation theory. I. Generalized generic eigenvalues*, *J. Differential Equations*, **71**, (1988), 71–92.
- [31] B. FIEDLER, *An index for global Hopf bifurcation in parabolic systems*, *J. Reine Angew. Math.*, **359**, (1985), 1–36.
- [32] P. M. FITZPATRICK, *Homotopy, linearization and bifurcation*, *Nonlinear Anal. T.M.A.*, **12**, (1988), 171–184.

- [33] P. M. FITZPATRICK, I. MASSABÓ, AND J. PEJSACHOWICZ, *Global several parameters bifurcation and continuation theorems, a unified approach via complementing maps*, Math. Ann., **263**, (1983), 61–73.
- [34] P. M. FITZPATRICK AND J. PEJSACHOWICZ, *Parity and generalized multiplicity*, Trans. Amer. Math. Soc., **326**, (1991), 281–305.
- [35] P. M. FITZPATRICK, J. PEJSACHOWICZ AND P. J. RABIER, *The degree for proper C^2 Fredholm mappings I*, J. Reine Angew. Math., **424**, (1992), 1–33.
- [36] F. B. FULLER, *An index of fixed point type for periodic orbits*, Amer. J. Math., **89**, (1967), 133–148.
- [37] M. FURI, M. MARTELLI, AND A. VIGNOLI, *On the Solvability of nonlinear operator equations in normed spaces*, Ann. Mat. Pura Appl., **124**, (1980), 321–343.
- [38] C. GARCIA-ASPEITIA AND J. IZE, *Global bifurcation of planar and spatial periodic solutions in the restricted n -body problem*, Celestial Mech. Dynam. Astronom., **110**, (2011), 217–237.
- [39] C. GARCIA-ASPEITIA AND J. IZE, *Global bifurcation of polygonal relative equilibria for masses, vortices and $dNLS$ oscillators*, J. Differential Equations, **251**, (2011), 3202–3227.
- [40] C. GARCIA-ASPEITIA AND J. IZE, *Global bifurcation of planar and spatial periodic solutions from the polygonal relative equilibria for the n -body problem*, Submitted.
- [41] C. GARCIA-ASPEITIA AND J. IZE, *Bifurcation of periodic solutions from a ring configuration in the vortex and filament problems*, Journal of Differential Equations, **252**, (2012), 5662–5678.
- [42] C. GARCIA-ASPEITIA AND J. IZE, *Bifurcation of periodic solutions from a ring configuration of discrete nonlinear oscillators*, accepted in Discrete and Contin. Dyn. Syst., (2012).
- [43] K. GEBA, *Degree for gradient equivariant maps and equivariant Conley index*, in Topological Nonlinear Analysis II, (1995), 247–272, Progr. Nonlinear Differential Equations Appl., **27**, Birkhäuser Boston, Boston (1997).
- [44] K. GEBA AND A. GRANAS, *Infinite dimensional cohomology theories*, J. Math. Pures Appl., **52**, (1973), 147–270.
- [45] K. GEBA, W. KRAWCEWICZ, AND J. H. WU, *An equivariant degree with applications to symmetric bifurcation problems. I. Construction of the degree*, Proc. London Math. Soc., **69**, (1994), 377–398.
- [46] K. GEBA, I. MASSABÓ, AND A. VIGNOLI, *Generalized topological degree and bifurcation*, Nonlinear Funct. Anal. Appl. Reidel (1986), 55–73.
- [47] J. GUCKENHEIMER AND P. HOLMES, *Nonlinear oscillations, dynamical systems and bifurcation of vector fields*, Springer Verlag (1983).
- [48] S. HILDEBRANDT AND A. TROMBA, *The parsimonious universe. Shape and form in the natural world*, Copernicus, New York (1996).
- [49] G. IOOSS AND D. D. JOSEPH, *Elementary Stability and Bifurcation Theory*, Springer Verlag (1980).
- [50] J. IZE, *Bifurcation theory for Fredholm operators*. Mem. Amer. Math. Soc. (1976) **174**.
- [51] J. IZE, *Periodic solutions for nonlinear parabolic equations*, Comm. in Partial Differential Equations, **12**, (1979), 1299–1387.
- [52] J. IZE, *Introduction to bifurcation theory*. Springer Verlag Lect. Notes in Math, **957**, (1982), 145–202.
- [53] J. IZE, *Obstruction theory and multiparameter Hopf bifurcation*, Trans. Amer. Math. Soc., **289**, (1985), 757–792.
- [54] J. IZE, *Necessary and sufficient conditions for multiparameter bifurcation*, Rocky Mountain J. of Math., **18**, (1988), 305–337.
- [55] J. IZE, *Topological Bifurcation*, Progress in Nonlinear Differential Equations and their Applications, Topological Nonlinear Analysis, M. Matzeu and A. Vignoli Ed. Birkhauser, **15**, (1995), 341–463.
- [56] J. IZE, *Equivariant degree*, Handbook of Topological fixed point theory. R. F. Brown, M. Furi, L. Gorniewicz and B. Jiang, Eds, (2005), Springer, 301–338.
- [57] J. IZE, I. MASSABÓ, J. PEJSACHOWICZ, AND A. VIGNOLI, *Structure and dimension of global branches of solutions to multiparameter nonlinear equations*, Trans. Amer. Math. Soc., **291**, (1985), 383–435.
- [58] J. IZE, I. MASSABÓ, AND A. VIGNOLI, *Global results on continuation and bifurcation for equivariant maps*, NATO-ASI, **173**, (1986), 75–111.
- [59] J. IZE, I. MASSABÓ, AND A. VIGNOLI, *Degree theory for equivariant maps I*, Trans. Amer. Math. Soc., **315**, (1989), 433–510.

- [60] J. IZE, I. MASSABÓ, AND A. VIGNOLI, *Degree theory for equivariant maps: The general S^1 -action*, Mem. Amer. Math. Soc., **481**, (1992).
- [61] J. IZE AND A. VIGNOLI, *Equivariant degree for abelian actions, Part I; equivariant homotopy groups*, Topol. Methods Nonlinear Anal., **2**, (1993), 367–413.
- [62] J. IZE AND A. VIGNOLI, *Equivariant degree for abelian actions, Part II. Index Computations*, Topol. Methods Nonlinear Anal., **7**, (1996), 369–430.
- [63] J. IZE AND A. VIGNOLI, *Equivariant degree for abelian actions. Part III: Orthogonal maps*, Topol. Methods Nonlinear Anal., **13**, (1999), 105–146.
- [64] J. IZE AND A. VIGNOLI, *Equivariant Degree Theory*, De Gruyter Ser. Nonlinear Anal. Appl., **8** (2003).
- [65] H. KIELHÖFER, *Bifurcation Theory. An introduction with applications to PDEs.*, Springer-Verlag, New York, (2004).
- [66] M. A. KRASNOSEL'SKII, *Topological methods in the theory of nonlinear integral equations.* Pergamon Press, MacMilan, New York, (1964).
- [67] M. A. KRASNOSEL'SKII AND P. P. ZABREIKO, *Geometrical Methods in Nonlinear Analysis*, Grundlehren Math. Wiss., **263**, (1984), Springer-Verlag.
- [68] KUSHKULEY AND Z. BALANOV, *Geometric Methods in Degree Theory for Equivariant Maps*, *Lect. Notes in Math.*, **1632**, (1996), Springer.
- [69] A. MACIEJEWSKI AND S. RYBICKI, *Global bifurcations of periodic solutions of Hénon-Heiles system via degree for S^1 -equivariant orthogonal maps*, Rev. Math. Phys., **10**, (1998), 1125–1146.
- [70] R. J. MAGNUS, *A generalization of multiplicity and the problem of bifurcation*, Proc. London Math. Soc., **32**, (1976), 251–278.
- [71] W. MARZANTOWICZ AND C. PRIETO, *Computation of the equivariant 1-stem*, Nonlinear Anal., **63**, (2005), 513–524.
- [72] J. MAWHIN AND M. WILLEM, *Critical point theory and Hamiltonian systems*, Applied Mathematical Sciences, **74**, Springer-Verlag (1989).
- [73] J. MALLET-PARET AND J. A. YORKE, *Snakes: oriented families of periodic orbits, their sources, sinks and continuation*, J. Differential Equations, **43**, (1982), 419–450.
- [74] L. NIRENBERG, *Topics in nonlinear functional analysis.* Lect. Notes Courant Institute, New York Univ (1974).
- [75] J. PEJSACHOWICZ, *K-theoretic methods in bifurcation theory*, Contemp. Math., **72**, (1988), 193–205.
- [76] J. PEJSACHOWICZ, *Bifurcation of Fredholm maps I; The index bundle and bifurcation*, Topol. Methods Nonlinear Anal., **38**, (2011), 115–168.
- [77] J. PEJSACHOWICZ, *The family index theorem and bifurcation of solutions of nonlinear elliptic BVP*, J. Differential Equations, **252**, (2012), 4942–4961.
- [78] G. PESCHKE, *Degree of certain equivariant maps into a representation sphere*, Topology Appl., **59**, (1994), 137–156.
- [79] P. H. RABINOWITZ, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal., **7**, (1971), 487–513.
- [80] P. H. RABINOWITZ, *Minimax methods in critical point theory with applications to differential equations*, CBMS Reg. Conf. Ser. Math., **65**. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, (1986).
- [81] P. J. RABIER, *Generalized Jordan chains and two bifurcation theorems of Krasnoselskii*, Nonlinear Anal. TMA., **13**, **8**, (1989), 903–934.
- [82] F. ROMERO RUIZ DEL PORTAL, *Teoría del grado topológico generalizado y aplicaciones*, Ph. Thesis (1990), Madrid.
- [83] S. RYBICKI, *A degree for S^1 -equivariant orthogonal maps and its applications to bifurcation theory*, Nonlinear Anal., **23**, (1994), 83–102.
- [84] S. RYBICKI, *Equivariant degree for gradient maps*, Millan J. Math., **73**, (2005), 1–35.
- [85] D. H. SATTINGER, *Branching in the Presence of Symmetry*, Wiley (1983).
- [86] A. E. TAYLOR, *Theorems on ascent, descent, nullity and defect of linear operators*, Math. Ann., **163**, (1966), 18–49.
- [87] M. M. VAINBERG AND V. A. TRENIGIN, *Theory of branching of solutions of nonlinear equations.* Noordhoff, Leyden (1974).
- [88] D. WESTREICH, *Bifurcation at eigenvalues of odd multiplicity*, Proc. Amer. Math. Soc., **41**, (1973), 609–614.

- [89] G. W. WHITEHEAD, *On the homotopy groups of spheres and rotation groups*, Ann. of Math., **43**, (1942), 634–640.
- [90] V. G. ZVYAGIN AND N. M. RATNER, *Oriented degree of Fredholm maps of nonnegative index and its application to global bifurcation of solutions*, in Global analysis—studies and applications, Lecture Notes in Math., **1520**, (1992), Springer, 111–137.

THE WORK OF JORGE IZE REGARDING THE n -BODY PROBLEM

C. GARCÍA-AZPEITIA

ABSTRACT. In this paper we present a summary of the last works of Jorge Ize regarding the global bifurcation of periodic solutions from the equilibria of a satellite attracted by n primary bodies. We present results on the global bifurcation of periodic solutions for the primary bodies from the Maxwell's ring, in the plane and in space, where n identical masses on a regular polygon and one central mass are turning in a plane at a constant speed. The symmetries of the problem are used in order to find the irreducible representations, and with the help of the orthogonal degree theory, all the symmetries of the bifurcating branches. The results presented in this paper were done during the Ph.D. of the author under the direction of Jorge Ize (see [16], [17], [18], [19], [20]). This paper is dedicated to his memory.

This paper is devoted to present the results of the author in collaboration to Jorge Ize regarding the movement of a satellite attracted by n primary bodies. In particular, when the primary bodies form the polygonal relative equilibrium corresponding to n identical masses arranged on a regular polygon with one mass in the centre. This model was posed by Maxwell in order to explain the stability of Saturn's rings.

For this polygonal relative equilibrium, we give also a description of bifurcation of planar and spatial periodic solutions. According to the value of the central mass, there are up to $2n$ branches of planar periodic solutions, with different symmetries, and up to n additional branches, with non trivial vertical components, if some non resonance condition is satisfied. The linearization of the system is degenerated due to rotational symmetries. These facts imply that the classical bifurcation results for periodic solutions may not be applied directly. The proof is carried on with the use of a topological degree for maps that commute with symmetries and are orthogonal to the infinitesimal generators for these symmetries.

We also expose the global bifurcation of periodic solutions for a satellite attracted by n primary bodies. These solutions will form a continuum in the plane of the primaries and other solutions outside the plane. A particular attention is given to the case where $n + 1$ primaries form the Maxwell's Saturn ring.

In order to explain the results, we give a short description of the steps to prove the bifurcation theorem. The ideas we follow are from the book [24], where general bifurcation theorems are proven. In addition, in [15] there is a systematic application to Hamiltonian systems. The results exposed here for the n -body problem and the satellite are from the papers [16], [17], and [19].

1. Orthogonal degree

A Hilbert space V is a Γ -representation if there is a morphism of groups

$$\rho : \Gamma \rightarrow GL(V).$$

The action of the group over a point generates one orbit denoted by Γx . A set $\Omega \subset V$ is Γ -invariant if it is made of orbits, this is $\Gamma x \subset \Omega$ for all $x \in \Omega$.

The *isotropy group* of a point x is defined by

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\},$$

and the *fixed point space* of the subgroup H is

$$X^H = \{x \in X : hx = x, \forall h \in H\}.$$

A space V is an *irreducible representation* when V does not have Γ -invariant proper subspaces. The irreducible representations of the action of a compact abelian Lie group are always two dimensional, and as such, equivalent to the complex space.

A function $f : \Omega \rightarrow W$ is Γ -equivariant if

$$f(\gamma x) = \gamma' f(x),$$

and Γ -invariant if the action in the range is trivial, $f(\gamma x) = f(x)$.

PROPOSITION (1.1). *A differentiable Γ -equivariant function at x satisfies*

$$df(\gamma x)\gamma = \gamma' df(x)$$

for all $\gamma \in \Gamma$. In particular, the derivative $f'(x)$ is a Γ_x -equivariant map. Moreover, the gradient of a Γ -invariant functional is a Γ -equivariant map when the action is orthogonal.

Proof. The first statement follows from the uniqueness of the derivative, and from the equality

$$\begin{aligned} df(\gamma x)\gamma y + o(y) &= f(\gamma(y+x)) - f(\gamma x) \\ &= \gamma'[f(y+x) - f(x)] = \gamma' df(x)y + o(y). \end{aligned}$$

The second statement is a consequence of

$$\gamma^T \nabla f(\gamma x) = [Df(\gamma x)\gamma]^T = Df(x)^T = \nabla f(x). \quad \square$$

Let γ be an element of a torus $\gamma = (\varphi_1, \dots, \varphi_n) \in T^n$, with $\varphi_j \in (-\pi, \pi)$. The j -th generator of the torus T^n is the vector fields tangent to the orbit

$$A_j x = \frac{\partial}{\partial \varphi_j} (\gamma x)|_{\gamma=0}.$$

The gradient of a Γ -invariant function is Γ -equivariant by the previous proposition. Moreover, this kind of gradient is orthogonal to the generators because

$$\langle \nabla f(x), A_j x \rangle = \frac{\partial}{\partial \varphi_j} f(\gamma x)|_{\gamma=0} = 0.$$

A general Γ -equivariant map is called Γ -orthogonal if it satisfy

$$\langle f(x), A_j x \rangle = 0 \text{ for all } x \in \Omega.$$

The following definition of Γ -orthogonal degree for compact abelian Lie groups is due to J. Ize and A. Vignoli, see [23].

Let Γ be a compact abelian Lie group, an Ω a Γ -invariant domain of V . Let f_0 and f_1 two Γ -orthogonal maps which are non-zero on the boundary $\partial\Omega$. It is said

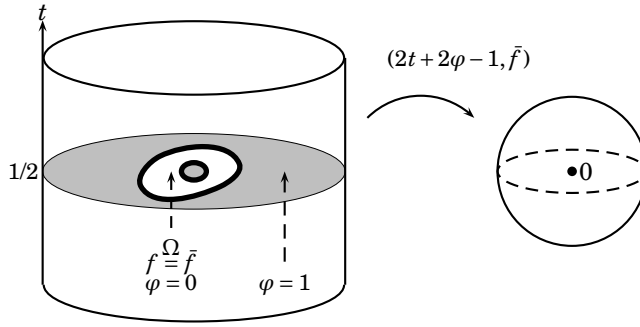


Figure 1. Degree definition.

that two maps f_0 and f_1 are Γ -orthogonal homotopic when there is a continuous deformation

$$f_t : \bar{\Omega} \times [0, 1] \rightarrow E,$$

where the map f_t is Γ -orthogonal and non-zero in the boundary $\partial\Omega$ for each step t .

The ball $B = \{x \in V : \|x\| \leq r\}$ is Γ -invariant when the representation in V is an isometry. In this case, let us define \mathcal{C} as the set of Γ -orthogonal maps of the form

$$f : \partial([0, 1] \times B) \rightarrow \mathbb{R} \times V - \{0\}.$$

Since the boundary of $[0, 1] \times B_r$ is isomorphic to the sphere S^V , and since the set $\mathbb{R} \times V - \{0\}$ is Γ -homotopic to S^V , then the map f may be thought from S^V into S^V .

Since the Γ -orthogonal homotopy forms an equivalent relation in \mathcal{C} , then one define $\Pi_{\perp}[S^V]$ as the set of equivalent classes of \mathcal{C} and

$$[f]_{\perp} \in \Pi_{\perp}[S^V]$$

as the equivalent class of f .

Shrinking the top $\{0\} \times B$ and the bottom $\{1\} \times B$ to the point $(1, 0)$, one may prove that all homotopy classes $[f]_{\perp}$ have one function such that $f(t, x) = (1, 0)$ for $t \in \{0, 1\}$. With these functions one may define the sum of homotopy classes as $[f]_{\perp} + [g]_{\perp} = [f \oplus g]_{\perp}$ with

$$f \oplus g = \begin{cases} f(2t, x) & \text{for } t \in [0, 1/2], \\ g(2t - 1, x) & \text{for } t \in [1/2, 1]. \end{cases}$$

With this sum, the set $\Pi_{\perp}[S^V]$ has a group structure. The identity is the map $[(1, 0)]_{\perp}$, and the inverse of some class $[f]_{\perp}$ is the class $[f(1 - t, x)]_{\perp}$. Moreover, one may prove that the group $\Pi_{\perp}[S^V]$ is abelian when V^{Γ} is non trivial.

To define the Γ -orthogonal degree of a map $f : \Omega \rightarrow V$, it is necessary to extend the function f to a ball, $\bar{f} : \Omega \subset B \rightarrow V$. Also, one needs a Urysohn Γ -invariant map with value 0 in $\bar{\Omega}$, and value 1 in $B \setminus N$, where N is a small neighborhood of $\bar{\Omega}$. The existence of the Urysohn map φ and the extension \bar{f} follows from the Γ -orthogonal extension theorem of Borsuk. The proof of this theorem is only for actions of compact abelian Lie groups on finite spaces, see [24].

Definition (1.2). The Γ -orthogonal degree of f is defined as the homotopy class

$$\text{deg}_{\perp}(f; \Omega) = [(2t + 2\varphi - 1, \bar{f})]_{\perp} \in \Pi_{\perp}[S^V].$$

When the domain is a ball, the degree is just the homotopy class of the suspension $\text{deg}_\perp(f; B) = [(2t - 1, f)]_\perp$, and this definition is equivalent to the Brouwer degree if the action of the group is trivial.

In [24] it is proven that for each isotropy group of Γ , $H \in \text{Iso}(\Gamma)$, the group $\Pi_\perp[S^V]$ has a copy of a group isomorphic to the group \mathbb{Z} , this is

$$\Pi_\perp[S^V] = \bigoplus_{H \in \text{Iso}(\Gamma)} \mathbb{Z}.$$

Moreover, the degree of the map f is

$$\text{deg}_\perp(f; \Omega) = \sum_{H \in \text{Iso}(\Gamma)} d_H [F_H]_\perp,$$

where $[F_H]_\perp$ is the generator of one \mathbb{Z} corresponding to each isotropy group $H \in \text{Iso}(\Gamma)$, and d_H is just an integer.

The orthogonal degree has the known properties of a degree: existence, excision and Γ -orthogonal homotopy invariance. In this case, the existence property means that the map f must have a zero in $\Omega \cap V^H$ if $d_H \neq 0$.

REMARK (1.3). *For a k -dimensional orbit, with a tangent space generated by k of the infinitesimal generators of the group, one uses a Poincaré section for the map augmented with k Lagrange-like multipliers for the generators. (See the construction in [24], Section 4.3). For instance, for the action of $SO(2)$, the study of zeros of the equivariant map $F(x)$, orthogonal to the generator Ax , is equivalent to the study of the zeros of $F(x) + \lambda Ax$; if x is not fixed by the group, i.e., if Ax is not 0, for which λ is 0. In this way, one has added an artificial parameter. This trick has been used very often and, in the context of a topological degree argument, was called “orthogonal degree” by Rybicki in [29]. See also [10] and [22] for the case of gradients. The general case of the action of abelian groups was treated in [23]. The complete study of the orthogonal degree theory is given in [24], Chapters 2 and 4. From the theoretical point of view, the theory has to be extended to the action of non-abelian groups and to abstract infinite dimensional spaces.*

2. Satellite

The restricted n -body problem is the study of the movement of a satellite attracted by n primary bodies which are rotating, at a constant angular speed, around an axis. Since the mass of the satellite is small, one assumes that the satellite does not perturb the trajectories of the primaries, which follow the trajectories of relative equilibrium and, as such, are in a plane.

Let $q(t) \in \mathbb{R}^3$ be the position of the satellite without mass, and let $(a_j, 0)$ be the position of a primary body with mass m_j . Let J be the standard symplectic matrix in \mathbb{R}^2 . In rotating coordinates $q(t) = (e^{\omega J t} u(t), z(t))$, $u \in \mathbb{R}^2$, Newton’s equations describing the movement of the satellite, with angular speed $\omega = 1$, are

$$(2.1) \quad \begin{aligned} \ddot{u} + 2J\dot{u} - u &= - \sum_{j=1}^n m_j \frac{u - a_j}{\|(u, z) - (a_j, 0)\|^3}, \\ \ddot{z} &= - \sum_{j=1}^n m_j \frac{z}{\|(u, z) - (a_j, 0)\|^3}. \end{aligned}$$

One may ask for existence of bifurcation of periodic solutions starting from the equilibria of the satellite. These solutions will form a continuum in the plane of

the primaries and there are other global branches outside of that plane. The proof is based on the use of the orthogonal degree.

(2.1) The orthogonal bifurcation map. Let $H_{2\pi}^2(\mathbb{R}^n)$ be the Sobolev space of 2π -periodic functions. Define the collision points set as $\Psi = \{a_1, \dots, a_n\}$, and the collision-free paths as

$$H_{2\pi}^2(\mathbb{R}^3 \setminus \Psi) = \{x \in H_{2\pi}^2(\mathbb{R}^3) : x(t) \neq a_j\}.$$

Changing variables from t to t/ν , the $2\pi/\nu$ -periodic solutions are zeros of the map

$$f : H_{2\pi}^2(\mathbb{R}^3 \setminus \Psi) \times \mathbb{R}^+ \rightarrow L_{2\pi}^2$$

$$f(x, \nu) = -\nu^2 \ddot{x} - 2\nu \operatorname{diag}(J, 0)\dot{x} + \nabla V(x).$$

where V is the potential

$$V(u, z) = |u|^2/2 - \sum_{j=1}^n m_j \frac{1}{\|(u, z) - (a_j, 0)\|}.$$

In view of the definitions, the collision-free 2π -periodic solutions are zeros of the bifurcation operator $f(x, \nu)$. Furthermore, the operator f is well defined and continuous.

Define the actions of the group $\mathbb{Z}_2 \times S^1$ on $H_{2\pi}^2(\mathbb{R}^3 \setminus \Psi)$ as

$$\rho(\kappa)x = \operatorname{diag}(1, 1, -1)x \text{ and } \rho(\varphi)x = x(t + \varphi).$$

Since the equation of the satellite is invariant by this reflection, and since the equation is autonomous, then f is $\mathbb{Z}_2 \times S^1$ -equivariant. The generator of the group S^1 in the space $H_{2\pi}^2$ is

$$Ax = \frac{d}{d\varphi}(\rho(\varphi)x)_{\varphi=0} = \dot{x}.$$

Moreover, the map f is $\mathbb{Z}_2 \times S^1$ -orthogonal because it satisfies the orthogonal condition

$$\langle f(x), \dot{x} \rangle_{L_{2\pi}^2} = \int_0^{2\pi} (-\nu^2 |\dot{x}|^2/2 + V(x))' dt = 0.$$

REMARK (2.2). For periodic and non-periodic solutions of the equations, the conservation of energy is written as

$$E = -\nu^2 |\dot{x}|^2/2 + V(x) = \text{cte}.$$

Thus, one may think that the orthogonal condition is equivalent to conservation of energy.

The Fourier transform of the bifurcation map is

$$f(x) = \sum_{l \in \mathbb{Z}} (l^2 \nu^2 x_l - 2il\nu \operatorname{diag}(J, 0)x_l + g_l) e^{ilt},$$

where x_l and g_l are the Fourier modes of x and $\nabla V(x)$ respectively.

Since the matrix

$$l^2 \nu^2 I - 2il\nu \operatorname{diag}(J, 0)$$

is invertible for all l 's, except a finite number. One may perform a global Lyapunov-Schmidt reduction using the global implicit function theorem for non-collision paths. In this way, one gets the reduced map $f_1(x_1, x_2(x_1, \nu), \nu)$, where x_1 corresponds to a finite number of modes and x_2 to the complement. Moreover, the reduced map is a Γ -orthogonal map, see [24] or [15] for details. Furthermore, for

bifurcation without resonances one may reduce the map to the principal Fourier mode $l = 1$.

For isolated orbits Γ_{x_0} , the degree is calculated in terms of the linearization at x_0 . Close to an equilibrium x_0 one has that $\nabla V(x_0 + h) = D^2V(x_0)h + o(h)$, then the linearization of the reduced map is

$$f'_1(x_0, \nu)x_1 = \sum_{\text{finite } l's} M(l\nu)x_l e^{ilt} \text{ with} \\ M(\nu) = \nu^2 I - 2i\nu \text{diag}(J, 0) + D^2V(x_0).$$

So the linearization of the reduced map is a diagonal matrix with blocks $M(l\nu)$ for a finite number of l 's. For bifurcation without resonances, it has only the block $M(\nu)$, for the 1-th Fourier mode.

(2.2) Symmetries. The action of the element $(\kappa, \varphi) \in \mathbb{Z}_2 \times S^1$ satisfy

$$\rho(\kappa, \varphi)x = \rho(\kappa)x(t + \varphi) = \sum_l \rho(\kappa) e^{il\varphi} x_l e^{ilt},$$

thus the action of the group is inherited on the Fourier modes as

$$\rho(\kappa, \varphi)x_l = \rho(\kappa) e^{il\varphi} x_l.$$

Since all the equilibria are planar, the isotropy subgroup of any equilibrium is $\mathbb{Z}_2 \times S^1$, this means that all equilibria are fixed by the action of $\mathbb{Z}_2 \times S^1$. When one apply orthogonal degree to the bifurcation problem, one need to know the irreducible representations of the action of $\Gamma_{x_0} = \mathbb{Z}_2 \times S^1$.

2.2.1. Planar symmetries. In order to simplify the exposition, only the symmetries of the group $\mathbb{Z}_2 \times S^1$ for the 1-th mode will be studied. This correspond to the case without resonances. The space \mathbb{C}^3 corresponding to the 1-th mode has two spaces of similar irreducible representations: $V_0 = \mathbb{C}^2 \times \{0\}$ and $V_1 = \{0\} \times \mathbb{C}$. This is, the group \mathbb{Z}_2 acts as $\rho(\kappa) = I$ on V_0 , and as $\rho(\kappa) = -1$ on V_1 . Consequently, the action of the group $\mathbb{Z}_2 \times S^1$ in V_0 for the 1-th mode is

$$\rho(\kappa, \varphi)x = e^{i\varphi}x.$$

Since $(\kappa, 0)$ is the only element that fix the points of V_0 , the isotropy subgroup of the points in V_0 is generated by $(\kappa, 0)$,

$$\mathbb{Z}_2 = \langle (\kappa, 0) \rangle.$$

Solutions $x = (u, z)$ to the equation (2.1) with isotropy group \mathbb{Z}_2 satisfy

$$x(t) = \rho(\kappa)x(t) = \text{diag}(1, 1, -1)x(t).$$

Therefore, solutions to the equation (2.1) with symmetry \mathbb{Z}_2 are just planar solutions, i.e. $z(t) = 0$.

2.2.2. Spatial symmetries. In V_1 the action of the group $\mathbb{Z}_2 \times S^1$ is

$$(\kappa, \varphi)x = -e^{i\varphi}x.$$

Since (κ, π) is the only element that fix the points of V_1 , thus the isotropy subgroup for V_1 is generated by (κ, π) ,

$$\tilde{\mathbb{Z}}_2 = \langle (\kappa, \pi) \rangle.$$

Solutions $x = (u, z)$ to the equation (2.1) with isotropy group $\tilde{\mathbb{Z}}_2$ satisfy

$$x(t) = \rho(\kappa, \pi)x(t) = \text{diag}(1, 1, -1)x(t + \pi),$$

this is

$$u(t) = u(t + \pi) \text{ and } z(t) = -z(t + \pi).$$

Solutions to the equation (2.1) with these symmetries follows twice the planar π -periodic curve u , one time with the spatial coordinate z and a second time with $-z$. Consequently, there is at least one t_0 where $z(t_0) = z(t_0 + \pi) = 0$. For instance, if only one of these zeros exists, then the solution looks like a spatial eight near the equilibrium. For this reason, these solutions will be called eight-solutions.

(2.3) Bifurcation theorem. For bifurcation without resonances, one may reduce the bifurcation study to the 1-th Fourier mode. In this case, the $\mathbb{Z}_2 \times S^1$ -orthogonal degree of the reduced map complemented by the right function is

$$\eta_{\mathbb{Z}_2}(v_0)[F_{\mathbb{Z}_2}] + \eta_{\tilde{\mathbb{Z}}_2}(v_0)[F_{\tilde{\mathbb{Z}}_2}],$$

where $[F_{\mathbb{Z}_2}]$ and $[F_{\tilde{\mathbb{Z}}_2}]$ are generators of one \mathbb{Z} in the homotopy group Π_{\perp} . The numbers $\eta_*(v_0)$ correspond to the change of Morse index of the block $M(v)$ in the space V_0 , for \mathbb{Z}_2 , and in the space V_1 , for $\tilde{\mathbb{Z}}_2$.

From the existence property of the degree, one has a zeros of the bifurcation map when $\eta(v_0) \neq 0$, this is, there is periodic solutions near (x_0, v_0) with isotropy group \mathbb{Z}_2 , if $\eta_{\mathbb{Z}_2}(v_0) \neq 0$, and with isotropy group $\tilde{\mathbb{Z}}_2$, if $\eta_{\tilde{\mathbb{Z}}_2}(v_0) \neq 0$. For resonances one may have more generators of Π_{\perp} corresponding to bifurcation of harmonic periods of the principal one.

What remains is to analyze the Morse index in the subspaces V_0 and V_1 . This is done in [16], where one arrives at the following conclusion.

THEOREM (2.3). *Let T and D be the trace and determinant of the Hessian of the potential in the plane, V , at the equilibrium x_0 . If $D < 0$, there is one global bifurcation of planar periodic solutions from x_0 . If $0 < D < (2 - T/2)^2$, there are two global bifurcations of planar periodic solutions.*

THEOREM (2.4). *Every equilibrium x_0 has a global bifurcation of periodic eight solutions*

$$u(t) = u(t + \pi) \text{ and } z(t) = -z(t + \pi).$$

Moreover, the local branch is truly spatial, $z(t) \neq 0$, provided that some nonresonant condition between the periods of the spatial and the planar solutions is satisfy.

By global branch, one means that there is a continuum of solutions starting at the equilibrium, where the continuum goes to infinity in the norm of the solution or in the period, or goes to collision, or otherwise goes to other relative equilibria in such a way that the sum of the jumps in the orthogonal degrees is zero.

2.3.1. A Morse potential. One may easily prove that all equilibria for the satellite are planar. Moreover, provided that the potential in the plane V is a Morse function, there are at least one global minimum and n saddle points, see [16]. For example, in the classical restricted three body problem, case $n = 2$, there are two minimums where the satellite form an equilateral triangle with the primaries, and three saddle points where the satellite is collinear with the two primaries.

THEOREM (2.5). *Provided that the potential in the plane V is a Morse function, each one of the n saddle points has one global bifurcation of planar periodic solutions, and one global bifurcation of periodic eight solutions.*

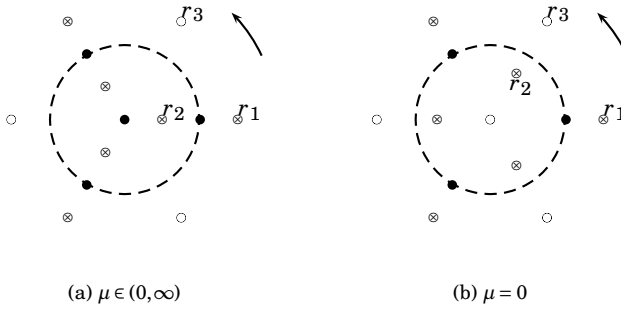


Figure 2. Example for $n = 3$.

THEOREM (2.6). *The minimum point satisfy one of the following options: (a) it has two global bifurcations of planar periodic solutions and one bifurcation of periodic eight solutions, or (b) it has only one bifurcation of spatial periodic eight solutions.*

2.3.2. The Maxwell's Saturn ring. One may apply these results when the primaries form the Maxwell's Saturn ring, see Proposition (3.4). This is a classical model for Saturn and one ring around it. In this case one has the following theorem.

THEOREM (2.7). *The potential has two \mathbb{Z}_n -orbits of saddle points ($r1$) and ($r2$), when $n \geq 2$, and one more \mathbb{Z}_n -orbit of saddle points when $n \geq 3$ and μ is near from zero. Furthermore, each saddle point has one global bifurcation of planar periodic solutions and one global bifurcation of periodic eight-solutions.*

THEOREM (2.8). *The potential has one \mathbb{Z}_n -orbit of minimum points ($r3$) for $n \geq 2$. Moreover, provided μ is big enough, each minimum point has two global bifurcations of planar periodic solutions, and one global bifurcation of periodic eight solutions. On the other hand, if μ is small and $n \geq 3$, there is another \mathbb{Z}_n -orbit of minimum points with only one bifurcation of spatial periodic eight-solutions.*

REMARK (2.9). *The equilibria of the \mathbb{Z}_n -orbit of minimum points ($r3$) are linearly stable if μ is big enough. This is proven in the paper [2]. The existence of the two extra orbits of equilibria for μ small was pointed out in the paper [1]. The stability and this fact is proven also in the paper [16]. The orthogonal degree has been used to prove bifurcation in the restricted three body problem also in the paper [25].*

REMARK (2.10). *The degree arguments, coupled with group representation ideas, give global information, i.e., an indication of where the bifurcation branches could go. Also, since the results are valid for problems which are deformation of the original problem, the method does not require high order computations and they may be applied in some degenerate cases (for instance it is not necessary that the bifurcation parameter crosses a critical value with non-zero speed; it is enough that it crosses it eventually). An immediate drawback of this approach is that topological methods do not provide a detailed information on the local behavior of the bifurcating branch, such as stability or the existence of other type of solutions, like KAM*

tori. Other methods, such as normal forms or special coordinates, should be used for these purposes but they only provide local information near the critical point. In a similar way, the degree arguments give only partial results on resonances and other tools should be used.

3. The n -body problem

Let $q_j(t) \in \mathbb{R}^3$ be the position of the j -th body with mass m_j , for $j \in \{0, 1, \dots, n\}$. Let J be the standard symplectic matrix in \mathbb{R}^2 . Newton's equations of the n bodies, in rotating coordinates $q_j(t) = (e^{\sqrt{\omega}tJ} u_j(t), z_j(t))$, are

$$(3.1) \quad m_j \ddot{u}_j + 2m_j \sqrt{\omega} J \dot{u}_j = \omega m_j u_j - \sum_{\substack{i=0 \\ i \neq j}}^n m_i m_j \frac{u_j - u_i}{\|(u_j, z_j) - (u_i, z_i)\|^3}$$

$$m_j \ddot{z}_j = - \sum_{\substack{i=0 \\ i \neq j}}^n m_i m_j \frac{z_j - z_i}{\|(u_j, z_j) - (u_i, z_i)\|^3}.$$

Relative equilibria of the n -body problem correspond to equilibria in these rotating coordinates. Since all relative equilibria are planar, the positions $(a_j, 0)$ correspond to a relative equilibrium if they satisfy the relations

$$(3.2) \quad \omega a_j = \sum_{\substack{i=0 \\ i \neq j}}^n m_i \frac{a_j - a_i}{\|a_j - a_i\|^3}.$$

REMARK (3.3). Actually, identifying the plane and the complex plane, solutions of (3.2) may give also homographic solutions of the form $q_j = qa_j$, where the function $q(t) \in \mathbb{C}$ satisfy the Kepler equation. In these general solutions, the bodies may move in ellipses, parabolas or hyperbolas, instead of circular orbits. One may have also solutions with total collapse or growing like $q(t) = (9\omega/2)^{1/3} t^{2/3}$.

PROPOSITION (3.4). Set the position of the bodies as: $a_0 = 0$ with mass μ , and $a_j = e^{ij\zeta}$ with mass 1 for $j \in \{1, \dots, n\}$, where $\zeta = 2\pi/n$. The a_j 's correspond to a relative equilibrium when $\omega = \mu + s_1$, where

$$s_1 = \sum_{j=1}^{n-1} \frac{1 - e^{ij\zeta}}{\|1 - e^{ij\zeta}\|^3} = \frac{1}{4} \sum_{j=1}^{n-1} \frac{1}{\sin(j\zeta/2)}.$$

Proof. For $j = 0$ the equality is $\omega a_0 - \mu \sum_{j=0}^{n-1} e^{ij\zeta} = 0$. For $j \neq 0$ the equality is

$$\sum_{\substack{i=1 \\ i \neq j}}^n \frac{a_j - a_i}{\|a_j - a_i\|^3} + \mu a_j = (\mu + s_1) a_j = \omega a_j. \quad \square$$

Therefore, the a_j 's form a relative equilibrium for the frequency $\omega = \mu + s_1$. This relative equilibrium was studied by Maxwell as a simplified model of Saturn and its rings.

In the paper [17], Proposition 23, one finds that for each $k \in \{1, \dots, n-1\}$, there is one mass μ_k with one global bifurcation of relative equilibria. Let h be the maximum common divisor of k and n , the bifurcation branch from μ_k has solutions where n bodies are arranged as n/h regular polygons of h sides. See the example for $n = 6$.

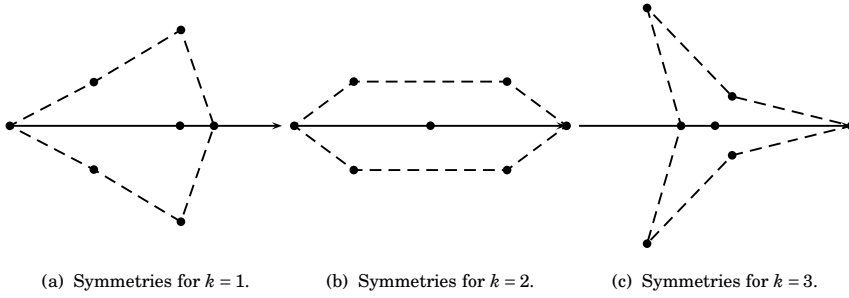


Figure 3. $n = 6$.

REMARK (3.5). *The n -body problem has been the object of many papers, with different techniques and different purposes. For the stability of the polygonal equilibrium, or the bifurcation of relative equilibria from it, one shall mention: [33], [30], [27], [31], among others.*

(3.1) The orthogonal bifurcation map. Changing variables from t to t/v , the $2\pi/v$ -periodic solutions of equation (3.1) are zeros of the bifurcation map f defined in the spaces

$$f : H_{2\pi}^2(\mathbb{R}^{3(n+1)} \setminus \Psi) \times \mathbb{R}^+ \rightarrow L_{2\pi}^2,$$

where $\Psi = \{x \in \mathbb{R}^{3(n+1)} : x_i = x_j\}$ is the collision set, corresponding to two or more of the bodies colliding, and $H_{2\pi}^2(\mathbb{R}^{3(n+1)} \setminus \Psi)$ is the open subset, consisting of the collision-free periodic (and continuous) functions of the Sobolev space $H_{2\pi}^2(\mathbb{R}^{3(n+1)})$.

Define the action of $(\kappa, \theta) \in \mathbb{Z}_2 \times SO(2)$ in $\mathbb{R}^{3(n+1)}$ as

$$\begin{aligned} \rho(\kappa)(u_j, z_j) &= (u_j, -z_j), \\ \rho(\theta)(u_j, z_j) &= (e^{-J\theta}u_j, z_j), \end{aligned}$$

where the group \mathbb{Z}_2 reflects the z -axis, and where $SO(2)$ rotates the (x, y) -plane. Since Newton's equations are invariant by isometries, the group $\mathbb{Z}_2 \times SO(2)$ represents the inherited isometries in rotating coordinates and the map f is $\mathbb{Z}_2 \times SO(2)$ -equivariant.

Let S_n be the group of permutations of the numbers $\{1, \dots, n\}$. Define the action of an element $\gamma \in S_n$ in $x \in \mathbb{R}^{3(n+1)}$ as $\rho(\gamma)x_0 = x_0$ for $j = 0$, and for $j \in \{1, \dots, n\}$ as

$$\rho(\gamma)x_j = x_{\gamma(j)}.$$

Since the action of S_n permutes the n bodies with equal mass, then the map f is S_n -equivariant.

The map f is S^1 -equivariant with the action $\rho(\varphi)x(t) = x(t + \varphi)$, because the equations are autonomous. As the orthogonal degree is defined only for abelian groups, the map f will be considered only as $\Gamma \times S^1$ -equivariant, where Γ is the abelian group

$$\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_n \times SO(2),$$

and \mathbb{Z}_n is the subgroup of S_n generated by $\zeta(j) = j + 1$.

The element $\kappa \in \mathbb{Z}_2$ always leaves an equilibrium fixed because all equilibria are planar, see [15] for a proof. Let $\tilde{\mathbb{Z}}_n$ be the subgroup of Γ generated by

$$(\zeta, \zeta) \in \mathbb{Z}_n \times SO(2),$$

where $\zeta = 2\pi/n \in SO(2)$. The actions of (ζ, ζ) send the point x_0 to $e^{-J\zeta}x_0$, and it sends x_j to $e^{-J\zeta}x_{j+1}$ for the other j 's. One may easily verify that the a_j 's are fixed by the action of (ζ, ζ) , thus the isotropy group of a is the group $\Gamma_a \times S^1$ with

$$\Gamma_a = \mathbb{Z}_2 \times \tilde{\mathbb{Z}}_n.$$

In each component, the infinitesimal generator of the action of S^1 is given by $A_0x_j = \dot{x}_j$, and the infinitesimal generators of $SO(2)$ is given by

$$A_1x_j = \frac{\partial}{\partial \theta} |_{\theta=0} (e^{-J\theta} u_j, z_j) = \text{diag}(-J, 0)x_j.$$

Thus, the equalities $\langle f(x), \dot{x} \rangle_{L^2_{2\pi}} = 0$ and $\langle f(x), A_1x \rangle_{L^2_{2\pi}} = 0$ follow as the proof of conservation of energy and angular momentum for Newton's equations, see [19] for a proof. Thus the map f is a $\Gamma \times S^1$ -orthogonal map.

REMARK (3.6). *The orbit of the polygonal equilibrium a consists of all the rotations in the (x, y) -plane. Since $f = 0$ on the orbit Γa , deriving the map f along a parametrization of this orbit one gets that the generator A_1a is tangent to the orbit, and must be in the kernel of $f'(a)$. This is a well known fact where symmetries imply degeneracies.*

(3.2) Symmetries.

3.2.1. Planar symmetries. In the paper [17], it is proven that there are n subspaces W_k for the similar irreducible representations of $\tilde{\mathbb{Z}}_n$, where the action of $\kappa \in \mathbb{Z}_2$ is $\rho(\kappa) = I$, and the action of $(\zeta, \zeta) \in \tilde{\mathbb{Z}}_n$ is given by

$$\rho(\zeta, \zeta, \varphi) = e^{ik\zeta}.$$

Moreover, since the action of S^1 on the fundamental Fourier mode is given by

$$\rho(\varphi) = e^{i\varphi},$$

the isotropy subgroup of $\Gamma_{\bar{a}} \times S^1$ in the space W_k is generated by $\kappa \in \mathbb{Z}_2$ and $(\zeta, \zeta, -k\zeta) \in \tilde{\mathbb{Z}}_n \times S^1$. This is, the points of W_k are fixed by the group

$$\tilde{\mathbb{Z}}_n(k) \times \mathbb{Z}_2 = \langle (\zeta, \zeta, -k\zeta) \rangle \times \langle \kappa \rangle.$$

As for the satellite, solutions with isotropy group \mathbb{Z}_2 must satisfy $z_j(t) = 0$, and solutions with isotropy group $\tilde{\mathbb{Z}}_n(k)$ satisfy the symmetries

$$u_j(t) = \rho(\zeta, \zeta, -k\zeta)u_j(t) = e^{-ik\zeta}u_{\zeta(j)}(t - k\zeta).$$

In this case, for the central body one has the symmetry

$$u_0(t) = e^{ij\zeta}u_0(t + jk\zeta).$$

Using the notation $u_j = u_{j+kn}$ for $j \in \{1, \dots, n\}$, one has that $\zeta(j) = j + 1$, then the n bodies with equal mass satisfy

$$u_{j+1}(t) = e^{ij\zeta}u_1(t + jk\zeta).$$

Thus, each one of the n bodies with equal mass follows the same planar curve, but with different phase and with some rotation in the (x, y) -plane.

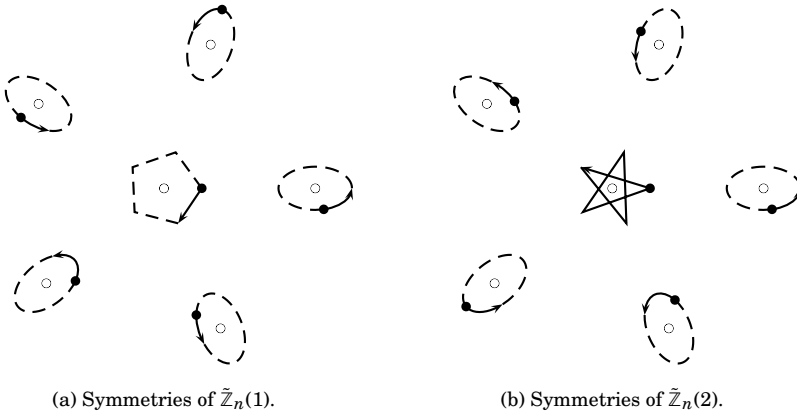


Figure 4. For $n = 5$.

REMARK (3.7). In fixed coordinates, the solutions are $q_j(t) = e^{i\sqrt{\omega}t} u_j(\omega t)$. Thus in fixed coordinates the solutions are in general quasiperiodic solutions. In particular, when the central body has mass zero, we are considering the n -body problem with equal masses. In this case, one has for $j \in \{1, \dots, n\}$ that

$$q_{j+1}(t) = e^{ij\zeta\Omega} q_1(t + jk\zeta)$$

with $\Omega = 1 - k\sqrt{\omega}/v$. If $\Omega \in n\mathbb{Z}$, then solutions with isotropy group $\tilde{Z}_n(k)$ satisfy

$$q_{j+1}(t) = q_1(t + jk\zeta).$$

These solutions where all the bodies follow the same path are known as choreographies, see [8].

3.2.2. Spatial symmetries. In the paper [19] it is proven that there are n subspaces W_k for the similar irreducible representations of \tilde{Z}_n , where the action of $\kappa \in \mathbb{Z}_2$ is given by $\rho(\kappa) = -I$, and the action of the element $(\zeta, \zeta) \in \tilde{Z}_n$ is

$$\rho(\zeta, \zeta, \varphi) = e^{ik\zeta}.$$

Since the action of S^1 on the fundamental mode is $\rho(\varphi) = e^{i\varphi}$, then the elements $(\zeta, \zeta, -k\zeta) \in \tilde{Z}_n \times S^1$ and $(\kappa, \pi) \in \mathbb{Z}_2 \times S^1$ act trivially on W_k . Thus, the isotropy group of W_k is generated by $(\zeta, \zeta, -k\zeta)$ and (κ, π) ,

$$\tilde{Z}_n(k) \times \tilde{Z}_2 = \langle (\zeta, \zeta, -k\zeta) \rangle \times \langle (\kappa, \pi) \rangle.$$

As we saw for the satellite, solutions with isotropy group \tilde{Z}_2 satisfy

$$u_j(t) = u_j(t + \pi) \text{ and } z_j(t) = -z_j(t + \pi),$$

thus the projection of this solution on the (x, y) -plane follows twice the π -periodic curve $u(t)$, one time with the spatial coordinate $z(t)$ and a second time with $-z(t)$. Thus solution looks like a spatial eight near the equilibrium.

Since the group $\tilde{Z}_n(k)$ is generated by $(\zeta, \zeta, -k\zeta)$, the solutions satisfy also the symmetries

$$\begin{aligned} u_j(t) &= e^{-i\zeta} u_{\zeta(j)}(t - k\zeta), \\ z_j(t) &= z_{\zeta(j)}(t - k\zeta). \end{aligned}$$

REMARK (3.8). *To see one example, suppose that $n = 2m$ and choose $k = m$. In this case the central body remains at the center. Moreover, the n bodies with equal masses satisfy*

$$u_{j+1}(t) = e^{ij\kappa} u_1(t + j\pi) = e^{ij\kappa} u_1(t)$$

and

$$z_{j+1}(t) = z_1(t + j\pi) = (-1)^j z_1(t).$$

Thus, there are two m -polygons which oscillate vertically, one with $z_1(t)$ and the other with $-z_1(t)$. Furthermore, the projection of the two m -polygons in the plane is always a $2m$ -polygon. These solutions are known as Hip-Hop orbits.

See [19] for a general description of the symmetries.

(3.3) Bifurcation theorem. The linearization of the system at the polygonal equilibrium is a $3(n + 1) \times 3(n + 1)$ matrix, which is non invertible due to the rotational symmetry. In [19] one finds a change of variables that organize the spaces W_k 's of similar irreducible representation of $\Gamma_a \times S^1$, and also simplify the analysis of the spectrum. This is, the arrange of the subspaces of similar irreducible representations gives a decomposition of the linearization in $2n$ blocks, n of them for the spatial coordinates, given in [19], and n of them for the planar coordinates, given in [17].

Applying orthogonal degree to the reduced bifurcation map, one finds that the degree has one component for each one of these $2n$ blocks, when there are no resonances. In the case of the satellite there were only two components. Each component has one number $\eta(v)$ which is the change of Morse index of the corresponding block. By the existence property of the degree, there is one bifurcation branch starting from (a, v_0) each time $\eta(v_0) \neq 0$, with the symmetries of the corresponding block.

In this way one get the following theorems, see [19] for details.

THEOREM (3.9). *For $n \geq 3$ and each $k \in \{2, \dots, n - 2\}$, the polygonal equilibrium has a global bifurcation of planar periodic solutions with symmetries $\tilde{Z}_n(k)$, if $\mu \in (-s_1, \mu_k)$, and two global bifurcations if $\mu \in (m_+, \infty)$.*

For $n \geq 7$ and each $k \in \{1, n - 1\}$, the polygonal equilibrium has two global bifurcation branches of planar periodic solutions with symmetries $\tilde{Z}_n(k)$ when $\mu > m_+$.

By global branch, one means that there is a continuum of solutions starting at the ring configuration, and the continuum goes to infinity in the norm of the solution or in the period, or goes to collision, or otherwise goes to other relative equilibria in such a way that the sum of the jumps in the orthogonal degrees is zero.

THEOREM (3.10). *The polygonal equilibrium has a global bifurcation of periodic solutions with symmetries $\tilde{Z}_n(k) \times \tilde{Z}_2$ for each $k \in \{1, \dots, n\}$. Except for a possible finite number of μ 's and frequencies, for μ positive, bounded and different from μ_k , due to resonances, these solutions are truly spatial, this is $z_j(t) \neq 0$ for some j -th body.*

REMARK (3.11). *Here, only the generic cases were exposed for simplicity. In [19] all cases of bifurcation from the polygonal equilibrium were studied for $n \geq 2$. In the paper [19], there is also a theorem for the general n -body problem, where it*

is proven that any relative equilibrium has one bifurcation of spatial like eight solutions, and that generically there are $n - 1$ of these bifurcations.

The planar bifurcation for $k = n$ consists of solutions with $u_0(t) = 0$ and $u_j(t) = e^{ij\zeta} u_n(t)$. This branch was constructed in an explicit way in [28], by reducing the problem to a 6-dimensional dynamical system and a normal form argument.

The spatial bifurcation for $k = n$ is made of solutions where the ring moves as a whole and the central body makes the contrary movement in order to stabilize the forces, that is $z_j(t) = z_1(t)$ for $j \in \{1, \dots, n\}$ and $z_0 = -nz_j$. This solution was called an oscillating ring in [28].

The spatial bifurcation for $k = n/2$ has the symmetries of the well known Hip-Hop orbits. This kind of solutions appears first in the paper [11] without the central body. Later on, in [28] for a big central body in order to explain the pulsation of the Saturn ring, where they are called kink solutions. Finally, there is a proof in [4] when there is no central body.

REMARK (3.12). *The same group of symmetries of the polygonal equilibrium for the n -body problem is present also in the papers: In [19] for charges instead of bodies. In [18] for vortices and traveling waves in almost parallel filaments, and in [20] for a periodic lattice of coupled nonlinear Schrödinger oscillators. Although there are many similarities with the n body problem, in particular in the change of variables, these results are of a quite different nature.*

As we saw before, due to the rotational symmetry, the linearization at any equilibrium has at least one dimensional kernel. In order to find bifurcation of relative equilibria, in the paper [17], one get rid of this degeneracy looking for solutions in fixed-point subspaces of some reflection, where one is able to use ordinary degree or another method.

For bifurcation of periodic solutions, the polygonal equilibrium is fixed only by the action of

$$\tilde{\kappa}x_j(t) = \text{diag}(1, -1, 1)x_{n-j}(-t),$$

which is a coupling between the reflection on the plane, a reversal of time, and a permutation of bodies. This is the only reflection able to get ride of the degeneracy.

However, when one restricts the problem to the fixed-point subspace of $\tilde{\kappa}$, one may prove bifurcation of periodic solutions only for the symmetries $k = n$ and $k = n/2$. For the remaining k 's, the linearization on the fixed-point subspace of $\tilde{\kappa}$ is a complex matrix with non-negative determinant as a real matrix. One could also use the gradient structure and apply the results for bifurcation based on Conley index. Actually, analytical studies with normal forms of high order and additional hypotheses of non-resonance are proposed in [7] for these cases. However, this approach do not provide the proof of the existence of a global continuum, something which follows from the application of the orthogonal degree. This fact implies that one may not use a classical degree argument or other simple analytical proofs to find the solutions presented here.

REMARK (3.13). *Variational techniques have been quite successful in treating the existence of closed solutions. In particular, [14], [12] and [13], classify all the possible groups which give periodic solutions which are minimizers of the action without collisions. Thus, the issue is different from ours, since one has the proof*

of the existence of a solution in the large, with a specific symmetry. For choreographies, following the seminal paper [8], with no central mass, there are studies with more than 3 bodies in [6] and [5], for instance. In the case of hip-hop solutions, these methods were successful in [9] and [32]. One of the advantages of the orthogonal degree is that it applies to problems which are not necessarily variational, but present conserved quantities.

Received September 18, 2012

Final version received October 23, 2012

C. GARCÍA-AZPEITIA
DEPARTMENT OF MATHEMATICS & STATISTICS
MCMASTER UNIVERSITY
1280 MAIN STREET WEST
HAMILTON, ONTARIO L8S 4K1
CANADA
cgazpe@hotmail.com

REFERENCES

- [1] M. ARRIBAS AND A. ELIPE, *Bifurcations and equilibria in the extended n -body ring problem*, Mech. Res. Comm., **31** (1), (2004), 1–8.
- [2] D. BANG AND B. ELMABSOUT, *Restricted $n + 1$ -body problem: existence and stability of relative equilibria*, Celestial Mech. Dynam. Astronom., **89** (4), (2004), 305–318.
- [3] Z. BALANOV, W. KRAWCEWICZ, AND H. STEINLEIN, *Applied equivariant degree*, AIMS Series on Differential Equations & Dynamical Systems, 1. American Institute of Mathematical Sciences (AIMS), 2006.
- [4] E. BARRABAS, J. M. CORS, C. PINYOL, AND J. SOLER, *Hip-hop solutions of the $2n$ -body problem*, Celestial Mech. Dynam. Astronom., **95** (1-4), (2006), 55–66.
- [5] V. BARUTELLO AND S. TERRACINI, *Action minimizing orbits in the n -body problem with simple choreography constraint*, Nonlinearity, **17** (6), (2004), 2015–2039.
- [6] K. C. CHEN, *Action-minimizing orbits in the parallelogram four-body problem with equal masses*, Arch. Ration. Mech. Anal., **158**, (2001), 293–318.
- [7] A. CHENCINER AND J. FEJOZ, *Unchained polygons and the n -body problem*, Regular and chaotic dynamics, **14** (1), (2009), 64–115.
- [8] A. CHENCINER AND R. MONTGOMERY, *A remarkable periodic solution of the three-body problem in the case of equal masses*, Ann. of Math., **152** (2), (2000), 881–901.
- [9] A. CHENCINER AND A. VENTURELLI, *Minima de l'integrale d'action du problème newtonien de 4 corps de masses égales dans \mathbb{R}^3 : orbites "hip-hop"*, Celest. Mech. Dyn. Astron., **77**, (2000), 139–152.
- [10] N. DANCER, *A new degree for S^1 -invariant gradient mappings and applications*, Ann. Inst. H. Poincaré, Anal. Non Linéaire, **2**, (1985), 329–370.
- [11] I. DAVIES, A. TRUMAN, AND D. WILLIAMS, *Classical periodic solutions of the equal-mass $2n$ -body problem, $2n$ -ion problem and the n -electron atom problem*, Physics Letters A., **99** (1), (1983), 15–18.
- [12] D. FERRARIO, *Symmetry groups and non-planar collisionless action-minimizing solutions of the three-body problem in three-dimensional space*, Arch. Ration. Mech. Anal., **179** (3), (2006), 389–412.
- [13] D. FERRARIO, *Transitive decomposition of symmetry groups for the n -body problem*, Adv. Math., **213** (2), (2007), 763–784.
- [14] D. FERRARIO AND S. TERRACINI, *On the existence of collisionless equivariant minimizers for the classical n -body problem*, Invent. Math., **155** (2), (2004), 305–362.
- [15] C. GARCÍA-AZPEITIA, *Aplicación del grado ortogonal a la bifurcación en sistemas hamiltonianos*, UNAM, Mexico, PhD thesis, 2010.

- [16] C. GARCÍA-AZPEITIA AND J. IZE, *Global bifurcation of planar and spatial periodic solutions in the restricted n -body problem*, *Celestial Mech. Dyn. Astr.*, **110**, (2011), 217-227.
- [17] C. GARCÍA-AZPEITIA AND J. IZE, *Global bifurcation of polygonal relative equilibria for masses, vortices and dNLS oscillators*, *J. Dif. Eq.*, **251**, (2011), 3202–3227.
- [18] C. GARCÍA-AZPEITIA AND J. IZE, *Bifurcation of periodic solutions from a ring configuration in the vortex and filament problems*, *J. Dif. Eq.*, **252**, (2012), 5662–5678.
- [19] C. GARCÍA-AZPEITIA AND J. IZE, *Global bifurcation of planar and spatial periodic solutions from the polygonal relative equilibria for the n -body problem*, To appear in *J. Dif. Eq.*
- [20] C. GARCÍA-AZPEITIA AND J. IZE, *Bifurcation of periodic solutions from a ring configuration of discrete nonlinear oscillators*, To appear in *Discrete and Continuous Dynamical Systems Series S*.
- [21] J. IZE, *Topological bifurcation* In *Topological nonlinear analysis*, *Progr. Nonlinear Differential Equations Appl.*, **15**, (1995), 341–463. Birkhäuser Boston.
- [22] J. IZE, I. MASSABO, AND A. VIGNOLI, *Degree theory for equivariant maps, I*, *Trans. AMS*, **315**, (1989), 433–510.
- [23] J. IZE AND A. VIGNOLI, *Equivariant degree for abelian actions. Part III: orthogonal maps*, *Top. Methods Nonlinear Anal.*, **13**, (1999), 105–146.
- [24] J. IZE AND A. VIGNOLI, *Equivariant degree theory*, De Gruyter Series in Nonlinear Analysis and Applications 8. Walter de Gruyter, Berlin, 2003.
- [25] A. MACIEJEWSKI AND S. RYBICKI, *Global bifurcation of periodic solutions of the restricted three-body problem*, *Celestial Mechanics and Dynamical Astronomy*, **88**, (2004), 293–324.
- [26] K.R MEYER AND G. R. HALL, *An Introduction to Hamiltonian Dynamical Systems*, Springer-Verlag, New York, (1991).
- [27] K. R. MEYER AND D. S. SCHMIDT, *Bifurcations of relative equilibria in the n -body and Kirchhoff problems*, *SIAM J. Math. Anal.*, **19** (6), (1988), 1295–1313.
- [28] K. MEYER AND D. S. SCHMIDT, *Librations of central configurations and braided saturn rings*, *Celestial Mech. Dynam. Astronom.*, **55** (3), (1993), 289–303.
- [29] S. RYBICKI, *S^1 -degree for orthogonal maps and its applications to bifurcation theory*, *Nonl. Anal. TMA.*, **23**, (1994), 83–102.
- [30] G. E. ROBERTS, *Linear stability in the $1+n$ -gon relative equilibrium* In *Hamiltonian systems and celestial mechanics. HAMSYS-98. Proceedings of the 3rd international symposium*, J. Delgado editor, World Sci. Monogr. Ser. Math. 6, (2000), 303–330.
- [31] D. S. SCHMIDT, *Central configurations and relative equilibria for the n -body problem* In *Classical and celestial mechanics*, (2003), 1–33. Princeton Univ. Press.
- [32] S. TERRACINI AND A. VENTURELLI, *Symmetric trajectories for the $2n$ -body problem with equal masses*, *Arch. Rat. Mech. Anal.*, **184** (3), (2007), 465–493.
- [33] R. J. VANDERBEI AND E. KOLEMEN, *Linear stability of ring systems*, *The astronomical journal*, **133**, (2007), 656–664.

THE ACADEMIC LEGACY OF ERNESTO A. LACOMBA

ERNESTO PÉREZ CHAVELA

Semblance

On 26th June 2012, México lost one of its most outstanding mathematicians, when Ernesto Alejandro Lacomba Zamora passed away. These lines are by way of a tribute to him.

Ernesto Lacomba was born on December 2, 1945 in Mexico City; his father, Antonio Lacomba, had a bookstore in downtown Mexico City, where Ernesto Lacomba passed many afternoons. He became a great reader and fan of mathematics, a discipline at which he shone from a very young age. At the Instituto Politécnico Nacional (IPN), he studied simultaneously for a bachelor's degree in mathematical physics and electronic engineering. At this time, just prior to 1968, the city was convulsed by student unrest and disquiet.

When he finished his bachelor degrees, he travelled to the United States to continue his studies in mathematics. He obtained his Ph. D. under the supervision of Stephen Smale (1966 Fields Medal winner); the results of his thesis appeared in a paper published in the Transactions of the American Mathematical Society [1]. His thesis won a prize which consisted of travel to Brazil (joint with Ruth, with whom he had just married in Berkeley) with a year's visiting position at the University of Brasilia.

He returned to Mexico City in 1973, joining a research group in the Applied Mathematics and Systems Institute (IIMAS), at the National Autonomous University of Mexico (UNAM). In 1974, a new University in Mexico City, the Universidad Autónoma Metropolitana (UAM), was founded. This new project had the intention of creating new and modern degree programmes, with a new academic calendar and a new administration and structure. The challenge of forming a new university excited and stimulated Ernesto. He became a founder member of this new university, and it became the place where he spent the rest of his academic life as a teacher and researcher. In the mathematics department of the UAM, campus Iztapalapa, Ernesto created an active research group in celestial mechanics which today, in this subject, is of world renown.

As can be ascertained from his Curriculum Vitae, throughout his life Ernesto wrote sixty three research papers published in top international journals with high impact factor, twenty three papers in proceedings and eight chapters in books. He also wrote ten expository papers and some pedagogical material; he was also co-editor of five books (corresponding to the Proceedings of the international symposia *HAMSYS*).

Ernesto was deeply involved in the design and later in the modification of many courses in the undergraduate and graduate programmes of mathematics at the UAM where he was a dedicated teacher. Many students, finishing one of his courses would ask him to continue with the following course. He was a speaker

in many conferences concerning Hamiltonian systems and celestial mechanics. In 1990, he and Jaume Llibre from the Universidad Autónoma de Barcelona decided to organize periodically the now famous series of *International Symposia on Hamiltonian Systems and Celestial Mechanics (HAMSYS)*. To date, six have taken place: Hamsys-91, at CIMAT, Guanajuato; Hamsys-94, at Cocoyoc, Morelos; Hamsys-98, at Pátzcuaro, Michoacan; Hamsys-2001, at CIMAT, Guanajuato; Hamsys-2006, at CIMAT, Guanajuato; and Hamsys-2010 in Mexico City; at this last conference, Ernesto's 65th birthday was celebrated. The Proceedings of all these symposia were published by prestigious editorial houses and all articles passed a strict review process.

A synthesis of part of his academic work

As I mentioned before, the results of Ernesto's Ph.D. thesis were published in the Transactions of the American Mathematical Society in 1973 (see [1]); in this article he generalized the classical results of S. Smale [17], [18]. In order to give the main results of his paper, I need to give some definitions:

We say that a *mechanical system with symmetry* is a quadruple (M, K, V, G) , where M is a manifold (M is the configuration space, the phase space is the tangent bundle TM) endowed with a Riemannian metric, K is the kinetic energy (the square of the norm of the Riemannian metric), V is the potential energy defined on M , and G is a Lie group acting on M and leaving K and V invariant. Let \mathcal{G} be the Lie algebra of G . The momentum of the system J , is a first integral which can be studied as a map $J : TM \rightarrow \mathcal{G}^*$, where \mathcal{G}^* is the dual of the Lie algebra \mathcal{G} associated to the Lie group G . Another first integral is the total energy defined by $E = K + V$. Then we can study the energy-momentum map defined by

$$I = (E, J) : TM \rightarrow \mathbf{R} \times \mathcal{G}^*.$$

The function I can also be seen as a first integral of the system, then given any $(c, p) \in \mathbf{R} \times \mathcal{G}^*$, the set $I_{c,p} = I^{-1}(c, p)$ is invariant under the flow of the respective vector field. In this way, the topological structure of the sets $I_{c,p}$ allows us to describe the global topology of the phase space of a mechanical system with symmetry. The relative equilibria are the orbits which are invariant by the action of one-parameter subgroup of G . The bifurcations of the system occur at the relative equilibria. Let H be a closed subgroup of the Lie group G ; G/H is called a homogeneous space. A transitive mechanical system with symmetry is a mechanical system with symmetry where $M = G/H$ is a homogeneous space where the potential $V = 0$, so in this case the total energy is $E = K$; then, the vector field describes the geodesic flow in the phase space. These transitive mechanical systems were tackled by Ernesto in his thesis, where in particular he described the bifurcation set in the phase space.

Another important paper, published during his first years of research, corresponds to the regularization by surgery in the restricted three body problem [2], where using the techniques introduced by R. Easton [15], he obtained a regularization of the singularities. Here the idea is to use isolating blocks (neighborhoods) around the singularity, then applying changes of coordinates, newly rescaled time and some topological methods it is possible to glue a solution entering in the block with one which is leaving the block, the solution so obtained is unique and analytic.

In a series of articles with the French mathematician Lucette Losco [3], [4], [5], Ernesto gave a topological analysis of the variational characterization of contact vector fields in the group of diffeomorphisms. With her, using McGehee coordinates (see [16] for more details) they also studied the triple collision in the isosceles three body problem.

In a nice paper with Carles Simó [6], Ernesto studied the escape of particles in celestial mechanics; for this, they used the blow up technique introduced some years before by R. McGehee [16] in order to prove that the escape of particles is qualitatively different depending on the sign of the total energy of the system. The change of coordinates in order to obtain the blow up of infinity is different if the total energy h is $h < 0$, $h = 0$ or $h > 0$. In this paper they applied these new techniques to a number of interesting problems.

In [7], also with Carles Simó, a complete analysis was done of some degenerate quadruple collisions, this being one of the first papers dealing with global dynamics in the four body problem. More specifically, they studied the trapezoidal four body problem, where two different masses are symmetrically located at the vertices of a trapezium, taking symmetric initial conditions with regard to position and velocity, and assuming that the masses are moving according to Newton's gravitational law, they gave the equations of motions and studied two degenerate cases. The first corresponded to the case where the trapezium is a rectangle, while the second concerned the case where all particles lie on a straight line, both cases having two degrees of freedom. Using the blow-up technique (see [16]), they glued an invariant manifold (of codimension 1 in the phase space) which represented the quadruple collision, then they studied the flow on it and using the continuity of the solutions with respect to the initial data they obtained properties of the global flow for the orbits which end in collision and for those which pass close to total collision.

One of Ernesto's most important results (from my viewpoint), appears in [7], where again with C. Simó, they obtained a nice regularization of the singularities due to simultaneous binary collisions in the n -body problem, a type of singularities which are not easy to understand. The main results stated that a solution suffering a simultaneous binary collision can be continued analytically in terms of $t^{1/3}$.

Another important result appears in [9], where using new changes introduced by the authors, they obtained a compact model for the rhomboidal 4-body problem. First they studied the total collision manifold using the classical McGehee coordinates [16], then they regularized the singularities due to double collision in an innovative way, which with a re-parametrization of time gave a compact model. Some years later, these ideas were generalized by Q. Wang to obtain a global solution of the n -body problem: I repeat, *a global solution of the n -body problem*. In [19], Q. Wang obtained a convergent power series solution for the n -body problem, where he omitted only the solutions leading to collision singularities. A paradox emerges from this result: With this result it would appear that we may describe the motion of all the celestial bodies in our Solar system. However, this is not really the case. Wang's work is correct, but there is still a big problem, the series solution obtained are convergent in the whole real axis, but they converge very slowly, we have to sum many millions of terms in the series to obtain information regarding the motion of the particles in a short time interval. Nevertheless the

result, although theoretical is curious and interesting. In [10], also concerning the rhomboidal problem, the authors showed the existence of chaos. The idea is to use the blow-up technique in two different ways, first to study the total collision, and second to study the escapes to infinity. In both case we glue invariant submanifolds on the borders of the phase space. The global flow is extended to these submanifolds. Taking the mass ratio as a parameter, the authors find a bifurcation value say α_0 . The main result states that if the mass ratio is lesser than α_0 , then the stable submanifold associated to the escapes intersects transversality to the unstable manifold associated to the total collision. From here they introduce symbolic dynamics in a small neighborhood of the transversal intersection, showing the existence of chaos.

The above results formed part of my Ph.D. thesis - I was Ernesto's second Ph.D. student.

Ernesto maintained an important collaboration with Jaume Llibre of the Universidad Aut3noma de Barcelona; with him, Ernesto wrote several articles. In [11], they studied a particular restricted collinear three body problem where the masses at the ends are positive and between them there is a massless particle, the question is simple, is it possible to show the existence of chaos and is it possible to introduce symbolic dynamics on it. Using several clever changes of coordinates, they gave an affirmative answer to the above questions. In two papers with J. Delgado, J. Llibre and E. P3rez-Chavela [12] and [13], the authors applied the Poincar3 compactification to study first the Kepler and the collinear three body problem. They regularized the singularities in such a way to obtain a polynomial vector field, then they studied the global flow of the above problems showing that all singularities are on the equator of the respective sphere (also a sphere of codimension 1 with respect to the previous one). Then they extended the above results to a general Hamiltonian polynomial vector fields.

In vortex dynamics, joint with M. Celli and E. P3rez-Chavela [14], they studied polygonal relative equilibria. In a planar incompressible fluid with zero viscosity, a relative equilibrium is a particular solution where the mutual distances among the vortices remain constant during all motions. The main results stated that in order to have a polygonal relative equilibrium, all vorticities at the edges of the polygon must be equal. These authors also proved that in the case of two nested concentric squares, where the vorticities on each square are equal, it is possible to have a relative equilibrium where the corresponding diagonals of the different squares form an angle of forty five degrees for any value of the vorticities; this was one the last articles written by Dr. Lacomba.

The results mentioned above show only a small part of Ernesto's work, slanted by the academic background and interests of the author of this note, but I must mention that Ernesto had many collaborators and with them he wrote many papers on applications to symplectic geometry, thermodynamics, celestial mechanics, electrical circuits, geometric mechanics and vortex theory. In chronological order his collaborators are: L. Losco, C. Sim3, J. Bryant, L. Ibort, J. Cariñena, J. Llibre, M. de Le3n, H. Cendra, W. M. Tulczyjew, F. Cantrijn, E. P3rez-Chavela, A. Verdiell, J. Delgado, F. Diacu, C. Stoica, V. Mioc, A. Mingarelli, H.K. Bhaskara, K.K. Rama, D. Martin, P. Pitanga, G. Hern3ndez, J.C. Marrero, J.G. Reyes, S. Craig, M. Falconi, C. Vidal, M. Medina, A. Hern3ndez, S. Kaplan, H. Jim3nez, M. Celli, A. Castro.

His students

Ernesto was an excellent teacher, a good speaker and a great motivator of young students. He was an invited speaker (frequently a plenary speaker) at many conferences in the area. He was thesis supervisor of 8 Ph. D. students:

- (1984) Felipe Peredo, Cinvestav, IPN.
- (1991) Ernesto Pérez-Chavela, UAM-Iztapalapa.
- (1991) Joaquín Delgado, UAM-Iztapalapa.
- (1993) José G. Reyes, UAM-Iztapalapa.
- (1996) Manuel Falconi, Facultad de Ciencias, UNAM.
- (2006) Mario Medina, UAM-Iztapalapa.
- (2010) Hugo Jiménez, UAM-Iztapalapa.
- (2012) Alberto Castro, UAM-Iztapalapa.

Awards

Ernesto received awards on many occasions: Since its foundation in 1984 he was a member of the Sistema Nacional de Investigadores (SNI), first at Level II, and from 1990 until his death at level III, the highest of the system. In 2011, he received the distinction of emeritus researcher of the SNI. He also formed part of the judging committee of SNI, from 1993 to 1995.

In 1993 he won the prize “Lázaro Cárdenas”, given by the Instituto Politécnico Nacional to the most distinguished graduates of this institution. In 1985 he received a Honoric Mention within the Research Prize in Exact Sciences “Noriega Morales”, granted by the Organization of American States and in 2007, he received the prize “Silvia Torres Castilleja” in basic sciences, being part of the “Ciudad de México, Heberto Castillo” prize.

He was named Distinguished Professor by the Universidad Autónoma Metropolitana (UAM) in 1991. In early 2012, for his excellence as a researcher and teacher, the Faculty of Basic Sciences and Engineering of the UAM, recommended to the University Council that he be named Professor Emeritus of the UAM (this being the highest honour that the UAM can confer on its faculty members in recognition of outstanding teaching, research and scholarship). At that time (April, 2012) Ernesto’s health was deteriorating and unfortunately this distinction was finally confirmed by the University Council the day after his death.

In the process of his nomination for this distinction he received the support of many professors from different academic departments of the UAM, and from members of the mathematical community, both in Mexico and elsewhere. The confirmation of the distinction was finally awarded to his widow Ruth Lacomba, daughter Roxana and grandson Daniel.

Acknowledgments

I would like to thank Richard G. Wilson for his help in the final redaction of this manuscript.

ERNESTO PÉREZ CHAVELA
 DEPARTAMENTO DE MATEMÁTICAS, UAM IZTAPALAPA
 AV. SAN RAFAEL ATLIXCO 186, COL. VICENTINA, C. P. 09340
 MEXICO, D. F., MEXICO
 epc@xanum.uam.mx

REFERENCES

- [1] E. A. LACOMBA, *Mechanical systems with symmetry on homogeneous spaces*, Transactions of the American Mathematical Society **185** (1973), 477–491.
- [2] E. A. LACOMBA, *Regularization by surgery in the restricted 3-body problem*, Journal of Differential Equations **24** (1977), 240–272.
- [3] E. A. LACOMBA AND L. LOSCO, *Triple collision in the isosceles 3-body problem*, Bulletin Amer. Math. Society (New Series) **3** (1980), 710–714.
- [4] E. A. LACOMBA AND L. LOSCO, *Variational characterization of contact vector fields in the group of contact diffeomorphisms*, Physica, **114 A** (1982), 124–128.
- [5] E. A. LACOMBA AND L. LOSCO, *Characterization variationnelle globale des flots canoniques et de contact dans leurs groupes de diffeomorphisms*, Annales de l'Institut Poincaré **45** (1986), 99–116.
- [6] E. A. LACOMBA AND C. SIMÓ, *Boundary manifolds for energy surfaces in celestial mechanics*, Celestial Mechanics and Dynamical Astronomy **28** (1982), 37–48.
- [7] E. A. LACOMBA AND C. SIMÓ, *Analysis of some degenerate quadruple collisions*, Celestial Mechanics and Dynamical Astronomy **28** (1982), 49–62.
- [8] E. A. LACOMBA AND C. SIMÓ, *Regularization of simultaneous binary collisions in the n -body problem*, Journal of Differential Equations **98** (1992), 241–259.
- [9] E. A. LACOMBA AND E. PÉREZ-CHAVELA, *A compact model for the planar rhomboidal 4-body problem*, Celestial Mechanics and Dynamical Astronomy **54** (1992), 343–355.
- [10] E. A. LACOMBA AND E. PÉREZ-CHAVELA, *Motions close to escapes in the rhomboidal 4-body problem*, Celestial Mechanics and Dynamical Astronomy **57** (1993), 411–437.
- [11] E. A. LACOMBA AND J. LLIBRE, *On the dynamics and topology of elliptic rectilinear restricted 3-body problem*, Celestial Mechanics and Dynamical Astronomy **77** (2000), 1–15.
- [12] J. DELGADO, E. A. LACOMBA, J. LLIBRE, AND E. PÉREZ-CHAVELA, *Poincaré compactification of the Kepler and the collinear 3-body problem*, Seminar on Dynamical Systems Birkhauser, Basel-Boston (1993), 117–128.
- [13] J. DELGADO, E. A. LACOMBA, J. LLIBRE, AND E. PÉREZ-CHAVELA, *Poincaré compactification of Hamiltonian polynomial vector fields*, Hamiltonian Dynamical Systems, History, Theory and Applications. The IMA Volumes in Mathematics and its Applications **63**, Springer-Verlag, (1994) 99–115.
- [14] M. CELLI, E. A. LACOMBA, AND E. PÉREZ-CHAVELA, *On polygonal relative equilibria in the N -vortex problem*, Journal of Mathematical Physics **52** (2011), 103101.
- [15] R. EASTON, *Regularization of vector fields by surgery*, Journal of Differential Equations **12** (1971), 92–99.
- [16] R. MCGEHEE, *Singularities and collisions of Newtonian gravitational systems*, Inventiones Math. **27** (1974), 191–227.
- [17] S. SMALE, *Topology and mechanics I*, Inventiones Math. **10** (1970), 305–331.
- [18] S. SMALE, *Topology and mechanics II*, Inventiones Math. **11** (1970), 45–64.
- [19] Q. WANG, *The global solutions of the N -body problem*, Celestial Mechanics and Dynamical Astronomy **50** (1991), 73–88.

ON THE ISOSCELES SOLUTIONS OF THE THREE-BODY PROBLEM

To Ernesto Lacomba Zamora, in memoriam

HILDEBERTO E. CABRAL

ABSTRACT. The goal of this note is to present the classification of the isosceles solutions of the Three-Body Problem. We follow Wintner's presentation and correct a point in his arguments at the end of § 345 of his book "The Analytical Foundations of Celestial Mechanics". Also, we make a comment on the fundamental problem that the masses at the base of an isosceles solution are necessarily equal.

1. Introduction

The particular solutions of the 3-body problem whose configurations form an isosceles triangle for every time without degenerating into an equilateral triangle, a Lagrangian solution, or into a line segment, an Eulerian solution, were discovered around the end of the XIX century by A. E. Fransén, [4]. He arrived at these solutions assuming that the masses at the base of the triangle are equal. In 1913 E. J. Wilczynski published a paper [14] containing a proof, attributed by him to W. D. MacMillan, of the fundamental fact that the masses at the base of the isosceles configuration have necessarily to be equal. Notice that this is not true if we allow the isosceles triangle to be equilateral, as the Lagrange equilateral solutions exist for any three masses. In 1921, J. Chazy [3] gave a new proof of the above fact claiming it to be simpler than that of MacMillan-Wilczynski. Both proofs use analytic function theory. In Chazy's proof he studies the singularities of the equations that he was led to in his analysis and imposing compatibility conditions on a system of five equations on four variables he got the equality of the masses at the base of the triangle. Chazy says that while MacMillan in his proof applies several theorems of analytic function theory he applies just one namely, Fuchs' theorem on the singular points of linear differential equations, see [5], p. 85. His analysis however is by no means trivial. In his book published in 1941, Wintner raises the question of whether there exists a dynamical proof of this fact. Quoting from § 389 of [15]: "It would, of course, be desirable to find a proof based on dynamical, rather than on function-theoretical, principles. But it is quite doubtful that such a proof exists". At the end of this note we come back to this point.

Once established the equality of the masses at the base of the isosceles configuration a complete classification of such motions can be easily done. Following Wintner [15], §§ 344-346, we present such a classification in Section 3, taking this opportunity to correct a mistake in his analysis.

2010 *Mathematics Subject Classification*: 34C14, 70F07.

Keywords and phrases: isosceles configuration, three-body problem, celestial mechanics.

2. The heliocentric equations of motion

Let $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ be the position vectors of masses m_1, m_2, m_3 , in the Newtonian three-body problem, relative to an inertial frame. We consider the *heliocentric vectors* Y_1, Y_2 defined by (mass m_3 as “sun”)

$$(2.1) \quad Y_1 = \mathbf{r}_1 - \mathbf{r}_3 \quad \text{and} \quad Y_2 = \mathbf{r}_2 - \mathbf{r}_3.$$

Considering the center of mass

$$\bar{\mathbf{r}} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3), \quad \text{where} \quad M = m_1 + m_2 + m_3,$$

we have from (2.1)

$$(2.2) \quad \begin{aligned} \mathbf{r}_1 &= \bar{\mathbf{r}} + \frac{m_2 + m_3}{M}Y_1 - \frac{m_2}{M}Y_2, \\ \mathbf{r}_2 &= \bar{\mathbf{r}} - \frac{m_1}{M}Y_1 + \frac{m_1 + m_3}{M}Y_2, \\ \mathbf{r}_3 &= \bar{\mathbf{r}} - \frac{m_1}{M}Y_1 - \frac{m_2}{M}Y_2. \end{aligned}$$

The dynamics of the three-body problem with center of mass fixed at the origin can be completely described by the motion of the heliocentric vectors Y_1, Y_2 .

The Newtonian equations of motion in terms of the inertial vectors

$$m_j \ddot{\mathbf{r}}_j = \sum_{i \neq j} \frac{m_i m_j}{\|\mathbf{r}_i - \mathbf{r}_j\|^3} (\mathbf{r}_i - \mathbf{r}_j),$$

give rise to the heliocentric equations of motion

$$(2.3) \quad \ddot{Y}_1 = q_{11}Y_1 + q_{12}Y_2, \quad \ddot{Y}_2 = q_{21}Y_1 + q_{22}Y_2,$$

where

$$(2.4) \quad q_{ii} = -\frac{m_i + m_3}{\|Y_i\|^3} - \frac{m_j}{\|Y_1 - Y_2\|^3}, \quad q_{ij} = \frac{m_j}{\|Y_1 - Y_2\|^3} - \frac{m_j}{\|Y_j\|^3} \quad (i \neq j = 1, 2).$$

Definition (2.5). An *isosceles solution* of the three-body problem is defined as a solution for which the configuration of the masses form an isosceles triangle for every time without degenerating identically into an equilateral triangle or a line segment.

We notice that for reasons of analyticity the configuration along an isosceles solution can eventually become equilateral or rectilinear at an isolated value t_0 of the time but cannot persist as such in any open interval around t_0 . In fact, this instant of time cannot even be an accumulation point of such degenerate configurations.

3. The classification of the isosceles solutions

The goal of this section is to classify all the isosceles solutions with the masses $m_1 = m_2$ at the base of the isosceles configuration. In our exposition we follow Wintner [15], §§ 343bis-346. See also Chazy [3], pp. 184-188.

So the goal is to classify all the motions for which

$$(3.1) \quad m_1 = m_2 \quad \text{and} \quad \|Y_1\| = \|Y_2\|,$$

under the assumption that

$$(3.2) \quad q_{12} \neq 0 \quad \text{and} \quad Y_1 \times Y_2 \neq 0.$$

In view of (2.4), the first condition in (3.2) excludes the equilateral configurations while the second excludes those which are collinear.

We see from (2.4) that the conditions (3.1) imply the symmetry conditions

$$(3.3) \quad q_{11} = q_{22} \quad \text{and} \quad q_{12} = q_{21}.$$

Conversely, under the fulfilment of the first inequality in (3.2), the equalities (3.3) imply (3.1).

Therefore, our problem is to classify all the motions for which the symmetry conditions (3.3) hold, under the assumptions stated in (3.2). Taking these into consideration, then in terms of the vectors¹

$$(3.4) \quad X_1 = \frac{1}{2}(Y_1 + Y_2), \quad X_2 = \frac{1}{2}(Y_1 - Y_2),$$

the heliocentric equations (2.3) become

$$(3.5) \quad \ddot{X}_1 = (q_{11} + q_{12})X_1, \quad \ddot{X}_2 = (q_{11} - q_{12})X_2.$$

It follows immediately from the equations of motion (3.5) that $X_1 \times \dot{X}_1$ and $X_2 \times \dot{X}_2$ are constants of motion, that is there are two constant vectors A_1 and A_2 satisfying the equations

$$(3.6) \quad X_1 \times \dot{X}_1 = A_1 \quad \text{and} \quad X_2 \times \dot{X}_2 = A_2,$$

where \times denotes the cross product in \mathbb{R}^3 .

PROPOSITION (3.7). *The following equalities hold among the four vectors X_1, X_2, A_1, A_2 ,*

$$(3.8) \quad X_1 \cdot X_2 = 0, \quad A_1 \cdot X_1 = 0, \quad A_2 \cdot X_2 = 0 \quad \text{and} \quad A_1 \cdot A_2 = 0.$$

Proof. The first equality follows immediately from the second equality in (3.1), while the second and third are obvious. It remains to prove the last one. We first notice that

$$(3.9) \quad (X_1 \cdot \dot{X}_2)^2 = A_1 \cdot A_2.$$

Indeed, this follows from (3.6) by using the vector identity

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

together with the orthogonality of X_1 and X_2 , expressed by the first equality in (3.8), which also gives

$$(3.10) \quad X_1 \cdot \dot{X}_2 = -X_2 \cdot \dot{X}_1.$$

Let us now prove that $A_1 \cdot A_2 = 0$. Differentiating (3.9) we get

$$2(X_1 \cdot \dot{X}_2)[\dot{X}_1 \cdot \dot{X}_2 + X_1 \cdot \ddot{X}_2] = 0.$$

If the first factor is zero, we have $A_1 \cdot A_2 = 0$ by (3.9). If the second factor is zero, then

$$\dot{X}_1 \cdot \dot{X}_2 = 0,$$

¹The vector X_1 determines the direction of the symmetry axis and X_2 that of the basis of the triangle

because $X_1 \cdot \ddot{X}_2 = 0$ by the second equation of motion (3.5) together with the orthogonality of X_1 and X_2 . Differentiating the left-hand side of this equation we get

$$\dot{X}_1 \cdot \ddot{X}_2 + \ddot{X}_1 \cdot \dot{X}_2 = 0.$$

Substituting into this the double derivatives from (3.5) and using (3.10) we get

$$q_{12}(X_1 \cdot \dot{X}_2) = 0.$$

Since $q_{12} \neq 0$ by (3.2), the second factor vanishes hence $A_1 \cdot A_2 = 0$ by (3.9). \square

Now, we have the following possibilities for the vectors A_1 and A_2 :²

$$(1) A_1 = A_2 = 0, \quad (2) A_1 \neq 0, A_2 = 0, \quad (3) A_1 = 0, A_2 \neq 0 \quad (4) A_1 \neq 0, A_2 \neq 0.$$

PROPOSITION (3.11). *One or both of the vectors X_1 , X_2 moves along a fixed direction.*

Proof. This is clear in cases (1)–(3) due to the identity for a vector function $\mathbf{x} = \mathbf{x}(t)$ in \mathbb{R}^3 ,

$$\frac{d}{dt} \left(\frac{\mathbf{x}}{\|\mathbf{x}\|} \right) = \frac{(\mathbf{x} \times \dot{\mathbf{x}}) \times \mathbf{x}}{\|\mathbf{x}\|^3}.$$

In case (4), because $A_1 \cdot A_2 = 0$ we can consider the orthonormal frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, where $\mathbf{e}_1 = A_1/\|A_1\|$, $\mathbf{e}_2 = A_2/\|A_2\|$ and $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. Writing

$$X_1 = \alpha_1 \mathbf{e}_1 + \beta_1 \mathbf{e}_2 + \gamma_1 \mathbf{e}_3, \quad X_2 = \alpha_2 \mathbf{e}_1 + \beta_2 \mathbf{e}_2 + \gamma_2 \mathbf{e}_3,$$

by Proposition (3.7), we have that

$$\alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2 = 0, \quad \alpha_1 = 0 \quad \text{and} \quad \beta_2 = 0,$$

hence $\gamma_1 \gamma_2 = 0$ also. If $\gamma_1 = 0$ we get $X_1 = \beta_1 \mathbf{e}_2$ while for $\gamma_2 = 0$ we get $X_2 = \alpha_2 \mathbf{e}_1$. \square

From (2.1) and (3.4) we see that

$$X_1 = -\frac{M}{2m_1} \mathbf{r}_3 \quad \text{and} \quad X_2 = \frac{1}{2}(\mathbf{r}_1 - \mathbf{r}_2).$$

In the cases when only one or none of the vectors A_1 or A_2 vanishes denote by \mathbf{e} the fixed direction of X_1 , or X_2 as the case may be. Taking \mathbf{e} directed upwards, which means that the nonzero angular momentum \mathbf{C} in these cases have this direction, the figures 1, 2, 3 illustrate the three types of isosceles motion at which we arrived through our analysis. Notice that the angular momentum of the planar solution is zero.

This exhausts all the possibilities so it gives the classification of all the isosceles motions of the three-body problem with the masses $m_1 = m_2$ at the base of the isosceles configuration.

Of these three types of isosceles solutions, the planar one (see figure 1) and the spatial solution with a symmetry axis (see figure 3) are the most studied. Beginning with the work of McGehee [10], many papers have appeared applying his coordinates to study the behaviour of near collision orbits in the planar model and related problems. The Mexican school under the leadership of Ernesto Lacomba

²At this stage there is a mistake in Wintner's arguments. See the end of § 345 of [15], where he says that the four conditions (3.8) imply that at least one of the vectors A_1 or A_2 is zero and proceeds with the conclusion that one, or both, of the vectors X_1 or X_2 moves along a fixed direction.

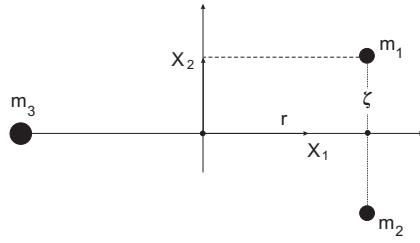


Figure 1. The planar isosceles solution.

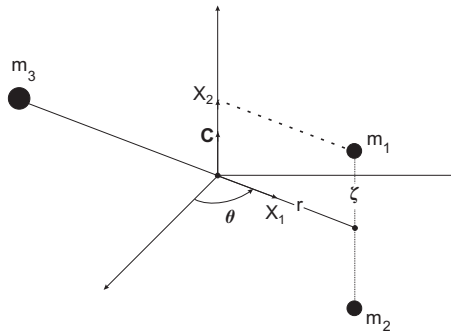


Figure 2. The isosceles solution with a fixed symmetry plane.

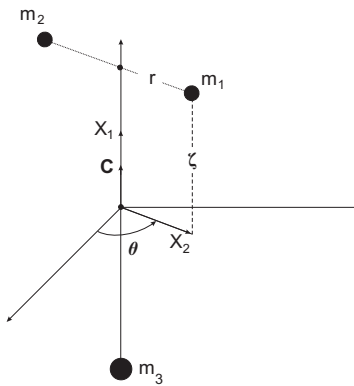


Figure 3. The isosceles solution with a fixed symmetry line.

has made important contributions in this area (see for instance the references [6], [7], [8]).

A limiting case of the spatial solution with a symmetry axis, the Sitnikov problem [13], was considered in detail in the applications of a general theory in a series of papers by Alekseev [1]. A very nice study of Sitnikov problem appears in the monograph by Moser [11]. After stating the main theorem on the symbolic dynamics of the zero mass particle $m_3 = 0$, [11], Theorem 3.5, he mentions an observation of Alekseev that the theorem would hold true even for small positive

values of m_3 (see figure 3). Recently in a joint paper with Daniel Offin [12], hyperbolicity for periodic orbits have been studied in this problem with general values of the masses. The case of the isosceles spatial solution with a symmetry plane, see figure 2, only now begins to be studied. A paper on the existence of periodic orbits, after regularization of the binary collisions, has been recently submitted for publication by Mateus-Venturelli-Vidal [9].

4. Comments on the problem of equal masses at the basis of an isosceles solution

In 1913 MacMillan gave a proof of the following theorem, published by Wilczynski [14].

THEOREM (4.1). *The masses at the basis of an isosceles solution are necessarily equal.*

In 1921 Jean Chazy gave another proof but did not avoid using analytical function theory to study the singularities of linear systems of differential equations in the complex plane. Wintner raises the question of whether there is a proof based on dynamical principles rather than on function theoretical arguments. In [2] we tried this approach but our proof has problems and does not settle the question. Let us examine this question again.

Differentiating the identity $\|Y_1\|^2 = \|Y_2\|^2$ twice and using the equations of motion (2.3) and (2.4), we get the identity

$$(4.2) \quad \|\dot{Y}_1\|^2 - \|\dot{Y}_2\|^2 = (m_2 - m_1) \left(\frac{1}{\|Y_1 - Y_2\|^3} - \frac{1}{\|Y_1\|^3} \right) (\langle Y_1, Y_2 \rangle - \|Y_1\| \|Y_2\|).$$

We know that the left-hand side vanishes because it is already established the equality of the masses. But if we could provide a direct dynamical proof that $\|\dot{Y}_1\| = \|\dot{Y}_2\|$ we would get $m_1 = m_2$. Indeed, the right-hand side of (4.2) would then be zero. The time-dependent factors in this equality cannot vanish identically because the annihilation of the second factor would give an equilateral solution of the three-body problem, while by the Cauchy-Schwarz inequality the annihilation of the third factor would give a collinear motion. Both of these possibilities are excluded from the definition of an isosceles solution, so it would remain to have $m_1 = m_2$.

Acknowledgments

We would like to thank Ernesto Perez-Chavela for helpful comments and suggestions.

Received August 21, 2012

Final version received October 16, 2012

HILDEBERTO E. CABRAL
 DEPARTAMENTO DE MATEMÁTICA - UFPE
 AV. PROF. LUIS FREIRE, S/N - CID. UNIV.
 RECIFE, PE - BRASIL
 hild@dmate.ufpe.br

REFERENCES

- [1] V. M. ALEKSEEV, *Quasirandom dynamical systems, I, II, III*, Math. USSR Sbornik **5** (1968), 73–128; **6** (1968), 505–560; **7** (1969), 1–43 .
- [2] H. E. CABRAL, *The Masses in an Isosceles Solution of the Three-Body Problem*, Celestial Mechanics **41** (1988), 175–177.
- [3] J. CHAZY, *Sur les solutions isocèles du Problème des Trois Corps*, Bull. Astr. (2) **1** (1921), 171–188.
- [4] A. E. FRANSEN, Öfv. Stockholm Akad. **52** (1895), 783–805.
- [5] P. HARTMAN, *Ordinary Differential Equations*, Birkhäuser, 1982.
- [6] E. A. LACOMBA AND L. LOSCO, *Triple collision in the isosceles 3-body problem*, Bulletin Amer. Math. Society (New Series) **3** (1980), 710–714.
- [7] E. A. LACOMBA AND E. PÉREZ-CHAVELA, *A compact model for the planar rhomboidal 4-body problem*, Celestial Mechanics and Dynamical Astronomy **54** (1992), 343–355.
- [8] E. A. LACOMBA AND E. PÉREZ-CHAVELA, *Motions close to escapes in the rhomboidal 4-body problem*, Celestial Mechanics and Dynamical Astronomy **57** (1993), 411–437.
- [9] E. MATEUS, A. VENTURELLI, AND C. VIDAL, *Collision quasi-periodic solutions in the spatial isosceles three-body problem*. Submitted to Archive for Rational Mechanics and Analysis.
- [10] R. MCGEHEE, *Triple collision in the collinear three-body problem*, Inventiones Math. **27** (1974), 191–227.
- [11] J. K. MOSER, *Stable and Random Motions in Dynamical Systems*, Princeton University Press, Princeton, 1973.
- [12] D. OFFIN AND H. E. CABRAL, *Hyperbolicity for symmetric periodic orbits in the isosceles three-body problem*, Discrete and Continuous Dynamical Systems, Series S (DCDS-S), **2** (2009), 379–392.
- [13] K. SITNIKOV, *Existence of oscillating motions for the three-body problem*, Dokl. Akad. Nauk, USSR, **133** (2) (1960), 303–306.
- [14] E. J. WILCZYNSKI, *Ricerche geometriche intorno al problema dei tre corpi*, Ann. di Mat., **21** (3) (1913), 1–21.
- [15] A. WINTNER, *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, New Jersey, 1941.

RADÓ'S THEOREM FOR FACTORISATIONS OF THE LAPLACE OPERATOR

To Ernesto Lacomba Zamora, in memoriam

C. GONZALEZ-FLORES AND EDUARDO S. ZERON

ABSTRACT. Let $\widehat{\mathcal{D}}$ and \mathcal{D} be a pair of first order differential operators with constant coefficients and such that the product $\widehat{\mathcal{D}}\mathcal{D}$ is equal to the Laplace operator ∇^2 . A classical theorem of Tibor Radó states that if a function h is continuous on an open set $\Omega \subset \mathbb{C}^n$ and holomorphic on the complement of its zero locus, then it is holomorphic everywhere on Ω . We show that a similar result holds, if one substitutes the condition that h is holomorphic by the assumption that the differential $\mathcal{D}h$ vanishes.

1. Introduction

The Laplace operator ∇^2 has played a prominent role in physics and mathematics since the eighteenth century. The fact that it can be factorised into the product of two first order differential operators has important consequences as well. We know for example that the real and imaginary parts of a holomorphic function are harmonic, because ∇^2 can be expressed in the complex plane as the product $4\partial\bar{\partial}$ of the Cauchy-Riemann operators ∂ and $\bar{\partial}$. Furthermore, physics became strongly interested in the factorisation of ∇^2 around 1925, when Dirac, Klein, and Gordon factorised the relativistic Schrödinger equation for a free particle:

$$(1.1) \quad \frac{\hat{m}^2 c^2}{\hbar^2} = \nabla^2 - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2} = \left(\frac{i\beta_4}{c} \cdot \frac{\partial}{\partial t} + \sum_{j=1}^3 \beta_j \frac{\partial}{\partial x_j} \right)^2.$$

The symbols β_k stand for real or complex square matrices that satisfy the conditions: β_k^2 is the identity matrix and $\beta_j\beta_k$ is equal to $-\beta_k\beta_j$ for all indices $j \neq k$. The first order factor at the right hand side of (1.1) is the classical example of a Dirac operator; i.e., the example of an operator that is the formal square root of the Laplace or d'Alembert operator. In the same way, the pair of Cauchy-Riemann operators 2∂ and $2\bar{\partial}$ is the classical example of a Dirac pair, because their product is equal to ∇^2 ; see for example [7]. We analyse in this paper factorisations of the Laplace operator ∇^2 into general homogeneous Dirac pairs with constant coefficients; i.e., we consider the following formula:

$$(1.2) \quad \nabla^2 = \widehat{\mathcal{D}}\mathcal{D}, \quad \text{where} \quad \widehat{\mathcal{D}} = \sum_{j=1}^n A_j \frac{\partial}{\partial x_j}, \quad \mathcal{D} = \sum_{k=1}^n B_k \frac{\partial}{\partial x_k},$$

2010 *Mathematics Subject Classification*: 35J46, 31B05, 35D30.

Keywords and phrases: Laplace operator, Radó's theorem for holomorphic functions.

Research supported by Cinvestav del IPN and Conacyt in México. The authors want to thank one of the referees for his enlightening suggestions and the Max-Planck-Institut für Mathematik in Germany, where part of this paper was written.

$\{A_j\}$ are $[\mu \times m]$ -complex matrices of full rank, $\mu \leq m$, and $\{B_k\}$ are some generalised inverse (pseudoinverse) $[m \times \mu]$ -matrices such that

$$(1.3) \quad A_k B_k = \text{Identity} \quad \text{and} \quad A_j B_k = -A_k B_j \quad \forall \quad j \neq k.$$

Notice that the choice of the matrices $\{A_k\}$ and $\{B_k\}$ is by no means unique, but they must all have full rank because the products $A_k B_k$ have to be equal to the identity matrix.

On the other hand, a classical theorem of Tibor Radó states that if a function h is continuous on an open set $U \subset \mathbb{C}^n$ and holomorphic on the complement of its zero locus, then it is holomorphic everywhere on U . A natural problem is to decide whether a similar result holds, if one substitutes the condition that h is holomorphic by the assumption that the differential $\mathcal{P}h$ vanishes for an elliptic differential operator \mathcal{P} . Several positive answers to this problem have been published since the middle of the twentieth century. It is quite interesting to analyse for example the works of Hounie and Tavares [4], Král [6], Tarkhanov [9], and Tavares [10]. Consider in particular the following result presented in [9], p. 40.

THEOREM (1.4) (Tarkhanov). *Let $\Omega \subset \mathbb{R}^n$ be an open domain, and h be a function in $C^p(\Omega)$ whose p -th derivatives are all locally Lipschitz. Given an elliptic differential operator \mathcal{P} of order $p+1$, if the differential $\mathcal{P}h$ vanishes on the complement of the zero locus $\Omega \setminus h^{-1}(0)$, then $\mathcal{P}h$ vanishes everywhere on Ω .*

The theorem above can be improved when \mathcal{P} is equal to the first order differential operator \mathcal{D} given in the factorisation (1.2)–(1.3) of the Laplace operator, so that $p = 0$. We proved in [3] that Theorem (1.4) holds when $\mathcal{P} \equiv \mathcal{D}$ and the function h has locally finite Dirichlet energy $\|Jh\|_2$ for the Jacobian J ; no Lipschitz condition on h was required at all. The main objective of this work is to remove completely the conditions that h is locally Lipschitz or has locally finite Dirichlet energy, so that we prove the following result.

THEOREM (1.5). *Let $\Omega \subset \mathbb{R}^n$ be an open domain, and \mathcal{D} be the first order differential operator defined in the factorisation (1.2)–(1.3). Given a continuous function F defined from Ω to \mathbb{R}^μ , if DF vanishes in the sense of distributions on the open set $\Omega \setminus F^{-1}(0)$, then F is harmonic and DF vanishes everywhere on Ω .*

Recall that F is harmonic if each of its entries is harmonic. The previous theorem is a direct generalisation of Radó's theorem, because the Laplace operator ∇^2 in the complex plane can be factorised as the product of the Cauchy-Riemann operators ∂ and $\bar{\partial}$. The following lemma is necessary in the proof of Theorem (1.5). This result is important, because it gives conditions for which the differential $\mathcal{D}F$ vanishes in the sense of distributions whenever $\mathcal{D}F$ is equal to zero almost everywhere (calculated in the strong sense).

LEMMA (1.6). *Let $U \subset \mathbb{R}^n$ be an open domain, and $E = F^{-1}(0)$ be the zero locus of a continuous function F defined from U to \mathbb{R}^μ . Assume that F is continuously differentiable on the open set $U \setminus E$ and consider the operator \mathcal{D} given in the factorisation (1.2)–(1.3). If the equation (1.7) below holds for every real smooth function φ with compact support in $U \setminus E$, then the differential DF exists almost everywhere*

(calculated in the strong sense) and is L^1 -integrable on every compact ball contained in the original set U :

$$(1.7) \quad \int_{x \in U} \left(\sum_{j=1}^n \frac{\partial \varphi(x)}{\partial x_j} A_j \right) \mathcal{D}F(x) dx = 0.$$

Moreover, the function F is harmonic in U if and only if the equation above holds for every real smooth function φ with compact support in U .

Recall that $\{A_j\}$ are $[\mu \times m]$ -matrices and $\mathcal{D}F$ is a function defined almost everywhere from U into \mathbb{R}^m . The lemma above implies in particular that the differential $\mathcal{D}F$ vanishes in the sense of distributions whenever $\mathcal{D}F$ is equal to zero almost everywhere. This result is not trivial at all, because the Cantor function is an example of a continuous function that is locally constant almost everywhere, but its derivative does not vanish in the sense of distributions. We may also take for example the non-continuous function $F(x)$ equal to zero when $x_1 > 0$ and equal to a fixed vector $v \neq 0$ everywhere else. We easily have that F is locally constant almost everywhere, but $\mathcal{D}F$ is not equal to zero in the sense of distributions when \mathcal{D} has a non-trivial component $B_1 \frac{\partial}{\partial x_1}$.

The results presented in this paper can be easily extended to consider factorisations of the Laplace operator into non-homogeneous Dirac pairs with constant coefficients, i.e.

$$\nabla^2 = \left(C_1 + \sum_{j=1}^n A_j \frac{\partial}{\partial x_j} \right) \left(C_2 + \sum_{k=1}^n B_k \frac{\partial}{\partial x_k} \right),$$

so that the matrices $\{A_j\}$ and $\{B_k\}$ should satisfy (1.3) and the products $C_1 C_2$, $A_j C_2$, and $C_1 B_k$ are all equal to the zero matrix. Nevertheless, we think that the above generalisation complicates the notation and does not really give a new perspective to the main theorem of this work. Theorem (1.5) is shown in the following chapter, while Lemma (1.6) is proved in the last section of this paper.

2. Proof of Main Theorem (1.5)

Let $\Omega \subset \mathbb{R}^n$ be an open domain, and $E = F^{-1}(0)$ be the zero locus of a continuous function F defined from Ω to \mathbb{R}^μ . Given the operator \mathcal{D} defined in the factorisation (1.2)–(1.3), assume that $\mathcal{D}F$ vanishes in the sense of distributions in the open set $\Omega \setminus E$. Then the Laplacian $\nabla^2 F$ also vanishes in the sense of distributions there; see for example [3] or directly analyse the last two lines in equation (3.1) in order to verify that all the integrals there vanish for every point y with Euclidean norm small enough and each real smooth function φ well defined and with compact support in $\Omega \setminus E$. Weyl's lemma then implies that F is harmonic in $\Omega \setminus E$; see for example [1], p. 27, [2], p. 33, or [5], p. 19.

We prove that both F is harmonic and $\mathcal{D}F$ vanishes everywhere in Ω . The result is trivial when E is empty or equal to Ω , so we suppose from now on that none of these cases holds. We need the following lemma originally presented in [3]; we include its proof for the sake of completeness.

LEMMA (2.1). *Let $f : \mathcal{I} \rightarrow \mathbb{R}$ be a continuous function defined on an open interval \mathcal{I} of the real line, and E be the zero locus of f . Assume that the derivative f' exists and it is continuous and L^1 -integrable on the open set $\mathcal{I} \setminus E$. Then f is absolutely*

continuous on \mathcal{I} and the derivative $f' \equiv 0$ almost everywhere on E . Finally, if f has compact support in the open interval \mathcal{I} , then the integral $\int_{\mathcal{I}} f' dt$ vanishes.

Proof. The result is trivial when E is empty or equal to \mathcal{I} . Hence, we assume from now on that the open set $U := \mathcal{I} \setminus E$ is not empty and that we may fix a point w in E . Define the functions

$$g(t) = \begin{cases} f'(t) & \text{if } t \in U, \\ 0 & \text{if } t \in E; \end{cases} \quad \text{and} \quad h(x) = \int_w^x g(t) dt.$$

The function g is L^1 -integrable on \mathcal{I} because f' is L^1 -integrable on the open set U . We assert that h and f coincide on \mathcal{I} . Every connected component of U is an open interval (b, c) with at least one of its end points in E , because E is not empty. The function $h(x)$ vanishes whenever x is contained in E , because

$$h(x) = \int_w^x g dt = \sum_{\theta} \int_{b_{\theta}}^{c_{\theta}} f' dt = \sum_{\theta} (f(c_{\theta}) - f(b_{\theta})) = 0.$$

The sums are calculated over all connected components (b_{θ}, c_{θ}) of U that lie between the points w and x of E , so that their respective end points b_{θ} and c_{θ} are all in the zero locus E of f . Moreover, when x is contained in U , let $s \in E$ be one of the end points of the connected component of U that contains x , so that

$$h(x) = \int_w^s g dt + \int_s^x g dt = \int_s^x f' dt = f(x) - f(s) = f(x).$$

Therefore, $f \equiv h$ is absolutely continuous on \mathcal{I} , and so the derivative f' exists and is also equal to g almost everywhere on \mathcal{I} ; see for example Theorem 8.17 of [8]. In particular, the derivative $f' \equiv 0$ almost everywhere on E , because g vanishes there. Assume that $d < e$ are the end points of the open interval \mathcal{I} . The following identities hold when f has compact support properly contained in \mathcal{I} ,

$$\int_{\mathcal{I}} f'(t) dt = \int_d^e g(t) dt = \lim_{t \rightarrow e} f(t) - \lim_{t \rightarrow d} f(t) = 0. \quad \square$$

Coming back to the proof of Theorem (1.5): We already have that each entry F_{ℓ} is continuous on Ω , harmonic on $\Omega \setminus E$, and equal to zero on E . Whence the positive and negative parts

$$(2.2) \quad \max\{F_{\ell}(x), 0\} \quad \text{and} \quad \max\{-F_{\ell}(x), 0\}$$

are all continuous and subharmonic on Ω . Lemma 1 in [6], p. 63, implies that the partial derivatives $\partial F_{\ell} / \partial x_k$ exist almost everywhere in Ω and are all L^1 -integrable (with respect to the Lebesgue measure) on every compact parallelogram

$$(2.3) \quad P = \prod_{k=1}^n [a_k, b_k] \quad \text{contained in } \Omega.$$

We obviously take $a_k < b_k$. Moreover, the entries F_{ℓ} are all finite and absolutely continuous on almost every line segment contained in P and oriented parallel to the coordinate axes. We assert that F is harmonic on the interior of any fixed parallelogram $P \subset \Omega$. The result is trivial when P is contained in or disjoint to E , so we suppose from now on that none of these cases holds. Define the following line segments in \mathbb{R}^n ,

$$L_{\xi} := [a_1, b_1] \times \{\xi\} \quad \text{with} \quad \xi \in \mathbb{R}^{n-1}.$$

The partial derivative $\partial F_\ell/\partial x_1$ is L^1 -integrable on L_ξ for every index ℓ and almost all points ξ such that $L_\xi \subset P$. For each of these points ξ we have that F_ℓ is continuous on L_ξ , smooth on $L_\xi \setminus E$, and equal to zero on $L_\xi \cap E$. Hence the partial derivative $\partial F_\ell/\partial x_1$ vanishes almost everywhere on $L_\xi \cap E$ according to Lemma (2.1). Fubini's theorem implies that $\partial F_\ell/\partial x_1$ is equal to zero almost everywhere on $P \cap E$ with respect to Lebesgue measure; see for example [8].

A similar analysis yields that the partially derivatives $\partial F_\ell/\partial x_k$ vanish almost everywhere on $P \cap E$ with respect to Lebesgue measure and for all indices k and ℓ . Notice that the differential DF is equal to zero in $P \setminus E$, because F is harmonic and DF vanishes in the sense of distributions there. Moreover, DF is a linear combination of the terms $\partial F_\ell/\partial x_k$ according to (1.2). Therefore, DF vanishes almost everywhere on P , and so equation (1.7) holds for U equal to the interior of P and every real smooth function φ well defined and with compact support in U . Lemma (1.6) implies then that F is harmonic and DF is identical to zero in the interior of P .

The desired result follows from the fact that the compact parallelogram P in Ω was chosen in an arbitrary form. We only need to prove Lemma (1.6) in order to conclude this paper.

3. Weak version of Weyl's lemma

The final section of this paper is devoted to prove (1.6). We restate the hypotheses in order to improve the presentation.

LEMMA (1.6). *Let $U \subset \mathbb{R}^n$ be an open domain, and $E = F^{-1}(0)$ be the zero locus of a continuous function F defined from U to \mathbb{R}^m . Assume that F is continuously differentiable on the open set $U \setminus E$ and consider the operator \mathcal{D} given in the factorisation (1.2)–(1.3). If the equation (1.7) below holds for every real smooth function φ with compact support in $U \setminus E$, then the differential DF exists almost everywhere (calculated in the strong sense) and is L^1 -integrable on every compact ball contained in the original set U :*

$$(1.7) \quad \int_{x \in U} \left(\sum_{j=1}^n \frac{\partial \varphi(x)}{\partial x_j} A_j \right) \mathcal{D}F(x) dx = 0.$$

Moreover, the function F is harmonic in U if and only if the equation above holds for every real smooth function φ with compact support in U .

Proof. First assume that F is continuously differentiable on U , so that DF is well defined and L^1 -integrable on every compact set in U . Recall that F is smooth whenever it is harmonic. The factorisation $\nabla^2 = \widehat{\mathcal{D}}\mathcal{D}$ given in (1.2)–(1.3) implies that F is harmonic if and only if the following formulae vanish for each point y in \mathbb{R}^n with Euclidean norm small enough and every real smooth functions φ well defined and with compact support in the open set U ,

$$\begin{aligned}
 (3.1) \quad & \int_{x \in U} \left(\sum_{j=1}^n \frac{\partial \varphi(x-y)}{\partial x_j} A_j \right) \mathcal{D}F(x) dx = \\
 & = - \sum_{j=1}^n A_j \frac{\partial}{\partial y_j} \int_{x \in U} \varphi(x-y) \left(\sum_{k=1}^n B_k \frac{\partial F(x)}{\partial x_k} \right) dx \\
 & = \sum_{j=1}^n A_j \frac{\partial}{\partial y_j} \int_{x \in U} \left(\sum_{k=1}^n \frac{\partial \varphi(x-y)}{\partial x_k} B_k \right) F(x) dx \\
 & = - \int_{x \in U} \left(\sum_{k=1}^n \frac{\partial^2 \varphi(x-y)}{\partial x_k^2} \right) F(x) dx.
 \end{aligned}$$

We only need to consider the case when $\nabla^2 F$ vanishes in the sense of distributions because of Weyl’s lemma; see for example [1], p. 27, [2], p. 33, or [5], p. 19. The equality between the second and third lines in (3.1) can be easily verified; we only need to observe that the following identities hold for all indexes k and ℓ ,

$$(3.2) \quad 0 = \int_{x \in U} \frac{\partial \varphi F_\ell}{\partial x_k} dx = \int_{x \in U} \left(\frac{\partial \varphi}{\partial x_k} F_\ell + \varphi \frac{\partial F_\ell}{\partial x_k} \right) dx.$$

On the other hand, consider the zero locus $E = F^{-1}(0)$ of the continuous function F defined on U . The original hypotheses yield that F is continuously differentiable in $U \setminus E$ and that equation (1.7) holds for every real smooth function φ well defined and with compact support in $U \setminus E$. The analysis done at the beginning of this proof implies that F is harmonic on $U \setminus E$. Hence the positive and negative parts of the entries F_ℓ are all continuous and subharmonic on U ; see (2.2). Let P be any compact parallelogram contained in U ; recall definition (2.3). Lemma 1 in [6], p. 63, implies that the partial derivatives $\partial F_\ell / \partial x_k$ and the differential $\mathcal{D}F$ exist almost everywhere in U and are all L^1 -integrable on P with respect to Lebesgue measure. This analysis concludes the first part of the proof.

We only need to prove now that F is harmonic whenever the integral in (1.7) vanishes for every smooth function φ with compact support in U . Let φ be any real smooth function well defined and with compact support in the interior of the parallelogram P . Define the following line segments in \mathbb{R}^n ,

$$L_\xi := [a_1, b_1] \times \{\xi\} \quad \text{with} \quad \xi \in \mathbb{R}^{n-1}.$$

The partial derivative $\frac{\partial \varphi F_\ell}{\partial x_1}$ is L^1 -integrable on L_ξ for every index ℓ and almost all points ξ such that $L_\xi \subset P$. For each of these points ξ we have that the product φF_ℓ is smooth on $L_\xi \setminus E$, equal to zero on $L_\xi \cap E$, and continuous with compact support in L_ξ . The integral of $\frac{\partial \varphi F_\ell}{\partial x_1}$ over L_μ vanishes according to Lemma (2.1). Fubini’s theorem implies that the integral of $\frac{\partial \varphi F_\ell}{\partial x_1}$ over P vanishes as well for every index ℓ ; see for example [8].

We obtain the same result if we derivate with respect to the variable x_k instead of x_1 , so that equation (3.2) holds for U equal to the interior of P , all indices k and ℓ , and every real smooth function φ with compact support in the interior of P . Hence, all equalities in (3.1) hold. Actually the first and third equalities in (3.1) hold because φ is smooth, while the second equality follows from (3.2). Since the integral in (1.7) vanishes according to the given hypotheses, the integrals in (3.1) are all equal to zero, and so the Laplacian $\nabla^2 F$ vanishes in the sense of

distributions in the interior of P . We can conclude that F is harmonic in U because of Weyl's lemma and the fact that the compact parallelogram P was chosen in an arbitrary form inside U . \square

Received January 2, 2012

Final version received April 18, 2012

C. GONZALEZ-FLORES
DEPARTAMENTO DE MATEMÁTICAS,
CINVESTAV DEL IPN,
APARTADO POSTAL 14-740,
MÉXICO D.F., 07000, MÉXICO
cgf@math.cinvestav.mx

EDUARDO S. ZERON
DEPARTAMENTO DE MATEMÁTICAS,
CINVESTAV DEL IPN,
APARTADO POSTAL 14-740,
MÉXICO D.F., 07000, MÉXICO
eszeron@math.cinvestav.mx

REFERENCES

- [1] YU. V. EGOROV AND B. W. SCHULZE, Pseudo-differential operators, singularities, applications, series: Oper. Theory Adv. Appl. **93**, Birkhäuser, Basel, 1997.
- [2] H. M. FARKAS AND I. KRA, Riemann surfaces, second edition, series: Grad. Texts in Math. **71**, Springer Verlag, New York, 1992.
- [3] C. GONZALEZ-FLORES AND E. S. ZERON, *Factorisations of the Helmholtz Operator, Radó's Theorem, and Clifford Analysis*, Adv. Appl. Clifford Algebr. **21** (2011), 89-101.
- [4] J. HOUNIE AND J. TAVARES, *Radó's theorem for locally solvable vector fields*, Proc. Amer. Math. Soc. **119**, 3 (1993), 829-836.
- [5] J. JOST, Partial differential equations, second edition, series: Grad. Texts in Math. **214**, Springer Verlag, New York, 2007.
- [6] J. KRÁL, *Some extension results concerning harmonic functions*, J. London Math. Soc. **28**, 1 (1983), 62-70.
- [7] M. MARTIN, *Deconstructing Dirac operators. III: Dirac and semi-Dirac pairs*, pp. 347-362 in: Topics in Operator Theory, Volume 2: Systems and Mathematical Physics, edited by J.A. Ball, V. Bolotnikov, L. Rodman, I.M. Spitkovsky and J.W. Helton, series: Oper. Theory Adv. Appl. **20**, Birkhäuser, Basel, 2010.
- [8] W. RUDIN, Real and complex analysis, second edition, McGraw-Hill, New York, 1974.
- [9] N. N. TARKHANOV, The Analysis of solutions of elliptic equations, series: Math. Appl. **406**, Kluwer Academic Publishers Group, Dordrecht, 1997.
- [10] J. TAVARES, *On the Radó's Theorem for locally integrable structures*, J. Math. Anal. Appl. **191**, 3 (1995), 490-496.

THE STRONG MATRIX STIELTJES MOMENT PROBLEM

ABDON E. CHOQUE RIVERO AND SERGEY M. ZAGORODNYUK

ABSTRACT. In this paper we study the strong matrix Stieltjes moment problem. We obtain necessary and sufficient conditions for its solvability. An analytic description of all solutions of the moment problem is derived. Necessary and sufficient conditions for the determinateness of the moment problem are given.

1. Introduction

In this paper we analyze the following problem: find a non-decreasing matrix function $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$, on $\mathbb{R}_+ = [0, +\infty)$, $M(0) = 0$, which is left-continuous on $(0, +\infty)$, and such that

$$(1.1) \quad \int_{\mathbb{R}_+} x^n dM(x) = S_n, \quad n \in \mathbb{Z},$$

where $\{S_n\}_{n \in \mathbb{Z}}$ is a prescribed sequence of Hermitian $(N \times N)$ complex matrices (moments), $N \in \mathbb{N}$. This problem is said to be a *strong matrix Stieltjes moment problem*. The problem is said to be *determinate* if it has a unique solution and *indeterminate* in the opposite case.

The scalar ($N = 1$) strong Stieltjes moment problem (in a slightly different setting) was introduced in 1980 by Jones, Thron and Waadeland [17]. Necessary and sufficient conditions for the existence of a solution with an infinite number of points of increase and for the uniqueness of such a solution were established in [17], Theorem 6.3. Also necessary and sufficient conditions for the existence of a unique solution with a finite number of points of increase were obtained [17], Theorem 5.2. The approach of Jones, Thron and Waadeland's investigation was made through the study of special continued fractions related to the moments.

In 1995, Njåstad described some classes of solutions of the scalar strong Stieltjes moment problem [35, 34]. He used properties of some associated Laurent polynomials.

In 1996, Kats and Nudelman obtained necessary and sufficient conditions for the existence of a solution of the scalar strong Stieltjes moment problem [19], Theorem 1.1 (without additional requirements for the solution). The degenerate case was studied in full: in this case the solution is unique, given explicitly and it has a finite number of points of increase. In the non-degenerate case, conditions for the determinacy were given and the unique solution was presented. In the (non-degenerate) indeterminate case a Nevanlinna-type parameterization for all solutions of the scalar strong Stieltjes moment problem was obtained [19], Theorem 4.1. Canonical solutions and Weyl-type lunes were studied, as well. Kats and Nudelman used the results of Krein on the semi-infinite string theory.

2010 *Mathematics Subject Classification*: 47A57, 30E05.

Keywords and phrases: Stieltjes moment problem, self-adjoint operator extension.

Various other results on the scalar strong Stieltjes moment problem can be found in papers [36, 7, 18, 37, 38] (see also references therein).

The moment problem (1.1) where the half-axis \mathbb{R}_+ is replaced by the whole axis \mathbb{R} is said to be the *strong matrix Hamburger moment problem*. The scalar ($N = 1$) strong matrix Hamburger moment problem has been intensively studied since 1980-th, see a survey [18], a recent paper [6] and references therein. For the matrix case, see papers [40, 43] and papers cited there.

The aim of our present investigation is threefold. Firstly, we obtain necessary and sufficient conditions for the solvability of the strong matrix Stieltjes moment problem (1.1). Consider the following block matrices constructed by moments:

$$(1.2) \quad \Gamma_n = (S_{i+j})_{i,j=-n}^n = \begin{pmatrix} S_{-2n} & \dots & S_{-n} & \dots & S_0 \\ \vdots & & \vdots & & \vdots \\ S_{-n} & \dots & S_0 & \dots & S_n \\ \vdots & & \vdots & & \vdots \\ S_0 & \dots & S_n & \dots & S_{2n} \end{pmatrix},$$

$$(1.3) \quad \tilde{\Gamma}_n = (S_{i+j+1})_{i,j=-n}^n = \begin{pmatrix} S_{-2n+1} & \dots & S_{-n+1} & \dots & S_1 \\ \vdots & & \vdots & & \vdots \\ S_{-n+1} & \dots & S_1 & \dots & S_{n+1} \\ \vdots & & \vdots & & \vdots \\ S_1 & \dots & S_{n+1} & \dots & S_{2n+1} \end{pmatrix},$$

where $n = 0, 1, 2, \dots$. We shall prove that conditions

$$(1.4) \quad \Gamma_n \geq 0, \quad \tilde{\Gamma}_n \geq 0 \quad n = 0, 1, 2, \dots,$$

are necessary and sufficient for the solvability of the moment problem (1.1).

Secondly, we obtain an analytic description of all solutions of the moment problem (1.1) using an operator approach.

The operator approach to the moment problems probably takes its origin in 1940-1943, in papers of Neumark [32, 33] for the case of the Hamburger moment problem. Neumark used an operator related to the corresponding Jacobi matrix. First, he obtained a description in terms of spectral functions of the operator. Then Neumark obtained the Nevanlinna formula using his results on the generalized resolvents of a symmetric operator with the deficiency index (1,1). Then this approach was developed by Krein and Krasnoselskiy [27], using the ideas of Krein of 1946-1948 [21, 24]. Various modifications appeared afterwards. Matrix or operator moment problems by the operator-theoretic approach were studied by (in the alphabetical order) Adamyán, Ando, Aleksandrov, Berezansky, Dudkin, Ershov, Ilmushkin, Inin, Kheifets, Krasnoselskiy, Krein, Luks, Simonov, Tkachenko, Turitsin (see, e.g., [1, 2, 5, 14, 6, 11, 13, 16, 20, 27, 25, 26, 30, 39, 40, 15]), among others. We shall use an abstract operator approach close to the “pure operator” approach of Szökefalvi-Nagy and Koranyi to the Nevanlinna-Pick interpolation problem, see [41, 42], and to the original approach of Neumark [32, 33].

We are not going to survey matricial and algebraic methods applied to various (not strong) truncated or full matrix moment problems, since these methods are quite different from the above-mentioned operator-theoretic methods, and

the problems under considerations by the methods are different. Instead of this, we shall illustrate the basic strategy, used in the present paper, on the example of the matrix trigonometric moment problem (see [44, 45] for more details). The truncated matrix trigonometric moment problem consists of finding a non-decreasing matrix-valued function $M(t) = (m_{k,l})_{k,l=0}^{N-1}$, $t \in [0, 2\pi]$, $M(0) = 0$, which is left-continuous in $(0, 2\pi]$, and such that

$$\int_0^{2\pi} e^{int} dM(t) = S_n, \quad n = 0, 1, \dots, d,$$

where $\{S_n\}_{n=0}^d$ is a prescribed sequence of $(N \times N)$ complex matrices (moments). Here $N \in \mathbb{N}$ and $d \in \mathbb{Z}_+$ are fixed numbers. Set

$$T_d = (S_{i-j})_{i,j=0}^d = \begin{pmatrix} S_0 & S_{-1} & S_{-2} & \dots & S_{-d} \\ S_1 & S_0 & S_{-1} & \dots & S_{-d+1} \\ S_2 & S_1 & S_0 & \dots & S_{-d+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S_d & S_{d-1} & S_{d-2} & \dots & S_0 \end{pmatrix},$$

where $S_k := S_{-k}^*$, $k = -d, -d + 1, \dots, -1$. Assume that $T_d \geq 0$ (this condition is necessary and sufficient for the solvability of the moment problem, see, e.g. [5]). The matrix T_d may be viewed as a usual complex matrix: $T_d = (\gamma_{n,m})_{n,m=0}^{(d+1)N-1}$. Then there exists a Hilbert space H and a sequence of elements $\{x_n\}_{n=0}^{(d+1)N-1}$ in H , such that

$$(1.5) \quad (x_n, x_m)_H = \gamma_{n,m}, \quad 0 \leq n, m \leq (d+1)N - 1,$$

and $\text{span}\{x_n\}_{n=0}^{(d+1)N-1} = \text{Lin}\{x_n\}_{n=0}^{(d+1)N-1} = H$. Then we define a linear operator A with $D(A) = \text{Lin}\{x_n\}_{n=0}^{dN-1}$ by equalities

$$Ax_n = x_{n+N}, \quad 0 \leq n \leq dN - 1.$$

All solutions of the moment problem can be obtained from the following relation:

$$M(t) = (m_{k,j}(t))_{k,j=0}^{N-1}, \quad t \in [0, 2\pi], \quad m_{k,j}(t) = (\mathbf{E}_t x_k, x_j)_H,$$

where \mathbf{E}_t is a left-continuous spectral function of the isometric operator A . By the important result of Chumakin on the generalized resolvents of isometric operators [9] it follows that all solutions are given by the following relations

$$M(t) = (m_{k,j}(t))_{k,j=0}^{N-1}, \quad t \in [0, 2\pi],$$

where $m_{k,j}$ are obtained from the following relation:

$$(1.6) \quad \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} dm_{k,j}(t) = ([E_H - \zeta(A \oplus \Phi_\zeta)]^{-1} x_k, x_j)_H, \quad \zeta \in \mathbb{D}.$$

Here Φ_ζ is an analytic in \mathbb{D} operator-valued function which values are linear contractions from $H \ominus D(A)$ into $H \ominus R(A)$, $\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$. The last step is to obtain the Nevanlinna-type formula. One should apply the Gram-Schmidt orthogonalization procedure to the vectors $x_0, x_1, \dots, x_{dN+N-1}$. During this procedure the numbers γ_\cdot are used, as well as relation (1.5). The elements of the obtained orthonormal basis are explicitly expressed as linear combinations of vectors x_n .

Calculating the matrix of the operator on the right in (1.6) with respect to this basis, we come to the following formula:

$$(1.7) \quad \int_0^{2\pi} \frac{1}{1-\zeta e^{it}} dM^T(t) = \frac{1}{h_\zeta} \mathbf{A}_\zeta - \frac{\zeta}{h_\zeta^2} \mathbf{B}_\zeta F_\zeta \left(I_\delta + \frac{1}{h_\zeta} \mathbf{C}_\zeta F_\zeta \right)^{-1} \mathbf{D}_\zeta, \quad \zeta \in \mathbb{D},$$

where $\mathbf{A}_\zeta, \mathbf{B}_\zeta, \mathbf{C}_\zeta, \mathbf{D}_\zeta$, are matrix polynomials, explicitly expressed in terms of the given moments, which values are matrices of sizes $N \times N, N \times \delta, \delta \times \delta, \delta \times N$, respectively. The polynomial h_ζ is scalar, and it is also explicitly calculated by the moments. Here F_ζ is an analytic in \mathbb{D} , $(\delta \times \delta)$ matrix-valued function which values are such that $F_\zeta^* F_\zeta \leq I_\delta, \forall \zeta \in \mathbb{D}, I_\delta = (\delta_{k,l})_{k,l=1}^\delta$.

We shall use this strategy for the strong Stieltjes moment problem, except the last step. In our case, when the moment problem is full, expressions in terms of moments are not so clearly effective. In fact, even in the case of the classical Hamburger moment problem, how to construct numerically the elements of the Nevanlinna matrix? On the other hand, operator expressions are compact. We shall also remark that the strategy was used in papers [43, 46] and we adapt ideas from these papers. However, after a description of solutions of the moment problem (1.1) in terms of spectral functions of the corresponding operator we shall go in another direction. We present a description of generalized Π -resolvents of a non-negative operator which does not use improper elements or relations as it was done in the original work of Krein [22] and in the paper of Derkach and Malamud [10]. We adapt some ideas from [8] of Chumakin who studied generalized resolvents of isometric operators. We shall need some properties of generalized Π -resolvents of non-negative operators and generalized sc -resolvents of Hermitian contractions, established by Krein and Ovcharenko in [23, 29].

Finally, we obtain necessary and sufficient conditions for the strong matrix Stieltjes moment problem to be determinate.

Notations

As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$, the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively; $\mathbb{R}_+ = [0, +\infty)$. The space of n -dimensional complex vectors $a = (a_0, a_1, \dots, a_{n-1})$, we denote by $\mathbb{C}^n, n \in \mathbb{N}$. If $a \in \mathbb{C}^n$, then a^* means the complex conjugate vector. By \mathbb{P}_L we denote the space of all complex Laurent polynomials, i.e. functions $\sum_{k=a}^b \alpha_k x^k, a, b \in \mathbb{Z}: a \leq b, \alpha_k \in \mathbb{C}$.

Let $M(x)$ be a left-continuous non-decreasing matrix function $M(x) = (m_{k,l}(x))_{k,l=0}^{N-1}$ on $\mathbb{R}_+, M(0) = 0$, and $\tau_M(x) := \sum_{k=0}^{N-1} m_{k,k}(x); \Psi(x) = (dm_{k,l}/d\tau_M)_{k,l=0}^{N-1}$. By $L^2(M)$ we denote a set (of equivalence classes) of vector-valued functions $f: \mathbb{R}_+ \rightarrow \mathbb{C}^N, f = (f_0, f_1, \dots, f_{N-1})$, such that (see, e.g., [31])

$$\|f\|_{L^2(M)}^2 := \int_{\mathbb{R}_+} f(x)\Psi(x)f^*(x)d\tau_M(x) < \infty.$$

The space $L^2(M)$ equipped with the inner product

$$(f, g)_{L^2(M)} := \int_{\mathbb{R}_+} f(x)\Psi(x)g^*(x)d\tau_M(x), \quad f, g \in L^2(M)$$

is a Hilbert space. We denote $\vec{e}_k = (\delta_{0,k}, \delta_{1,k}, \dots, \delta_{N-1,k}), 0 \leq k \leq N-1$, where $\delta_{j,k}$ is the Kronecker delta.

If H is a Hilbert space then $(\cdot, \cdot)_H$ and $\|\cdot\|_H$ mean the scalar product and the norm in H , respectively. Indices may be omitted in obvious cases. For a linear operator A in H , we denote by $D(A)$ its domain, by $R(A)$ its range, by $\text{Ker } A$ its kernel, and A^* means the adjoint operator if it exists. If A is invertible then A^{-1} means its inverse. \bar{A} means the closure of the operator, if the operator is closable. If A is self-adjoint, by $R_z(A)$ we denote the resolvent of A , $z \in \mathbb{C} \setminus \mathbb{R}$. If A is bounded then $\|A\|$ denotes its norm. For an arbitrary set of elements $\{x_n\}_{n \in \mathbb{Z}}$ in H , we denote by $\text{Lin}\{x_n\}_{n \in \mathbb{Z}}$ and $\text{span}\{x_n\}_{n \in \mathbb{Z}}$ the linear span and the closed linear span (in the norm of H), respectively. For a set $M \subseteq H$ we denote by \bar{M} the closure of M in the norm of H . By E_H we denote the identity operator in H , i.e. $E_H x = x$, $x \in H$. If H_1 is a subspace of H , then $P_{H_1} = P_{H_1}^H$ is an operator of the orthogonal projection on H_1 in H . By $[H]$ we denote the set of all bounded linear operators A in H , $D(A) = H$. If A and B are some linear operators in a Hilbert space H then $A \supseteq B$ ($A \subseteq B$) means that A is an extension of B (respectively B is an extension of A).

2. The solvability of the strong matrix Stieltjes moment problem

In this section we are going to establish the following theorem.

THEOREM (2.1). *Let the strong matrix Stieltjes moment problem (1.1) with a set of moments $\{S_n\}_{n \in \mathbb{Z}}$ be given. The moment problem has a solution if and only if the conditions in (1.4) are satisfied.*

Proof. Necessity. Let the strong matrix Stieltjes moment problem (1.1) have a solution $M(x)$. Choose an arbitrary vector function $f(x) = \sum_{k=-n}^n \sum_{j=0}^{N-1} f_{j,k} x^k \vec{e}_j$, $f_{j,k} \in \mathbb{C}$. This function belongs to $L^2(M)$ and

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}_+} f(x)x^s dM(x)f^*(x) = \sum_{k,r=-n}^n \sum_{j,l=0}^{N-1} f_{j,k} \overline{f_{l,r}} \int_{\mathbb{R}_+} x^{k+r+s} \vec{e}_j dM(x) \vec{e}_l^* \\ &= \sum_{k,r=-n}^n \sum_{j,l=0}^{N-1} f_{j,k} \vec{e}_j S_{k+r+s} \overline{f_{l,r}} \vec{e}_l^* = \sum_{k,r=-n}^n (f_{0,k}, f_{1,k}, \dots, f_{N-1,k}) S_{k+r+s} \\ &\quad * (f_{0,r}, f_{1,r}, \dots, f_{N-1,r})^* = \begin{cases} v \Gamma_n v^*, & s = 0 \\ v \tilde{\Gamma}_n v^*, & s = 1 \end{cases}, \end{aligned}$$

where $v = (f_{0,-n}, f_{1,-n}, \dots, f_{N-1,-n}, f_{0,-n+1}, f_{1,-n+1}, \dots, f_{N-1,-n+1}, \dots, f_{0,n}, f_{1,n}, \dots, f_{N-1,n})$. Since we can choose the complex numbers $f_{j,k}$ arbitrarily, it follows (1.4).

Sufficiency. Let the strong matrix Stieltjes moment problem (1.1) be given and (1.4) be satisfied. Let $S_j = (S_{j;k,l})_{k,l=0}^{N-1}$, $S_{j;k,l} \in \mathbb{C}$, $j \in \mathbb{Z}$. Consider the following infinite block matrix:

$$(2.2) \quad \Gamma = (S_{i+j})_{i,j=-\infty}^{\infty} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & S_{-2n} & \dots & S_{-n} & \dots & S_0 & \dots \\ \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots \\ \dots & S_{-n} & \dots & \boxed{S_0} & \dots & S_n & \dots \\ \dots & \vdots & \dots & \vdots & \dots & \vdots & \dots \\ \dots & S_0 & \dots & S_n & \dots & S_{2n} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

where the element in the box corresponds to the indices $i = j = 0$.

We assume that the left upper entry of the element in the box stands in row 0, column 0. Let us numerate rows (columns) in the increasing order to the bottom (respectively to the right). Then we numerate rows (columns) in the decreasing order to the top (respectively to the left). Thus, the matrix Γ may be viewed as a numerical matrix: $\Gamma = (\gamma_{k,l})_{k,l=-\infty}^{\infty}$, $\gamma_{k,l} \in \mathbb{C}$. Observe that the following equalities hold

$$(2.3) \quad \gamma_{rN+j,tN+n} = S_{r+t;j,n}, \quad r, t \in \mathbb{Z}, 0 \leq j, n \leq N-1.$$

From conditions (1.4) it easily follows that

$$(2.4) \quad (\gamma_{k,l})_{k,l=-r}^r \geq 0, \quad (\gamma_{k+N,l})_{k,l=-r}^r \geq 0, \quad \forall r \in \mathbb{Z}_+.$$

The first inequality in the latter relation implies that there exist a Hilbert space H and a set of elements $\{x_n\}_{n \in \mathbb{Z}}$ in H such that

$$(2.5) \quad (x_n, x_m)_H = \gamma_{n,m}, \quad n, m \in \mathbb{Z},$$

and $\text{span}\{x_n\}_{n \in \mathbb{Z}} = H$, see Lemma in [42], p. 177. The latter fact is well known and goes back to the paper of Gelfand, Naimark [12].

By (2.3) we get

$$(2.6) \quad \gamma_{a+N,b} = \gamma_{a,b+N}, \quad a, b \in \mathbb{Z}.$$

Set $L = \text{Lin}\{x_n\}_{n \in \mathbb{Z}}$. Choose an arbitrary element $x \in L$. Let $x = \sum_{k=-\infty}^{\infty} \alpha_k x_k$, $x = \sum_{k=-\infty}^{\infty} \beta_k x_{k+N}$, where $\alpha_k, \beta_k \in \mathbb{C}$. Here only a finite number of coefficients α_k, β_k are non-zero. In what follows, this will be assumed in analogous situations with elements of the linear span. By (2.5), (2.6) we may write

$$\begin{aligned} \left(\sum_{k=-\infty}^{\infty} \alpha_k x_{k+N}, x_l \right) &= \sum_{k=-\infty}^{\infty} \alpha_k \gamma_{k+N,l} = \sum_{k=-\infty}^{\infty} \alpha_k \gamma_{k,l+N} \\ &= \sum_{k=-\infty}^{\infty} \alpha_k (x_k, x_{l+N}) = \left(\sum_{k=-\infty}^{\infty} \alpha_k x_k, x_{l+N} \right) = (x, x_{l+N}), \quad l \in \mathbb{Z}. \end{aligned}$$

Similarly we conclude that $(\sum_{k=-\infty}^{\infty} \beta_k x_{k+N}, x_l) = (x, x_{l+N})$, $l \in \mathbb{Z}$. Since $\bar{L} = H$, we get $\sum_{k=-\infty}^{\infty} \alpha_k x_{k+N} = \sum_{k=-\infty}^{\infty} \beta_k x_{k+N}$.

Set

$$(2.7) \quad A_0 x = \sum_{k=-\infty}^{\infty} \alpha_k x_{k+N}, \quad x \in L, x = \sum_{k=-\infty}^{\infty} \alpha_k x_k, \alpha_k \in \mathbb{C}.$$

The above considerations ensure us that the operator A_0 is defined correctly. Choose arbitrary elements $x, y \in L$, $x = \sum_{k=-\infty}^{\infty} \alpha_k x_k$, $y = \sum_{n=-\infty}^{\infty} \beta_n x_n$, $\alpha_k, \beta_n \in \mathbb{C}$, and write

$$\begin{aligned} (A_0 x, y)_H &= \left(\sum_{k=-\infty}^{\infty} \alpha_k x_{k+N}, \sum_{n=-\infty}^{\infty} \beta_n x_n \right)_H = \sum_{k,n=-\infty}^{\infty} \alpha_k \overline{\beta_n} (x_{k+N}, x_n)_H = \\ &= \sum_{k,n=-\infty}^{\infty} \alpha_k \overline{\beta_n} (x_k, x_{n+N})_H = \left(\sum_{k=-\infty}^{\infty} \alpha_k x_k, \sum_{n=-\infty}^{\infty} \beta_n x_{n+N} \right)_H = (x, A_0 y)_H. \end{aligned}$$

Moreover, we have

$$(2.8) \quad (A_0 x, x)_H = \sum_{k,n=-\infty}^{\infty} \alpha_k \overline{\alpha_n} (x_{k+N}, x_n)_H = \sum_{k,n=-\infty}^{\infty} \alpha_k \overline{\alpha_n} \gamma_{k+N,n} \geq 0.$$

Thus, the operator A_0 is a non-negative symmetric operator in H . Set $A = \overline{A_0}$. The operator A always has a non-negative self-adjoint extension \tilde{A} in a Hilbert space $\tilde{H} \supseteq H$ [23], Theorem 7, p. 450. We may assume that $\text{Ker } \tilde{A} = \{0\}$. In the opposite case, since $\text{Ker } \tilde{A} \perp R(\tilde{A})$, $R(\tilde{A}) \supseteq L$, we conclude that $\text{Ker } \tilde{A} \perp H$. Therefore the operator \tilde{A} , restricted to $\tilde{H} \ominus \text{Ker } \tilde{A}$, also will be a self-adjoint extension of the operator A , with a null kernel.

Let $\{\tilde{E}_\lambda\}_{\lambda \in \mathbb{R}}$ be the left-continuous orthogonal resolution of unity of the operator \tilde{A} . By the induction argument it is easy to check that

$$x_{rN+j} = A^r x_j, \quad r \in \mathbb{Z}, \quad 0 \leq j \leq N-1.$$

By (2.3), (2.5) we may write

$$\begin{aligned} S_{r;j,n} &= \gamma_{rN+j,n} = (x_{rN+j}, x_n)_H = (A^r x_j, x_n)_H = (\tilde{A}^r x_j, x_n)_{\tilde{H}} \\ &= \int_{\mathbb{R}_+} \lambda^r d(\tilde{E}_\lambda x_j, x_n)_{\tilde{H}} = \int_{\mathbb{R}_+} \lambda^r d\left(P_{\tilde{H}}^{\tilde{H}} \tilde{E}_\lambda x_j, x_n\right)_H, \quad 0 \leq n \leq N-1. \end{aligned}$$

Therefore we get

$$(2.9) \quad S_r = \int_{\mathbb{R}_+} \lambda^r d\tilde{M}(\lambda), \quad r \in \mathbb{Z},$$

where $\tilde{M}(\lambda) := \left(\left(P_{\tilde{H}}^{\tilde{H}} \tilde{E}_\lambda x_j, x_n \right)_H \right)_{j,n=0}^{N-1}$. Therefore the matrix function $\tilde{M}(\lambda)$ is a solution of the moment problem (1.1) (From the properties of the orthogonal resolution of unity it easily follows that $\tilde{M}(\lambda)$ is left-continuous on $(0, +\infty)$, non-decreasing and $\tilde{M}(0) = 0$). \square

3. An analytic description of solutions of the strong matrix Stieltjes moment problem

Let A be an arbitrary closed Hermitian operator in a Hilbert space H , $D(A) \subseteq H$. Let \hat{A} be an arbitrary self-adjoint extension of A in a Hilbert space $\hat{H} \supseteq H$. Denote by $\{\hat{E}_\lambda\}_{\lambda \in \mathbb{R}}$ its orthogonal resolution of unity. Recall that an operator-valued function $\mathbf{R}_z = P_{\hat{H}}^{\hat{H}} R_z(\hat{A})$ is said to be a *generalized resolvent* of A , $z \in \mathbb{C} \setminus \mathbb{R}$. A function $\mathbf{E}_\lambda = P_{\hat{H}}^{\hat{H}} \hat{E}_\lambda$, $\lambda \in \mathbb{R}$, is said to be a *spectral function* of A . There exists a bijective correspondence between generalized resolvents and left-continuous (or

normalized in some other way) spectral functions established by the following relation [3]:

$$(3.1) \quad (\mathbf{R}_z f, g)_H = \int_{\mathbb{R}} \frac{1}{\lambda - z} d(\mathbf{E}_\lambda f, g)_H, \quad f, g \in H, z \in \mathbb{C} \setminus \mathbb{R}.$$

If the operator A is densely defined symmetric and non-negative ($A \geq 0$), and the extension \hat{A} is self-adjoint and non-negative, then the corresponding generalized resolvent \mathbf{R}_z and the spectral function \mathbf{E}_λ are said to be a *generalized Π -resolvent* and a *Π -spectral function* of A . Relation (3.1) establishes a bijective correspondence between generalized Π -resolvents and left-continuous Π -spectral functions.

If the operator A is a Hermitian contraction ($\|A\| \leq 1$), and the extension \hat{A} is a self-adjoint contraction, then the corresponding generalized resolvent \mathbf{R}_z and the spectral function \mathbf{E}_λ are said to be a *generalized sc -resolvent* and a *sc -spectral function* of A . Relation (3.1) establishes a bijective correspondence between generalized sc -resolvents and left-continuous sc -spectral functions, as well.

If a generalized Π -resolvent (a generalized sc -resolvent) is generated by an extension *inside* H , i.e. $\hat{H} = H$, then it is said to be a *canonical Π -resolvent* (respectively a *canonical sc -resolvent*).

Firstly, we shall obtain a description of solutions of the strong matrix Stieltjes moment problem by virtue of Π -spectral functions.

THEOREM (3.2). *Let the strong matrix Stieltjes moment problem (1.1) be given and (1.4) be satisfied. Suppose that the operator $A = \overline{A_0}$ in a Hilbert space H is constructed for the moment problem by (2.7) and the preceding procedure. All solutions of the moment problem have the following form*

$$(3.3) \quad M(\lambda) = (m_{k,j}(\lambda))_{k,j=0}^{N-1}, \quad m_{k,j}(\lambda) = (\mathbf{E}_\lambda x_k, x_j)_H,$$

where \mathbf{E}_λ is a left-continuous Π -spectral function of the operator A . On the other hand, each left-continuous Π -spectral function of the operator A generates by (3.3) a solution of the moment problem. Moreover, the correspondence between all left-continuous Π -spectral functions of the operator A and all solutions of the moment problem is bijective.

Proof. Let \mathbf{E}_λ be an arbitrary Π -spectral function of the operator A . It corresponds to a self-adjoint operator $\tilde{A} \supseteq A$ in a Hilbert space $\tilde{H} \supseteq H$. Then we repeat considerations after (2.8) to obtain that $M(\lambda)$, given by (3.3), is a solution of the moment problem (1.1).

Let $\tilde{M}(x) = (\tilde{m}_{k,l}(x))_{k,l=0}^{N-1}$ be an arbitrary solution of the moment problem (1.1). Consider the space $L^2(\tilde{M})$. A set (of classes of equivalence) of functions $f \in L^2(\tilde{M})$ such that (the corresponding class includes) $f = (f_0, f_1, \dots, f_{N-1})$, $f \in \mathbb{P}_L$, we denote by $\mathbb{P}_L^2(\tilde{M})$. Set $L_L^2(\tilde{M}) := \overline{\mathbb{P}_L^2(\tilde{M})}$.

For an arbitrary vector Laurent polynomial $f = (f_0, f_1, \dots, f_{N-1})$, $f_j \in \mathbb{P}_L$, there exists a unique representation of the following form:

$$(3.4) \quad f(x) = \sum_{k=0}^{N-1} \sum_{j=-\infty}^{\infty} \alpha_{k,j} x^j \vec{e}_k, \quad \alpha_{k,j} \in \mathbb{C},$$

where all except for a finite number of coefficients $\alpha_{k,j}$ are zero. Choose another vector Laurent polynomial g with a representation

$$(3.5) \quad g(x) = \sum_{l=0}^{N-1} \sum_{r=-\infty}^{\infty} \beta_{l,r} x^r \bar{e}_l, \quad \beta_{l,r} \in \mathbb{C}.$$

We may write

$$(3.6) \quad \begin{aligned} (f, g)_{L^2(\widehat{M})} &= \sum_{k,l=0}^{N-1} \sum_{j,r=-\infty}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} \int_{\mathbb{R}_+} x^{j+r} \bar{e}_k d\widehat{M}(x) \bar{e}_l^* \\ &= \sum_{k,l=0}^{N-1} \sum_{j,r=-\infty}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} \int_{\mathbb{R}_+} x^{j+r} d\widehat{m}_{k,l}(x) = \sum_{k,l=0}^{N-1} \sum_{j,r=-\infty}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} S_{j+r;k,l}. \end{aligned}$$

On the other hand, we have

$$(3.7) \quad \begin{aligned} \left(\sum_{j=-\infty}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k}, \sum_{r=-\infty}^{\infty} \sum_{l=0}^{N-1} \beta_{l,r} x_{rN+l} \right)_H &= \sum_{k,l=0}^{N-1} \sum_{j,r=-\infty}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} \\ * (x_{jN+k}, x_{rN+l})_H &= \sum_{k,l=0}^{N-1} \sum_{j,r=-\infty}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} \gamma_{jN+k,rN+l} \\ &= \sum_{k,l=0}^{N-1} \sum_{j,r=-\infty}^{\infty} \alpha_{k,j} \overline{\beta_{l,r}} S_{j+r;k,l}. \end{aligned}$$

By (3.6), (3.7) we get

$$(3.8) \quad (f, g)_{L^2(\widehat{M})} = \left(\sum_{j=-\infty}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k}, \sum_{r=-\infty}^{\infty} \sum_{l=0}^{N-1} \beta_{l,r} x_{rN+l} \right)_H.$$

Set

$$(3.9) \quad Vf = \sum_{j=-\infty}^{\infty} \sum_{k=0}^{N-1} \alpha_{k,j} x_{jN+k},$$

for a vector Laurent polynomial $f(x) = \sum_{k=0}^{N-1} \sum_{j=-\infty}^{\infty} \alpha_{k,j} x^j \bar{e}_k$. If f, g are vector Laurent polynomials with representations (3.4), (3.5), such that $\|f - g\|_{L^2(\widehat{M})} = 0$, then by (3.8) we may write

$$\|Vf - Vg\|_H^2 = (V(f - g), V(f - g))_H = (f - g, f - g)_{L^2(\widehat{M})} = \|f - g\|_{L^2(\widehat{M})}^2 = 0.$$

Thus, V is correctly defined as an operator from $\mathbb{P}^2(\widehat{M})$ to H . Relation (3.8) shows that V is an isometric transformation from $\mathbb{P}_L^2(\widehat{M})$ on L . We extend V by continuity to an isometric transformation from $L_L^2(\widehat{M})$ on H . Observe that

$$(3.10) \quad Vx^j \bar{e}_k = x_{jN+k}, \quad j \in \mathbb{Z}; \quad 0 \leq k \leq N-1.$$

Let $L_1^2(\widehat{M}) := L^2(\widehat{M}) \oplus L_L^2(\widehat{M})$, and $U := V \oplus E_{L_L^2(\widehat{M})}$. The operator U is an isometric transformation from $L^2(\widehat{M})$ on $H \oplus L_1^2(\widehat{M}) =: \widehat{H}$. Let Q be the operator of multiplication by an independent variable in $L^2(\widehat{M})$. Set

$$\widehat{A} := UQU^{-1}.$$

The operator \widehat{A} is a self-adjoint operator in \widehat{H} . Let $\{\widehat{E}_\lambda\}_{\lambda \in \mathbb{R}}$ be its left-continuous orthogonal resolution of unity. Notice that

$$\begin{aligned} UQU^{-1}x_{jN+k} &= VQV^{-1}x_{jN+k} = VQx^j\vec{e}_k = Vx^{j+1}\vec{e}_k = x_{(j+1)N+k} = \\ &= x_{jN+k+N} = Ax_{jN+k}, \quad j \in \mathbb{Z}; \quad 0 \leq k \leq N-1. \end{aligned}$$

By linearity we get: $UQU^{-1}x = Ax$, $x \in L$, and therefore $\widehat{A} \supseteq A$. Choose an arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$. Using the properties of integrals with respect to the resolution of unity we may write

$$\begin{aligned} \int_{\mathbb{R}_+} \frac{1}{\lambda - z} d(\widehat{E}_\lambda x_k, x_j)_{\widehat{H}} &= \left(\int_{\mathbb{R}_+} \frac{1}{\lambda - z} d\widehat{E}_\lambda x_k, x_j \right)_{\widehat{H}} \\ &= \left(U^{-1} \int_{\mathbb{R}_+} \frac{1}{\lambda - z} d\widehat{E}_\lambda x_k, U^{-1}x_j \right)_{L^2(\widehat{M})} \\ &= \left(\int_{\mathbb{R}_+} \frac{1}{\lambda - z} dU^{-1}\widehat{E}_\lambda U\vec{e}_k, \vec{e}_j \right)_{L^2(\widehat{M})}, \end{aligned}$$

Using the definition of the scalar product in $L^2(\widehat{M})$ we write

$$\begin{aligned} \left(\int_{\mathbb{R}_+} \frac{1}{\lambda - z} dU^{-1}\widehat{E}_\lambda U\vec{e}_k, \vec{e}_j \right)_{L^2(\widehat{M})} &= \left(\int_{\mathbb{R}_+} \frac{1}{\lambda - z} dE_\lambda \vec{e}_k, \vec{e}_j \right)_{L^2(\widehat{M})} \\ &= ((Q - z)^{-1}\vec{e}_k, \vec{e}_j)_{L^2(\widehat{M})} = \int_{\mathbb{R}_+} \frac{1}{\lambda - z} \vec{e}_k d\widehat{M}(\lambda)\vec{e}_j = \int_{\mathbb{R}_+} \frac{1}{\lambda - z} d\widehat{m}_{k,j}(\lambda), \end{aligned}$$

where E_λ is a left-continuous orthogonal resolution of unity of the operator Q (see e.g. [4]). By the Stieltjes-Perron inversion formula we conclude that

$$\widehat{m}_{k,j}(\lambda) = (P_H^{\widehat{H}} \widehat{E}_\lambda x_k, x_j)_H, \quad \lambda \in \mathbb{R}.$$

Thus, \widehat{M} is generated by a Π -spectral function of A .

Let us check that an arbitrary element $u \in L$ can be represented in the following form

$$(3.11) \quad u = u_z + u_0, \quad u_z \in H_z, \quad u_0 \in L_N,$$

where $L_N := \text{Lin}\{x_n\}_{n=0}^{N-1}$, $H_z := (A - zE_H)D(A)$. Let $u = \sum_{k=-\infty}^{\infty} c_k x_k$, $c_k \in \mathbb{C}$, and choose a number $z \in \mathbb{C} \setminus \mathbb{R}$. Suppose that $c_k = 0$, if $k \leq r$ or $k \geq R$, where $r \leq -2$; $R \geq N+1$. Set $d_k := 0$, if $k \leq r$ or $k \geq R - N$. Then we set

$$\begin{aligned} d_k &:= \frac{1}{z} (d_{k-N} - c_k), \quad k = r+1, \dots, -1; \\ d_{k-N} &:= z d_k + c_k, \quad k = R-1, R-2, \dots, N. \end{aligned}$$

Set $v := \sum_{k=-\infty}^{\infty} d_k x_k \in L$. Then we directly calculate that

$$(A - zE_H)v - u = \sum_{k=0}^{N-1} (d_{k-N} - z d_k - c_k) x_k,$$

and relation (3.11) holds. From the latter equality it easily follows that the deficiency index of A is equal to (n, n) , $0 \leq n \leq N$.

Let us check that different left-continuous Π -spectral functions of the operator A generate different solutions of the moment problem (1.1). Suppose that two different left-continuous Π -spectral functions generate the same solution of the

moment problem. This means that there exist two self-adjoint operators $A_j \supseteq A$, in Hilbert spaces $H_j \supseteq H$, such that $P_H^{H_1} E_{1,\lambda} \neq P_H^{H_2} E_{2,\lambda}$, and

$$(P_H^{H_1} E_{1,\lambda} x_k, x_j)_H = (P_H^{H_2} E_{2,\lambda} x_k, x_j)_H, \quad 0 \leq k, j \leq N-1, \quad \lambda \in \mathbb{R},$$

where $\{E_{n,\lambda}\}_{\lambda \in \mathbb{R}}$ are orthogonal left-continuous resolutions of unity of operators A_n , $n = 1, 2$. By the linearity we get

$$(3.12) \quad (P_H^{H_1} E_{1,\lambda} x, y)_H = (P_H^{H_2} E_{2,\lambda} x, y)_H, \quad x, y \in L_N, \quad \lambda \in \mathbb{R}.$$

Set $\mathbf{R}_{n,\lambda} := P_H^{H_n} R_\lambda(A_n)$, $n = 1, 2$. By (3.12), (3.1) we get

$$(3.13) \quad (\mathbf{R}_{1,\lambda} x, y)_H = (\mathbf{R}_{2,\lambda} x, y)_H, \quad x, y \in L_N, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Since

$$R_z(A_j)(A - zE_H)x = (A_j - zE_{H_j})^{-1}(A_j - zE_{H_j})x = x, \quad x \in L = D(A_0),$$

then $R_z(A_1)u = R_z(A_2)u \in H$, $u \in H_z$;

$$(3.14) \quad \mathbf{R}_{1,z}u = \mathbf{R}_{2,z}u, \quad u \in H_z, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

We may write

$$(\mathbf{R}_{n,z}x, u)_H = (R_z(A_n)x, u)_{H_n} = (x, R_{\bar{z}}(A_n)u)_{H_n} = (x, \mathbf{R}_{n,\bar{z}}u)_H,$$

where $x \in L_N$, $u \in H_{\bar{z}}$, $n = 1, 2$, and therefore

$$(3.15) \quad (\mathbf{R}_{1,z}x, u)_H = (\mathbf{R}_{2,z}x, u)_H, \quad x \in L_N, \quad u \in H_{\bar{z}}.$$

By (3.11) an arbitrary element $y \in L$ can be represented in the following form $y = y_{\bar{z}} + y'$, $y_{\bar{z}} \in H_{\bar{z}}$, $y' \in L_N$. Using (3.13) and (3.15) we obtain

$$(\mathbf{R}_{1,z}x, y)_H = (\mathbf{R}_{1,z}x, y_{\bar{z}} + y')_H = (\mathbf{R}_{2,z}x, y_{\bar{z}} + y')_H = (\mathbf{R}_{2,z}x, y)_H,$$

where $x \in L_N$, $y \in L$. Since $\bar{L} = H$, we obtain

$$(3.16) \quad \mathbf{R}_{1,z}x = \mathbf{R}_{2,z}x, \quad x \in L_N, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

For arbitrary $x \in L$, $x = x_z + x'$, $x_z \in H_z$, $x' \in L_N$, using relations (3.14), (3.16) we get

$$\mathbf{R}_{1,z}x = \mathbf{R}_{1,z}(x_z + x') = \mathbf{R}_{2,z}(x_z + x') = \mathbf{R}_{2,z}x, \quad x \in L, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

and therefore $\mathbf{R}_{1,z} = \mathbf{R}_{2,z}$, $z \in \mathbb{C} \setminus \mathbb{R}$. By (3.1) this means that the corresponding Π -spectral functions coincide. The obtained contradiction completes the proof. \square

We shall use some known important facts about sc-resolvents, see [29]. Let B be an arbitrary Hermitian contraction in a Hilbert space H . Set $\mathcal{D} = D(B)$, $\mathcal{R} = H \ominus \mathcal{D}$. A set of all self-adjoint contractive extensions of B inside H , we denote by $\mathcal{B}_H(B)$. A set of all self-adjoint contractive extensions of B in a Hilbert space $\tilde{H} \supseteq H$, we denote by $\mathcal{B}_{\tilde{H}}(B)$. By Krein's theorem [23], Theorem 2, p. 440, there always exists a self-adjoint extension \hat{B} of the operator B in H with the norm $\|B\|$. Therefore the set $\mathcal{B}_H(B)$ is non-empty. There are the "minimal" element B^μ and the "maximal" element B^M in this set, such that $\mathcal{B}_H(B)$ coincides with the operator segment

$$(3.17) \quad \{\tilde{B} : B^\mu \leq \tilde{B} \leq B^M\}.$$

In the case $B^\mu = B^M$ the set $\mathcal{B}_H(B)$ consists of a unique element. This case is said to be *determinate*. The case $B^\mu \neq B^M$ is called *indeterminate*. The case $B^\mu x \neq B^M x$, $x \in \mathcal{R} \setminus \{0\}$, is said to be *completely indeterminate*. The indeterminate case can be

always reduced to the completely indeterminate. If $\mathcal{R}_0 = \{x \in \mathcal{R} : B^\mu x = B^M x\}$, we may set

$$(3.18) \quad B_e x = Bx, \quad x \in \mathcal{D}; \quad B_e x = B^\mu x, \quad x \in \mathcal{R}_0.$$

The sets of generalized sc-resolvents for B and for B_e coincide ([29], p. 1039).

Elements of $\mathcal{B}_H(B)$ are *canonical* (i.e. inside H) extensions of B and their resolvents are said to be *canonical sc-resolvents* of B . On the other hand, elements of $\tilde{\mathcal{B}}_{\tilde{H}}(B)$ for all possible $\tilde{H} \supseteq H$ generate generalized sc-resolvents of B (here the space \tilde{H} is not fixed). The set of all generalized sc-resolvents we denote by $\mathcal{R}^c(B)$. Set

$$(3.19) \quad C = B^M - B^\mu,$$

$$(3.20) \quad Q_\mu(z) = \left(C^{\frac{1}{2}} R_z^\mu C^{\frac{1}{2}} + E_H \right) \Big|_{\mathcal{R}}, \quad z \in \mathbb{C} \setminus [-1, 1],$$

where $R_z^\mu = (B^\mu - zE_H)^{-1}$.

An operator-valued function $k(z)$ with values in $[\mathcal{R}]$ belongs to the class $R_{\mathcal{R}}[-1, 1]$ if

- 1) $k(z)$ is analytic in $z \in \mathbb{C} \setminus [-1, 1]$ and

$$\frac{\text{Im } k(z)}{\text{Im } z} \leq 0, \quad z \in \mathbb{C} : \text{Im } z \neq 0;$$

- 2) For $z \in \mathbb{R} \setminus [-1, 1]$, $k(z)$ is a self-adjoint non-negative contraction.

Notice that functions from the class $R_{\mathcal{R}}[-1, 1]$ admit a special integral representation, see [29].

THEOREM (3.21). ([29], p. 1053). *Let B be a Hermitian contraction in a Hilbert space H with $D(B) = \mathcal{D}$; $\mathcal{R} = H \ominus \mathcal{D}$. Suppose that for B it takes place the completely indeterminate case and that the corresponding operator C , as an operator in \mathcal{R} , has an inverse in $[\mathcal{R}]$. Then the following equality:*

$$(3.22) \quad \tilde{R}_z^c = R_z^\mu - R_z^\mu C^{\frac{1}{2}} k(z) (E_{\mathcal{R}} + (Q_\mu(z) - E_{\mathcal{R}})k(z))^{-1} C^{\frac{1}{2}} R_z^\mu,$$

where $k(z) \in R_{\mathcal{R}}[-1, 1]$, $\tilde{R}_z^c \in \mathcal{R}^c(B)$, establishes a bijective correspondence between the set $R_{\mathcal{R}}[-1, 1]$ and the set $\mathcal{R}^c(B)$.

Moreover, the canonical resolvents correspond in (3.22) to the constant functions $k(z) \equiv K$, $K \in [0, E_{\mathcal{R}}]$.

Let A be an arbitrary non-negative symmetric operator in a Hilbert space H , $\overline{D(A)} = H$. We are going to obtain a formula for the generalized Π -resolvents of A , by virtue of Theorem (3.21). Set

$$(3.23) \quad T = (E_H - A)(E_H + A)^{-1} = -E_H + 2(E_H + A)^{-1}, \quad D(T) = (A + E_H)D(A).$$

Then

$$(3.24) \quad A = (E_H - T)(E_H + T)^{-1} = -E_H + 2(E_H + T)^{-1}, \quad D(A) = (T + E_H)D(T).$$

The latter transformations were introduced and intensively studied by Krein [23]. The operator T is a Hermitian contraction in H . In fact, for an arbitrary $h \in (A + E_H)f$, $f \in D(A)$ we may write

$$\begin{aligned} \|Th\|_H^2 &= \|(-E_H + 2(E_H + A)^{-1})(A + E_H)f\|_H^2 = \|-Af + f\|_H^2 \\ &= \|Af\|_H^2 + \|f\|_H^2 - 2(Af, f)_H \leq \|Af\|_H^2 + \|f\|_H^2 + 2(Af, f)_H = \|h\|_H^2. \end{aligned}$$

Let $\tilde{A} \supseteq A$ be a non-negative self-adjoint extension of A in a Hilbert space $\tilde{H} \supseteq H$. Then the operator

$$(3.25) \quad \tilde{T} = (E_{\tilde{H}} - \tilde{A})(E_{\tilde{H}} + \tilde{A})^{-1} = -E_{\tilde{H}} + 2(E_{\tilde{H}} + \tilde{A})^{-1}, \quad D(\tilde{T}) = (\tilde{A} + E_{\tilde{H}})D(\tilde{A}),$$

is a self-adjoint contraction $\tilde{T} \supseteq T$ in \tilde{H} , and

$$(3.26) \quad \tilde{A} = (E_{\tilde{H}} - \tilde{T})(E_{\tilde{H}} + \tilde{T})^{-1} = -E_{\tilde{H}} + 2(E_{\tilde{H}} + \tilde{T})^{-1}, \quad D(\tilde{A}) = (\tilde{T} + E_{\tilde{H}})D(\tilde{T}).$$

Consider the following fractional linear transformation:

$$(3.27) \quad z = \frac{1-\lambda}{1+\lambda} = -1 + 2\frac{1}{1+\lambda}; \quad \lambda = \frac{1-z}{1+z} = -1 + 2\frac{1}{1+z}.$$

Choose an arbitrary $z \in \mathbb{C} \setminus \mathbb{R}$ and set $\lambda := \frac{1-z}{1+z}$. Observe that $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then

$$\begin{aligned} R_z(\tilde{T}) &= (\tilde{T} - zE_{\tilde{H}})^{-1} = \left(-E_{\tilde{H}} + 2(E_{\tilde{H}} + \tilde{A})^{-1} - \frac{1-\lambda}{1+\lambda}E_{\tilde{H}} \right)^{-1} \\ &= \left(\frac{(-2)}{1+\lambda}(E_{\tilde{H}} + \tilde{A})(E_{\tilde{H}} + \tilde{A})^{-1} + 2(E_{\tilde{H}} + \tilde{A})^{-1} \right)^{-1} \\ &= \left(\left(\frac{2\lambda}{1+\lambda}E_{\tilde{H}} - \frac{2}{1+\lambda}\tilde{A} \right) (E_{\tilde{H}} + \tilde{A})^{-1} \right)^{-1} \\ &= -\frac{\lambda+1}{2} \left((\tilde{A} - \lambda E_{\tilde{H}})(E_{\tilde{H}} + \tilde{A})^{-1} \right)^{-1} = -\frac{\lambda+1}{2} (E_{\tilde{H}} + \tilde{A})(\tilde{A} - \lambda E_{\tilde{H}})^{-1} \\ &= -\frac{(\lambda+1)^2}{2} (\tilde{A} - \lambda E_{\tilde{H}})^{-1} - \frac{\lambda+1}{2} E_{\tilde{H}} = -\frac{(\lambda+1)^2}{2} R_\lambda(\tilde{A}) - \frac{\lambda+1}{2} E_{\tilde{H}}. \end{aligned}$$

Therefore

$$(3.28) \quad R_\lambda(\tilde{A}) = -\frac{2}{(\lambda+1)^2} R_{\frac{1-\lambda}{1+\lambda}}(\tilde{T}) - \frac{1}{\lambda+1} E_{\tilde{H}}, \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Applying the orthogonal projection on H , we get

$$(3.29) \quad \mathbf{R}_\lambda(A) = -\frac{2}{(\lambda+1)^2} \mathbf{R}_{\frac{1-\lambda}{1+\lambda}}(T) - \frac{1}{\lambda+1} E_H, \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Here $\mathbf{R}_\lambda(A)$ is the generalized Π -resolvent corresponding to \tilde{A} , and $\mathbf{R}_z(T)$ is the generalized sc -resolvent corresponding to \tilde{T} . Thus, an arbitrary generalized Π -resolvent of A can be constructed by a generalized sc -resolvent of T by relation (3.29).

On the other hand, choose an arbitrary sc -resolvent $\mathbf{R}'_z(T)$ of T . It corresponds to a self-adjoint contractive extension $\hat{T} \supseteq T$ in a Hilbert space $\hat{H} \supseteq H$. Observe that

$$\text{Ker}(E_{\hat{H}} + \hat{T}) \perp R(E_{\hat{H}} + \hat{T}) \supseteq R(E_H + T) = D(A),$$

and therefore $\text{Ker}(E_{\hat{H}} + \hat{T}) \perp H$. We may assume that $H_1 := \text{Ker}(E_{\hat{H}} + \hat{T}) = \{0\}$, since in the opposite case one may consider the operator \hat{T} restricted to $\hat{H} \ominus H_1 \supseteq H$. Then we set

$$(3.30) \quad \hat{A} = (E_{\hat{H}} - \hat{T})(E_{\hat{H}} + \hat{T})^{-1} = -E_{\hat{H}} + 2(E_{\hat{H}} + \hat{T})^{-1}, \quad D(\hat{A}) = (\hat{T} + E_{\hat{H}})D(\hat{T}).$$

The operator \hat{A} is densely defined since $\hat{A} \supseteq A$, and it is self-adjoint. For an arbitrary $u \in D(\hat{T})$ we may write

$$(\hat{A}(\hat{T} + E_{\hat{H}})u, (\hat{T} + E_{\hat{H}})u)_{\hat{H}} = (-\hat{T}u + u, \hat{T}u + u)_{\hat{H}} = \|u\|_{\hat{H}}^2 - \|\hat{T}u\|_{\hat{H}}^2 \geq 0.$$

Thus, the operator \widehat{A} is non-negative. Observe that

$$(3.31) \quad \widehat{T} = (E_{\widehat{H}} - \widehat{A})(E_{\widehat{H}} + \widehat{A})^{-1} = -E_{\widehat{H}} + 2(E_{\widehat{H}} + \widehat{A})^{-1}.$$

Repeating the considerations after relation (3.27), we obtain that

$$(3.32) \quad \mathbf{R}'_{\lambda}(A) = -\frac{2}{(\lambda + 1)^2} \mathbf{R}'_{\frac{1-\lambda}{1+\lambda}}(T) - \frac{1}{\lambda + 1} E_H, \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R},$$

gives a generalized Π -resolvent of A (corresponding to \widehat{A}).

Consequently, the relation (3.29) establishes a bijective correspondence between the set of all *sc*-resolvents of T and the set of all Π -resolvents of A . It is not hard to see that the canonical *sc*-resolvents are related to the canonical Π -resolvents.

For the operator A it takes place a **completely indeterminate case**, if for the corresponding operator T it takes place the completely indeterminate case [28].

It is known that all self-adjoint contractive extensions of T are extensions of the extended operator T_e defined by (3.18), [29], Theorem 1.4. Set

$$(3.33) \quad A_e = (E_H - T_e)(E_H + T_e)^{-1} = -E_H + 2(E_H + T_e)^{-1}, \quad D(A_e) = (T_e + E_H)D(T_e).$$

It is easily seen that the above operator \widetilde{A} is an extension of A_e . Therefore the sets of generalized Π -resolvents for A and for A_e coincide.

THEOREM (3.34). *Let A be a non-negative symmetric operator in a Hilbert space H , $\overline{D(A)} = H$. Suppose that for A it takes place the completely indeterminate case. Let T be given by (3.23); $D = D(T)$, $\mathcal{R} = H \ominus D$. Suppose that the corresponding operator $C = T^M - T^\mu$, as an operator in \mathcal{R} , has an inverse in $[\mathcal{R}]$. Then the following equality:*

$$(3.35) \quad \begin{aligned} \mathbf{R}_{\lambda}(A) = & -\frac{2}{(\lambda + 1)^2} R_{\frac{1-\lambda}{1+\lambda}}^{\mu} - \frac{1}{\lambda + 1} E_H \\ & + \frac{2}{(\lambda + 1)^2} R_{\frac{1-\lambda}{1+\lambda}}^{\mu} C^{\frac{1}{2}} \mathbf{k}(\lambda) (E_{\mathcal{R}} + (\mathbf{Q}_{\mu}(\lambda) - E_{\mathcal{R}}) \mathbf{k}(\lambda))^{-1} C^{\frac{1}{2}} R_{\frac{1-\lambda}{1+\lambda}}^{\mu}, \end{aligned}$$

where $\mathbf{Q}_{\mu}(\lambda) = \mathbf{Q}_{\mu}(\frac{1-\lambda}{1+\lambda})$, $\mathbf{k}(\lambda) = k(\frac{1-\lambda}{1+\lambda})$; $k(\cdot) \in \mathcal{R}_{\mathcal{R}}[-1, 1]$, establishes a bijective correspondence between the set $\mathcal{R}_{\mathcal{R}}[-1, 1]$ and the set of all generalized Π -resolvents of A . Here \mathbf{Q}_{μ} is defined by (3.20) for T , $R_z^{\mu} = (T^{\mu} - zE_H)^{-1}$, and $\mathbf{R}_{\lambda}(A)$ is a generalized Π -resolvent of A .

Moreover, the canonical resolvents correspond in (3.35) to the constant functions $k(z) \equiv K$, $K \in [0, E_{\mathcal{R}}]$.

Proof. It follows directly from the preceding considerations, formula (3.29) and by applying Theorem (3.21). □

Let the strong matrix Stieltjes moment problem be given and (1.4) be satisfied. Consider an arbitrary Hilbert space H and a sequence of elements $\{x_n\}_{n \in \mathbb{Z}}$ in H , such that relation (2.5) holds. Let $A = \overline{A_0}$, where the operator A_0 is defined by (2.7). Denote $L_N = \text{Lin}\{x_k\}_{k=0}^{N-1}$. Define a linear transformation G from \mathbb{C}^N onto L_N by the following relation:

$$(3.36) \quad G\tilde{u}_k = x_k, \quad k = 0, 1, \dots, N - 1,$$

where $\tilde{u}_k = (\delta_{0,k}, \delta_{1,k}, \dots, \delta_{N-1,k})$.

THEOREM (3.37). Let the strong matrix Stieltjes moment problem (1.1) be given and (1.4) be satisfied. Let $\{x_n\}_{n \in \mathbb{Z}}$ be a sequence of elements of a Hilbert space H such that relation (2.5) holds. Let $A = \overline{A_0}$, where the operator A_0 is defined by relation (2.7). Let $T = -E_H + 2(E_H + A)^{-1}$. The following statements are true:

- 1) If $T^\mu = T^M$, then the moment problem (1.1) has a unique solution. This solution is given by

$$(3.38) \quad M(t) = (m_{j,n}(t))_{j,n=0}^{N-1}, \quad m_{j,n}(t) = (E_t^\mu x_j, x_n)_H, \quad 0 \leq j, n \leq N-1,$$

where $\{E_t^\mu\}$ is the left-continuous orthogonal resolution of unity of the operator $A^\mu = -E_H + 2(E_H + T^\mu)^{-1}$.

- 2) If $T^\mu \neq T^M$, define the extended operator T_e by (3.18); $\mathcal{R}_e = H \ominus D(T_e)$, $C = T^M - T^\mu$, and $R_z^\mu = (T^\mu - zE_H)^{-1}$, $Q_{\mu,e}(z) = \left(C^{\frac{1}{2}} R_z^\mu C^{\frac{1}{2}} + E_H \right) \Big|_{\mathcal{R}_e}$, $z \in \mathbb{C} \setminus [-1, 1]$. An arbitrary solution $M(\cdot)$ of the moment problem can be found by the Stieltjes-Perron inversion formula from the following relation

$$(3.39) \quad \int_{\mathbb{R}_+} \frac{1}{t-z} dM^T(t) = A(z) - C(z) \mathbf{k}(z) (E_{\mathcal{R}_e} + D(z) \mathbf{k}(z))^{-1} \mathcal{B}(z),$$

where $\mathbf{k}(\lambda) = k\left(\frac{1-\lambda}{1+\lambda}\right)$, $k(z) \in \mathcal{R}_{\mathcal{R}_e}[-1, 1]$, and on the right-hand side one means the matrix of the corresponding operator in \mathbb{C}^N . Here $A(z), \mathcal{B}(z), C(z), D(z)$ are analytic operator-valued functions given by

$$(3.40) \quad A(z) = -\frac{2}{(\lambda+1)^2} G^* R_{\frac{1-\lambda}{1+\lambda}}^\mu G - \frac{1}{\lambda+1} G^* G : \mathbb{C}^N \rightarrow \mathbb{C}^N,$$

$$(3.41) \quad \mathcal{B}(z) = C^{\frac{1}{2}} R_{\frac{1-\lambda}{1+\lambda}}^\mu G : \mathbb{C}^N \rightarrow \mathcal{R}_e,$$

$$(3.42) \quad C(z) = \frac{2}{(\lambda+1)^2} G^* R_{\frac{1-\lambda}{1+\lambda}}^\mu C^{\frac{1}{2}} : \mathcal{R}_e \rightarrow \mathbb{C}^N,$$

$$(3.43) \quad D(z) = Q_{\mu,e} \left(\frac{1-\lambda}{1+\lambda} \right) - E_{\mathcal{R}_e} : \mathcal{R}_e \rightarrow \mathcal{R}_e.$$

Moreover, the correspondence between all solutions of the moment problem and $k(z) \in \mathcal{R}_{\mathcal{R}_e}[-1, 1]$ is bijective.

Proof. Consider the case 1). In this case all self-adjoint contractions $\tilde{T} \supseteq T$ in a Hilbert space $\tilde{H} \supseteq H$ coincide on H with T^μ , see [29], p. 1039. Thus, the corresponding sc-spectral functions are spectral functions of the self-adjoint operator T^μ , as well. However, a self-adjoint operator has a unique (normalized) spectral function. Thus, a set of sc-spectral functions of T consists of a unique element. Therefore the set of Π -resolvents of A consists of a unique element, as well. This element is the spectral function of A^μ .

Consider the case 2). By Theorem (3.2) and relation (3.1) it follows that an arbitrary solution $M(t) = (m_{j,n}(t))_{j,n=0}^{N-1}$ of the moment problem (1.1) can be found from the following relation:

$$\int_{\mathbb{R}_+} \frac{1}{t-z} dm_{j,n}(t) = (\mathbf{R}_z x_j, x_n)_H, \quad 0 \leq j, n \leq N-1; \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where \mathbf{R}_z is a generalized Π -resolvent of the operator A . Moreover, the correspondence between the set of all generalized Π -resolvents of A (which is equal to the set of all generalized Π -resolvents of A_e) and the set of all solutions of the moment problem is bijective. Notice that $T^\mu = T_e^\mu$ and $T^M = T_e^M$. The operator $(T^M - T^\mu)$, as an operator in \mathcal{R}_e , has an inverse. Since \mathcal{R}_e is finite-dimensional, the inverse is bounded. By Theorem (3.34) (applied to the operator A_e) we may rewrite the latter relation in the following form:

$$\int_{\mathbb{R}_+} \frac{1}{t-z} dm_{j,n}(t) = \left(\left\{ -\frac{2}{(\lambda+1)^2} R_{\frac{1-\lambda}{1+\lambda}}^\mu - \frac{1}{\lambda+1} E_H + \frac{2}{(\lambda+1)^2} R_{\frac{1-\lambda}{1+\lambda}}^\mu C^{\frac{1}{2}} \mathbf{k}(\lambda) (E_{\mathcal{R}_e} + (\mathbf{Q}_{\mu,e}(\lambda) - E_{\mathcal{R}_e}) \mathbf{k}(\lambda))^{-1} C^{\frac{1}{2}} R_{\frac{1-\lambda}{1+\lambda}}^\mu \right\} x_j, x_n \right)_H,$$

where $\mathbf{k}(\lambda) = k\left(\frac{1-\lambda}{1+\lambda}\right)$, $k(z) \in R_{\mathcal{R}_e}[-1, 1]$, $\mathbf{Q}_{\mu,e}(\lambda) = \mathbf{Q}_{\mu,e}\left(\frac{1-\lambda}{1+\lambda}\right)$. Then

$$\int_{\mathbb{R}_+} \frac{1}{t-z} dm_{j,n}(t) = \left(\left\{ -\frac{2}{(\lambda+1)^2} G^* R_{\frac{1-\lambda}{1+\lambda}}^\mu G - \frac{1}{\lambda+1} G^* G + \frac{2}{(\lambda+1)^2} G^* * R_{\frac{1-\lambda}{1+\lambda}}^\mu C^{\frac{1}{2}} \mathbf{k}(\lambda) (E_{\mathcal{R}_e} + (\mathbf{Q}_{\mu,e}(\lambda) - E_{\mathcal{R}_e}) \mathbf{k}(\lambda))^{-1} C^{\frac{1}{2}} R_{\frac{1-\lambda}{1+\lambda}}^\mu G \right\} u_j, u_n \right)_{\mathbb{C}^N}.$$

Introducing functions $\mathcal{A}(z), \mathcal{B}(z), \mathcal{C}(z), \mathcal{D}(z)$ by formulas (3.40)-(3.43) one easily obtains relation (3.39). □

THEOREM (3.44). *Let the strong matrix Stieltjes moment problem (1.1) be given and (1.4) be satisfied. Let $\{x_n\}_{n \in \mathbb{Z}}$ be a sequence of elements of a Hilbert space H such that relation (2.5) holds. Let $A = \overline{A_0}$, where the operator A_0 is defined by relation (2.7). The moment problem is determinate if and only if $T^\mu = T^M$, where T^μ, T^M are the extremal extensions of the operator $T = -E_H + 2(E_H + A)^{-1}$.*

Proof. The sufficiency follows from Statement 1 of Theorem (3.37). The necessity follows from Statement 2 of Theorem (3.37), if we take into account that the class $R_{\mathcal{R}_e}([-1, 1])$, where $\dim \mathcal{R}_e > 0$, has at least two different elements. In fact, from the definition of the class $R_{\mathcal{R}_e}([-1, 1])$ it follows that $k_1(z) \equiv 0$, and $k_1(z) \equiv E_{\mathcal{R}_e}$, belong to $R_{\mathcal{R}_e}([-1, 1])$. □

Example (3.45). Consider the moment problem (1.1) with $N = 2$ and

$$S_n = \begin{pmatrix} 1 & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & 1 \end{pmatrix}, \quad n \in \mathbb{Z}.$$

In this case we have

$$\Gamma = (S_{i+j})_{i,j=-\infty}^\infty = (\gamma_{n,m})_{n,m=-\infty}^\infty,$$

where

$$\gamma_{2k,2l} = \gamma_{2k+1,2l+1} = 1, \quad \gamma_{2k,2l+1} = \gamma_{2k+1,2l} = \frac{3}{\sqrt{10}}, \quad k, l \in \mathbb{Z}.$$

Consider the space \mathbb{C}^2 and elements $u_0, u_1 \in \mathbb{C}^2$:

$$u_0 = \frac{1}{\sqrt{2}}(1, 1), \quad u_1 = \frac{1}{\sqrt{5}}(1, 2).$$

Set

$$x_{2k} = u_0, \quad x_{2k+1} = u_1, \quad k \in \mathbb{Z}.$$

Then relation (2.5) holds. Define by (2.7) the operator A_0 . In this case $A = A_0 = E_{\mathbb{C}^2}$. Therefore the operators A and $T = -E_H + 2(E_H + A)^{-1}$ are self-adjoint and

have unique spectral functions. Hence, $T^M = T^\mu$, and by Theorem (3.44) we conclude that the moment problem has a unique solution. By Theorem (3.2) it has the following form

$$M(\lambda) = (m_{k,j}(\lambda))_{k,j=0}^{N-1}, \quad m_{k,j}(\lambda) = (\mathbf{E}_\lambda x_k, x_j)_H,$$

where \mathbf{E}_λ is the left-continuous spectral function of the operator $E_{\mathbb{C}^2}$. Consequently, the matrix function $M(t)$ is equal to 0, for $t \leq 1$, and $M(t) = \begin{pmatrix} 1 & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & 1 \end{pmatrix}$, for $t > 1$.

Received February 22, 2012

Final version received September 10, 2012

ABDON E. CHOQUE RIVERO
 INSTITUTO DE FÍSICA Y MATEMÁTICAS,
 UNIVERSIDAD MICHOACANA DE SAN NICOLÁS DE HIDALGO,
 C.U. EDIFIO C3A, C.P. 58040,
 MORELIA, MICH., MÉXICO
 abdon@ifm.umich.mx

SERGEY M. ZAGORODNYUK
 SCHOOL OF MATHEMATICS AND MECHANICS,
 KARAZIN KHARKIV NATIONAL UNIVERSITY,
 KHARKIV 61077, UKRAINE
 Sergey.M.Zagorodnyuk@univer.kharkov.ua

REFERENCES

- [1] V. M. ADAMYAN AND I. M. TKACHENKO, *Solution of the truncated matrix Hamburger moment problem according to M. G. Krein in Operator Theory and Related Topics, Vol. II*, (Odessa, 1997), Oper. Theory Adv. Appl., Birkhäuser, Basel, (2000), 33–51.
- [2] V. M. ADAMYAN AND I. M. TKACHENKO, *General solution of the Stieltjes truncated matrix moment problem*, Oper. Theory Adv. Appl., Birkhäuser, Basel, **163**, (2006), 1–22.
- [3] N. I. AKHIEZER AND I. M. GLAZMAN, *Theory of linear operators in a Hilbert space*, (Russian) Gos. izdat. teh.-teor. lit., Moscow, Leningrad, 1950.
- [4] N. I. AKHIEZER, *Classical moment problem*, (Russian) Fizmatlit, Moscow, 1961.
- [5] T. ANDO, *Truncated moment problems for operators*, Acta Sci. Math. (Szeged), **31**, (1970), 319–334.
- [6] YU. M. BEREZANSKY AND M. E., DUDKIN, *The strong Hamburger moment problem and related direct and inverse spectral problems for block Jacobi-Laurent matrices*, Methods Funct. Anal. Topology, **16**, 3 (2010), 203–241.
- [7] C. M. BONAN-HAMADA AND W. B. JONES, *Para-orthogonal Laurent polynomials and the strong Stieltjes moment problem*, J. Comput. Appl. Math. **105**, 1-2 (1999), 175–185.
- [8] M. E. CHUMAKIN, *On a class of the generalized resolvents of an isometric operator*, (Russian) Uchenye zapiski Ul'yanovskogo pedagog. instituta, **20**, 4 (1966), 361–372.
- [9] M. E. CHUMAKIN, *Generalized resolvents of isometric operators*, Sibir. Mat. Zh. **8** (1967), 876–892. Engl. Transl.: Siberian Math. J. **8** (1967), 666–678.
- [10] V. A. DERKACH AND M. M. MALAMUD, *Generalized resolvents and the boundary value problems for Hermitian operators with gaps*, Journal of Functional Analysis, **95**, 1 (1991), 1–95.
- [11] V. G. ERSHOV, *The truncated matrix power moment problem*, (Russian) Izvestiya Vysshyyh Uchebnyh Zavedeniy, Matematika **76**, 9 (1968), 36–43.
- [12] I. M. GELFAND AND M. A. NAIMARK, *On the inclusion of a normed ring into the ring of operators in a Hilbert space*, (Russian) Matem. Sbornik, **12**, (1943), 197–213.
- [13] G. M. ILMUSHKIN, *On solutions of the operator power moment problem*, (Russian) Funktsionalnij analiz (Ulyanovsk), **14** (1980), 74–79.

- [14] G. M. ILMUSHKIN AND E. L. ALEKSANDROV, *On solutions of the operator power moment problem*, (Russian) *Izvestiya vuzov, Matematika*, **3**, (1989), 18–24.
- [15] G. M. ILMUSHKIN AND A. B. TURITSIN, *The truncated operator trigonometric moment problem*, (Russian) *Izvestiya Vysshyyh Uchebnyh Zavedeniy, Matematika*, **242**, 7 (1982), 17–21.
- [16] O. T. ININ, *The truncated matrix trigonometric moment problem*, (Russian) *Izvestiya Vysshyyh Uchebnyh Zavedeniy, Matematika*, **84**, 5 (1969), 49–57.
- [17] W. B. JONES, W. J. THRON, AND H. WADELAND, *A strong Stieltjes moment problem*, *Trans. Amer. Math. Soc.*, **261**, 2 (1980), 503–528.
- [18] W. B. JONES AND O. NJÅSTAD, *Orthogonal Laurent polynomials and strong moment theory: a survey*, *J. Comput. Appl. Math.*, **105**, 1-2 (1999), 51–91.
- [19] I. S. KATS AND A. A. NUDELMAN, *Strong Stieltjes moment problem*, *St. Petersburg Math. J.*, **8**, 6 (1997), 931–950; Russian original: *Algebra i analiz* **8**, 6 (1996).
- [20] A. KHEIFETS, *Hamburger moment problem: Parseval equality and A-singularity*, *Journal of Functional Analysis*, **141**, (1996), 374–420.
- [21] M. G. KREIN, *On a general method of decomposing Hermite-positive nuclei into elementary products*, (Russian) *Doklady Acad. Sci. URSR*, **53**, (1946), 3–6.
- [22] M. KREIN, *Concerning the resolvents of an Hermitian operator with the deficiency-index (m, m)* , *Doklady Akad. Nauk SSSR*, **LII**, 8 (1946), 651–654.
- [23] M. G. KREIN, *The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications*, (Russian) *Matematicheskiy sbornik*, **20 (62)**, 3 (1947), 431–495.
- [24] M. G. KREIN, *On Hermitian operators with directed functionals*, (Ukrainian) *Akad. Nauk Ukrain. RSR. Zbirnyk Prac' Inst. Mat.*, 10 (1948), 83–106.
- [25] M. G. KREIN, *The fundamental propositions of the theory of representations of Hermitian operators with deficiency index (m, m)* , (Russian) *Ukr. mat. zhurnal*, **1**, 2 (1949), 3–66.
- [26] M. G. KREIN, *Infinite J -matrices and the matrix moment problem*, (Russian) *Doklady AN SSSR*, **LXIX**, 2 (1949), 125–128.
- [27] M. G. KREIN AND M. A. KRASNOSELSKIY, *Basic theorems on an extension of Hermitian operators and some their applications to the theory of orthogonal polynomials and the moment problem*, (Russian) *Uspehi matem. nauk T.2* **3 (19)**, (1947), 60–106.
- [28] M. G. KREIN AND I. E. OVCHARENKO, *On generalized resolvents and resolvent matrices of positive Hermitian operators*, (Russian) *Doklady AN SSSR*, **231**, 5 (1976), 1063–1066.
- [29] M. G. KREIN AND I. E. OVCHARENKO, *On Q -functions and sc -resolvents of nondensely defined Hermitian contractions*, (Russian) *Sibirskiy matem. zhurnal*, **XVIII**, 5 (1977), 1032–1056.
- [30] L. N. LUKS, *On a solution of the matrix truncated power moment problem on a segment and the half-axis*, (Russian) *Funktsionalniy analiz (Ulyanovsk)*, **7** (1976), 116–124.
- [31] M. M. MALAMUD AND S. M. MALAMUD, *Operator measures in a Hilbert space*, (Russian) *Algebra i analiz*, **15**, 3 (2003), 1–52.
- [32] M. A. NEUMARK, *Spectral functions of a symmetric operator*, (Russian) *Izvestiya AN SSSR*, **4** (1940), 277–318.
- [33] M. A. NEUMARK, *On spectral functions of a symmetric operator*, *Izvestiya AN SSSR*, **7**, (1943), 285–296. (Russian).
- [34] O. NJÅSTAD, *Extremal solutions of the strong Stieltjes moment problem*, *J. of Computational and Applied Mathematics*, **65**, 1-3 (1995), 309–318.
- [35] O. NJÅSTAD, *Solutions of the strong Stieltjes moment problem*, *Methods and Applications of Analysis*, **2**, 3 (1995), 320–347.
- [36] O. NJÅSTAD, *Nevanlinna matrices for the strong Stieltjes moment problem*, *J. of Computational and Applied Mathematics*, **99**, 1-2 (1998), 311–318.
- [37] O. NJÅSTAD, *Lens-shaped regions for Strong Stieltjes moment problems*, *Methods and Applications of Analysis*, **6**, 2 (1999), 195–208.
- [38] A. A. NUDELMAN, *Extension of the Stieltjes moment sequences to the left and related problems of the spectral theory of inhomogeneous string*, *Ukrainian Mathematical Journal*, **59**, 6 (2007), 894–906.
- [39] K. K. SIMONOV, *Strong truncated matrix moment problem of Hamburger*, *Sarajevo Journal of Mathematics*, **2(15)**, (2006), 181–204.
- [40] K. K. SIMONOV, *Strong matrix moment problem of Hamburger*, *Methods Funct. Anal. Topology*, **12**, 2 (2006), 183–196.

- [41] B. SZÖKEFALVI-NAGY AND A. KORANYI, *Relations d'un problème de Nevanlinna et Pick avec la théorie de l'espace hilbertien*, Acta Math. Acad. Sci. Hungar., **7**, (1957), 295–302.
- [42] B. SZÖKEFALVI-NAGY AND A. KORANYI, *Operatortheoretische Behandlung und Verallgemeinerung eines Problemkreises in der komplexen Funktionentheorie*, Acta Math., **100**, (1958), 171–202.
- [43] S. M. ZAGORODNYUK, *On the strong matrix Hamburger moment problem*, Ukrainian Mathematical Journal, **62**, 4 (2010), 537–551.
- [44] S. M. ZAGORODNYUK, *The truncated matrix trigonometric moment problem: the operator approach*, (Russian) Ukrainian Mathematical Journal, **63**, 6 (2011), 786–797.
- [45] S. M. ZAGORODNYUK, *The Nevanlinna-type formula for the truncated matrix trigonometric moment problem*, (Russian) Ukrainian Mathematical Journal, **64**, 8 (2012), 1053–1066.
- [46] S. M. ZAGORODNYUK, *The matrix Stieltjes moment problem: a description of all solutions*, New York J. Math., **18**, (2012), 479–497.

THE ALMOST SURE CENTRAL LIMIT THEOREM FOR RANDOMLY INDEXED SUMS OF ASSOCIATED RANDOM VARIABLES

MARCIN DUDZIŃSKI AND PRZEMYSŁAW GÓRKA

ABSTRACT. Suppose that $\{X_n\}_{n=1}^\infty$ is a sequence of associated, zero mean random variables, the sums $S_n := \sum_{i=1}^n X_i$ have bounded, continuous densities, and $\{N_n\}_{n=1}^\infty$ denotes a sequence of independent, random indexes, independent of $\{X_n\}_{n=1}^\infty$. We prove the almost sure central limit theorem for suitably normalized, randomly indexed sums S_{N_n} . Some example of application of our result is also given.

1. Introduction

Starting with the discovery of the almost sure central limit theorem (ASCLT) by Brosamler [5] and Schatte [27], a vast literature on this subject has developed over the past two decades. The early results concerning the ASCLT dealt mostly with partial sums of i.i.d. r.v.'s - among the celebrated papers of Brosamler [5] and Schatte [27], we also refer to Lacey and Philipp [20] and Fisher [17] in this context. For independent, but not necessarily identically distributed, r.v.'s a general result for sums has been proved by Berkes and Dehling [3]. The mentioned results have been later generalized for sums of some weakly dependent r.v.'s - we cite in this context the papers of: Peligrad and Shao [25], Matuła [22], [23], Mielniczuk [24], Rodzik and Rychlik [26], and Dudziński [10]. Except for the ASCLTs for sums, the ASCLTs for some other functions of r.v.'s have been studied as well. Namely, in Fahrner and Stadtmuller [14] and Cheng et al. [7], the ASCLTs for maxima of i.i.d. r.v.'s have been proved. On the other hand, the ASCLTs for maxima of some dependent, stationary Gaussian sequences have been obtained by Csaki and Gonchigdanzan [8] and Dudziński [9], while the ASCLT for maxima of some dependent, but not necessarily stationary, Gaussian sequences has been established by Chen and Lin [6]. Furthermore, the ASCLTs for some order statistics have also been proved - see the papers of Stadtmuller [28] and Dudziński [12]. In addition, the ASCLT in the joint version for maxima and sums of some stationary Gaussian sequences has been established as well - we refer to Dudziński [11] in this context. Some other works, which are also worthwhile to mention in this place are: the paper of Berkes and Csaki [2], where several interesting proofs of the ASCLTs for some functions of independent r.v.'s have been given, the article of Gonchigdanzan and Rempała [18], where the ASCLT for the product of partial sums has been established, and the papers of Fazekas and Rychlik [16], [15], which are devoted to the ASCLT for random fields and to the functional ASCLT, respectively.

2010 *Mathematics Subject Classification*: 60F15, 60F05.

Keywords and phrases: almost sure central limit theorem, randomly indexed sums, associated sequences.

The above mentioned papers relate to the ASCLTs for non-randomly indexed sequences. There are not many works devoted to the ASCLTs for randomly indexed functions of r.v.'s. A notable exception is the paper of Krajka and Wasiura [19], where the ASCLT for suitably normalized, randomly indexed sums $S_{N_n} = \sum_{i=1}^{N_n} X_i$ of i.i.d. r.v.'s has been established (N_n denotes, here and in further considerations, a certain sequence of positive, integer-valued r.v.'s). We shall pursue this direction and prove that this type of theorem occurs for randomly indexed sums of associated r.v.'s. Our approach differs from the one in [19] that: firstly, we consider the random version of the ASCLT for sums of some dependent r.v.'s, and secondly, our assumptions on the sequence of random indexes $\{N_n\}$ are milder and more general, since we omit the restrictive condition that $N_n/l_n \xrightarrow{a.s.} \infty$ for some sequence of real numbers $\{l_n\}$.

Let us recall that (cf. [13]) $\{X_n\}_{n=1}^\infty$ is a sequence of associated r.v.'s, if for every finite subcollection $X_{n_1}, X_{n_2}, \dots, X_{n_k}$ and any coordinatewise, nondecreasing functions $f, g: \mathbb{R}^k \rightarrow \mathbb{R}$, the inequality

$$\text{Cov}(f(X_{n_1}, X_{n_2}, \dots, X_{n_k}), g(X_{n_1}, X_{n_2}, \dots, X_{n_k})) \geq 0$$

holds, whenever the given covariance is defined.

Associated processes belong to the class of weakly dependent r.v.'s. They play a significant role in mathematical physics and statistics. The ASCLTs for non-randomly indexed sums of associated r.v.'s have been intensively studied by Peligrad and Shao [25] and Matula [22], [23].

Except for the current Section, our paper consists of the four other Sections. It is organized as follows. In Section 2, we present our main result. In Section 3, we state and prove some auxiliary result, we make an extensive use of in the proof of the our main result. Section 4 contains the proof of the main result, while an example of its application is shown in Section 5.

Notations

Throughout the paper $\{X_n\}_{n=1}^\infty$ is a sequence of associated, zero mean r.v.'s and:

$$u(n) := \sup_{k \in \mathbb{N}} \sum_{j: |j-k| \geq n} \text{Cov}(X_j, X_k), \quad S_n := \sum_{i=1}^n X_i, \quad \sigma_n^2 := ES_n^2.$$

Furthermore, $f(j, n) \ll h(j, n)$ denotes that $f(j, n) \leq C \cdot h(j, n)$ for all sufficiently large j, n and some absolute constant $C > 0$ (i.e., $f(j, n) = \mathcal{O}(h(j, n))$), $I(A)$ stands for the indicator function of the set A and $a \wedge b := \min(a, b)$.

2. Main result

Our aim is to prove the following ASCLT for randomly indexed sums of some associated r.v.'s.

THEOREM (2.1). *Let $\{X_n\}_{n=1}^\infty$ be a sequence of associated, zero mean r.v.'s, such that the sums $S_n := X_1 + X_2 + \dots + X_n$, $n \in \mathbb{N}$, have bounded, continuous densities. Suppose that:*

$$(2.2) \quad u(n) \ll e^{-\lambda n} \quad \text{for some } \lambda > 0,$$

$$(2.3) \quad \inf_{n \in \mathbb{N}} \sigma_n^2/n > 0, \quad \sup_{n \in \mathbb{N}} E|X_n|^3 < \infty.$$

Assume moreover that $\{N_n\}_{n=1}^\infty$ is a sequence of independent random indexes, independent of the sequence $\{X_n\}_{n=1}^\infty$. In addition, suppose that, there exists $0 < \mu < 1$, such that:

$$(2.4) \quad \sum_{j=1}^n \frac{1}{j} E \left\{ \left(\left(\sqrt[6]{\frac{N_j N_n}{jn}} \wedge 1 \right) \sqrt{\frac{N_j}{N_n} + \frac{(\log N_n)^2}{\sqrt{N_n}}} \right) I(N_j < N_n) \right\} \ll (\log n)^{1-\mu},$$

$$(2.5) \quad \sum_{j=1}^n \frac{1}{j} E \left\{ \left(\left(\sqrt[6]{\frac{N_j N_n}{jn}} \wedge 1 \right) \sqrt{\frac{N_n}{N_j} + \frac{(\log N_j)^2}{\sqrt{N_j}}} \right) I(N_j > N_n) \right\} \ll (\log n)^{1-\mu},$$

$$(2.6) \quad \sum_{j=1}^n \frac{1}{j} P(N_j = N_n) \ll (\log n)^{1-\mu}.$$

Then, we have

$$(2.7) \quad P \left\{ \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} \left(I \left(\frac{S_{N_n}}{\sqrt{\frac{n}{N_n}} \sigma_{N_n}} \leq x \right) - \Phi \left(x \sqrt{\frac{n}{N_n}} \right) \right) = 0 \right\} = 1 \text{ for any } x \in \mathbb{R}.$$

3. Auxiliary result

We need the following lemma for the proof of Theorem (2.1).

LEMMA (3.1). *Let*

$$g_{jn}(k_j, k_n, x) := E \left\{ I \left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}} \sigma_{k_j}} \leq x \right) - \Phi \left(x \sqrt{\frac{j}{k_j}} \right) \right\} \left\{ I \left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n}} \sigma_{k_n}} \leq x \right) - \Phi \left(x \sqrt{\frac{n}{k_n}} \right) \right\}.$$

Then, under the assumptions of Theorem (2.1) on the sequence $\{X_n\}_{n=1}^\infty$, we have that, there exists an absolute constant C , such that:

(i) *If $k_j < k_n$, then*

$$(3.2) \quad |g_{jn}(k_j, k_n, x)| \leq C \left\{ \left(\sqrt[6]{\frac{k_j k_n}{jn}} \wedge 1 \right) \sqrt{\frac{k_j}{k_n} + \frac{(\log k_n)^2}{\sqrt{k_n}}} \right\} \text{ for any } x \in \mathbb{R},$$

(ii) *If $k_j > k_n$, then*

$$(3.3) \quad |g_{jn}(k_j, k_n, x)| \leq C \left\{ \left(\sqrt[6]{\frac{k_j k_n}{jn}} \wedge 1 \right) \sqrt{\frac{k_n}{k_j} + \frac{(\log k_j)^2}{\sqrt{k_j}}} \right\} \text{ for any } x \in \mathbb{R},$$

(iii) *If $k_j = k_n$ and $j \leq n$, then*

$$(3.4) \quad |g_{jn}(k_j, k_n, x)| \ll \begin{cases} \frac{\sqrt{k_n/n}}{\exp(n/k_n)} + \frac{(\log k_n)^2}{\sqrt{k_n}}, & \text{if } x \geq \sqrt{2} \text{ and } n/k_n \rightarrow \infty, \\ 1, & \text{otherwise.} \end{cases}$$

Proof of Lemma (3.1) (i). Assume that $k_j < k_n$. It follows from the definition of $g_{jn}(k_j, k_n, x)$ that

$$\begin{aligned} |g_{jn}(k_j, k_n, x)| &= \left| P \left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}} \sigma_{k_j}} \leq x, \frac{S_{k_n}}{\sqrt{\frac{n}{k_n}} \sigma_{k_n}} \leq x \right) - P \left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}} \sigma_{k_j}} \leq x \right) \Phi \left(x \sqrt{\frac{n}{k_n}} \right) \right. \\ &\quad \left. - \Phi \left(x \sqrt{\frac{j}{k_j}} \right) P \left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n}} \sigma_{k_n}} \leq x \right) + \Phi \left(x \sqrt{\frac{j}{k_j}} \right) \Phi \left(x \sqrt{\frac{n}{k_n}} \right) \right|. \end{aligned}$$

We can write that

$$\begin{aligned} |g_{jn}(k_j, k_n, x)| &= \left| P \left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}} \sigma_{k_j}} \leq x, \frac{S_{k_n}}{\sqrt{\frac{n}{k_n}} \sigma_{k_n}} \leq x \right) - P \left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}} \sigma_{k_j}} \leq x \right) P \left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n}} \sigma_{k_n}} \leq x \right) \right. \\ &\quad \left. + P \left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}} \sigma_{k_j}} \leq x \right) P \left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n}} \sigma_{k_n}} \leq x \right) - P \left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}} \sigma_{k_j}} \leq x \right) \Phi \left(x \sqrt{\frac{n}{k_n}} \right) \right. \\ &\quad \left. - \Phi \left(x \sqrt{\frac{j}{k_j}} \right) P \left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n}} \sigma_{k_n}} \leq x \right) + \Phi \left(x \sqrt{\frac{j}{k_j}} \right) \Phi \left(x \sqrt{\frac{n}{k_n}} \right) \right|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} (3.5) \quad |g_{jn}(k_j, k_n, x)| &\leq \left| P \left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}} \sigma_{k_j}} \leq x, \frac{S_{k_n}}{\sqrt{\frac{n}{k_n}} \sigma_{k_n}} \leq x \right) - P \left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}} \sigma_{k_j}} \leq x \right) P \left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n}} \sigma_{k_n}} \leq x \right) \right| \\ &\quad + 2 \left| P \left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n}} \sigma_{k_n}} \leq x \right) - \Phi \left(x \sqrt{\frac{n}{k_n}} \right) \right| =: A_1 + A_2. \end{aligned}$$

Our goal now is to give the bounds for the components A_1, A_2 in (3.6).

In order to estimate A_1 , we shall rewrite it as follows

$$\begin{aligned} A_1 &= \left| P \left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}} \sigma_{k_j}} \leq x, \frac{S_{k_n}}{\sqrt{\frac{n}{k_n}} \sigma_{k_n}} \leq x \right) - P \left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}} \sigma_{k_j}} \leq x \right) P \left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n}} \sigma_{k_n}} \leq x \right) \right| \\ &= \left| P \left(\frac{S_{k_j}}{\sigma_{k_j}} \leq x \sqrt{\frac{j}{k_j}}, \frac{S_{k_n}}{\sigma_{k_n}} \leq x \sqrt{\frac{n}{k_n}} \right) - P \left(\frac{S_{k_j}}{\sigma_{k_j}} \leq x \sqrt{\frac{j}{k_j}} \right) P \left(\frac{S_{k_n}}{\sigma_{k_n}} \leq x \sqrt{\frac{n}{k_n}} \right) \right|. \end{aligned}$$

Notice that $\left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}\sigma_{k_j}}, \frac{S_{k_n}}{\sqrt{\frac{n}{k_n}\sigma_{k_n}}}\right)$ and $\left(\frac{S_{k_j}}{\sigma_{k_j}}, \frac{S_{k_n}}{\sigma_{k_n}}\right)$ are associated, as $\{X_n\}_{n=1}^\infty$ is the sequence of associated r.v.'s. This and Lemma 2.2 in Bagai and Prakasa Rao [1] (see also Lemma 2 in Matuła [22]) imply that for some positive, absolute constant C_1 and any $x \in \mathbb{R}$

$$(3.6) \quad A_1 \leq C_1 \left\{ \left(\text{Cov} \left(\frac{S_{k_j}}{\sqrt{\frac{j}{k_j}\sigma_{k_j}}, \frac{S_{k_n}}{\sqrt{\frac{n}{k_n}\sigma_{k_n}}} \right) \right)^{1/3} \wedge \left(\text{Cov} \left(\frac{S_{k_j}}{\sigma_{k_j}}, \frac{S_{k_n}}{\sigma_{k_n}} \right) \right)^{1/3} \right\}$$

$$= C_1 \left(\sqrt[6]{\frac{k_j k_n}{jn}} \wedge 1 \right) \left(\text{Cov} \left(\frac{S_{k_j}}{\sigma_{k_j}}, \frac{S_{k_n}}{\sigma_{k_n}} \right) \right)^{1/3}.$$

We now give the bound for the covariance $\left(\text{Cov} \left(\frac{S_{k_j}}{\sigma_{k_j}}, \frac{S_{k_n}}{\sigma_{k_n}} \right) \right)^{1/3}$. We have

$$\left(\frac{\text{Cov} \left(S_{k_j}, S_{k_j} + (S_{k_n} - S_{k_j}) \right)}{\sigma_{k_j}\sigma_{k_n}} \right)^{1/3} = \left(\frac{\sigma_{k_j}^2 + \text{Cov} \left(X_1 + \dots + X_{k_j}, X_{k_j+1} + \dots + X_{k_n} \right)}{\sigma_{k_j}\sigma_{k_n}} \right)^{1/3}$$

$$\leq \left(\frac{\sigma_{k_j}^2 + u(1) + u(2) + \dots + u(k_n - k_j)}{\sigma_{k_j}\sigma_{k_n}} \right)^{1/3}.$$

Observe that, by (2.2), there exists a positive constant C_2 , independent of k_n and k_j , such that

$$u(1) + u(2) + \dots + u(k_n - k_j) \leq C_2 \quad \text{for any } k_j < k_n.$$

Furthermore, the assumptions in (2.3) and Theorem 1 in Birkel [4] imply that, there exist the positive, absolute constants C_3, C_4 , satisfying

$$C_3 k_n \leq \sigma_{k_n}^2 \leq C_4 k_n \quad \text{for any } n \in \mathbb{N}.$$

Thus, we may write that, there exists the positive, absolute constant C_5 , such that

$$(3.7) \quad \left(\text{Cov} \left(\frac{S_{k_j}}{\sigma_{k_j}}, \frac{S_{k_n}}{\sigma_{k_n}} \right) \right)^{1/3} \leq C_5 \left(\frac{k_j}{\sqrt{k_j}\sqrt{k_n}} \right)^{1/3} = C_5 \sqrt[6]{\frac{k_j}{k_n}}.$$

Due to (3.6), (3.7), we obtain

$$(3.8) \quad A_1 \leq C_1 C_5 \left(\sqrt[6]{\frac{k_j k_n}{jn}} \wedge 1 \right) \sqrt[6]{\frac{k_j}{k_n}}.$$

Our purpose now is to estimate the component A_2 in (3.6).

It follows from our assumptions and Theorem 2.1 in Birkel [4] (see also Matuła [22], p. 343) that, for some absolute constant C_6 ,

$$\sup_{-\infty < u < \infty} \left| P \left(\frac{S_n}{\sigma_n} \leq u \right) - \Phi(u) \right| \leq C_6 \frac{(\log n)^2}{\sqrt{n}},$$

which yields

$$\sup_{-\infty < x < \infty} \left| P\left(\frac{S_{k_n}}{\sigma_{k_n}} \leq x\sqrt{\frac{n}{k_n}}\right) - \Phi\left(x\sqrt{\frac{n}{k_n}}\right) \right| \leq C_6 \frac{(\log k_n)^2}{\sqrt{k_n}}.$$

Therefore, we have for any $x \in \mathbb{R}$

$$(3.9) \quad A_2 = 2 \left| P\left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n}}\sigma_{k_n}} \leq x\right) - \Phi\left(x\sqrt{\frac{n}{k_n}}\right) \right| \leq 2C_6 \frac{(\log k_n)^2}{\sqrt{k_n}}.$$

By the relations in (3.6), (3.8) and (3.9), we conclude that (3.2) holds, which completes the proof of Lemma (3.1) (i). \square

Proof of Lemma (3.1) (ii). In order to prove (3.3), it remains to replace k_j by k_n and k_n by k_j in the proof of Lemma (3.1) (i) and proceed analogously as in that proof. \square

Proof of Lemma (3.1) (iii). Assume that $k_j = k_n$ and $j \leq n$. Then

$$\begin{aligned} |g_{jn}(k_j, k_n, x)| &= |g_{jn}(k_n, k_n, x)| \\ &= \left| P\left(\frac{S_{k_n}}{\sqrt{\frac{j}{k_n}}\sigma_{k_n}} \leq x, \frac{S_{k_n}}{\sqrt{\frac{n}{k_n}}\sigma_{k_n}} \leq x\right) - P\left(\frac{S_{k_n}}{\sqrt{\frac{j}{k_n}}\sigma_{k_n}} \leq x\right) \Phi\left(x\sqrt{\frac{n}{k_n}}\right) \right. \\ &\quad \left. - \Phi\left(x\sqrt{\frac{j}{k_n}}\right) P\left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n}}\sigma_{k_n}} \leq x\right) + \Phi\left(x\sqrt{\frac{j}{k_n}}\right) \Phi\left(x\sqrt{\frac{n}{k_n}}\right) \right| \\ &= \left| P\left(\frac{S_{k_n}}{\sqrt{\frac{j}{k_n}}\sigma_{k_n}} \leq x\right) - P\left(\frac{S_{k_n}}{\sqrt{\frac{j}{k_n}}\sigma_{k_n}} \leq x\right) \Phi\left(x\sqrt{\frac{n}{k_n}}\right) \right. \\ &\quad \left. - \Phi\left(x\sqrt{\frac{j}{k_n}}\right) P\left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n}}\sigma_{k_n}} \leq x\right) + \Phi\left(x\sqrt{\frac{j}{k_n}}\right) \Phi\left(x\sqrt{\frac{n}{k_n}}\right) \right|. \end{aligned}$$

Therefore, we may write that

$$(3.10) \quad |g_{jn}(k_n, k_n, x)| \leq \left(1 - \Phi\left(x\sqrt{\frac{n}{k_n}}\right)\right) + \left| P\left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n}}\sigma_{k_n}} \leq x\right) - \Phi\left(x\sqrt{\frac{n}{k_n}}\right) \right|.$$

It follows from (1.5.4) in Leadbetter et al. [21] that, if $x > 0$ and $n/k_n \rightarrow \infty$, then

$$1 - \Phi\left(x\sqrt{\frac{n}{k_n}}\right) \leq \exp\left(-\frac{x^2 n}{2k_n}\right) / x\sqrt{\frac{n}{k_n}} \quad \text{for all sufficiently large } n.$$

Thus, we may write that

$$(3.11) \quad 1 - \Phi\left(x\sqrt{\frac{n}{k_n}}\right) \ll \frac{\sqrt{k_n/n}}{\exp(n/k_n)}, \quad \text{if } x \geq \sqrt{2} \text{ and } n/k_n \rightarrow \infty.$$

In addition, we have (see the relation above (3.9))

$$(3.12) \quad \sup_{-\infty < x < \infty} \left| P \left(\frac{S_{k_n}}{\sqrt{\frac{n}{k_n} \sigma_{k_n}}} \leq x \right) - \Phi \left(x \sqrt{\frac{n}{k_n}} \right) \right| \leq C_6 \frac{(\log k_n)^2}{\sqrt{k_n}}.$$

The relations in (3.10)-(3.12) imply the desired result in (3.4). □

4. Proof of the main result

In this Section, we shall give the proof of our main assertion.

Proof of Theorem (2.1). Let

$$(4.1) \quad L(N, x) := \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} \left(I \left(\frac{S_{N_n}}{\sqrt{\frac{n}{N_n} \sigma_{N_n}}} \leq x \right) - \Phi \left(x \sqrt{\frac{n}{N_n}} \right) \right).$$

We have

$$\begin{aligned} \text{Var}(L(N, x)) &\leq E \left\{ \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} \left(I \left(\frac{S_{N_n}}{\sqrt{\frac{n}{N_n} \sigma_{N_n}}} \leq x \right) - \Phi \left(x \sqrt{\frac{n}{N_n}} \right) \right) \right\}^2 \leq \\ &\frac{2}{(\log N)^2} \sum_{n=1}^N \sum_{j=1}^n \frac{1}{jn} E \left\{ I \left(\frac{S_{N_j}}{\sqrt{\frac{j}{N_j} \sigma_{N_j}}} \leq x \right) - \Phi \left(x \sqrt{\frac{j}{N_j}} \right) \right\} \\ &\qquad \qquad \qquad \left\{ I \left(\frac{S_{N_n}}{\sqrt{\frac{n}{N_n} \sigma_{N_n}}} \leq x \right) - \Phi \left(x \sqrt{\frac{n}{N_n}} \right) \right\}. \end{aligned}$$

Since $\{N_n\}_{n=1}^\infty$ is a sequence of independent random indexes, independent of the sequence $\{X_n\}_{n=1}^\infty$, we get

$$\text{Var}(L(N, x)) \leq \frac{2}{(\log N)^2} \sum_{k_1=1}^\infty \dots \sum_{k_N=1}^\infty \sum_{n=1}^N \sum_{j=1}^n \frac{1}{jn} |g_{jn}(k_j, k_n, x)| \prod_{i=1}^N P(N_i = k_i).$$

This and Lemma (3.1) imply

$$\begin{aligned} &\text{Var}(L(N, x)) \\ &\leq \frac{2C}{(\log N)^2} \sum_{k_1=1}^\infty \dots \sum_{k_N=1}^\infty \sum_{n=1}^N \sum_{j=1}^n \frac{1}{jn} \left(\left(\sqrt{\frac{k_j k_n}{jn}} \wedge 1 \right) \sqrt{\frac{k_j}{k_n} + \frac{(\log k_n)^2}{\sqrt{k_n}}} \right) P(N_j < N_n) \prod_{i=1}^N P(N_i = k_i) \\ &+ \frac{2C}{(\log N)^2} \sum_{k_1=1}^\infty \dots \sum_{k_N=1}^\infty \sum_{n=1}^N \sum_{j=1}^n \frac{1}{jn} \left(\left(\sqrt{\frac{k_j k_n}{jn}} \wedge 1 \right) \sqrt{\frac{k_n}{k_j} + \frac{(\log k_j)^2}{\sqrt{k_j}}} \right) P(N_j > N_n) \prod_{i=1}^N P(N_i = k_i) \\ &\qquad \qquad \qquad + \frac{2}{(\log N)^2} \sum_{k_1=1}^\infty \dots \sum_{k_N=1}^\infty \sum_{n=1}^N \sum_{j=1}^n \frac{1}{jn} P(N_j = N_n) \prod_{i=1}^N P(N_i = k_i). \end{aligned}$$

Therefore, since $\{N_n\}_{n=1}^\infty$ is an independent sequence, we obtain

$$\begin{aligned} & \text{Var}(L(N, x)) \\ & \leq \frac{2C}{(\log N)^2} \sum_{n=1}^N \frac{1}{n} \sum_{j=1}^n \frac{1}{j} E \left\{ \left(\left(\sqrt[6]{\frac{N_j N_n}{jn}} \wedge 1 \right) \sqrt[6]{\frac{N_j}{N_n} + \frac{(\log N_n)^2}{\sqrt{N_n}}} \right) I(N_j < N_n) \right\} \\ & \quad + \frac{2C}{(\log N)^2} \sum_{n=1}^N \frac{1}{n} \sum_{j=1}^n \frac{1}{j} E \left\{ \left(\left(\sqrt[6]{\frac{N_j N_n}{jn}} \wedge 1 \right) \sqrt[6]{\frac{N_n}{N_j} + \frac{(\log N_j)^2}{\sqrt{N_j}}} \right) I(N_j > N_n) \right\} \\ & \quad + \frac{2}{(\log N)^2} \sum_{n=1}^N \frac{1}{n} \sum_{j=1}^n \frac{1}{j} P(N_j = N_n). \end{aligned}$$

The last relation and the assumptions in (2.4)-(2.6) yield that for any $x \in \mathbb{R}$

$$\text{Var}(L(N, x)) \leq \frac{2C_1}{(\log N)^2} \sum_{n=1}^N \frac{1}{n} (\log n)^{1-\mu} \ll (\log N)^{-\mu} \quad \text{for some } 0 < \mu < 1.$$

This and Chebyshev's inequality imply that for any $x \in \mathbb{R}$

$$(4.2) \quad P(|L(N, x)| > \varepsilon) \ll \frac{(\log N)^{-\mu}}{\varepsilon^2} \quad \text{for any } \varepsilon > 0 \text{ and some } 0 < \mu < 1.$$

Continuing our proof, we put $N(l) := \lceil e^{l^{1+1/\mu}} + 1 \rceil$, where μ is such as in (4.2). Then, we obtain that for any $x \in \mathbb{R}$

$$P(|L(N(l), x)| > \varepsilon) \ll \frac{(l^{1+1/\mu})^{-\mu}}{\varepsilon^2} = \frac{1}{\varepsilon^2 l^{\mu+1}} \quad \text{for some } 0 < \mu < 1,$$

which yields

$$\sum_{l=1}^\infty P(|L(N(l), x)| > \varepsilon) < \infty \quad \text{for any } x \in \mathbb{R}.$$

Thus, by the Borel-Cantelli Lemma,

$$(4.3) \quad |L(N(l), x)| \rightarrow 0 \quad \text{a.s. for any } x \in \mathbb{R}, \text{ if } l \rightarrow \infty.$$

Furthermore, for $N(l) < N \leq N(l+1)$, we have

$$(4.4) \quad |L(N, x)| \leq |L(N(l), x)| + \frac{2}{\log N(l)} \sum_{n=N(l)+1}^{N(l+1)} \frac{1}{n}.$$

It is easily seen that

$$(4.5) \quad \frac{2}{\log N(l)} \sum_{n=N(l)+1}^{N(l+1)} \frac{1}{n} \ll \frac{\log N(l+1)}{\log N(l)} - 1 \rightarrow 0, \text{ if } l \rightarrow \infty.$$

The relations in (4.3)-(4.5) imply

$$(4.6) \quad \lim_{N \rightarrow \infty} |L(N, x)| = 0 \quad \text{a.s. for any } x \in \mathbb{R}.$$

Due to (4.1), (4.6), we have

$$\lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{1}{n} \left(I \left(\frac{S_{N_n}}{\sqrt{\frac{n}{N_n}} \sigma_{N_n}} \leq x \right) - \Phi \left(x \sqrt{\frac{n}{N_n}} \right) \right) \quad \text{a.s. for any } x \in \mathbb{R},$$

which yields the desired result in (2.7). □

5. Example of application

In this Section, we aim to show that, there exists a large class of indexes $\{N_n\}_{n=1}^\infty$, satisfying the assumptions of Theorem (2.1).

PROPOSITION (5.1). *Let $\{N_n\}_{n=1}^\infty$ be a sequence of independent, positive-integer valued r.v.'s. Assume moreover that, there exist the positive constants c_1, c_2, c_3 and the nonnegative constants β_1, β_2 , such that $\beta_1 + \beta_2 < 1$, and*

$$P(N_n \notin J_n) \leq c_3 (\log n)^{-\beta_1 - \beta_2} \quad \text{for all sufficiently large } n,$$

where $J_n = (c_1 n (\log n)^{-1 + \beta_1}, c_2 n (\log n)^{-\beta_2})$.

Then, the relations in (2.4)-(2.6) hold.

Proof. First, let us consider the events of the form $\{N_j = N_n\}$. Note that

$$\begin{aligned} \sum_{j=1}^n \frac{1}{j} P(N_j = N_n) &= \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} P(N_j = N_n) + \sum_{\{j: J_j \cap J_n \neq \emptyset\}} \frac{1}{j} P(N_j = N_n) \\ &\leq \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} P(\{N_j \notin J_j\} \cup \{N_n \notin J_n\}) + \sum_{\{j: J_j \cap J_n \neq \emptyset\}} \frac{1}{j}. \end{aligned}$$

Hence, we have

$$(5.2) \quad \sum_{j=1}^n \frac{1}{j} P(N_j = N_n) \ll \sum_{j=2}^n \frac{1}{j} P(N_j \notin J_j) + \sum_{j=2}^n \frac{1}{j} P(N_n \notin J_n) + \sum_{\{j: J_j \cap J_n \neq \emptyset\}} \frac{1}{j}.$$

It follows from the assumptions on $\{N_n\}, \{J_n\}$ that

$$(5.3) \quad \sum_{\{j: J_j \cap J_n \neq \emptyset\}} \frac{1}{j} \leq \sum_{\left\{j: \frac{c_1 n}{c_2 (\log n)^{1 - \beta_1}} < j \leq n\right\}} \frac{1}{j} \leq \frac{c_2}{c_1} (\log n)^{1 - \beta_1}.$$

Due to (5.2) and (5.3), we obtain

$$(5.4) \quad \sum_{j=1}^n \frac{1}{j} P(N_j = N_n) \ll \sum_{j=2}^n \frac{1}{j} P(N_j \notin J_j) + \sum_{j=2}^n \frac{1}{j} P(N_n \notin J_n) + (\log n)^{1 - \beta_1}.$$

Now, let us consider the events of the form $\{N_j > N_n\}$. We have

$$\begin{aligned} \sum_{j=1}^n \frac{1}{j} EI(N_j > N_n) &= \sum_{j=1}^n \frac{1}{j} P(N_j > N_n) \\ &= \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} P(N_j > N_n) + \sum_{\{j: J_j \cap J_n \neq \emptyset\}} \frac{1}{j} P(N_j > N_n) \\ &\leq \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} P(\{N_j \notin J_j\} \cup \{N_n \notin J_n\}) + \sum_{\{j: J_j \cap J_n \neq \emptyset\}} \frac{1}{j}. \end{aligned}$$

Therefore, we obtain that

$$(5.5) \quad \sum_{j=1}^n \frac{1}{j} EI(N_j > N_n) \ll \sum_{j=2}^n \frac{1}{j} P(N_j \notin J_j) + \sum_{j=2}^n \frac{1}{j} P(N_n \notin J_n) + \sum_{\{j: J_j \cap J_n \neq \emptyset\}} \frac{1}{j}.$$

The relations in (5.5) and (5.3) yield

$$(5.6) \quad \sum_{j=1}^n \frac{1}{j} EI(N_j > N_n) \ll \sum_{j=2}^n \frac{1}{j} P(N_j \notin J_j) + \sum_{j=2}^n \frac{1}{j} P(N_n \notin J_n) + (\log n)^{1 - \beta_1}.$$

Note that

$$(5.7) \quad \sum_{j=1}^n \frac{1}{j} P(N_j = N_n) + \sum_{j=1}^n \frac{1}{j} E \left\{ \left(\left(\sqrt[6]{\frac{N_j N_n}{jn}} \wedge 1 \right) \sqrt[6]{\frac{N_n}{N_j}} + \frac{(\log N_j)^2}{\sqrt{N_j}} \right) I(N_j > N_n) \right\} \\ \leq \sum_{j=1}^n \frac{1}{j} P(N_j = N_n) + c_4 \sum_{j=1}^n \frac{1}{j} E I(N_j > N_n),$$

where $c_4 := 1 + \max_{k \in \mathbb{N}} \{(\log k)^2 / \sqrt{k}\}$.

Furthermore, by (5.4) and (5.6), we have

$$(5.8) \quad \sum_{j=1}^n \frac{1}{j} P(N_j = N_n) + c_4 \sum_{j=1}^n \frac{1}{j} E I(N_j > N_n) \\ \ll \sum_{j=2}^n \frac{1}{j} P(N_j \notin J_j) + \sum_{j=2}^n \frac{1}{j} P(N_n \notin J_n) + (\log n)^{1-\beta_1}.$$

Since $P(N_k \notin J_k) \leq c_3 (\log k)^{-\beta_1 - \beta_2}$ for all sufficiently large k , we obtain

$$(5.9) \quad \sum_{j=2}^n \frac{1}{j} P(N_j \notin J_j) + \sum_{j=2}^n \frac{1}{j} P(N_n \notin J_n) \ll (\log n)^{1-(\beta_1 + \beta_2)}.$$

It follows from (5.8) and (5.9) that

$$(5.10) \quad \sum_{j=1}^n \frac{1}{j} P(N_j = N_n) + c_4 \sum_{j=1}^n \frac{1}{j} E I(N_j > N_n) \ll (\log n)^{1-\beta_1}.$$

Thus, due to (5.7) and (5.10),

$$\sum_{j=1}^n \frac{1}{j} P(N_j = N_n) + \sum_{j=1}^n \frac{1}{j} E \left\{ \left(\left(\sqrt[6]{\frac{N_j N_n}{jn}} \wedge 1 \right) \sqrt[6]{\frac{N_n}{N_j}} + \frac{(\log N_j)^2}{\sqrt{N_j}} \right) I(N_j > N_n) \right\} \\ \ll (\log n)^{1-\beta_1},$$

and the relations in (2.5), (2.6) hold.

Our purpose now is to show (2.4). Put

$$X_{j,n} := \left(\sqrt[6]{\frac{N_j N_n}{jn}} \wedge 1 \right) \sqrt[6]{\frac{N_j}{N_n}} + \frac{(\log N_n)^2}{\sqrt{N_n}},$$

and denote by A' the complement of the set A . Obviously

$$\sum_{j=1}^n \frac{1}{j} E X_{j,n} I(N_j < N_n) = \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} E X_{j,n} I(N_j < N_n) \\ + \sum_{\{j: J_j \cap J_n \neq \emptyset\}} \frac{1}{j} E X_{j,n} I(N_j < N_n).$$

Therefore, we may write that

$$\sum_{j=1}^n \frac{1}{j} E X_{j,n} I(N_j < N_n) \leq \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} E X_{j,n} I(N_j < N_n, (N_j \in J_j, N_n \in J_n)) \\ + \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} E X_{j,n} I(N_j < N_n, (N_j \in J_j, N_n \in J_n)') + \sum_{\{j: J_j \cap J_n \neq \emptyset\}} \frac{1}{j} E X_{j,n}.$$

Consequently

$$\begin{aligned} \sum_{j=1}^n \frac{1}{j} \mathbf{E} X_{j,n} I(N_j < N_n) &\leq \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} \mathbf{E} X_{j,n} I(N_j < N_n, N_j \in J_j, N_n \in J_n) \\ &+ \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} \mathbf{E} X_{j,n} I(N_j < N_n, (\{N_j \notin J_j\} \cup \{N_n \notin J_n\})) + \sum_{\{j: J_j \cap J_n \neq \emptyset\}} \frac{1}{j} \mathbf{E} X_{j,n}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \sum_{j=1}^n \frac{1}{j} \mathbf{E} X_{j,n} I(N_j < N_n) &\leq \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} \mathbf{E} X_{j,n} I(N_j < N_n, N_j \in J_j, N_n \in J_n) \\ &+ c_4 \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} P(N_j \notin J_j) + c_4 \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} P(N_n \notin J_n) + c_4 \sum_{\{j: J_j \cap J_n \neq \emptyset\}} \frac{1}{j}, \end{aligned}$$

where, for recollection, $c_4 := 1 + \max_{k \in \mathbb{N}} \{(\log k)^2 / \sqrt{k}\}$.

This yields

$$\begin{aligned} (5.11) \quad &\sum_{j=1}^n \frac{1}{j} \mathbf{E} \left\{ \left(\left(\sqrt[6]{\frac{N_j N_n}{jn}} \wedge 1 \right) \sqrt[6]{\frac{N_j}{N_n}} + \frac{(\log N_n)^2}{\sqrt{N_n}} \right) I(N_j < N_n) \right\} \\ &\ll \left(\sum_{j=2}^n \frac{1}{j} P(N_j \notin J_j) + \sum_{j=2}^n \frac{1}{j} P(N_n \notin J_n) + (\log n)^{1-\beta_1} \right) \\ &+ \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} \mathbf{E} \left\{ \left(\sqrt[6]{\frac{N_j}{N_n}} \right) I(N_j < N_n, N_j \in J_j, N_n \in J_n) \right\} \\ &+ \sum_{\{j: J_j \cap J_n = \emptyset\}} \frac{1}{j} \mathbf{E} \left\{ \left(\frac{(\log N_n)^2}{\sqrt{N_n}} \right) I(N_j < N_n, N_j \in J_j, N_n \in J_n) \right\} \\ &=: B_1 + B_2 + B_3. \end{aligned}$$

It follows from the previous derivations that

$$(5.12) \quad B_1 \ll (\log n)^{1-\beta_1}.$$

Our goal now is to give the bound for B_2 in [\(5.12\)](#). We have

$$\begin{aligned} B_2 &\ll \sum_{j=2}^n \frac{c_1 n}{c_2 (\log n)^{1-\beta_1}} \frac{1}{j} \mathbf{E} \left\{ \left(\sqrt[6]{\frac{N_j}{N_n}} \right) I(N_j < N_n, N_j \in J_j, N_n \in J_n) \right\} \\ &\ll \sum_{\left\{j: j \leq \frac{c_1 n}{c_2 (\log n)^{1-\beta_1}} \wedge j^{1/6} (\log j) < n^{1/6}\right\}} \frac{1}{j} \mathbf{E} \left\{ \left(\sqrt[6]{\frac{N_j}{N_n}} \right) I(N_j < N_n, N_j \in J_j, N_n \in J_n) \right\} \\ &+ \sum_{\left\{j: j \leq \frac{c_1 n}{c_2 (\log n)^{1-\beta_1}} \wedge j^{1/6} (\log j) \geq n^{1/6}\right\}} \frac{1}{j} \mathbf{E} \left\{ \left(\sqrt[6]{\frac{N_j}{N_n}} \right) I(N_j < N_n, N_j \in J_j, N_n \in J_n) \right\}. \end{aligned}$$

Thus, it follows from the definition of the sets J_j, J_n that

$$\begin{aligned}
 (5.13) \quad B_2 &\ll \sum_{\left\{j: j \leq \frac{c_1 n}{c_2 (\log n)^{1-\beta_1}} \wedge j^{1/6} (\log j) < n^{1/6}\right\}} \frac{1}{j} \sqrt[6]{\frac{c_2 j (\log j)^{-\beta_2}}{c_1 n (\log n)^{-1+\beta_1}}} \\
 &+ \sum_{\left\{j: j \leq \frac{c_1 n}{c_2 (\log n)^{1-\beta_1}} \wedge j^{1/6} (\log j) \geq n^{1/6}\right\}} \frac{1}{j} \sqrt[6]{\frac{c_2 j (\log j)^{-\beta_2}}{c_1 n (\log n)^{-1+\beta_1}}} \\
 &=: B_{21} + B_{22}.
 \end{aligned}$$

We now give the bound for the component B_{21} in (5.14). Note that

$$\begin{aligned}
 B_{21} &\ll \sum_{\left\{j: j \leq \frac{c_1 n}{c_2 (\log n)^{1-\beta_1}} \wedge j^{1/6} (\log j) < n^{1/6}\right\}} \frac{1}{j^{5/6} n^{1/6}} \frac{(\log n)^{(1-\beta_1)/6}}{(\log j)^{\beta_2/6}} \\
 &\leq \sum_{j=2}^n \frac{1}{j^{5/6} j^{1/6} (\log j)} \frac{(\log n)^{(1-\beta_1)/6}}{(\log j)^{\beta_2/6}} = (\log n)^{(1-\beta_1)/6} \sum_{j=2}^n \frac{1}{j (\log j)^{1+\beta_2/6}}.
 \end{aligned}$$

Since in addition, $\sum_{j=2}^\infty \frac{1}{j (\log j)^{1+\beta_2/6}} < \infty$, we get

$$(5.14) \quad B_{21} \ll (\log n)^{(1-\beta_1)/6} \leq (\log n)^{1/6}.$$

In order to estimate the component B_{22} in (5.14), observe that

$$\begin{aligned}
 B_{22} &\ll \sum_{\left\{j: j \leq \frac{c_1 n}{c_2 (\log n)^{1-\beta_1}} \wedge j^{1/6} (\log j) \geq n^{1/6}\right\}} \frac{1}{j^{5/6} n^{1/6}} \frac{(\log n)^{(1-\beta_1)/6}}{(\log j)^{\beta_2/6}} \\
 &\leq (\log n)^{(1-\beta_1)/6} \sum_{\left\{j: \frac{n}{(\log j)^6} \leq j \leq \frac{c_1 n}{c_2 (\log n)^{1-\beta_1}}\right\}} \frac{1}{j} \leq (\log n)^{(1-\beta_1)/6} \sum_{\left\{j: \frac{n}{(\log n)^6} \leq j \leq \frac{c_1 n}{c_2 (\log n)^{1-\beta_1}}\right\}} \frac{1}{j},
 \end{aligned}$$

which implies that

$$\begin{aligned}
 B_{22} &\ll (\log n)^{(1-\beta_1)/6} \left(\log \frac{c_1 n}{c_2 (\log n)^{1-\beta_1}} - \log \frac{n}{(\log n)^6} \right) \\
 &= (\log n)^{(1-\beta_1)/6} \log \left(\frac{c_1}{c_2} (\log n)^{5+\beta_1} \right) \ll (\log n)^{(1-\beta_1)/6} \log(\log n).
 \end{aligned}$$

Therefore, we may write that

$$(5.15) \quad B_{22} \ll (\log n)^{1/6}.$$

Due to (5.14)-(5.15), we obtain

$$(5.16) \quad B_2 \ll (\log n)^{1/6}.$$

Thus, it remains to estimate the component B_3 in (5.12). It is clear that for any $0 < \gamma < 1/2$, there exists $C(\gamma)$, dependent only on γ , such that

$$\frac{(\log n)^2}{\sqrt{n}} \leq C(\gamma) \frac{1}{n^{1/2-\gamma}} \quad \text{for any } n \in \mathbb{N}.$$

Hence, for any fixed $0 < \gamma < 1/2$,

$$B_3 \ll \sum_{j=2}^{\frac{c_1 n}{c_2 (\log n)^{1-\beta_1}}} \frac{1}{j} \mathbf{E} \left\{ \frac{1}{(N_n)^{1/2-\gamma}} I(N_j < N_n, N_j \in J_j, N_n \in J_n) \right\},$$

and consequently

$$\begin{aligned} B_3 &\ll \sum_{j=2}^{\frac{c_1 n}{c_2 (\log n)^{1-\beta_1}}} \frac{1}{j} \frac{1}{(c_1 n (\log n)^{-1+\beta_1})^{1/2-\gamma}} \ll \sum_{j=2}^{\frac{c_1 n}{c_2 (\log n)^{1-\beta_1}}} \frac{(\log n)^{(1-\beta_1)(1/2-\gamma)}}{j n^{1/2-\gamma}} \\ &\leq (\log n)^{(1-\beta_1)(1/2-\gamma)} \sum_{j=2}^{\frac{c_1 n}{c_2 (\log n)^{1-\beta_1}}} \frac{1}{j^{1+(1/2-\gamma)}}. \end{aligned}$$

Therefore, we have

$$(5.17) \quad B_3 \ll (\log n)^{(1-\beta_1)(1/2-\gamma)} < (\log n)^{1/2}.$$

The relations in (5.12), (5.12), (5.16) and (5.17) yield (2.4) for some $0 < \mu < 1$. \square

Acknowledgements

P. Górka is partially supported by FONDECYT grant #3100019;

Received July 25, 2011

Final version received July 23, 2012

MARCIN DUDZIŃSKI

FACULTY OF APPLIED INFORMATICS AND MATHEMATICS,
WARSAW UNIVERSITY OF LIFE SCIENCES,
UL. NOWOURSYNOWSKA 159, 02-776 WARSAW, POLAND,
marcin_dudzinski@sggw.pl

PRZEMYSŁAW GÓRKA

INSTITUTO DE MATEMÁTICA Y FÍSICA, UNIVERSIDAD DE TALCA,
CASILLA 747, TALCA, CHILE

AND

DEPARTMENT OF MATHEMATICS AND INFORMATION SCIENCES,
WARSAW UNIVERSITY OF TECHNOLOGY,
PL. POLITECHNIKI 1, 00-661 WARSAW, POLAND
P.Gorka@mini.pw.edu.pl

REFERENCES

- [1] I. BAGAI AND B.L.S. PRAKASA RAO, *Estimation of the survival function for stationary associated processes*, Statist. Probab. Lett. **12**, (1991), 385-391.
- [2] I. BERKES AND E. CSAKI, *A universal result in almost sure central limit theory*, Stoch. Process. Appl. **94**, (2001), 105-134.
- [3] I. BERKES AND H. DEHLING, *Some limit theorems in log density*, Ann. Probab. **21**, (1993), 1640-1670.
- [4] T. BIRKEL, *Moment bounds for associated sequences*, Ann. Probab. **16**, (1988), 1184-1193.
- [5] G. A. BROSAMLER, *An almost everywhere central limit theorem*, Math. Proc. Cambridge Philos. Soc. **104**, (1988), 561-574.
- [6] S. CHEN AND Z. LIN, *Almost sure max-limits for nonstationary Gaussian sequence*, Statist. Probab. Lett. **76**, (2006), 1175-1184.

- [7] S. CHENG, L. PENG, AND Y. QI, *Almost sure convergence in extreme value theory*, Math. Nachr. **190**, (1998), 43-50.
- [8] E. CSAKI AND K. GONCHIGDANZAN, *Almost sure limit theorems for the maximum of stationary Gaussian sequences*, Statist. Probab. Lett. **58**, (2002), 195-203.
- [9] M. DUDZIŃSKI, *An almost sure maximum limit theorem for certain class of dependent stationary Gaussian sequences*, Demonstratio Mathematica **35**, (2002), 879-890.
- [10] M. DUDZIŃSKI, *A note on the almost sure central limit theorem for some dependent random variables*, Statist. Probab. Lett. **61**, (2003), 31-40.
- [11] M. DUDZIŃSKI, *The almost sure central limit theorems in the joint version for the maxima and sums of certain stationary Gaussian sequences*, Statist. Probab. Lett. **78**, (2008), 347-357.
- [12] M. DUDZIŃSKI, *The almost sure central limit theorems for certain order statistics of some stationary Gaussian sequences*, Annales Universitatis Mariae Curie-Skłodowska. Sectio A: Mathematica, **63**, (2009), 63-81.
- [13] J. ESARY, F. PROSCHAN, AND D. WALKUP, *Association of random variables with applications*, Ann. Math. Statist. **38**, (1967), 1466-1474.
- [14] I. FAHRNER AND U. STADTMULLER, *On almost sure central max-limit theorems*, Statist. Probab. Lett. **37**, (1998), 229-236.
- [15] I. FAZEKAS AND Z. RYCHLIK, *Almost sure functional limit theorems*, Universitatis Mariae Curie-Skłodowska. Sectio A: Mathematica. **56**, (2002), 1-18.
- [16] I. FAZEKAS AND Z. RYCHLIK, *Almost sure central limit theorems for random fields*, Math. Nachr. **259**, (2003), 12-18.
- [17] A. FISHER, *A pathwise central limit theorem for random walks*, Preprint, (1989).
- [18] K. GONCHIGDANZAN AND G. REMPALA, *A note on the almost sure limit theorem for the product of partial sums*, Appl. Math. Lett. **19**, (2006), 191-196.
- [19] A. KRAJKA AND J. WASIURA, *On the almost sure central limit theorem for randomly indexed sums*, Math. Nachr. **282**, (2009), 569-580.
- [20] M. LACEY AND W. PHILIPP, *A note on the almost everywhere central limit theorem*, Statist. Probab. Lett. **9**, (2009), 201-205.
- [21] M. R. LEADBETTER, G. LINDGREN, AND H. ROOTZEN, *Extremes and Related Properties of Random Sequences and Processes*. Springer, New York, 1983.
- [22] P. MATUŁA, *Convergence of weighted averages of associated random variables*, Probab. Math. Statist. **16**, (1996), 337-343.
- [23] P. MATUŁA, *On the almost sure central limit theorem for associated random variables*, Probab. Math. Statist. **18**, (1998), 411-416.
- [24] J. MIELNICZUK, *Some remarks on the almost sure central limit theorem for dependent sequences* In: Limit theorems in Probability and Statistics, vol. 2. Janos Bolyai Mathematical Society, Budapest, (2002), 391-403.
- [25] M. PELIGRAD AND Q. SHAO, *A note on the almost sure central limit theorem for weakly dependent random variables*, Statist. Probab. Lett. **22**, (1995), 131-136.
- [26] B. RODZIK AND Z. RYCHLIK, *Almost sure central limit theorems for weakly dependent random variables*, Demonstratio Mathematica **34**, (2001), 375-384.
- [27] P. SCHATTE, *On strong versions of the central limit theorem*, Math. Nachr. **137**, (1988), 249-256.
- [28] U. STADTMUELLER, *Almost sure versions of distributional limit theorems for certain order statistics*, Statist. Probab. Lett. **58**, (2002), 413-426.