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AUTOMORPHISMES POLYNOMIAUX PRÉSERVANT UNE ACTION DE GROUPE

STÉPHANE LAMY

RÉSUMÉ. Nous donnons une description précise du groupe des automorphismes polynomiaux de $\mathbb{C}^3$ préservant la forme quadratique y^2+xz, à l'aide de la notion d'automorphisme de $\mathbb{C}^2$ à paramètre. Nous en déduisons une description des automorphismes polynomiaux de $\mathfrak{gl}(2,\mathbb{C})=\mathbb{C}^4$ (resp. $\mathfrak{sl}(2,\mathbb{C})=\mathbb{C}^3$) laissant invariant les orbites de l'action par application adjointe de $\mathrm{GL}(2,\mathbb{C})$ (resp. $\mathrm{SL}(2,\mathbb{C})$).

ABSTRACT. We give a description of the group of polynomial automorphisms of $\mathbb{C}^3$ that preserve the quadratic form y^2+xz, using the notion of automorphism of $\mathbb{C}^2$ with parameter. It follows a description of the polynomial automorphisms of $\mathfrak{gl}(2,\mathbb{C})=\mathbb{C}^4$ (resp. $\mathfrak{sl}(2,\mathbb{C})=\mathbb{C}^3$) that preserve the orbits of the natural action of $\mathrm{GL}(2,\mathbb{C})$ (resp. $\mathrm{SL}(2,\mathbb{C})$).

1. Introduction

Dans l'étude du groupe $\mathrm{Aut}[\mathbb{C}^n]$ des automorphismes polynomiaux de $\mathbb{C}^n$, le cas $n=2$ est très particulier. On dispose en effet d'un théorème de structure qui décrit $\mathrm{Aut}[\mathbb{C}^2]$ comme le produit amalgamé de deux de ses sous-groupes : les groupes affine et élémentaire. Nous noterons A_n le groupe des automorphismes affines de $\mathbb{C}^n$ (autrement dit les automorphismes qui se prolongent en des biholomorphismes de $\mathbb{P}^n_{\mathbb{C}}$), et E_n le groupe des automorphismes élémentaires c'est-à-dire ceux de la forme

$$(x_1,\cdots,x_n)\mapsto(\alpha_1x_1+f_1,\cdots,\alpha_nx_n+f_n)$$

avec $\alpha_i\in\mathbb{C}^*$, $f_i\in\mathbb{C}[x_{i+1},\cdots,x_n]$. En dimension 2 nous avons donc le

THÉORÈME (Jung-Van der Kulk). *Le groupe* $\mathrm{Aut}[\mathbb{C}^2]$ *est le produit amalgamé des sous-groupes affine et élémentaire suivant leur intersection :*

$$\mathrm{Aut}[\mathbb{C}^2]=A_2*_{\cap}E_2\,.$$

Cela revient à dire que A_2 et E_2 engendrent $\mathrm{Aut}[\mathbb{C}^2]$, et que toutes les relations dans $\mathrm{Aut}[\mathbb{C}^2]$ sont induites par les relations dans A_2 et E_2 (pour une démonstration de ce théorème on pourra se référer à [7], [9], [11] ou [12]). Ce résultat ne paraît pas pouvoir se généraliser en dimension 3 (ni a fortiori en dimension supérieure). D'une part il est facile de voir que $\langle A_3,E_3\rangle$ n'est pas

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Mots clés : automorphismes polynomiaux, forme quadratique invariante.
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REDUCTION FUNCTORS AND EXACT STRUCTURES FOR BOCSES

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ABSTRACT. Here we introduce a generalization of free bocses in the sense of Roiter, t-bocses. We study the category of representations of such bocses and full and faithful functors (reduction functors) between its representation categories. An exact structure is introduced on the category of representations of triangular t-bocses. This exact structure is hereditary. We also see the behaviour of the reduction functors in terms of the corresponding exact structures.

1. Introduction

For the systematic study of problems of classification of matrices, M. Kleiner and A.V. Roiter introduced in [10] the Differential Graded Categories (dgc), later A.V. Roiter ([12]) introduced the Bimodules over Categories with Structure of Coalgebra (bocs), these two concepts are closely related. The main feature of these structures is the existence of procedures which allows us to go from the representations of a bocs to the representations of smaller dimension of another bocs. This gives a powerful method of descend, which has been succesfully used in proving central results in the theory of representations of finite dimensional algebras over algebraically closed fields (see [7], [4],[5]). Reduction procedures which can be iterated exist until now only for the semi-free triangular bocses in the sense of [7]. In this paper we introduce a version of bocs, called t-bocs, close to the original free dgc, this version is suitable for dealing with cases where k is not algebraically closed. The relation of t-bocses, bocses and dgc is discussed at the end of sections 2 and 3.

Given a t-bocs $\mathcal{A}$ we will see several constructions of new t-bocses $\mathcal{B}$ and full and faithfull functors $F : \operatorname{Rep}\mathcal{B} \to \operatorname{Rep}\mathcal{A}$, where $\operatorname{Rep}\mathcal{C}$ is the category of representations of the t-bocs $\mathcal{C}$. These functors include the usual reduction functors used in the literature. For the category of representations of certain t-bocses (triangular t-bocses) we study an exact structure (introduced in [11]), for which our reduction functors become exact, this structure is hereditary. For a triangular t-bocs $\mathcal{A}$ and $M, N \in \operatorname{Rep}\mathcal{A}$, we denote by $\operatorname{Ext}_{\mathcal{A}}(M, N)$ the corresponding extension group. We will see a stronger version of triangular t-bocs, free triangular t-bocs, for these t-bocses if $F : \operatorname{Rep}\mathcal{B} \to \operatorname{Rep}\mathcal{A}$ is one of our reduction functors and $\mathcal{A}$ is free triangular, the same holds for $\mathcal{B}$. In this case we will see for $M, N \in \operatorname{Rep}\mathcal{B}$ some relations between $\operatorname{Ext}_{\mathcal{B}}(M, N)$ and $\operatorname{Ext}_{\mathcal{A}}(F(M), F(N))$.

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SMALL SUMSETS IN A PRIME ORDER GROUP

BERNARDO LLANO

ABSTRACT. In this paper, we study pairs of subsets of a prime order group for which the cardinality of their sum is relatively small compared to the sum of the sets cardinality. The smallest case is the theorem of Vosper. We characterize the subsequent cases, that is, pairs (A, B) with $A, B \subseteq \mathbb{Z}_p$ such that $|A+B| = |A| + |B| + i$ $(i = 0, 1)$.

1. Introduction

Addition theorems appeared in the solution of problems of number theory. They were extended to group and semigroup theory. A number of theorems on set sums are known now, some of them are generalizations of the famous Cauchy-Davenport's theorem [3], [4], others are related to the well-known combinatorial problem of Erdös, Ginzburg and Ziv [5], [11], to maximal sum-free sets [14], [15] or to density problems of integers. In the same way, interesting developments have been made in solving problems in the theory of digraphs, which have led to the proof of new addition theorems as well as generalizations of known results [6], [7], [8]. This subject has an extensive literature, an appropriate reference to the theory is [10] for a general view. The main goal of this theory is the description of a sum of sets in terms of certain properties of the summands, as the cardinality or the measure of the sets.

The results to be proved in the next sections are in a sense, a continuation of a work of A. G. Vosper [12], [13] and they were applied to classify tight and untight equations [9]. This classification is part of the study of tightness in uniform hypergraphs as a natural generalization of connectedness in graphs. For details, see [1], [2]. Our purpose of this paper is to study pairs of subsets of a prime order group for which the cardinality of their sum is relatively small compared to the sum of the sets cardinality. The smallest case is the already cited theorem of Vosper. We characterize the subsequent cases, that is, pairs (A, B) such that $|A+B| = |A| + |B| + i$ $(i = 0, 1)$. As corollaries of these theorems, we give the structure of a set $A \subset \mathbb{Z}_p$ such that $|hA| = h|A|$ and $|hA| = h|A| + (h-1)$, where h is a positive integer, $h \geq 2$ and hA denotes the sum of h copies of the set A.

2. Preliminaries

Let $\mathbb{Z}_p$ denote the additive group of residues modulo p, a prime number. The cardinality and the complement of the set $A \subseteq \mathbb{Z}_p$ will be denoted by $|A|$ and $\overline{A}$ respectively. If $\emptyset \neq A, B \subseteq \mathbb{Z}_p$, then the sum of these subsets is

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A MACKEY-ARENS THEOREM FOR TOPOLOGICAL ABELIAN GROUPS

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ABSTRACT. If G is an Abelian group with a collection X of homomorphisms into the usual torus $\mathbb{T}$ such that (a) X is a group under pointwise operation, and (b) the weakest topology on G that makes the elements of X continuous is Hausdorff. We give necessary and sufficient conditions for the existence of the finest locally quasi-convex group topology on G, having exactly X as the group of continuous homomorphisms into $\mathbb{T}$. Such topology is the topology of uniform convergence on an appropriate, sometimes proper, subfamily of weakly compact quasi-convex subsets of X. These results are motivated by the usual Mackey-Arens theorem for locally convex spaces.

1. Introduction and motivation

Given an Abelian topological group (G, t), with underlying group G and topology t, we say that a group topology λ on G is *compatible* with t if the sets of t- and λ-continuous homomorphisms from G into $\mathbb{T}$ coincide, *i.e.,* if $(G,t)^\frown = (G,\lambda)^\frown$. For example, it is a theorem of Comfort and Ross [8] that σ, the weakest topology on G that makes the elements of $(G,t)^\frown$ continuous, is compatible with (G, t). Following the lead from the theory of locally convex spaces (LCSs), it is natural to ask (a) if (G, t) has its *Mackey* topology, *i.e.,* the *finest* group topology μ on G compatible with t, in the sense that if λ is a compatible group topology with t, then $\lambda \leq \mu$, and (b) if such μ exists, is it given as the topology of uniform convergence on a certain family of subsets of the character group? These questions have been undertaken by Chasco, Martín-Peinador and Tarieladze (hereafter Chasco et al) [5]. For example, they prove that there is a topological group with no Mackey topology, thus answering (a) negatively. After giving the reasons for studying question (a) in the realm of locally quasi-convex group topologies (defined below) –a natural generalization of local convexity in the theory of topological vector spaces (TVSs)– they give conditions for its existence, and show that for certain types of groups the Mackey topology exists. As for question (b) the contributions of [5] are significant as they look at the family $\mathfrak{C}$ of weakly compact quasiconvex subsets of the character group -the exact counterpart in the theory of LCS- giving necessary and sufficient conditions for $\mathfrak{C}$ to be the collection requested in (b).

In this paper we give a positive answer to question (b), showing that if the (locally quasi-convex) Mackey topology μ exists, then it is given as the topology

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Keywords and phrases: Abelian topological group, character, compact, compatible topologies, dual group, equicontinuity, locally quasi-convex group, Mackey topology, polar, weak topology.

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EXTREMAL LENGTHS FOR MAPPINGS WITH BOUNDED s-DISTORTION ON CARNOT GROUPS

IRINA MARKINA

ABSTRACT. In 1957 B. Fuglede [8] has introduced the notion of the exceptional measure system. A system of measures E is said to be exceptional of order p if its p-module $M_p(E)$ vanishes. E. Poletsky [28] was the first who applied this notion to the description of the behaviour of a family of curves under nonhomeomorphic quasiconformal mappings. In the present paper we generalize this result by E. Poletsky and study the behaviour of the p-module of a family of horizontal curves under mappings with bounded s-distortion on Carnot groups.

1. Introduction and statement of main results

In 1957 B. Fuglede [8] has introduced the notion of the exceptional measure system. A system of measures E is said to be exceptional of order p if its p-module $M_p(E)$ vanishes. If we consider as a system of measure a family of curves in the Euclidean space $\mathbb{R}^n$, then one of the results by B. Fuglede can be stated as follows: *an absolutely continuous function $u : \mathbb{R}^n \to \mathbb{R}$ with p-integrable partial derivatives is absolutely continuous on p-almost all curves.* A direct consequence of this result is that a quasiconformal mapping is absolutely continuous on p-almost all curves. A quasiconformal mapping is homeomorphic, therefore the inverse mapping possesses the same property. The situation is not so simple for nonhomeomorphic quasiconformal mappings, so called mappings with bounded distortion, or in another terminology, quasiregular mappings [24, 30, 31, 33]. The absence of an inverse mapping does not permit to apply directly Fuglede's result. The first group of inequalities between corresponding modules of families of curves and their images under a nonhomeomorphic quasiconformal mapping was obtained by E. Poletsky [28, 29]. His result has given the start to applications of the method of p-module to the investigation of quasiregular mappings.

Recently, analysis on the homogeneous groups (the Carnot groups $\mathbb{G}$) has been intensively developed. The fundamental role of such groups in analysis was pointed out by E. M. Stein [36], in his address to the International Congress of Mathematicians in 1970, see also his monograph [37]. Briefly, a homogeneous group is a simply connected nilpotent Lie group, whose Lie algebra admits a grading. There is a natural family of dilations on the group under which the metric behaves like the Euclidean metric under the Euclidean dilation [2, 9]. The analysis on homogeneous groups is also a test ground for

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INVERSIBILITÉ TOPOLOGIQUE ET PROBLÈME DE L'IDÉAL FERMÉ

R. CHOUKRI, A. EL KINANI, M. OUDADESS

ABSTRACT. We give algebraic and topological properties of topologically invertible elements. Also, a link is established between the notion involved and the closed ideal problem.

Préliminaires et introduction

Toutes les algèbres considérées sont supposées complexes. On dit qu'une algèbre A est topologique si elle est munie d'une topologie d'espace vectoriel séparée pour laquelle le produit est séparement continu. Si le produit est continu, on dira que A est à produit continu. Une telle algèbre est dite une F-algèbre si elle est metrisable et complète. Une B_0-algèbre est une F-algèbre localement convexe. Une algèbre localement multiplicativement convexe ($a.l.m.c.$) est une algèbre topologique dont la topologie est définie par une famille de semi-normes sous-multiplicatives. Elle est dite de Fréchet si elle est metrisable et complète. Soit $(e_\lambda)_{\lambda\in\Lambda}$, une famille d'éléments d'une algèbre topologique A, où Λ est un ensemble filtrant croissant. On dit que $(e_\lambda)_{\lambda\in\Lambda}$ est une unité approchée à gauche (resp. à droite), de A, si, pour tout $x \in A$, $\lim_\lambda e_\lambda x = x$ (resp. $\lim_\lambda xe_\lambda = x$). Si de plus Λ est dénombrable, $(e_\lambda)_{\lambda\in\Lambda}$ est dite une unité approchée (à gauche ou à droite) séquentielle. Un élément $a \in A$ est dit un diviseur topologique de zéro à droite (resp. à gauche) s'il existe une suite généralisée $(x_\lambda)_\lambda$ de A, ne tendant pas vers zéro, telle que $(ax_\lambda)_\lambda$ (resp. $(x_\lambda a)_\lambda$ tend vers zéro.

Soit A une algèbre topologique complexe. Un élément a, de A, est dit topologiquement inversible si $\overline{aA} = \overline{Aa} = A$, où, pour toute partie X de A, $\overline{X}$ désigne l'adhérence de X dans A. Cette définition différe de celle adoptée par Bhatt ([4]) et Abel ([1]). Ces auteurs s'intéressent surtout à la notion de complétude advertible (advertible completeness). Ici, nous nous intéressons aux éléments topologiquement inversibles pour eux-mêmes. Plus précisement, en notant par $S(A)$ l'ensemble des éléments topologiquement inversibles de A, nous dégageons certaines propriétés, aussi bien algébriques que topologiques, de la partie $S(A)$. Ces propriétés différent suivant la classe d' algèbres topologiques dans laquelle on se place. L'ensemble $S(A)$ a un comportement "semblable" à celui du groupe des éléments inversibles. En fait, ces deux ensembles coincident pour certaine classe d'algèbres topologiques. Dans le cas normé, nous établissons une propriété topologique de $S(A)$ liée à l'existence d'une algèbre

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SOME CLASSES OF p-SUMMING TYPE OPERATORS

OSCAR BLASCO AND TERESA SIGNES

ABSTRACT. Let X, Y be Banach spaces and denote by $\ell_p^w(X,Y)$, $\ell_p^s(X,Y)$ and $\ell_p(X,Y)$ the spaces of sequences of operators (T_n) from X into Y such that $\sup_{||x||=1, ||y^*||=1} \sum | < T_n(x), y^* > | < \infty$, $\sup_{||x||=1} \sum ||T_n(x)|| < \infty$ and $\sum ||T_n|| < \infty$ respectively. Given Banach spaces X, Y, U and V, we introduce and study the classes of bounded linear operators $\Phi : \mathcal{L}(X,Y) \to \mathcal{L}(U,V)$ such that $(T_n) \to (\Phi(T_n))$ maps $\ell_p^s(X,Y)$ into $\ell_p(U,V)$, $\ell_p^s(X,Y)$ into $\ell_p^s(U,V)$ and $\ell_p^w(X,Y)$ into $\ell_p^s(U,V)$.

1. Introduction

Throughout this paper X, Y, Z, W will stand for Banach spaces, $\mathcal{L}(X,Y)$ for the space of bounded linear operators from X into Y and X^* for the dual of X. As usual B_X denotes the closed unit ball of X.

For $1 \leq p < \infty$ we write $\ell_p(X)$ for the Banach space of all *absolutely p-summable* sequences $(x_n)_{n=1}^\infty$ in X, i.e., the space of sequences such that $\|(x_n)\|_{\ell_p(X)} = (\sum_{n=1}^\infty \|x_n\|_X^p)^{\frac{1}{p}} < \infty$ and $\ell_p^w(X)$ for the space of all *weakly p-summable* sequences in X, i.e., the space of sequences $(x_n)_{n=1}^\infty$ such that $(\langle x^*, x_n\rangle)_{n=1}^\infty \in \ell_p$ for every $x^* \in X^*$, which becomes a Banach space with respect to the norm $\|(x_n)\|_{\ell_p^w(X)} = \sup_{x^* \in B_{X^*}} (\sum_{n=1}^\infty |\langle x^*, x_n\rangle|^p)^{\frac{1}{p}}$.

Recall that the space of p-summing operators $\Pi_p(X,Y)$ consists in those bounded linear operators $T \in \mathcal{L}(X,Y)$ such that $\tilde{T}((x_n)_{n=1}^\infty) = (T(x_n))_{n=1}^\infty$ defines a bounded linear operator from $\ell_p^w(X)$ into $\ell_p(Y)$. The reader is referred to [2], [3], [7], [11], [12] or [14] for definitions and results about these classes and their applications in Banach space theory.

We shall simply recall here two related notions which are the main motivations for our future attention.

The first one appears when analyzing operators acting on spaces of vector-valued continuous functions $T : C(\Omega, X) \to Y$, where Ω is a compact Haussdorf space. An operator is called p-dominated operator (see [4], III.19.3) if there exist a constant $C > 0$ and a probability measure μ on Ω such that

$$\|T(f)\|^p \leq C \int_\Omega \|f(t)\|^p d\mu(t)$$

for all $f \in C(\Omega, X)$. The connection between p-summing and p-dominated operators was first given by A. Pietsch, who showed that for finite dimensional Banach spaces X both notions are the same. For infinite dimensional Banach

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INVERSE LIMITS AND A PROPERTY OF J. L. KELLEY, II

W. T. INGRAM

ABSTRACT. In this paper we continue an investigation of the Property of Kelley in inverse limits. In particular, we show every permutation map arising from a permutation in S_3, S_4 or S_5 produces a continuum with the Property of Kelley. The tools to acomplish this include a ray theorem and a union theorem for continua having the Property of Kelley. We round out the investigation with some theorems on confluent maps and a proof that the hereditarily decomposable continuum arising as an inverse limit on [0, 1] using as a single bonding map the logistic map $f_\lambda(x) = 4\lambda x(1-x)$ where λ is the Feigenbaum limit has the Property of Kelley.

Introduction

In an earlier paper [7], the author began a study of the Property of Kelley in inverse limits. That paper along with the results presented here grew out of consideration of a question asked of the author by W. J. Charatonik. In [6], the author considered inverse limits on [0, 1] using a single bonding map determined in a natural way by a permutation – see the end of this section for a full description. Charatonik asked if all the continua which arise as such inverse limits have the Property of Kelley. A partial answer was provided in [7, Section 2]. In this paper, we provide further partial results. However, Charatonik's original question remains unanswered.

By a *continuum* we mean a compact, connected subset of a metric space. By a *mapping* we mean a continuous function. If M is a continuum, a subcontinuum H of M is said to be *irreducible about* a closed subset E of M if H contains E but no proper subcontinuum of H contains E. If M is a continuum, and E is a subset of M, we denote the diameter of E by diam(E). If M is a continuum, we denote by $C(M)$ the space of all subcontinua of M with the Hausdorff metric $\mathcal{H}$ [3, p. 11]. A continuum M with metric d is said to have the *Property of Kelley* (or *Property* κ) provided if $\varepsilon > 0$ there is a positive number δ such that if p and q are points of M and $d(p,q) < \delta$ and H is a subcontinuum of M containing p then there is a subcontinuum K of M containing q such that $\mathcal{H}(H,K) < \varepsilon$. Suppose $(D, \preceq)$ is a directed set, for each α in D, X_α is a topological space and for α and β in D with $\alpha \preceq \beta$, f_α^β is a mapping from X_β to X_α with f_α^α being the identity on X_α for each α in D, and if $\alpha \preceq \beta \preceq \gamma$ then $f_\alpha^\gamma = f_\alpha^\beta \circ f_\beta^\gamma$. By the *inverse limit of the inverse system* $\{X_\alpha, f_\alpha^\beta, D\}$ is meant the subset of $\prod_D X_\alpha$ to which the point x belongs if and only if $f_\alpha^\beta(x_\beta) = x_\alpha$ for each α and β in D such that $\alpha \preceq \beta$. Most inverse systems we consider in this paper will be

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GENERALIZED AXIAL MAPS AND EUCLIDEAN IMMERSIONS OF LENS SPACES

LUIS ASTEY, DONALD M. DAVIS AND JESÚS GONZÁLEZ

ABSTRACT. Generalized axial maps are introduced and related to the immersion problem for 2-torsion lens spaces to produce a characterization of this problem in the metastable range. A direct analysis gives a complete solution of the immersion problem for the missing cases. Other variations of the problem, which extend classical constructions for projective spaces, are discussed.

1. Introduction

The results in this paper are motivated by the well known relationship between the existence of Euclidean immersions of real projective spaces and the existence of axial maps of type (n, k), that is, maps of the form

$$P^n \times P^n \to P^{n+k} \tag{1.1}$$

which are homotopically nontrivial over each axis. B. J. Sanderson observed in [21] that an axial map as above can be obtained from a given immersion $P^n \subseteq \mathbb{R}^{n+k}$ and that, at least in the metastable range (that is, when $n < 2k$), such an immersion can homotopically be recovered from (1.1). Sanderson's work relies on results of Hirsch and Haefliger ([12], also considered by James in [14]) and takes advantage of the so-called *twisted normal bundle* associated to an immersion $P^n \subseteq \mathbb{R}^{n+k}$. The success of the technique depends on the fact that the canonical real line bundle over P^n has multiplicative order 2. However, this is precisely the main drawback in a first attempt to generalize the ideas for higher 2-torsion lens spaces. Indeed, on the one hand, Hirsch's basic result on immersing manifolds [13] implies that the codimension in an optimal immersion for $L^{2n+1}(2^e)$ —the $(2n + 1)$-dimensional 2^e-torsion lens space— agrees with the geometric dimension of $-(n + 1)\xi_{n,e}$, where $\xi_{n,e}$ is the realification of the canonical complex line bundle over $L^{2n+1}(2^e)$; but on the other hand, $\xi_{n,e}$ is not even a unit in $\mathrm{KO}(L^{2n+1}(2^e))$. We straighten out the situation by following a path, first suggested in [1] by Adem, Gitler and James, which naturally leads to the concept of generalized e-axial maps (to be formalized in Definition (2.2)). Our main result is as follows.

THEOREM (1.2). *If $L^{2n+1}(2^e)$ immerses in Euclidean codimension k, then there is an e-axial map of the form $a\colon S^{2n+1} \times_e P^{2n+1} \to P^{2n+k+1}$. The converse holds except perhaps for $n = 2$, 3 or 5.*

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Keywords and phrases: axial maps, immersions of lens spaces, obstruction theory, modified Postnikov towers.

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ON INCOMPLETENESS OF THE DELETED PRODUCT OBSTRUCTION FOR EMBEDDABILITY

J. MALEŠIČ, D. REPOVŠ AND A. SKOPENKOV

ABSTRACT. Let $\tilde{N} = N \times N \setminus (\Delta N)$, where ΔN denotes the diagonal. The purpose of this paper is to construct counterexamples to the deleted product criterion for embeddability into $\mathbb{R}^m$ for certain dimensions. Two counterexamples are constructed: (1) an example of a 3–dimensional manifold N with boundary which is not embeddable in $\mathbb{R}^3$, but for which there exists an equivariant mapping $\varphi\colon \Sigma\tilde{N} \to \Sigma S^2$ and (2) an example of a closed smooth $4k$–dimensional manifold N which does not smoothly embed into $\mathbb{R}^{6k-1}$, but for which there exists an equivariant mapping $\tilde{N} \to S^{6k-2}$.

1. Introduction

Recall that *the deleted product* of a space N is $\tilde{N} = N \times N \setminus (\Delta N)$, where ΔN is the diagonal. If $f\colon N \hookrightarrow \mathbb{R}^m$ is an embedding, then define the mapping $\tilde{f}\colon \tilde{N} \to S^{m-1}$ by the formula

$$\tilde{f}(x,y) = \frac{f(x)-f(y)}{||f(x)-f(y)||} \tag{1.1}$$

The mapping $\tilde{f}$ is equivariant with respect to the action of $\mathbb{Z}_2$:

- on $\tilde{N}$, acting as the symmetry $(x,y) \to (y,x)$, and
- on the sphere S^{m-1}, taking a point into its antipode.

Consider the following assertion (for PL or DIFF categories):

(*) *For a smooth n-manifold or an n-polyhedron N, if there exists an equivariant map $\tilde{N} \to S^{m-1}$, then N piecewise-linearly or smoothly embeds into* $\mathbb{R}^m$.

The existence of an equivariant map $\tilde{N} \to S^{m-1}$ can be checked for many cases ([2, beginning of §2], [4], [5, 1.7.1], [1, 7.1], [17]). Thus if the assertion (*) is true, the embedding problem is reduced to manageable (although not trivial) algebraic problems. Therefore a problem appeared in the 1960's to find conditions under which the asserion (*) above is true.

The assertion (*) is true for:

- $m = 2, n = 1$ (see [9], [24]);
- $m \geq \frac{3(n+1)}{2}$ (see [4], [23]); and
- $m \geq n + 3$, a PL $(3n - 2m + 2)$-connected closed n-manifold N and in the PL category (see [21], [22]).

However, in general, the assertion (*) above is false when the dimension m is too low. More precisely, it is known to be false in the following cases:

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INTEGRATED DIFFUSION PROCESSES INSIDE RECTANGLES

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ABSTRACT. Let $dX(t) = Y(t)dt$, where $Y(t)$ is a one-dimensional diffusion process. The process $(X(t), Y(t))$ is considered inside a rectangle located in the upper half-plane. The aim is to compute explicitly the probability that the two-dimensional process will exit the rectangle through a given side. Exact expressions are obtained in the cases when $Y(t)$ is a Wiener process, an Ornstein-Uhlenbeck process, a geometric Brownian motion and a Bessel process. The solutions are expressed as generalized Fourier series.

1. Introduction

Let $(X(t), Y(t))$ be a two-dimensional diffusion process defined by the system of stochastic differential equations

$$\begin{aligned} dX(t) &= Y(t)\,dt, \\ dY(t) &= f[Y(t)]\,dt + \{v[Y(t)]\}^{1/2}\,dW(t), \end{aligned} \tag{1.1}$$

where $v(\cdot)$ is nonnegative and $W(t)$ is a standard Brownian motion. That is, $X(t)$ is an integrated diffusion process. First-passage problems for this type of processes have been considered by many authors. For instance, problems for the case when $Y(t)$ is a standard Brownian motion have been treated by [20], [6] and [7], as well as by the author (see [14], in particular), by [19], and especially by Lachal in a series of papers (see, for example, [10] and, more recently, [13]). The author [15] and [11] were also interested in the case when $Y(t)$ is an Ornstein-Uhlenbeck process. Finally, the author [16] and [18] considered stochastic control problems involving first-passage times for integrated geometric Brownian motions and integrated Bessel processes, respectively.

The papers mentioned above are but a small number of the papers dedicated to first-passage problems for integrated diffusion processes. However, the vast majority of these papers deal with first-passage *time* problems; the number of publications on first-passage *place* problems, for diffusion processes in general and especially for integrated diffusion processes, is much more limited. Lachal computed [12], in particular, the probability that the integral of a Brownian motion will exit the interval (a, b) at a (or at b) first.

Now, let

$$\tau(y) := \inf\{t > 0 : Y(t) \notin (b_1, b_2) \mid Y(0) = y \in (b_1, b_2)\}.$$

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Keywords and phrases: hitting place, Brownian motion, Kolmogorov equation, Sturm-Liouville equation

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THE AVERAGE COST OPTIMALITY EQUATION: A FIXED POINT APPROACH

OSCAR VEGA-AMAYA

ABSTRACT. We are concerned with the *expected average cost* optimal control problem for discrete-time Markov control processes with Borel state and action spaces and possibly unbounded costs. We show, under a Lyapunov stability condition and a growth condition on the costs, the existence of a stationary optimal policy using the well-known Banach's fixed point theorem.

1. Introduction

The *expected average cost* (EAC) criterion is among the most studied optimality criteria for discrete-time Markov control processes (MCPs) and there are several approaches to analyze it, for instance, value and policy iteration algorithms, the vanishing discount factor approach, linear programming, the convex analytic approach, etc. (see, for instance, [1], [2], [6], [7],[8], [22], and their extensive bibliographies).

In recent years, some variants of Lyapunov-like stability conditions have been used in several papers to handle the EAC optimal control problem with unbounded costs for MCPs on Borel spaces ([4], [5], [12], [13]) and, more recently, for semi-Markov control processes ([14], [18], [24]), as well as for zero-sum Markov and semi-Markov games ([10], [15], [16], [17], [21]). A key property used in all of these papers is that the imposed stability conditions yield the so-called *weighted geometric ergodicity* for the Markov chains induced by stationary policies. (The weighted geometric ergodicity is a generalization of the standard uniform geometric ergodicity in Markov chain theory; see, [8, Ch. 7] and [19, Ch. 16] for a detailed discussion of these concepts). This fact makes the main difference with our approach since we use "fixed point arguments" and do not need to use, at least explicitly, the W-geometric ergodicity. Fixed point arguments have been used in several previous paper (see, for instance, [6, Lemma 3.5 and Comments 3.7, pp. 59 and 61], [11], [3]), but under a stronger form of the Doeblin condition.

In the present paper, we show the existence of an optimal stationary policy for the EAC control problem with *unbounded costs* for MCPs on *Borel spaces* using a new variant of the Lyapunov condition—Assumption (3.2)—and a growth condition on the costs—Assumption (3.1)—besides a standard continuity/compactness requirements—Assumption (3.8). To do this, we first show that some operator, which is closely related to the *average cost optimality*

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LIMITING AVERAGE COST ADAPTIVE CONTROL PROBLEM FOR TIME-VARYING STOCHASTIC SYSTEMS

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ABSTRACT. We consider a class of time-varying stochastic control systems, with Borel state and action spaces, and possibly unbounded costs. The processes evolve according to a discrete-time equation $x_{n+1} = G_n(x_n, a_n, \xi_n)$, $n = 0, 1, \ldots$, where the ξ_n are i.i.d. $\mathbb{R}^k$-valued random vectors whose common density is unknown, and the G_n are given functions converging, in a restricted way, to some function G_∞ as $n \to \infty$. Assuming observability of ξ_n, we construct an adaptive policy which is average cost optimal for the *limiting control system* $x_{n+1} = G_\infty(x_n, a_n, \xi_n)$.

1. Introduction

We are concerned with a discrete-time, time-varying stochastic control system of the form

$$x_{n+1} = G_n(x_n, a_n, \xi_n), \quad n \in \mathbb{N}_0 := \{0, 1, ...\}, \tag{1.1}$$

where x_n and a_n denote the state and control variables respectively, and $\{\xi_n\}$, the so-called "disturbance" or "driving" process, is a sequence of independent and identically distributed (i.i.d.) random vectors in $\mathbb{R}^k$ having an unknown density ρ. In addition, $\{G_n\}$ is a sequence of given functions converging to some function G_∞ in the following way:

$$E1_B\left[G_n(x, a, \xi_0)\right] \to E1_B\left[G_\infty(x, a, \xi_0)\right] \quad \text{for all } (x, a) \text{ and Borel set } B, \tag{1.2}$$

where $1_B(.)$ denotes the indicator function of the set B (See Assumption (2.4) for more details on this condition). This type of systems appears, for instance, in some time-varying controlled biotechnological processes [1], [13]. We will illustrate the main results of this paper with a generic model of bioreaction.

Assuming that the realizations of the processes $\{\xi_t\}$ and $\{x_t\}$ are completely observable, our main objective is to introduce average cost optimal adaptive policies for the general limiting system

$$x_{t+1} = G_\infty(x_t, a_t, \xi_t), \quad t \in \mathbb{N}_0, \tag{1.3}$$

considering possibly unbounded one-stage costs. This work is motivated by a previous paper [16] which deals with the construction of asymptotically discounted optimal adaptive policies for (1.3). We take advantage of these results for studying the average optimality via the average cost optimality inequality, and using a variant of the well-known vanishing discount factor approach (see

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