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THE JACOBSON RADICAL POWER SERIES OF MODULE CATEGORIES AND THE REPRESENTATION TYPE

DANIEL SIMSON AND ANDRZEJ SKOWROŃSKI

Abstract

Let R be an artinian ring with an identity element. Representation type properties of the category $\operatorname{mod} R$ of finitely generated right R-modules are studied by means of the vanishing properties of the sequence of powers of the infinite Jacobson radical $\operatorname{rad}^{\infty}(\operatorname{mod} R))$ of $\operatorname{mod} R$ defined below. One of our main results of the paper is Theorem (3.3) asserting that for a strongly simply connected finite dimensional algebra R over an algebraically closed field K the following conditions are equivalent: (i) R is representation-tame and domestic, (ii) the infinite Jacobson radical $\operatorname{rad}^{\infty}(\operatorname{mod} R)$ of the category $\operatorname{mod} R$ is right T-nilpotent, (iii) the square of $(\operatorname{rad}^{\infty}(\operatorname{mod} R))^{\infty}$ is zero. It is also shown that if R is an artinian hereditary ring, or R is an artinian PI-ring then the following conditions are equivalent: (a) the infinite radical $\operatorname{rad}^{\infty} = \operatorname{rad}^{\infty}(\operatorname{mod} R)$ is zero, (b) $\operatorname{rad}^{\infty}(R_R, -) = 0$ and every right R-module is a direct sum of finitely generated modules, (c) the ring R is of finite representation type. A list of open problems is presented in Section 4.

1. Introduction

Throughout this paper we assume that R is a connected basic right artinian ring with an identity element. We denote by $J(R)$ the Jacobson radical of R. We recall that R is said to be connected if R is not decomposable into a product of rings; and R is said to be basic, if $R/J(R) \cong F_1 \times \cdots \times F_m$, where $F_1, \ldots, F_m$ are division rings. We denote by $\operatorname{mod} R$ the category of finitely generated right R-modules.

We recall that a ring R is said to be of *finite representation type* if R is artinian and the number of the isomorphism classes of finitely generated indecomposable right (and left) R-modules is finite.

Following [11] by the *Jacobson radical* of the category $\operatorname{mod} R$ we shall mean the two-sided ideal $\operatorname{rad}(\operatorname{mod} R)$ of $\operatorname{mod} R$ generated by all non-invertible homomorphisms between indecomposable modules in $\operatorname{mod} R$ (see [3], [25]).

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Keywords and phrases: Auslander-Reiten quiver, infinite Jacobson radical, *pg*-critical algebra, pure semisimple ring, tame algebra, domestic algebra, tubular algebra.

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TOPOLOGICAL GROUPS FOR TOPOLOGISTS: PART I

MIKHAIL G. TKACHENKO

Contents

1. Introduction

We present here the first part of a relatively concise (and hence incomplete) survey of the theory of topological groups; the final part will appear in the next issue of this journal. The survey is addressed primarily to the General Topology-inclined reader, because we focus our attention to topological ideas and methods in the area and completely omit the very rich and profound algebraic part of the theory of locally compact groups and the theory of integration on locally compact groups.

The application of topological methods to the study of Lie groups and various kinds of transformation groups goes back the second half of the nineteenth century. After the basics of General Topology had been developed, it was natural

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Keywords and phrases: Topological group, normal subgroup, direct product, quotient group, metrizable, first countable, sequential, complete, locally compact, $\aleph_0$- bounded invariant basis, thin set.

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STABILITY OF PENCILS OF QUADRICS IN $\mathbf{P}^4$

D. AVRITZER AND R. MIRANDA

Abstract

In this article the stability (in the sense of Geometric Invariant Theory) is analyzed for pencils of quadrics in $\mathbf{P}^4$. The base locus of such a pencil is in general a Del Pezzo surface of degree 4, and we relate the description of the unstable and strictly semi-stable pencils to the alternate descriptions of the Del Pezzo surfaces as blowups of the plane at a subscheme of length five.

1. Introduction

In this article we present the Geometric Invariant Theory related to pencils of quadrics in $\mathbf{P}^4$, Del Pezzo surfaces of degree 4, and finite subschemes of the plane of length 5. Although the stability issues for linear systems of curves and quadrics has been studied before (see for example [12], [6], [13] and [14]) to our knowledge this has not been directly related to the invariant theory for Del Pezzo surfaces and the finite subschemes of the plane.

These three problems are, as is well-known, three aspects of the same classical subject. Given a general pencil of quadrics in $\mathbf{P}^4$, its base locus is a Del Pezzo surface of degree four; and conversely, the quadrics through such a Del Pezzo surface form a pencil. Given a finite subscheme S of the plane of length 5, the cubics through S embed the plane birationally as a Del Pezzo surface of degree 4 in $\mathbf{P}^4$; and conversely, every Del Pezzo surface arises this way. Thus to a planar subscheme of length 5 there is a naturally associated pencil of quadrics in $\mathbf{P}^4$, the quadrics through the embedded Del Pezzo surface.

We begin this paper by reviewing the classical material about pencils of quadrics, as treated by Corrado Segre (see [10]) and exposed for example in [5]. Pencils of quadrics are classified by their *Segre Symbol*, which we review. Using results of Wall, we give an explicit description of the ring of invariants of pencils of quadrics in $\mathbf{P}^4$, leading to an explicit construction of its two-dimensional moduli space; it is a double cover of a weighted projective plane, with weights $(4, 8, 12)$. We also determine the orbit structure of the moduli space; since it is only a coarse moduli space, the strictly semistable orbits are

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A CONSTRUCTION OF HOMOGENEOUS MATRIX PROBLEMS

DIETER VOSSIECK

Abstract

We describe a class of matrix problems such that all indecomposable matrices fit into homogeneous tubes.

Introduction

In their investigation of minimal wild bocses, Bautista, Lei and Zhang discovered an unexpected phenomenon: for one of these bocses, every indecomposable representation X is homogeneous in the sense that it admits an almost split sequence of the form $0 \to X \to E \to X \to 0$ ([ZLB2]). The original proof was based on lengthy calculations ([ZLB1]). In search of a direct explanation, we noticed that the bocs under consideration has a more familiar interpretation: its category of representations is equivalent to the category of finite-dimensional projective modules over the algebra of dual numbers equipped with an endomorphism. Replacing in an intermediate step the algebra of dual numbers by an arbitrary symmetric algebra, we finally arrived at a general and surprisingly simple construction of matrix problems for which every indecomposable matrix is homogeneous (Theorem (2.1)).

We have been informed that in the meantime Bautista, Lei and Zhang, in collaboration with Crawley-Boevey, also found a short proof of their original result which applies as well to matrix problems derived from symmetric algebras.

1. Matrix problems

In the sequel, the base field k is assumed to be algebraically closed for simplicity of notation. The terminology, if not explained here, is that of [GR].

(1.1). For our purposes, a *matrix problem* $(\mathcal{A}, \mathcal{B})$ consists of an aggregate $\mathcal{A}$ (over k) and an $\mathcal{A}$-bimodule $\mathcal{B}$ (= k-bilinear functor from $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$ into mod_k, the category of finite-dimensional k-vector spaces). It gives rise to a new aggregate $\mathcal{A}\mathcal{B}$ whose objects are the $\mathcal{B}$-*matrices*, i.e. the pairs (U, x) with $U \in \mathcal{A}$ and $x \in \mathcal{B}(U, U)$; a morphism $(U, x) \to (V, y)$ is given by an $f \in \mathcal{A}(U, V)$ such that $fx = yf$.

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AUSLANDER-REITEN COMPONENTS FOR CLANS

CHRISTOF GEIß AND JOSÉ ANTONIO DE LA PEÑA

Dedicated to Professor Helmut Lenzing
on the occasion of his 60th birthday

Abstract

We show how results of the Auslander-Reiten theory of special biserial algebras can be carried over to clannish algebras. Also, we find a class of clannish algebras whose repetitive algebra is again clannish, in parallelity with the well-known case of gentle algebras. As an application, we show that the stable Auslander-Reiten quiver $\Gamma_{\widehat{\Lambda},s}$ of the repetitive algebra of a gentle algebra Λ contains a finite number of $\mathbb{Z}\mathbb{A}_\infty$-components if $\widehat{\Lambda}$ is not domestic. This carries over to the clannish situation.

Introduction

Let k be a field. Clannish algebras were introduced in [CB] as a special case of "clans," a class of tame matrix-problems; in the same paper there is an explicit description of the indecomposable representations of clans, which is basically independent of the field. In [Ge] there was obtained an explicit description of Auslander-Reiten sequences for clans, which is based on a description of the indecomposables which is independent of the base-field [Bd], [Dn]. In this paper we give a description of the possible shapes of components of the stable Auslander-Reiten $\Gamma_{\Lambda,s}$ quiver of a quasi-clannish algebra Λ (quasi-clannish algebras, defined in (4.2) are a minor generalization of clannish algebras). The point is that quasi-clannish algebras can be obtained in case char $k \neq 2$ as skew-group constructions (see [ReRd]) from special-biserial algebras [SkWa]. Special biserial algebras are tame, and the description of the indecomposable modules, as well as the Auslander-Reiten sequences are basically independent of the field; moreover the description of the stable components of the Auslander-Reiten quiver is well-known, see for example [BuRi]. The theory developed in [ReRd] allows one to determine the shape of the stable

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FACTORING GROUPS HAVING PERIODIC MAXIMAL SUBGROUPS

SÁNDOR SZABÓ AND COBURN WARD

Abstract

It has been shown that if every normed factorization of a finite abelian group contains a periodic factor (the Hajós property), then every factorization must have a factor which does not span the entire group (the Rédei property.) We ask the following: if some maximal subgroup has the Hajós property, then does the group have the Rédei property? We verify this in four new cases. The arguments are based on using characters and we turn to a computer when the number of cases we have to inspect is large.

1. Introduction

Let G be a finite abelian group, written multiplicatively with identity element e. If A and B are subsets of G, then AB will represent the product of A and B, which is the collection of all ab such that $a \in A$ and $b \in B$. If each element of AB is uniquely expressible as the product of some $a \in A$ and some $b \in B$, then we say that the product of A and B is *direct*. If G is the direct product of subsets A and B, then we say that $G = AB$ is a *factorization* of G. If $e \in A$ and $e \in B$, we say that the factorization $G = AB$ is *normed*. The concept of a group factorization has many connections with other parts of mathematics; see [13] for Number Theory, [5] for Functional Analysis, [4] for Ramsey Numbers, [10] for Tilings and [2] for Coding Theory.

The smallest subgroup of G which contains a subset A of G, the *span* of A, will be denoted by $\langle A \rangle$. If for each factorization $G = AB$, either $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$, we will say that G has the *Rédei property*. A subset A of a finite abelian group G is called *periodic* if there is an element $g \in G \setminus \{e\}$ such that $Ag = A$. Each element g with this property is called a period of A. All the periods of A together with the identity element form a subgroup H of G. In addition there is a subset C of G such that $A = HC$, where the product is direct. If for every factorization $G = AB$, it follows that either A or B is periodic, then we say that G has the *Hajós property*. The Rédei and Hajós properties are related; by Lemma 1 of [11], if G has the Hajós property, then it has the Rédei property.

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STUDY OF A HILBERT SPACE OF HOLOMORPHIC FUNCTIONS

JOSÉ A. CANAVATI

Abstract

There is an extensive literature devoted to the study of Banach and Hilbert spaces of holomorphic functions on the unit disk, the whole plane, and half-spaces. It is the purpose of this work, to show that a corresponding theory can be developed for the domain $\Omega = \{z \in \mathbb{C} : |\arg z| < \pi\}$, where many important functions like the principal branches of $\log z$ and arbitrary powers z^a of the complex variable z are defined.

1. Introduction

In a very interesting work [2], Bargmann considered the Hilbert space $\mathfrak{F}_1$ consisting of all entire functions $f(z)$ such that

$$\|f\|^2 = \int_{\mathbb{C}} |f(z)|^2 \, d\nu < \infty, \tag{1.1}$$

where $d\nu = \frac{1}{\pi} \exp(-|z|^2) \, dx \, dy$ ($z = x + iy$) is the Gaussian measure on the plane $\mathbb{C}$.

Here, the inner product on $\mathfrak{F}_1$ is given by

$$(f, g) = \int_{\mathbb{C}} f(z)\overline{g(z)} \, d\nu \, . \tag{1.2}$$

Among other things (as a matter of fact, he dealt with the more general Hilbert space $\mathfrak{F}_n$ consisting of all entire functions f on $\mathbb{C}^n$), he gave a characterization of those entire functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ satisfying condition (1.1): he showed that $f \in \mathfrak{F}_1$ iff $\sum_{k=0}^{\infty} k! \, |a_k|^2 < \infty$, and in this case one has $\|f\|^2 = \sum_{k=0}^{\infty} k! \, |a_k|^2$. Also, if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ belong to $\mathfrak{F}_1$, then the inner product

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Keywords and phrases: Gaussian measure, Hilbert space, holomorphic function, reproducing kernel.

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CHAMPS COMPLETS AVEC SINGULARITÉS NON ISOLÉES SUR LES SURFACES COMPLEXES

JULIO C. REBELO

Abstract

We provide a full list of holomorphic normal forms for complete holomorphic vector fields around a non isolated singularity in complex dimension 2. This list enables us to show that the analytic curve formed by the singularities of a holomorphic vector field in a complex compact surface has at most nodal singularities.

Résumé

Nous donnons la liste des possibles formes normales holomorphes d'un champ de vecteurs holomorphe complet au voisinage d'une singularité non-isolée en dimension complexe 2. Cette liste nous permet de montrer que la courbe analytique formée par les singularités d'un champ de vecteurs holomorphe sur une surface complexe compacte ne peut avoir que des singularités nodales.

1. Introduction

Le but de cet article est compléter la classification des germes de champs de vecteurs semi-complets en dimension complexe 2 (initiée dans [Reb1] et [Gh-Reb]), en considérant le cas des singularités non isolées. On obtient comme corollaire les résultats correspondant pour les champs de vecteurs holomorphes complets sur les surfaces complexes. Évidemment le long de ce travail nous ne considérons pas des champs de vecteurs identiquement nuls. La classification des germes de champs semi-complets à singularité non isolée s'énonce de la façon suivante :

THÉORÈME A. *Soit X un germe de champ de vecteurs holomorphe défini et semi-complet au voisinage de l'origine de $\mathbb{C}^2$. Supposons que l'origine n'est pas une singularité isolée de X. Alors, un changement de coordonnées près, X s'écrit sous l'une des formes ci-dessous :*

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WEIGHTED $L^P - L^Q$ INEQUALITIES FOR THE FRACTIONAL MAXIMAL OPERATOR WHEN $1 < q < p < \infty$

Y. Rakotondratsimba

Abstract

We give conditions, some necessary and some sufficient on weights $u(\cdot)$ and $v(\cdot)$ for which the fractional maximal operator M_α is bounded from the weighted Lebesgue spaces L^p_v into L^q_u whenever $1 < q < p < \infty$ and $0 \leq \alpha < n$. Actually, for a large class of weights, such a boundedness is characterized.

1. Introduction

The fractional maximal operator M_α of order α, $0 \leq \alpha < n$, acts on locally integrable functions of $\mathbb{R}^n$ as

$$(M_\alpha f)(x) = \sup_{t>0}\left\{t^{\alpha-n}\int_{|x-y|<t}|f(y)|dy\right\}.$$

Our purpose in this work is to derive conditions on weight functions $u(\cdot)$ and $v(\cdot)$ for which there is a constant $C > 0$ such that

$$(1.1)\qquad \left(\int_{\mathbb{R}^n}(M_\alpha f)^q(x)u(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n}f^p(x)v(x)dx\right)^{\frac{1}{p}} \qquad \text{for all } f(\cdot) \geq 0$$

whenever $1 < q < p < \infty$. The boundedness (1.1) will be also denoted by $M_\alpha : L^p_v \to L^q_u$.

Since inequalities (1.1) have a fundamental role in Analysis, they have been studied extensively by many authors for the range $p \leq q$ (see for instance [Sa-Wh-Zh], [Ge-Go-Ko]).

A characterization for $M_\alpha : L^p_v \to L^q_u$, with $q < p$, was due to I. Verbitski [Ve] (Theorem 2, p. 128) and asserts that this boundedness is equivalent to

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HEREDITARILY INDECOMPOSABLE CONTINUA HAVE UNIQUE HYPERSPACE 2^X

SERGIO MACÍAS

Abstract

We prove that if X is a hereditarily indecomposable continuum and Y is a continuum such that 2^X (the hyperspace of compact subsets of X) is homeomorphic to 2^Y then X is homeomorphic to Y

A *continuum* is a nonempty, compact, connected, metric space. A *subcontinuum* of a continuum X is a continuum contained in X. A continuum X is said to be *decomposable* provided that $X = A \cup B$, where A and B are proper subcontinua of X. X is *indecomposable* if it is not decomposable. A continuum is *hereditarily indecomposable* provided that each subcontinuum of it is indecomposable. An *arc* is any space homeomorphic to $[0, 1]$. If A is a subset of the continuum X and $\varepsilon > 0$ is given, then $\mathcal{V}_\varepsilon(A)$ will denote the open ball about A of radius ε. Given a continuum X and a point x in X, then the *composant* of x in X is the union of all the proper subcontinua of X containing x.

Given a continuum X, we define its *hyperspaces* as the following sets:

$$2^X = \{A \subset X \mid A \text{ is closed and nonempty}\},$$
$$\mathcal{C}(X) = \{A \in 2^X \mid A \text{ is a connected}\}.$$

It is known that 2^X is a metric space with the Hausdorff metric, $\mathcal{H}$, defined as follows:

$$\mathcal{H}(A, B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}_\varepsilon(B) \text{ and } B \subset \mathcal{V}_\varepsilon(A)\},$$

(see [N, (0.1)]), and in fact, 2^X and $\mathcal{C}(X)$ are arcwise connected continua (see [N, (1.13)]). Let us observe that there is an isometric copy of X contained in $\mathcal{C}(X)$, namely: $\mathcal{F}_1(X) = \{\{x\} \mid x \in X\}$.

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TWO DIMENSIONAL WEAK PSEUDOMANIFOLDS ON SEVEN VERTICES

Basudeb Datta

Abstract

In this article we have explicitly determined all the 2-dimensional weak pseudomanifolds on 7 vertices. We have proved that there are (up to isomorphism) 13 such weak pseudomanifolds. The geometric carriers of them are 6 topological spaces, three of which are not manifolds.

Introduction

Recall that a *simplicial complex* is a collection of non-empty finite sets (sets of vertices) such that every non-empty subset of an element is also an element. For $i \geq 0$, the elements of size $i+1$ are called the i-simplices of the complex. For $i = 1, 2, 3$, the i-simplices are also called the *edges*, *triangles* and *tetrahedra* of the complex, respectively. A k-simplex $\{v_0, \ldots, v_k\}$ is also denoted by $v_0 \cdots v_k$. For a simplicial complex X, the maximum k such that X has a k-simplex is called the *dimension* of X. The set $V(X) := \bigcup_{\lambda \in X} \lambda$ is called the *vertex-set* of X. A simplicial complex X is called *finite* if $\#(V(X)) < +\infty$. A simplicial complex X may be thought of as a prescription for the construction of a topological space by pasting together geometric simplices. The topological space thus obtained from a simplicial complex X is called the *geometric carrier* of X and is denoted by $|X|$.

If X_1, X_2 are two simplicial complexes, then a *simplicial isomorphism* from X_1 to X_2 is a bijection $\pi : V(X_1) \to V(X_2)$ such that for $\sigma \subseteq V(X_1)$, σ is a simplex of X_1 if and only if $\pi(\sigma)$ is a simplex of X_2. The complexes X_1, X_2 are called (simplicially) *isomorphic* when such an isomorphism exists. We identify two simplicial complexes if they are isomorphic. An isomorphism from a simplicial complex X to itself is called an *automorphism* of X.

A simplicial complex X is called a *weak pseudomanifold* of dimension d if (i) all the maximal simplices of X are d-simplices and (ii) each $(d-1)$-simplex of X is contained in exactly two d-simplices of X. A d-dimensional weak pseudomanifold is called a *pseudomanifold* (without boundary) if (iii) for any two distinct d-simplices σ and τ, there exists a sequence $\sigma_1, \ldots, \sigma_n$ of

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Keywords and phrases: two dimensional complexes, pseudomanifolds, triangulated 2-manifolds.

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NOTE ON THE FUNCTIONAL LAW OF THE ITERATED LOGARITHM FOR THE BOOTSTRAP EMPIRICAL PROCESS

TIM ZAJIC

Abstract

We prove a functional law of the iterated logarithm for the bootstrap empirical process. Our proof adapts the approach taken by Dembo and Zajic in deriving a functional law of the iterated logarithm for the empirical process and that of Giné and Zinn in their treatment of the law of large numbers and central limit theorem for the bootstrap empirical process.

1. Introduction and statement of results

Let $\{X_i\}$ be an i.i.d. sequence of random variables defined on a probability space $(\Omega, \mathcal{B}, P)$ with values in a Polish space (Σ, d) and common law μ. We equip Σ with the metric topology and Borel σ-field and denote by $B(\Sigma)$ the bounded measurable functions on Σ. Given a class of functions $\mathcal{F} \subset B(\Sigma)$ such that $0 \leq f \leq 1$ for all $f \in \mathcal{F}$, we may consider the *functional centered, scaled empirical measure*

$$\widehat{L}_n(t) \equiv n^{-1/2} \sum_{i=1}^{[nt]} (\delta_{X_i} - \mu) , \quad t \in [0, \infty)$$

as taking values in the Banach space $l_\infty(\mathcal{F})$, the space of bounded functionals over $\mathcal{F}$ equipped with the norm $\|F\|_{\mathcal{F}} = \sup_{f\in\mathcal{F}} |F(f)|$, in which case we refer to $\widehat{L}_n(\cdot)$ as the *functional centered, scaled empirical process*. We denote $\widehat{L}_n(1)$ by $\widehat{L}_n$.

With $\beta(t) = \sqrt{2(t \vee 1) \log\log(t \vee 3)}$, let $D_\beta[l_\infty(\mathcal{F})]$ denote the Banach space of cadlag functions $F : \mathbb{R}_+ \to l_\infty(\mathcal{F})$ such that

$$\|F\|_{\mathcal{F},2} = \sup_{t\in\mathbb{R}_+} \frac{\|F_t\|_{\mathcal{F}}}{\beta(t)} < \infty ,$$

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Keywords and phrases: functional law of the iterated logarithm, bootstrap empirical process, moderate deviations principle.