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DUALITY OF TOTALLY BOUNDED ABELIAN GROUPS

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ABSTRACT. Let G be a totally bounded Abelian (Hausdorff) group and denote by $\widetilde{G}$ the group of *characters, i.e.,* continuous homomorphisms from G into the usual Torus $\mathbb{T}$, equipped with operation defined pointwise, and endowed with the topology of *pointwise convergence* on G. Define the *evaluation mapping* $\varphi : G \longrightarrow \widetilde{\widetilde{G}}$ by the relation $\varphi(x)(\lambda) := \lambda(x)$ for $x \in G$, $\lambda \in \widetilde{G}$. We show that φ is a topological isomorphism of G onto $\widetilde{\widetilde{G}}$. We compare this with the usual duality on locally compact Abelian (LCA) groups. As an application, a new proof is presented of the fact that LCA groups respect compactness when equipped with their Bohr topology.

1. Introduction and motivation

All groups considered in this note are commutative. If G is a topological group, denote by $\widehat{G}$ the group of characters of G (as defined in the abstract) equipped with the topology of uniform convergence on the compact subsets of G. So defined $\widehat{G}$ becomes a topological group. Let the evaluation map $\Omega : G \to \widehat{\widehat{G}}$ be defined in a similar way as in the abstract. The *Pontryagin-van Kampen (P-vK) Duality* Theorem states that if G is locally compact and Hausdorff, then Ω is a topological isomorphism of G onto $\widehat{\widehat{G}}$. Notice that in this case, even though $\widehat{G}$ is also locally compact (and Hausdorff), G and $\widehat{G}$ can be topologically quite different, as $\widehat{G}$ is compact whenever G is discrete and *vice versa*.

The class of totally bounded Hausdorff Abelian groups, *i.e.,* of subgroups of compact Hausdorff groups, is quite stable as it is closed under forming subgroups, products and Hausdorff quotients. Moreover, their topological structure is very easy to describe: It is a theorem of Comfort and Ross [4] that the topology of such a group is the weakest that makes the elements of the character group continuous. This in particular implies that the character group A of G is *point-separating, i.e.,* if $g \in G$ is not the identity, then there is $a \in A$ such that $a(g) \neq 1$. The converse is evident: If G is an Abelian group and A is any *point-separating* group of homomorphisms from G into $\mathbb{T}$, then since $\mathbb{T}$ is compact, the weakest topology on G that makes the elements of A continuous makes of it a totally bounded Hausdorff group.

Therefore an equivalent formulation of the theorem of Comfort and Ross is the following: *If G is a topological group, then it carries the topology of pointwise convergence on its character group if and only if G is totally bounded.* Thus

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Keywords and phrases: Ascoli's Theorem, Baire space, Bohr topology, character, compact-open topology, equicontinuity, finite-open topology, pointwise convergence, Pontryagin-van Kampen duality, totally bounded group.

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A CHARACTERIZATION OF POSITIVE UNIT FORMS, PART II

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ABSTRACT. This paper concludes the work begun in [1]. It considers unit forms, i.e. positive definite integral quadratic froms with unitary coefficients in the quadratic terms. The equivalence classes of connected unit forms are given by Dynkin diagrams. The paper presents a characterization of positive unit forms which are equivalent to $\mathbb{D}_n$ for some integer n in terms of the associated bigraphs and gives a list for the case $\mathbb{E}_6$.

1. Introduction and result

We consider *unit forms*, that are integral quadratic forms

$$q : \mathbb{Z}^n \to \mathbb{Z}, \quad q(x) = \sum_{i=1}^{n} q_i x(i)^2 + \sum_{i<j} q_{ij} x(i)x(j)$$

such that $q_i = 1$ for all i. Unit forms play an important role in the theory of representations of algebras as associated forms to a finite dimensional algebra over an algebraically closed field: the Tits form and in case the algebra has finite global dimension also the Euler form. Their properties, such as (weak) positivity or (weak) non-negativity, reflect properties of the algebras, see for example [4, 6, 2, 3].

A unit form is called *positive* if $q(x) > 0$ for all non-zero x. Two unit forms, p and q, are called *$\mathbb{Z}$-equivalent* if there exists a $\mathbb{Z}$-invertible linear transformation T such that $p = qT$. It is well known, that positive unit forms can be classified, up to $\mathbb{Z}$-equivalence, by Dynkin diagrams. Namely, one associates to each unit form q a bigraph $\mathrm{B}(q)$ with vertices $1, \dots, n$ and edges of two types, full and broken ones, according to the following. Between i and j, there are $-q_{ij}$ full edges, if $q_{ij} < 0$, else there are q_{ij} broken edges. Conversely, to any bigraph B (without loops and not both, broken and full edges, between two fixed vertices) we may associate a unit form q_B such that $\mathrm{B}(q_B) = B$. A unit form is called *connected* if its bigraph is connected. Each connected, positive unit form q is $\mathbb{Z}$-equivalent to q_Δ, where Δ is a Dynkin diagram, called the *Dynkin-type* of q and denoted by $\mathrm{Dyn}(q)$. A bigraph is called a *cycle* if it is connected and every vertex has exactly two neighbours.

We denote by $\Phi(q)$ the *frame* of a unit form q, that is the graph obtained from $\mathrm{B}(q)$ by turning the broken edges into full ones. In [1] it was shown that a connected unit form q is positive of Dynkin type $\mathbb{A}_n$ if and only if $\mathrm{B}(q)$ *satisfies the cycle condition* (that is, each cycle contains an odd number of broken edges) and $\Phi(q)$ is *a tree assemblage of complete graphs* (that is, $\Phi(q)$ is obtained from the disjoint union of complete graphs $\Sigma_1, \dots, \Sigma_n$ by identifying $\sigma_i(\alpha)$ with $\sigma_j(\alpha)$,

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DIMENSION OF THE FIXED POINT SET OF A NILPOTENT ENDOMORPHISM ON THE FLAG VARIETY

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ABSTRACT. By means of a combinatorial optimization approach, we reduce to a procedure of polynomial complexity the problem of computing the dimension of the projective algebraic variety of flags fixed by a nilpotent endomorphism. We also recover a necessary and sufficient condition to decide when this dimension attains a well known upper bound. This last result is related to the celebrated Gale-Ryser Theorem on the existence of (0, 1)-matrices.

1. Introduction

Let V denote an n-dimensional vector space and let N be a nilpotent endomorphism of V. A subspace $W \subseteq V$ is said to be *fixed* by N if $N(W) \subset W$. For a positive integer k let $G(n, k)^N$ denote the family of subspaces W of dimension k that are fixed by N. It is well known that $G(n, k)^N$ is a projective algebraic variety, i.e., it is the set of common zeros of a collection of homogeneous polynomials. The dimension of this variety is, at least intuitively, the minimum number of parameters needed to describe any of its elements; for instance, when $N = 0$, $G(n, k)^0$ results the Grassmann variety of dimension $k(n - k)$; and more particularly, $G(n, 1)^0$ is the projective space of dimension $n - 1$. The dimension of $G(n, k)^N$ was computed in [1] by means of a greedy algorithm. In this paper we go one step further: first we generalize subspaces to flags, and then the problem of computing the dimension of the variety of flags fixed by N is reduced to a procedure of polynomial complexity (Theorem (5.11)).

The paper is organized as follows: Sections 2–3 introduce flags, tableaux and the dimension problem; Sections 4–5 bring out our combinatorial optimization approach; and Sections 6–8 deal with a special case and some upper bounds for the dimension.

2. Preliminaries

Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be a *partition* of n, i.e., each λ_i is a positive integer, $\lambda_1 \geq \cdots \geq \lambda_p$, and $\lambda_1 + \cdots + \lambda_p = n$. Once we fix a base $\{e_1, \dots, e_n\}$ of V, λ determines a nilpotent endomorphism N_λ of V as follows: first consider the Young diagram of shape λ (i.e., a diagram with λ_i cells left-justified in its i-th

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SYMMETRIC RADICALS OVER NOETHERIAN RINGS

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ABSTRACT. Symmetric radicals over a Noetherian ring R that satisfies the second layer condition are characterized in terms of stability, link closure and other properties of certain classes of prime ideals. The notions of a *saturated* ring and a τ-*rigid* ring are introduced and used to characterize radicals that are symmetric on Noetherian bimodules in case R has finite classical Krull dimension.

1. Introduction

A *pair* of radicals for a ring R will always mean an ordered pair (σ, τ) where τ is a torsion radical defined on right R-modules and σ is a torsion radical defined on left R-modules. A pair (σ, τ) is called *weakly symmetric* provided $\tau(R/P) = \sigma(R/P)$ for all prime ideals P. A pair (σ, τ) is called *symmetric* provided $\tau(A/B) = \sigma(A/B)$ for all ideals $B \subseteq A$. Clearly, any symmetric pair is weakly symmetric. On the other hand, for a well known pair $(\sigma_\alpha, \tau_\alpha)$ defined in terms of modules with Krull dimension α over a two-sided Noetherian ring R (see Section 5), it is an open question if symmetry follows from weak symmetry. Even assuming that $(\sigma_\alpha, \tau_\alpha)$ is symmetric, it is an open question if $\tau_\alpha(M) = \sigma_\alpha(M)$ for a Noetherian $R-R$ bimodule M. A pair (σ, τ) with the property that $\tau(M) = \sigma(M)$ for all Noetherian $R - R$ bimodules M will be called a *strongly symmetric pair*.

The first part of this paper is concerned with characterizing a symmetric pair in terms of the behavior of certain classes of prime ideals that are naturally associated with a torsion radical. The starting point of our investigation, in Section 4, is a result of Beachy's in [2] that connects symmetry with weak symmetry via the following property: A torsion radical τ is called *stably bounded* provided $R/r_R(M)$ is τ-torsion for all finitely generated modules M_R that contain an essential submodule N with $R/r_R(N)$ τ-torsion. Beachy's result [2, Prop. 6] asserts that for a Noetherian ring R that satisfies the strong second layer condition, a weakly symmetric pair (σ, τ) is symmetric if and only if both τ and σ are stably bounded. From [2, pg.245], stable boundedness of τ amounts to requiring that the associated bounded radical of τ is stable. A torsion radical τ is *stable* provided any essential extension of a τ-torsion module is τ-torsion.

It is shown, in Corollary (4.7), that Beachy's assertion remains true if R is only required to satisfy the second layer condition and both σ and τ are stable on tame modules. Since the latter is characterized as stable boundedness restricted to tame modules in Proposition (4.1), Beachy's result is an easy

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ON ARTIN RINGS WHOSE IDEMPOTENT IDEALS HAVE FINITE PROJECTIVE DIMENSION

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ABSTRACT. Let Λ be a weakly triangular artin ring such that its idempotent ideals have finite projective dimension. The main purpose of this paper is to show that a Λ-module M has finite projective dimension if and only if M has a filtration with factors in a special (finite) set of modules.

1. Introduction

The study of the idempotent ideals of an artin ring can give important information on the ring itself and on the category of its finitely generated modules. In [1], M. Auslander, M. I. Platzeck and G. Todorov have studied homological properties of the idempotent ideals of an artin algebra Λ, and gave there a characterization of the idempotent two sided ideals which are projective left Λ-modules. Also, artin rings and algebras whose idempotent ideals are projective were studied in [6] and [11] (see also [10]). Observe that an idempotent ideal of an artin ring Λ can be characterized as the trace $\tau_P(\Lambda)$ of a projective Λ-module P in Λ.

The purpose of these notes is to generalize some results from [11]. It is proven there that if Λ is an artin ring such that all its idempotent ideals are projective ((iip)-rings for short), then the finitistic projective dimension of Λ is at most one. Moreover, if Λ is an artin algebra, then the subcategory $\mathcal{P}^{<\infty}$ consisting of the Λ-modules of finite projective dimension has relative almost split sequences. In order to prove the results above, the second author has shown that for such a class of algebras, there exists a finite set of modules $\mathcal{A}$ such that a module has finite projective dimension if only if it has a filtration with factors in $\mathcal{A}$.

Here, we are going to relax the condition on the idempotent ideals allowing them to have higher projective dimension but on the other hand we shall have to impose a further condition to show the existence of a filtration as above. We say that an artin ring Λ is *weakly triangular* provided the non-isomorphic indecomposable projective Λ-modules $P_1, \cdots, P_n$ can be ordered in such a way that $\mathrm{Hom}_\Lambda(P_i, P_j) = 0$ whenever $i > j$. Observe that an (iip)-ring is weakly triangular (note that, by [11] (2.1), the latter condition is automatic if all idempotent ideals of Λ are projective). Our main result is as follows. Denote, for each i, by A_i the quotient of P_i by the idempotent ideal $\tau_{P_1\oplus\cdots\oplus P_{i-1}}(P_i)$.

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Keywords and phrases: weakly triangular rings, idempotent ideals.

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EXACT FACTORS OF EXACT CATEGORIES

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ABSTRACT. In the representation theory of algebras situations occur where one has to transport an exact structure $\mathcal{E}$ on a category $\mathcal{A}$ to a factor category by a relation $\mathcal{R}$. We characterize when this is possible and discuss how almost split pairs are transferred to the factor category. As an illustration, we give applications for the category of finitely presented functors on modules over an artin algebra and for the category of regular modules over a wild hereditary algebra.

Introduction

In [5] it was studied how one can form new exact categories by looking at subfunctors of the functor Ext. The motivating examples for this investigations came from the representation theory of algebras where certain reduction functors only happen to be exact when looking at the appropriate exact structure. In fact, in some examples in addition to choosing the correct exact structure one also had to pass to a factor category by a relation (see [5]). For this purpose, in [5, Proposition 1.11] a sufficient condition for a relation $\mathcal{R}$ of an exact category $\mathcal{A}$ was given which ensures that the factor category $\mathcal{A}/\mathcal{R}$ together with the appropriate induced exact structure becomes exact again. A different but similar problem has been considered by M. Schlichting, namely, when is a localization of an exact category by a class of morphisms again an exact category. In his thesis [9] he gives a sufficient condition for this.

It is the aim of the first section of this paper to prove a necessary and sufficient condition for $\mathcal{A}/\mathcal{R}$ to be exact. Moreover, we discuss how our conditions simplify if the relation $\mathcal{R}$ is generated by projective and injective objects. Finally, we study when $\mathcal{A}/\mathcal{R}$ becomes even abelian.

The modern representation theory of algebras was strongly pushed forward by the detection of Auslander and Reiten that almost split sequences exist in the category of finitely generated modules over artin algebras. Meanwhile it turned out that almost split pairs exist in many more exact categories. Therefore we devote the second section to showing how almost split pairs are pushed down to exact factor categories. As a preparation we investigate how projective and injective objects are transferred.

In the third section we use our results to reprove that the category mod (mod Λ) of finitely presented functors from the category mod Λ to the category of abelian groups is abelian and has almost split sequences (it follows from [2, Proposition 3.2]), where Λ is an artin algebra and mod Λ is the category of

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DISTINGUISHED SLOPES FOR QUIVER REPRESENTATIONS

LUTZ HILLE AND JOSÉ ANTONIO DE LA PEÑA

ABSTRACT. Let Q be a quiver without oriented cycles of wild representation type. A slope $\mu = \theta/\kappa$ is a function on the Grothendieck group $K_0(Q)$, where $\theta, \kappa : K_0(Q) \to \mathbb{Z}$ are linear and $\kappa(M) > 0$ for every nonzero module M. We consider properties of slopes for Q. In particular, we consider the existence of slopes of the form $(b^-\theta^- - b^+\theta^+)/(a^+\theta^+ + a^-\theta^-)$, with a^+, a^- positive numbers and b^+, b^- non-negative numbers, at least one non-zero, where $\theta^+ = \langle -, y^+\rangle$ and $\theta^- = \langle y^-, -\rangle$ are linear functionals defined by the positive eigenvectors y^+ and y^- of the Coxeter matrix Φ of Q with eigenvalues ρ and $1/\rho$ respectively, where $1 < \rho$ is the spectral radius of Φ.

1. Introduction

Let Q be a finite connected quiver without oriented cycles. We consider the category $\operatorname{mod} Q$ of finite dimensional representations of Q over a field k. Recall the notion of a slope μ introduced in [HP]: let θ and κ be two $\mathbb{Z}$–linear functions on the Grothendieck group $K_0(Q)$ of $\operatorname{mod} Q$, where the latter one is positive on all non-zero objects, and define $\mu(M) := \theta(M)/\kappa(M)$ (here we write a function on $K_0(Q)$ as taking values on the objects of $\operatorname{mod} Q$). Recall that a representation M is μ–stable, respectively μ–semistable, if for all proper non-zero subrepresentations N of M we have $\mu(N) < \mu(M)$, respectively $\mu(N) \leq \mu(M)$. The μ–stable objects have remarkable geometric properties (see [K]) and in addition also remarkable categorical properties (see [HP] for more details, and [Ru] for a different approach). Thus, it is desirable to study μ–stable representations. Since we already have considered tame hereditary algebras in [HP], Section 5, in this work we shall only consider wild quivers.

The aim of this note is to consider slopes μ which have the natural properties (S1) to (S5) below. Note that any non-trivial slope for a representation-infinite quiver with two vertices (that is wild generalized Kronecker quiver) has all these properties (see [HP], Section 5, Example 2). We refer to [R1] for the definition of preprojective, regular and preinjective indecomposable modules. Observe that a module is called preprojective if it is the direct sum of indecomposable preprojective modules.

(S1) Each indecomposable preprojective and each indecomposable preinjective representation is μ–stable.

(S2) Let P be a preprojective representation, let R be a regular representation, and let I be a preinjective representation. Then $\mu(P) < \mu(R) < \mu(I)$.

(S3) An indecomposable regular representation M is μ–stable precisely if its Auslander-Reiten translation τM is also μ–stable.

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Keywords and phrases: quiver representations, wild representation type, slope.

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ON GENERIC MODULES FOR STRING ALGEBRAS

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ABSTRACT. We consider a string algebra A. In case A is domestic, and G, G' are non-isomorphic generic modules, then G' neither generates nor cogenerates G. On the other hand, in case A is non-domestic, we are going to construct sequences $G_1, G_2, \dots$ of pairwise non-isomorphic generic modules such that there are monomorphisms $G_i \to G_{i+1}$ and epimorphisms $G_{i+1} \to G_i$, for all i. The proof of the existence of such sequences answers a question raised by Bautista.

In particular, we see that any non-domestic string algebra has generic modules whose endomorphism rings have a non-zero radical. Actually, we will show that in the non-domestic case there always do exist generic modules with nilpotent endomorphisms of arbitrary large nilpotency index. Of course, in the domestic case the nilpotency index of a nilpotent endomorphism of a generic modules is bounded; in fact, it is bounded by the nilpotency index of the radical of A.

Let k be a field and A a finite dimensional k-algebra which is a string algebra. Recall that this means that $A = kQ/\mathfrak{I}$ where $Q = (Q_0, Q_1)$ is a finite quiver and $\mathfrak{I}$ is an admissible ideal generated by monomials, with the following properties: every vertex of Q is endpoint of at most two arrows and starting point of at most two arrows, and second, for any arrow β, there is at most one arrow α such that $\alpha\beta$ does not belong to $\mathfrak{I}$, and at most one arrow γ such that $\beta\gamma$ does not belong to $\mathfrak{I}$.

The finite-dimensional indecomposable A-modules are well-known, this classification is essentially due to Gelfand and Ponomarev (see [GP]; for the notation used here and for more detailed information, we refer to [R]). There are two types of finite-dimensional indecomposable A-modules: First of all, there are the string modules $M(w)$ described by words w. Second, there are the band modules $M(w, \phi, n)$, where w is a primitive cyclic word, ϕ an irreducible polynomial in $k[T]$ different from T and n a natural number: if λ is a non-zero element of k, we write $M(w, \lambda, n)$ instead of $M(w, T-\lambda, n)$. The words considered here use as letters the arrows of the quivers (the 'direct' letters) and formal inverses of these arrows (the 'inverse' letters); such a word may be interpreted as walking around in the quiver, avoiding the given zero relations. We denote by $|w|$ the length of the word w. Recall that a word w is called *cyclic* provided it contains both direct and inverse letters and such that also $w^2 = ww$ is a word; a cyclic word is said to be *primitive* provided it is not a proper power of some other word.

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BIFURCATION OF DISCONTINUOUS MAPS OF THE INTERVAL AND PALINDROMIC NUMBERS

RAFAEL LABARCA AND SERGIO PLAZA S.

ABSTRACT. We study the bifurcation produced by a rotation of a not expansive discontinuous map (a Lorenz map) of the interval. We show the special pattern of bifurcation which occurs in this case.

1. Introduction

In a remarkable contribution of the meteorologist E. N. Lorenz [9], numerical evidence of the existence of a strange attractor for a quadratic system of differential equations in three variables is shown. Later J. Guckenheimer [6] introduced symbolic dynamics in order to understand the topologically equivalence classes for nearly similar attractors. R. F. Williams introduced a geometrical model in order to understand the dynamics of these Lorenz attractors in [18]. Using this geometrical model, the dynamics of the three-dimensional vector field may be reduced to the dynamical behavior of a one-dimensional map with one discontinuity. Guckenheimer and Williams [7] used this fact to show that there exist uncountable many classes of nonequivalent Lorenz attractors. We have to point out that W. Tucker [17] recently announced that, in fact the Lorenz attractor is geometric.

Hence it seems natural to start with a three-dimensional vector field that preserves a two-dimensional foliation and obtain the bifurcation theory in a neighborhood of it throughout the bifurcation theory of the one-dimensional map defined by the foliation.

In this article we consider a vector field X defined on a three-dimensional solid torus having the following critical elements (singularities and periodic orbits): two hyperbolic periodic orbits σ (repulsive) and γ (attracting), and two hyperbolic singularities of saddle type s_1 and s_2. We assume that all the critical elements are hyperbolic (see Figure 1).

We suppose that a component of the unstable manifold $W^u(s_2)$ of the singularity s_2 transversally intersects a component of the stable manifold $W^s(s_1)$ of the singularity s_1. Define the first return map from the disk D_1 (which is identified with the disk D_2 to obtain the solid torus).

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IDEALS OF FUNCTIONS WHICH ACHIEVE ZERO ON A COMPACT SET

To Professor Krzysztof Maurin

ANTONI WAWRZYŃCZYK

ABSTRACT. Let A be a complex unital commutative Banach algebra. If K is a compact set of maximal ideals of A, we denote $S(K) = \bigcup_{I \in K} I$. The principal result of the paper asserts that for every ideal J of A contained in $S(K)$ there is a maximal ideal I of A such that $J \subset I \subset S(K)$. The consequences in the theory of joint spectra are studied.

1. Introduction

Let K be a compact Hausdorff space. Denote by $C(K)$ the Banach algebra of continuous functions on K equipped with the supremum norm.

The following theorem was proved in [3].

THEOREM (1.1). *Let $\mathcal{A}$ be a unital subalgebra of $C(K)$. Let $f_1, \dots, f_k \in \mathcal{A}$. Assume that every function which belongs to the ideal generated in $\mathcal{A}$ by the elements $f_1, \dots, f_k$ vanishes somewhere on K. Then for an arbitrary $g \in \mathcal{A}$ there exists $\mu \in \mathbb{C}$ such that the ideal generated by $f_1, \dots, f_k, g - \mu$ also consists of functions which attain zero on K.*

The algebraic content of this theorem is studied in section 1, where we take as $\mathcal{A}$ the algebra of polynomial functions restricted to a compact set $K \subset \mathbb{C}^n$. We obtain a presumably unknown fact about the rationally convex hull of K.

Section 2 contains the principal results of the paper. For a given compact set K of maximal ideals in a commutative unital Banach algebra A we consider the set $S(K) = \bigcup_{I \in K} I$. We refer to the elements of $S(K)$ as K-singular elements. Our Theorem (3.5) asserts that every ideal of A consisting of K-singular elements is contained in a maximal ideal of A also consisting of K-singular elements.

The special case of this situation was studied in [3], where K is the cortex of A, that is the set of the maximal ideals of A consisting of the joint topological zero divisors. The corresponding set $S(K)$ consists of all topological zero divisors. Applying Theorem (1.1) we prove that every ideal consisting of topological zero divisors is contained in some maximal ideal also consisting of topological zero divisors.

In the present paper we show that this phenomenon is of more general character.

The author is very greateful to the reviewers for stimulating remarks.

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Keywords and phrases: maximal ideal, subspectrum, topological zero divisor.

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GROEBNER BASES AND THE COHOMOLOGY OF GRASSMANN MANIFOLDS WITH APPLICATION TO IMMERSION

KENNETH G. MONKS

ABSTRACT. Let $G_{k,n}$ be the Grassmann manifold of k-planes in $\mathbb{R}^{n+k}$. Borel showed that $H^*\left(G_{k,n};\mathbb{Z}_2\right)=\mathbb{Z}_2\left[w_1,\ldots,w_k\right]/I_{k,n}$ where $I_{k,n}$ is the ideal generated by the dual Stiefel-Whitney classes $\overline{w}_{n+1},\ldots,\overline{w}_{n+k}$. We compute Groebner bases for the ideals $I_{2,2^i-3}$ and $I_{2,2^i-4}$ and use these results along with the theory of modified Postnikov towers to prove immersion results, namely that $G_{2,2^i-3}$ immerses in $R^{2^{i+2}-15}$. As a benefit of the Groebner basis theory we also obtain a simple description of $H^*\left(G_{2,2^i-3};\mathbb{Z}_2\right)$ and $H^*\left(G_{2,2^i-4};\mathbb{Z}_2\right)$ and use these results to give a simple proof of some non-immersion results of Oproui.

1. Introduction

Let $G_{k,n}$ denote the Grassmann manifold of unoriented k-planes in $\mathbb{R}^{n+k}$. Define the immersion dimension of $G_{k,n}$ to be the smallest j such that $G_{k,n}$ immerses in $\mathbb{R}^j$. There have been many results in the literature which obtain upper and lower bounds on the immersion dimension of $G_{k,n}$. Lower bounds are obtained for certain families of $G_{k,n}$ in [6], [9], [12], [13] through the use of Stiefel-Whitney classes. Upper bounds have been obtained by Lam [7] who showed $G_{k,n}$ has immersion dimension less than or equal to $\binom{n+k}{2}$. For many values of n, k Lam's result improves on the standard upper bound of $2nk-1$ given by Whitney's theorem [14] and the stronger upper bound of $2nk-\alpha(nk)$ (where $\alpha(nk)$ is the number of ones in the binary expansion of nk) given by Cohen's theorem [3]. However, except for the results of Hiller-Stong [6] who showed the immersion dimension $G_{k,k}$ and $G_{k,k+1}$ is equal to Lam's upper bound, the exact immersion dimension of $G_{k,n}$ is largely not known. There are also many results on the immersion and non-immersion of projective spaces which is the case $G_{1,n}$ (or since $G_{k,n}$ is diffeomorphic to $G_{n,k}$, we could also say $G_{k,1}$). As we are not interested in the projective space case in this article, we shall henceforth assume that $1<k\leq n$ when discussing $G_{k,n}$.

Many common techniques for computing bounds on the immersion dimension of $G_{k,n}$ require a good working understanding of the structure of the cohomology ring $H^*\left(G_{k,n};\mathbb{Z}_2\right)$. Borel [1] gave the following description of this ring. It is well known that

$$H^*\left(G_{k,\infty};\mathbb{Z}_2\right)=H^*\left(BO(k);\mathbb{Z}_2\right)=\mathbb{Z}_2\left[w_1,\ldots,w_k\right]$$

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