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SUR LA CONJECTURE DE REMMERT

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ABSTRACT. This paper is a survey of the main works on the so-called Remmert's conjecture, *if X^n is a closed, homogeneous complex manifold with automorphism group* Aut(X) *then* dim(Aut(X)) $\leq n^2 + 2n$. We describe the structure of a closed, homogeneous complex manifold X, prove Remmert's conjecture for Kähler homogeneous manifolds, then describe the counterexamples constructed by Snow and Winkelman with dim(X) $= 3m + 1$ and dim(Aut(X)) $= 3^m + 3m$, and finally show Akhiezer's theorem (which gives a bound on dim(Aut(X)) for fixed n, being thus a weak version of Remmert's conjecture).

0. Introduction

Si X est une variété complexe, son groupe d'automorphismes Aut(X) peut être un groupe de Lie complexe ($X = \mathbb{C}$), un groupe de Lie réel ($X = \mathbb{H}$), ou un groupe de dimension infinie ($X = \mathbb{C}^2$). Mais si X est *compacte*, Aut(X) est un groupe de Lie de dimension finie, dont l'algèbre de Lie est celle des champs de vecteurs holomorphes sur X [B-M]. À n fixé, le groupe d'automorphismes d'une variété complexe compacte X^n de dimension n peut être de dimension arbitrairement grande. Des exemples simples sont les surfaces de Hirzebruch F_m dont le groupe d'automorphismes est de dimension $m + 5$ (voir exemple (1.8)). Cependant, lorsque X est homogène, c'est-à-dire que tout point peut être envoyé sur n'importe quel autre par un automorphisme de X, Akhiezer [Ak1] a démontré que la dimension de Aut(X) est majorée par une fonction $\delta(n)$ ne dépendant que de la dimension n de X. Si la borne $\delta(n)$ donnée par Akhiezer est très grossière, de l'ordre de $(5n)^{\frac{n(n+2)}{4}}$, elle ne dépasse pas $n(n+2)$ pour beaucoup de classes de variétés complexes compactes homogènes X. Par exemple Borel et Remmert [B-R] ont démontré que si X est une variété kählérienne compacte holomorphiquement homogène, alors Aut(X) est de dimension inférieure à $n(n+2)$, avec égalité si et seulement si X est biholomorphe à l'espace projectif complexe $\mathbb{C}P^n$. C'est donc aussi le cas de toutes les variétés projectives homogènes. D'autres exemples sont les variétés compactes holomorphiquement parallélisables, c'est-à-dire celles dont le fibré tangent est holomorphiquement trivial, qui sont homogènes et dont le groupe d'automorphismes est de même dimension que la variété [Wa]. C'est une ancienne question posée par Remmert que de savoir si parmi les variétés complexes compactes de dimension n, l'espace projectif complexe est celle qui a le plus d'automorphismes.

2000 *Mathematics Subject Classification*: 32M05, 32M10.

Keywords and phrases: Automorphism group acting on complex manifold, homogeneous complex manifold, complex Lie group.

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ON THE LARGEST PRIME FACTOR OF $(ab+1)(ac+1)(bc+1)$

SANTOS HERNÁNDEZ AND FLORIAN LUCA

ABSTRACT. We prove a more general form of a conjecture of Győry, Sárközy and Stewart concerning the largest prime factor of expressions of the form $(ab+1)(ac+1)(bc+1)$ with distinct positive integers $a,\ b,\ c$.

1. Introduction

Recently, Bugeaud, Corvaja, and Zannier (see [2]) proved the following result.

THEOREM (1.1). *Let $a>1$ and $b>1$ be multiplicatively independent positive integers. Then, for every $\varepsilon>0$ there exists a positive integer n_ε such that*

$$\gcd(a^n-1,\ b^n-1)<\exp(\varepsilon n) \tag{1.2}$$

holds for all $n>n_\varepsilon$.

They also mention in their paper that the method they used to prove the above Theorem can also be employed to get non-trivial upper bounds on expressions of the form $\gcd(u_n, v_n)$, where $(u_n)_{n\ge 0}$ and $(v_n)_{n\ge 0}$ are power sums with integer coefficients and positive integer roots (i.e., linear combinations with non-zero integer coefficients of functions of the type $n\mapsto a_i^n$ with a_i positive integers) provided that $(u_n)_{n\ge 0}$ and $(v_n)_{n\ge 0}$ satisfy some mild technical assumptions (such as, for example, that $(u_n)_{n\ge 0}$ admits a *dominant root* and that $(u_n)_{n\ge 0}$ does not divide $(v_n)_{n\ge 0}$ in the ring of such power sums).

In a different direction, but of a somewhat similar flavour, the second author (see [6]) has recently proved that if

$$u_n := ca^n + db^n \qquad \text{for } n=0,1,\ldots \tag{1.3}$$

is such that a,b,c,d are non-zero integers with a and b coprime and a/b and c/d multiplicatively independent, then

$$\gcd(u_n, u_m)<\exp(c_1\sqrt{m}) \tag{1.4}$$

holds for all positive integers $m>n>0$, with an explicit constant c_1 depending on $a,\ b,\ c,\ d$, and a similar type of inequality as (1.4) above is also valid for large $m>n$ for an expression $(u_n)_{n\ge 0}$ as shown in formula (1.3) but with the pair (c,d) replaced by $(c(n), d(n))$, where $c(x)$ and $d(x)$ are polynomials with integer coefficients such that the rational function c/d is not a constant (see [7])..

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Keywords and phrases: subspace theorem, $\mathcal{S}$-units.

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ON THE FLOW OUTSIDE AN UNSTABLE EQUILIBRIUM POINT OR INVARIANT SET

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ABSTRACT. The flow of a dynamical or semidynamical system near an unstable equilibrium point or invariant set is studied in the nontrivial case where instability is not determined by a single solution ("Zubov's condition"). The principal result describes the structure of the prolongation of the point or set in question, improving and extending an older criterion of R. W. Bass.

Introduction

A compact positively invariant set M in the state space X of a dynamical or semidynamical system is stable (in the sense of Lyapunov) if every neighbourhood of M contains a positively invariant neighbourhood of the set. The negation of stability is called instability. There are two ways of characterizing stability and instability. One is using Lyapunov functions (in the case of instability usually called Chetayev functions), the other is by describing the structural characteristics of the flow in the vicinity of the point or set in question. The subject matter of this paper is of the second kind.

In the trivial case (negation of "Zubov's condition" [10], [2]) instability is determined by the existence of a single orbit outside of M with the property that an arbitrary point of it belongs to positive semiorbits with initial points in arbitrarily small neighbourhoods of M. This condition is sufficient for instability, but not necessary, as the example below shows. In general, instability of M can be characterized by the fact that the prolongation of M (in the sense of [7], [3], [6]; see definition in sect. 2) contains M as a proper subset. This holds for differential systems in a finite dimensional Euclidean space [7], or, in general, for dynamical systems in a locally compact metric space [3]. In order to be true for a (semi-)dynamical system in a general metric space, an additional condition must be imposed such as asymptotic compactness, which is the one we adopt in this paper. With the prolongation determining instability, the natural question is what can be said about the structure of the prolongation. This subject was first studied by R. W. Bass in [2], but the characterization given by him has certain limitations. In particular, it does not give an adequate description of certain relatively simple cases such as

$$\dot{x} = xy^2, \quad \dot{y} = -y^3,$$

which represents a semidynamical system (also, a local dynamical system), the origin being unstable though it satisfies Zubov's condition.

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Keywords and phrases: structural characterization of instability.

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THE FOURIER-BESSEL TRANSFORMATION AND THE GELFAND-SHILOV SPACES OF TYPE W

I. MARRERO

ABSTRACT. The even functions in the Gelfand-Shilov spaces of type W are known to determine basic spaces invariant under the Fourier-Bessel transformation. In this paper such spaces are characterized by symmetric decay conditions on their elements and their Fourier-Bessel transforms. As a byproduct, an intrinsic characterization of the even functions in the Gelfand-Shilov spaces of type S is obtained.

1. Introduction and motivation

Let $I = (0, \infty)$ and consider the set

$$K = \left\{M \in C^2[0,\infty) : M(0) = M'(0) = 0,\ M'(\infty) = \infty,\ M''(x) > 0\ (x \in I)\right\}.$$

Interesting properties of this class of functions have been collected in [11] and [7]. For every $M \in K$ we may consider its *Young conjugate* $M^{\times}$, given by

$$M^{\times}(\rho) = \sup_{x\in I}\left(x\rho - M(x)\right) \qquad (\rho \in I)$$

([11], p. 19). Then $M^{\times} \in K$ and $(M^{\times})^{\times} = M$. Also in [11], with the purpose of investigating uniqueness of solutions to the Cauchy problem for partial differential equations, the spaces of type W were defined as follows.

Definitions (1.1). Let $M, \Omega \in K$.
(i) The space W_M consists of all $\varphi \in C^{\infty}(\mathbb{R})$ such that

$$|D^m\varphi(x)| \leq C_m \exp\{-M(a|x|)\} \qquad (m \in \mathbb{N}_0,\ x \in \mathbb{R})$$

for some $C_m, a > 0$.
(ii) The space W^{Ω} consists of all entire functions $\varphi = \varphi(z)$ on $\mathbb{C}$ satisfying

$$\left|z^k\varphi(z)\right| \leq C_k \exp\{\Omega(b\,|\mathrm{Im}\, z|)\} \qquad (k \in \mathbb{N}_0,\ z \in \mathbb{C})$$

for some $C_k, b > 0$.
(iii) The space W_M^{Ω} consists of all entire functions $\varphi = \varphi(z)$ on $\mathbb{C}$ such that

$$|\varphi(z)| \leq C \exp\left\{-M(a\,|\mathrm{Re}\, z|) + \Omega(b\,|\mathrm{Im}\, z|)\right\} \qquad (z \in \mathbb{C})$$

for some $C, a, b > 0$.

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Keywords and phrases: Fourier-Bessel transformation, Hankel transformation, spaces of type W, spaces of type S.

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AN EXTENSION OF THE TOEPLITZ-HAUSDORFF THEOREM

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ABSTRACT. The Toeplitz-Hausdorff Theorem asserts that for any operator A acting on a complex Hilbert space $\mathbb{H}$, the set of numbers of the form $\langle Az, z\rangle$, where z varies over the unit sphere of $\mathbb{H}$, is always a convex subset of $\mathbb{C}$. In this paper we obtain the same result for non-homogeneous quadratic functions of the form $\langle Az, z\rangle + \langle \alpha, z\rangle + \langle z, \beta\rangle + c$. This implies, in particular, that the set of numbers of the form $\langle Az, z\rangle$, where z varies over any sphere in $\mathbb{H}$, centered or not at the origin, is always convex. We also show by an example that the corresponding result is not true for pairs of operators on a real Hilbert space.

1. Introduction

The famous Toeplitz-Hausdorff Theorem states that the numerical range of an operator A is always convex, where the numerical range of A is, by definition, the set of complex numbers of the form $f(z) = \langle Az, z\rangle$, where z varies over the unit sphere of a complex Hilbert space. See the book by Halmos ([2], sections 166 of the first edition, 210 of the second one) for a detailed description and history of this theorem. Several proofs are known, of which the simplest is probably the one given by Halmos himself in the second edition of his book (pp. 314–315).

In this paper we will extend this theorem by showing that the image under f of any sphere in a Hilbert space, centered or not at the origin, is also convex, and that this holds also for a more general type of non-homogeneous quadratic functions. The proof follows the same pattern as Halmos' proof of the Toeplitz-Hausdorff Theorem, but the details are more complicated. It is based on some standard ideas of the theory of singularities of functions on a manifold and on an elementary fact about minima of functions of two variables.

2. Statement of results

Let $\mathbb{H}$ denote a Hilbert space over the set of complex numbers $\mathbb{C}$ with Hermitian product $\langle\ ,\ \rangle$ and let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be a function of the form

$$f(z) = \langle Az, z\rangle + \langle \alpha, z\rangle + \langle z, \beta\rangle + c$$

where A is a linear operator from $\mathbb{H}$ to itself, α, β are elements of $\mathbb{H}$ and c is a complex number. Let S_p^r be the sphere with center at $p \in \mathbb{H}$ and radius r:

$$S_p^r = \{z \in \mathbb{H} \mid \langle z - p, z - p\rangle = r^2\}.$$

THEOREM (2.1). *If $\mathbb{H}$ has dimension greater than 1, then for every $p \in \mathbb{H}$ and $r > 0$, $f(S_p^r)$ is a convex subset of $\mathbb{C}$.*

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Keywords and phrases: numerical range, convexity.

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NULL GENERAL HELICES AND SUBMANIFOLDS

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ABSTRACT. In this paper we give some characterizations for a Cartan framed null curve in the Lorentzian manifold to be a general helix. Also, we prove the following theorem: "Let $M_1(\dim M_1 \geq 3)$ be a Lorentzian submanifold of a pseudo-Riemannian manifold $\overline{M_i}$. If every Cartan framed null general helix in M_1 is also a Cartan framed null general helix in $\overline{M_i}$, then M_1 is a totally geodesic submanifold in $\overline{M_i}$".

1. Introduction

A general helix in Euclidean space $\mathbb{R}^3$ is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 is: *A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant* [5], [8].

The curve is called a circular helix if the curvature and torsion are constant. A characterization for a time-like curve to be a circular helix in a Lorentzian manifold M_1 ($\dim M_1 \geq 3$) is given by T. Ikawa in [4].

Afterwards, this characterization was generalized for a general helix by N. Ekmekci and H. H. Hacısalihoğlu in [2].

2. Preliminaries

A curve in a Lorentzian manifold M_1 is a smooth mapping

$$\gamma : I \to M_1,$$

where I is an open interval in the real line $\mathbb{R}$.

If γ is a space-like or a time-like curve, we can reparametrize it such that $\langle \gamma'(t), \gamma'(t) \rangle = \varepsilon_o$ (where $\varepsilon_o = +1$ if α is space-like, $\varepsilon_o = -1$ if γ is time-like and $\varepsilon_o = 0$ if γ is null (light-like) respectively). In this case γ is said to be unit speed, or it has arc length parametrization [1], [4], [7].

Now let M_α be an n-dimensional pseudo-Riemannian manifold of index α $(0 \leq \alpha \leq n)$ isometrically immersed into an m-dimensional pseudo-Riemannian manifold $\overline{M}_i$ of index i. Then M_α is called a pseudo-Riemannian submanifold of $\overline{M}_i$. If $\alpha = 1$, M_1 is called a Lorentzian submanifold of $\overline{M}_i$. We denote the metrics of M_1 and $\overline{M}_i$ by the symbol $\langle\ ,\ \rangle$ and the covariant differentiation of M_1 (*resp.* $\overline{M}_i$) by D (*resp.* $\overline{D}$). The *Gauss formula* is:

$$\overline{D}_X Y = D_X Y + B(X, Y),$$

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Keywords and phrases: null curves, general helix, Lorentzian manifold, totally geodesic submanifold.

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THE SPACE OF SIZE MAPS IS HOMEOMORPHIC TO THE HILBERT SPACE l_2

ALICJA SAMULEWICZ

ABSTRACT. A theorem about order preserving maps on compact partially ordered spaces is proved that yields the answers to questions, posed by Illanes and Nadler ([7], Question 83.16, p.472 and [7], Questions, p. 458), about the spaces of size maps and Whitney maps on hyperspaces.

All spaces considered in the paper are assumed to be metric separable. By a *continuum* we mean a compact connected space. By 2^X (or $C(X)$) we denote the space of nonempty compact subsets (subcontinua, respectively), equipped with the Hausdorff metric. By a *hyperspace* of X we mean a compact $\mathcal{H}$ such that $C(X) \subseteq \mathcal{H} \subseteq 2^X$. A *partially ordered space* is a topological space P endowed with a partial order $\leqslant$ whose graph is a closed subset of $P \times P$. $\operatorname{Min} P$ and $\operatorname{Max} P$ denote the sets of minimal and maximal elements of P, respectively. It is known that, given a continuum X, 2^X and $C(X)$ are continua and partially ordered spaces with respect to the inclusion, thus every hyperspace $\mathcal{H}$ of X is partially ordered. A *size map* for a partially ordered compact space P is a continuous order-preserving function $\sigma : P \to [0, \infty)$ with $\sigma(\operatorname{Min} P) = \{0\}$ and $\sigma(\operatorname{Max} P) = \{\max \sigma(P)\}$. A *Whitney map* for P is a size map for P such that for every $p, q \in P$ the conditions $p \leqslant q$ and $p \neq q$ imply $\sigma(p) < \sigma(q)$. We use the following notation:

$$S(P) = \{\sigma : P \to [0, \infty) : \sigma \text{ is a size map for } P\},$$

$$S_1(P) = \{\sigma \in S(P) : \sigma(\operatorname{Max} P) = \{1\}\},$$

$$W(P) = \{\sigma \colon P \to [0, \infty) \colon \sigma \text{ is a Whitney map for } P\},$$

$$W_1(P) = \{\sigma \in W(P) \colon \sigma(\operatorname{Max} P) = \{1\}\}.$$

We consider $S(P)$, $S_1(P)$, $W(P)$ and $W_1(P)$ as subspaces of the function space $\mathcal{C}(P, \mathbb{R})$ consisting of all continuous functions from P to $\mathbb{R}$ with the "sup metric" induced by the "sup norm" $\| \ \|$. If P is a compact partially ordered space such that $\operatorname{Min} P$ and $\operatorname{Max} P$ are disjoint closed sets then $W_1(P) \subset S_1(P)$ is nonempty (see [9], Theorem 2.1).

LEMMA (0.1). *Let P be a compact, partially ordered space. Then*
($*$) *$W(P)$ is homeomorphic to the space $W_1(P) \times (0, \infty)$;*
($**$) *$S(P)$ is homeomorphic to the open cone over $S_1(P)$, i.e., the quotient space $(S_1(P) \times [0, \infty))/(S_1(P) \times \{0\})$.*

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Keywords and phrases: continuum, Hilbert space l_2, hyperspace of subcontinua, size map, Whitney map.

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ON SOME GENERALIZATIONS OF COMPACTNESS IN SPACES $C_p(X, 2)$ AND $C_p(X, \mathbb{Z})$

A. CONTRERAS-CARRETO AND A. TAMARIZ-MASCARÚA

ABSTRACT. We discuss topological properties of a space X which imply that the spaces $C_p(X, 2)$ and $C_p(X, \mathbb{Z})$ have properties similar to compactness, such as σ-compactness and σ-countable compactness. In particular, for a zero-dimensional space X, we prove: (1) X is normal and $C_p(X, 2)$ is σ-compact iff X is an Eberlein-Grothendieck space and the set of non-isolated points in X is Eberlein compact, and (2) $C_p(X, \mathbb{Z})$ is σ-compact iff X is an Eberlein compact space.

1. Introduction

All spaces are assumed to be Tychonoff unless otherwise stated. Given two spaces X and Y, we denote by $C(X, Y)$ the set of all continuous functions from X to Y, and $C_p(X, Y)$ is the set $C(X, Y)$ equipped with the topology of pointwise convergence (that is, the topology inherited by $C(X, Y)$ as a subspace of the space Y^X of all functions from X to Y with the Tychonoff product topology). The space $C_p(X, \mathbb{R})$ is denoted as $C_p(X)$, and $C_p^*(X)$ stands for the subset of all bounded elements in $C_p(X)$. For points $x_0, \ldots, x_n$ in X and subsets $A_1, \ldots, A_n$ of Y, we will denote by $[x_0, \ldots, x_n; A_0, \ldots, A_n]$ the subset of Y^X of those functions f such that $f(x_i) \in A_i$ for every $i \in \{0, \ldots, n\}$.

The symbols $\mathbb{R}$, I, ω, $\mathbb{Z}$ and 2 stand for the real line, interval $[0, 1]$, the natural numbers, the discrete group of the integer numbers and the discrete group $\{0, 1\}$, respectively. The letters t, n, m, k will denote natural numbers; and if t is a natural number, we will use the same symbol t to denote the discrete space of cardinality t. For topological spaces X and Y, the symbol $X \cong Y$ means that X and Y are homeomorphic. The space $\beta(\omega)$ is the Stone-Čech compactification of the natural numbers, and ω^* is equal to $\beta(\omega) \setminus \omega$. If $\mathcal{P}$ is a topological property, then a space X is σ-$\mathcal{P}$ if X is the countable union of subspaces satisfying $\mathcal{P}$. A space X is a *P-space* if the intersection of a countable family of open subsets of X is still an open set. A subspace Y of X is *bounded* in X if for every $f \in C(X)$, $f \restriction Y$ is a bounded function, or equivalently, if every sequence of open sets in X, which meets Y, has an accumulation point in X. A subspace Y of a space X is C^*-*embedded* in X if for every $f \in C^*(Y)$ there is $g \in C^*(X)$ such that $g \restriction Y = f$. A space X is ω-*discrete* if every subset Y of X of cardinality $\leq \aleph_0$ is discrete; and X is *b-discrete* if every subset Y of X of cardinality $\leq \aleph_0$ is discrete and C^*-embedded in X.

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Keywords and phrases: spaces of continuous functions; C_α-compact spaces; α-pseudocompact space; ultracompact spaces; pseudocompact spaces; *sk*-directed properties; Eberlein-Grothendieck space; Eberlein compact space.

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COUNTABLY COMPACT GROUPS AND p-LIMITS

S. GARCIA-FERREIRA AND A. H. TOMITA

ABSTRACT. For $\emptyset \neq M \subseteq \omega^*$, a space X is said to be quasi M-compact, if for every sequence $(x_n)_{n<\omega}$ in X there are $p \in M$ and $x \in X$ such that for every neighborhood V of x in X, $\{n < \omega : x_n \in V\} \in p$. This concept strengthens countable compactness. Assuming $\mathfrak{p} = \mathfrak{c}$, we construct a selective ultrafilter $p \in \omega^*$ and a quasi $T(p)$-compact topological group G whose square is not countably compact, where $T(p)$ is the type of p in ω^*. We also construct, via forcing, a countably compact group which is not quasi M-compact for any $M \in [\omega^*]^{<2^{\mathfrak{c}}}$; and a family of topological groups $\{G_\alpha : \alpha < 2^{\mathfrak{c}}\}$ such that for a subset I of $2^{\mathfrak{c}}$, $\prod_{\alpha \in I} G_\alpha$ is countably compact if and only if $|I| < 2^{\mathfrak{c}}$.

0. Introduction

In this paper, the spaces are Tychonoff, and the topological groups are Hausdorff (hence, they are Tychonoff). $\beta(\omega)$ is identified with the set of all ultrafilters on ω, and $\omega^* = \beta(\omega) \setminus \omega$ with the set of all free ultrafilters on ω. The Rudin-Keisler pre-ordering on ω^* is defined as follows: For $p, q \in \omega^*$, $p \leq_{RK} q$ if there is a function $f : \omega \longrightarrow \omega$ such that $\{f^{-1}(A) : A \in q\} \subseteq p$, and $p \approx q$ means that $p \leq_{RK} q$ and $q \leq_{RK} p$. The type of $p \in \omega^*$ is the set $T(p) = \{q \in \omega^* : p \approx q\}$. If $f, g : \omega \to \omega$ are two functions, then $f \leq^* g$ will mean that there is $k < \omega$ such that $f(n) \leq g(n)$, for every $k \leq n < \omega$. For the definition of the cardinal numbers $\mathfrak{b}$ and $\mathfrak{p}$ the reader is referred to [Va].

The following concept was introduced by A. R. Bernstein [B].

Definition (0.1). [B] Let $p \in \omega^*$ and let $(x_n)_{n<\omega}$ be a sequence in a space X. We say that x is a $p-$limit point of $(x_n)_{n<\omega}$ and we write $x = p - \lim_{n \to \omega} x_n$, if for every neighborhood V of x, $\{n < \omega : x_n \in V\} \in p$.

It is not difficult to see that a space X is countably compact iff every sequence of points in X has a $p-$limit point for some $p \in \omega^*$. Thus it is natural to consider the following class of spaces (also introduced by A. R. Bernstein):

Definition (0.2). [B] Let $p \in \omega^*$. A space X is said to be $p-$compact if for every sequence $(x_n)_{n<\omega}$ of points of X there is $x \in X$ such that $x = p-\lim_{n \to \omega} x_n$.

The authors of [GS] proved that all powers of a space X are countably compact iff there is $p \in \omega^*$ such that X is $p-$compact. As $p-$compactness is preserved under arbitrary products, for every $p \in \omega^*$ ([B]), there are countably compact spaces which are not $p-$compact for any $p \in \omega^*$ (see [GJ]). The main notion of this paper is the following:

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Keywords and phrases: p-limit, p-compact, almost p-compact, quasi M-compact, countably compact, topological group.

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RATIONAL FUNCTION SPACES

KUNG-KUEN TSE

ABSTRACT. We calculate the size of the rationalization of the function space Map(X, E) for E being the total space of a principal fibration induced by a map between two loop spaces. We then discuss the rational homotopy types among the path components of the function space with $\mathbb{C}P^n$ as the target space.

1. Introduction

One of the central problems in homotopy theory is to classify the maps from a topological space X to another topological space Y up to homotopy.

The problem of the homotopy classification of the function (mapping) space Map(X, Y) is the problem of finding the path components of Map(X, Y). In general, $[X, Y]$, the set of path components of Map(X, Y), has more than one element. In this case, if we are able to describe $[X, Y]$, then a finer and more interesting problem is to describe the space of all homotopic functions to a given one, f, namely the path component, $\mathrm{Map}_f(X, Y)$, of Map(X, Y) containing f.

When Y is an H-group (a group up to homotopy), then the path components are all of the same homotopy types (the homotopy inverse of Y gives the homotopy equivalence). But when Y is not an H-group, then in general, the path components may be of different homotopy types.

The homotopy types of the path components of Map(X, Y) were studied in [9], for the case X being an n-dimensional sphere. Later, Federer in [5] described a spectral sequence for computing the homotopy groups of the path components of the function space. Then, Thom in [14] studied the function space, for the case Y being an Eilenberg-Mac Lane space. More recently, Hansen in [8] studied the case X being a closed connected oriented n-dimensional manifold and Y being an n-dimensional sphere.

The rational homotopy of the path components of Map(X, Y) has been studied by Møller and Raussen [10], for the case X and Y being complex projective spaces. Recently, S. Smith [11] considered the case $X = Y$.

Federer's spectral sequence collapses for the rationalization of the path component containing the constant map, and the homotopy groups can be expressed in terms of the cohomology groups of X and the homotopy groups of Y. For the other components, the computation of the homotopy groups is not as easy. All we know is they are, in each grading, sub-vector spaces of the homotopy group of the path component containing the constant map.

The objective of this paper is two-fold. First, to study the number of path components of Map(X, Y), when Y is the pullback of a map between two loop

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SOME NEW IMMERSIONS AND NONIMMERSIONS OF 2^r-TORSION LENS SPACES

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ABSTRACT. We obtain new results, four of which are optimal, for the immersion problem for 2^r-torsion lens spaces. We do this using obstruction theory to prove the possibility or impossibility of lifting the map which classifies the stable normal bundle over the lens space $L^{2n+1}(2^r)$ to a map $L^{2n+1}(2^r) \longrightarrow BO(k)$, for the relevant values of n, r, and k. These results compare nicely with recent work of J. González in [11].

1. Statement of results

For positive integers n and r, let $L^{2n+1}(2^r)$ be the $(2n+1)$-dimensional 2^r-torsion lens space, i.e., the orbit space of $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ by the restriction to $\mathbb{Z}_{2^r}$ of the diagonal action of S^1. Let $\mathbb{R}P^{2n+1}$ be $L^{2n+1}(2)$, the $(2n+1)$-dimensional real projective space, and let $\mathbb{C}P^n$ be the $2n$-dimensional complex projective space. In [11] González shows that if $\mathbb{C}P^n$ immerses in $\mathbb{R}^l$, then $L^{2n+1}(2^r)$ immerses in $\mathbb{R}^{l+1}$, and if $L^{2n+1}(2^{r+1})$ immerses in $\mathbb{R}^l$, then $L^{2n+1}(2^r)$ immerses in $\mathbb{R}^l$. Thus, known results of immersions of $\mathbb{C}P^n$ (see [3] and [5]) and nonimmersions of $\mathbb{R}P^{2n+1}$ (see [4]) yield immersion and nonimmersion results for $L^{2n+1}(2^r)$. We use obstruction theory involving calculations with modified Postnikov towers (MPTs) to improve upon some of these results or González's result in [11] that if $r \geq \alpha(n)$, the number of 1's in the binary expansion of n, then $L^{2n+1}(2^r)$ does not immerse in $\mathbb{R}^{4n-2\alpha(n)}$.

Our new results are the following:

THEOREM (1.1). *If n is odd and $\alpha(n) = 2$, then $L^{2n+1}(4)$ immerses in $\mathbb{R}^{4n-2\alpha(n)+1}$ and, for $r \geq 3$, $L^{2n+1}(2^r)$ does not immerse in $\mathbb{R}^{4n-2\alpha(n)+1}$.*

THEOREM (1.2). *If n is even, $\alpha(n) = 2$ and $r \geq 2$, then $L^{2n+1}(2^r)$ does not immerse in $\mathbb{R}^{4n-2\alpha(n)+2}$.*

COROLLARY (1.3). *If n is odd, $\alpha(n) = 3$ and $r \geq 2$, then $L^{2n+1}(2^r)$ does not immerse in $\mathbb{R}^{4n-2\alpha(n)}$.*

THEOREM (1.4). *If $n \equiv 2 \bmod 4$, $\alpha(n) = 3$ and $r \geq 2$, then $L^{2n+1}(2^r)$ does not immerse in $\mathbb{R}^{4n-2\alpha(n)}$.*

THEOREM (1.5). *If n is even and $\alpha(n) = 3$, then $L^{2n+1}(4)$ immerses in $\mathbb{R}^{4n-2\alpha(n)+2}$.*

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