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SOCIEDAD MATEMATICA MEXICANA

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# TRIANGULAR PARTITIONS WITH AUGMENTED FIRST ROWS AND WEIGHTS FOR THE SYMMETRIC GROUPS IN CHARACTERISTIC TWO

LUIS VALERO-ELIZONDO

ABSTRACT. Alperin's weight conjecture for the symmetric groups has been proved using an enumeration of the weights and the simple modules (see [2]), but so far there is no explicit way to associate weights with simple modules. In this paper we prove that some weights for the symmetric groups in characteristic two can be found inside the Brauer quotients of the simple modules parameterized by partitions consisting of a triangle with an enlarged first row. Furthermore, we find subgroups of $S_n$ which are minimal such that their Brauer quotients have a simple projective summand.

## 1. Introduction

One of the most important (and difficult) open problems in the representation theory of finite groups is Alperin's weight conjecture. Even though this conjecture has already been established for several families of groups (see Section 3), some of these proofs are just enumerations of the sets of weights and irreducibles, with no explicit correspondence between them. Such is the case of the symmetric groups.

Alperin and Fong proved in [2] that Alperin's conjecture holds for the symmetric groups, so we know that the number of weights for $kS_n$ equals the number of simple $kS_n$-modules, where $k$ is a field of characteristic $p > 0$. In [11] we proved that three infinite families of 2-regular partitions - which, as is well-known, parameterize the irreducible $kS_n$-modules in characteristic two - have Brauer quotients which are simple and projective. Hence these Brauer quotients represent weights for $kS_n$, and at least for the simple modules parameterized by these special partitions we have a way of assigning explicit weights.

In this paper we prove that for another family of partitions, the Brauer quotients of the simple modules parameterized by the partitions in the family have simple projective summands. Although we cannot guarantee that their Brauer quotients are themselves simple (and hence projective), the existence of a simple projective summand is enough to once again assign specific weights to the simple modules parameterized by the partitions in our family.

We shall work with an algebraically closed field $k$ of characteristic two. The partitions we shall consider are the ones that can be obtained by adding $n$ nodes to the first row of a triangular partition of size $t$. Since triangular partitions

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# ON A $q$-EXTENSION OF THE HERMITE POLYNOMIALS $H_n(x)$ WITH THE CONTINUOUS ORTHOGONALITY PROPERTY ON $\mathbb{R}$

R. ÁLVAREZ-NODARSE, M. K. ATAKISHIYEVA, AND N. M. ATAKISHIYEV

ABSTRACT. We study a polynomial sequence of $q$-extensions of the classical Hermite polynomials $H_n(x)$, which satisfies continuous orthogonality on the whole real line $\mathbb{R}$ with respect to the positive weight function. This sequence can be expressed either in terms of the $q$-Laguerre polynomials $L_n^{(\alpha)}(x;q)$, $\alpha = \pm 1/2$, or through the discrete $q$-Hermite polynomials $\tilde{h}_n(x;q)$ of type II.

## 1. Introduction

There is a well-known family of the continuous $q$-Hermite polynomials of Rogers [1, 2]

$$H_n(\cos\theta|q) := \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q e^{i(n-2k)\theta} = e^{in\theta}{}_2\phi_0\left( \begin{matrix} q^{-n}, 0 \\ - \end{matrix} \,\middle|\, q, q^n e^{-2i\theta} \right), \tag{1.1}$$

which are $q$-extensions of the classical Hermite polynomials $H_n(x)$ for $q \in (0,1)$. Throughout this paper we will employ the standard notations of the $q$-special functions theory, see [3]-[5]. In particular,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}} = \begin{bmatrix} n \\ n-k \end{bmatrix}_q \tag{1.2}$$

stands for the $q$-binomial coefficient and $(a;q)_0 = 1$ and $(a;q)_n = \prod_{j=0}^{n-1}(1 - aq^j)$, $n = 1, 2, 3, \ldots$, is the $q$-shifted factorial. Besides, explicit forms of $q$-polynomials from the Askey-scheme [4] are often expressed in terms of the terminating basic hypergeometric polynomial

(1.3)

$$\begin{aligned} {}_r\phi_s&\left( \begin{matrix} q^{-n}, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} \,\middle|\, q, z \right) \\ &:= \sum_{k=0}^{n} \frac{(q^{-n};q)_k (a_2;q)_k \cdots (a_r;q)_k}{(b_1;q)_k (b_2;q)_k \cdots (b_s;q)_k} \frac{z^k}{(q;q)_k} \left[ (-1)^k q^{k(k-1)/2} \right]^{s-r+1} \end{aligned}$$

of degree $n$ in the variable $z$. So the continuous $q$-Hermite polynomials in (1.1) correspond to the case when $r = 2$ and $s = 0$.

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# THE INDEX OF NON ALGEBRAICALLY ISOLATED SINGULARITIES

VÍCTOR CASTELLANOS VARGAS

ABSTRACT. An important goal of the theory of the index of real analytic vector fields with a (real) isolated singularity is to prove an analogue of the Eisenbud–Levine theorem for such fields for which the zero set of the corresponding complexified vector field has arbitrary dimension.

In this paper we give an algebraic formula to compute the Poincaré–Hopf index of a class of vector fields that have an isolated singularity which is non algebraically isolated. We have called those vector fields of **Type $\mathbf{A}_k$**. In the main theorem we give a finite dimensional module similar to that found by Eisenbud and Levine with a non–degenerate, symmetric bilinear form whose signature is the desired Poincaré–Hopf index. We give an intrinsic characterization of this module in terms of the homology of the Koszul complex of the vector field.

## 1. Introduction

Let $X = \sum_{i=1}^{n} X^i \dfrac{\partial}{\partial x_i}$ be a (germ of a) real analytic vector field (at 0 in $\mathbb{R}^n$) with an isolated singularity at 0. We will denote by $X_{\mathbb{C}}$ the complexified vector field, which is a (germ of a) holomorphic vector field (at 0) in $\mathbb{C}^n$ and is defined by the same power series as $X$; the zeroes of $X$ and $X_{\mathbb{C}}$ are denoted by $Z_{\mathbb{R}}(X)$ and $Z_{\mathbb{C}}(X)$ respectively.

We remember that an isolated singularity of a real analytic vector field, is called algebraically isolated, if it is also an isolated singularity of the complexified vector field. When the (real) isolated singularity is non isolated singularity of complexified vector fields, then it is called non algebraically isolated singularity.

The Poincaré–Hopf index, $\mathrm{Ind}_0 X$, of a real analytic vector field $X$ at the isolated singular point 0 is the degree of the map $X/\|X\| \colon S_\epsilon^{n-1} \to S_1^{n-1}$ of a sufficiently small sphere $\|x\| = \epsilon$ in the source space to the unit sphere in the target space. When the isolated singularity is algebraically isolated, Eisenbud and Levine showed how one can compute the Poincaré–Hopf index through a bilinear form, [EL]. This result, was also proved by Khimshiashvili [AGV], [K].

Let $\mathcal{A} = \mathbb{R}[[t_1, \ldots, t_n]]$ (respectively $\mathcal{O} = \mathcal{A} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[[t_1, \ldots, t_n]]$) be the local ring of germs of real (respectively complex) analytic germs at $0 \in \mathbb{R}^n$ (respectively $0 \in \mathbb{C}^n$). The Eisenbud and Levine theorem states the following.

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# A NOTE ON A DIVERGENT SERIES RELATED TO THE RIEMANN ZETA FUNCTION

LUIS G. GOROSTIZA

ABSTRACT. An approximation for a class of divergent series in given. A special case is the series $\sum n^{-s}$, $0 < s \leq 1$, related to the Riemann Zeta function. The approximation is also related the behavior of random walks on an ultrametric group.

## 1. Results and proofs

Consider the series $\sum n^{-s}$ with real $s > 0$, which for $s > 1$ represents the Riemann Zeta function $\zeta(s)$. Approximations of $\zeta(s)$ for $0 < s < 1$ involving the partial sums $\sum_{n=1}^{N} n^{-s}$ are well known (e.g. [1], [4], [6]), and they may also be regarded as approximations (or renormalizations) of the divergent series $\sum n^{-s}$, $0 < s < 1$. We give here an approximation of the series $\sum n^{-s}$, $0 < s \leq 1$, by means of exponential convergence summands.

PROPOSITION (1.1). *For any $a > 1$,*

$$\sum_{n=1}^{\infty} \frac{1}{n^s}\left(1 - \exp\left\{-\frac{n^s}{a^n}t\right\}\right) \sim \begin{cases} \dfrac{1}{1-s}\left(\dfrac{\log t}{\log a}\right)^{1-s}, & 0 < s < 1 \\ \log\log t, & s = 1 \end{cases}$$

*as $t \to \infty$.*

This is special case of the following general setting. Let $c_n$, $n = 1, 2, \ldots$ be positive real numbers such that $c_n \leq c_{n+1}$ for all $n$ and $\sum_{n=1}^{\infty} c_n^{-1} = \infty$, $a > 1$ and

$$A(t) = \sum_{n=1}^{\infty} \frac{1}{c_n}\left(1 - \exp\left\{-\frac{c_n}{a^n}t\right\}\right), \quad t > 0.$$

Clearly this series converges (since $1 - e^{-x} \leq x$) and $A(t) \nearrow \sum c_n^{-1} = \infty$ as $t \to \infty$. The question that interests us is how fast does $A(t)$ approach $\infty$ as $t \to \infty$. The next proposition gives the answer.

PROPOSITION (1.2).

$$A(t) \sim \sum_{n=1}^{\log t/\log a} \frac{1}{c_n} \quad as \quad t \to \infty.$$

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# THE PROJECTION PROPERTY OF A FAMILY OF IDEALS IN SUBALGEBRAS OF BANACH ALGEBRAS

ARMANDO VELÁZQUEZ GONZÁLEZ AND ANTONI WAWRZYŃCZYK

ABSTRACT. Let $B$ be unital Banach algebra and $A$ its unital subalgebra. The family $\mathcal{I}^l_B(A)$ of left ideals of $A$ consisting of elements non invertible in $B$ is studied. We prove that under the condition that $aAa^{-1} \subset A$ for all $a \in A$ which are invertible in $B$, the family $\mathcal{I}^l_B(A)$ has the following projection property: for every $k$-tuple of mutually commuting $a_1, \dots, a_k \in I \in \mathcal{I}^l_B(A)$ and for every $c \in A$ commuting with all $a_i$ there exists $\lambda \in \mathbb{C}$ and $J \in \mathcal{I}^l_B(A)$ such that $a_1, \dots, a_k, c - \lambda e \in J$.

It follows that the mapping defined for $a_1, \dots, a_k \in A$ by the formula

$$\sigma_B(a_1, \dots, a_k) = \{(\lambda_1, \dots, \lambda_k) \in \mathbb{C}^k | \exists I \in \mathcal{I}_B(A),\ a_i - \lambda_i e \in I,\ 1 \le i \le k\}$$

is a subspectrum on $A$.

## 1. Introduction

This paper presents the most suitable approach to problems studied in the papers [3], [4], [5] in the case of commutative Banach algebras. Besides the considerable simplification of proofs we obtain an important generalization of the principal results on the noncommutative case.

Let $B$ be a Banach algebra over $\mathbb{C}$ with the unit denoted by $e$. By $G(B)$ we denote the set of all invertible element of $B$. By $\mathcal{I}^l(A)$ is denoted the family of all proper left ideals of $A$. Let $\mathcal{I}^l_B(A)$ be the set of left ideals of $A$ whose elements are not invertible in $B$. If $A$ is commutative we use the notation $\mathcal{I}(A)$ and $\mathcal{I}_B(A)$, respectively.

By $R(A)$ we mean the smaller closed subalgebra of $B$ which contains $A$ and the set $\{a^{-1} | \ a \in A \cap G(B)\}$.

Let $\mathcal{U}$ be a family of left ideals in $A$. We say that the family $\mathcal{U}$ has *the projection property* if for every ideal $I \in \mathcal{U}$ for all mutually commuting $a_1, \dots, a_k \in I \in \mathcal{U}$ and for every $c \in A$ commuting with all $a_i$ there exist $\lambda \in \mathbb{C}$ and $J \in \mathcal{U}$ such that $a_1, \dots, a_n, c - \lambda e \in J$. The fact that $\mathcal{I}^l(A)$ has the projection property was proved by Harte in [1] Theorem 4.2.

The papers mentioned at the beginning are related, explicitly or not, to the fact that for a commutative Banach algebra $B$ and for any subalgebra $A \subset B$ the family $\mathcal{I}_B(A)$ has the projection property.

In the present paper we consider the same problem for $B$ being an arbitrary Banach algebra with unit and $A$ its unital subalgebra such that $aAa^{-1} \subset A$ for all $a \in A \cap G(B)$. The subalgebra $A$ is not assumed to be closed in $B$. $A$ is unital subalgebra of $B$ if the unit of $B$ is contained in $A$. The inverse $a^{-1}$ need not to be an element of $A$.

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# ESPACE GÉODÉSIQUE, ORTHOGONALITÉ ENTRE GÉODÉSIQUES ET NON EXISTENCE DES POINTS FOCAUX DANS LES ESPACES DE HADAMARD

TAOUFIK BOUZIANE

ABSTRACT. The purposes of the present work are, firstly, to consider the problem of defining the concepts of orthogonal geodesics in general geodesic spaces and to investigate, under reasonable assumptions, the existence of such geodesics (theorems (3.2), (3.7). Secondly, to introduce, by analogy with riemannian geometry, the notion of focal points in geodesic spaces and to demonstrate the nonexistence of such points in the case of Hadamard spaces. Finally, we give a conjecture (4.6) about the nonexistence of focal points, under suitable hypothesis, in the case of $CAT_k$ spaces.

## 1. Introduction

La notion de courbure bornée sur des espaces métriques revient à Alexandrov [Ale] et Busemann [Bus]. Ce dernier a initié la théorie des espaces à courbure non positive. Plusieurs autres mathématiciens s'y sont également intéressés par la suite ; nous citerons en particulier Gromov [Gr] dont le travail a permis une explosion de la théorie des espaces singuliers. Pour en avoir un aperçu, nous renvoyons le lecteur à la bibliographie de Ballmann [Ba]. Dans cette dernière référence, une généralisation du fameux théorème de Cartan-Hadamard y est décrite. Cette généralisation s'inspire de celle faite, dans le cadre des espaces métriques, géodésiques, complets et localement convexes, par S.B. Alexander et R.L. Bishop [AlBi].

Citons quelques exemples d' espaces à courbure majorée :

(1) Les varietés Riemaniennes complètes à courbure sectionnelle majorée.

(2) Les immeubles de Tits euclidiens, sphériques et hyperboliques [Ti], [Ro].

(3) Les complexes avec courbure constante par morceaux [ChDa], [DaJa].

En restant dans le même esprit et après une brève section (2) de rappels des notions fondamentales de la géometrie des espaces géodésiques, nous consacrerons la section suivante (3) à la notion d'orthogonalité entre géodésiques. Après avoir naturellement défini cette notion d'orthogonalité entre géodésiques, nous considèrerons ensuite le cas d'une géodésique complète et d'un point extérieur. Nous analyserons alors, suivant les conditions posées sur l'espace et (ou) sur la géodésique, les possibles existence et unicité d'une seconde géodésique passant par ce point extérieur et orthogonale à la première géodésique.

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# SAMUEL UNIFORMITIES

ADALBERTO GARCÍA-MÁYNEZ AND STEPHEN WATSON

ABSTRACT. Every $T_2$-uniformity $\mathcal{U}$ on a set $X$ determines a totally bounded uniformity $\mathcal{S}(\mathcal{U})$ on $X$ which determines the same topology and satisfies a universal property. This is the so-called *Samuel uniformity* of $\mathcal{U}$. If $\mathcal{V}$ is another uniformity on $X$ and $\mathcal{S}(\mathcal{U}) = \mathcal{S}(\mathcal{V})$, we describe a homeomorphic copy of the completion $\widehat{(X, \mathcal{V})}$ as a subspace of the Samuel compactification $\widehat{(X, \mathcal{S}(\mathcal{U}))}$. If $\mathcal{V}_1, \mathcal{V}_2$ are two uniformities on $X$ lying between $\mathcal{S}(\mathcal{U})$ and $\mathcal{U}$, we find conditions which insure that the completions $\widehat{(X, \mathcal{V}_1)}$, $\widehat{(X, \mathcal{V}_2)}$ are homeomorphic. We find conditions for two subsets $A$ and $B$ of $(X, \mathcal{U})$ to be completely separated in $\widehat{(X, \mathcal{S}(\mathcal{U}))}$; for $(X, \mathcal{U})$ to be $Z$-embedded, $G_\mathcal{S}$-dense or $C_1^1$-embedded in its Samuel compactification; for a subset $A$ of $(X, \mathcal{U})$ to be $G_\mathcal{S}$-dense in its closure in $\widehat{(X, \mathcal{S}(\mathcal{U}))}$ and for the existence of a normal Wallman basis $\mathcal{B}$ on $X$, consisting of cozero sets of $X$ and whose corresponding uniformity coincides with $\mathcal{S}(\mathcal{U})$.

## 1. Introduction

The theory of Hausdorff compactifications of Tychonoff spaces may be viewed as a part of the theory of completions of totally bounded uniform spaces. In fact, every Hausdorff compactification $Z$ of a Tychonoff space $X$ is the completion of a $T_2$-uniform space $(X, \mathcal{U})$, where $\mathcal{U}$ is a certain totally bounded uniformity on $X$. In case $Z = \beta X$, the Stone-Čech compactification of $X$, the corresponding uniformity on $X$ is the one generated by finite cozero covers of $X$. If we denote this uniformity by $\mathcal{U}_0(X)$, it is clear that every continuous map of $(X, \mathcal{U}_0(X))$ into any other totally bounded uniform space $(Y, \mathcal{V})$ is uniformly continuous and hence the map can be extended to the completions $\beta X$ and $\widehat{(Y, \mathcal{V})}$, where $\widehat{(Y, \mathcal{V})}$ stands for the completion of $(Y, \mathcal{V})$. This is an alternative way to understand the universal property of $\beta X$.

If we start with an arbitrary $T_2$-uniform space $(X, \mathcal{U})$, the finite covers in $\mathcal{U}$ generate a totally bounded uniformity on $X$, denoted by $\mathcal{S}(\mathcal{U})$ and called the *Samuel uniformity* on $X$, which has the following properties:

i) $\mathcal{U}$ and $\mathcal{S}(\mathcal{U})$ generate the same topology on $X$.
ii) The identity map $j\colon (X, \mathcal{U}) \to (X, \mathcal{S}(\mathcal{U}))$ is uniformly continuous.
iii) Any uniformly continuous map $\varphi\colon (X, \mathcal{U}) \to (Z, \mathcal{U}_0(Z))$, where $Z$ is a compact Hausdorff space, yields a uniformly continuous map $\widehat{\varphi}\colon \widehat{(X, \mathcal{S}(\mathcal{U}))} \to (Z, \mathcal{U}_0(Z))$ such that $\varphi = \widehat{\varphi} \circ \lambda \circ j$, where $j\colon (X, \mathcal{U}) \to (X, \mathcal{S}(\mathcal{U}))$ is the identity map and $\lambda\colon (X, \mathcal{S}(\mathcal{U})) \to \widehat{(X, \mathcal{S}(\mathcal{U}))}$ is the canonical embedding.

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*Keywords and phrases*: Samuel uniformity, round, co-round, stable, weakly stable, uniformly separated, uniformly compact, uniformly discrete, Wallman basis.

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# MULTIPLE STOCHASTIC FRACTIONAL INTEGRALS: A TRANSFER PRINCIPLE FOR MULTIPLE STOCHASTIC FRACTIONAL INTEGRALS

VICTOR PÉREZ-ABREU AND CONSTANTIN TUDOR

ABSTRACT. Multiple integrals with respect to a fractional Brownian motion are defined explicitely in terms of multiple Wiener integrals and fractional integrals and derivatives of deterministic functions of several variables. As applications we introduce the chaos form of the fractional stochastic integral and we derive the strong law of large numbers for the fractional Brownian motion.

## 1. Introduction

Fractional Brownian motion (fBm) is playing an increasing role in modeling long-range dependence phenomena in many fields such as telecommunication network, economics, hydrology and biology. From the theoretical point of view a fBm $B = \{B_t^H\}_{t\in T}$ is a Gaussian process which is a suitable generalization of the Brownian motion $W = \{W_t\}_{t\in T}$ but exhibiting long-range dependence.

Since the pioneering work by Itô [15], multiple integrals with respect to a Brownian motion and their corresponding chaos expansions have been studied by many authors, being standard tools for dealing with nonlinear functionals of $W$ as well as with anticipating stochastic calculus for $W$. This is true for both the Wiener-Itô and the Stratonovich multiple integrals (see for example the book by Nualart [23]).

It seems then natural to study multiple fractional integrals and in particular chaos expansions for a fractional Brownian motion. This has recently been started by Dasgupta and Kallianpur [4], [5] and Duncan, Hu and Pasik-Duncan [8] for the fBm of Hurst parameter $H > \frac{1}{2}$, and for general Gaussian process twenty years ago by Huang and Cambanis [14]. The techniques used in these papers involve Wick product and reproducing kernel Hilbert space theory.

In this paper, our starting point of view is the representation

$$B_t^H = \int K(t,s)dW_s$$

of the fractional Brownian motion as a (single) Wiener integral of a nonrandom *fractional kernel* $K(t,s)$, with respect to the Wiener process $W$ defined in the same probability space of the fractional Brownian motion $B^H$ (see Barton and Poor [2], Decreusefond and Üstunel [7], Norros et al. [22] and Pipiras and Taqqu [24], [25]). Using this representation we are able to define in an

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# SEMILINEAR FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

JORGE A. LEÓN AND CONSTANTIN TUDOR

ABSTRACT. The purpose of this paper is to study the existence of a unique solution to stochastic differential equations with linear diffusion coefficients driven by a fractional Brownian motion (fBm) with Hurst parameter $H \in (1/2, 1)$. The coefficients are random and the stochastic integral is a forward one. The method that we use to obtain our results is based on the stochastic calculus for the fBm via the Malliavin calculus developed by Alòs and Nualart [2], and on the pathwise approach given by Kohatsu–Higa et al [4] to analyze Stratonovich stochastic differential equations driven by a continuous local martingale.

## 1. Introduction

In recent years several authors have used different interpretations of fractional stochastic integrals to investigate the existence and uniqueness of a solution for stochastic differential equations of the form

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \quad 0 \le t \le T. \tag{1.1}$$

Here $B$ is a fractional Brownian motion (fBm) of Hurst parameter $H \in (0, 1)$. That is, it is a centered Gaussian process such that $B_0 = 0$ and $\mathrm{Var}(B_t - B_s) = |t - s|^{2H}$, $t, s \ge 0$ (see [6]). This implies that the paths of $B$ are Hölder–continuous with all Hölder exponents less than $H$.

Some properties of the fBm $B$ have allowed equations of the form (1.1) to become appropriate models for phenomena that exhibit scale–invariant and long–range correlated noise (see [6]). However, when the Hurst parameter $H \neq 1/2$, $B$ is *not* a semimartingale. Hence we cannot apply the classical Ito's stochastic calculus to construct a stochastic integral with respect to the fBm $B$, with $H \neq 1/2$.

In the case $H \in (1/2, 1)$, it is reasonable to interpret (1.1) as a path–by–path ordinary differential equation due to $B$ having zero quadratic variation (see Lin [5]), and $\int_0^T Y_s dB_s$ existing as a pathwise Riemann–Stieltjes integral for any $\lambda$–Hölder continuous stochastic process $Y$ with $\lambda > 1 - H$ (see Young [11]). This pathwise equation has been considered by different researchers under special assumptions on the diffusion coefficient $\sigma$ (see for instance [3], [5], [10] and [12]). In order to improve this trajectory–by trajectory approach, Zähle [12] has introduced an extension of the Lebesgue–Stieltjes integral via

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# ZERO-SUM SEMI-MARKOV GAMES IN BOREL SPACES: DISCOUNTED AND AVERAGE PAYOFF

FERNANDO LUQUE–VÁSQUEZ

ABSTRACT. We study two-person zero-sum semi-Markov games in Borel spaces with possibly unbounded payoff function, under discounted and average payoff criteria. Conditions are given for the existence of the *value* of the game, the existence of optimal strategies for both players, and for a characterization of the discounted optimal stationary strategies.

## 1. Introduction

This paper deals with two-person zero-sum semi-Markov games with Borel spaces and possibly unbounded payoff function, under discounted and average payoff criteria. Under suitable conditions on the transition law, the payoff function and the distribution of the transition times, we show the existence of the value of the game and the existence of pairs of optimal stationary strategies for both the discounted and the average criteria. We also obtain a characterization for a pair of stationary strategies to be discounted optimal.

In the discounted case, our approach is based on contraction conditions. Our assumptions ensure that the minimax operator is a contraction map with respect to a weighted norm, and we show that the (unique) fixed point is the value of the game. Fan's Minimax Theorem [3] and measurable selection theorems are used to prove the existence of optimal stationary strategies. This approach has been used for several authors in Markov games (for example, [1], [14], [17]) and for semi-Markov games in [11], which considers a countable state space and a bounded payoff function. For the analysis of the average criterion, we suppose, besides the usual continuity/compactness requirements, that the payoff function satisfies a growth condition, and also that the embedded Markov chains have suitable stability properties. A key step, is the use of a data transformation to associate a Markov game with our original semi-Markov game. This transformation has been used by several authors (for example, [4], [5], [11] and [12]). Markov stochastic games with discounted and average payoff have been widely studied by several authors (for example, [1], [8], [13], [14], [16] [17] and [19]) but, to the best of our knowledge, the only papers that study zero-sum *semi-Markov* stochastic games are [10] and [11]. Our main results generalize to the semi-Markov context, some results in [14], [16], [17] and [11].

The remainder of the paper is organized as follows. In Section 2 the semi-Markov game model is described. Next, in Section 3, the discounted and the

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