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HAUSDORFF DIMENSION OF BOUNDARIES OF SELF-AFFINE TILES IN $\mathbb{R}^N$

J. J. P. Veerman

Abstract

We present a new method to calculate the Hausdorff dimension of a certain class of fractals: boundaries of self-affine tiles. Among the interesting aspects are that even if the affine contraction underlying the iterated function system is not conjugated to a similarity we obtain an upper- and and lower-bound for its Hausdorff dimension. In fact, we obtain the exact value for the dimension if the moduli of the eigenvalues of the underlying affine contraction are all equal (this includes Jordan blocks). The tiles we discuss play an important role in the theory of wavelets. We calculate the dimension for a number of examples.

1. Introduction

The object of this study is a class of self-affine (or self-similar) sets generated by self-affine pairs. In what follows, we abbreviate this to pair.

Definition (1.1). A pair (M, R) is a linear isomorphism $M : \mathbb{R}^n \to \mathbb{R}^n$ with all eigenvalues outside the unit circle together with a finite subset R of $\mathbb{R}^n$.

The space of closed and a priori bounded subsets of $\mathbb{R}^n$ will be denoted by $H(\mathbb{R}^n)$. Endow this space with the usual Hausdorff distance between two compact sets (the infimum of ϵ such that an ϵ-neighborhood of each one of the two sets contains the other). This distance induces a topology on $H(\mathbb{R}^n)$ with respect to which $H(\mathbb{R}^n)$ is a complete compact metric space. In $H(\mathbb{R}^n)$, we define

$$\tau : H(\mathbb{R}^n) \to H(\mathbb{R}^n)$$

by

$$\tau(A) := \cup_{r\in R} M^{-1}(A + r).$$

Such systems are affine examples of what are known as iterated function systems (see [2]). It is easy to prove that τ is a contraction (see [13]) and its unique fixed

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Keywords and phrases: Hausdorff dimension, boundaries of self affine tiles, iterated function system.

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MODULI SPACES FOR ENDOMORPHISMS OF FINITE DIMENSIONAL VECTOR SPACES

HERBERT KANAREK

Abstract

Consider the set End_n of endomorphisms of vector spaces of dimension $n < \infty$ over a field K. D. Mumford showed that there is no moduli space for End_n. He also showed that for the set of semi-simple endomorphisms there is a coarse moduli space and that for the set of cyclic endomorphisms one has a fine moduli space given by the characteristic polynomial.

What we present here is a stratification of End_n into a family of subsets in which for every element of this family there exists a fine moduli space and where one of these subsets is composed of the cyclic endomorphisms.

Introduction

Consider the pair (V, T) where V is an n-dimensional vector space over a field K and T is an endomorphism of V. Apply to this set the equivalence relation of isomorphism, this is, $(V, T) \sim (V', T')$ if and only if there exists an isomorphism $h: V \to V'$ of vector spaces such that

$$h \circ T = T' \circ h\,.$$

We denote by (End_n) the corresponding moduli problem.

The problem now is to find a natural structure of a variety for (End_n). Mumford showed that this is not possible in general, but he gave two cases in which he succeeds. The first one is when one only takes the set of semi-simple endomorphisms, this is, the endomorphisms that can be represented by a diagonal matrix for some choice of basis of V. In this case one has a coarse moduli space: one can give a structure of variety to the set of semi-simple endomorphisms but this structure is not unique. The second case is when we take the cyclic endomorphisms. In this case Mumford shows that one has a fine moduli space which is the nicest situation for a moduli problem giving a unique structure of variety for our objects.

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Keywords and phrases: family, moduli space, stratification, vector spaces, endomorphisms, canonical forms.

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ON AN ELEMENTARY DECOMPOSITION THEOREM FOR POLARIZED COMPLEX TORI WITH AUTOMORPHISMS

Gustavo Labbé M.

Abstract

In this paper we study the action of the group of automorphisms on polarized abelian varieties. We obtain a decomposition theorem of a polarized abelian variety as a product of polarized abelian varieties of smaller dimension via an isogeny. Moreover we find conditions for this isogeny to be an isomorphism. First we study the case of a cyclic group of automorphisms and then the general case. We finish with the construction of examples of some one dimensional families of polarized abelian varieties of different dimension (two, three, four and five) so that the polarization types and the group of automorphisms of each of them are different.

1. Preliminaries

Let V denote a complex vector space of dimension g and Λ a lattice in V (i. e., Λ is a discrete subgroup of rank $2g$ of V). The lattice Λ acts on V by addition. The quotient $T = V/\Lambda$ is called a *complex torus*.

An *isogeny* from a complex torus T_1 to a complex torus T_2 is a surjective homomorphism $f : T_1 \to T_2$ with finite kernel. Equivalently, a homomorphism $T_1 \to T_2$ is an isogeny if an only if it is surjective and dim $T_1 =$ dim T_2.

A *polarized abelian variety* is a pair (T, H) where $T = V/\Lambda$ is a complex torus of dimension g and $H : V \times V \to \mathbb{C}$ is a positive definite hermitian form whose alternating form $E = \Im H$ is integer-valued on the lattice, i. e., $E(\Lambda \times \Lambda) \subset \mathbb{Z}$. In this case, H or E are called a *polarization* on T.

According to the elementary divisor theorem (see [3] or [1]), there is a basis $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots, \mu_g$ of Λ with respect to which E is given by the matrix $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$, where $D = \mathrm{diag}(d_1, \ldots, d_g)$, with integers $d_j \geq 0$ satisfying $d_j | d_{j+1}$ for $j = 1, \ldots, g-1$. The numbers $d_1, \ldots, d_g$ are uniquely determined by E and Λ. The vector $(d_1, \ldots, d_g)$ as well as the matrix D are called the *type* of the

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SEMISTANDARD k–TABLEAUX: ELEMENTARY OPERATIONS

José Martínez–Bernal

Abstract

Semistandard k-tableaux are combinatorial objects, related to Young diagrams, that parametrize a partition of the variety of k-dimensional subspaces fixed by a nilpotent endomorphism. These tableaux are partially ordered in a natural way. We give an algorithm that takes as its input two semistandard k-tableaux, $\alpha < \beta$, and returns as its output a third semistandard k-tableau γ such that $\alpha < \gamma \leq \beta$, and it is obtained by applying to α an elementary operation. A consequence of this result is an affirmative answer to a conjecture about the inclusion relations of the Zariski closures of the members in the referred partition.

Introduction

Let u be a nilpotent endomorphism of a finite n-dimensional vector space defined over an algebraically closed field. Let λ be the Jordan partition of n determined by u. Semistandard k-tableaux of shape λ are combinatorial objects that arise naturally in the study of the geometrical structure of the projective variety G_k^u of k-dimensional subspaces fixed by u. They parametrize a partition of this variety into locally closed affine spaces. The points in a member of this partition are the k-dimensional subspaces in a Bruhat cell S_α that are fixed by u (α being a semistandard k-tableau); so we denote it by S_α^u.

Let cl S_α^u denote the Zariski closure of S_α^u in G_k^u. One can verify that the irreducible components of G_k^u are the subsets $\mathrm{cl}\, S_\alpha^u$ that are inclusionwise maximal. It is well-known that when u is the zero endomorphism, that is, when G_k^u is the Grassmann variety, the inclusion relations of the $\mathrm{cl}\, S_\alpha^u$ are described by a Bruhat order and G_k^u has just one irreducible component, G_k^u itself. To find those α corresponding to components, for arbitrary u an k, is an open problem.

An important step toward the solution of this problem was given by Shimomura in [5]. He introduced two operations defined on semistandard k-tableaux to obtain new k-tableaux. Some of his results are: If β is obtained from α by applying the first operation, then $\mathrm{cl}\, S_\alpha^u \subset \mathrm{cl}\, S_\beta^u$. So, $\mathrm{cl}\, S_\alpha^u$ cannot be a component.

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Keywords and phrases: Young diagram, semistandard k-tableau, fixed subspace, nilpotent endomorphism, irreducible component.

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SEMISTANDARD k–TABLEAUX: COVERING RELATIONS

HÉCTOR DÍAZ–LEAL AND JOSÉ MARTÍNEZ–BERNAL

Abstract

Semistandard k–tableaux are combinatorial objects, related to Young diagrams, that parametrize a decomposition into affine spaces of the variety G_k^u of k-dimensional subspaces fixed by a nilpotent endomorphism u. They are partially ordered in a natural way, here we characterize its covering relations. As a first application we show how to compute the dimension of the variety G_k^u by means of a greedy algorithm. As a second application we show that if the Jordan partition of the endomorphism u is of the form $\lambda = (p, \dots, p)$, then the corresponding poset is lexicographically shellable. This last result generalizes the well-known case when u is the zero endomorphism, that is, $\lambda = (1, \dots, 1)$.

Introduction

Semistandard k-tableaux are combinatorial objects that arise naturally in the study of the geometrical structure of the projective variety G_k^u of k-dimensional subspaces fixed by a nilpotent endomorphism u. Let λ denote the Jordan partition of u. In [6] is shown that semistandard k-tableaux of shape λ parametrize a decomposition of G_k^u into affine spaces. It was conjectured there, and proved in [3], that if u is of rectangular type; that is, $\lambda = (p, \dots, p)$, then G_k^u is an irreducible variety. This result was obtained, partially, as a consequence of a purely combinatorial argument.

In this paper we continue the combinatorial study of the poset $[\lambda : k]$, whose elements are the semistandard k-tableaux of shape λ; the partial order was defined in [3]. Our main result here is a characterization of the covering relations of this poset: Theorems (1.3), (1.4) and (1.5) below. As a first application of this result we show how to compute the dimension, as an algebraic variety, of G_k^u by means of a greedy algorithm. As a second application we show that if λ is of rectangular type, then the poset $[\lambda : k]$ is lexicographically shellable.

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Keywords and phrases: Young diagram, semistandard k–tableau, fixed subspace, greedy algorithm, nilpotent endomorphism, lexicographically shellable.

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SOBRE LA CLASE DE LAS $\{1,2,4\}$-INVERSAS COMO SOLUCIÓN DE UN PROBLEMA LINEAL

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Abstract

We present the class of $\{1,2,4\}$ generalized inverse of a real $m \times n$ matrix A as the linear variety which is solution of a linear problem. We also obtain the solution space of the associate homogeneous problem and its relations with the column spaces of A and A^T.

Resumen

A partir de caracterizaciones del conjunto de las $\{1,2,4\}$-inversas de una matriz real A $m \times n$, se presenta dicho conjunto como la variedad solución de un problema lineal. Se determina, además, el espacio solución del problema homogéneo asociado y se establecen sus relaciones con los espacios columna de A y de A^T.

1. Introducción

El concepto de inversa generalizada para matrices arbitrarias $m \times n$ fue presentado en 1935 por Moore [1] [2], cuya definición es esencialmente la siguiente:

Definición. A^+ es la inversa generalizada de A si

$$A\ A^+ = P_{R(A)}$$
$$A^+\ A = P_{R(A^+)}$$

A es real $m \times n$, $R(A)$ denota el espacio columna de A, $P_{R(A)}$ y $P_{R(A^+)}$ denotan a las matrices de proyección ortogonal de $\mathbb{R}^m$ y $\mathbb{R}^n$ sobre $R(A)$ y $R(A^+)$, respectivamente.

En su trabajo de 1955, Penrose [3], caracterizó la inversa generalizada de una matriz como la solución única de un conjunto de ecuaciones matriciales:
Para cualquier matriz real A, $X = A^+$ si y sólo si:
(1) $AXA = A$
(2) $XAX = X$
(3) $(AX)^T = AX$

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MORSIFICATION OF $\mathcal{D}$-MODULES

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Abstract

A proof is given of the equality conjectured by P. Deligne about the dimension of the space of vanishing cyles at a point x in V of a holonomic $\mathcal{D}$-module $\mathcal{M}$ on a complex manifold V with respect to an holomorphic function f with the characteristic cycle of $\mathcal{M}$.

Introduction

Let $f: \mathcal{U} \to \mathbf{C}$ be a complex analytic function defined on an open neighborhood of 0 in $\mathbf{C}^n$. The point 0 is an ordinary quadratic critical point (or a non-degenerate critical point) of f if $df_0 = 0$ and the Hessian of f at 0 is not zero. A complex analogue of Morse lemma shows that there are local complex analytic coordinates $x_1 \dots, x_n$ of $\mathbf{C}^n$ at 0 such that $f = \sum x_i^2$ in an open neighborhood $\mathcal{V}$ of 0. Such a critical point is stable in the following sense: for any small analytic perturbation $f + g$ of f, the function $f + g$ has only one critical point in $\mathcal{V}$ which is still ordinary quadratic.

More generally, let $x_1, \dots, x_n$ be local holomorphic coordinates of $\mathbf{C}^n$ at 0. To the function f we associate the $\mathbf{C}$-algebra

$$\mathbf{C}\{x_1, \dots, x_n\}/J(f),$$

quotient of the local $\mathbf{C}$-algebra $\mathbf{C}\{x_1, \dots, x_n\}$ of complex analytic series in the coordinates $x_1, \dots, x_n$ convergent in some neighborhood of 0 by the ideal generated by the partial derivatives of f in those coordinates. Weierstrass preparation Theorem implies that the function f has an isolated singular point at 0, if and only if $\mathbf{C}\{x_1, \dots, x_n\}/J(f)$ is a finite dimensional $\mathbf{C}$-vector space. In this case, the complex dimension of the $\mathbf{C}$-vector space $\mathbf{C}\{x_1, \dots, x_n\}/J(f)$ is called the Milnor number of f at 0. If 0 is an isolated critical point of f and $\overline{\mathrm{B}}$ is a small closed ball which contains no other critical point of f than 0, another application

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A MULTIWAVELET BASED ON PIECEWISE C^1 FRACTAL FUNCTIONS AND RELATED APPLICATIONS TO DIFFERENTIAL EQUATIONS

Peter R. Massopust

Abstract

A multiwavelet approach to solving partial differential equations based on fractal functions is presented. The scaling vector and multiwavelet for this method consist of piecewise C^1 fractal functions supported on intervals of length at most 2. Some properties such as weak regularity and Fourier transforms are discussed. As the underlying fractal functions are interpolatory and vanish on the boundary, this approach is ideal for solving variational boundary value problems using Galerkin or collocation methods.

1. Introduction

Recently scaling functions and wavelets have been used to obtain (weak) numerical solutions of partial differential equations. An incomplete list of references is [2,3,6,10,18,23,24,25,30,32]. In some of these papers only the scaling functions are used for Galerkin-type solution techniques and the associated wavelets are not assumed to be fully orthogonal.

If linear differential operators of polynomial type are considered then it is found that in almost all cases the stiffness matrix in the underlying iterative procedure is ill-conditioned. However, this problem can be solved, and its solution is presented in the paper by Dahmen and Kunoth [8].

In this paper, we present a piecewise C^1 scaling vector and associated multiwavelet and consider related applications to differential equations. The existence of such a scaling vector and multiwavelet was first announced in [14]. The authors based the construction on techniques developed earlier in [16,17]. This particular scaling vector and multiwavelet consists each of three piecewise fractal functions interpolating a finite set of nodes in $\mathbb{Z}/2$ and $\mathbb{Z}/4$, respectively. The support of

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Keywords and phrases: fractal functions, scaling vector, multiwavelet, multiresolution analysis, Sobolev spaces, weak solution, collocation method.

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CONES AND SUSPENSIONS THAT ARE HILBERT CUBES

Dedicated to Professor Henry W. Gould on the occasion of his seventieth birthday

SAM B. NADLER, JR.

Abstract

Let Q denote the Hilbert cube; let Cone(Y) and $\Sigma(Y)$ denote, respectively, the cone over Y and the suspension over Y. It is shown that the following are equivalent: Cone(Y) is Q; $Y \times [0,1]$ is Q; $\Sigma(Y)$ is Q. Some consequences are given.

1. Introduction

The paper is motivated by recent finite-dimensional results: Assume that the space Y is not an $(n-1)$-sphere; if the cone over Y is an n-cell, then $Y \times [0,1]$ is an n-cell ((3.3) of [1]). The converse is false for each $n \geq 5$ even when Y is a manifold ((4.3) of [1]).

We prove that the implication for n-cells stated above and its converse are true for the Hilbert cube Q and that the analogous equivalence for suspensions is also true (2.2). This result provides us with a number of non-Q-manifolds whose cones and suspensions are Q (2.5). In contrast, Q is the only Q-manifold whose cone or suspension is Q (2.6). Finally, we show that Q is the only compact Q-manifold that is a suspension (2.7).

We use the following notation: $I = [0,1]$; Q denotes the Hilbert cube; AR and ANR stand for absolute retract and absolute neighborhood retract (respectively) without assuming compactness; $\times$ and $\Pi_{i=1}^{\infty}$ are used in denoting cartesian products; $Z/_K$ denotes the quotient space obtained by shrinking $K \subset Z$ to a point [8, p. 125]; $\approx$ means "is homeomorphic to".

The *cone over* Y, denoted by Cone(Y), is the quotient space $Y \times I/_{Y \times \{1\}}$. The letter v will always denote the vertex, $\{Y \times \{1\}\}$, of Cone(Y).

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Keywords and phrases: absolute neighborhood retract, absolute retract, CE map, compactification, cone, Hilbert cube, Q-manifold, quotient space, shape, simply connected, suspension, Z-set.